A 14-dimensional module for the symplectic group: orbits on vectors

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Abstract

Let $\mathbb{F}$ be a field, V a 6-dimensional $\mathbb{F}$-vector space and f a nondegenerate alternating bilinear form on V. We consider a 14-dimensional module for the symplectic group $Sp(V,f) \cong Sp(6,\mathbb{F})$ associated with (V,f), and classify the orbits on vectors. For characteristic distinct from 2, this module is irreducible and isomorphic to the Weyl module of $Sp(V,f)$ for the fundamental weight λ_3. If the characteristic is 2, then the module is reducible as it contains an 8-dimensional submodule isomorphic to the spin module of $Sp(V,f)$.

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1 Introduction

Let $\mathbb{F}$ be a field and V a 6-dimensional $\mathbb{F}$-vector space equipped with a nondegenerate alternating bilinear form f. The symplectic group $G = Sp(V,f) \cong Sp(6,\mathbb{F})$ associated with the symplectic space (V,f) has a natural action on the third exterior power $\bigwedge^3 V$ of V. The corresponding 20-dimensional $\mathbb{F}G$-module has two nontrivial submodules, one of dimension 14 which we will denote by $W$and another one of dimension 6 which we will denote by $\widetilde{W}$. The 14-dimensional submodule W is generated by all decomposable trivectors of the form $v_1 \wedge v_2 \wedge v_3$, where $\langle v_1, v_2, v_3 \rangle$ is a 3-space totally isotropic for f. This module is the Weyl module of $Sp(V,f)$ for the fundamental weight λ_3, see Premet and Suprunenko [27]. The 6-dimensional submodule $\widetilde{W}$ consists of all trivectors $\alpha \in \bigwedge^3 V$ such that $\alpha \wedge \beta = 0$ for all $\beta \in W$. The module $\widetilde{W}$ is isomorphic to V, regarded (in a natural way) as an $\mathbb{F}G$-module.

For characteristic 0, Maschke's theorem guarantees that the $\mathbb{F}G$-module $\bigwedge^3 V$ can be written as the direct sum of irreducible submodules. In fact, this property holds as soon as the characteristic of $\mathbb{F}$ is distinct from 2. Indeed, in case the characteristic is distinct from 2, the submodules W and $\widetilde{W}$ are irreducible and $\bigwedge^3 V = W \oplus \widetilde{W}$. On the other hand, if $\mathsf{char}(\mathbb{F}) = 2$, then $\widetilde{W} \subseteq W$ and so W cannot be irreducible. Besides W and $\widetilde{W}$, we can also consider the $\mathbb{F}G$-modules on the quotient spaces $\bigwedge^3 V/W$ and $\bigwedge^3 V/\widetilde{W}$. The first module is isomorphic to V, and latter module is isomorphic to W if the characteristic is distinct from 2. If the characteristic equals 2, then the submodule $\bigwedge^3 V/\widetilde{W}$ is reducible as it contains $\{w + \widetilde{W} \,|\, w \in W\}$ as an 8-dimensional submodule. This submodule is isomorphic to the spin module for $Sp(V, f)$, see Gow [19].

The aim of this paper is to classify the orbits on vectors of the $\mathbb{F}G$-module $\bigwedge^3 V/\widetilde{W}$. The difficulty of the problem and the methods to solve it heavily depend on the characteristic of $\mathbb{F}$. The case where $\mathsf{char}(\mathbb{F}) \neq 2$ is the easiest one. In this case, the module is isomorphic to W and the orbits on vectors of W were already described in the literature. For algebraically closed fields of characteristic distinct from 2, these orbits were determined by Igusa [24, p. 1027]. For general fields, these orbits can be extracted from a series of four papers [13, 14, 15, 16] by the authors, where they succeeded in obtaining a complete classification of all orbits on vectors of the $Sp(V, f)$-module $\bigwedge^3 V$, hereby extending a result of Popov [26] who succeeded in the same goal, but under the extra assumption that the underlying field $\mathbb{F}$ is algebraically closed of characteristic distinct from 2. The results of the papers [13, 14, 15, 16], which will be recalled in Section 2, will play an important role the present paper to obtain the desired classification results in the characteristic 2 case.

The problem of classifying orbits on vectors (or on subspaces) of certain group modules has already been considered before in the literature. All finite-dimensional irreducible rational $\mathbb{K}H$-modules on which a group H has a finite number of orbits on vectors have been determined in Guralnick et al. [20] in case H is a connected linear algebraic group over an algebraically closed field $\mathbb{K}$. For the purpose of studying the subgroup structure of the Chevalley groups of type E_6, Aschbacher studied the 27-dimensional modules for these groups. In particular, he classified the orbits on vectors and hyperplanes of these modules, see [1]. Cooperstein [7] classified orbits on vectors of the 57-dimensional modules for the Chevalley groups of type E_7, also with the intention to use this information to study the subgroup structure. There are a number of other papers dealing with the problem of classifying orbits on vectors and subspaces of certain group modules, see e.g. [3, 5]. For group modules involving a general linear group $GL(V)$ acting on an exterior power of V, we also have a number of results dealing with the classification of orbits on vectors, see [2, 4, 18, 21, 22, 23, 28, 29, 30, 32, 33, 34]. Some of these results however impose certain restrictions on the underlying field.

One of the motivations for studying the problem under consideration in this paper is the so-called isomorphism problem for hyperplanes of symplectic dual polar spaces. With the pair (V, f), there is associated a *symplectic dual polar space* $DW(5, \mathbb{F})$. This is the point-line geometry whose points are the 3-spaces of V totally isotropic for f, with each line being the collection of all totally isotropic 3-spaces that contain a given totally isotropic 2-space. A *hyperplane* of a point-line geometry is a set of points, distinct from the whole point-set, meeting each line in either one or all of its points. If the point-line geometry is fully embeddable in a projective space, then there is a standard way of constructing hyperplanes, namely by intersecting the embedded geometry with hyperplanes of the ambient projective space. Hyperplanes that arise in this way are called *classical*. If the field $\mathbb{F}$ has at least three elements, then theoretical results of Cooperstein [8], Kasikova and Shult [25] and Ronan [31] imply that all classical hyperplanes of $DW(5,\mathbb{F})$ must arise from the so-called Grassmann embedding of $DW(5,\mathbb{F})$. Without going into technical details, this amounts to saying that if $|\mathbb{F}| \geq 3$, then there exists some one-to-one correspondence between the classical hyperplanes of $DW(5,\mathbb{F})$ and the 1-spaces of the quotient module $\bigwedge^3 V/\widetilde{W}$ under consideration in the present paper. The knowledge of the orbits of vectors of this quotient module seems indispensable to obtain a classification of the isomorphism classes of hyperplanes of $DW(5,\mathbb{F})$. Such classifications for hyperplanes have already been obtained in the case the underlying field is perfect of characteristic 2 (De Bruyn [10]) or finite and of odd characteristic (Cooperstein and De Bruyn [9]). The results of the present paper will allow to generalize some of the results contained in these papers.

The group $GL(V)$ has a natural action on $\bigwedge^3 V$. Indeed, for every $\theta \in GL(V)$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$ such that $\bigwedge^3(\theta)(v_1 \wedge v_2 \wedge v_3) = \theta(v_1) \wedge \theta(v_2) \wedge \theta(v_3)$ for all $v_1, v_2, v_3 \in V$. In the sequel, we will often write θ instead of $\bigwedge^3(\theta)$, accepting this abuse of notation for the gain of readability. Using this notation, we say that two trivectors χ_1 and χ_2 are *G-equivalent* for some subgroup G of $GL(V)$ if $\chi_2 = \theta(\chi_1)$ for some $\theta \in G$. We can define an equivalence relation on the vectors of $\bigwedge^3 V$ which is coarser than $Sp(V,f)$-equivalence. We say that two trivectors χ_1 and χ_2 are *quasi-$Sp(V,f)$-equivalent* if there exists a $\theta \in Sp(V,f)$ and a $\chi \in \widetilde{W}$ such that $\chi_2 = \theta(\chi_1) + \chi$. Obviously, there is a bijective correspondence between the quasi-$Sp(V,f)$-equivalence classes and the orbits on vectors of the $\mathbb{F}G$-module $\bigwedge^3 V/\widetilde{W}$. In view of this, we prefer to state our main results in terms of this quasi-$Sp(V,f)$-equivalence relation. Before we can do that, we still need to discuss a result of Revoy regarding the classification of the $GL(V)$-equivalence classes of trivectors of V.

Put $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$ and let $\overline{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}$. For every separable quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, we choose one pair $(a_{\mathbb{F}'}, b_{\mathbb{F}'}) \in \mathbb{F}^2$ such that the quadratic polynomial $X^2 - a_{\mathbb{F}'}X - b_{\mathbb{F}'} \in \mathbb{F}[X]$ is irreducible and $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by this polynomial. In general, there are many possibilities for $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$, but throughout this paper $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$ will be a fixed choice among all these possibilities. For every nonseparable quadratic extension $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$, we put $a_{\mathbb{F}'} := 0$ and we choose a nonsquare $b_{\mathbb{F}'}$ in $\mathbb{F}$ such that $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $X^2 + b_{\mathbb{F}'} \in \mathbb{F}[X]$. There are many possibilities for $b_{\mathbb{F}'}$, but throughout this paper $b_{\mathbb{F}'}$ will be a fixed choice among all these possibilities. Put

$$\begin{aligned}
\Psi &:= \{(a_{\mathbb{F}'}, b_{\mathbb{F}'}) \,|\, \mathbb{F}' \subseteq \overline{\mathbb{F}} \text{ is a separable quadratic extension of } \mathbb{F}\}, \\
\Psi' &:= \{b_{\mathbb{F}'} \,|\, \mathbb{F}' \subseteq \overline{\mathbb{F}} \text{ is a nonseparable quadratic extension of } \mathbb{F}\}.
\end{aligned}$$

Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be a fixed basis of V. For every quadratic extension $\mathbb{F}'$ of $\mathbb{F}$

contained in $\overline{\mathbb{F}}$, we have $b_{\mathbb{F}'} \neq 0$ and we define

$$\chi^*_{\mathbb{F}'} := \lambda_{\mathbb{F}'} \cdot v_1 \wedge v_2 \wedge v_3 + \mu_{\mathbb{F}'} \cdot v_4 \wedge v_5 \wedge v_6 + (v_1 + v_4) \wedge (v_2 + v_5) \wedge (v_3 + v_6),$$

where $\lambda_{\mathbb{F}'} := a_{\mathbb{F}'} + b_{\mathbb{F}'} - 1$ and $\mu_{\mathbb{F}'} := \frac{1-a_{\mathbb{F}'}-b_{\mathbb{F}'}}{b_{\mathbb{F}'}}$.

A complete classification of all $GL(V)$-equivalence classes of trivectors of V was obtained by Revoy [29].

Proposition 1.1 ([29]) *Let $\{v_1, v_2, \ldots, v_6\}$ be the fixed basis of V as above. Then every nonzero trivector of V is $GL(V)$-equivalent with precisely one of the following vectors:*
(*A*) $v_1 \wedge v_2 \wedge v_3$;
(*B*) $v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_4 \wedge v_5$;
(*C*) $v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$;
(*D*) $v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_3 \wedge v_5 + v_2 \wedge v_3 \wedge v_6$;
(*E*) $\chi^*_{\mathbb{F}'}$ *for some quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$.*

If $\mathbb{F}'_1$ and $\mathbb{F}'_2$ are two distinct quadratic extensions of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, then the trivectors $\chi^*_{\mathbb{F}'_1}$ and $\chi^*_{\mathbb{F}'_2}$ are not $GL(V)$-equivalent.

Let $X \in \{A, B, C, D, E\}$. A nonzero trivector of V is said to be of *Type (X)* if it is $GL(V)$-equivalent with (one of) the trivector(s) described in (X) of Proposition 1.1. It should be mentioned that the description of the trivectors of Type (E) in terms of the parameters $\lambda_{\mathbb{F}'}$ and $\mu_{\mathbb{F}'}$ is not taken from Revoy's paper [29], but from the paper [12] of one of the authors. A complete classification of all $GL(V)$-equivalence classes of trivectors of V was also obtained by a number of other people under certain assumptions of the underlying field $\mathbb{F}$, see for instance Cohen and Helminck [4] and Reichel [28].

The next two theorems give a complete classification of all quasi-$Sp(V, f)$-equivalence classes in the case the characteristic of $\mathbb{F}$ is distinct from 2. These two theorems will be proved in Section 2 and are an almost immediate consequence of the classification of the $Sp(V, f)$-equivalence classes of trivectors. In the remainder of this introductory section, $(e_1, f_1, e_2, f_2, e_3, f_3)$ will denote a fixed hyperbolic basis of the symplectic space (V, f), that means that $f(e_i, e_j) = f(f_i, f_j) = 0$ and $f(e_i, f_j) = \delta_{ij}$ for all $i, j \in \{1, 2, 3\}$.

Theorem 1.2 (Section 2) *Suppose* $\mathsf{char}(\mathbb{F}) \neq 2$. *Then every trivector of V is quasi-$Sp(V, f)$-equivalent with (at least) one of the following trivectors:*

$(Q1)$ *the zero vector of* $\bigwedge^3 V$;

$(Q2)$ $\chi_{A1} = e_1 \wedge e_2 \wedge e_3$;

$(Q3)$ $\chi_{B4}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot e_1 \wedge f_2 \wedge f_3$ *for some* $\lambda \in \mathbb{F}^*$;

$(Q4)$ $\chi_{C1}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot f_1 \wedge f_2 \wedge f_3$ *for some* $\lambda \in \mathbb{F}^*$;

$(Q5)$ $\chi_{D3}(\lambda_1, \lambda_2) = e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge f_1 + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$;

(Q6) $\chi_{E1}(a,b,h_1,h_2,h_3) = 2 \cdot e_1 \wedge e_2 \wedge e_3 + a \cdot \Big(h_1 \cdot f_1 \wedge e_2 \wedge e_3 + h_2 \cdot e_1 \wedge f_2 \wedge e_3 + h_3 \cdot e_1 \wedge e_2 \wedge f_3\Big) + (a^2+2b) \cdot \Big(h_1h_2 \cdot f_1 \wedge f_2 \wedge e_3 + h_1h_3 \cdot f_1 \wedge e_2 \wedge f_3 + h_2h_3 \cdot e_1 \wedge f_2 \wedge f_3\Big) + h_1h_2h_3a(a^2+3b) \cdot f_1 \wedge f_2 \wedge f_3$ *for some* $(a,b) \in \Psi$ *and some* $h_1, h_2, h_3 \in \mathbb{F}^*$.

In the case $\mathsf{char}(\mathbb{F}) \neq 2$, a trivector of V is said to be of *Type (Qi)*, $i \in \{1,2,\ldots,6\}$, if it is quasi-$Sp(V,f)$-equivalent with (one of) the trivector(s) defined in (Qi) of Theorem 1.2.

Theorem 1.3 (Section 2) *Suppose* $\mathsf{char}(\mathbb{F}) \neq 2$.

- *Let* $i, j \in \{1,2,\ldots,6\}$ *with* $i \neq j$. *Then no trivector of Type (Qi) is quasi-*$Sp(V,f)$*-equivalent with a trivector of Type (Qj).*
- *Let* $\lambda, \lambda' \in \mathbb{F}^*$. *Then the two trivectors* $\chi_{B4}(\lambda)$ *and* $\chi_{B4}(\lambda')$ *of* V *are quasi-*$Sp(V,f)$*-equivalent if and only if* $\frac{\lambda}{\lambda'}$ *is a square in* $\mathbb{F}$.
- *Let* $\lambda, \lambda' \in \mathbb{F}^*$. *Then the two trivectors* $\chi_{C1}(\lambda)$ *and* $\chi_{C1}(\lambda')$ *of* V *are quasi-*$Sp(V,f)$*-equivalent if and only if* $\lambda' \in \{\lambda, -\lambda\}$.
- *Let* $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. *Then the two trivectors* $\chi_{D3}(\lambda_1,\lambda_2)$ *and* $\chi_{D3}(\lambda_1',\lambda_2')$ *of* V *are quasi-*$Sp(V,f)$*-equivalent if and only if the matrices* $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_1\lambda_2)$ *and* $\mathrm{diag}(\lambda_1',\lambda_2',\lambda_1'\lambda_2')$ *are congruent, i.e. if and only if there exists a nonsingular* (3×3)*-matrix* A *over* $\mathbb{F}$ *such that* $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_1\lambda_2) = A \cdot \mathrm{diag}(\lambda_1',\lambda_2',\lambda_1'\lambda_2') \cdot A^T$.
- *Let* $h_1,h_2,h_3,h_1',h_2',h_3' \in \mathbb{F}^*$ *and* $(a,b),(a',b') \in \Psi$. *Then the two trivectors* $\chi_{E1}(a,b,h_1,h_2,h_3)$ *and* $\chi_{E1}(a',b',h_1',h_2',h_3')$ *of* V *are quasi-*$Sp(V,f)$*-equivalent if and only if* $(a,b)=(a',b')$ *and there exists a* 3×3*-matrix* A *over* $\mathbb{F}'$ *with determinant equal to* 1 *such that* $A \cdot \mathrm{diag}(h_1,h_2,h_3) \cdot (A^\psi)^T$ *is equal to* $\mathrm{diag}(h_1',h_2',h_3')$ *or* $\mathrm{diag}(-h_1',-h_2',-h_3')$. *Here,* $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ *is the quadratic extension of* $\mathbb{F}$ *determined by the irreducible quadratic polynomial* X^2-aX-b *of* $\mathbb{F}[X]$ *and* ψ *is the unique nontrivial element of the Galois group* $\mathsf{Gal}(\mathbb{F}'/\mathbb{F})$.

In Theorem 1.3, $\mathrm{diag}(h_1,h_2,h_3)$ denotes the (3×3)-diagonal matrix whose (i,i)-th entry is equal to h_i for every $i \in \{1,2,3\}$. In the next two theorems, we describe the obtained classification results for the quasi-$Sp(V,f)$-equivalence classes in the case the characteristic of the field $\mathbb{F}$ is equal to 2. These two theorems will be proved in Section 4.

Theorem 1.4 (Section 4) *Suppose* $\mathsf{char}(\mathbb{F}) = 2$. *Let* χ *be a trivector of* V *which is quasi-*$Sp(V,f)$*-equivalent with a trivector of Type (A), (B), (C) or (D). Then* χ *is quasi-*$Sp(V,f)$*-equivalent with (at least) one of the following trivectors:*

(Q1′) *the zero vector of* $\bigwedge^3 V$;

(Q2′) $\chi_{A1} = e_1 \wedge e_2 \wedge e_3$;

$(Q3')$ $\chi_{A2} = e_1 \wedge e_2 \wedge f_2$;

$(Q4')$ $\chi_{B4}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot e_1 \wedge f_2 \wedge f_3$ *for some nonsquare* λ *of* $\mathbb{F}$;

$(Q5')$ $\chi_{B5}(\lambda) = \lambda \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge (e_2 - e_3) \wedge (f_2 + f_3)$ *for some* $\lambda \in \mathbb{F}$ *such that the polynomial* $X^2 + \lambda X + 1 \in \mathbb{F}[X]$ *is irreducible;*

$(Q6')$ $\chi_{C1}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot f_1 \wedge f_2 \wedge f_3$ *for some* $\lambda \in \mathbb{F}^*$;

$(Q7')$ $\chi_{D2}(\lambda) = \lambda \cdot e_1 \wedge e_2 \wedge f_3 + e_2 \wedge f_1 \wedge e_3 + f_1 \wedge e_1 \wedge f_2$ *for some* $\lambda \in \mathbb{F}^*$;

$(Q8')$ $\chi_{D3}(\lambda_1, \lambda_2) = e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge f_1 + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$ *such that the equation* $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ *has no solutions for* $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$;

$(Q9')$ $\chi_{D4}(\lambda_1, \lambda_2) = e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge (f_1 + f_3) + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$.

In the case $\mathsf{char}(\mathbb{F}) = 2$, a trivector of V is said to be of *Type (Qi')*, $i \in \{1, 2, \ldots, 9\}$, if it is quasi-$Sp(V, f)$-equivalent with (one of) the trivector(s) defined in (Qi') of Theorem 1.4. Two (3×3)-matrices A_1 and A_2 over $\mathbb{F}$ are called *pseudo-congruent* if there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that the matrix $A_1 - MA_2M^T$ is *alternating*, i.e. skew-symmetric and having all diagonal elements equal to 0. The relation of being pseudo-congruent defines an equivalence relation of the set of all (3×3)-matrices over $\mathbb{F}$.

Theorem 1.5 (Section 4) *Suppose* $\mathsf{char}(\mathbb{F}) = 2$.

- *Let* $i, j \in \{1, 2, \ldots, 9\}$ *with* $i \neq j$. *Then no trivector of Type (Qi') is quasi-*$Sp(V, f)$*-equivalent with a trivector of Type (Qj').*
- *Let* λ *and* λ' *be two nonsquares of* $\mathbb{F}$. *Then the two trivectors* $\chi_{B4}(\lambda)$ *and* $\chi_{B4}(\lambda')$ *are quasi-*$Sp(V, f)$*-equivalent if and only if the polynomials* $X^2 + \lambda$ *and* $X^2 + \lambda'$ *define the same quadratic extension of* $\mathbb{F}$ *in* $\overline{\mathbb{F}}$.
- *Let* λ *and* λ' *be two elements of* $\mathbb{F}$ *such that the polynomials* $X^2 + \lambda X + 1 \in \mathbb{F}[X]$ *and* $X^2 + \lambda' X + 1 \in \mathbb{F}[X]$ *are irreducible. Then the two trivectors* $\chi_{B5}(\lambda)$ *and* $\chi_{B5}(\lambda')$ *are quasi-*$Sp(V, f)$*-equivalent if and only if the polynomials* $X^2 + \lambda X + 1$ *and* $X^2 + \lambda' X + 1$ *define the same quadratic extension of* $\mathbb{F}$ *in* $\overline{\mathbb{F}}$.
- *Let* $\lambda, \lambda' \in \mathbb{F}^*$. *Then the two trivectors* $\chi_{C1}(\lambda)$ *and* $\chi_{C1}(\lambda')$ *are quasi-*$Sp(V, f)$*-equivalent if and only if* $\lambda = \lambda'$.
- *Let* $\lambda, \lambda' \in \mathbb{F}^*$. *Then the two trivectors* $\chi_{D2}(\lambda)$ *and* $\chi_{D2}(\lambda')$ *are quasi-*$Sp(V, f)$*-equivalent if and only if* $\lambda = \lambda'$.
- *Let* $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$ *such that none of the equations* $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ *and* $\lambda_1' X^2 + \lambda_2' Y^2 + Z^2 = 0$ *has solutions for* $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$. *Then the two trivectors* $\chi_{D3}(\lambda_1, \lambda_2)$ *and* $\chi_{D3}(\lambda_1', \lambda_2')$ *are quasi-*$Sp(V, f)$*-equivalent if and only if there exists a* $\mu \in \mathbb{F}^*$ *such that the matrices* $\mathrm{diag}(\mu\lambda_1, \mu\lambda_2, \mu)$ *and* $\mathrm{diag}(\lambda_1', \lambda_2', 1)$ *are pseudo-congruent.*

- *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. Then the two trivectors $\chi_{D4}(\lambda_1, \lambda_2)$ and $\chi_{D4}(\lambda_1', \lambda_2')$ are quasi-$Sp(V,f)$-equivalent if and only if there exists a $\mu \in \mathbb{F}^*$ such that the matrices*

$$\begin{bmatrix} \mu\lambda_1 & 0 & \mu\lambda_1 \\ 0 & \mu\lambda_2 & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \lambda_1' & 0 & \lambda_1' \\ 0 & \lambda_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  are pseudo-congruent.

It should be mentioned that the classification of the quasi-$Sp(V,f)$-equivalence classes is incomplete in the characteristic 2 case, as it does not include the case of trivectors of Type (E). It could be that (for certain fields) some (if not all) of the trivectors of Type (E) are quasi-$Sp(V,f)$-equivalent with a trivector of Type (A), (B), (C) or (D). In the final remark of this paper, we will show that this is always the case for some of the trivectors of Type (E).

Suppose $\mathbb{F}$ is a finite field of order q. For q odd, a classification of all quasi-$Sp(V,f)$-equivalence classes (or equivalently, $Sp(V,f)$-equivalence classes contained in W) can be found in [9] along with information about the stabilizers of the trivectors. For q even, $\mathbb{F}$ is a perfect field. For perfect fields of characteristic 2, it can be shown that any nonzero trivector is quasi-$Sp(V,f)$-equivalent with a trivector of Type (A), (B) or (C) (see [10] for a discussion using the connection with hyperplanes of $DW(5,\mathbb{F})$) and so our results here along with [10] offer a complete classification of the quasi-$Sp(V,f)$-equivalence classes for those fields.

The classification results obtained in the present paper rely on the classification of the $Sp(V,f)$-equivalence classes of trivectors of V. This lengthy classification, which was realized in a series of four papers [13, 14, 15, 16] by the authors, will be recalled in Section 2. This classification will immediately be used in Section 2 to determine all quasi-$Sp(V,f)$-equivalence classes in the case $\mathsf{char}(\mathbb{F}) \neq 2$. The case $\mathsf{char}(\mathbb{F}) = 2$ is more complicated and will be treated in Section 4. Section 3 is devoted to developing the tools that will be necessary to obtain the classification results in the characteristic 2 case.

2 The classification of the $Sp(V,f)$-equivalence classes of trivectors

We continue with the notation introduced in Section 1. So, V denotes a 6-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f, and $(e_1, f_1, e_2, f_2, e_3, f_3)$ denotes a fixed hyperbolic basis of (V,f). In a series of four papers ([13, 14, 15, 16]), the authors obtained a complete classification of all $Sp(V,f)$-equivalence classes of nonzero trivectors of V. This classification is summarized in Tables 1 and 2.

A trivector of V is said to be of *Type* $(X) \in \{(A1), (A2), \dots, (E2'), (E3')\}$ if it is $Sp(V,f)$-equivalent with (one of) the trivector(s) mentioned in (X) of the tables. The trivectors

(A1)	$\chi_{A1} := e_1 \wedge e_2 \wedge e_3$
(A2)	$\chi_{A2} := e_1 \wedge e_2 \wedge f_2$
(B1)	$\chi_{B1} := e_1 \wedge e_2 \wedge e_3 + e_1 \wedge f_1 \wedge f_3$
(B2)	$\chi_{B2} := e_1 \wedge e_2 \wedge f_2 + e_1 \wedge f_1 \wedge e_3$
(B3)	$\chi_{B3} := e_1 \wedge e_2 \wedge f_2 + e_1 \wedge e_3 \wedge f_3$
(B4)	$\chi_{B4}(\lambda) := e_1 \wedge e_2 \wedge e_3 + \lambda \cdot e_1 \wedge f_2 \wedge f_3$ for some $\lambda \in \mathbb{F}^*$
	λ/λ' is a square in $\mathbb{F}$
(B5)	$\chi_{B5}(\lambda) := \lambda \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge (e_2 - e_3) \wedge (f_2 + f_3)$ for some $\lambda \in \mathbb{F}^*$
	$\lambda = \lambda'$
(C1)	$\chi_{C1}(\lambda) := e_1 \wedge e_2 \wedge e_3 + \lambda \cdot f_1 \wedge f_2 \wedge f_3$ for some $\lambda \in \mathbb{F}^*$
	$\lambda' \in \{\lambda, -\lambda\}$
(C2)	$\chi_{C2}(\lambda) := f_1 \wedge (e_2 + e_3) \wedge (f_2 - f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$
	$\lambda = \lambda'$
(C3)	$\chi_{C3}(\lambda) := e_1 \wedge e_2 \wedge f_2 + \lambda \cdot f_1 \wedge e_3 \wedge f_3$ for some $\lambda \in \mathbb{F}^*$
	$\lambda' \in \{\lambda, -\lambda\}$
(C4)	$\chi_{C4}(\lambda) := f_1 \wedge e_3 \wedge (e_2 + f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$
	$\lambda' \in \{\lambda, -\lambda\}$
(C5)	$\chi_{C5}(\lambda) := e_1 \wedge e_3 \wedge (f_3 + f_2) + \lambda \cdot e_2 \wedge f_3 \wedge (f_1 + e_3)$ for some $\lambda \in \mathbb{F}^*$
	$\lambda' \in \{\lambda, -\lambda\}$
(C6)	$\chi_{C6}(\lambda, \epsilon) := f_1 \wedge (e_2 + e_3) \wedge (f_2 + \epsilon f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$ and some $\epsilon \in \mathbb{F} \setminus \{0, -1\}$
	$\epsilon = \epsilon'$ and $\lambda' \in \{\lambda, -\lambda\}$
(D1)	$\chi_{D1} := e_1 \wedge e_2 \wedge f_2 + e_2 \wedge f_1 \wedge e_3 + f_1 \wedge e_1 \wedge f_3$
(D2)	$\chi_{D2}(\lambda) := \lambda \cdot e_1 \wedge e_2 \wedge f_3 + e_2 \wedge f_1 \wedge e_3 + f_1 \wedge e_1 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$
	$\lambda = \lambda'$
(D3)	$\chi_{D3}(\lambda_1, \lambda_2) := e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge f_1 + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$
	the matrices $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_1\lambda_2)$ and $\mathrm{diag}(\lambda_1', \lambda_2', \lambda_1'\lambda_2')$ are congruent
(D4)	$\chi_{D4}(\lambda_1, \lambda_2) := e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge (f_1 + f_3) + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$
	$\lambda_1 = \lambda_1'$ and there exist $X, Y \in \mathbb{F}$ such that $X^2 + \lambda_1 XY + \lambda_1 Y^2 = \lambda_2'/\lambda_2$
(D5)	$\chi_{D5}(\lambda) := e_1 \wedge e_2 \wedge f_3 + \lambda \cdot e_2 \wedge e_3 \wedge (f_1 + f_2 + f_3) - e_3 \wedge e_1 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$
	$1/\lambda + 1/\lambda'$ is of the form $X^2 + X$ for some $X \in \mathbb{F}$
(D6)	(only if $\mathsf{char}(\mathbb{F}) \neq 2$) $\chi_{D6} := -e_1 \wedge e_2 \wedge f_2 + e_2 \wedge e_3 \wedge f_1 + e_3 \wedge e_1 \wedge f_3$
(D7)	(only if $\lvert\mathbb{F}\rvert = 2$) $\chi_{D7} := e_1 \wedge e_2 \wedge f_2 + e_2 \wedge e_3 \wedge (f_1 + f_3) + e_3 \wedge e_1 \wedge f_3$

Table 1: The $Sp(V, f)$-equivalence classes of trivectors of Type A, B, C and D

(E1)	$\chi_{E1}(a,b,h_1,h_2,h_3) := 2\cdot e_1\wedge e_2\wedge e_3 + h_1h_2h_3a(a^2+3b)\cdot f_1\wedge f_2\wedge f_3 + (a^2+2b)\cdot \Big(h_1h_2\cdot f_1\wedge f_2\wedge e_3 + h_1h_3\cdot f_1\wedge e_2\wedge f_3 + h_2h_3\cdot e_1\wedge f_2\wedge f_3\Big) + a\cdot \Big(h_1\cdot f_1\wedge e_2\wedge e_3 + h_2\cdot e_1\wedge f_2\wedge e_3 + h_3\cdot e_1\wedge e_2\wedge f_3\Big)$ for some $(a,b)\in\Psi$ and some $h_1,h_2,h_3\in\mathbb{F}^*$	$(a,b)=(a',b')$ and there exists a 3×3-matrix A over $\mathbb{F}'$ with $\det(A)=1$ and an $\epsilon\in\{1,-1\}$ such that $A\cdot\mathrm{diag}(h_1,h_2,h_3)\cdot(A^\psi)^T=\epsilon\cdot\mathrm{diag}(h_1',h_2',h_3')$. Here, $\mathbb{F}'\subseteq\overline{\mathbb{F}}$ is the extension of $\mathbb{F}$ determined by $X^2-aX-b\in\mathbb{F}[X]$ and $1\neq\psi\in\mathsf{Gal}(\mathbb{F}'/\mathbb{F})$
(E2)	$\chi_{E2}(a,b,k) := k\cdot\Big(f_1\wedge e_2\wedge f_3 - b\cdot f_1\wedge f_2\wedge e_3 + a\cdot f_1\wedge e_3\wedge f_3\Big) + e_1\wedge e_2\wedge f_2 + e_1\wedge e_3\wedge f_3$ for some $(a,b)\in\Psi$ and some $k\in\mathbb{F}^*$	$(a,b)=(a',b')$ and $k'\in\{k,-k\}$
(E3)	$\chi_{E3}(a,b,k,h) := k\cdot\Big(f_1\wedge e_2\wedge f_3 - b\cdot f_1\wedge f_2\wedge e_3 + a\cdot f_1\wedge e_3\wedge f_3\Big) + e_1\wedge e_2\wedge f_2 + e_1\wedge e_3\wedge f_3 + h\cdot e_1\wedge f_2\wedge f_3$ for some $(a,b)\in\Psi$ and some $k,h\in\mathbb{F}^*$	$(a,b)=(a',b')$ and there exist a $\sigma\in\{1,-1\}$ and $X,Y\in\mathbb{F}$ such that $k'=\sigma k$ and $h'=\sigma h(X^2+aXY-bY^2)$
(E4)	$\chi_{E4}(a,b,k,h_1,h_2) := (1-h_1h_2(a^2+4b))\cdot e_1\wedge e_2\wedge f_2 + (1+h_1h_2(a^2+4b))\cdot e_1\wedge e_3\wedge f_3 + h_1(1-h_1h_2(a^2+4b))\cdot e_1\wedge f_2\wedge f_3 + (a^2+4b)h_2\cdot e_1\wedge e_2\wedge e_3 + k\cdot\Big(f_1\wedge e_2\wedge f_3 - b(1-h_1h_2(a^2+4b))\cdot f_1\wedge f_2\wedge e_3 + a\cdot f_1\wedge e_3\wedge f_3\Big)$ for some $(a,b)\in\Psi$ and some $k,h_1,h_2\in\mathbb{F}^*$ satisfying $h_1h_2(a^2+4b)\neq 1$	$(a,b)=(a',b')$, $h_1h_2=h_1'h_2'$ and there exist $X,Y,Z,U\in\mathbb{F}$ and a $\sigma\in\{1,-1\}$ such that $k'=\sigma k$ and $\sigma h_1'=h_1(X^2+aXY-bY^2)+h_2(Z^2+aZU-bU^2)$
(E5)	$\chi_{E5}(a,b,k) := f_1\wedge e_2\wedge f_3 + 2\cdot e_1\wedge f_1\wedge e_2 - a\cdot f_1\wedge e_2\wedge f_2 + a\cdot f_1\wedge e_3\wedge f_3 + (a^2+b)\cdot f_1\wedge f_2\wedge e_3 + k\cdot\Big(a\cdot e_1\wedge f_2\wedge e_3 - e_1\wedge e_2\wedge f_2 + e_1\wedge e_3\wedge f_3\Big) + a\cdot e_1\wedge f_1\wedge e_3$ for some $(a,b)\in\Psi$ and some $k\in\mathbb{F}^*$	$(a,b)=(a',b')$ and $k'\in\{k,-k\}$
(E1’)	$\chi'_{E1}(a,h_1,h_2,h_3) := \frac{a+1}{a}\cdot e_1\wedge e_2\wedge e_3 + (e_1+h_1f_1)\wedge(e_2+h_2f_2)\wedge(e_3+h_3f_3) + (a+1)h_1h_2h_3\cdot f_1\wedge f_2\wedge f_3$ for some $a\in\Psi'$ and some $h_1,h_2,h_3\in\mathbb{F}^*$	$a=a'$ and there exists a 3×3-matrix A over $\mathbb{F}'$ with $\det(A)=1$ such that $\mathrm{diag}(h_1',h_2',h_3')=A\cdot\mathrm{diag}(h_1,h_2,h_3)\cdot A^T$. Here, $\mathbb{F}'\subseteq\overline{\mathbb{F}}$ is the extension of $\mathbb{F}$ determined by $X^2+a\in\mathbb{F}[X]$
(E2’)	$\chi'_{E2}(a,k,h_1,h_2) := \frac{1}{a}\cdot e_1\wedge(e_2+h_1(a+1)f_3)\wedge f_2 + k\cdot f_1\wedge e_3\wedge(h_2(a+1)e_2+f_3) + \frac{1}{(a+1)^2}\cdot(e_1+kf_1)\wedge(e_2+(a+1)e_3+h_1(a+1)f_3)\wedge(h_2(a+1)e_2+(a+1)f_2+f_3)$ for some $a\in\Psi'$ and some $k,h_1,h_2\in\mathbb{F}$ satisfying $k\neq 0$ and $h_1h_2(a+1)^2\neq 1$	$a=a'$, $k=k'$, $h_1h_2=h_1'h_2'$ and there exist $X,Y,Z,U\in\mathbb{F}$ such that $h_1'=h_1(X^2+aY^2)+h_2(Z^2+aU^2)+(XU+YZ)$
(E3’)	$\chi'_{E3}(a,h_1,h_2) := \frac{1}{a}\cdot e_1\wedge(e_2+e_3)\wedge f_2 + e_2\wedge f_1\wedge(e_1+h_1f_3) + \frac{1}{a+1}\cdot(e_1+e_2)\wedge(e_3+h_1f_3)\wedge\Big((a+1)^2h_2e_1+f_1+f_2\Big)$ for some $a\in\Psi'$ and some $(h_1,h_2)\in\mathbb{F}^*\times\mathbb{F}$	$a=a'$, $h_1=h_1'$ and h_2+h_2' is of the form $h_1(X^2+aY^2)+Y$ for some $X,Y\in\mathbb{F}$

Table 2: The $Sp(V,f)$-equivalence classes of trivectors of Type E

of Type (D7) only exist if $|\mathbb{F}| = 2$, which seems somewhat unnatural. The reason is that we have aimed to give a classification which is valid for all characteristics. If we had only restricted to the characteristic 2 case, a "more natural" description can be given by replacing (D5) and (D7) by the following:

($D5'$) $\chi_{D5}(\lambda) := e_1 \wedge e_2 \wedge f_3 + \lambda \cdot e_2 \wedge e_3 \wedge (f_1 + f_2 + f_3) - e_3 \wedge e_1 \wedge f_2$ for some $\lambda \in \mathbb{F}^*$ such that λ^{-1} is not of the form $X^2 + X$ for some $X \in \mathbb{F}$;

($D7'$) $\chi_{D7} := e_1 \wedge e_2 \wedge f_2 + e_2 \wedge e_3 \wedge (f_1 + f_3) + e_3 \wedge e_1 \wedge f_3$.

The two tables divide the nonzero trivectors of V into 28 families (which are subfamilies of those mentioned in Proposition 1.1; hence the names for the types). In [13, 14, 15, 16], it was also shown that trivectors belonging to distinct families can never be $Sp(V, f)$-equivalent. In case a family of trivectors has at least two members, the tables also mention a condition that indicates when two trivectors of the same family are equivalent. We illustrate the interpretation of that condition by means of a concrete example, namely that of the (E3)-family: If $(a, b), (a', b') \in \Psi$ and $k, h, k', h' \in \mathbb{F}^*$, then the two trivectors $\chi_{E3}(a, b, k, h)$ and $\chi_{E3}(a', b', k', h')$ are $Sp(V, f)$-equivalent if and only if $(a, b) = (a', b')$ and there exist a $\sigma \in \{1, -1\}$ and $X, Y \in \mathbb{F}$ such that $k' = \sigma k$ and $h' = \sigma h(X^2 + aXY - bY^2)$ (we use accents for the parameters of the second member of the family).

The subspace W defined in Section 1 is the subspace of $\bigwedge^3 V$ generated by the 14 trivectors $e_1 \wedge e_2 \wedge e_3$, $e_1 \wedge e_2 \wedge f_3$, $e_1 \wedge f_2 \wedge e_3$, $e_1 \wedge f_2 \wedge f_3$, $f_1 \wedge e_2 \wedge e_3$, $f_1 \wedge e_2 \wedge f_3$, $f_1 \wedge f_2 \wedge e_3$, $f_1 \wedge f_2 \wedge f_3$, $e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)$, $f_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)$, $e_2 \wedge (e_1 \wedge f_1 - e_3 \wedge f_3)$, $f_2 \wedge (e_1 \wedge f_1 - e_3 \wedge f_3)$, $e_3 \wedge (e_1 \wedge f_1 - e_2 \wedge f_2)$, $f_3 \wedge (e_1 \wedge f_1 - e_2 \wedge f_2)$. So, the nonzero trivectors contained in W are precisely the trivectors of Type A1, B4, C1, D3, E1, E1' and (only if the characteristic of $\mathbb{F}$ is equal to 2) B3.

The subspace $\widetilde{W}$ defined in Section 1 is the subspace of $\bigwedge^3 V$ generated by the 6 trivectors $e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3)$, $f_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3)$, $e_2 \wedge (e_1 \wedge f_1 + e_3 \wedge f_3)$, $f_2 \wedge (e_1 \wedge f_1 + e_3 \wedge f_3)$, $e_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2)$, $f_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2)$. The nonzero trivectors contained in $\widetilde{W}$ are precisely the trivectors of Type B3.

Suppose now that $\mathsf{char}(\mathbb{F}) \neq 2$. Then every trivector $\chi \in \bigwedge^3 V$ can be written in a unique way as $\chi_1 + \chi_2$ where $\chi_1 \in W$ and $\chi_2 \in \widetilde{W}$. We define $\pi_W(\chi) := \chi_1$ and $\pi_{\widetilde{W}}(\chi) := \chi_2$. For every $\theta \in Sp(V, f)$ and every $\chi \in \bigwedge^3 V$, we have $\theta(\chi) = \theta(\pi_W(\chi)) + \theta(\pi_{\widetilde{W}}(\chi))$ where $\theta(\pi_W(\chi)) \in W$ and $\theta(\pi_{\widetilde{W}}(\chi)) \in \widetilde{W}$. So, if $\theta \in Sp(V, f)$, then $\theta \circ \pi_W = \pi_W \circ \theta$ and $\theta \circ \pi_{\widetilde{W}} = \pi_{\widetilde{W}} \circ \theta$.

Proposition 2.1 *Suppose* $\mathsf{char}(\mathbb{F}) \neq 2$. *Then every trivector* χ *of* V *is quasi-*$Sp(V, f)$*-equivalent with* $\pi_W(\chi)$. *Moreover, two trivectors* χ_1 *and* χ_2 *of* V *are quasi-*$Sp(V, f)$*-equivalent if and only if* $\pi_W(\chi_1)$ *and* $\pi_W(\chi_2)$ *are* $Sp(V, f)$*-equivalent.*

Proof. The first claim follows from the fact that $\chi = \pi_W(\chi) + \pi_{\widetilde{W}}(\chi)$, where $\pi_{\widetilde{W}}(\chi) \in \widetilde{W}$. As for the second claim, the two trivectors $\pi_W(\chi_1)$ and $\pi_W(\chi_2)$ are $Sp(V, f)$-equivalent if and only if $\pi_W(\chi_2) = \theta(\pi_W(\chi_1)) = \pi_W(\theta(\chi_1))$ for some $\theta \in Sp(V, f)$, i.e. if and only if there exist a $\theta \in Sp(V, f)$ and a $\chi \in \widetilde{W}$ such that $\chi_2 = \theta(\chi_1) + \chi$. ∎

Theorems 1.2 and 1.3 are immediate consequences of Proposition 2.1 and the classification mentioned in Tables 1 and 2.

3 Tools

We continue with the notation introduced in Section 1. Recall that V is a 6-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f.

The following lemma will be useful during the classification of the quasi-$Sp(V, f)$-equivalence classes of trivectors in the case $\mathsf{char}(\mathbb{F}) = 2$. A proof of it can be found in De Bruyn and Kwiatkowski [13, Lemma 2.9].

Lemma 3.1 ([13]) *Let U be a 4-dimensional vector space over the field $\mathbb{F}$ and let $\{u_1, u_2, u_3, u_4\}$, $\{u_1', u_2', u_3', u_4'\}$ be two bases of U such that $u_1 \wedge u_2 + u_3 \wedge u_4 = u_1' \wedge u_2' + u_3' \wedge u_4'$. Then $u_1 \wedge u_2 \wedge u_3 \wedge u_4 = u_1' \wedge u_2' \wedge u_3' \wedge u_4'$.*

If $(e_1, f_1, e_2, f_2, e_3, f_3)$ is a hyperbolic basis of (V, f), then

(1) for every permutation σ of $\{1, 2, 3\}$, also $(e_{\sigma(1)}, f_{\sigma(1)}, e_{\sigma(2)}, f_{\sigma(2)}, e_{\sigma(3)}, f_{\sigma(3)})$ is a hyperbolic basis of (V, f);

(2) for every $\lambda \in \mathbb{F}^*$, also $(\frac{e_1}{\lambda}, \lambda f_1, e_2, f_2, e_3, f_3)$ is a hyperbolic basis of (V, f);

(3) for every $\lambda \in \mathbb{F}$, also $(e_1 + \lambda e_2, f_1, e_2, -\lambda f_1 + f_2, e_3, f_3)$ is a hyperbolic basis of (V, f);

(4) for every $\lambda \in \mathbb{F}$, also $(e_1, f_1, e_2, f_2, e_3, f_3 + \lambda e_3)$ is a hyperbolic basis of (V, f);

(5) for every $\lambda \in \mathbb{F}$, also $(e_1, f_1, e_2, f_2, e_3 + \lambda f_3, f_3)$ is a hyperbolic basis of (V, f).

For every $i \in \{1, 2, \ldots, 5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of (V, f) such that B_2 can be obtained from B_1 as described in (i) above. The following lemma was proved in De Bruyn [11, Lemma 2.1].

Lemma 3.2 ([11]) *If B and B' are two hyperbolic bases of (V, f), then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ of (V, f) for some $k \geq 0$ such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_5$ for every $i \in \{1, 2, \ldots, k\}$.*

The properties mentioned in Lemmas 3.3 and 3.4 below are known. (One could give explicit descriptions of π and π' with respect to those bases of $\bigwedge^3 V$ and $\bigwedge^4 V$ defined by some specific hyperbolic basis $(e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) and use Lemma 3.2 to show that these descriptions are independent of the chosen hyperbolic basis.)

Lemma 3.3 *There exists a unique linear map* $\pi : \bigwedge^3 V \to V$ *for which* $\pi(W) = \{o\}$ *and* $\pi(e_1 \wedge e_2 \wedge f_2) = e_1$ *for any three linearly independent vectors* $e_1, e_2, f_2 \in V$ *satisfying* $f(e_1, e_2) = f(e_1, f_2) = 0$ *and* $f(e_2, f_2) = 1$.

Lemma 3.4 *There exists a unique linear map* $\pi' : \bigwedge^4 V \to \bigwedge^2 V$ *mapping*

- $e_1 \wedge f_1 \wedge e_2 \wedge e_3$ *to* $e_2 \wedge e_3$ *for any four linearly independent vectors* e_1, f_1, e_2 *and* e_3 *of* V *satisfying* $f(e_1, f_1) = 1$, $f(e_2, e_3) = 0$ *and* $\{e_1, f_1\} \subseteq e_2^{\perp_f} \cap e_3^{\perp_f}$;
- $e_1 \wedge f_1 \wedge e_2 \wedge f_2$ *to* $e_1 \wedge f_1 + e_2 \wedge f_2$ *for any four linearly independent vectors* e_1, f_1, e_2 *and* f_2 *of* V *satisfying* $f(e_1, f_1) = f(e_2, f_2) = 1$ *and* $\{e_1, f_1\} \subseteq e_2^{\perp_f} \cap f_2^{\perp_f}$.

Above, we have already mentioned that V and $\widetilde{W}$ are isomorphic as $Sp(V, f)$-modules. We now describe an explicit isomorphism. If v is a nonzero vector of V, then we define

$$\phi(v) := v \wedge (e_2 \wedge f_2 + e_3 \wedge f_3),$$

where the vectors e_2, f_2, e_3, f_3 are chosen in such a way that $(v, w, e_2, f_2, e_3, f_3)$ is a hyperbolic basis of (V, f) for a certain vector $w \in V$. It can be shown that $\phi(v)$ is independent of the chosen hyperbolic basis $(v, w, e_2, f_2, e_3, f_3)$ of (V, f). We also put $\phi(o)$ equal to the zero vector of $\bigwedge^3 V$. Then $\phi : V \to \widetilde{W}$ is a linear isomorphism between the 6-dimensional vector spaces V and $\widetilde{W}$, and $\phi \circ \theta = \theta \circ \phi$ for every $\theta \in Sp(V, f)$.

The following lemma is a combination of Lemma 5.4 and Corollary 5.5 of De Bruyn and Kwiatkowski [15].

Lemma 3.5 ([15]) *Let* A *and* A' *be two nonsingular* (3×3)*-matrices over* $\mathbb{F}$*, and let* $(e_1, f_1, e_2, f_2, e_3, f_3)$ *be a hyperbolic basis of* (V, f). *Put* $[w_1, w_2, w_3]^T := A \cdot [f_1, f_2, f_3]^T$ *and* $[w_1', w_2', w_3']^T := A' \cdot [f_1, f_2, f_3]^T$. *Then* $e_1 \wedge e_2 \wedge w_3 + e_2 \wedge e_3 \wedge w_1 + e_3 \wedge e_1 \wedge w_2$ *and* $e_1 \wedge e_2 \wedge w_3' + e_2 \wedge e_3 \wedge w_1' + e_3 \wedge e_1 \wedge w_2'$ *are* $Sp(V, f)$*-equivalent if and only if one of the following two equivalent properties are satisfied:*

- *there exists a nonsingular* (3×3)*-matrix* M *over* $\mathbb{F}$ *such that* $A' = \frac{1}{\det(M)} \cdot MAM^T$;
- *the matrices* $\frac{A}{\det(A)}$ *and* $\frac{A'}{\det(A')}$ *are congruent.*

For every $v \in V \setminus \{o\}$, let W_v denote the subspace of $\bigwedge^3 V$ generated by all trivectors $v_1 \wedge v_2 \wedge v_3$, where $v_1, v_2, v_3 \in V$ such that $\langle v_1, v_2, v_3 \rangle$ is a totally isotropic subspace containing v.

Lemma 3.6 *For every* $v \in V \setminus \{o\}$*, we have* $\dim(W_v) = 5$.

Proof. This is a known fact. If we choose a hyperbolic basis $(e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) such that $v = e_1$, then we would have $W_v = \langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge f_3, e_1 \wedge f_2 \wedge e_3, e_1 \wedge f_2 \wedge f_3, e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3) \rangle$. ∎

Lemma 3.7 *Let $v \in V\setminus\{o\}$ and $\langle v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\rangle$ a totally isotropic 3-dimensional subspace containing v. Then there exist three 3-dimensional totally isotropic subspaces $\langle v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\rangle$, $\langle v_1^{(3)}, v_2^{(3)}, v_3^{(3)}\rangle$ and $\langle v_1^{(4)}, v_2^{(4)}, v_3^{(4)}\rangle$ containing v and intersecting $\langle v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\rangle$ in a subspace of dimension 2 such that $\dim(\langle v_1^{(j)}\wedge v_2^{(j)}\wedge v_3^{(j)}\,|\,j \in \{1,2,3,4\}\rangle) = 4$. Moreover, for any three totally isotropic subspaces that have been chosen in this way the following holds: if $\langle v_1^{(5)}, v_2^{(5)}, v_3^{(5)}\rangle$ is a 3-dimensional totally isotropic subspace intersecting $\langle v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\rangle$ in $\langle v\rangle$, then $W_v = \langle v_1^{(j)} \wedge v_2^{(j)} \wedge v_3^{(j)}\,|\,j \in \{1,2,3,4,5\}\rangle$.*

Proof. If the claim of the lemma is valid if $(v, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}) = (w, w_1, w_2, w_3)$, then the claim is also valid if $(v, v_1^{(1)}, v_2^{(1)}, v_3^{(1)})$ were equal to (w, w_1', w_2', w_3'), where w_1', w_2' and w_3' are three vectors of V such that $\langle w_1', w_2', w_3'\rangle = \langle w_1, w_2, w_3\rangle$. So, without loss of generality, we may suppose that $v = v_1^{(1)}$. Now, we can choose a hyperbolic basis $(e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) such that $v_1^{(1)} = e_1$, $v_2^{(1)} = e_2$ and $v_3^{(1)} = e_3$. Then $W_v = \langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge f_3, e_1 \wedge f_2 \wedge e_3, e_1 \wedge f_2 \wedge f_3, e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)\rangle$. If $\langle v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\rangle$, $\langle v_1^{(3)}, v_2^{(3)}, v_3^{(3)}\rangle$ and $\langle v_1^{(4)}, v_2^{(4)}, v_3^{(4)}\rangle$ are three 3-dimensional totally isotropic subspaces containing v and intersecting $\langle v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\rangle$ in subspaces of dimension 2, then we have $\langle v_1^{(j)}\wedge v_2^{(j)}\wedge v_3^{(j)}\,|\,j \in \{1,2,3,4\}\rangle \subseteq \langle e_1\wedge e_2\wedge e_3, e_1\wedge e_2\wedge f_3, e_1\wedge f_2\wedge e_3, e_1\wedge(e_2\wedge f_2 - e_3\wedge f_3)\rangle$. So, we must show that we can choose these subspaces in such a way that we have equality. This is realized by making the following choices:

$$v_1^{(2)} = v_1^{(3)} = v_1^{(4)} = e_1,\ v_2^{(2)} = e_2,\ v_3^{(2)} = f_3,\ v_2^{(3)} = f_2,\ v_3^{(3)} = e_3,\ v_2^{(4)} = e_2+e_3,\ v_3^{(4)} = f_2-f_3.$$

If $\langle v_1^{(5)}, v_2^{(5)}, v_3^{(5)}\rangle$ is a 3-dimensional totally isotropic subspace intersecting $\langle v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\rangle$ in $\langle v\rangle$, then $v_1^{(5)} \wedge v_2^{(5)} \wedge v_3^{(5)}$, written as a linear combination of the trivectors $e_1 \wedge e_2 \wedge e_3$, $e_1 \wedge e_2 \wedge f_3$, $e_1 \wedge f_2 \wedge e_3$, $e_1 \wedge f_2 \wedge f_3$ and $e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)$, should have a nonzero component in $e_1 \wedge f_2 \wedge f_3$, implying that $W_v = \langle v_1^{(j)} \wedge v_2^{(j)} \wedge v_3^{(j)}\,|\,j \in \{1,2,3,4,5\}\rangle$. ■

For every $\chi \in \bigwedge^3 V$, let $\mathcal{D}(\chi)$ denote the set of all $v \in V$ such that $\chi \wedge \chi' = 0$ for every $\chi' \in W_v$, i.e. the set of all $v \in V$ such that $\chi \wedge v_1 \wedge v_2 \wedge v_3 = 0$ for all $v_1, v_2, v_3 \in V$ such that $\langle v_1, v_2, v_3\rangle$ is a totally isotropic subspace containing v. Notice that $v \in \mathcal{D}(\chi)$ for every $v \in V$ such that $\chi \wedge v = 0$. However, it is also possible that $v \in \mathcal{D}(\chi)$ while $\chi \wedge v \neq 0$.

Lemma 3.8 *For every $\chi \in \bigwedge^3 V$ and every $\chi' \in \widetilde{W}$, we have $\mathcal{D}(\chi) = \mathcal{D}(\chi + \chi')$.*

Proof. This follows from the fact that $\chi' \wedge v_1 \wedge v_2 \wedge v_3 = 0$ for all vectors v_1, v_2, v_3 of V such that $\langle v_1, v_2, v_3\rangle$ is totally isotropic. ■

Lemma 3.9 *If χ_1 and χ_2 are two quasi-$Sp(V, f)$-equivalent trivectors of V, then there exists a $\theta \in Sp(V, f)$ such that $\mathcal{D}(\chi_2) = \theta(\mathcal{D}(\chi_1))$.*

Proof. Suppose χ_1 and χ_2 are two quasi-$Sp(V, f)$-equivalent trivectors. Then there exists a $\theta \in Sp(V, f)$ and a $\chi \in \widetilde{W}$ such that $\chi_2 = \theta(\chi_1) + \chi$. Observe that $\chi_1 \wedge v_1 \wedge v_2 \wedge v_3 =$

$0 \Leftrightarrow \theta(\chi_1) \wedge \theta(v_1) \wedge \theta(v_2) \wedge \theta(v_3) = 0$, $v \in \langle v_1, v_2, v_3 \rangle \Leftrightarrow \theta(v) \in \langle \theta(v_1), \theta(v_2), \theta(v_3) \rangle$ and $f(v_i, v_j) = 0 \Leftrightarrow f(\theta(v_i), \theta(v_j)) = 0$ $(i, j \in \{1, 2, 3\})$. So, $\mathcal{D}(\chi_2) = \mathcal{D}(\theta(\chi_1)) = \theta(\mathcal{D}(\chi_1))$. ■

In the sequel of this section, we will suppose that $\mathsf{char}(\mathbb{F}) = 2$. For every hyperbolic basis $B = (e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) and every trivector

$$\begin{aligned}\chi = {} & \lambda_1 \cdot e_1 \wedge e_2 \wedge e_3 + \mu_1 \cdot f_1 \wedge f_2 \wedge f_3 + \lambda_2 \cdot e_1 \wedge e_2 \wedge f_3 + \mu_2 \cdot f_1 \wedge f_2 \wedge e_3 \\ & + \lambda_3 \cdot e_1 \wedge f_2 \wedge e_3 + \mu_3 \cdot f_1 \wedge e_2 \wedge f_3 + \lambda_4 \cdot e_1 \wedge f_2 \wedge f_3 + \mu_4 \cdot f_1 \wedge e_2 \wedge e_3 \\ & + \lambda_5 \cdot e_1 \wedge e_2 \wedge f_2 + \mu_5 \cdot f_1 \wedge e_3 \wedge f_3 + \lambda_6 \cdot e_1 \wedge e_3 \wedge f_3 + \mu_6 \cdot f_1 \wedge e_2 \wedge f_2 \\ & + \lambda_7 \cdot e_2 \wedge e_1 \wedge f_1 + \mu_7 \cdot f_2 \wedge e_3 \wedge f_3 + \lambda_8 \cdot e_2 \wedge e_3 \wedge f_3 + \mu_8 \cdot f_2 \wedge e_1 \wedge f_1 \\ & + \lambda_9 \cdot e_3 \wedge e_1 \wedge f_1 + \mu_9 \cdot f_3 \wedge e_2 \wedge f_2 + \lambda_{10} \cdot e_3 \wedge e_2 \wedge f_2 + \mu_{10} \cdot f_3 \wedge e_1 \wedge f_1\end{aligned}$$

of V, we define

$$\eta_B(\chi) \;:=\; \sum_{i=1}^{10} \lambda_i \mu_i.$$

For every hyperbolic basis $B = (e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f), let $\mathcal{B}_B$ denote the ordered basis $(e_1 \wedge e_2 \wedge e_3, f_1 \wedge f_2 \wedge f_3, e_1 \wedge e_2 \wedge f_3, f_1 \wedge f_2 \wedge e_3, e_1 \wedge f_2 \wedge e_3, f_1 \wedge e_2 \wedge f_3, e_1 \wedge f_2 \wedge f_3, f_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge f_2, f_1 \wedge e_3 \wedge f_3, e_1 \wedge e_3 \wedge f_3, f_1 \wedge e_2 \wedge f_2, e_2 \wedge e_1 \wedge f_1, f_2 \wedge e_3 \wedge f_3, e_2 \wedge e_3 \wedge f_3, f_2 \wedge e_1 \wedge f_1, e_3 \wedge e_1 \wedge f_1, f_3 \wedge e_2 \wedge f_2, e_3 \wedge e_2 \wedge f_2, f_3 \wedge e_1 \wedge f_1)$ of $\bigwedge^3 V$.

Proposition 3.10 *Suppose* $\mathsf{char}(\mathbb{F}) = 2$. *Then for any two hyperbolic bases* B *and* B' *of* (V, f), *we have* $\eta_B = \eta_{B'}$.

Proof. In view of Lemma 3.2, it suffices to show that $\eta_B = \eta_{B'}$ if $(B, B') \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_5$. This clearly holds if $(B, B') \in \Omega_1$. We will now also deal with the four remaining cases. Suppose $B = (e_1, f_1, e_2, f_2, e_3, f_3)$ and let χ be an arbitrary vector of $\bigwedge^3 V$.

(1) Suppose $(B, B') \in \Omega_2$. Then $B' = (\frac{e_1}{\lambda}, \lambda f_1, e_2, f_2, e_3, f_3)$ for some $\lambda \in \mathbb{F}^*$. If $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{10}, \mu_{10})$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_{B'}$, then $(\frac{\lambda_1}{\lambda}, \lambda\mu_1, \frac{\lambda_2}{\lambda}, \lambda\mu_2, \frac{\lambda_3}{\lambda}, \lambda\mu_3, \frac{\lambda_4}{\lambda}, \lambda\mu_4, \frac{\lambda_5}{\lambda}, \lambda\mu_5, \frac{\lambda_6}{\lambda}, \lambda\mu_6, \lambda_7, \mu_7, \lambda_8, \mu_8, \lambda_9, \mu_9, \lambda_{10}, \mu_{10})$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_B$. So, we see that $\eta_{B'}(\chi) = \eta_B(\chi)$.

(2) Suppose $(B, B') \in \Omega_3$. Then $B' = (e_1 + \lambda e_2, f_1, e_2, \lambda f_1 + f_2, e_3, f_3)$ for some $\lambda \in \mathbb{F}$. If $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{10}, \mu_{10})$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_{B'}$, then $(\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3 + \lambda^2\lambda_4 + \lambda\mu_9 + \lambda\mu_{10}, \lambda_4, \lambda_3\lambda^2 + \mu_4 + \lambda\lambda_9 + \lambda\lambda_{10}, \lambda_5, \mu_5 + \lambda\mu_7, \lambda_6, \mu_6 + \lambda\mu_8, \lambda\lambda_5 + \lambda_7, \mu_7, \lambda\lambda_6 + \lambda_8, \mu_8, \lambda\lambda_3 + \lambda_9, \mu_9 + \lambda\lambda_4, \lambda\lambda_3 + \lambda_{10}, \mu_{10} + \lambda\lambda_4)$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_B$. One verifies that $\eta_B(\chi) = \eta_{B'}(\chi)$.

(3) Suppose $(B, B') \in \Omega_4$. Then $B' = (e_1, f_1, e_2, f_2, e_3, f_3 + \lambda e_3)$ for some $\lambda \in \mathbb{F}$. If $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{10}, \mu_{10})$ are the coordinates of χ with respect to the ordered bases $\mathcal{B}_{B'}$, then $(\lambda_1 + \lambda\lambda_2, \mu_1, \lambda_2, \mu_2 + \lambda\mu_1, \lambda_3 + \lambda\lambda_4, \mu_3, \lambda_4, \mu_4 + \lambda\mu_3, \lambda_5, \mu_5, \lambda_6, \mu_6, \lambda_7, \mu_7, \lambda_8, \mu_8, \lambda_9 + \lambda\mu_{10}, \mu_9, \lambda_{10} + \lambda\mu_9, \mu_{10})$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_B$. One verifies that $\eta_B(\chi) = \eta_{B'}(\chi)$.

(4) Suppose $(B, B') \in \Omega_5$. Then $B' = (e_1, f_1, e_2, f_2, e_3 + \lambda f_3, f_3)$ for some $\lambda \in \mathbb{F}$. If $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{10}, \mu_{10})$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_{B'}$, then $(\lambda_1, \mu_1+\lambda\mu_2, \lambda\lambda_1+\lambda_2, \mu_2, \lambda_3, \mu_3+\lambda\mu_4, \lambda\lambda_3+\lambda_4, \mu_4, \lambda_5, \mu_5, \lambda_6, \mu_6, \lambda_7, \mu_7, \lambda_8, \mu_8, \lambda_9, \mu_9+\lambda\lambda_{10}, \lambda_{10}, \mu_{10} + \lambda\lambda_9)$ are the coordinates of χ with respect to the ordered basis $\mathcal{B}_B$. One verifies that $\eta_B(\chi) = \eta_{B'}(\chi)$. ■

Put $\eta := \eta_B$ where B is any hyperbolic basis of (V, f).

Corollary 3.11 *Suppose* $\mathsf{char}(\mathbb{F}) = 2$. *Then for every trivector* χ *and every* $\theta \in Sp(V, f)$, *we have* $\eta(\chi) = \eta(\theta(\chi))$.

Proof. Let B be an arbitrary hyperbolic basis of (V, f). Then we have $\eta(\theta(\chi)) = \eta_{\theta(B)}(\theta(\chi)) = \eta_B(\chi) = \eta(\chi)$. ■

Lemma 3.12 *Suppose* $\mathsf{char}(\mathbb{F}) = 2$. *Then* $\eta(\chi + \chi') = \eta(\chi)$ *for every* $\chi \in W$ *and every* $\chi' \in \widetilde{W}$.

Proof. Obviously, this holds if $\chi' = 0$. So, suppose $\chi' \neq 0$ and put $\chi' = \phi(v)$ where v is some nonzero vector of V. There exists a hyperbolic basis $B = (e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) such that $v = e_1$. Let $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{10}, \mu_{10})$ denote the coordinates of χ with respect to the ordered basis $\mathcal{B}_B$. Since $\chi \in W$, we have

$$\lambda_5 = \lambda_6,\ \mu_5 = \mu_6,\ \lambda_7 = \lambda_8,\ \mu_7 = \mu_8,\ \lambda_9 = \lambda_{10},\ \mu_9 = \mu_{10}.$$

Now, the coordinates of $\chi + \phi(v)$ with respect to $\mathcal{B}_B$ are $(\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3, \lambda_4, \mu_4, \lambda_5 + 1, \mu_5, \lambda_6 + 1, \mu_6, \lambda_7, \mu_7, \lambda_8, \mu_8, \lambda_9, \mu_9, \lambda_{10}, \mu_{10})$. So, we have $\eta(\chi + \chi') = \eta_B(\chi + \phi(v)) = \eta_B(\chi) + (\mu_5 + \mu_6) = \eta_B(\chi) = \eta(\chi)$. ■

Corollary 3.11 and Lemma 3.12 then implies the following.

Corollary 3.13 *Suppose* $\mathsf{char}(\mathbb{F}) = 2$ *and* χ_1, χ_2 *are two quasi-*$Sp(V, f)$*-equivalent trivectors of* W. *Then* $\eta(\chi_1) = \eta(\chi_2)$.

Remark. The form η defines a quadratic form on $\bigwedge^3 V$, left invariant by $Sp(V, f)$. Denote by $b : \bigwedge^3 V \times \bigwedge^3 V \to \mathbb{F}; (\chi_1, \chi_2) \mapsto \eta(\chi_1 + \chi_2) - \eta(\chi_1) - \eta(\chi_2)$ the nondegenerate alternating bilinear form associated to η. If η' and b' are the restrictions of η and b to W and $W \times W$ respectively, then $\widetilde{W}$ is the radical of b' and is totally singular for η'.

Also the following lemma will be useful in our classification.

Lemma 3.14 *Suppose* $\mathsf{char}(\mathbb{F}) = 2$. *Let* A *and* A' *be two nonsingular* (3×3)*-matrices over* $\mathbb{F}$, *and* $(e_1, f_1, e_2, f_2, e_3, f_3)$ *a hyperbolic basis of* (V, f). *Put* $[w_1, w_2, w_3]^T := A\cdot[f_1, f_2, f_3]^T$ *and* $[w_1', w_2', w_3']^T := A'\cdot[f_1, f_2, f_3]^T$. *If there exists a* $\mu \in \mathbb{F}^*$ *such that the matrices* μA *and* A' *are pseudo-congruent, then the two trivectors* $\chi = e_1\wedge e_2\wedge w_3 + e_2\wedge e_3\wedge w_1 + e_3\wedge e_1\wedge w_2$ *and* $\chi' = e_1 \wedge e_2 \wedge w_3' + e_2 \wedge e_3 \wedge w_1' + e_3 \wedge e_1 \wedge w_2'$ *are quasi-*$Sp(V, f)$*-equivalent.*

Proof. Let M be a nonsingular (3×3)-matrix over $\mathbb{F}$ such that $\mu A - MA'M^T$ is an alternating matrix. If we put $A'' := \frac{1}{\det(M)} \cdot (MA'M^T)$ and $[w_1'', w_2'', w_3'']^T := A'' \cdot [f_1, f_2, f_3]^T$, then $\chi'' := e_1 \wedge e_2 \wedge w_3'' + e_2 \wedge e_3 \wedge w_1'' + e_3 \wedge e_1 \wedge w_2''$ is $Sp(V, f)$-equivalent with χ' by Lemma 3.5. Since $\frac{\mu}{\det(M)} \cdot A - A''$ is alternating, χ'' is quasi-$Sp(V, f)$-equivalent with $\frac{\mu}{\det(M)} \cdot \chi$. Now, χ and $\frac{\mu}{\det(M)} \cdot \chi$ are $Sp(V, f)$-equivalent: if θ denotes the element of $Sp(V, f)$ mapping $(e_1, f_1, e_2, f_2, e_3, f_3)$ to $(\frac{\mu e_1}{\det(M)}, \frac{\det(M) \cdot f_1}{\mu}, \frac{\mu e_2}{\det(M)}, \frac{\det(M) \cdot f_2}{\mu}, \frac{\mu e_3}{\det(M)}, \frac{\det(M) \cdot f_3}{\mu})$, then θ maps the former trivector to the latter. We conclude that χ and χ' are quasi-$Sp(V, f)$-equivalent. ■

4 Classification in the case $\mathsf{char}(\mathbb{F}) = 2$

We continue with the notation introduced in Section 1. So, V denotes a 6-dimensional vector space over a field $\mathbb{F}$ which is equipped with a nondegenerate alternating bilinear form f, and $(e_1, f_1, e_2, f_2, e_3, f_3)$ denotes a fixed hyperbolic basis of (V, f). Throughout this section, we will suppose that $\mathsf{char}(\mathbb{F}) = 2$.

Our first goal is to prove Theorem 1.4. This will be achieved in a series of lemmas (4.1 till 4.14). Table 1 shows that the trivectors of V whose type is either (A), (B), (C) or (D) can be divided into 19 families when one studies the $Sp(V, f)$-equivalence between them. After studying the coarser relation of being quasi-$Sp(V, f)$-equivalent, it will turn out that a description using only nine families is already sufficient, see Corollary 4.15 (which is precisely Theorem 1.4). Our next goal will be to show that none of these nine families is superfluous in the description. In Lemma 4.23 (which is precisely the first claim of Theorem 1.5), we will indeed show that trivectors belonging to distinct families (among these nine) can never be quasi-$Sp(V, f)$-equivalent. This lemma implies that it suffices to study quasi-$Sp(V, f)$-equivalence between trivectors belonging the same family. This will be done in the six lemmas at the end of this section, and will prove the remaining claims of Theorem 1.5.

Lemma 4.1 *The trivector χ_{B1} is quasi-$Sp(V, f)$-equivalent with the trivector $\chi_{D4}(1, 1)$.*

Proof. Let θ be the element of $Sp(V, f)$ mapping the hyperbolic basis $(e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) to the hyperbolic basis $(e_2 + e_3, f_1 + f_3, e_1 + e_2 + e_3, f_1, f_2 + f_3, e_2)$ of (V, f). Then θ maps the trivector $\chi_{B1} = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge f_1 \wedge f_3$ to the trivector

$$
\begin{aligned}
& (e_2 + e_3) \wedge (e_1 + e_2 + e_3) \wedge (f_2 + f_3) + (e_2 + e_3) \wedge (f_1 + f_3) \wedge e_2 \\
= \; & e_1 \wedge (e_2 + e_3) \wedge (f_2 + f_3) + e_2 \wedge e_3 \wedge (f_1 + f_3) \\
= \; & e_1 \wedge e_2 \wedge f_3 + e_2 \wedge e_3 \wedge (f_1 + f_3) + e_1 \wedge e_3 \wedge f_2 + e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) \\
= \; & \chi_{D4}(1, 1) + e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3).
\end{aligned}
$$

It follows that χ_{B1} is quasi-$Sp(V, f)$-equivalent with $\chi_{D4}(1, 1)$. ■

Lemma 4.2 *The trivector χ_{B2} is quasi-$Sp(V, f)$-equivalent with χ_{A2}.*

Proof. The trivector $\chi_{B2} = e_1 \wedge e_2 \wedge f_2 + e_1 \wedge f_1 \wedge e_3$ is quasi-$Sp(V, f)$-equivalent with $e_1 \wedge e_2 \wedge f_2 + e_1 \wedge f_1 \wedge e_3 + e_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2) = (e_1 + e_3) \wedge e_2 \wedge f_2$ and the latter trivector is $Sp(V, f)$-equivalent with χ_{A2}. ∎

The following result is obvious.

Lemma 4.3 *The trivector χ_{B3} is quasi-$Sp(V, f)$-equivalent with the 0-vector of $\bigwedge^3 V$. In general, the trivectors quasi-$Sp(V, f)$-equivalent with the 0-vector of $\bigwedge^3 V$ are precisely the vectors of $\widetilde{W}$.*

Lemma 4.4 *For every $\lambda \in \mathbb{F}^*$, the trivector $\chi_{C2}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with the trivector $\chi_{D2}(\lambda^2)$.*

Proof. The trivector $\chi_{C2}(\lambda) = f_1 \wedge (e_2 + e_3) \wedge (f_2 - f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ is quasi-$Sp(V, f)$-equivalent with $f_1 \wedge e_2 \wedge f_3 + f_1 \wedge e_3 \wedge f_2 + \lambda \cdot e_1 \wedge e_2 \wedge f_2 = \lambda^2 \cdot e'_2 \wedge e'_1 \wedge f'_3 + f'_1 \wedge e'_2 \wedge e'_3 + e'_1 \wedge f'_1 \wedge f'_2$, where $(e'_1, f'_1, e'_2, f'_2, e'_3, f'_3)$ is the hyperbolic basis $(e_2, f_2, \frac{f_1}{\lambda}, \lambda e_1, \lambda e_3, \frac{f_3}{\lambda})$ of (V, f). So, $\chi_{C2}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{D2}(\lambda^2)$. ∎

Lemma 4.5 *For every $\lambda \in \mathbb{F}^*$, the trivector $\chi_{C3}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with χ_{A2}.*

Proof. The trivector $\chi_{C3}(\lambda) = e_1 \wedge e_2 \wedge f_2 + \lambda \cdot f_1 \wedge e_3 \wedge f_3$ is quasi-$Sp(V, f)$-equivalent with $e_1 \wedge e_3 \wedge f_3 + \lambda \cdot f_1 \wedge e_3 \wedge f_3 = (e_1 + \lambda f_1) \wedge e_3 \wedge f_3$, and the latter trivector is $Sp(V, f)$-equivalent with χ_{A2}. ∎

Lemma 4.6 *For every $\lambda \in \mathbb{F}^*$, the trivector $\chi_{C4}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with χ_{B1} and hence also with $\chi_{D4}(1, 1)$.*

Proof. The trivector $\chi_{C4}(\lambda) = f_1 \wedge e_3 \wedge (e_2 + f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ is quasi-$Sp(V, f)$-equivalent with $f_1 \wedge e_3 \wedge (e_2 + f_3) + \lambda \cdot e_1 \wedge e_3 \wedge f_3 = e_3 \wedge e_2 \wedge f_1 + e_3 \wedge f_3 \wedge (f_1 + \lambda e_1) = e'_1 \wedge e'_2 \wedge e'_3 + e'_1 \wedge f'_1 \wedge f'_3$, where $(e'_1, f'_1, e'_2, f'_2, e'_3, f'_3)$ is the hyperbolic basis $(e_3, f_3, \lambda e_2, \frac{f_2}{\lambda}, \frac{f_1}{\lambda}, f_1 + \lambda e_1)$ of (V, f). So, $\chi_{C4}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with χ_{B1} and hence also with $\chi_{D4}(1, 1)$ by Lemma 4.1. ∎

Lemma 4.7 *For every $\lambda \in \mathbb{F}^*$, the trivector $\chi_{C5}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{C2}(\lambda)$ and hence also with $\chi_{D2}(\lambda^2)$.*

Proof. Let $(e'_1, f'_1, e'_2, f'_2, e'_3, f'_3)$ be the hyperbolic basis $(\frac{e_1}{\lambda} + e_2, \lambda f_1, e_3, \lambda f_1 + f_2 + f_3, e_2 + e_3, \lambda e_2 + \lambda e_3 + \lambda f_1 + f_2)$ of (V, f). Then $f'_1 \wedge (e'_2 + e'_3) \wedge (f'_2 - f'_3) + \lambda \cdot e'_1 \wedge e'_2 \wedge f'_2$ is $Sp(V, f)$-equivalent with $\chi_{C2}(\lambda)$ and equal to

$$e_1 \wedge e_3 \wedge (f_3 + f_2) + \lambda \cdot e_2 \wedge f_3 \wedge (f_1 + e_3) + \lambda \cdot e_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2) = \chi_{C5}(\lambda) + \phi(\lambda e_3).$$

So, $\chi_{C5}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{C2}(\lambda)$ and hence also with $\chi_{D2}(\lambda^2)$ by Lemma 4.4. ∎

Lemma 4.8 *For every $\lambda \in \mathbb{F}^*$ and every $\epsilon \in \mathbb{F} \setminus \{0, -1\}$, the trivector $\chi_{C6}(\lambda, \epsilon)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{D2}(\lambda^2\epsilon)$.*

Proof. The trivector $\chi_{C6}(\lambda, \epsilon) = f_1 \wedge (e_2 + e_3) \wedge (f_2 + \epsilon f_3) + \lambda \cdot e_1 \wedge e_2 \wedge f_2 = f_1 \wedge e_2 \wedge f_2 + \epsilon \cdot f_1 \wedge e_2 \wedge f_3 + f_1 \wedge e_3 \wedge f_2 + \epsilon \cdot f_1 \wedge e_3 \wedge f_3 + \lambda \cdot e_1 \wedge e_2 \wedge f_2$ is quasi-$Sp(V, f)$-equivalent with $((1+\epsilon)f_1 + \lambda e_1) \wedge e_2 \wedge f_2 + f_1 \wedge e_2 \wedge (\epsilon f_3) + f_1 \wedge f_2 \wedge e_3 = \lambda^2\epsilon \cdot e_1' \wedge e_2' \wedge f_3' + e_2' \wedge f_1' \wedge e_3' + f_1' \wedge e_1' \wedge f_2'$, where $(e_1', f_1', e_2', f_2', e_3', f_3')$ is the hyperbolic basis $(f_2, e_2, \frac{f_1}{\lambda}, (1 + \epsilon)f_1 + \lambda e_1, \lambda\epsilon f_3, \frac{e_3}{\lambda\epsilon})$ of (V, f). So, $\chi_{C6}(\lambda, \epsilon)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{D2}(\lambda^2\epsilon)$. ■

Lemma 4.9 *The trivector χ_{D1} is quasi-$Sp(V, f)$-equivalent with χ_{B1} and hence also with $\chi_{D4}(1, 1)$.*

Proof. The trivector $\chi_{D1} = e_1 \wedge e_2 \wedge f_2 + e_2 \wedge f_1 \wedge e_3 + f_1 \wedge e_1 \wedge f_3$ is quasi-$Sp(V, f)$-equivalent with $e_1 \wedge e_2 \wedge f_2 + e_2 \wedge f_1 \wedge e_3 + e_2 \wedge f_2 \wedge f_3 = e_1' \wedge e_2' \wedge e_3' + e_1' \wedge f_1' \wedge f_3'$, where $(e_1', f_1', e_2', f_2', e_3', f_3')$ is the hyperbolic basis $(e_2, f_2, f_1 + e_3, e_1, e_3, e_1 + f_3)$ of (V, f). So, χ_{D1} is quasi-$Sp(V, f)$-equivalent with χ_{B1} and hence also with $\chi_{D4}(1, 1)$ by Lemma 4.1. ■

Lemma 4.10 *For every $\lambda \in \mathbb{F}^*$, the trivector $\chi_{D5}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with $\chi_{B5}(\lambda)$.*

Proof. Let θ be the unique element of $Sp(V, f)$ mapping the hyperbolic basis $(e_1, f_1, e_2, f_2, e_3, f_3)$ of (V, f) to the hyperbolic basis $(e_2 + e_3, f_2, e_3, f_1 + f_2 + f_3, e_1 + e_3, f_1)$ of (V, f). Then θ maps $\chi_{B5}(\lambda) = \lambda \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge (e_2 - e_3) \wedge (f_2 + f_3)$ to

$$\begin{aligned} & \lambda \cdot (e_2 + e_3) \wedge e_3 \wedge (f_1 + f_2 + f_3) + (e_2 + e_3) \wedge e_1 \wedge (f_2 + f_3) \\ = \; & \lambda \cdot e_2 \wedge e_3 \wedge (f_1 + f_2 + f_3) + e_1 \wedge e_2 \wedge f_3 + e_1 \wedge e_3 \wedge f_2 + e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) \\ = \; & \chi_{D5}(\lambda) + e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3). \end{aligned}$$

So, the trivectors $\chi_{D5}(\lambda)$ and $\chi_{B5}(\lambda)$ are indeed quasi-$Sp(V, f)$-equivalent. ■

Lemma 4.11 *If $|\mathbb{F}| = 2$, then the trivector χ_{D7} is quasi-$Sp(V, f)$-equivalent with the trivector χ_{A2}.*

Proof. The trivector $\chi_{D7} = e_1 \wedge e_2 \wedge f_2 + e_2 \wedge e_3 \wedge (f_1 + f_3) + e_3 \wedge e_1 \wedge f_3$ is quasi-$Sp(V, f)$-equivalent with the trivector $e_2 \wedge e_3 \wedge (f_1 + f_3)$, which is itself $Sp(V, f)$-equivalent with χ_{A2}. ■

Lemma 4.12 *If λ is a square in $\mathbb{F}^*$, then the trivector $\chi_{B4}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with χ_{A1}.*

Proof. Suppose $\lambda = \mu^2$ where $\mu \in \mathbb{F}^*$. Then $\chi_{B4}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot e_1 \wedge f_2 \wedge f_3 = e_1 \wedge (e_2 + \mu f_3) \wedge (e_3 + \mu f_2) + \mu \cdot e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3)$ is quasi-$Sp(V, f)$-equivalent with the trivector $e_1 \wedge (e_2 + \mu f_3) \wedge (e_3 + \mu f_2)$ which itself is $Sp(V, f)$-equivalent with χ_{A1}. ■

Lemma 4.13 *Let $\lambda \in \mathbb{F}^*$. If the equation $X^2 + \lambda X + 1$ has a solution for $X \in \mathbb{F}$, then $\chi_{B5}(\lambda)$ is quasi-$Sp(V, f)$-equivalent with χ_{A2}.*

Proof. Suppose $\mu^2 + \mu\lambda + 1 = 0$ for some $\mu \in \mathbb{F}$. Then $\mu \neq 0$ and $\lambda = \mu + \frac{1}{\mu}$. Now, $\chi_{B5}(\lambda) = (\mu + \frac{1}{\mu}) \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge (e_2 + e_3) \wedge (f_2 + f_3) = (\lambda e_1) \wedge \frac{e_2+\mu e_3}{\lambda} \wedge (\frac{1}{\mu} f_2 + f_3) + (\mu+1) \cdot e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3)$ is quasi-$Sp(V,f)$-equivalent with $(\lambda e_1) \wedge \frac{e_2+\mu e_3}{\lambda} \wedge (\frac{1}{\mu} f_2 + f_3)$ which is a trivector of Type (A2) since $f(\frac{e_2+\mu e_3}{\lambda}, \frac{f_2}{\mu} + f_3) = 1$. ∎

Lemma 4.14 *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. If the equation $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ has a nonzero solution for $(X,Y,Z) \in \mathbb{F}^3$, then the trivector $\chi_{D3}(\lambda_1, \lambda_2)$ is quasi-$Sp(V,f)$-equivalent with $\chi_{B4}(\lambda_1)$ or $\chi_{B4}(\lambda_2)$.*

Proof. Let $(\alpha, \beta, \gamma) \in \mathbb{F}^3 \setminus \{(0,0,0)\}$ such that $\lambda_1\alpha^2 + \lambda_2\beta^2 + \gamma^2 = 0$. It is impossible that $\alpha = \beta = 0$. So, $\alpha \neq 0$ or $\beta \neq 0$.

Suppose $\alpha \neq 0$. Then $\chi_{D3}(\lambda_1, \lambda_2) = e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge f_1 + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2$ is equal to

$$\frac{e_1'}{\alpha} \wedge (e_2' \wedge e_3' + \lambda_2 \cdot f_2' \wedge f_3') + \frac{\gamma}{\alpha} \cdot e_2' \wedge (e_1' \wedge f_1' + e_3' \wedge f_3') + \frac{\lambda_2 \beta}{\alpha} \cdot f_3' \wedge (e_1' \wedge f_1' + e_2' \wedge f_2'),$$

where $(e_1', f_1', e_2', f_2', e_3', f_3')$ is the hyperbolic basis $(\alpha e_1 + \beta e_2 + \gamma e_3, \frac{f_1}{\alpha}, e_2, f_2 + \frac{\beta}{\alpha} f_1, f_3 + \frac{\gamma}{\alpha} f_1, e_3)$ of (V,f). So, $\chi_{D3}(\lambda_1, \lambda_2)$ is quasi-$Sp(V,f)$-equivalent with $\frac{e_1'}{\alpha} \wedge (e_2' \wedge e_3' + \lambda_2 \cdot f_2' \wedge f_3')$ which itself is $Sp(V,f)$-equivalent with $\chi_{B4}(\lambda_2)$.

So, if $\alpha \neq 0$, then we know that $\chi_{D3}(\lambda_1, \lambda_2)$ is quasi-$Sp(V,f)$-equivalent with $\chi_{B4}(\lambda_2)$. If $\beta \neq 0$, then by reasons of symmetry, we know that $\chi_{D3}(\lambda_1, \lambda_2)$ is quasi-$Sp(V,f)$-equivalent with $\chi_{B4}(\lambda_1)$. ∎

The following corollary, which is precisely Theorem 1.4, is a consequence of Lemmas 4.1 – 4.14.

Corollary 4.15 *Let χ be a trivector of V quasi-$Sp(V,f)$-equivalent with a trivector of Type (A), (B), (C) or (D). Then χ is quasi-$Sp(V,f)$-equivalent with (at least) one of the following trivectors:*
($Q1'$) the zero vector of $\bigwedge^3 V$;
($Q2'$) χ_{A1};
($Q3'$) χ_{A2};
($Q4'$) $\chi_{B4}(\lambda)$ for some nonsquare λ of $\mathbb{F}$;
($Q5'$) $\chi_{B5}(\lambda)$ for some $\lambda \in \mathbb{F}$ such that the polynomial $X^2 + \lambda X + 1 \in \mathbb{F}[X]$ is irreducible;
($Q6'$) $\chi_{C1}(\lambda)$ for some $\lambda \in \mathbb{F}^*$;
($Q7'$) $\chi_{D2}(\lambda)$ for some $\lambda \in \mathbb{F}^*$;
($Q8'$) $\chi_{D3}(\lambda_1, \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$ such that the equation $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ has no solutions for $(X,Y,Z) \in \mathbb{F}^3 \setminus \{(0,0,0)\}$;
($Q9'$) $\chi_{D4}(\lambda_1, \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$.

Our next goal will be to show that trivectors belonging to distinct families (as occurring in Corollary 4.15) can never be quasi-$Sp(V,f)$-equivalent.

Lemma 4.16 *Let $\chi_1 \in W$ and $\chi_2 \notin W$. Then χ_1 and χ_2 are not quasi-$Sp(V,f)$-equivalent.*

Proof. Suppose χ_1 and χ_2 are quasi-$Sp(V,f)$-equivalent. Then there exists a $\theta \in Sp(V,f)$ and a $\chi \in \widetilde{W}$ such that $\chi_2 = \theta(\chi_1) + \chi$. Since $\theta(\chi_1) \in W$ and $\chi \in \widetilde{W} \subset W$, we must have that $\chi_2 \in W$, a contradiction. ∎

Corollary 4.17 *Let $i \in \{1,2,4,6,8\}$ and $j \in \{3,5,7,9\}$. Then no trivector of Type (Qi') is quasi-$Sp(V,f)$-equivalent with a trivector of Type (Qj').*

Proof. This follows from Lemma 4.16, taking into account that a trivector of Type (Qi') belongs to W, while a trivector of Type (Qj') does not belong to W. ∎

Lemma 4.18 *Let $i \in \{2,3,\ldots,9\}$. Then no trivector of Type (Qi') is quasi-$Sp(V,f)$-equivalent with the zero vector.*

Proof. The nonzero trivectors quasi-$Sp(V,f)$-equivalent with the zero trivector are precisely the trivectors $Sp(V,f)$-equivalent with χ_{B3} and none of these trivectors is of Type (Qi'). ∎

Recall that $\overline{\mathbb{F}}$ is a fixed algebraic closure of $\mathbb{F}$. For every field $\mathbb{K}$ satisfying $\mathbb{F} \subseteq \mathbb{K} \subseteq \overline{\mathbb{F}}$, the vector space V naturally embeds into a $\mathbb{K}$-vector space $V_{\mathbb{K}}$ by allowing the coordinates (with respect to $\{e_1, f_1, \ldots, e_3, f_3\}$) to be elements of $\mathbb{K}$. The nondegenerate alternating bilinear form f naturally extends to a nondegenerate alternating bilinear form $f_{\mathbb{K}}$ of $V_{\mathbb{K}}$. Every trivector of V can also be regarded as a trivector of $V_{\mathbb{K}}$. For every trivector $\chi \in \bigwedge^3 V_{\mathbb{K}}$, we can define a set $\mathcal{D}_{\mathbb{K}}(\chi) \subseteq V_{\mathbb{K}}$ in a similar way as in Section 3. Notice that we have used a subindex to indicate the underlying field in order to avoid confusion. Indeed, if $\chi \in \bigwedge^3 V$, then χ can also be regarded as an element of $\bigwedge^3 V_{\mathbb{K}}$ and the sets $\mathcal{D}(\chi)$ and $\mathcal{D}_{\mathbb{K}}(\chi)$ need not to be equal.

Lemma 4.19 *Let $\mathbb{K}$ be a field such that $\mathbb{F} \subseteq \mathbb{K} \subseteq \overline{\mathbb{F}}$. Suppose $\chi = e_1 \wedge e_2 \wedge w_3 + e_2 \wedge e_3 \wedge w_1 + e_3 \wedge e_1 \wedge w_2$, where $[w_1, w_2, w_3]^T = A \cdot [f_1, f_2, f_3]^T$ for some (3×3)-matrix $A = (a_{ij})_{1 \leq i,j \leq 3}$ over $\mathbb{K}$. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{K}^3 \setminus \{(0,0,0)\}$, and put $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$. Then the following are equivalent:*

(1) *$v \in \mathcal{D}_{\mathbb{K}}(\chi)$;*

(2) *$\sum_{1 \leq i,j \leq 3} a_{ij} \alpha_i \alpha_j = 0$;*

(3) *there exist vectors $v', v'' \in V_{\mathbb{K}}$ such that $\langle v, v', v'' \rangle_{\mathbb{K}}$ is a 3-space totally isotropic for $f_{\mathbb{K}}$, $\langle v, v', v'' \rangle_{\mathbb{K}} \cap \langle e_1, e_2, e_3 \rangle_{\mathbb{K}} = \langle v \rangle_{\mathbb{K}}$ and $\chi \wedge v \wedge v' \wedge v'' = 0$.*

Proof. Put $v_1^{(1)} = e_1$, $v_2^{(1)} = e_2$ and $v_3^{(1)} = e_3$, and let $v_i^{(j)}$ with $i \in \{1,2,3\}$ and $j \in \{2,3,4,5\}$ be vectors of $V_{\mathbb{K}}$ as in Lemma 3.7. By Property (v) of Lemma 3.7, $\chi \wedge v_1^{(j)} \wedge v_2^{(j)} \wedge v_3^{(j)} = 0$ for every $j \in \{1,2,3,4\}$. So, we have that $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathcal{D}_{\mathbb{K}}(\chi)$ if and only if $\chi \wedge v_1^{(5)} \wedge v_2^{(5)} \wedge v_3^{(5)} = 0$. The equivalence of (1) and (3) follows.

If $\alpha_2 \neq 0$, then we can take $v_1^{(5)} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, $v_2^{(5)} = \alpha_2 f_1 + \alpha_1 f_2$ and $v_3^{(5)} = \alpha_3 f_2 + \alpha_2 f_3$. In this case, we put $\chi' := \frac{1}{\alpha_2} \cdot (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) \wedge (\alpha_2 f_1 + \alpha_1 f_2) \wedge (\alpha_3 f_2 + \alpha_2 f_3)$.

If $\alpha_2 = 0$, then we can take $v_1^{(5)} = \alpha_1 e_1 + \alpha_3 e_3$, $v_2^{(5)} = f_2$ and $v_3^{(5)} = \alpha_3 f_1 + \alpha_1 f_3$. In this case, we put $\chi' := (\alpha_1 e_1 + \alpha_3 e_3) \wedge f_2 \wedge (\alpha_3 f_1 + \alpha_1 f_3)$.

In any case, we have $\chi' = \alpha_1\alpha_3 \cdot e_1 \wedge f_1 \wedge f_2 + \alpha_1\alpha_2 \cdot e_1 \wedge f_1 \wedge f_3 + \alpha_1^2 \cdot e_1 \wedge f_2 \wedge f_3 + \alpha_2\alpha_3 \cdot e_2 \wedge f_1 \wedge f_2 + \alpha_2^2 \cdot e_2 \wedge f_1 \wedge f_3 + \alpha_1\alpha_2 \cdot e_2 \wedge f_2 \wedge f_3 + \alpha_3^2 \cdot e_3 \wedge f_1 \wedge f_2 + \alpha_2\alpha_3 \cdot e_3 \wedge f_1 \wedge f_3 + \alpha_1\alpha_3 \cdot e_3 \wedge f_2 \wedge f_3$. Now, $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathcal{D}_{\mathbb{K}}(\chi)$ if and only if $\chi \wedge \chi' = 0$. This happens precisely when $\sum_{1 \leq i,j \leq 3} a_{ij}\alpha_i\alpha_j = 0$. ■

Lemma 4.20 (1) *We have* $\mathcal{D}(\chi_{A1}) = \langle e_1, e_2, e_3 \rangle$.

(2) *For every nonsquare* λ *of* $\mathbb{F}$, *we have* $\mathcal{D}(\chi_{B4}(\lambda)) = \langle e_1 \rangle$.

(3) *For every* $\lambda \in \mathbb{F}^*$, *we have* $\mathcal{D}(\chi_{C1}(\lambda)) = \mathcal{D}_{\overline{\mathbb{F}}}(\chi_{C1}(\lambda)) = \{o\}$.

(4) *For all* $\lambda_1, \lambda_2 \in \mathbb{F}^*$ *such that the equation* $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ *has no solution for* $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0,0,0)\}$, *we have* $\mathcal{D}(\chi_{D3}(\lambda_1, \lambda_2)) = \{o\}$ *and* $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D3}(\lambda_1, \lambda_2)) \neq \{o\}$.

Proof. In [17], a method was described to determine the sets $\mathcal{D}(\chi)$ for trivectors χ of V. In Section 3 of that paper, this method was applied to some particular cases. These cases include all trivectors of Type A1, B4 and C1. Claims (1), (2) and (3) of the lemma follow from that treatment. We will now also prove Claim (4).

Put $\chi := \chi_{D3}(\lambda_1, \lambda_2)$. By Lemma 4.19, the elements of $\mathcal{D}(\chi)$ and $\mathcal{D}_{\overline{\mathbb{F}}}(\chi)$ of the form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ are determined by the equation $\lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 + \alpha_3^2 = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ in the former case and $\alpha_1, \alpha_2, \alpha_3 \in \overline{\mathbb{F}}$ in the latter case. This shows that $\mathcal{D}_{\overline{\mathbb{F}}}(\chi) \neq \{o\}$ and that $\mathcal{D}(\chi)$ does not contain nonzero vectors of the form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$.

We will now show that $\mathcal{D}(\chi) = \{o\}$. Suppose that this is not the case. Then there exists a vector $v_1 \in \mathcal{D}(\chi) \setminus \langle e_1, e_2, e_3 \rangle$. Let v_2 be a nonzero vector of $\langle e_1, e_2, e_3 \rangle \cap v_1^{\perp_f}$ and let v_3 be a vector of V such that $\langle v_1, v_2, v_3 \rangle$ is a 3-dimensional totally isotropic subspace intersecting $\langle e_1, e_2, e_3 \rangle$ in $\langle v_2 \rangle$. Since $v_1 \in \mathcal{D}(\chi)$, we have $\chi \wedge v_1 \wedge v_2 \wedge v_3 = 0$. Lemma 4.19 would now imply that $v_2 \in \mathcal{D}(\chi)$, which is impossible as $v_2 \in \langle e_1, e_2, e_3 \rangle$. ■

Lemma 4.21 *Let* $\lambda_1, \lambda_2 \in \mathbb{F}^*$ *such that the equation* $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ *has no solutions for* $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0,0,0)\}$. *Then the following are equivalent for two linearly independent vectors* v_1, v_2 *of* V *satisfying* $f(v_1, v_2) = 0$:

(1) $v_1, v_2 \in \langle e_1, e_2, e_3 \rangle$;

(2) $\chi_{D3}(\lambda_1, \lambda_2) \wedge v_1 \wedge v_2 \wedge v_3 = 0$ *for every vector* $v_3 \in V$ *such that* $\langle v_1, v_2, v_3 \rangle$ *is a totally isotropic 3-space.*

Proof. Clearly, (1) implies (2) (as $\chi_{D3}(\lambda_1, \lambda_2) \wedge v_1 \wedge v_2 = 0$ in that case). Conversely, suppose that (2) holds and that not both of v_1, v_2 are contained in $\langle e_1, e_2, e_3 \rangle$. Then we can choose a vector v_3 such that $\langle v_1, v_2, v_3 \rangle$ is a totally isotropic 3-space intersecting

$\langle e_1, e_2, e_3\rangle$ in a one-dimensional subspace $\langle e\rangle$. By Lemma 4.19, we would then have that $e \in \mathcal{D}(\chi_{D3}(\lambda_1, \lambda_2))$. But this is in contradiction with the fact that $\mathcal{D}(\chi_{D3}(\lambda_1, \lambda_2)) = \{o\}$ (see Lemma 4.20). ∎

Lemma 4.22 (1) *The set $\mathcal{D}(\chi_{A2})$ is the union of all onedimensional subspaces of the form $\langle e_1\rangle$, $\langle be_2 + \beta f_2\rangle$, $\langle ce_3 + \gamma f_3\rangle$, $\langle e_1 + be_2 + \beta f_2\rangle$, $\langle e_1 + ce_3 + \gamma f_3\rangle$, where (b, β) and (c, γ) belong to $\mathbb{F}^2 \setminus \{(0, 0)\}$.*

(2) *For every $\lambda \in \mathbb{F}$ such that the polynomial $X^2 + \lambda X + 1 \in \mathbb{F}[X]$ is irreducible, we have $\mathcal{D}(\chi_{B5}(\lambda)) = \langle e_1\rangle$.*

(3) *For every $\lambda \in \mathbb{F}^*$, we have $\mathcal{D}(\chi_{D2}(\lambda)) = \mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D2}(\lambda)) = \{o\}$.*

(4) *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then $\mathcal{D}(\chi_{D4}(\lambda_1, \lambda_2))$ is either $\{o\}$ or the union of $|\mathbb{F}| + 1$ onedimensional subspaces contained in $U := \langle e_1, e_2, e_3\rangle$ defining a nonempty nonsingular conic of the projective plane $\mathrm{PG}(U)$, and $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D4}(\lambda_1, \lambda_2))$ is the union of $|\overline{\mathbb{F}}| + 1$ onedimensional subspaces contained in $\widetilde{U} := \langle e_1, e_2, e_3\rangle_{\overline{\mathbb{F}}}$, defining a nonempty nonsingular conic of $\mathrm{PG}(\widetilde{U})$.*

Proof. In [17], a method was described to determine the sets $\mathcal{D}(\chi)$ for trivectors χ of V. In Section 3 of that paper, this method was applied to determine these sets for some particular trivectors. Claims (1), (2) and (3) of the lemma follow from that treatment. We will now also prove Claim (4).

Put $\chi := \chi_{D4}(\lambda_1, \lambda_2)$ and let $\mathbb{K} \in \{\mathbb{F}, \overline{\mathbb{F}}\}$. The elements of $\mathcal{D}_{\mathbb{K}}(\chi)$ of the form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ are determined by the equation $\lambda_1\alpha_1^2 + \lambda_1\alpha_1\alpha_3 + \alpha_3^2 + \lambda_2\alpha_2^2 = 0$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}$. This equation determines a nonsingular conic of $\mathrm{PG}(\langle e_1, e_2, e_3\rangle_{\mathbb{K}})$. If this conic is nonempty (which is always the case if $\mathbb{K} = \overline{\mathbb{F}}$), then it contains precisely $|\mathbb{K}| + 1$ points. Observe that regardless of whether $\mathbb{K} = \mathbb{F}$ or $\mathbb{K} = \overline{\mathbb{F}}$, the conic can never contain a line of $\mathrm{PG}(\langle e_1, e_2, e_3\rangle_{\mathbb{K}})$. In view of what we need to prove, it now suffices to show that the set $\mathcal{D}_{\mathbb{K}}(\chi) \setminus \langle e_1, e_2, e_3\rangle_{\mathbb{K}}$ is empty. Suppose that this is not the case and let v_1 be a vector belonging to this set. Then we will show that every nonzero vector v_2 belonging to the 2-space $v_1^{\perp_{f_{\mathbb{K}}}} \cap \langle e_1, e_2, e_3\rangle_{\mathbb{K}}$ belongs to $\mathcal{D}_{\mathbb{K}}(\chi)$, which is in contradiction with the above-mentioned fact that the conic does not contain lines. Having chosen such a v_2, we choose a vector $v_3 \in V_{\mathbb{K}}$ such that $\langle v_1, v_2, v_3\rangle_{\mathbb{K}}$ is a 3-dimensional subspace, totally isotropic for $f_{\mathbb{K}}$, intersecting $\langle e_1, e_2, e_3\rangle_{\mathbb{K}}$ in $\langle v_2\rangle_{\mathbb{K}}$. Since $v_1 \in \mathcal{D}_{\mathbb{K}}(\chi)$, we have $\chi \wedge v_1 \wedge v_2 \wedge v_3 = 0$. Lemma 4.19 would now imply that $v_2 \in \mathcal{D}_{\mathbb{K}}(\chi)$. As this holds for every $v_2 \in v_1^{\perp_{f_{\mathbb{K}}}} \cap \langle e_1, e_2, e_3\rangle_{\mathbb{K}}$, we have obtained our desired contradiction. ∎

Lemma 4.23 *Let $i, j \in \{1, 2, \ldots, 9\}$ with $i \neq j$. Then no trivector of Type (Qi′) is quasi-$Sp(V, f)$-equivalent with a trivector of Type (Qj′).*

Proof. This is a consequence of Corollary 4.17 and Lemmas 3.9, 4.18, 4.20, 4.22, also taking into account that if two trivectors of V are $Sp(V, f)$-equivalent, then they are also $Sp(V_{\overline{\mathbb{F}}}, f_{\overline{\mathbb{F}}})$-equivalent. ∎

So, it remains to study the quasi-$Sp(V,f)$-equivalence between trivectors having the same type (Qi'), where $i \in \{4,5,6,7,8,9\}$. This will be done in the next six lemmas. These six lemmas correspond to the six claims of Theorem 1.5 that still need to be proved.

Lemma 4.24 *Let λ and λ' be two nonsquares of $\mathbb{F}$. Then the trivectors $\chi_{B4}(\lambda)$ and $\chi_{B4}(\lambda')$ are quasi-$Sp(V,f)$-equivalent if and only if the polynomials $X^2+\lambda$ and $X^2+\lambda'$ define the same quadratic extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$.*

Proof. Let $\mathbb{K} = \mathbb{F}(\sqrt{\lambda}) \subseteq \overline{\mathbb{F}}$ and $\mathbb{K}' = \mathbb{F}(\sqrt{\lambda'}) \subseteq \overline{\mathbb{F}}$ be the quadratic extensions of $\mathbb{F}$ defined by the respective irreducible quadratic polynomials $X^2+\lambda$ and $X^2+\lambda'$ of $\mathbb{F}[X]$.

Suppose the trivectors $\chi_{B4}(\lambda)$ and $\chi_{B4}(\lambda')$ are quasi-$Sp(V,f)$-equivalent. Then they are also quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent. By Lemma 4.12, $\chi_{B4}(\lambda)$ is quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent with χ_{A1} and hence also $\chi_{B4}(\lambda')$ should be quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent with χ_{A1}. This implies by Lemma 3.9 and Lemma 4.20(1)+(2) that λ' should have a root in $\mathbb{K}$, i.e. $\mathbb{K}' = \mathbb{F}(\sqrt{\lambda'}) \subseteq \mathbb{K}$. By symmetry, we also have $\mathbb{K} \subseteq \mathbb{K}'$. So, $\mathbb{K} = \mathbb{K}'$.

Conversely, suppose that $\mathbb{K} = \mathbb{K}'$. Then there exist $a,b \in \mathbb{F}$ with $a \neq 0$ such that $\lambda' = a^2\lambda + b^2$. The element of $Sp(V,f)$ mapping the hyperbolic basis $(e_1,f_1,e_2,f_2,e_3,f_3)$ of (V,f) to the hyperbolic basis $(ae_1, \frac{f_1}{a}, \frac{1}{a}(e_2+bf_3), af_2, e_3+bf_2, f_3)$ of (V,f) maps $\chi_{B4}(\lambda) = e_1 \wedge e_2 \wedge e_3 + \lambda \cdot e_1 \wedge f_2 \wedge f_3$ to the trivector

$$e_1 \wedge e_2 \wedge e_3 + (a^2\lambda + b^2) \cdot e_1 \wedge f_2 \wedge f_3 + (be_1) \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) = \chi_{B4}(\lambda') + \phi(be_1).$$

We conclude that $\chi_{B4}(\lambda)$ and $\chi_{B4}(\lambda')$ are quasi-$Sp(V,f)$-equivalent. ∎

Lemma 4.25 *Let $\lambda, \lambda' \in \mathbb{F}$ such that the polynomials $X^2+\lambda X+1$ and $X^2+\lambda' X+1$ of $\mathbb{F}[X]$ are irreducible. Then the trivectors $\chi_{B5}(\lambda)$ and $\chi_{B5}(\lambda')$ are quasi-$Sp(V,f)$-equivalent if and only if the polynomials $X^2+\lambda X+1$ and $X^2+\lambda' X+1$ define the same quadratic extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$.*

Proof. Let $\mathbb{K} \subseteq \overline{\mathbb{F}}$ and $\mathbb{K}' \subseteq \overline{\mathbb{F}}$ be the quadratic extensions of $\mathbb{F}$ defined by the respective irreducible quadratic polynomials $X^2+\lambda X+1$ and $X^2+\lambda' X+1$.

Suppose the trivectors $\chi_{B5}(\lambda)$ and $\chi_{B5}(\lambda')$ are quasi-$Sp(V,f)$-equivalent. Then they are also quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent. By Lemma 4.13, $\chi_{B5}(\lambda)$ is quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent with χ_{A2} and hence also $\chi_{B5}(\lambda')$ should be quasi-$Sp(V_{\mathbb{K}},f_{\mathbb{K}})$-equivalent with χ_{A2}. This implies by Lemma 3.9 and Lemma 4.22(1)+(2) that the quadratic polynomial $X^2+\lambda' X+1$ should have its roots in $\mathbb{K}$. Hence, $\mathbb{K}' \subseteq \mathbb{K}$. By symmetry, we also have $\mathbb{K} \subseteq \mathbb{K}'$. So, $\mathbb{K} = \mathbb{K}'$.

Conversely, suppose that $\mathbb{K} = \mathbb{K}'$. Then there exist $a,b \in \mathbb{F}$ with $a \neq 0$ such that $\lambda' = a\lambda$ and $a^2+b^2+ab\lambda = 1$ (or equivalently, $(aX+b)^2+\lambda'(aX+b)+1 = a^2(X^2+\lambda X+1)$). We need to prove that the trivectors $\chi_{B5}(\lambda)$ and $\chi_{B5}(\lambda')$ are quasi-$Sp(V,f)$-equivalent, or equivalently, that the trivectors $\chi := \lambda \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge e_2 \wedge f_3 + e_1 \wedge e_3 \wedge f_2$ and $\chi' := \lambda' \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge e_2 \wedge f_3 + e_1 \wedge e_3 \wedge f_2$ are quasi-$Sp(V,f)$-equivalent. Now, let θ be the element of $Sp(V,f)$ mapping the hyperbolic basis $(e_1,f_1,e_2,f_2,e_3,f_3)$ of (V,f)

to the hyperbolic basis $(ae_1, \frac{f_1}{a}, e_2 + be_3, f_2, ae_3, \frac{bf_2+f_3}{a})$ of (V, f). Then θ maps χ to the trivector

$$(\lambda a) \cdot e_1 \wedge e_2 \wedge f_2 + e_1 \wedge e_2 \wedge f_3 + e_1 \wedge e_3 \wedge f_2 + be_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) = \chi' + \phi(be_1).$$

It follows that the trivectors χ and χ' are quasi-$Sp(V, f)$-equivalent and hence also the trivectors $\chi_{B5}(\lambda)$ and $\chi_{B5}(\lambda')$. ■

Lemma 4.26 *Let $\lambda, \lambda' \in \mathbb{F}^*$. Then $\chi_{C1}(\lambda)$ and $\chi_{C1}(\lambda')$ are quasi-$Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

Proof. If $\chi_{C1}(\lambda)$ and $\chi_{C1}(\lambda')$ are quasi-$Sp(V, f)$-equivalent, then $\lambda = \eta(\chi_{C1}(\lambda)) = \eta(\chi_{C1}(\lambda')) = \lambda'$ by Corollary 3.13. ■

Lemma 4.27 *Let $\lambda, \lambda' \in \mathbb{F}^*$. Then $\chi_{D2}(\lambda)$ and $\chi_{D2}(\lambda')$ are quasi-$Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

Proof. Suppose the trivectors $\chi_{D2}(\lambda)$ and $\chi_{D2}(\lambda')$ are quasi-$Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(e_1', f_1', e_2', f_2', e_3', f_3')$ of (V, f) and a $v \in V$ such that

$$\begin{aligned} & \lambda' \cdot e_1 \wedge e_2 \wedge f_3 + e_2 \wedge f_1 \wedge e_3 + f_1 \wedge e_1 \wedge f_2 + \phi(v) \\ = \; & \lambda \cdot e_1' \wedge e_2' \wedge f_3' + e_2' \wedge f_1' \wedge e_3' + f_1' \wedge e_1' \wedge f_2'. \end{aligned} \tag{1}$$

If we let the map π act on both sides of the equality (1), then we find $f_2 = f_2'$. Now, put $v = a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 f_1 + b_2 f_2 + b_3 f_3$ where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}$. Then $\phi(v)$ is equal to

$$a_1 \cdot e_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) + a_2 \cdot e_2 \wedge (e_1 \wedge f_1 + e_3 \wedge f_3) + a_3 \cdot e_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2)$$

$$+ b_1 \cdot f_1 \wedge (e_2 \wedge f_2 + e_3 \wedge f_3) + b_2 \cdot f_2 \wedge (e_1 \wedge f_1 + e_3 \wedge f_3) + b_3 \cdot f_3 \wedge (e_1 \wedge f_1 + e_2 \wedge f_2).$$

So, if we let the map η act on both sides of the equality (1), then we find that $a_2 = 0$. Now, take the wedge product of both sides of (1) with $f_2 = f_2'$. Then we find

$$\lambda' \cdot e_1 \wedge e_2 \wedge f_3 \wedge f_2 + e_2 \wedge f_1 \wedge e_3 \wedge f_2 + a_1 \cdot e_1 \wedge e_3 \wedge f_3 \wedge f_2 + a_3 \cdot e_3 \wedge e_1 \wedge f_1 \wedge f_2 +$$

$$b_1 \cdot f_1 \wedge e_3 \wedge f_3 \wedge f_2 + b_3 \cdot f_3 \wedge e_1 \wedge f_1 \wedge f_2 = \lambda \cdot e_1' \wedge e_2' \wedge f_3' \wedge f_2' + e_2' \wedge f_1' \wedge e_3' \wedge f_2'.$$

If we let π' act on both sides of the latter equality, then we find

$$\begin{aligned} & \lambda' \cdot e_1 \wedge f_3 + f_1 \wedge e_3 + a_1 \cdot e_1 \wedge f_2 + a_3 \cdot e_3 \wedge f_2 + b_1 \cdot f_1 \wedge f_2 + b_3 \cdot f_3 \wedge f_2 \\ = \; & \lambda \cdot e_1' \wedge f_3' + f_1' \wedge e_3'. \end{aligned} \tag{2}$$

Since $(f_2')^{\perp_f} = f_2^{\perp_f} = \langle e_1, e_3, f_1, f_3, f_2 \rangle = \langle e_1', e_3', f_1', f_3', f_2' \rangle$, there exist unique vectors $e_1'', e_3'', f_1'', f_3'' \in U := \langle e_1, e_3, f_1, f_3 \rangle$ and unique $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ such that $e_1' = e_1'' + \alpha f_2$, $e_3' = e_3'' + \beta f_2$, $f_1' = f_1'' + \gamma f_2$ and $f_3' = f_3'' + \delta f_2$. Since $f(e_1', e_3') = f(e_1', f_3') = f(f_1', e_3') = f(f_1', f_3') = 0$ and $f(e_1', f_1') = f(e_3', f_3') = 1$, also $f(e_1'', e_3'') = f(e_1'', f_3'') = f(f_1'', e_3'') =$

$f(f_1'', f_3'') = 0$ and $f(e_1'', f_1'') = f(e_3'', f_3'') = 1$. So, both $(e_1'', f_1'', e_3'', f_3'')$ and (e_1, f_1, e_3, f_3) are hyperbolic bases of $(U, f_{|U})$. Now, equation (2) becomes

$$\lambda' \cdot e_1 \wedge f_3 + f_1 \wedge e_3 + a_1 \cdot e_1 \wedge f_2 + a_3 \cdot e_3 \wedge f_2 + b_1 \cdot f_1 \wedge f_2 + b_3 \cdot f_3 \wedge f_2$$

$$= \lambda \cdot e_1'' \wedge f_3'' + f_1'' \wedge e_3'' + (\lambda\delta) \cdot e_1'' \wedge f_2 + (\lambda\alpha) \cdot f_2 \wedge f_3'' + \beta \cdot f_1'' \wedge f_2 + \gamma \cdot f_2 \wedge e_3''.$$

Since $e_1'', f_1'', e_3'', f_3'' \in \langle e_1, f_1, e_3, f_3 \rangle$, this implies that

$$\lambda' \cdot e_1 \wedge f_3 + f_1 \wedge e_3 = \lambda \cdot e_1'' \wedge f_3'' + f_1'' \wedge e_3'',$$

and

$$a_1 e_1 + a_3 e_3 + b_1 f_1 + b_3 f_3 = (\lambda\delta) e_1'' + \beta f_1'' + \gamma e_3'' + (\lambda\alpha) f_3''.$$

By Lemma 3.1, the former equation implies that

$$\lambda' \cdot e_1 \wedge f_1 \wedge e_3 \wedge f_3 = \lambda \cdot e_1'' \wedge f_1'' \wedge e_3'' \wedge f_3''.$$

Since (e_1, f_1, e_3, f_3) and $(e_1'', f_1'', e_3'', f_3'')$ are two hyperbolic bases of $(U, f_{|U})$, we have $e_1 \wedge f_1 \wedge e_3 \wedge f_3 = e_1'' \wedge f_1'' \wedge e_3'' \wedge f_3''$ (Cohn [6, Corollary 3.6.4]). We conclude that $\lambda' = \lambda$. ■

Lemma 4.28 *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$ such that none of the equations $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$ and $\lambda_1' X^2 + \lambda_2' Y^2 + Z^2 = 0$ has solutions for $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0,0,0)\}$. Then the trivectors $\chi_{D3}(\lambda_1, \lambda_2)$ and $\chi_{D3}(\lambda_1', \lambda_2')$ are quasi-$Sp(V, f)$-equivalent if and only if there exists a $\mu \in \mathbb{F}^*$ such that the matrices $\mu \cdot \mathrm{diag}(\lambda_1, \lambda_2, 1)$ and $\mathrm{diag}(\lambda_1', \lambda_2', 1)$ are pseudo-congruent.*

Proof. If the matrices $\mu \cdot \mathrm{diag}(\lambda_1, \lambda_2, 1)$ and $\mathrm{diag}(\lambda_1', \lambda_2', 1)$ are pseudo-congruent for some $\mu \in \mathbb{F}^*$, then Lemma 3.14 implies that the trivectors $\chi_{D3}(\lambda_1, \lambda_2)$ and $\chi_{D3}(\lambda_1', \lambda_2')$ are quasi-$Sp(V, f)$-equivalent.

Conversely, suppose that the trivectors $\chi_{D3}(\lambda_1, \lambda_2)$ and $\chi_{D3}(\lambda_1', \lambda_2')$ are quasi-$Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(e_1', f_1', e_2', f_2', e_3', f_3')$ of (V, f) and a vector $v \in V$ such that

$$e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge f_1 + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2 + \phi(v)$$

$$= \chi := e_1' \wedge e_2' \wedge f_3' + \lambda_1' \cdot e_2' \wedge e_3' \wedge f_1' + \lambda_2' \cdot e_3' \wedge e_1' \wedge f_2'.$$

By Lemma 4.21, there exists a (necessarily unique) totally isotropic 3-space U such that the following are equivalent for two linearly independent vectors v_1 and v_2 of V for which $f(v_1, v_2) = 0$:

(1) $v_1, v_2 \in U$;

(2) $\chi \wedge v_1 \wedge v_2 \wedge v_3 = \chi_{D3}(\lambda_1, \lambda_2) \wedge v_1 \wedge v_2 \wedge v_3 = 0$ for every vector $v_3 \in V$ such that $\langle v_1, v_2, v_3 \rangle$ is a totally isotropic 3-space.

Lemma 4.21 implies moreover that $U = \langle e_1, e_2, e_3\rangle$ and $U = \langle e'_1, e'_2, e'_3\rangle$. Hence, $\langle e_1, e_2, e_3\rangle = \langle e'_1, e'_2, e'_3\rangle$.

By Lemma 4.19, the set $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D3}(\lambda_1, \lambda_2)) \cap \langle e_1, e_2, e_3\rangle_{\overline{\mathbb{F}}}$ is the union of all onedimensional subspaces of the form $\langle \alpha e_1 + \beta e_2 + \gamma e_3\rangle$, where $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ satisfies $\sqrt{\lambda_1}\cdot\alpha + \sqrt{\lambda_2}\cdot\beta + \gamma = 0$. Similarly, the set $\mathcal{D}_{\overline{\mathbb{F}}}(\chi) \cap \langle e_1, e_2, e_3\rangle_{\overline{\mathbb{F}}}$ is the union of all onedimensional subspaces of the form $\langle \alpha' e'_1 + \beta' e'_2 + \gamma' e'_3\rangle$, where $(\alpha', \beta', \gamma') \neq (0, 0, 0)$ satisfies $\sqrt{\lambda'_1}\cdot\alpha' + \sqrt{\lambda'_2}\cdot\beta' + \gamma' = 0$.

Now, since $\langle e_1, e_2, e_3\rangle = \langle e'_1, e'_2, e'_3\rangle$, there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that $[e'_1, e'_2, e'_3]^T = M \cdot [e_1, e_2, e_3]^T$. We have $\alpha e_1 + \beta e_2 + \gamma e_3 = \alpha' e'_1 + \beta' e'_2 + \gamma' e'_3$, where $[\alpha, \beta, \gamma] = [\alpha', \beta', \gamma'] \cdot M$. Moreover, if $[\alpha, \beta, \gamma] = [\alpha', \beta', \gamma'] \cdot M$, then $[\alpha, \beta, \gamma] \cdot [\sqrt{\lambda_1}, \sqrt{\lambda_2}, 1]^T = [\alpha', \beta', \gamma'] \cdot M \cdot [\sqrt{\lambda_1}, \sqrt{\lambda_2}, 1]^T$.

Since $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D3}(\lambda_1, \lambda_2)) = \mathcal{D}_{\overline{\mathbb{F}}}(\chi)$, the equations $[\alpha', \beta', \gamma'] \cdot M \cdot [\sqrt{\lambda_1}, \sqrt{\lambda_2}, 1]^T = 0$ and $[\alpha', \beta', \gamma'] \cdot [\sqrt{\lambda'_1}, \sqrt{\lambda'_2}, 1]^T = 0$ should be equivalent. So, there should exist a $\mu \in \overline{\mathbb{F}} \setminus \{0\}$ such that $[\sqrt{\lambda'_1}, \sqrt{\lambda'_2}, 1]^T = \mu \cdot M \cdot [\sqrt{\lambda_1}, \sqrt{\lambda_2}, 1]^T$. Hence,

$$\begin{cases} \sqrt{\lambda'_1} &= \mu M_{11}\sqrt{\lambda_1} + \mu M_{12}\sqrt{\lambda_2} + \mu M_{13}, \\ \sqrt{\lambda'_2} &= \mu M_{21}\sqrt{\lambda_1} + \mu M_{22}\sqrt{\lambda_2} + \mu M_{23}, \\ 1 &= \mu M_{31}\sqrt{\lambda_1} + \mu M_{32}\sqrt{\lambda_2} + \mu M_{33}, \end{cases}$$

or equivalently,

$$\begin{cases} \lambda'_1 &= \mu^2 M_{11}^2\lambda_1 + \mu^2 M_{12}^2\lambda_2 + \mu^2 M_{13}^2, \\ \lambda'_2 &= \mu^2 M_{21}^2\lambda_1 + \mu^2 M_{22}^2\lambda_2 + \mu^2 M_{23}^2, \\ 1 &= \mu^2 M_{31}^2\lambda_1 + \mu^2 M_{32}^2\lambda_2 + \mu^2 M_{33}^2. \end{cases}$$

This implies that $\mu^2 \in \mathbb{F}^*$ and that $\mathrm{diag}(\lambda'_1, \lambda'_2, 1) - \mu^2 \cdot (M \cdot \mathrm{diag}(\lambda_1, \lambda_2, 1) \cdot M^T)$ is an alternate matrix. Hence, the matrices $\mu^2 \cdot \mathrm{diag}(\lambda_1, \lambda_2, 1)$ and $\mathrm{diag}(\lambda'_1, \lambda'_2, 1)$ are pseudo-congruent. ■

Lemma 4.29 *Let $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{D4}(\lambda_1, \lambda_2)$ and $\chi_{D4}(\lambda'_1, \lambda'_2)$ are quasi-$Sp(V, f)$-equivalent if and only if there exists a $\mu \in \mathbb{F}^*$ such that the matrices μA and A' are pseudo-congruent, where*

$$A := \begin{bmatrix} \lambda_1 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A' := \begin{bmatrix} \lambda'_1 & 0 & \lambda'_1 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. If the matrices μA and A' are pseudo-congruent for some $\mu \in \mathbb{F}^*$, then Lemma 3.14 implies that the trivectors $\chi_{D4}(\lambda_1, \lambda_2)$ and $\chi_{D4}(\lambda'_1, \lambda'_2)$ are quasi-$Sp(V, f)$-equivalent.

Conversely, suppose that the trivectors $\chi_{D4}(\lambda_1, \lambda_2)$ and $\chi_{D4}(\lambda'_1, \lambda'_2)$ are quasi-$Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(e'_1, f'_1, e'_2, f'_2, e'_3, f'_3)$ of (V, f) and a vector $v \in V$ such that

$$e_1 \wedge e_2 \wedge f_3 + \lambda_1 \cdot e_2 \wedge e_3 \wedge (f_1 + f_3) + \lambda_2 \cdot e_3 \wedge e_1 \wedge f_2 + \phi(v)$$

$$= \chi := e'_1 \wedge e'_2 \wedge f'_3 + \lambda'_1 \cdot e'_2 \wedge e'_3 \wedge (f'_1 + f'_3) + \lambda'_2 \cdot e'_3 \wedge e'_1 \wedge f'_2.$$

By Lemma 4.22(4), $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D4}(\lambda_1, \lambda_2)) = \mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D4}(\lambda_1, \lambda_2) + \phi(v))$ generates the subspace $\langle e_1, e_2, e_3\rangle_{\overline{\mathbb{F}}}$, and $\mathcal{D}_{\overline{\mathbb{F}}}(\chi)$ generates the subspace $\langle e'_1, e'_2, e'_3\rangle_{\overline{\mathbb{F}}}$. Hence, $\langle e_1, e_2, e_3\rangle = \langle e'_1, e'_2, e'_3\rangle$.

So, there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that $[e_1, e_2, e_3]^T = M \cdot [e'_1, e'_2, e'_3]^T$. Let $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ be an arbitrary nonzero vector of $V_{\overline{\mathbb{F}}}$ contained in $\langle e_1, e_2, e_3 \rangle_{\overline{\mathbb{F}}}$. We have $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \alpha'_1 e'_1 + \alpha'_2 e'_2 + \alpha'_3 e'_3$, where $[\alpha_1, \alpha_2, \alpha_3] \cdot M = [\alpha'_1, \alpha'_2, \alpha'_3]$. By Lemma 4.19, the onedimensional subspace $\langle \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \rangle_{\overline{\mathbb{F}}} = \langle \alpha'_1 e'_1 + \alpha'_2 e'_2 + \alpha'_3 e'_3 \rangle_{\overline{\mathbb{F}}}$ is contained in $\mathcal{D}_{\overline{\mathbb{F}}}(\chi_{D4}(\lambda_1, \lambda_2)) = \mathcal{D}_{\overline{\mathbb{F}}}(\chi)$ if and only if one of the following (necessarily equivalent) conditions is satisfied:

- $[\alpha_1, \alpha_2, \alpha_3] \cdot A \cdot [\alpha_1, \alpha_2, \alpha_3]^T = 0$,
- $[\alpha'_1, \alpha'_2, \alpha'_3] \cdot A' \cdot [\alpha'_1, \alpha'_2, \alpha'_3]^T = 0$.

Now, $[\alpha'_1, \alpha'_2, \alpha'_3] \cdot A' \cdot [\alpha'_1, \alpha'_2, \alpha'_3]^T = [\alpha_1, \alpha_2, \alpha_3] \cdot (MA'M^T) \cdot [\alpha_1, \alpha_2, \alpha_3]^T$. Since the equations $[\alpha_1, \alpha_2, \alpha_3] \cdot A \cdot [\alpha_1, \alpha_2, \alpha_3]^T = 0$ and $[\alpha_1, \alpha_2, \alpha_3] \cdot (MA'M^T) \cdot [\alpha_1, \alpha_2, \alpha_3]^T = 0$ describe the same nondegenerate nonempty conic of $\mathrm{PG}(\langle e_1, e_2, e_3 \rangle_{\overline{\mathbb{F}}})$, there should exist a $\mu \in \mathbb{F}^*$ such that $\mu A - MA'M^T$ is an alternate matrix. Hence, the matrices μA and A' are pseudo-congruent. ■

Remark. We were also able to deal with some but not all of the trivectors of Type (E). Indeed, every trivector of Type (E2), (E3) or (E5) is quasi-$Sp(V,f)$-equivalent with a trivector of Type (X) with $(X) \in \{(B),(C),(D)\}$ Note that since $\mathsf{char}(\mathbb{F}) = 2$, we must have that $a \neq 0 \neq b$ for every $(a,b) \in \Psi$.

- Suppose $(a,b) \in \Psi$ and $k \in \mathbb{F}^*$. Then the trivector $\chi_{E2}(a,b,k)$ of Type (E2) is quasi-$Sp(V,f)$-equivalent with $k \cdot (f_1 \wedge e_2 \wedge f_3 + b \cdot f_1 \wedge f_2 \wedge e_3 + a \cdot f_1 \wedge e_3 \wedge f_3) = f_1 \wedge f_3 \wedge (ke_2 + kae_3) + f_1 \wedge f_2 \wedge (kbe_3)$ which is a trivector of Type (B).
- Suppose $(a,b) \in \Psi$ and $k, h \in \mathbb{F}^*$. Then the trivector $\chi_{E3}(a,b,k,h)$ of Type (E3) is quasi-$Sp(V,f)$-equivalent with $k \cdot (f_1 \wedge e_2 \wedge f_3 + b \cdot f_1 \wedge f_2 \wedge e_3 + a \cdot f_1 \wedge e_3 \wedge f_3) + h \cdot e_1 \wedge f_2 \wedge f_3 = f_1 \wedge f_3 \wedge (ke_2 + kae_3) + f_1 \wedge f_2 \wedge (kbe_3) + f_2 \wedge f_3 \wedge (he_1)$ which is a trivector of Type (D).
- Suppose $(a,b) \in \Psi$ and $k \in \mathbb{F}^*$. Then the trivector $\chi_{E5}(a,b,k)$ of Type (E5) is quasi-$Sp(V,f)$-equivalent with $f_1 \wedge e_2 \wedge f_3 + a \cdot e_2 \wedge f_2 \wedge e_3 + (a^2+b) \cdot f_1 \wedge f_2 \wedge e_3 + ka \cdot e_1 \wedge f_2 \wedge e_3 = f_1 \wedge e_2 \wedge f_3 + (ae_2 + (a^2+b)f_1 + kae_1) \wedge f_2 \wedge e_3$ which is a trivector of Type (C).

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