Advances in Mathematics of Communications Volume 5, No. 2, 2011, 157–160 $\,$

doi:10.3934/amc.2011.5.157

A GEOMETRIC PROOF OF THE UPPER BOUND ON THE SIZE OF PARTIAL SPREADS IN $H(4n+1,q^2)$

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(Communicated by Ivan Landjev)

ABSTRACT. We give a geometric proof of the upper bound of $q^{2n+1} + 1$ on the size of partial spreads in the polar space $H(4n+1,q^2)$. This bound is tight and has already been proved in an algebraic way. Our alternative proof also yields a characterization of the partial spreads of maximum size in $H(4n+1,q^2)$.

1. INTRODUCTION

A classical finite polar space is an incidence structure, consisting of the totally isotropic subspaces of a projective space with respect to a non-degenerate sesquilinear form or a non-degenerate quadratic form. All dimensions will be assumed to be projective from now on, and we will also refer to *m*-dimensional subspaces as simply *m*-spaces. In particular, the 0- and 1-dimensional subspaces of such a polar space are known as its *points* and *lines*, respectively. The *generators* are its subspaces of maximal dimension. A *partial spread* of a classical finite polar space is a set of generators with no two incident with a common point. If a partial spread actually partitions the point set of the polar space, it is said to be a *spread*.

The Hermitian variety $H(n, q^2)$ is a particular type of classical finite polar space, consisting of the subspaces in $\mathsf{PG}(n, q^2)$, the points of which all have homogeneous coordinates (x_0, \ldots, x_n) satisfying the equation $x_0^{q+1} + \ldots + x_n^{q+1} = 0$. In this polar space, the generators are (n-1)/2-dimensional, if n is odd, or (n-2)/2-dimensional, if n is even, and the number of points is given by $|H(n, q^2)| = (q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$. We refer to [4] for proofs and much more information on Hermitian varieties and polar spaces in general.

Thas [6] proved that in $H(2n + 1, q^2)$ spreads, or thus partial spreads of size $q^{2n+1} + 1$, cannot exist, which has made the question on the size of a partial spread in such a polar space, an intriguing question. Improved upper bounds on the size of partial spreads in $H(2n + 1, q^2)$ were proved in [2].

On the other hand, partial spreads of size $q^{n+1} + 1$ in $H(2n + 1, q^2)$ were constructed for all $n \ge 1$ in [1], by use of a symplectic polarity of the projective space $PG(2n + 1, q^2)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which these two polarities coincide, a (regular) spread of the induced symplectic polar space W(2n + 1, q) can always be found, and these

²⁰⁰⁰ Mathematics Subject Classification: 05B25, 51E23.

Key words and phrases: Hermitian varieties, partial spreads.

This research is supported by the Research Foundation Flanders-Belgium (FWO-Vlaanderen).

 $q^{n+1} + 1$ generators extend to pairwise disjoint generators of $H(2n + 1, q^2)$. Maximality of partial spreads of $H(2n + 1, q^2)$ constructed in this way was also shown for n = 1, 2 in [1] and for all even n in [5].

In [3], De Beule and Metsch proved that the maximum size of a partial spread in $H(5, q^2)$ is $q^3 + 1$, and they also obtained additional information on partial spreads meeting that tight bound. In particular, they found that every generator of $H(5, q^2)$, not meeting any element of such a partial spread S in a line or more, meets exactly $q^2 - q + 1$ elements of S in a point.

Using techniques from algebraic graph theory, we recently proved in [7] that the size of a partial spread in $H(4n+1, q^2)$ is at most $q^{2n+1}+1$, and this bound is thus tight as well. It turns out that a geometric property of partial spreads of maximum size in $H(5, q^2)$ can be generalized, and in fact paves the way for a new, completely geometric proof of the upper bound in $H(4n+1, q^2)$.

2. Tools

We first state a lemma by Thas [6].

Lemma 2.1. Let π_1, π_2 and π be three mutually disjoint generators in $H(2n+1, q^2)$. The set of points on π_1 , that are on a (necessarily unique) line of $H(2n+1, q^2)$ meeting both π and π_2 , form a non-singular Hermitian variety in π_1 .

Corollary 2.2. Let π_1, π_2 and π be three mutually disjoint generators in $H(2n + 1, q^2)$. The number of generators meeting π in an (n-1)-space, and meeting both π_1 and π_2 in a point is $|H(n, q^2)| = \frac{(q^{n+1}+(-1)^n)(q^n-(-1)^n)}{q^2-1}$.

Proof. We let \perp denote the Hermitian polarity of $\mathsf{PG}(2n+1,q^2)$, associated with the polar space. It is obvious that every generator meeting π in an (n-1)-space, can meet π_1 and π_2 in at most one point. On the other hand, through any point $p_1 \in \pi_1$, there is a unique generator $\langle p_1, p_1^{\perp} \cap \pi \rangle$ meeting π in an (n-1)-space. Hence we have to determine the number of points $p_1 \in \pi_1$ such that the generator $\langle p_1, p_1^{\perp} \cap \pi \rangle$ also meets π_2 in a point.

First suppose that a point $p_1 \in \pi_1$ is such that the generator $\langle p_1, p_1^{\perp} \cap \pi \rangle$ meets π_2 in a point p_2 . In that case, the line p_1p_2 is a line of $H(2n+1,q^2)$, meeting π as well, as $p_1^{\perp} \cap \pi$ is a hyperplane of $\langle p_1, p_1^{\perp} \cap \pi \rangle$. Conversely, suppose a point $p_1 \in \pi_1$ is on a line of $H(2n+1,q^2)$, meeting π in p and π_2 in p_2 . In that case, both p_1 and p are in the generator $\langle p_1, p_1^{\perp} \cap \pi \rangle$, and hence so is the entire line p_1p , including the point p_2 . The desired result thus follows from Lemma 2.1.

3. The proof

Theorem 3.1. The size of a partial spread S in $H(4n + 1, q^2)$, $n \ge 1$, is at most $q^{2n+1}+1$. If |S| > 1 and $\pi \in S$, then every generator meeting π in a (2n-1)-space, will meet the same number of other elements of S in just a point, if and only if $|S| = q^{2n+1} + 1$. In that case, that number must be q^{2n} .

Proof. Let S be a partial spread of size at least 2 in $H(4n+1,q^2)$. Consider a fixed element $\pi \in S$. Let $\{N_i | i \in I\}$ be the set of generators meeting π in a (2n-1)-space. As the number of (2n-1)-spaces in a generator equals $(q^{4n+2}-1)/(q^2-1)$, and the number of generators through any (2n-1)-space in $H(4n+1,q^2)$ is given by q+1, the cardinality of I is $\frac{q^{4n+2}-1}{q^2-1}q$.

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Note that any generator N_i and any generator in $S \setminus \{\pi\}$, are either disjoint or meet in a point. For every $N_i, i \in I$, let t_i denote the number of generators in $S \setminus \{\pi\}$, meeting N_i in a point. We now count the number of pairs (N_i, π') , with π' an element of $S \setminus \{\pi\}$ meeting N_i in a point, in two ways. As through every point p'on an element π' of $S \setminus \{\pi\}$, there is a unique generator meeting π in a (2n-1)-space, we obtain:

(1)
$$\sum_{i \in I} t_i = (|S| - 1) \frac{q^{4n+2} - 1}{q^2 - 1}.$$

Now we count the number of ordered triples (N_i, π_1, π_2) , with π_1 and π_2 two distinct elements of $S \setminus \{\pi\}$, both meeting N_i in a point. We know from Corollary 2.2 that for every two distinct elements of $S \setminus \{\pi\}$, there will be exactly $|H(2n, q^2)|$ generators N_i , meeting both of them in a point. Hence we obtain:

(2)
$$\sum_{i \in I} t_i(t_i - 1) = (|S| - 1)(|S| - 2)\frac{(q^{2n+1} + 1)(q^{2n} - 1)}{q^2 - 1}.$$

Combining (1) and (2), we find:

(3)
$$\sum_{i \in I} t_i^2 = (|S| - 1) \frac{q^{2n+1} + 1}{q^2 - 1} \left((q^{2n+1} - 1) + (|S| - 2)(q^{2n} - 1) \right).$$

As $(\sum_{i \in I} t_i)^2 \leq (\sum_{i \in I} t_i^2) |I|$, with equality if and only if all t_i are equal, this implies: $(|S|-1)^2 \left(\frac{q^{4n+2}-1}{q^2-1}\right)^2 \leq (|S|-1) \frac{q^{2n+1}+1}{q^2-1} \left((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\right) \frac{q^{4n+2}-1}{q^2-1}q$,

with equality if and only if all t_i are equal. Since we assumed that |S| > 1, we can cancel factors on both sides to obtain:

$$(|S|-1)(q^{2n+1}-1) \le \left((q^{2n+1}-1) + (|S|-2)(q^{2n}-1) \right) q,$$

implying that $|S| \leq q^{2n+1} + 1$, with equality if and only if all t_i are equal. In that case, their constant value must equal $(\sum_{i \in I} t_i)/|I| = q^{2n}$.

4. Remark

This technique fails when applied to partial spreads in $H(4n + 3, q^2)$, where it yields a negative lower bound on the size instead.

Acknowledgements

The author is grateful to Jan De Beule for his helpful suggestions while preparing this manuscript.

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Received March 2010; revised August 2010.

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