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Gegenbauer polynomials and the Fueter theorem

David Eelbode[a*], V. Souček[b†] and P. Van Lancker[c]

[a] *Antwerp University, Dept. of Mathematics and Computer Science Middelheimlaan 1, 2020 Antwerp, Belgium*;
[b] *Mathematics Institute of Charles University Sokolovská 83, 186 75 Prague 8, Czech Republic*;
[c] *University College Ghent (Member of Ghent University Association) Dept. of Engineering Sciences Schoonmeersstraat 52, 9000 Ghent, Belgium*

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The Fueter theorem states that regular (resp. monogenic) functions in quaternionic (resp. Clifford) analysis can be constructed from holomorphic functions $f(z)$ in the complex plane, hereby using a combination of a formal substitution and the action of an appropriate power of the Laplace operator. In this paper we interpret this theorem on the level of representation theory, as an intertwining map between certain $\mathfrak{sl}(2)$-modules.

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1. Introduction

The original Fueter theorem ([8]) is a classical result in quaternionic analysis, which tells you how to obtain regular functions (solutions for a generalised Cauchy-Riemann operator), starting from holomorphic functions $f(z)$ in the complex plane $\mathbb{C}$. This result has later been generalized, on several occasions, within the setting of Clifford analysis (far from claiming completeness, we e.g. refer to [10, 13, 16]). This is a branch of classical analysis in which null solutions for spin-invariant differential operators are studied from a function theoretical point of view. The main object of study in this subdomain of classical analysis is the Dirac operator, see e.g. [3, 4, 9]. This first-order operator is the unique (up to a constant) conformally invariant operator acting on Clifford algebra-valued functions $f(x)$ on the Euclidean space $\mathbb{R}^m$. This (real or complex) Clifford algebra ($\mathbb{R}_m$ or $\mathbb{C}_m$) is the associative algebra generated by an orthonormal basis $\{e_1, \cdots, e_m\}$ for $\mathbb{R}^m$, endowed with the multiplication rules $e_pe_q + e_qe_p = -2\delta_{pq}$. The generalized Fueter theorem then yields a particular method to construct null solutions (monogenics) for this Dirac operator, starting from holomorphic functions. Note that the Dirac operator $\partial_x = \sum_j e_j\partial_{x_j}$ and the vector variable $x \in \mathbb{R}^m$ can be seen as the (odd) generators of the Lie

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*Corresponding author. Email: david.eelbode@ua.ac.be
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superalgebra $\mathfrak{osp}(1,2)$, a crucial fact which allows to interpret results in Clifford analysis as concrete realisations for more abstract properties about representations for the orthosymplectic Lie superalgebra.
The aim of this paper is to show how the Fueter theorem, and some generalisations thereof, can be understood in terms of particular properties of Gegenbauer polynomials on $\mathbb{R}^m$. These polynomials play a crucial role in the representation theory for the spin group $\mathrm{Spin}(m)$, in particular within the setting of branching rules and axially monogenic polynomials on $\mathbb{R}^m$, which means that we will find an alternative interpretation for Fueter's theorem using classical representation theoretical techniques. This will lead to a sharper result than the one that has appeared in the literature. Moreover, the methods used in the present paper can be generalised to more complicated settings, such as for Dunkl operators or higher spin versions of the classical Dirac operator.

2. Gegenbauer polynomials and series expansions

First of all we list some standard properties of Gegenbauer polynomials which will be useful in the sequel. Let $\Re(\lambda) > -\frac{1}{2}$. The Gegenbauer polynomial $C_k^\lambda(t)$ of degree k can be defined by means of the following generating function:

$$(1-2tz+z^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(t) z^k\,, \quad |z|<1\,,\ t\in[-1,1]\,.$$

A more explicit expression in terms of the hypergeometric ${}_2F_1$-function is given by:

$$C_k^\lambda(t) = \frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)} F(-k, k+2\lambda; \lambda+\frac{1}{2}, \frac{1-t}{2})\,. \tag{1}$$

From the definition via the generating function it is clear that $C_k^0(t) = 0$ for $k \geq 1$ and that Gegenbauer polynomials satisfy the parity condition

$$C_k^\lambda(-t) = (-1)^k C_k^\lambda(t)\ .$$

Let us then introduce spherical coordinates $(r,\omega) \in \mathbb{R}^+ \times S^{m-1}$ on the vector space $\mathbb{R}^m$. For two vectors $x, y \in \mathbb{R}^m$ with $x = |x|\omega$, $y = |y|\xi$ and $\omega, \xi \in S^{m-1}$, one has:

$$|x-y|^{-2\lambda} = \left(|x|^2 - 2\langle x,y\rangle + |y|^2\right)^{-\lambda} = \left(|x|^2 - 2|x||y|\langle\omega,\xi\rangle + |y|^2\right)^{-\lambda}\ .$$

We thus obtain two expansions:

$$|x-y|^{-2\lambda} = \begin{cases} |x|^{-2\lambda} \displaystyle\sum_{k=0}^{\infty} C_k^\lambda(\langle\omega,\xi\rangle) \left(\frac{|y|}{|x|}\right)^k & |y|<|x|\,, \\ |y|^{-2\lambda} \displaystyle\sum_{k=0}^{\infty} C_k^\lambda(\langle\omega,\xi\rangle) \left(\frac{|x|}{|y|}\right)^k & |x|<|y|\,. \end{cases} \tag{2}$$

Definition 2.1: For $\Re(\lambda) > -\frac{1}{2}$ and $k \in \mathbb{N}_0$ we define the following function:

$$K_k^\lambda(\omega,\xi) = C_k^\lambda(\langle\omega,\xi\rangle) + \omega\xi\, C_{k-1}^\lambda(\langle\omega,\xi\rangle) \in C^\infty(S^{m-1}\times S^{m-1})\,. \tag{3}$$

If $k=0$, then $K_0^\lambda(\omega,\xi)=C_0^\lambda(\langle\omega,\xi\rangle)=1$. The Gegenbauer polynomial $C_k^\lambda(t)$ is an even (odd) polynomial of t if k is even (odd). Therefore $\omega\mapsto K_k^\lambda(\omega,\xi)$ admits a k-homogeneous polynomial extension to $\mathbb{R}^m$ which we will denote as:

$$K_k^\lambda(x,\xi)=|x|^k K_k^\lambda(\omega,\xi)\in\mathcal{P}(\mathbb{R}^m)\otimes C^\infty(S^{m-1})\,. \tag{4}$$

Remark that for $\lambda=0$, we get:

$$K_k^0(\omega,\xi)=\begin{cases}1\,, & k=0\\ \omega\xi\,, & k=1\\ 0\,, & k\geq 2\end{cases}\,. \tag{5}$$

Clearly $K_k^\lambda(x,\xi)$ is of the form $f_k(|x|^2,\langle x,\xi\rangle)+x\,\xi\,f_{k-1}(|x|^2,\langle x,\xi\rangle)$ where $f_k(u,v)$ is a polynomial in the real variables u and v. Multiplying the identities in (2) with x and y we obtain the expansions:

$$\frac{x-y}{|x-y|^{2\lambda}}=\begin{cases}\dfrac{x}{|x|^{2\lambda}}\displaystyle\sum_{k=0}^{\infty}K_k^\lambda(\omega,\xi)\left(\frac{|y|}{|x|}\right)^k & |y|<|x|\,,\\ -\dfrac{y}{|y|^{2\lambda}}\displaystyle\sum_{k=0}^{\infty}K_k^\lambda(\xi,\omega)\left(\frac{|x|}{|y|}\right)^k & |x|<|y|\,,\\ -\left(\displaystyle\sum_{k=0}^{\infty}K_k^\lambda(\omega,\xi)\left(\frac{|x|}{|y|}\right)^k\right)\dfrac{y}{|y|^{2\lambda}} & |x|<|y|\,,\end{cases} \tag{6}$$

For $\lambda=\frac{m}{2}$, the left hand side is (up to a constant multiple) the Cauchy kernel of the Dirac operator ∂_x in $\mathbb{R}^m$. The series expansion of the Cauchy kernel, being a basic result in Clifford analysis of the Dirac operator ∂_x in $\mathbb{R}^m$, appears at numerous places in the literature (see e.g. [3, 4]).

The expansions (2) can be rewritten in a more symmetric way as

$$\left(1-2\langle x,y\rangle+|x|^2|y|^2\right)^{-\lambda}=|1+xy|^{-2\lambda}=\sum_{k=0}^{\infty}C_k^\lambda(\langle\omega,\xi\rangle)\,(|x||y|)^k\quad(|x||y|<1)\,.$$

Hereby $|1+xy|^2$ has to be understood as the Clifford norm of $(1+xy)\in\mathbb{R}_m$:

$$|1+xy|^2=|1-\langle x,y\rangle+x\wedge y|^2=(1-\langle x,y\rangle)^2+|x|^2|y|^2-\langle x,y\rangle^2\,.$$

In a similar way:

$$\begin{aligned}\frac{1+xy}{|1+xy|^{2\lambda}}&=C_0^\lambda(\langle\omega,\xi\rangle)+\sum_{k=1}^{\infty}\Big(C_k^\lambda(\langle\omega,\xi\rangle)+\omega\xi\,C_{k-1}^\lambda(\langle\omega,\xi\rangle)\Big)\,(|x||y|)^k\\&=\sum_{k=0}^{\infty}K_k^\lambda(\omega,\xi)\,(|x||y|)^k\,,\quad|x||y|<1\,.\end{aligned} \tag{7}$$

Definition 2.2: Let $\alpha\in\mathbb{C}$. Define the inversion operator $\mathcal{I}_\alpha$ on functions $f(x)$:

$$(\mathcal{I}_\alpha f)(x)=\frac{x}{|x|^{m+\alpha}}f\Big(\frac{x}{|x|^2}\Big)\,.$$

Obviously $\mathcal{I}_\alpha^2 = -\mathrm{id}$ and $\mathcal{I} := \mathcal{I}_0$ is the (conformal) inversion operator mapping monogenic functions to monogenic functions. In particular, the inversion $\mathcal{I}[1]$ of the constant function $f = 1$ yields (up to a multiple) the Cauchy kernel (with singularity in the origin) for the Dirac operator ∂_x in $\mathbb{R}^m$. In the same way, for $k \in \mathbb{N}$ the inversion $\mathcal{I}_{-2k}[1]$ of $f = 1$ gives rise to (a multiple of) the fundamental solution $|x|^{2k}\frac{x}{|x|^m}$ of ∂_x^{2k+1}.

Let $K_{k,\xi}^\lambda(x) = K_k^\lambda(x,\xi)$. This then gives:

$$(\mathcal{I}_{-m+2\lambda}K_{k,\xi}^\lambda)(x) = \frac{x}{|x|^{2\lambda}}K_{k,\xi}^\lambda(\frac{x}{|x|^2}) = \frac{x}{|x|^{2\lambda+k}}K_{k,\xi}^\lambda(\omega)\,.$$

Hence for $x, y \in \mathbb{R}^m$ with $y = |y|\xi,\ \xi \in S^{m-1}$:

$$\frac{x-y}{|x-y|^{2\lambda}} = \sum_{k=0}^{\infty}(\mathcal{I}_{-m+2\lambda}K_{k,\xi}^\lambda)(x)|y|^k \quad (|y| < |x|)\ .$$

Lemma 2.3: *Let $2l + \Re(\alpha) > -m-1$ and let $P_l(\underline{x})$ be an arbitrary function on $\mathbb{R}_0^{m-1}$ which is homogeneous of degree $l \in \mathbb{R}$. Then*

$$K_k^{l+\frac{m+\alpha}{2}}(x,e_1)P_l(\underline{x}) = \frac{1}{k!}\left(\mathcal{I}_\alpha\partial_{x_1}\mathcal{I}_\alpha\right)^k P_l(\underline{x})\,. \tag{8}$$

In particular we get that

$$K_k^{\frac{m+\alpha}{2}}(x,e_1) = \frac{1}{k!}\left(\mathcal{I}_\alpha\partial_{x_1}\mathcal{I}_\alpha\right)^k[1]\ . \tag{9}$$

Note that relation (8) is also valid if $l + \frac{m+\alpha}{2} = 0$, in which case $K_k^0(x,e_1)$ is given by (5); consequently we have that $\left(\mathcal{I}_{-(2l+m)}\partial_{x_1}\mathcal{I}_{-(2l+m)}\right)^k P_l(\underline{x}) = 0$ for $k \geq 2$.

Proof: For $y = te_1$ with $t \in \mathbb{R}$ we have the following power series in t:

$$\begin{aligned}\frac{x-te_1}{|x-te_1|^{m+\alpha+2l}}P_l(\underline{x}) &= \Big(\sum_{k=0}^{\infty}\big(\mathcal{I}_{\alpha+2l}K_k^{l+\frac{m+\alpha}{2}}\big)(x,e_1)t^k\Big)P_l(\underline{x})\,, \quad t<|x| \\ &= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\partial_{x_1}^k\left(\mathcal{I}_\alpha P_l(\underline{x})\right)\,, \quad t<|x|\ .\end{aligned}$$

Uniqueness of the Taylor series therefore implies:

$$\begin{aligned}\frac{(-1)^k}{k!}\partial_{x_1}^k\left(\mathcal{I}_\alpha[P_l(\underline{x})]\right) &= \frac{(-1)^k}{k!}\partial_{x_1}^k\left(\frac{x}{|x|^{m+\alpha+2l}}\right)P_l(\underline{x}) \\ &= \Big((\mathcal{I}_{\alpha+2l}K_k^\lambda)(x,e_1)\Big)P_l(\underline{x}) \\ &= \mathcal{I}_\alpha\Big(K_k^\lambda(x,e_1)P_l(\underline{x})\Big)\ ,\end{aligned}$$

where $\lambda = l + \frac{m+\alpha}{2}$. Since $(\mathcal{I}_\alpha)^2 = -\mathrm{id}$, the statement easily follows. $\square$

3. $\mathfrak{sl}(2)$-actions in Clifford analysis

In this section we introduce a particular realisation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, in terms of the inversion operator $\mathcal{I}_\alpha$. For $\alpha = 0$, we then get a subalgebra of the full conformal symmetry algebra $\mathfrak{so}(1, m+1)$ of the Dirac equation. This subalgebra generates, through repeated action of the creation operator on the scalar $1 \in \mathbb{R}$, the Gegenbauer polynomials from the previous section (see lemma 2.3). Note that this lemma is actually more general, in the sense that it creates *all* axially monogenic polynomials, see also [3, 11].

Lemma 3.1: *Let $\alpha \in \mathbb{C}$. The maps $\pi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}(\mathcal{P}(\mathbb{R}^m))$ defined by*

$$\begin{aligned}
\pi_\alpha(H) &= 2\mathbb{E}_x + \alpha + m - 1\\
\pi_\alpha(E_+) &= \mathcal{I}_\alpha \partial_{x_1} \mathcal{I}_\alpha = -|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + \alpha + m - 1) - e_1 \wedge x\\
\pi_\alpha(E_-) &= -\partial_{x_1}
\end{aligned}$$

define a one-parameter family of Lie algebra homomorphisms.

Proof: Let us fix the notation $\mathcal{I} := \mathcal{I}_0$ in what follows. First of all, one has the fundamental relation $\mathcal{I}\partial_x\mathcal{I} = |x|^2\partial_x$, which follows from the conformal invariance of the Dirac operator. Since $2\partial_{x_1} = -\{e_1, \partial_x\}$, it is clear that $2\mathcal{I}\partial_{x_1}\mathcal{I} = \{\mathcal{I}e_1\mathcal{I}, \mathcal{I}\partial_x\mathcal{I}\}$. Using the notation e_1 for the multiplication operator $f(x) \mapsto e_1 f(x)$, we get:

$$\mathcal{I}e_1\mathcal{I} = \frac{xe_1x}{|x|^2} = e_1 - 2\frac{x_1x}{|x|^2}\,.$$

Invoking the identities: $\{x, \partial_x\} = -m - 2\mathbb{E}_x$ and $|x|^2\partial_x|x|^{-2} = -2x|x|^{-2} + \partial_x$, we obtain:

$$\begin{aligned}
\mathcal{I}\partial_{x_1}\mathcal{I} &= \frac{1}{2}\Big\{e_1 - 2\frac{x_1x}{|x|^2}, |x|^2\partial_x\Big\}\\
&= -|x|^2\partial_{x_1} - x_1x\partial_x - |x|^2\partial_x|x|^{-2}x_1x\\
&= -|x|^2\partial_{x_1} - x_1x\partial_x - 2x_1 - \partial_x x_1 x\\
&= -|x|^2\partial_{x_1} - x_1(x\partial_x + \partial_x x) - 2x_1 - e_1x\\
&= -|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + m - 1) - e_1 \wedge x\,.
\end{aligned}$$

Noting that $|x|^{-\alpha}|x|^2\partial_x|x|^\alpha = |x|^2\partial_x + \alpha x_1$ we also get:

$$\begin{aligned}
\mathcal{I}_\alpha\partial_{x_1}\mathcal{I}_\alpha &= |x|^{-\alpha}\mathcal{I}\partial_{x_1}\mathcal{I}|x|^\alpha\\
&= |x|^{-\alpha}\big(-|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + m - 1) - e_1 \wedge x\big)|x|^\alpha\\
&= -|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + \alpha + m - 1) - e_1 \wedge x\,.
\end{aligned}$$

The $\mathfrak{sl}(2,\mathbb{R})$-relations with the shifted Euler operator are obvious and

$$\big[\partial_{x_1}, x_1(2\mathbb{E}_x + \alpha + m - 1) - |x|^2\partial_{x_1} - e_1 \wedge x\big] = -2x_1\partial_{x_1} + (2\mathbb{E}_x + \alpha + m - 1) + 2x_1\partial_{x_1}\,,$$

which proves the statement. □

As was noted in a series of recent papers, see e.g. [2, 6], the operators obtained through the Lie algebra homomorphism can be used to obtain explicit expressions for the embedding factors appearing in the branching problem for harmonic (resp. monogenic) polynomials.

4. Intertwining relations

In this section, we prove a few (rather technical) operator identities which will allow us to consider intertwining maps between certain $\mathfrak{sl}(2)$-modules in the following section. We first introduce some definitions and notations. Consider the H-action of $s \in \mathrm{Spin}(m)$ on $P(x) \in \mathcal{P}(\mathbb{R}^m)$ given by $H(s)P(x) = P(s^{-1}xs)$. This corresponds to the standard action of $\mathrm{SO}(m)$ on polynomials. The derived action of H then gives rise to the Lie algebra $\mathfrak{so}(m)$ generated by the (angular momentum) operators $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$, with $i, j = 1, \ldots, m$ and $i \neq j$. The Gamma operator Γ_x can now be defined as the operator

$$\Gamma_x := -\big[x \wedge \partial_x\big]_2 = -\sum_{i<j} e_{ij}\big(x_i\partial_{x_j} - x_j\partial_{x_i}\big) = -\sum_{i<j} e_{ij}L_{ij}\ ,$$

with $[\cdot]_2$ denoting the bi-vectorial part (see e.g. [4]). In terms of polar coordinates $x = r\omega$, with $\omega \in S^{m-1}$, the Dirac operator can now be written as

$$\partial_x = \frac{1}{r}\omega(\mathbb{E}_x + \Gamma_x) \quad \text{or} \quad x\partial_x = -\mathbb{E}_x - \Gamma_x\ .$$

The Laplace Beltrami operator Δ_{LB} on the sphere S^{m-1} is defined as the operator $\Delta_{LB} := \sum_{i<j} L_{ij}^2$ and can also be expressed in terms of the Gamma operator:

$$\Delta_{LB} = |x|^2\Delta - \mathbb{E}_x(\mathbb{E}_x + m - 2) = \Gamma_x(-\Gamma_x + m - 2)\,.$$

As was mentioned in the introduction, the Dirac operator ∂_x and the vector variable x generate the Lie superalgebra $\mathfrak{osp}(1|2)$. The Scasimir operator (in one vector variable) is the first order differential operator $Sc \in \mathcal{U}(\mathfrak{osp}(1,2))$, defined by

$$Sc := \frac{1}{2}[x, \partial_x] - \frac{1}{2} = \frac{1}{2}(x\,\partial_x - \partial_x x) - \frac{1}{2} = \frac{m-1}{2} - \Gamma_x\,.$$

Because of the extra constant in the definition of Sc, this operator has better intertwining properties than the standard Gamma operator, a fact that was fully exploited in [7] in order to define higher spin operators exhibiting full $\mathfrak{osp}(1,2)$-symmetry. The distinguishing feature here is that Sc has the following properties: $\{Sc, x\} = \{Sc, \partial_x\} = 0$. This shows that Sc anti-commutes with the odd part and commutes with the even part of $\mathfrak{osp}(1,2)$, see also [1] where the Scasimir operator for the more general Lie superalgebra $\mathfrak{osp}(1,2k)$ is considered. In the sequel we will frequently use the following short-hand notation (dropping the subscript x):

$$\mathbb{E}' := \mathbb{E}_x + \frac{m-1}{2}\,, \quad \Gamma' := \Gamma_x + \frac{1-m}{2} = -Sc\ .$$

Consider the operators $T := -x\partial_x$ and $U := -\partial_x x - 1 = x\partial_x + 2\mathbb{E}_x + m - 1$. Then:

$$\begin{aligned} T &= \mathbb{E}_x + \Gamma_x = \mathbb{E}' + \Gamma' = \mathbb{E}' - Sc \\ U &= \mathbb{E}_x + m - 1 - \Gamma_x = \mathbb{E}' - \Gamma' = \mathbb{E}' + Sc\,. \end{aligned} \tag{10}$$

In the following lemma we list some useful identities in $\mathcal{U}(\mathfrak{osp}(1,2))$. Note that we hereby use the Pochammer symbol $(a)_j = a(a+1)\dots(a+j-1)$.

Lemma 4.1: *(Factorization of powers of $|x|^2\partial_x$)*
Let $j \in \mathbb{N}$ be a positive integer. The following holds:

(1) The operators T and U satisfy the intertwining relations

$$Tx = x(U+1)\ , \quad Ux = x(T+1)\ , \quad Tx^2 = x^2(T+2)\ , \quad Ux^2 = x^2(U+2) \tag{11}$$

Hence: $P(T)\,x = x\,P(U+1)$ and $P(U)\,x = x\,P(T+1)$ for $P(t) \in \mathbb{R}[t]$. Moreover, we also have that

$$\begin{aligned}\Delta^j |x|^{2j} &= (T+2)\dots(T+2j)(U+1)(U+3)\dots(U+2j-1) \qquad (12)\\ &= 2^{2j}\left(\frac{\mathbb{E}_x+\Gamma_x+2}{2}\right)_j\left(\frac{\mathbb{E}_x-\Gamma_x+m}{2}\right)_j\end{aligned}$$

(2) The odd powers of $|x|^2\partial_x$ admit the factorization

$$\begin{aligned}\left(|x|^2\partial_x\right)^{2j+1} &= x^{2j+1}T(T+2)\dots(T+2j)(U+1)(U+3)\dots(U+2j-1) \qquad (13)\\ &= (2x)^{2j+1}\left(\frac{\mathbb{E}_x+\Gamma_x}{2}\right)_{j+1}\left(\frac{\mathbb{E}_x-\Gamma_x+m}{2}\right)_j\\ &= |x|^{2j+2}\partial_x^{2j+1}|x|^{2j}\,. \qquad (14)\end{aligned}$$

(3) The even powers of $|x|^2\partial_x$ admit the factorization

$$\begin{aligned}\left(|x|^2\partial_x\right)^{2j} &= x^{2j}T(T+2)\dots(T+2j-2)(U+1)(U+3)\dots(U+2j-1) \qquad (15)\\ &= (2x)^{2j}\left(\frac{\mathbb{E}_x+\Gamma_x}{2}\right)_j\left(\frac{\mathbb{E}_x-\Gamma_x+m}{2}\right)_j\\ &= (-1)^{j-1}|x|^{2j}x\,\Delta^j x|x|^{2j-2}\,. \qquad (16)\end{aligned}$$

Proof: The first statement follows from

$$Tx = (\mathbb{E}'+\Gamma')x = x(\mathbb{E}'+1-\Gamma')\,, \quad Ux = (\mathbb{E}'-\Gamma')x = x(\mathbb{E}'+1+\Gamma').$$

For the second part, notice that $|x|^2\partial_x = x(-x\partial_x) = x(E+\Gamma) = xT$. Hence

$$\left(|x|^2\partial_x\right)^{2j+1} = (xT)^{2j+1}\ .$$

Using the intertwining relations of the previous part, the vector variable x can be brought in front and we obtain the identity (13). Invoking the formulae for Δ_{LB}

mentioned before and $\left[\Delta, |x|^2\right] = 4\mathbb{E}_x + 2m$, one has

$$\begin{aligned}\Delta|x|^2 &= \Delta_{LB} + (\mathbb{E}+m)(\mathbb{E}+2)\\ &= -(\mathbb{E}+\Gamma+2)(\Gamma-\mathbb{E}-m)\\ &= (T+2)(U+1)\ .\end{aligned}$$

Therefore, using the intertwining relations:

$$\begin{aligned}\Delta^j|x|^{2j} &= \Delta^{j-1}(T+2)(U+1)|x|^{2j-2}\\ &= \Delta^{j-1}|x|^{2j-2}(T+2j)(U+2j-1)\ .\end{aligned}$$

Repeated application of this identity leads finally to (12). On the other hand

$$|x|^{2j+2}\partial_x^{2j+1}|x|^{2j} = -x^{2j+1}x\partial_x\Delta^j|x|^{2j} = x^{2j+1}T\Delta^j|x|^{2j}\ .$$

Using (12) we can reduce the right hand side of this identity to the expression in (13). This shows that

$$\left(|x|^2\partial_x\right)^{2j+1} = |x|^{2j+2}\partial_x^{2j+1}|x|^{2j}\ .$$

The proof of part (3) is similar. □

We now come to the basic result we were after:

Theorem 4.2: (*Intertwining relation for odd powers of the Dirac operator* ∂_x)

(1) The operator $|x|^2\partial_x$ *commutes with the positive root* $\pi_0(E_+)$ *.*
(2) Let $j \in \mathbb{N}$ *and* $v \in \mathfrak{sl}(2,\mathbb{R})$*. We have the following intertwining property on* $\mathcal{P}(\mathbb{R}^m)$*:*

$$\partial_x^{2j+1}\pi_{-2j}(v) = \pi_{2j+2}(v)\partial_x^{2j+1}\ . \tag{17}$$

(3) The kernel of ∂_x^{2j+1} *is an* $\mathfrak{sl}(2,\mathbb{R})$*-module under the action* π_{-2j}*.*

Proof: (1) Since $|x|^2\partial_x = \mathcal{I}\partial_x\mathcal{I}$ and $\pi_0(E_+) = \mathcal{I}\partial_{x_1}\mathcal{I}$, the statement follows from $[\partial_{x_1}, \partial_x] = 0$.
(2) Notice that for $\alpha \in \mathbb{C}$, one has:

$$\left[\mathcal{I}_\alpha\partial_{x_1}\mathcal{I}_\alpha, |x|^2\right] = \left[-|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + \alpha + m - 1) - e_1 \wedge x, |x|^2\right] = 2x_1|x|^2\ .$$

Hence $\mathcal{I}_\alpha\partial_{x_1}\mathcal{I}_\alpha|x|^2 = |x|^2\mathcal{I}_{\alpha+2}\partial_{x_1}\mathcal{I}_{\alpha+2}$, or equivalently:

$$\pi_\alpha(\mathbb{E}_+)|x|^2 = |x|^2\pi_{\alpha+2}(E_+)$$

The intertwining relation (17) is straightforward for $v = H$ or E_-. If $v = E_+$, we use the identity (14) of lemma 4.1:

$$\left(|x|^2\partial_x\right)^{2j+1} = |x|^{2j+2}\partial_x^{2j+1}|x|^{2j}\ .$$

By part (1), each power of $|x|^2\partial_x$ commutes with $\pi_0(E_+)$:

$$\pi_0(E_+)|x|^{2j+2}\partial_x^{2j+1}|x|^{2j} = |x|^{2j+2}\partial_x^{2j+1}|x|^{2j}\pi_0(E_+)$$

or, using (4):

$$|x|^{2j+2}\pi_{2j+2}(E_+)\partial_x^{2j+1}|x|^{2j} = |x|^{2j+2}\partial_x^{2j+1}\pi_{-2j}(E_+)|x|^{2j} \ .$$

Omitting the factors $|x|^{2j+2}$ and $|x|^{2j}$, we obtain (17) for $v = E_+$.
Formula (3) then follows immediately from part (2). □

5. Fueter's Theorem revisited: paravector formalism

We now formulate the generalization of Fueter's theorem as in [10]. Their approach is based on the so-called paravector formalism where $\mathbb{R}^{m+1} \cong \mathbb{R} \oplus \mathbb{R}^1_m$, thus identifying points $x \in \mathbb{R}^{m+1}$ with paravectors $x_0 + x \in \mathbb{R} \oplus \mathbb{R}^1_m$. The analogue in this setting of the usual Dirac operator is the generalized Cauchy-Riemann operator $D := \partial_{x_0} + \partial_x$ in $\mathbb{R}^{m+1}$, which also appears in e.g. [4]. This formalism fixes from the start a chosen direction in the Euclidean space $\mathbb{R}^{m+1}$. The Fueter theorem in $(m+1)$ dimensions (with $(m+1)$ even) provides a way to construct monogenic functions (with respect to the D-operator) starting from holomorphic functions.

Set $x = r\omega$, $\omega \in S^{m-1} \subset \mathbb{R}^m$ and $z = \xi + i\eta$. Let f be holomorphic in $\Omega \subset \mathbb{C}$ and consider the splitting in its real and imaginary part: $f(z) = u(\xi,\eta) + iv(\xi,\eta)$. Let $P_l(x) \in \mathcal{M}_l(\mathbb{R}^m)$ be an l-homogeneous monogenic polynomial on $\mathbb{R}^m$. Fueter's theorem ([10]) states that the substitution $(\xi,\eta;i) \mapsto (x_0,r;\omega)$ in $f(z)P_l(x) = \big(u(\xi,\eta) + iv(\xi,\eta)\big)P_l(x)$, followed by the action of a fixed power of the Laplace operator to this expression leads to a non-trivial monogenic function (for D) on $\mathbb{R}^{m+1}$. More precisely:

Theorem 5.1 : (*Paravector formulation of Fueter theorem in* $\mathbb{R}^{m+1}$*, see [10]*) *Let m be odd and $P_l(x) \in \mathcal{M}_l(\mathbb{R}^m, \mathbb{R}_m)$. One then has:*

$$\Delta^{l+\frac{m-1}{2}}\big((u(x_0,r) + \omega v(x_0,r))P_l(x)\big)$$

is monogenic in $\widetilde{\Omega} = \{x \in \mathbb{R}^{m+1} : (x_0,r) \in \Omega\}$.

Remark: Note that the case m even has also been treated, using the notion of Fourier multipliers (see the work of Q. Tao [13]).

6. Fueter's Theorem: vector formalism

The aim of this section is to formulate and prove Fueter's theorem in the vector formalism. Here, vectors in $\mathbb{R}^m$ are identified with vector variables $x \in \mathbb{R}^1_m$. Instead of the Cauchy-Riemann operator D we will use the standard Dirac operator ∂_x in $\mathbb{R}^m$. Since we are now working on $\mathbb{R}^m$ instead of $\mathbb{R}^{m+1}$, m should be even. The main ingredient of our approach of Fueter's theorem is that the *odd* powers ∂_x^{2j+1} intertwine certain $\mathfrak{sl}(2)$-actions on function spaces on $\mathbb{R}^m$ (cf. theorem 4.2). In particular:

$$\partial_x^{2l+m-1}\pi_{-(2l+m-2)}(v) = \pi_{2l+m}(v)\partial_x^{2l+m-1} \, .$$

This allows us to consider the kernel of the map $\partial_x^{2l+m-1} : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m)$ as a representation for $\pi_{-(2l+m-2)}$ and Fueter's theorem can then be regarded as a statement about a special type of null solutions for the operator ∂_x^{2l+m-1} in $\mathbb{R}^m$.

We will now introduce some particular $\mathfrak{sl}(2)$-modules. Let $\alpha \in \mathbb{R}$ and take an *arbitrary* (not necessarily monogenic) polynomial $P_l(\underline{x}) \in \mathcal{P}_l(\mathbb{R}^{m-1})$. Consider the lowest weight $\mathfrak{sl}(2,\mathbb{R})$-module generated by the lowest weight vector $P_l(\underline{x})$:

$$V_\alpha(P_l(\underline{x})) = \bigoplus_{s\geq 0} \big(\pi_\alpha(E_+)\big)^s P_l(\underline{x}) = \bigoplus_{s\geq 0} (\mathcal{I}_\alpha \partial_{x_1} \mathcal{I}_\alpha)^s P_l(\underline{x}) \ .$$

Since $\pi_\alpha(H)P_l(\underline{x}) = (2\mathbb{E}_x + \alpha + m - 1)P_l(\underline{x}) = (2l + \alpha + m - 1)P_l(\underline{x})$, this module will be infinite-dimensional if the weight $(2l + \alpha + m - 1) > 0$. We can express the summands in the module $V_\alpha(P_l(\underline{x}))$ by means of (2.3) as

$$V_\alpha(P_l(\underline{x})) = \bigoplus_{s\geq 0} (\mathcal{I}_\alpha \partial_{x_1} \mathcal{I}_\alpha)^s P_l(\underline{x}) = \bigoplus_{s\geq 0} K_s^{l+\frac{m+\alpha}{2}}(x, e_1) P_l(\underline{x}) \ .$$

If e.g. $\alpha = -(2l + m)$, the condition for infinite dimensionality is not fulfilled and we obtain the finite-dimensional $\mathfrak{sl}(2,\mathbb{R})$-module

$$V_{-(2l+m)}(P_l(\underline{x})) = \bigoplus_{s\geq 0} K_s^0(x, e_1) P_l(\underline{x}) = (\mathbb{R} \oplus \mathbb{R} x e_1) P_l(\underline{x}) \ .$$

Fix $j \in \mathbb{N}$ and consider the solutions of ∂_x^{2j+1} which are also annihilated by the negative root $-\partial_{x_1}$. In particular, one can take the trivial solutions given by polynomials $R(\underline{x})$ of degree $d \leq 2j$ in $\mathbb{R}^{m-1}$, i.e. $R(\underline{x}) = \sum_{l=0}^{2j} P_l(\underline{x})$ with $P_l(\underline{x}) \in \mathcal{P}_l(\mathbb{R}^{m-1})$. Each $P_l(\underline{x})$ then generates a lowest-weight $\mathfrak{sl}(2,\mathbb{R})$-module $V_{-2j}(P_l(\underline{x}))$ with lowest weight $(2(l - j) + m - 1)$, which is infinite-dimensional for $2j < 2l + m - 1$. The largest value for $2j$ having this property is $2j = m - 2 + 2l$.
We will now focus our attention on this case, also requiring that m is even (unless stated otherwise). We denote the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ as $\mathbb{C}[(z)]$. It turns out that the $\mathfrak{sl}(2)$-module $V_\lambda(P_l(\underline{x}))$ has a very simple form in terms of holomorphic polynomials.

Lemma 6.1: *Put $\lambda = -(2l + m - 2)$ and let $P_l(\underline{x}) \in \mathcal{P}_l(\mathbb{R}^{m-1})$ be an arbitrary polynomial. One has:*

(1) The mapping $z \mapsto -e_1 x$ extends to an algebra isomorphism

$$\zeta : \mathbb{C}[(z)] \to \mathbb{C}[(-e_1 x)] \ .$$

(2) The space $V_\lambda(P_l(\underline{x}))$ is an irreducible lowest-weight module with weight 1 which, up to the factor $P_l(\underline{x})$, can be expressed in terms of holomorphic polynomials:

$$V_\lambda(P_l(\underline{x})) = \bigoplus_{s\geq 0} (\mathcal{I}_\lambda \partial_{x_1} \mathcal{I}_\lambda)^s P_l(\underline{x}) = \mathbb{C}[-e_1 x] P_l(\underline{x}) \cong \mathbb{C}[z] P_l(\underline{x}) \ . \quad (18)$$

Moreover, $F(x) \in V_\lambda(P_l(\underline{x}))$ if $F(x)$ satisfies the Vekua-type equation

$$\frac{1}{2}\Big(\partial_{x_1} - e_1 \underline{\omega}\big(\partial_r - \frac{l}{r}\big)\Big) F(x) = 0 \ .$$

Proof: Put $\alpha = \lambda := -(2l+m-2)$ in (6). Then we obtain that

$$V_\lambda(P_l(\underline{x})) = \bigoplus_{s\geq 0} (\mathcal{I}_\lambda \partial_{x_1} \mathcal{I}_\lambda)^s P_l(\underline{x}) = \bigoplus_{s\geq 0} K_s^1(x,e_1) P_l(\underline{x})$$

is a lowest-weight $\mathfrak{sl}(2,\mathbb{R})$-module of weight 1 under the action $\pi_{-(2l+m-2)}$. Consider for fixed x the Laurent series in t, given by

$$-\frac{x-te_1}{|x-te_1|^2} = \sum_{k=0}^{\infty} K_k^1(x,e_1) t^{-k-1} e_1 \ , \quad |x| < t \ .$$

This series can be rewritten in the form

$$\big(e_1(x-te_1)\big)^{-1} = (x-te_1)^{-1} e_1^{-1} = \sum_{k=0}^{\infty} K_k^1(x,e_1) t^{-k-1} \ , \quad |x| < t \ .$$

On the other hand, for $|e_1 x| = |x| < t$ one has the binomial series expansion:

$$\big(e_1(x-te_1)\big)^{-1} = (t+e_1x)^{-1} = \frac{1}{t}\Big(1+\frac{e_1 x}{t}\Big)^{-1} = \frac{1}{t}\sum_{k=0}^{\infty}\Big(\frac{-e_1 x}{t}\Big)^k \ .$$

By the uniqueness of this expansion

$$K_k^1(x,e_1) = (-e_1x)^k = (x_1 - e_1\underline{x})^k = (x_1 - e_1\underline{\omega}|\underline{x}|)^k = |x|^k \exp(-e_1\underline{\omega}\, k\theta) \quad (19)$$

where we set $\tan(\theta) = |\underline{x}|/x_1$. Notice that the bivector $-e_1\underline{\omega}$ defines a complex structure in the two-dimensional $(e_1, -\underline{\omega})$-plane. Under the substitution $i \mapsto -e_1\underline{\omega}$, $u \mapsto x_1$ and $v \mapsto |\underline{x}|$, the complex variable z is transformed into $(-e_1 x)$. By (19) it is clear that the monomial $(-e_1x)^k$ behaves in a similar way as the k-th power of the standard complex variable z. In fact, the map $z \mapsto -e_1 x$ extends to an algebra isomorphism between $\mathbb{C}[z]$ and $\mathbb{C}[-e_1x]$. Putting $|\underline{x}| = r$, we can also identify $\mathbb{C}[-e_1x]$ with the polynomial null solutions $f(-e_1x)$ of the Cauchy-Riemann type operator $\frac{1}{2}(\partial_{x_1} - e_1\underline{\omega}\partial_r)$ and $F(x) \in V_\lambda(P_l(\underline{x}))$ if

$$r^l\Big(\frac{1}{2}(\partial_{x_1} - e_1\underline{\omega}\partial_r)\Big) r^{-l} F(x) = \frac{1}{2}\Big(\partial_{x_1} - e_1\underline{\omega}(\partial_r - \frac{l}{r})\Big) F(x) = 0 \, .$$

This proves the lemma. □

In the following lemma we collect some further results which will be used in the proof of Fueter's theorem.

Lemma 6.2: (*Properties of inversions*)
Set $\lambda = -(2l+m-2)$. Let $P_l(\underline{x}) \in \mathcal{P}_l(\mathbb{R}^{m-1})$. Then:

(1) The map $-L(e_1)\mathcal{I}_\lambda : \mathbb{C}[(-e_1x)]P_l(\underline{x}) \to \mathbb{C}[(-e_1x)]P_l(\underline{x})$:

$$f(-e_1x)P_l(\underline{x}) \mapsto \frac{1}{e_1x} f\Big(\frac{1}{e_1x}\Big) P_l(\underline{x}) = \frac{xe_1}{|x|^2} f\Big(\frac{xe_1}{|x|^2}\Big) P_l(\underline{x})$$

and the map

$$\iota : \mathbb{C}[(z)] \to \mathbb{C}[(z)] : f(z) \mapsto -\frac{1}{z} f\Big(-\frac{1}{z}\Big)$$

are intertwinable by ζ and satisfy $(-L(e_1)\mathcal{I}_\lambda)^2 = -id$ and $\iota^2 = -id$.

(2) Let m be even. If $P(x) \in \mathcal{P}(\mathbb{R}^m)$ is a polynomial satisfying the condition $\partial_x^{2l+m-1} P(x) = 0$, then

$$(\mathcal{I}_\lambda P)(x) = (\mathcal{I}_{-(2l+m-2)} P)(x) = \frac{x}{|x|^2} |x|^{2l} P\Big(\frac{x}{|x|^2}\Big) \in C^\infty(\mathbb{R}_0^m)$$

belongs to the kernel of ∂_x^{2l+m-1} and $\partial_x^{2l+m-1}\big(L(e_1)\mathcal{I}_\lambda P\big) = 0$. In particular, if $P_l(x)$ belongs to $\mathcal{P}_l(\mathbb{R}^m)$, it follows that

$$(\mathcal{I}_\lambda P_l)(x) = \frac{x}{|x|^2} P_l(x)$$

belongs to the kernel of ∂_x^{2l+m-1}.

(3) Let m be odd. If $P(x) \in \mathcal{P}(\mathbb{R}^m)$ is a polynomial such that $\partial_x^{2l+m} P(x) = 0$, then also

$$(\mathcal{I}_{\lambda-1} P)(x) = (\mathcal{I}_{-(2l+m-1)} P)(x) = \frac{x}{|x|} |x|^{2l} P\Big(\frac{x}{|x|^2}\Big) \in C^\infty(\mathbb{R}_0^m)$$

belongs to the kernel of ∂_x^{2l+m} and $\partial_x^{2l+m}\big(L(e_1)\mathcal{I}_\lambda P\big) = 0$. In particular, if $P_l(x)$ belongs to $\mathcal{P}_l(\mathbb{R}^m)$, it follows that

$$(\mathcal{I}_{\lambda-1} P_l)(x) = \frac{x}{|x|} P_l(x) = \omega P_l(x) \tag{20}$$

belongs to the kernel of ∂_x^{2l+m}.

Proof: Let $F = f(-e_1 x) P_l(\underline{x})$. Consider the inversion operator $\mathcal{I}_\lambda$ acting on F:

$$(\mathcal{I}_\lambda F)(x) = \frac{x}{|x|^{m+\lambda}} F\Big(\frac{x}{|x|^2}\Big) = x|x|^{2l-2} F\Big(\frac{x}{|x|^2}\Big) = \frac{x}{|x|^2} f\Big(-\frac{e_1 x}{|x|^2}\Big) P_l(\underline{x}) \ .$$

In particular, for $f = (-e_1 x)^k$, the identities

$$\frac{x}{|x|^2}\left(-\frac{e_1 x}{|x|^2}\right)^k = \left(-\frac{x e_1}{|x|^2}\right)^k \frac{x}{|x|^2} = \left(-\frac{x e_1}{|x|^2}\right)^{k+1} e_1 = (-e_1 x)^{-(k+1)} e_1$$

leads to the following formula, which holds on each summand $\mathbb{C}[-e_1 x] P_l(\underline{x})$:

$$\mathcal{I}_\lambda : (-e_1 x)^k P_l(\underline{x}) \to (-e_1 x)^{-(k+1)} e_1 P_l(\underline{x}) \ .$$

This is equivalent to the transformation

$$-L(e_1)\mathcal{I}_\lambda : (-e_1 x)^k P_l(\underline{x}) \to (e_1 x)^{-(k+1)} P_l(\underline{x}) \ ,$$

which under the identification $z = -e_1 x$ corresponds to the classical complex inversion $f(z) \mapsto -\frac{1}{z} f(-\frac{1}{z})$ for holomorphic functions. Hence,

$$-L(e_1)\mathcal{I}_\lambda : \mathbb{C}[(-e_1 x)] P_l(\underline{x}) \to \mathbb{C}[(-e_1 x)] P_l(\underline{x})$$

corresponds to an inversion satisfying $\big(-L(e_1)\mathcal{I}_\lambda\big)^2 = -\text{id}$ and the intertwining property is also clear. In order to prove (2), let us suppose that $\partial_x^{2l+m-1} P(x) = 0$.

This implies that $P(x)$ has a Fischer decomposition in x of the form

$$P(x) = \sum_{j=0}^{2l+m-2} x^j M_j(x)\,, \quad M_j(x) \in \mathcal{M}(\mathbb{R}^m)\ .$$

The action of $\mathcal{I}_\lambda$ on $P(x)$ yields

$$\begin{aligned}(\mathcal{I}_\lambda P)(x) &= \sum_{j=0}^{2l+m-2} x^j |x|^{2l+m-2-2j} \frac{x}{|x|^m} M_j\Big(\frac{x}{|x|^2}\Big) \\ &= \sum_{j=0}^{2l+m-2} \epsilon_j\, x^{2l+m-2-j} \widetilde{M_j}(x) \end{aligned} \tag{21}$$

where $\widetilde{M_j}(x) := \mathcal{I} M_j(x)$ is left monogenic in x, $\epsilon_j = \pm 1$, and m is supposed to be even. Hence, the right hand side of (21) is polymonogenic of order $2l+m-1$. Since $L(e_1)$ anti-commutes with the action of ∂_x, the statement in (2) is proved. The proof of (3) is analogous to (2). □

This leads now immediately to the following version of the generalization of Fueter's Theorem.

Theorem 6.3: (*Fueter theorem in $\mathbb{R}^m$ with m even: vector formulation*) *Fix a unit vector $e_1 \in \mathbb{R}^m$, with m an even dimension, and identify $e_1^\perp$ with $\mathbb{R}^{m-1}$. Let $P_l(\underline{x}) \in \mathcal{P}_l(\mathbb{R}^{m-1})$ be an arbitrary polynomial and let $f(z) \in \mathbb{C}[(z)]$. Consider the algebra homomorphism $\zeta : \mathbb{C}[(z)] \to \mathbb{C}[(-e_1 x)] : f(z) \mapsto f(-e_1 x)$. We then have:*

$$\begin{aligned} \partial_x^{2l+m-1}\big(f(-e_1 x) P_l(\underline{x})\big) &= 0 \\ \partial_x^{2l+m-1}\big(f(-e_1 x) e_1 \underline{\omega} P_l(\underline{x})\big) &= 0\ . \end{aligned}$$

Proof: The intertwining property (17) gives for all $v \in \mathfrak{sl}(2,\mathbb{R})$ that

$$\partial_x^{2l+m-1} \pi_{-(2l+m-2)}(v) = \pi_{2l+m}(v) \partial_x^{2l+m-1}\ . \tag{22}$$

Recall that we have put $\lambda = -(2l+m-2)$. By lemma 6.1, we have the graded sum

$$V_\lambda(P_l(\underline{x})) = \bigoplus_{s\geq 0} (\mathcal{I}_\lambda \partial_{x_1} \mathcal{I}_\lambda)^s P_l(\underline{x}) = \bigoplus_{s \geq 0} (-e_1 x)^s P_l(\underline{x})\ ,$$

consisting of homogeneous polynomials. Let s be an arbitrary positive integer. By repeated application of identity (22) for $v = E_+$ we find that

$$\begin{aligned} \partial_x^{2l+m-1}\big(s!(-e_1 x)^s P_l(\underline{x})\big) &= \partial_x^{2l+m-1}\big((\mathcal{I}_\lambda \partial_{x_1} \mathcal{I}_\lambda)^s P_l(\underline{x})\big) \\ &= (\pi_{2l+m}(E_+))^s \partial_x^{2l+m-1} P_l(\underline{x}) = 0\ . \end{aligned}$$

This proves the statement for $f(z) \in \mathbb{C}[z]$. Let now $h(z)$ have a Laurent series which only consists of negative powers of z, then $h(z) = \iota f(z)$ with $f(z) \in \mathbb{C}[z]$ and

$$\partial_x^{2l+m-1}\big(h(-e_1 x) P_l(\underline{x})\big) = \partial_x^{2l+m-1}\big((-L(e_1)\mathcal{I}_\lambda) f(-e_1 x) P_l(\underline{x})\big) = 0$$

because $\partial_{x}^{2l+m-1}\big(f(-e_1x)P_l(\underline{x})\big)=0$.
For the second part we apply (20) to $P_l(\underline{x})\in\mathcal{P}(\mathbb{R}^{m-1})$. Since $(m-1)$ is odd, it follows that

$$\underline{\omega}P_l(\underline{x})\in \mathrm{Ker}\big(\partial_{\underline{x}}^{2l+m-1}\big).$$

We can then regard the function $\underline{\omega}P_l(\underline{x})$ as a C^∞-function on $\mathbb{R}^m\setminus\mathbb{R}e_1$ which is annihilated by ∂_{x_1}. Hence, we obtain another lowest weight vector $\underline{\omega}P_l(\underline{x})$ which is moreover in the kernel of ∂_x^{2l+m-1}. Again with $\lambda=-(2l+m-2)$, we obtain that

$$V_\lambda(\underline{\omega}P_l(\underline{x}))=\bigoplus_{s\geq 0}(\mathcal{I}_\lambda\partial_{x_1}\mathcal{I}_\lambda)^s\underline{\omega}P_l(\underline{x})=\bigoplus_{s\geq 0}K_s^1(x,e_1)\underline{\omega}P_l(\underline{x})=\mathbb{C}[-e_1x]\underline{\omega}P_l(\underline{x})$$

is a lowest weight module for $\mathfrak{sl}(2,\mathbb{R})$ of lowest weight 1 under the action $\pi_{-(2l+m-2)}$ and each function in this module is annihilated by ∂_x^{2l+m-1}. In view of the fact that $\mathcal{I}_\lambda\partial_{x_1}\mathcal{I}_\lambda$ and $(-e_1x)$ anti-commute with $L(e_1)$, one can also replace $\underline{\omega}$ by $(e_1\underline{\omega})$. □

Remark that the identification in lemma 6.1, given by

$$V_\lambda(P_l(\underline{x}))=\mathbb{C}[-e_1x]P_l(\underline{x})\overset{\zeta}{\cong}\mathbb{C}[z]P_l(\underline{x})\ ,$$

also explains where the substitution: $z\rightarrow x_1-e_1\underline{\omega}|\underline{x}|$ in Fueter's theorem has its origin. Moreover, our approach immediately yields in a natural and unified way the result for general polynomials $P_l(\underline{x})\in\mathcal{P}_l(\mathbb{R}^{m-1})$. In case of theorem 5.1 formulated in the paravector formalism, this result was proved over the years in several rather technical papers: first the case $l=0$, then $P_l(x)\in\mathcal{M}_l(\mathbb{R}^m)$ and finally also for $P_l(x)\in\mathcal{P}_l(\mathbb{R}^m)$. In chronological order, we mention the papers [8, 10, 12, 14, 16]. Remark that the same statement remains true when $P_l(\underline{x})\in\mathcal{P}_l(\mathbb{R}^{m-1})$ is replaced by a polynomial $P_l(x)\in\mathcal{P}_l(\mathbb{R}^m)$:

Corollary 6.4: (*Fueter theorem for $P_l(x)\in\mathcal{P}_l(\mathbb{R}^m)$ in $\mathbb{R}^m$ with m even*)
Let $P_l(x)\in\mathcal{P}_l(\mathbb{R}^m)$ be an arbitrary polynomial and let $f(z)\in\mathbb{C}[(z)]$. We then have:

$$\partial_x^{2l+m-1}\big(f(-e_1x)P_l(x)\big)=0$$

Proof: Consider the expansion

$$P_l(x)=\sum_{j=0}^{l}x_1^jP_{l-j}(\underline{x})\ ,\quad P_{l-j}(\underline{x})\in\mathcal{P}_{l-j}(\mathbb{R}^{m-1})\ .$$

Since $-e_1x=x_1-e_1\underline{x}$ (with e_1x and $e_1\underline{x}$ commuting variables), we find that

$$\begin{aligned}f(-e_1x)P_l(x)&=\sum_{j=0}^{l}\sum_{s=0}^{j}\binom{j}{s}f(-e_1x)(-e_1x)^{j-s}(e_1\underline{x})^sP_{l-j}(\underline{x})\,,\\&=\sum_{j=0}^{l}\sum_{s=0}^{j}f_{js}(-e_1x)(e_1\underline{x})^sP_{l-j}(\underline{x})\,,\quad f_{sj}(z)\in\mathbb{C}[z]\,.\end{aligned}\qquad(23)$$

Since $(e_1\underline{x})^sP_{l-j}(\underline{x})\in\mathcal{P}_{l-j+s}(\mathbb{R}^{m-1})$ we can apply Fueter's theorem 6.3 to each of the summands, hence

$$\partial_x^{2(l-j+s)+m-1}\big(f_{js}(-e_1 x)(e_1\underline{x})^s P_{l-j}(\underline{x})\big) = 0\,, \quad j = 0,\dots,l\,,\ s = 0,\dots,j\,.$$

The highest power of the Dirac operator which can occur is $2l+m-1$. This proves the statement. □

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