On hyperbolic sets of maxes in dual polar spaces

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Abstract

Suppose Δ is a fully embeddable thick dual polar space of rank $n \geq 3$. It is known that a hyperplane H of Δ is classical if all its nontrivial intersections with quads are classical. In order to conclude that a hyperplane H is classical, it is perhaps not necessary to require in advance that all these intersections are classical. In fact, in this paper we show that for dual polar spaces admitting hyperbolic sets of maxes, the existence of certain classical quad-hyperplane intersections implies that other quad-hyperplane intersections need to be classical as well. We will also derive necessary and sufficient conditions for two disjoint maxes to be contained in a (necessarily unique) hyperbolic set of maxes. Dual polar spaces admitting hyperbolic sets of maxes include all members of a class of embeddable dual polar spaces related to quadratic alternative division algebras.

Keywords: dual polar space, hyperbolic set of maxes, (classical) hyperplane, universal embedding, generalized quadrangle
MSC2000: 51A50, 51A45, 51E12

1 Introduction

Let Π be a thick polar space of rank $n \geq 3$ (Tits [13, Chapter 7]). The maximal singular subspaces of Π then have (projective) dimension $n-1$. With Π, there is associated a dual polar space Δ of rank n. This is a point-line geometry whose points are the maximal singular subspaces of Π and whose lines are certain sets of maximal singular subspaces. Specifically, there exists a bijective correspondence between the singular subspaces of dimension $n-2$ of Π and the lines of Δ: if α is a singular subspace of dimension $n-2$, then the set L_α of all maximal singular subspaces containing α is a line of Δ. If F is a convex subspace of diameter $\delta \in \{2, 3, \ldots, n\}$ of Δ, then we denote by $\widetilde{F}$ the point-line geometry induced on F by the lines of Δ that are contained in F. The geometry $\widetilde{F}$ is a dual polar space of rank δ. A convex subspace of diameter δ is called a *quad* if $\delta = 2$ and a *max* if $\delta = n-1$. The point-line geometry induced on a quad is a generalized quadrangle. There exists a bijective correspondence between the points of Π and the maxes of Δ. If x is a point of Π, then the set M_x of all maximal singular subspaces of Π containing x is a max of Δ. If M is a max of Δ, then we denote by x_M the unique point of Π corresponding to M.

A first class of objects under study in this paper are the hyperbolic sets of maxes. A set $\mathcal{H}$ of mutually disjoint maxes of Δ is called a *hyperbolic set of maxes* if the following two properties are satisfied:

(H1) every line of Δ meeting two distinct maxes of $\mathcal{H}$ meets all maxes of $\mathcal{H}$;

(H2) $L = \bigcup_{M \in \mathcal{H}} (M \cap L)$ for every line L of Δ meeting all maxes of $\mathcal{H}$.

In Section 2, we will determine necessary and sufficient conditions for two disjoint maxes of Δ to be contained in a (necessarily unique) hyperbolic set of maxes.

Hyperbolic sets of maxes have already been considered in the literature for symplectic dual polar spaces. In Section 3, we will indicate a larger class of dual polar spaces admitting hyperbolic sets of maxes. The dual polar spaces of this class are related to certain quadratic alternative division algebras.

In Section 4, we will discuss an application of hyperbolic sets of maxes to hyperplanes of dual polar spaces. A *(full) projective embedding* of a point-line geometry $\mathcal{S}$ into a projective space Σ is an injective map e from the point set of $\mathcal{S}$ to the point set of Σ mapping lines of $\mathcal{S}$ to full lines of Σ such that the image of e generates the whole projective space Σ. A *hyperplane* of $\mathcal{S}$ is a set H of points, distinct from the whole point set, such that every line of $\mathcal{S}$ has either one or all its points in H. If $e : \mathcal{S} \to \Sigma$ is a full projective embedding of $\mathcal{S}$ and U is a hyperplane of the projective space Σ, then the set of all points of $\mathcal{S}$ that are mapped by e into U is a hyperplane of $\mathcal{S}$. Any hyperplane of $\mathcal{S}$ that can be obtained in this way is said to *arise from* e. A hyperplane of $\mathcal{S}$ is called *classical* if it arises from some full projective embedding. If H is a hyperplane of a dual polar space Δ and Q is a quad, then either $Q \subset H$ or $Q \cap H$ is a hyperplane of $\widetilde{Q}$. If $Q \subset H$, then the intersection $Q \cap H$ (which is equal to Q) is called *trivial*.

Combining results of Cardinali, De Bruyn & Pasini [2] and McInroy & Shpectorov [10] regarding simple connectedness of hyperplane complements in dual polar spaces and results of Ronan regarding hyperplanes and projective embeddings of point-line geometries (Corollaries 2 & 4 on page 180 and Corollary 4 on page 184 of [11]), we know that the following must hold (see [2] for more details):

Proposition 1.1 ([2, 10, 11]) *Suppose Δ is a fully embeddable thick dual polar space. Then the following are equivalent for a hyperplane H of Δ:*

(1) *H is classical;*

(2) *for every quad Q of Δ not contained in H, the intersection $Q \cap H$ is a classical hyperplane of $\widetilde{Q}$.*

One can now wonder whether it is possible to prove a stronger version of Proposition 1.1 by relaxing the condition (2). More precisely, one can wonder about the existence of a set $\mathcal{Q}$ of quads - not containing all quads and preferably as small as possible - such that Proposition 1.1 still remains valid if condition (2) is replaced by the following:

(2') for every quad $Q \in \mathcal{Q}$ not contained in H, the intersection $Q \cap H$ is a classical hyperplane of $\widetilde{Q}$.

In Section 4, we show that such sets $\mathcal{Q}$ exist if the embeddable dual polar space Δ admits hyperbolic sets of maxes. We show in this case that the existence of certain classical quad-hyperplane intersections implies that other quad-hyperplane intersections need to be classical as well. Among other things, we will prove the following in Section 4.

Proposition 1.2 *Let H be a hyperplane of a fully embeddable thick dual polar space Δ of rank at least 3, and let M_1 and M_2 be two disjoint maxes of Δ. Suppose M_1 and M_2 are contained in a (necessarily unique) hyperbolic set $\mathcal{H}$ of maxes. Suppose also that the following hold:*

(*a*) *For every $i \in \{1,2\}$, $H_i := H \cap M_i$ is either M_i or a classical hyperplane of $\widetilde{M_i}$.*

(*b*) *For every quad Q meeting M_1 and M_2 (necessarily in lines), $H \cap Q$ is either Q or a classical hyperplane of $\widetilde{Q}$.*

Then for every $M \in \mathcal{H}$, we have that $H \cap M$ is either M or a classical hyperplane of $\widetilde{M}$.

2 Hyperbolic sets of maxes

Let Π denote a thick polar space of rank $n \geq 3$, and Δ its associated dual polar space. If A is a set of points of Π, then $A^\perp$ denotes the set of all points of Π collinear with all points of A. We also define $A^{\perp\perp} := (A^\perp)^\perp$. Two points of Δ are called *opposite* if they lie at maximal distance n from each other. Here, we follow the convention that distances $\mathrm{d}(\cdot,\cdot)$ in Δ will always be measured in its collinearity graph. If x is a point and L a line of Δ, then L contains a unique point $\pi_L(x)$ nearest to x. Two lines of Δ are called *opposite* if they lie at maximal distance $n-1$ from each other. If L_1 and L_2 are two opposite lines of Δ, then the maps $L_1 \to L_2; x \mapsto \pi_{L_2}(x)$ and $L_2 \to L_1; x \mapsto \pi_{L_1}(x)$ are bijections which are each other's inverses. If x_1 and x_2 are two points of Δ at distance δ from each other, then x_1 and x_2 are contained in a unique convex subspace $\langle x_1, x_2 \rangle$ of diameter δ. If M is a max of Δ, then every point x not contained in M is collinear with a unique point $\pi_M(x)$ of M. If F is a convex subspace of diameter δ meeting a max M, then either $F \subseteq M$ or $F \cap M$ is a convex subspace of diameter $\delta - 1$.

Suppose M_1 and M_2 are two disjoint maxes of Δ. Then the map $M_1 \to M_2; x \mapsto \pi_{M_2}(x)$ is an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$. If x_1 and y_1 are two points of M_1 and if x_2 and y_2 denote the respective points of M_2 collinear with x_1 and y_1, then the distance between the lines $L_1 = x_1x_2$ and $L_2 = y_1y_2$ is equal to $\mathrm{d}(x_1, y_1)$. Moreover, every point x of L_1 lies at distance $\mathrm{d}(L_1, L_2)$ from a unique point of L_2, namely the point $\pi_{L_2}(x)$, and the maps $L_1 \to L_2; x \mapsto \pi_{L_2}(x)$ and $L_2 \to L_1; x \mapsto \pi_{L_1}(x)$ are bijections which are each other's inverses.

Lemma 2.1 *If L_1 and L_2 are two opposite lines of Δ, then $\{\langle u, \pi_{L_2}(u) \rangle \,|\, u \in L_1\}$ is a set of mutually disjoint maxes of Δ.*

Proof. Notice that if $u \in L_1$, then $\mathrm{d}(u, \pi_{L_2}(u)) = n-1$ and hence $\langle u, \pi_{L_2}(u) \rangle$ is a max.

Let u_1 and u_2 be two distinct points of L_1. Then $\pi_{L_2}(u_1) \neq \pi_{L_2}(u_2)$. Put $M_i := \langle u_i, \pi_{L_2}(u_i) \rangle$, $i \in \{1,2\}$. Then $M_i \cap L_1 = \{u_i\}$, since every point of $L_1 \setminus \{u_i\}$ lies at distance n from $\pi_{L_2}(u_i)$. Suppose v is a point of $M_1 \cap M_2$. Since $\pi_{L_1}(v)$ is contained on a shortest path from $v \in M_1$ to $u_1 \in M_1$, we have $\pi_{L_1}(v) \in M_1$ and hence $\pi_{L_1}(v) = u_1$. A similar argument allows us to conclude that $\pi_{L_1}(v) \in M_2$ and $\pi_{L_1}(v) = u_2$. Since $u_1 \neq u_2$, this is not possible. So, the maxes M_1 and M_2 should be disjoint. ■

Proposition 2.2 (1) *If L_1 and L_2 are two opposite lines of Δ, then there exists at most one hyperbolic set $\mathcal{H}$ of maxes of Δ such that L_1 and L_2 meet each max of $\mathcal{H}$. If $\mathcal{H}$ is such a hyperbolic set, then $\mathcal{H} = \{\langle u, \pi_{L_2}(u) \rangle \mid u \in L_1\}$.*

(2) *Every two disjoint maxes M_1 and M_2 of Δ are contained in at most one hyperbolic set of maxes.*

Proof. (1) Every max meeting L_1 and L_2 is of the form $\langle u, \pi_{L_2}(u) \rangle$ for some point u of L_1. So, if $\mathcal{H}$ is a hyperbolic set of maxes of Δ such that L_1 and L_2 meet each max of $\mathcal{H}$, then necessarily $\mathcal{H} = \{\langle u, \pi_{L_2}(u) \rangle \mid u \in L_1\}$.

(2) Let x_1 and x_2 be two opposite points of M_1, and let y_1 and y_2 be the respective points of M_2 collinear with x_1 and x_2. Put $L_1 := x_1y_1$ and $L_2 := x_2y_2$. Then L_1 and L_2 are opposite lines of Δ. If $\mathcal{H}$ is a hyperbolic set of maxes containing M_1 and M_2, then every max of $\mathcal{H}$ must meet L_1 and L_2. The claim now follows from part (1). ■

Proposition 2.3 *Let M_1 and M_2 be two disjoint maxes of Δ. If M_1 and M_2 are contained in a hyperbolic set $\mathcal{H}$ of maxes, then $\{x_M \mid M \in \mathcal{H}\} = \{x_{M_1}, x_{M_2}\}^{\perp\perp}$.*

Proof. Since M_1 and M_2 are disjoint maxes, the points x_{M_1} and x_{M_2} of Π are noncollinear. So, the singular subspaces of Π contained in $\{x_{M_1}, x_{M_2}\}^\perp$ define a polar space of rank $n-1$. Let α_1 and α_2 be two disjoint $(n-2)$-dimensional singular subspaces of Π contained in $\{x_{M_1}, x_{M_2}\}^\perp$, and let L_i, $i \in \{1,2\}$, be the line L_{α_i} of Δ. We have $L_i \cap M_j = \langle \alpha_i, x_{M_j} \rangle$ if $i, j \in \{1,2\}$. The lines L_1 and L_2 are opposite lines of Δ meeting M_1 and M_2. By Proposition 2.2(1), $\mathcal{H}$ consists of all maxes meeting L_1 and L_2, i.e. $\mathcal{H} = \{M_x \mid x \in (\alpha_1 \cup \alpha_2)^\perp\}$. So, we have $\{x_{M_1}, x_{M_2}\}^{\perp\perp} \subseteq (\alpha_1 \cup \alpha_2)^\perp = \{x_M \mid M \in \mathcal{H}\}$.

We will now prove the inclusion in the other direction. So, we need to prove that $x_M \in y^\perp$ for every $M \in \mathcal{H}$ and every $y \in \{x_{M_1}, x_{M_2}\}^\perp$. The fact that $y \in \{x_{M_1}, x_{M_2}\}^\perp$ implies that the max M_y meets M_1 and M_2. So, there exists a line K in M_y meeting M_1 and M_2. Since this line meets M, the quads M and M_y meet, showing that $x_M \in y^\perp$ as we needed to prove. ■

Remark. If x_1 and x_2 are two noncollinear points of Π, then $\{x_1, x_2\}^{\perp\perp}$ is a set of mutually noncollinear points of Π containing x_1 and x_2.

Proposition 2.4 *Let M_1 and M_2 be two disjoint maxes of Δ. Then the following are equivalent:*

(1) *M_1 and M_2 are contained in a hyperbolic set of maxes;*

(2) *for every* $(n-2)$*-dimensional singular subspace* α *of* Π *contained in* $\{x_{M_1}, x_{M_2}\}^\perp$*, the maximal singular subspaces of* Π *containing* α *are precisely the singular subspaces* $\langle \alpha, x\rangle$ *where* $x \in \{x_{M_1}, x_{M_2}\}^{\perp\perp}$*;*

(3) *there exists an* $(n-2)$*-dimensional singular subspace* α *of* Π *contained in* $\{x_{M_1}, x_{M_2}\}^\perp$ *such that the maximal singular subspaces of* Π *containing* α *are precisely the singular subspaces* $\langle \alpha, x\rangle$*, where* $x \in \{x_{M_1}, x_{M_2}\}^{\perp\perp}$*.*

Proof. (1) $\Rightarrow$ (2). Suppose M_1 and M_2 are contained in a hyperbolic set $\mathcal{H}$ of maxes. Then $\{x_{M_1}, x_{M_2}\}^{\perp\perp} = \{x_M \mid M \in \mathcal{H}\}$ by Proposition 2.3. Let α be an $(n-2)$-dimensional singular subspace of Π contained in $\{x_{M_1}, x_{M_2}\}^\perp$. Then the line L_α of Δ meets M_1 and M_2. The points of the line L_α are precisely the points contained in the singletons $L_\alpha \cap M$, where $M \in \mathcal{H}$. Hence, the maximal singular subspaces of Π containing α are precisely the singular subspaces $\langle \alpha, x\rangle$, where $x \in \{x_{M_1}, x_{M_2}\}^{\perp\perp}$.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): Let $\mathcal{H}$ denote the set of all maxes M for which $x_M \in \{x_{M_1}, x_{M_2}\}^{\perp\perp}$. Since $\{x_1, x_2\}^{\perp\perp}$ is a set of mutually noncollinear points of Π, the set $\mathcal{H}$ is a set of mutually disjoint maxes of Δ. As $x_M \in \beta^\perp$ for every $(n-2)$-dimensional singular subspace β of Π contained in $\{x_{M_1}, x_{M_2}\}^\perp$, every line meeting M_1 and M_2 also meets every max $M \in \mathcal{H}$. Suppose now that K is some line meeting M_1 and M_2. Then we still need to show that $K = \bigcup_{M\in\mathcal{H}}(M \cap K)$. As (3) holds, this is certainly the case if K is the line L_α of Δ. If $\bigcup_{M\in\mathcal{H}}(M \cap K)$ would be properly contained in K, there would exist a point $k^* \in K$ not contained in $\bigcup_{M\in\mathcal{H}}(M \cap K)$. Recall that $\mathrm{d}(k, L_\alpha) = \mathrm{d}(K, l) = \mathrm{d}(K, L_\alpha)$ for all $k \in K$ and all $l \in L_\alpha$. Moreover, the maps $K \to L_\alpha; k \mapsto \pi_{L_\alpha}(k)$ and $L_\alpha \to K; l \mapsto \pi_K(l)$ are bijections which are each other's inverses. Now, put $l^* := \pi_{L_\alpha}(k^*)$. The unique max of $\mathcal{H}$ containing l^* meets K and must therefore contain the unique point k^* of K nearest to l^*, which is however impossible. So, we should have $\bigcup_{M\in\mathcal{H}}(M \cap K) = K$. ∎

Proposition 2.5 *The following conditions are equivalent:*

(1) *every two disjoint maxes of* Δ *are contained in a hyperbolic set of maxes;*

(2) *if* L_1 *and* L_2 *are two disjoint lines of* Δ *contained in the same quad of* Δ*, then* L_1 *and* L_2 *are contained in a full subgrid.*

Proof. Suppose (1) holds. Let L_1 and L_2 be two disjoint lines of Δ that are contained in some quad Q of Δ. For every $i \in \{1, 2\}$, let M_i be a max through L_i such that $L_i = Q \cap M_i$. Then M_1 and M_2 are disjoint. Let $\mathcal{H}$ denote the unique hyperbolic set of maxes containing M_1 and M_2. Let $\mathcal{L}_1$ denote the set of lines meeting L_1 and L_2. Every line of $\mathcal{L}_1$ is contained in Q and meets every max M of $\mathcal{H}$ implying that $Q \cap M$ is a line. Let $\mathcal{L}_2$ denote the set of lines of the form $Q \cap M$ where $M \in \mathcal{H}$. Then every line of $\mathcal{L}_1$ intersects every line of $\mathcal{L}_2$ in a unique point. Since every line of $\mathcal{L}_1$ is contained in $\bigcup_{M\in\mathcal{H}} M$, the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ define a full subgrid of Δ. Observe that $L_1, L_2 \in \mathcal{L}_2$.

Conversely, suppose that (2) holds. Let M_1 and M_2 be two disjoint maxes of Δ. Let x_1 and y_1 be two opposite points of $\widetilde{M_1}$ and let x_2, y_2 be the two (opposite) points of M_2

collinear with respectively x_1 and y_1. Put $K := x_1x_2$ and $L := y_1y_2$. Then K and L are opposite lines of Δ. Let $\mathcal{H}$ denote the set of all maxes meeting K and L. Then $\mathcal{H}$ is a set of mutually disjoint maxes of Δ. For every point z of M_1, let L_z denote the unique line through z meeting M_2. We call the point $z \in M_1$ *nice* if the following hold:

- L_z meets every max of $\mathcal{H}$ (necessarily in a point);
- $L_z \subseteq \bigcup_{M\in\mathcal{H}} M$.

We see that the points x_1 and y_1 are nice. Observe also that if x and y are nice points of M_1 at maximal distance $n-1$ from each other, then $\mathcal{H}$ consists of all maxes meeting L_x and L_y. We will prove the following:

> (∗) Suppose x and y are nice points of M_1 at maximal distance $n-1$ from each other. If x' is a point of M_1 at distance 1 from x, then x' is also nice.

Suppose (∗) holds. Take then a point y' in M_1 collinear with y and at maximal distance $n-1$ from x'. Then (∗) would also imply that y' is nice. So, x' and y' are then nice points of M_1 at maximal distance $n-1$ from each other, implying that $\mathcal{H}$ consists of all maxes meeting $L_{x'}$ and $L_{y'}$. Knowing that, we can again apply (∗), with (x, y) replaced by (x', y'). An inductive argument relying on the connectedness of M_1 will thus show that every point of M_1 is nice. This then implies that $\mathcal{H}$ is a hyperbolic set of maxes containing M_1 and M_2.

So, it suffices to show that (∗) holds. For convenience, suppose that $x = x_1$, $y = y_1$ and put $x' = u_1$. Call z_1 the unique point on the line x_1u_1 at distance $n-2$ from y_1. Let u_2 and z_2 be the respective points of M_2 collinear with u_1 and z_1. For every point $z \in z_1z_2 = L_{z_1}$, put $z' = \pi_K(z)$ and $z'' \in \pi_L(z)$. Then $\mathrm{d}(z, z') = 1$, $\mathrm{d}(z, z'') = n-2$, $\mathrm{d}(z', z'') = n-1$ and so $\langle z', z''\rangle$ is the unique element of $\mathcal{H}$ containing z. Since the map $L_{z_1} \to K; z \mapsto z'$ is bijective, we see that every element of $\mathcal{H}$ intersects L_{z_1} in a singleton. The point z_1 is thus nice. Since $x_1z_1 = x_1u_1$ and $x_2z_2 = x_2u_2$ are contained in a full subgrid, the line L_{u_1} meets every line zz' with $z \in L_{z_1}$ in a singleton. Since the map $L_{z_1} \to \mathcal{H}; z \mapsto \langle z', z''\rangle$ is bijective, we see that the point $x' = u_1$ should also be nice. ∎

3 A class of dual polar spaces admitting hyperbolic sets of maxes

Suppose $\mathcal{Q}$ is a nonsingular quadric of Witt index 2 of a projective space $\mathrm{PG}(V)$, where V is some vector space over a field $\mathbb{F}$. Then the points and lines of $\mathrm{PG}(V)$ contained in $\mathcal{Q}$ define a generalized quadrangle. We call a generalized quadrangle of *quadric-type* if it arises in this fashion. If K_1 and K_2 are two disjoint lines of $\mathrm{PG}(V)$ contained in $\mathcal{Q}$, then the 3-dimensional subspace $\alpha = \langle K_1, K_2\rangle$ intersects $\mathcal{Q}$ in a quadric of α which contains two disjoint lines, but no planes. So, $\alpha \cap \mathcal{Q}$ is a hyperbolic quadric, which has the structure of a grid fully embedded into α. From Proposition 2.5 and the above discussion, we immediately have:

Proposition 3.1 *Let Δ be a thick dual polar space of rank $n \geq 3$. If one (and hence all) quad(s) of Δ is/are of quadric-type, then every two disjoint maxes of Δ are contained in a (necessarily unique) hyperbolic set of maxes.*

Remark. If Q_1 and Q_2 are two quads of a thick dual polar space of rank $n \geq 3$, then $\widetilde{Q_1} \cong \widetilde{Q_2}$. So, if one quad of Δ is a quadric-type, then all quads of Δ are of quadric-type.

We will now describe a class of dual polar spaces all whose quads are of quadric-type. Suppose $(\mathbb{O}, +, \cdot)$ is a ring with $|\mathbb{O}| \geq 2$ having a neutral element 1 for the multiplication. Let $0 \neq 1$ denote the neutral element for the addition. Then $(\mathbb{O}, +, \cdot)$ is called an *alternative division ring* if for every $a \in \mathbb{O} \setminus \{0\}$, there exists a (necessarily unique) element $a^{-1} \in \mathbb{O}$ such that $a^{-1} \cdot (a \cdot b) = b = (b \cdot a) \cdot a^{-1}$ for every $b \in \mathbb{O}$. It is costume to denote the product $a \cdot b$ of two elements $a, b \in \mathbb{O}$ by ab. The *center* $Z(\mathbb{O})$ of an alternative division ring $\mathbb{O}$ is defined to be the set of all $a \in \mathbb{O}$ such that $ab = ba$, $a(bc) = (ab)c$, $(ba)c = b(ac)$ and $(bc)a = b(ca)$ for all $b, c \in \mathbb{O}$. Clearly, $Z(\mathbb{O})$ is a field and $\mathbb{O}$ can be regarded as an algebra over $Z(\mathbb{O})$.

Suppose $\mathbb{F}$ is a subfield of $Z(\mathbb{O})$. We say that $\mathbb{O}$ is *quadratic over* $\mathbb{F}$ if there exist (necessarily unique) functions $T : \mathbb{O} \to \mathbb{F}$ and $N : \mathbb{O} \to \mathbb{F}$ such that

- $a^2 - T(a) \cdot a + N(a) = 0$ for every $a \in \mathbb{O}$;
- $T(a) = 2a$ and $N(a) = a^2$ for every $a \in \mathbb{F}$.

The following proposition is precisely Theorem 20.3 of Tits & Weiss [14].

Proposition 3.2 ([14]) *Suppose $\mathbb{O}$ is an alternative division ring that is quadratic over some subfield $\mathbb{F}$ of its center $Z(\mathbb{O})$. Let $T : \mathbb{O} \to \mathbb{F}$ and $N : \mathbb{O} \to \mathbb{F}$ be the unique functions as defined above and put $a^\sigma := T(a) - a$ for every $a \in \mathbb{O}$. Then exactly one of the following holds:*

(a) *$\mathbb{O} = \mathbb{F}$ is a field and $\sigma = 1$;*

(b) *$\mathbb{O}$ and $\mathbb{F}$ are fields, $\mathbb{O}$ is a separable quadratic extension of $\mathbb{F}$ and σ is the unique nontrivial element of the Galois group $\mathsf{Gal}(\mathbb{O}/\mathbb{F})$;*

(c) *$\mathbb{O}$ is a field of characteristic 2, $\sigma = 1$ and $\mathbb{O}^2 \subseteq \mathbb{F} \neq \mathbb{O}$;*

(d) *$\mathbb{O}$ is a quaternion division algebra, $\mathbb{F} = Z(\mathbb{O})$ and σ is the so-called standard involution of $\mathbb{O}$;*

(e) *$\mathbb{O}$ is a Cayley-Dickson division algebra over $\mathbb{F} = Z(\mathbb{O})$ and σ is the so-called standard involution of $\mathbb{O}$.*

In each case, σ is an involution of $\mathbb{O}$ and $N(a) = a^\sigma a \in \mathbb{F}$ for all $a \in \mathbb{O}$.

Suppose now that $\mathbb{O}$ is an alternative division ring that is quadratic over some subfield $\mathbb{F}$ of its center $Z(\mathbb{O})$. By using coordinates, De Bruyn & Van Maldeghem showed in [6] that with the pair $\mathcal{T} = (\mathbb{O}, \mathbb{F})$, there is associated a polar space $\Pi_{\mathcal{T}}$. If case (a) of Proposition 3.2 occurs, then $\Pi_{\mathcal{T}}$ is isomorphic to the symplectic polar space $W(5, \mathbb{F})$. If case (b) occurs, then $\Pi_{\mathcal{T}}$ is isomorphic to the Hermitian polar space $H(5, \mathbb{O}/\mathbb{F})$. In case (c), $\Pi_{\mathcal{T}}$ is a so-called polar space of mixed type, and in case (d), $\Pi_{\mathcal{T}}$ is a so-called quaternionic polar space. Finally, if case (e) occurs, then $\Pi_{\mathcal{T}}$ is isomorphic to the unique (up to isomorphism) nonembeddable polar space for which all planes are non-Desarguesian Moufang planes having $\mathbb{O}$ as associated Cayley-Dickson division algebra.

Let $\Delta_{\mathcal{T}}$ be the dual polar space associated with $\Pi_{\mathcal{T}}$. In De Bruyn & Van Maldeghem [7], it was shown that this dual polar space admits a full projective embedding and that its quads are of quadric-type. So, from Proposition 3.1, we can conclude that every two disjoint quads of $\Delta_{\mathcal{T}}$ are contained in a unique hyperbolic set of quads.

In fact, in each of the cases (a), (b), (c) and (d), there exists a set $\{\Delta_{\mathcal{T}}^{(3)}, \Delta_{\mathcal{T}}^{(4)}, \Delta_{\mathcal{T}}^{(5)}, \ldots\}$ of dual polar spaces such that:

- $\Delta_{\mathcal{T}}^{(n)}$ is a dual polar space of rank n for every $n \geq 3$;
- $\Delta_{\mathcal{T}}^{(3)} = \Delta_{\mathcal{T}}$;
- if $n_1 \geq n_2 \geq 3$ and F is a convex subspace of diameter n_2 of $\Delta_{\mathcal{T}}^{(n_1)}$, then $\widetilde{F} \cong \Delta_{\mathcal{T}}^{(n_2)}$.

For each $n \geq 4$, the quads of $\Delta_{\mathcal{T}}^{(n)}$ are still of quadric-type and so it remains valid that every two disjoint maxes of $\Delta_{\mathcal{T}}^{(n)}$ are contained in a unique hyperbolic set of maxes.

4 An application to hyperplanes of dual polar spaces

Throughout this section, Δ denotes a fully embeddable thick dual polar space of rank $n \geq 3$. Results of Dienst [8], Kasikova & Shult [9, Section 4.6] and Tits [13, 8.6] guarantee that Δ must admit the so-called absolutely universal embedding $\widetilde{e} : \Delta \to \widetilde{\Sigma}$. Every classical hyperplane of Δ must arise from $\widetilde{e}$. If F is a convex subspace of diameter at least 2 of Δ, then by De Bruyn [4, Theorem 1.4] the full projective embedding of the dual polar space $\widetilde{F}$ induced by $\widetilde{e}$ is isomorphic to the absolutely universal embedding of $\widetilde{F}$. If U is a subset of $\widetilde{\Sigma}$, then we denote by $\langle U \rangle$ the subspace of $\widetilde{\Sigma}$ generated by U.

Lemma 4.1 *If Δ' is a thick dual polar space of rank $n' \geq 2$ and H' is a hyperplane of Δ' arising from some full projective embedding $e' : \Delta' \to \Sigma'$ of Δ', then $U' = \langle e'(H') \rangle_{\Sigma'}$ is a hyperplane of Σ' and $H' = {e'}^{-1}(e'(\mathcal{P}') \cap U')$, where $\mathcal{P}'$ denotes the point set of Δ'.*

Proof. This is a known property. It follows from the fact that hyperplanes of thick dual polar spaces are maximal proper subspaces, see Blok & Brouwer [1, Theorem 7.3] or Shult [12, Lemma 6.1]. ■

Proposition 4.2 *If M_1 and M_2 are two disjoint maxes of Δ, then $\langle \widetilde{e}(M_1) \rangle$ and $\langle \widetilde{e}(M_2) \rangle$ are disjoint subspaces of $\widetilde{\Sigma}$.*

Proof. Suppose that $U := \langle \widetilde{e}(M_1)\rangle \cap \langle \widetilde{e}(M_2)\rangle$ is nonempty. Then let V denote a hyperplane of $\langle \widetilde{e}(M_2)\rangle$ not containing U. Let G_2 denote the classical hyperplane of $\widetilde{M_2}$ consisting of all points of M_2 that are mapped by $\widetilde{e}$ into V, and let G_1 be the classical hyperplane $\pi_{M_1}(G_2)$ of $\widetilde{M_1}$. Let H be the hyperplane of Δ consisting of all points at distance at most 1 from G_1. The hyperplane H is called the *extension* of G_1 and arises from the embedding $\widetilde{e}$ by De Bruyn [4, Theorem 1.2(1)]. So, there exists a necessarily unique hyperplane W of $\widetilde{\Sigma}$ such that $H = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap W)$, where $\mathcal{P}$ denotes the point set of Δ. Since $H \cap M_2 = G_2$, the hyperplane W cannot contain $\langle \widetilde{e}(M_2)\rangle$ and hence $V' := W \cap \langle \widetilde{e}(M_2)\rangle$ is a hyperplane of $\langle \widetilde{e}(M_2)\rangle$. Since $V' \cap \widetilde{e}(M_2) = (W \cap \widetilde{e}(\mathcal{P})) \cap \widetilde{e}(M_2) = \widetilde{e}(H) \cap \widetilde{e}(M_2) = \widetilde{e}(H \cap M_2) = \widetilde{e}(G_2)$, we have $V' = V$. Now, since $M_1 \subseteq H$, we have $\langle \widetilde{e}(M_1)\rangle \subseteq W$ and hence $U = \langle \widetilde{e}(M_1)\rangle \cap \langle \widetilde{e}(M_2)\rangle \subseteq W \cap \langle \widetilde{e}(M_2)\rangle = V' = V$, contrary to our assumption that V does not contain U. ∎

Remark. If Δ' is a thick dual polar space of rank $n' \geq 2$ and $\{H_i \,|\, i \in I\}$ is a set of hyperplanes of Δ' covering the whole point set of Δ', then by Lemma 3.1 of Cardinali, De Bruyn & Pasini [2], $|I|$ is at least equal to the total number of points on a line (which is a constant, as Δ' is thick). This remark shows that the set X occurring in the following proposition is nonempty.

Proposition 4.3 *Suppose M_1 and M_2 are two disjoint maxes of Δ that are contained in a (necessarily unique) hyperbolic set $\mathcal{H}$ of maxes. Let H be a hyperplane of Δ, and put $H_1 := M_1 \cap H$, $H_2 := M_2 \cap H$ and $H_1' := \pi_{M_1}(H_2)$. Suppose $H_1 \neq M_1$ and $H_2 \neq M_2$. Then $X := M_1 \setminus (H_1 \cup H_1') \neq \emptyset$. Let $\mathcal{L}_1$ denote the set of lines of M_1 disjoint from $H_1 \cap H_1'$ and let $\mathcal{L}_2$ denote the set of lines of M_1 meeting $H_1 \cap H_1'$ nontrivially. Suppose $\mathcal{L}_1'$ is a subset of $\mathcal{L}_1$ such that $\mathcal{L}_1' \cup \mathcal{L}_2$ defines a connected geometry[1] on X. For every line L of M_1, the lines L and $\pi_{M_2}(L)$ are contained in a unique quad Q_L and we define $\mathcal{Q} := \{Q_L \,|\, L \in \mathcal{L}_1'\}$. Suppose the following holds:*

(*a*) *For every $i \in \{1, 2\}$, $H_i = H \cap M_i$ is a classical hyperplane of $\widetilde{M_i}$.*

(*b*) *For every $Q \in \mathcal{Q}$, $H \cap Q$ is a classical hyperplane of $\widetilde{Q}$.*

Then for every $M \in \mathcal{H}$, we have that $H \cap M$ is either M or a classical hyperplane of $\widetilde{M}$.

Proof. Put $U_i = \langle \widetilde{e}(M_i)\rangle$, $i \in \{1, 2\}$. For every point x of M_1, let L_x denote the unique line through x meeting M_2, i.e. $L_x = x\pi_{M_2}(x)$. Let $\mathcal{M}$ denote the union of all maxes of $\mathcal{H}$, and let $\widetilde{\mathcal{M}}$ denote the point-line geometry induced on $\mathcal{M}$ by those lines of Δ that are contained in $\mathcal{M}$. The embedding $\widetilde{e}$ induces an embedding of $\widetilde{\mathcal{M}}$ into the subspace $U := \langle U_1, U_2\rangle$ of $\widetilde{\Sigma}$. Let x^* be an arbitrary point of X and let y^* denote the unique point of H on the line L_{x^*}. Since $U_1 \cap U_2 = \emptyset$ (Proposition 4.2) and $\langle \widetilde{e}(H_i)\rangle$, $i \in \{1, 2\}$, is a hyperplane of U_i, we have that $V := \langle \widetilde{e}(H_1 \cup H_2 \cup \{y^*\})\rangle$ is a hyperplane of U. Let $\mathcal{G}$ denote the set of points of $\mathcal{M}$ which are mapped by $\widetilde{e}$ into V. Then $\mathcal{G}$ is a hyperplane of $\widetilde{\mathcal{M}}$ for which the following holds:

[1]Since $\mathcal{L}_1 \cup \mathcal{L}_2$ consists of all lines of $\widetilde{M_1}$, the set $\mathcal{L}_1'$ can always be chosen in such a way.

If $M \in \mathcal{H}$, then $\mathcal{G} \cap M$ is either M or a classical hyperplane of $\widetilde{M}$.

So, in order to prove the proposition, it suffices to show that $\mathcal{G} = H \cap \mathcal{M}$, or equivalently that $\mathcal{G} \cap L_x = H \cap L_x$ for every point x of M_1.

- Suppose $x \in H_1 \cap H_1'$. Then $L_x \subset H$ and $L_x \subset \mathcal{G}$ since x and $\pi_{M_2}(x)$ are two distinct points of L_x contained in $\mathcal{G} \cap H$.
- Suppose $x \in H_1 \setminus H_1'$. Then $L_x \cap H = \{x\} = L_x \cap \mathcal{G}$ since $x \in H \cap \mathcal{G}$ and $\pi_{M_2}(x) \notin H \cup \mathcal{G}$.
- Suppose $x \in H_1' \setminus H_1$. Then $L_x \cap H = \{\pi_{M_2}(x)\} = L_x \cap \mathcal{G}$ since $\pi_{M_2}(x) \in H \cap \mathcal{G}$ and $x \notin H \cup \mathcal{G}$.
- Suppose $x = x^*$. Then $L_x \cap H = \{y^*\} = L_x \cap \mathcal{G}$ since $y^* \in H \cap \mathcal{G}$ and $x \notin H \cup \mathcal{G}$.

In view of the above and the connectedness of $(X, \mathcal{L}_1' \cup \mathcal{L}_2)$, it thus suffices to prove the following:

If x_1 and x_2 are two distinct collinear points of X such that $x_1x_2 \in \mathcal{L}_1' \cup \mathcal{L}_2$ and $L_{x_1} \cap \mathcal{G} = L_{x_1} \cap H$, then also $L_{x_2} \cap \mathcal{G} = L_{x_2} \cap H$.

So, let x_1 and x_2 be points as above. Let L be the line x_1x_2, let Q be the quad Q_L and let G be the full subgrid $Q \cap \mathcal{M}$. Let u denote the unique point in $L_{x_1} \cap \mathcal{G} = L_{x_1} \cap H$.

Suppose first that $L \in \mathcal{L}_2$. Let v denote the unique point in $L \cap H_1 = L \cap H_1'$ and let u' denote the unique point of L_v collinear with u. Then $uu' \cup L_v$ is contained in $\mathcal{G} \cap H$. Since the grid G is not completely contained in $\mathcal{G}$, nor in H, we must have $\mathcal{G} \cap G = uu' \cup L_v = H \cap G$. Hence, $\mathcal{G} \cap L_{x_2} = H \cap L_{x_2}$ as we needed to show.

Suppose next that $L \in \mathcal{L}_1'$. Let v denote the unique point in $L \cap H_1$, let v' denote the unique point in $L \cap H_1'$ and put $v'' := \pi_{M_2}(v')$. Let α be the 3-dimensional subspace $\langle \widetilde{e}(G) \rangle$ of $\widetilde{\Sigma}$. Then $\beta = \langle \widetilde{e}(u), \widetilde{e}(v), \widetilde{e}(v'') \rangle$ is a plane of α. Since u, v and v'' are contained in $\mathcal{G}$ but G is not contained in $\mathcal{G}$, we have $\mathcal{G} \cap G = \widetilde{e}^{-1}(\widetilde{e}(G) \cap \beta)$.

Now, the embedding of $\widetilde{Q}$ induced by $\widetilde{e}$ is isomorphic to the absolutely universal embedding of $\widetilde{Q}$. Since $H \cap Q$ is a classical hyperplane of $\widetilde{Q}$, there must exist a necessarily unique hyperplane γ of $\langle \widetilde{e}(Q) \rangle$ such that $H \cap Q = \widetilde{e}^{-1}(\widetilde{e}(Q) \cap \gamma)$. This implies that there exists a subspace β' in α such that $H \cap G$ is equal to $\widetilde{e}^{-1}(\widetilde{e}(G) \cap \beta')$. Since u, v, v'' belong to $H \cap G$, but G itself is not completely contained in H, we have $\beta' = \langle \widetilde{e}(u), \widetilde{e}(v), \widetilde{e}(v'') \rangle = \beta$. Hence, $H \cap G = \widetilde{e}^{-1}(\widetilde{e}(G) \cap \beta') = \widetilde{e}^{-1}(\widetilde{e}(G) \cap \beta) = \mathcal{G} \cap G$. Hence, $L_{x_2} \cap \mathcal{G} = L_{x_2} \cap H$ as we needed to show. ■

Proposition 4.4 *Let H be a hyperplane of Δ, and let M_1 and M_2 be two disjoint maxes of Δ. Suppose M_1 and M_2 are contained in a (necessarily unique) hyperbolic set $\mathcal{H}$ of maxes. Suppose also that the following hold:*

(a) *For every $i \in \{1, 2\}$, $H_i := H \cap M_i$ is either M_i or a classical hyperplane of $\widetilde{M_i}$.*

(*b*) *For every quad Q meeting M_1 and M_2 (necessarily in lines), $H \cap Q$ is either Q or a classical hyperplane of $\widetilde{Q}$.*

Then for every $M \in \mathcal{H}$, we have that $H \cap M$ is either M or a classical hyperplane of $\widetilde{M}$.

Proof. If $M_1 \subseteq H$ and $M_2 \subseteq H$, then every max $M \in \mathcal{H}$ is contained in H and we are done. If precisely one of M_1, M_2 is contained in H, say M_1, then for every $M \in \mathcal{H} \setminus \{M_1\}$, the intersection $M \cap H = \pi_M(M_2 \cap H)$ is a classical hyperplane of $\widetilde{M}$ isomorphic to the classical hyperplane $M_2 \cap H$ of $\widetilde{M_2}$. So, we may suppose that neither M_1 nor M_2 is contained in H.

By Ronan [11, Corollary 2, p. 180], every hyperplane of a fully embeddable point-line geometry with three points per line is classical. So, the proposition is certainly valid if every line of Δ is incident with precisely three points.

Suppose every line of Δ is incident with precisely four points. Then the existence of hyperbolic sets of maxes implies that Δ is isomorphic to either the symplectic dual polar space $DW(2n-1, 3)$ or the Hermitian dual polar space $DH(2n-1, 9)$. In each of these two cases, all hyperplanes of Δ and its maxes are classical. For the symplectic case, this follows from De Bruyn [3, Corollary p. 1385]. For the Hermitian case, this follows from Cardinali, De Bruyn & Pasini [2, Corollary 1.6] and De Bruyn & Pralle [5].

Suppose now that every line of Δ is incident with at least five points. Then the claim would follow from Proposition 4.3 if we were able to show that $\mathcal{L}_1 \cup \mathcal{L}_2$ defines a connected geometry on the set X. Here, X, $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same meaning as in Proposition 4.3. In fact, this can be shown using a similar argument by means of which the connectedness of hyperplane complements has been shown in Blok & Brouwer [1, Theorem 7.3] and Shult [12, Lemma 6.1]. Specifically, we need to show that any two points x_1 and x_2 of X are connected by a path such that any two succeeding points are incident with some line of $\mathcal{L}_1 \cup \mathcal{L}_2$. This can be achieved by using an inductive argument on the distance $\delta := \mathrm{d}(x_1, x_2)$ between x_1 and x_2. The cases $\delta = 0$ and $\delta = 1$ are trivial. If $\delta \geq 2$, then two lines L_1 and L_2 in M_1 can be chosen such that $x_1 \in L_1$, $x_2 \in L_2$ and every point of L_1 has distance $\delta - 1$ from L_2. Then the maps $L_1 \to L_2; x \mapsto \pi_{L_2}(x)$ and $L_2 \to L_1; x \mapsto \pi_{L_1}(x)$ are bijections which are each other's inverses. Since L_1 and L_2 are incident with at least five points, there exists a point $u \in L_1$ such that neither u nor $\pi_{L_2}(u)$ is contained in $H_1 \cup H_1'$. Since $\mathrm{d}(u, \pi_{L_2}(u)) = \delta - 1$, the induction hypothesis implies that u and u' are connected by a suitable path, implying that also x_1 and x_2 must be connected by a suitable path. ∎

Propositions 4.3 and 4.4 thus imply that if all intersections of H with certain well-chosen quads are classical, then the intersections with certain other quads need to be classical as well (more precisely, those quads $Q \not\subseteq H$ that are contained in some max of $\mathcal{H}$). In fact, Propositions 4.3 and 4.4 can not only be applied to the hyperplane H of Δ, but in principle also to the hyperplane $H \cap F$ of $\widetilde{F}$, where F is any convex subspace of diameter at least 3 of Δ which is not contained in H.

References

[1] R. J. Blok and A. E. Brouwer. The geometry far from a residue. *Groups and geometries* (Siena, 1996), 29–38, Trends Math., Birkhäuser, Basel, 1998.

[2] I. Cardinali, B. De Bruyn and A. Pasini. On the simple connectedness of hyperplane complements in dual polar spaces. *Discrete Math.* 309 (2009), 294–303.

[3] B. De Bruyn. The hyperplanes of finite symplectic dual polar spaces which arise from projective embeddings. *European J. Combin.* 32 (2011), 1384–1393.

[4] B. De Bruyn. On extensions of hyperplanes of dual polar spaces. *J. Combin. Theory Ser. A* 118 (2011), 949–961.

[5] B. De Bruyn and H. Pralle. The hyperplanes of $DH(5, q^2)$. *Forum Math.* 20 (2008), 239–264.

[6] B. De Bruyn and H. Van Maldeghem. Non-embeddable polar spaces. *Münster J. Math.*, to appear.

[7] B. De Bruyn and H. Van Maldeghem. Universal and homogeneous embeddings of dual polar spaces of rank 3 defined over quadratic alternative division algebras. *J. Reine Angew. Math.*, to appear.

[8] K. J. Dienst. Verallgemeinerte Vierecke in projektiven Räumen. *Arch. Math.* 35 (1980), 177–186.

[9] A. Kasikova and E. Shult. Absolute embeddings of point-line geometries. *J. Algebra* 238 (2001), 265–291.

[10] J. McInroy and S. Shpectorov. On the simple connectedness of hyperplane complements in dual polar spaces. II. *Discrete Math.* 310 (2010), 1381–1388.

[11] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179–185.

[12] E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[13] J. Tits. *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics 386. Springer-Verlag, Berlin-New York, 1974.

[14] J. Tits and R. M. Weiss. *Moufang polygons.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.