# A GEOMETRIC PROOF OF THE UPPER BOUND ON THE SIZE OF PARTIAL SPREADS IN $H\left(4 n+1, q^{2}\right)$ 

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#### Abstract

We give a geometric proof of the upper bound of $q^{2 n+1}+1$ on the size of partial spreads in the polar space $H\left(4 n+1, q^{2}\right)$. This bound is tight and has already been proved in an algebraic way. Our alternative proof also yields a characterization of the partial spreads of maximum size in $H\left(4 n+1, q^{2}\right)$.


## 1. Introduction

A classical finite polar space is an incidence structure, consisting of the totally isotropic subspaces of a projective space with respect to a non-degenerate sesquilinear form or a non-degenerate quadratic form. All dimensions will be assumed to be projective from now on, and we will also refer to $m$-dimensional subspaces as simply $m$-spaces. In particular, the 0 - and 1-dimensional subspaces of such a polar space are known as its points and lines, respectively. The generators are its subspaces of maximal dimension. A partial spread of a classical finite polar space is a set of generators with no two incident with a common point. If a partial spread actually partitions the point set of the polar space, it is said to be a spread.

The Hermitian variety $H\left(n, q^{2}\right)$ is a particular type of classical finite polar space, consisting of the subspaces in $\operatorname{PG}\left(n, q^{2}\right)$, the points of which all have homogeneous coordinates $\left(x_{0}, \ldots, x_{n}\right)$ satisfying the equation $x_{0}^{q+1}+\ldots+x_{n}^{q+1}=0$. In this polar space, the generators are $(n-1) / 2$-dimensional, if $n$ is odd, or $(n-2) / 2$-dimensional, if $n$ is even, and the number of points is given by $\left|H\left(n, q^{2}\right)\right|=\left(q^{n+1}+(-1)^{n}\right)\left(q^{n}-\right.$ $\left.(-1)^{n}\right) /\left(q^{2}-1\right)$. We refer to [4] for proofs and much more information on Hermitian varieties and polar spaces in general.

Thas [6] proved that in $H\left(2 n+1, q^{2}\right)$ spreads, or thus partial spreads of size $q^{2 n+1}+1$, cannot exist, which has made the question on the size of a partial spread in such a polar space, an intriguing question. Improved upper bounds on the size of partial spreads in $H\left(2 n+1, q^{2}\right)$ were proved in [2].

On the other hand, partial spreads of size $q^{n+1}+1$ in $H\left(2 n+1, q^{2}\right)$ were constructed for all $n \geq 1$ in [1], by use of a symplectic polarity of the projective space $\mathrm{PG}\left(2 n+1, q^{2}\right)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which these two polarities coincide, a (regular) spread of the induced symplectic polar space $W(2 n+1, q)$ can always be found, and these

[^0]$q^{n+1}+1$ generators extend to pairwise disjoint generators of $H\left(2 n+1, q^{2}\right)$. Maximality of partial spreads of $H\left(2 n+1, q^{2}\right)$ constructed in this way was also shown for $n=1,2$ in [1] and for all even $n$ in [5].

In [3], De Beule and Metsch proved that the maximum size of a partial spread in $H\left(5, q^{2}\right)$ is $q^{3}+1$, and they also obtained additional information on partial spreads meeting that tight bound. In particular, they found that every generator of $H\left(5, q^{2}\right)$, not meeting any element of such a partial spread $S$ in a line or more, meets exactly $q^{2}-q+1$ elements of $S$ in a point.

Using techniques from algebraic graph theory, we recently proved in [7] that the size of a partial spread in $H\left(4 n+1, q^{2}\right)$ is at most $q^{2 n+1}+1$, and this bound is thus tight as well. It turns out that a geometric property of partial spreads of maximum size in $H\left(5, q^{2}\right)$ can be generalized, and in fact paves the way for a new, completely geometric proof of the upper bound in $H\left(4 n+1, q^{2}\right)$.

## 2. Tools

We first state a lemma by Thas [6].
Lemma 2.1. Let $\pi_{1}, \pi_{2}$ and $\pi$ be three mutually disjoint generators in $H\left(2 n+1, q^{2}\right)$. The set of points on $\pi_{1}$, that are on a (necessarily unique) line of $H\left(2 n+1, q^{2}\right)$ meeting both $\pi$ and $\pi_{2}$, form a non-singular Hermitian variety in $\pi_{1}$.

Corollary 2.2. Let $\pi_{1}, \pi_{2}$ and $\pi$ be three mutually disjoint generators in $H(2 n+$ $\left.1, q^{2}\right)$. The number of generators meeting $\pi$ in an $(n-1)$-space, and meeting both $\pi_{1}$ and $\pi_{2}$ in a point is $\left|H\left(n, q^{2}\right)\right|=\frac{\left(q^{n+1}+(-1)^{n}\right)\left(q^{n}-(-1)^{n}\right)}{q^{2}-1}$.
Proof. We let $\perp$ denote the Hermitian polarity of $\operatorname{PG}\left(2 n+1, q^{2}\right)$, associated with the polar space. It is obvious that every generator meeting $\pi$ in an $(n-1)$-space, can meet $\pi_{1}$ and $\pi_{2}$ in at most one point. On the other hand, through any point $p_{1} \in \pi_{1}$, there is a unique generator $\left\langle p_{1}, p_{1}^{\perp} \cap \pi\right\rangle$ meeting $\pi$ in an ( $n-1$ )-space. Hence we have to determine the number of points $p_{1} \in \pi_{1}$ such that the generator $\left\langle p_{1}, p_{1}^{\perp} \cap \pi\right\rangle$ also meets $\pi_{2}$ in a point.

First suppose that a point $p_{1} \in \pi_{1}$ is such that the generator $\left\langle p_{1}, p_{1}^{\perp} \cap \pi\right\rangle$ meets $\pi_{2}$ in a point $p_{2}$. In that case, the line $p_{1} p_{2}$ is a line of $H\left(2 n+1, q^{2}\right)$, meeting $\pi$ as well, as $p_{1}^{\perp} \cap \pi$ is a hyperplane of $\left\langle p_{1}, p_{1}^{\perp} \cap \pi\right\rangle$. Conversely, suppose a point $p_{1} \in \pi_{1}$ is on a line of $H\left(2 n+1, q^{2}\right)$, meeting $\pi$ in $p$ and $\pi_{2}$ in $p_{2}$. In that case, both $p_{1}$ and $p$ are in the generator $\left\langle p_{1}, p_{1}^{\perp} \cap \pi\right\rangle$, and hence so is the entire line $p_{1} p$, including the point $p_{2}$. The desired result thus follows from Lemma 2.1.

## 3. The proof

Theorem 3.1. The size of a partial spread $S$ in $H\left(4 n+1, q^{2}\right), n \geq 1$, is at most $q^{2 n+1}+1$. If $|S|>1$ and $\pi \in S$, then every generator meeting $\pi$ in a $(2 n-1)$-space, will meet the same number of other elements of $S$ in just a point, if and only if $|S|=q^{2 n+1}+1$. In that case, that number must be $q^{2 n}$.

Proof. Let $S$ be a partial spread of size at least 2 in $H\left(4 n+1, q^{2}\right)$. Consider a fixed element $\pi \in S$. Let $\left\{N_{i} \mid i \in I\right\}$ be the set of generators meeting $\pi$ in a ( $2 n-1$ )-space. As the number of $(2 n-1)$-spaces in a generator equals $\left(q^{4 n+2}-1\right) /\left(q^{2}-1\right)$, and the number of generators through any $(2 n-1)$-space in $H\left(4 n+1, q^{2}\right)$ is given by $q+1$, the cardinality of $I$ is $\frac{q^{4 n+2}-1}{q^{2}-1} q$.

Note that any generator $N_{i}$ and any generator in $S \backslash\{\pi\}$, are either disjoint or meet in a point. For every $N_{i}, i \in I$, let $t_{i}$ denote the number of generators in $S \backslash\{\pi\}$, meeting $N_{i}$ in a point. We now count the number of pairs $\left(N_{i}, \pi^{\prime}\right)$, with $\pi^{\prime}$ an element of $S \backslash\{\pi\}$ meeting $N_{i}$ in a point, in two ways. As through every point $p^{\prime}$ on an element $\pi^{\prime}$ of $S \backslash\{\pi\}$, there is a unique generator meeting $\pi$ in a ( $2 n-1$ )-space, we obtain:

$$
\begin{equation*}
\sum_{i \in I} t_{i}=(|S|-1) \frac{q^{4 n+2}-1}{q^{2}-1} \tag{1}
\end{equation*}
$$

Now we count the number of ordered triples $\left(N_{i}, \pi_{1}, \pi_{2}\right)$, with $\pi_{1}$ and $\pi_{2}$ two distinct elements of $S \backslash\{\pi\}$, both meeting $N_{i}$ in a point. We know from Corollary 2.2 that for every two distinct elements of $S \backslash\{\pi\}$, there will be exactly $\left|H\left(2 n, q^{2}\right)\right|$ generators $N_{i}$, meeting both of them in a point. Hence we obtain:

$$
\begin{equation*}
\sum_{i \in I} t_{i}\left(t_{i}-1\right)=(|S|-1)(|S|-2) \frac{\left(q^{2 n+1}+1\right)\left(q^{2 n}-1\right)}{q^{2}-1} \tag{2}
\end{equation*}
$$

Combining (1) and (2), we find:

$$
\begin{equation*}
\sum_{i \in I} t_{i}^{2}=(|S|-1) \frac{q^{2 n+1}+1}{q^{2}-1}\left(\left(q^{2 n+1}-1\right)+(|S|-2)\left(q^{2 n}-1\right)\right) \tag{3}
\end{equation*}
$$

As $\left(\sum_{i \in I} t_{i}\right)^{2} \leq\left(\sum_{i \in I} t_{i}^{2}\right)|I|$, with equality if and only if all $t_{i}$ are equal, this implies:
$(|S|-1)^{2}\left(\frac{q^{4 n+2}-1}{q^{2}-1}\right)^{2} \leq(|S|-1) \frac{q^{2 n+1}+1}{q^{2}-1}\left(\left(q^{2 n+1}-1\right)+(|S|-2)\left(q^{2 n}-1\right)\right) \frac{q^{4 n+2}-1}{q^{2}-1} q$,
with equality if and only if all $t_{i}$ are equal. Since we assumed that $|S|>1$, we can cancel factors on both sides to obtain:

$$
(|S|-1)\left(q^{2 n+1}-1\right) \leq\left(\left(q^{2 n+1}-1\right)+(|S|-2)\left(q^{2 n}-1\right)\right) q
$$

implying that $|S| \leq q^{2 n+1}+1$, with equality if and only if all $t_{i}$ are equal. In that case, their constant value must equal $\left(\sum_{i \in I} t_{i}\right) /|I|=q^{2 n}$.

## 4. Remark

This technique fails when applied to partial spreads in $H\left(4 n+3, q^{2}\right)$, where it yields a negative lower bound on the size instead.

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