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Relevance in Reasoning
The Adaptive Logics Approach

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Preface

This dissertation is the result of my work at the *Centre for Logic and Philosophy of Science* at Ghent University,[1] where I worked on the GOA–project entitled *Development of adaptive logics for the study of central topics in contemporary philosophy of science.*[2]

First of all, I would like to thank Diderik "Dirk" Batens, Erik Weber and Joke Meheus, for giving me the opportunity to work on this dissertation by granting me a phd–sholarship related to the GOA–project.

In relation to this dissertation, I would like to thank Dirk for reading some of the chapters, and for the valuable comments he made. They have definitely improved the final result.

Many thanks go to my colleagues, family and friends for their friendship and support. Finally, and not in the least, I want to thank my wife Lobke for her endless support and faith in me, especially on those days when I found it quite hard to take place behind my computer for just another day of endless writing.

Hans Lycke

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[1]The website of the Centre is situated at `http://logica.ugent.be/centrum/`.

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Part I

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Chapter 1

Introduction

> However, the logician's commitment to truth–functional connectives is not without its reasons. How is one to characterize such an obscure notion as that of [logical] dependence?[1]

1.1 Introduction

Although the title might have suggested otherwise, this dissertation does not in the first place deal with relevance in reasoning. It deals with the famous problem of *Chrysippus' Dog* in Relevance Logic.[2] However, as what is investigated in Relevance Logic is the dependence relation between premises and conclusions in real deductive reasoning, this dissertation does indirectly deals with relevance in reasoning.

In this chapter, I will both introduce Relevance Logic (in section 1.2) and the problem related to *Chrysippus' Dog* (in section 1.3).

Preliminary Remark. Before I can start, I need to make an important preliminary remark: I will restrict myself to the branch of Relevance Logic that sprung from the work of Anderson & Belnap's [5]. This means that I will not consider other kinds of Relevance Logic, such as for example Parry's Analytical Implication (see e.g. Parry [81], Dunn [51]), Verhoeven's Relevantly Assertable Disjunction (see e.g. Verhoeven [121, 122]),...

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[1] See Suppes [104, p. 7].

[2] Besides "Relevance Logic", also the term "Relevant Logic" is used in the literature and it is often claimed that the preference of one term over the other is a geographical matter. Americans would prefer "Relevance Logic", while Australians and Europeans would prefer "Relevant Logic". Anyway, I will use "Relevance Logic" to denote the scientific research area directed at the investigation of relevance in reasoning, while I will use "relevant logics" to denote the particular logical systems.

1.2 Relevance Logic

In Relevance Logic, a formula is only considered a relevant consequence of a premise set whenever there is a proof expressing a *substantial connection* between premises and conclusion:

> So *what* is it all *about*? A short answer: *connection*. One statement implies another [...], only if it is connected with it, only if the statements have *enough to do with* one another;[3]

1.2.1 The Paradoxes of the Material Implication

The derivability relation of *Classical Logic* (**CL**) doesn't express a substantial connection between premises and conclusion, which is obvious from the so–called *paradoxes of the material implication*.[4] The best–known ones are the following:

(1) $\vdash_{\mathbf{CL}} A \supset (B \supset A)$
(2) $\vdash_{\mathbf{CL}} \neg A \supset (A \supset B)$
(3) $\vdash_{\mathbf{CL}} (A \wedge \neg A) \supset B$
(4) $\vdash_{\mathbf{CL}} A \supset (B \vee \neg B)$

The paradoxes of the material implication are interpreted as *fallacies of relevance*, because they express an implicative connection between formulas where there should be none. For example, (1) allows for the derivation of the formula $B \supset A$ from the formula A, while there is obviously no connection between B and A. Hence, accepting (1) would make it possible to derive "if I died yesterday, I'm writing my PhD today" from "I'm writing my PhD today," which is clearly absurd.[5]

In fact, Relevance Logic began as an attempt to avoid the paradoxes of the material implication. As a consequence, in relevant logics (**RL**), the material implication is replaced by a relevant implication (usually denoted by $\rightarrow$), an implication that doesn't commit the fallacies of relevance.

More CL–Paradoxes. Besides the paradoxes of the material implication, two other fallacies of relevance obtain in **CL**. The first one is the **EQV**–paradox (Ex Quodlibet Verum), which states that some formulas (usually called logical truths, tautologies or theorems) are derivable from any premise set, even the empty one.

[3] Sic, see Sylvan and Norman [106, p. 3].

[4] Also *Modal Logic* (**ML**) doesn't express a substantial connection between premises and conclusion, as is shown by the *the paradoxes of the strict implication*. I will however not consider the paradoxes of the strict implication.

[5] For a host of examples concerning the paradoxes of the material implication, see Routley and Meyer [101, ch. 1].

[EQV] $\forall A$: If $\vdash_{\mathbf{CL}} A$ then $\forall\Gamma : \Gamma \vdash_{\mathbf{CL}} A$.

This is obviously a fallacy of relevance, as it can not be claimed that a logical truth really follows *from* a premise set. In other words, the derivation of logical truths is not dependent upon the premises.

The second additional fallacy of relevance is the **EFQ**–paradox (Ex Falso Quodlibet), which states that anything follows from an inconsistency.[6]

[EFQ] When $\Gamma \vdash_{\mathbf{CL}} B$ and $\Gamma \vdash_{\mathbf{CL}} \neg B$, then $\forall A : \Gamma \vdash_{\mathbf{CL}} A$.

This is a fallacy of relevance, as the consequences might have nothing to do with the premises. Despite the fact that those consequences do follow from the premises (in contradistinction with the **EQV**–paradoxes), no meaningful connection between them and the premises is guaranteed.

The additional **CL**–paradoxes are avoided by going paraconsistent and paracomplete. Usually, paraconsistency and paracompleteness are obtained by extending the set of models, so that inconsistencies are true in some of them (resulting in paraconsistency) and that no formula is true in all of them (resulting in paracompleteness).

At first sight, **RL** do not seem to avoid all of the **EQV**– and **EFQ**–paradoxes. For example, they allow for the derivation of logical truths:

(5) $\vdash_{\mathbf{RL}} A \to A$
(6) ...

However, **RL** also avoid the **EQV**– and **EFQ**–paradoxes, albeit not in the usual way. This will be shown in the next section. As a consequence, relevant logicians also try to capture relevant deduction, and not solely relevant implication.

1.2.2 Relevant Deduction

In relevant logics (**RL**), *relevant deduction* (or relevant derivability) is not a primitive, but a derivative notion. More specifically, it is defined as follows:[7]

Definition 1.1 *$A_1, ..., A_n \vdash\!\!\!\to_{\mathbf{RL}} B$ is a relevant deduction of B from $A_1, ..., A_n$ iff $\vdash_{\mathbf{RL}} (A_1 \wedge ... \wedge A_n) \to B$.*

Definition 1.1 clearly shows that in **RL**, relevant deduction is dependent upon the implicational theorems. This in fact means that relevant logicians

[6] Some would prefer to state that anything follows from a *contradiction*. I however take the difference to be immaterial.

[7] See Brady and Bunder [43, pp. 302–308] for an explicit treatment of this point. Remark also that this definition is not possible for logics not containing the conjunction connective. As I will not consider such logics, I am happy with definition 1.1.

interpret implicational theorems — which they usually call entailments[8] — as “inference rules”. They are used to derive conclusions from a premise set, but cannot be added to the premises in order to derive the conclusions.

Remark that this way of characterizing derivability clearly avoids the **EQV**– and **EFQ**–paradoxes, as the **CL**–paradoxes (3) and (4) from the foregoing section, are not **RL**–entailments.

Classical and Relevant Deduction. By taking relevant deduction to be riding piggyback on the notion of entailment, relevant logicians actually split up derivability into *classical derivability* (denoted by $\vdash$) and *relevant derivability* (denoted by $\vdash\!\!\rightarrow$).[9]

Implication and Deduction. Because of the fact that relevant derivability is dependent upon classical derivability, and by consequence on **RL**–entailments, relevant logician take their implication symbol to be equivalent in meaning to the turnstile symbol (the one for relevant derivability obviously). As such, the implication symbol is to be read as “is relevantly derivable from”.[10]

Personally, I don’t like the identity of the turnstile and the implication. I take deduction and implication to be two distinct notions. Moreover, it is probably a good idea to state that in this dissertation I will be concerned with relevant deduction and not with relevant implication. I take the latter to be captured quite nicely in **RL** (in contradistinction to the latter).

1.2.3 Characterizing Entailments

As entailments are used to characterize relevant deduction, it should be stated clearly what the criteria are for counting as an entailment. Although other people have provided different criteria (for example Lance [63] and Brady [41]), I will present the two relevance criteria that were provided by Anderson & Belnap in [5].

The Use–Criterium

Anderson and Belnap’s first relevance criterium is the so–called *use–criterium* (**UC**). It was first presented by Church in [50], and Meyer [75, p. 54] summarized it as follows:

[8]Remark that calling the implicational theorems entailments is actually quite appropriate, as they are taken to express a deductive relation.

[9]This distinction was introduced by Batens and Van Bendegem in [27]. They also discuss the relation between both kinds of derivability and give an alternative characterization of relevant derivability. I will stick with the one from the Relevance Logic literature.

[10]Maybe I am generalizing to quickly here. It might well be the case that not all relevant logicians are adherents of this identity. In any case, Anderson & Belnap were (as they explain in the grammatical propaedeutic of their [5]).

(**UC**) A formula B is *relevantly derivable* from the *bunch* of premisses $A_1, ..., A_n$ just in case there is a *deduction* of B from the premisses in which each of the A_i is *used*.[11]

This criterium was first only applied to the purely implicational fragment of the language and lead to the relevant logic $\mathbf{R}_{\rightarrow}$, the implicational fragment of the relevant logic **R**.

In [5], Anderson & Belnap presented their famous Fitch–style proof theory for the logic $\mathbf{R}_{\rightarrow}$. It captures **UC** in a fairly straightforward way. Shortly stated, all introduced hypotheses are given an index set, corresponding to their depth (meaning that the hypothesis of the n–th subproof gets the index set $\{n\}$). When the inference rule Modus Ponens (**MP**) is applied, the index set of the resulting formula is the union of the index sets of its premises. Finally, the Conditional Proof–rule (**CP**) may only be applied when the index set of the hypothesis of the subproof is a subset of the index set of the formula on the last line of that subproof, which expresses that the hypothesis has been *used* in the derivation of that formula.[12]

In order to illustrate the proof theory and the use–criterium on which it is based, let's consider some examples. The first example shows us that the formula $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$ is an **RL**–entailment.

1	$\mid p \rightarrow (p \rightarrow q)_{\{1\}}$	HYP$_1$
2	$\mid \mid p_{\{2\}}$	HYP$_2$
3	$\mid \mid p \rightarrow (p \rightarrow q)_{\{1\}}$	1;REIT
4	$\mid \mid p \rightarrow q_{\{1,2\}}$	2,3;MP
5	$\mid \mid q_{\{1,2\}}$	2,4;MP
6	$\mid p \rightarrow q_{\{1\}}$	2,5;CP
7	$(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)_{\{\emptyset\}}$	1,7;CP

It should be immediately clear that this indeed captures some of our intuitions about how hypothetical reasoning proceeds.

Moreover, the paradoxes of the material implication are avoided, as is shown by the proof below. In it, the implicational formula $p \rightarrow (q \rightarrow p)$ is not derivable, as the hypothesis q on line 2 was not used in the derivation of the formula p on line 3.

1	$\mid p_{\{1\}}$	HYP$_1$
2	$\mid \mid q_{\{2\}}$	HYP$_2$
3	$\mid \mid p_{\{1\}}$	1;REIT

[11] Sic.

[12] Obviously the proof theory differs for **RL** different from **R**, but those differences should not concern us here.

Although the use–criterium works quite nicely when the language is restricted to the purely implicational fragment, it breaks down when the other connectives are introduced in the language. At least, if the classical rules governing those connectives are retained. In that case, the proof below would allow us to consider $(p \wedge \neg p) \rightarrow q$ as an **RL**–entailment.

1	$p \wedge \neg p_{\{1\}}$	HYP_1
2	$p_{\{1\}}$	1;SIM
3	$\neg p_{\{1\}}$	1;SIM
4	$p \vee q_{\{1\}}$	2;ADD
5	$q_{\{1\}}$	1;DS
6	$(p \wedge \neg p) \rightarrow q_{\emptyset}$	1,5;CP

Although the formula on line 1 was clearly used in the derivation of the formula on line 5, the resulting entailment can hardly be called a relevant entailment. So, it seems that some extra relevance criterium is needed in order to cope with the additional connectives. This criterium is the so–called *variable–sharing criterium.*

Variable–Sharing Criterium

Intuitively, the *variable-sharing criterium* (**VC**) states that $A \rightarrow B$ should only be an **RL**–entailment, when A and B share at least one sentential letter.[13] Consequently, $(p \wedge \neg p) \rightarrow q$ is not considered a valid entailment.

This criterium is defended by referring to the fact that in order for an implication to be an entailment, some of the "meaning content" of the antecedent should also be present in the consequent, which simply comes down to the fact that both have to be related to each other in some way. In order to grasp this "common meaning content" in a formal way, Anderson & Belnap state that "commonality of meaning in propositional logic is carried by commonality of propositional variables."[14]

Tautological Entailments. The importance of the **VC** is to be situated in the fact that it is the sole and sufficient criterium for tautological entailments. Tautological entailments are entailments of the form $A \rightarrow B$ in which A and B do not contain any implications (the only connectives occurring in A and B are $\wedge, \vee$ and $\sim$).[15] As tautological entailments are also called first degree entailments, I will take **VC** to capture first degree relevance.

However, the variable–sharing criterium does not demarcate relevant from irrelevant tautological entailments by simply checking whether some

[13] This is of course my terminology. Anderson & Belnap use "propositional variables".

[14] See Anderson & Belnap [5, p. 33].

[15] From now on, I will use $\sim$ in order to denote the negation of relevant logics, while $\neg$ will be used for other types of negation, such as for example classical negation.

sentential letter occurs both in the antecedent and in the consequent. The method is somewhat more demanding. First, a tautological entailment is transformed into *normal form*. This comes down to the fact that the antecedent is placed in *disjunctive normal form*, while the consequent is placed in *conjunctive normal form*.[16] The result is an entailment of the following form:

$$A_1 \vee ... \vee A_n \rightarrow B_1 \wedge ... \wedge B_m$$

Such an entailment in normal form is now considered a tautological entailment iff for all its A_i and B_j share an atomic formulas (a sentential letter or a negated sentential letter).

This characterization of tautological entailments is important as the tautological entailments were used to determine the behavior of the truthfunctional connectives[17] in general. As such, **VC** provided the solution to the problem with the use–criterium stated above.

Combining the Criteria

From the above, we can conclude that the *use–criterium* characterizes relevant implication, while the *variable-sharing criterium* characterizes the behavior of the truthfunctional connectives. In the Fitch–style proof theory, they are combined in the following way: the characterization of the implication remains the same, while inside a subproof, the truthfunctional connectives behave as is determined by the tautological entailments.

1.2.4 Conclusion

In this section, I have shown how standard Relevance Logic incorporated certain relevance criteria in order to obtain a formal theory of relevant deduction. I have however not said anything about whether Relevance Logic also succeeded in giving a satisfactory account of relevant deduction. To this I will turn now, by considering the parable of Chrysippus' dog.

1.3 Chrysippus' Dog

> [Chrysippus] declares that the dog makes use of the fifth complex indemonstrable syllogism when, on arriving at a spot where three ways meet, after smelling at the two roads by which the quarry did not pass, he rushes off at once by the third without stopping to smell. For, says the old writer, the dog implicitly reasons thus: "The creature either went by this road, or by that, or by the other: but it did not go by this road or by that: therefore it went by the other.'[18]

[16] I take these notions to be well–known, so that they don't need any explanation.
[17] Anderson & Belnap refer to $\wedge, \vee, \sim$ as to the truthfunctional connections.
[18] Sextus Empiricus, cited in [5, p. 296].

1.3.1 To Muzzle the Dog

In modern formal logic, the parable of Chrysippus' dog — presented in the quote above — became famous because of its appearance in [5], in which Anderson & Belnap stringently reject the validity of the dog's reasoning.

> Nevertheless, we do hold that the inference from $\neg A$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make. Such an inference commits nothing less than a fallacy of relevance.

The particular element in the dog's reasoning that is rejected by Anderson & Belnap, is the application of the inference rule *Disjunctive Syllogism* (**DS**), which they do not consider a valid step in the deduction process.[19]

Why did Anderson & Belnap, and with them so many others, reject **DS**? In this section, I will present two reasons they themselves have given, and also two additional ones.

Relevance Considerations

Because of the equivalence of deduction and implication (see section 1.2.2), the rejection of the inference rule **DS** in fact comes down to the fact that the entailment **EDS** is not considered as a **RL**–entailment, as it clearly expresses **DS**.

[**EDS**] $\not\models_{\mathbf{RL}} ((A \vee B) \wedge {\sim}A) \rightarrow B$

Why should **EDS** not be considered as a relevant **RL**–entailment? As Anderson & Belnap themselves mention in the quote above, the primary reason for rejecting (**E**)**DS** is that it commits "a fallacy of relevance". At first, this might seem odd, but when we recall the variable–sharing criterium, then it becomes plainly obvious that **EDS** indeed commits a fallacy of relevance. For, consider **EDS** in normal form:

$$(B \wedge {\sim}A) \vee (A \wedge {\sim}A) \rightarrow B$$

As the second disjunct of the antecedent $(A \wedge {\sim}A)$ doesn't share a variable with the consequent (B), **EDS** can safely be rejected by means of the variable–sharing criterium.

Moreover, if we would plainly add **EDS** to standard **RL**, without bothering about the variable–sharing criterium (after all, $(A \vee B) \wedge {\sim}A$ and B do share a variable), we would end up with a lot of irrelevant entailments, as the following proof makes clear:

[19] In fact, Anderson and Belnap not only consider **DS** as fallacious, but *material detachment* (**MD**) in general. Besides **DS**, this also includes Modus Ponens and Modus Tollens for the material implication. However, because of the identity in **CL** of an implicative formula with a disjunctive formula ($A \supset B =_{df} \neg A \vee B$), all of these inference rules can also be considered as instances of **DS**.

1	$(A \wedge {\sim}A) \rightarrow A$	**RL**–axiom
2	$(A \wedge {\sim}A) \rightarrow {\sim}A$	**RL**–axiom
3	$A \rightarrow (A \vee B)$	**RL**–axiom
4	$(A \wedge {\sim}A) \rightarrow (A \vee B)$	1,3;Transitivity
5	$(A \wedge {\sim}A) \rightarrow ((A \vee B) \wedge {\sim}A))$	2,4;**RL**–axiom and MP
6	$((A \vee B) \wedge {\sim}A) \rightarrow B$	**EDS**
7	$(A \wedge {\sim}A) \rightarrow B$	5,6;Transitivity

As it is the explicit aim of Relevance Logic to avoid the fallacies of relevance obtaining in **CL**, it should be clear from the example that adding **EDS** to relevant logics is not an option.

Deduction–Implication Equivalence

The arguments above made clear that **EDS** cannot be added to the axiom system of **RL**. But, it might nevertheless be possible to add **DS** to the proof theory of **RL** as a primitive inference rule, an inference rule that is not represented by an entailment.[20]

> when The Man accepts $A \wedge (\neg A \vee B) \rightarrow B$, he is making a s simple inferential blunder. But surely The Man has *something* in mind, and we may charitably suppose him to have been believing that, whereas B clearly is not entailed by A and $\neg A \vee B$, B on the other hand *is* derivable from A and $\neg A \vee B$, in the sense that from A and $\neg A \vee B$ as premisses we can find a deduction of B."[21]

However, this strategy obviously forces one to make a distinction between relevant deduction and relevant implication, a distinction Anderson & Belnap are not willing to make.

> This charity, though welcome, is misplaced, at least for a plausible understanding of what The Dog means by "derivable." For as the Entailment theorem of §23.6 teaches us, if there were a proof that A and $\neg A \vee B$ entailed B, then we would have $\vdash (A \wedge (\neg A \vee B)) \rightarrow B$, which we know is not so.[22]

Above, I already mentioned that I do not like the equivalence of relevant deduction and relevant implication. As a consequence, I do not really consider this a good reason for not adding **DS** to the **RL**–proof theory. However, there are other reasons as well. I will present them below.

[20] Let it be clear that the proof theory I here refer to, is the proof theory for relevant derivability, and not the one for classical derivability (see section 1.2.2).

[21] See Anderson & Belnap [5, p. 297], sic.

[22] See Anderson & Belnap [5, p. 297], sic.

The Paraconsistent Turn

The first reason for not adding **DS** as a primitive rule to the **RL**–proof theory is the following: if **DS** can be applied unrestrictedly, it leads to triviality in the face of inconsistencies. Consequently, adding **DS** to the **RL**–proof theory would lead to the **EFQ**–paradoxes, which is most clearly shown by Lewis' independent proof:

1	A	Premise
2	$\sim A$	Premise
3	$A \vee B$	1;Addition
4	B	3,4;DS

In order to avoid the **EFQ**–fallacies, so one might reason, the inference rule **DS** should be rejected. However, is it necessary to blame **DS**? In fact, in Lewis' independent proof two inference rules were used, Addition (**ADD**) and **DS**, but it is not at all clear that **DS** is the one to blame for the **EFQ**–fallacies. After all, it is **ADD** which allows to add possibly irrelevant parts to formulas. As a consequence, it seems quite arbitrary to blame (only) **DS**. However, let's for the moment agree that **DS** is to blame.

Inconsistent Theories. Not all reasons in favor of the rejection of **DS** are based on relevance considerations. In fact, all reasons given in favor of paraconsistent logics will also do in this case. Paraconsistent logics are logics that allow for inconsistent theories, which means that they do not lead to triviality when applied to an inconsistent theory.

An obvious question to all this, is why this should concern us. Is it necessary to allow for inconsistent theories? As a matter of fact, it is. In the literature, several reasons have been given (see for example Priest [84, 86], Priest et al. [87]), but the most convincing one is that there actually are theories that are inconsistent, but non–trivial (a much–referred to example is Bohr's theory of the atom). Hence, the logic underlying these theories will be paraconsistent.

I think that this should indeed be a sufficient reason to convinced anyone of the importance of paraconsistent logics. However, the demand for paraconsistency is not the same as the rejection of **DS**. So, the question remains whether it is justified to reject **DS** in order to obtain paraconsistency.

Garfield's Dog

The second additional reason to reject **DS** as a valid rule of inference is provided by Jay Garfield [56, 57]. Garfield dismisses **DS** because, as he states, "it is rational to reason Relevantly, but irrational to reason classically."[23] I

[23]See Garfield [56, pp. 97–98], sic.

agree that it is irrational to reason classically, but this doesn't automatically mean that it is rational to reject the use of **DS**. Anyway, let's have a closer look at his argument.

In short, Garfield's claim comes down to the following: as humans, we live in an *epistemically hostile environment*, which means that

> We are imperfect; we have false beliefs; we have limited inferential powers; we have limited memory; we have suboptimal belief fixations mechanisms; we inhabit an environment which does not go out of its way to deliver the truth to us, and in fact often goes out of its way to deceive us. I refer to this unfortunate state of inner and outer affairs as an "epistemically hostile environment."[24]

In such an environment, Garfield claims, it is not safe to reason by means of the inference rule **DS**, as it must lead to a lot of unjustified beliefs.

> Suppose that in these unfortunate but all too common circumstances you come to believe on the misleading information of a normally reliable source (A) that Albuquerque is the capital of Arizona. Under the spell of the evil classical logician you freely disjoin (B) Belnap is a classical logician (with no relevance index, of course). Since A is justified, so is $A \vee B$. Now, suppose that a bit later your geographical source corrects himself, and you now come to believe $\sim A$. Now $\sim A$ and $A \vee B$ are both in your belief set, you have no positive reason to reject $A \vee B$, you are still classical, and so conclude B. B, of course, is manifestly false. What's more, from a suitably distant perspective (ours) you have no real reason to believe it. What went wrong? The answer is plain: you used classical disjunction rules.[25]

Garfield's claim looks quite appealing at first. It is indeed true that people do not keep track of all justifications of their beliefs, and this might indeed lead to some faulty conclusions.

The problem with Garfield's account is however that if the epistemic hostility of the environment is a reason not to use **DS**, it is as good a reason not to use Modus Ponens, Conjunction, or any other inference rule. For all of those rules it is possible to construct an example as the one above, see for example Mares [67, pp. 177–178]. Also Meyer made this point when he stated the following:

> Similarly, if there are reasons — in terms of unwelcome consequences — to avoid $\supset$E sometimes, the same will ultimately hold of $\rightarrow$E as well. The advantages of $\rightarrow$ over $\supset$ are relative, not absolute.[26]

As a consequence, following Garfield's argumentation, we should conclude that under conditions of epistemic hostility, it is probably best not to reason

[24] See Garfield [56, p. 99].

[25] See Garfield [56, p. 104].

[26] See Meyer [76, p. 606]. Meyer is here not talking about **DS**, but about **MD**. However, as I stated in an earlier footnote, I consider these equivalent.

at all. Of course, this is exactly how Odie behaves, but it definitely is not how humans behave (no, not even some of them).

Conclusion

In conclusion, it can be stated that there are two convincing reasons to reject the dog's reasoning. First of all, the entailment expressing **DS** (**EDS**) does not pass the variable–sharing criterium. Moreover, if it is added to the axiom system of **RL**, it leads to the derivation of irrelevant entailments. Secondly, because of the need for paraconsistency, it is not even possible to add **DS** as a primitive rule to the **RL**–proof theory, as this leads to triviality in the case of inconsistencies.

1.3.2 The Dog Bites Back

The rejection of **DS** however encountered a lot of opposition. First of all, because it seems extremely counterintuitive not to consider **DS** as a valid reasoning step, a fact that is even recognized by Graham Priest in [83], despite the fact that he also favors a logic (called **LP**) that doesn't validate **DS**.

> The most obvious thing about the logic of paradox [**LP**] is that it forces us to give up as invalid certain principles of deduction that one would not normally suspect.[27]

However, an appeal to the intuitions is certainly not enough to grant **DS** its right of existence. A lot of things which are at first counterintuitive, are nevertheless true, for example that the earth is spinning around at high speed. The main problem with the rejection of **DS** is that it is clearly in contradiction with human practice. Even more, it is clearly in contradiction with *rational* human practice.

> In view of the fact that everyday arguments and mathematical proofs abound in instances of disjunctive syllogism, one may wonder how Anderson & Belnap could hope to reconcile their rejection of disjunctive syllogism with the claim that their "relevant logic" is compatible with commonsense and accepted mathematical practice.[28]

As logic is taken to explicate rational reasoning, this really constitutes a problem.[29] It means that **RL** do not adequately capture deductive relevance,

[27]See Priest [83, p. 231].

[28]See Burgess [46, p. 98]. Obviously, he is referring to **DS**.

[29]Explication consists in turning an unclear, intuitive concept into a very precise one. In fact, the notion originated from the work of Carnap. In [49, p. 3], he defined it as follows:

> The task of explication consists in transforming a given more or less inexact concept into an exact one or, rather, in replacing the first by the second. We call

the way in which premises and conclusions are connected in real deductive reasoning.

1.3.3 What the Dog Should Do

At this point, it seems that we are stuck in between the relevantists and the classicalists. It doesn't seem true that **DS** is unrestrictedly valid (the classical claim), but it also doesn't seem true that it is a fallacy of relevance (the relevantist claim). This is what I will call the **DS**–problem.

Now, given the **DS**–problem, what should the dog do? Or better, what should a logician do, when confronted with this problem? My answer is plain and simple: he should try to find an alternative way to characterize relevant deduction.

Being a logician, this is exactly what I will do in this dissertation. However, I will not start all over again and construct a completely new theory of relevant deduction. As the **DS**–problem is situated at the first degree, I will only present a new theory of first degree relevance, which will afterwards be combined with the relevant implication of **RL**. That this is possible should not surprise us, as it was shown above that relevant implication and first degree relevance were handled by different relevance criteria (resp. **UC** and **VC**). As such, they remained partly independent.

Remark that this means that I will not present a theory of relevant implication. First of all, I do not consider it necessary, as I find that relevant implication is captured quite nicely in **RL**. And secondly, this dissertation is long enough as it is now.

1.4 The Aim of this Dissertation

As should be clear by now, this dissertation can be considered as a long–winded afterthought to the parable of Chrysippus' dog. More specifically, I will show that it is possible to reintroduce **DS** in **RL** without reintroducing any of the fallacies of relevance.

Overview of This Dissertation

In chapter 2, the second chapter of this part, I will discuss some of the solutions to the **DS**–problem, proposed in the literature. It will turn out

> the given concept (or the term used for it) the explicandum, and the exact concept proposed to take the place of the first (or the term proposed for it) the explicatum. The explicandum may belong to everyday language or to a previous stage in the development of scientific language. The explicatum must be given by explicit rules for its use, for example, by a definition which incorporates it into a well–constructed system of scientific either logico–mathematical or empirical concepts.

Reasoning is taken to be explicated by a logic when its proof theory reflects the actual reasoning processes.

that none of the proposed solutions has really solved the problem, despite the fact that some of them seem intuitively appealing.

In part II, I will present the logic background of this dissertation. First of all, in chapter 3, I will present the adaptive logics programme (**ALP**). Moreover, also the standard format of adaptive logics will be presented, which is necessary as in later chapters, relevant deduction will be characterized by means of adaptive logics. Next, in chapters 4 and 5, I will introduce respectively those paralogics and those relevant logics that will be used at a certain point in this dissertation.

In part III, I will characterize relevant deduction at the first degree. In chapter 6, an intuitive characterization of first degree relevance will be given. Moreover, I will show that it is possible distinguish between two kinds of relevant deduction, namely classical relevance (**CR**) and first degree relevance (**FDR**). In chapters 7 and 8, I will present two adaptive logics that both explicate **CR**, the first one in a straightforward way, the second one in a way that can be extended to other logics as well. Finally, in chapter 9, the adaptive logic that explicates **FDR** will be presented.

In order to obtain a complete theory of relevant deduction, the theory of first degree relevance presented in part III has to be combined with the relevant implication from standard Relevance Logic. This is done in part IV. More specifically, in chapter 10, some inconsistency–adaptive relevant logics will be presented, which are used in chapter 11 to characterize the adaptive logics that capture relevant deduction in an adequate way.

Finally, part V consists of two chapters that are rather unrelated to the rest of this dissertation. However, as they make use of the logical systems presented in part III, they show that the presented logics are not only suited to characterize relevant deduction, but can be used for other purposes as well. In chapter 12, it is shown that (inconsistency–)adaptive logics are based on only a partial insight in the premises, and it is shown how adaptive logics can be constructed that are based on a complete insight in the premises. Moreover, this is done by making use of the logic that was presented in chapter 8. To conclude, in chapter 13, I will present some adaptive logics for explicating abductive reasoning. This will be done both for abduction based on consistent and on inconsistent theories. To characterize the latter, I will make use of the logic presented in chapter 7.

Chapter 2

Proposed Solutions

2.1 Introduction

The overt rejection by Anderson & Belnap of the inference rule disjunctive syllogism, did not pass unnoticed. As was mentioned in chapter 1, it encountered a lot of opposition, not only from classically oriented logicians such as Burgess [46, 47, 48], but even from within the Relevance Logic community, see for example Meyer [75, 76].

In this chapter, I will discuss some of the solutions for the **DS**–problem, proposed by relevant logicians. It will turn out that they are all incapable to satisfactorily cope with the problem. However, by considering these proposals, it will become clear what the conditions are for a satisfactory solution.

Preliminary Remark. Below, I have included all proposed solutions known to me. However, it is not improbable that I overlooked some of the proposals. I nevertheless do remain quite confident that these will resemble (one or more of) the discussed proposals.

2.2 Ambiguity of the Disjunction

Probably the best known attempt to solve the **DS**–problem is the one first made by Anderson & Belnap themselves in [5, §16], and most firmly defended by Stephen Read in [89, 90, 91, 92]. It consists in interpreting the natural language disjunction as an ambiguous connective.

2.2.1 The Ambiguous Disjunction

The proponents of this approach claim that disjunction has two possible interpretations, an extensional and an intensional one.[1] The former, denoted

[1]Obviously, they completely leave aside the inclusive vs. exclusive discussion.

by " $\vee$ ", is the usual truthfunctional disjunction, while the latter, denoted by " $+$ ", is actually an implication "in disguise":

Definition 2.1 $A + B =_{df} {\sim}A \to B.$

It is now easy to check that in relevant logics, the inference rule **DS** is valid for the intensional disjunction (**IDS**), while it is invalid for the extensional disjunction (**EDS**):

EDS $\quad A \vee B, {\sim}A \nvdash_{\mathbf{RL}} B$
IDS $\quad A + B, {\sim}A \vdash_{\mathbf{RL}} B$

Moreover, the inference rule **ADD** is only valid for the extensional disjunction:

EADD $\quad A \vdash_{\mathbf{RL}} A \vee B$
IADD $\quad A \nvdash_{\mathbf{RL}} A + B$

As a consequence, applying the inference rule **IDS** can never lead to irrelevant consequences.

Disjunctive Syllogism. How this approach actually solves the **DS**–problem is now quite straightforward:

> Furthermore, in rejecting the principle of the disjunctive syllogism, we intend to restrict our rejection to the case in which the "or" is taken truth functionally. in general and with respect to our ordinary reasonings this would not be the case; perhaps always when the principle is used in reasoning one has in mind an intensional meaning of "or," where there is relevance between the disjuncts.[2]

As a consequence, irrelevant deductions that make use of **DS** — such as for example, the Lewis Proof — should be interpreted, not as a fallacy of relevance, but as a fallacy of ambiguity: the disjunction in the proof was interpreted as an intensional disjunction, but should have been interpreted as an extensional disjunction.

2.2.2 Problems with this Approach

There are nevertheless some problems with the above attempt at solving the **DS**-problem. I will present three of them below, namely those which I consider the most important ones.

[2]See Anderson & Belnap [5, p. 165].

Relevant Connection. It is not possible to convincingly argue for the fact that all instances of **DS** in real–life reasoning are instances of **IDS**, at least if one takes the intensional disjunction to express "relevance between the disjuncts" as in the quote above. This was convincingly shown by Burgess in [46, 47, 48], who points to the fact that in order for $A + B$ to be true, not only the truth of $A \vee B$ is required, but also some objective, "relevant" connection between A and B. In order to make this clearer, consider the following example:

> Suppose that X has an insurance policy that pays off if X loses either an arm or a leg. And suppose moreover that one knows both that X is receiving payments and that he hasn't lost an arm. "Well, then," one concludes, "he must have lost a leg."[3]

Obviously, as Burgess correctly point out, there is no objective, "relevant" connection between losing an arm and losing a leg. As a consequence, the disjunction in this example should be interpreted as an extensional disjunction.

There are two possible reactions against this kind of counterargument. The first one consists in taking the relevant connection as merely subjective (or psychological). As such, it might be triggered by the contextual knowledge, which in the example above is the knowledge about the insurance policy. This is a sensible reaction, but it nevertheless runs into trouble, which will be made clear below. The second possible reaction is Stephen Read's. He reacted to Burgess' objections by claiming that the only connection required between both disjuncts of an intensional disjunction, is not an objective connection, but merely a deductive connection, expressed in the object language by a relevant implication:

> One cannot challenge a purported derivation of q from p by the assertion that q is not in some sense relevant to p. If q has indeed been derived from p, what greater connection of relevance could a logician desire? None.

This indeed refutes Burgess' objection, and justifies the interpretation of each instance of **DS** as an instance of **IDS**. However, Read's solution is nevertheless not completely convincing, which is shown below.

Irrelevant Consequences. The second problem for the 'ambiguous disjunction'–approach consists in the fact that interpreting the intensional disjunction as a relevant implication allows to derive some irrelevant consequences. This is most easily shown by means of an example from Barker [6, pp. 372–375]. First, consider a device consisting of a button B and two light bulbs R and L. Whenever the button is pushed, one (and only one!)

[3] Apparently this is an example from Dunn, that was mentioned by Meyer in [75]. I actually read it in Burgess [47, p. 46].

of the light bulbs will light up. For each light bulb, the chance that it will light up when the button is pushed, is 1/2. Next, consider the following two arguments:

A1 If button B is pressed, then R or L will light up. Button B is pressed and R does not light up. Hence, L lights up.

A2 If button B is pressed, then R or L will light up. Moreover, if R lights up, then L will not light up. As a consequence, if button B is pressed then R will light up, or if button B is pressed then L will light up.

It should be immediately clear that given the set–up described above, A1 is clearly valid, while A2 is not. The latter is not valid, as it is not the case that "if B is pressed R will light" is true or that "if B is pressed L will light" is true. The only sensible hypothetical statement that can be made is that "if B is pressed then R will light or L will light".

As proponents of the "ambiguous disjunction"–approach claim that all valid instances of **DS** should be interpreted as instances of **IDS**, the first argument should be formalized as IA1 below, and not as IA2:

IA1 $B \rightarrow (R + L),\ B,\ {\sim}R \vdash_{\mathbf{R_d}} L$
EA1 $B \rightarrow (R \vee L),\ B,\ {\sim}R \nvdash_{\mathbf{R_d}} L$

But, now consider the second argument. It should be invalid, but this is only possible if it is formalized as EA2 below and not as IA2:

IA2 $B \rightarrow (R + L),\ R \rightarrow {\sim}L \vdash_{\mathbf{R_d}} (B \rightarrow R) \vee (B \rightarrow L)$
EA2 $B \rightarrow (R \vee L),\ R \rightarrow {\sim}L \nvdash_{\mathbf{R_d}} (B \rightarrow R) \vee (B \rightarrow L)$

Obviously, the unsound argument A2 can only be avoided when the disjunction is interpreted as extensional, but then the sound argument A1 becomes invalid. But, as the information included in both arguments is based on the same set–up, the disjunction should be interpreted for both arguments in the same way. Consequently, the "ambiguous disjunction"–approach is facing a serious problem here: if the disjunction is taken to be an intensional disjunction (as it should according to the proponents of this approach), then some irrelevant consequences follow.

Incoherence. The third problem for this proposed solution to the **DS**–problem is given by Meyer in [76], where he states that not for all theories disjunction can be coherently interpreted as an intensional disjunction. He gives the following example:

> In the system of relevant arithmetic introduced in Meyer (1975), one finds the theorem $x \neq 0 \subset 0 < x$. The system becomes incoherent if one strengthens this to $x \neq 0 \rightarrow 0 < x$. So we are stuck, at best, with a material $\subset$. But $7 \neq 0$. It seems to me a reasonable conclusion that $0 < 7$ (which is, thank goodness, a theorem). And the task is to find some systematic Relevant way of drawing such reasonable conclusions.

Conclusion. Although the "ambiguous disjunction"–account might seem intuitively appealing at first, it nevertheless breaks down, most importantly because in some cases, it allows for the derivation of irrelevant consequences, and because it seems not possible to interpret all theories in an intensional way.

2.3 The Normality Assumption

Some relevant logicians solve the **DS**–problem by stating that the validity of (extensional) **DS** as an inference rule, is dependent upon some of the features of the deductive situation. More specifically, it is claimed that

> disjunctive syllogism is valid in a proper subclass of reasoning contexts [or deductive situations], namely the "normal" ones.[4]

Obviously, not all logicians agreed on how to interpret the notions "deductive situation" and "normal deductive situation". Hence, these notions were given numerous different interpretations, corresponding to different ways to cope with **DS**. Nevertheless, in all of them **DS** is only applicable under demand of the normality of the deductive situation.

Moreover, in all of them this normality assumption serves as a hidden premise. This is why this approach is also called the *Enthymematical Approach*.

In this section, I will discuss two proposals from the literature. The first one contains what I will call a *global normality assumption* and was presented by Mortensen in [78, 79]. The second one on the contrary, only contains a *local normality assumption*. It was presented by Lavers in [64].

2.3.1 Global Normality

As far as I know, Chris Mortensen was the first to propose that the application of **DS** should be dependent upon the deductive situation. He stated it as follows:

> Human beings are often in the position of deducing sentences from other sentences. Disputes as to the validity of a deduction from certain premisses can, I propose, be thought of as disputes as to the exact nature of the deductive situation containing those premisses.[5]

More specifically, Mortensen claims that a deductive situation, which he interpreted as a theory (the deductive closure of a premise set under a certain logic), will only be closed under **DS** when it is *consistent* and *prime*. But, as people cannot know in advance whether or not a theory is consistent and

[4] See Lavers [64, p. 35].
[5] See Mortensen [78, p. 36].

prime (which would presuppose logical omniscience). Hence, it should be the case that people presuppose consistency and primeness for some deductive situations.

> The examples of **DS** which seem intuitive are often instances of **EDS** [= extensional **DS**]; but this does not make **EDS** valid, and it is not. Whenever it seems intuitive to infer using **EDS**, it is because there is an extra assumption, that things are "normal", which ensures the truth of the conclusion and which explains the apparent intuitiveness of **EDS**.[6]

This presupposition is taken to be metatheoretical. As such, it doesn't appear in the object language. Nevertheless, it has often been represented as follows:

$$[\mathbf{C\&P}] \quad (Con(Th) \wedge Pr(Th) \wedge (A \vee B) \in Th \wedge (\sim A) \in Th) \rightarrow (B \in Th).$$

There are two problems with Mortensen's approach. First of all, presupposing the deductive situation to be consistent and prime doesn't make it consistent and prime. As a consequence, if $Con(Th)$ and $Pr(Th)$ are supposed to be true, a fact which later turns out to be false, then this will allow for the derivation of a lot of irrelevant and plainly false consequences. So, two options are open for Mortensen, which are both not very attractive.[7]

The first option is to claim that people are omniscient. But, this is clearly absurd, and I don't think Mortensen would have chosen this option. The second option is to claim that when confronted with the falsity of their presupposition, people stop applying **DS** and retract all consequences that were obtained by the use of **DS**. Although this seems a reasonable strategy, it nevertheless limits reasoning capacities to an unjustified extent. A lot of consequences obtained by **DS** might be considered safe consequences, namely those that have nothing to do with the found abnormalities.[8]

2.3.2 Local Normality

Also Peter Lavers is a proponent of the enthymematical approach. He also claims that when **DS** is used, some hidden premises were presupposed. Those hidden premises are formulas of the form $b(A, B)$, defined in the following way:[9]

$$b(A, B) =_{df} (A \wedge B) + (A \wedge \sim B) + (\sim A \wedge B) + (\sim A \wedge \sim B).$$

[6] See Mortensen [79, p. 195].

[7] Mortensen doesn't mention these options!

[8] In adaptive terms, reasoning would behave in a way resembling flip–flop logics.

[9] The "+"–symbol again refers to the intensional disjunction. However, it is here used for completely different purposes than in section 2.2.

First of all, remark that these are object–language formulas, which means that Lavers, in contradistinction with Mortensen, doesn't interpret the normality assumption at the metatheoretical level.

> So rather than moving a level up in order to address the question of normality of a reasoning context, the question of normality is regarded as simply part of the reasoning context.[10]

Next, Lavers shows that the hidden premises express local consistency and completeness, which is shown by proving that the following entailments are valid:

- $\vdash_{\mathbf{R}} b(A, B) \to ((A \wedge ({\sim}A \vee B)) \to B)$
- $\vdash_{\mathbf{R}} b(A, B) \to ((A \wedge {\sim}A) \to B)$
- $\vdash_{\mathbf{R}} b(A, B) \to (A \to (B \vee {\sim}B))$

As a consequence, adding a formula $b(A, B)$ to a premise makes it "locally Boolean" with respect to A and B.

Although I prefer Lavers' account over Mortensen's because it treats normality as a local phenomenon instead of a global one, it nevertheless faces the same problem as Mortensen's: presupposing normality does not guarantee normality. In Lavers' own words:

> Whilst the $b(A, B)$ are object-language expressions of local consistency and completeness, assuming all instances of $b(A, B)$ to be true is of course no guarantee that the reasoning context is in fact consistent and complete.

Remarkable as it may be, Lavers doesn't consider this a disadvantage of his approach.

> (This does not constitute a weakness of the account. It simply means that some theories are false in that they affirm $b(A, B)$ when in fact the theory is not consistent and complete, or they fail to affirm $b(A, B)$ (or they affirm its denial) when in fact they are consistent and complete. So: some theories are false.) Whether or not one is in a normal reasoning situation is simply another piece of information to be deliberated by the reasoner and used in the reasoning process.

There is also a second problem with Lavers' proposal. As some fragments of the language will behave in a classical (Boolean) way, a lot of irrelevant consequences (e.g. paradoxes of the material implication) will be derivable in those fragments. As a consequence, one can hardly consider this a solution to the **DS**–problem that takes relevance seriously.

[10]See Lavers [64, p. 35].

2.3.3 Conclusion

I have shown that the enthymematical approach faces one general problem: adding hidden premises is not enough to cope relevantly with the **DS**–problem. In the end, it remains necessary to rely on extra–logical means in order to save the approach.

2.4 Pragmatic Disjunctive Syllogism

A final possible strategy to cope with the **DS**–problem is to treat **DS** as a pragmatic inference rule, related to belief revision. This solution was proposed by Ed Mares in [69] and [67, ch. 10].

Mares' Pragmatism. First of all, Mares introduces the notions of denial and assertion. These refer to speech acts which come down to respectively the explicit rejection and the explicit acceptance of a formula. Next, he states that rejecting a formula is not the same as accepting the negation of that formula. More specifically, rejecting a formula is stronger, as it is not possible to both reject and accept a formula at the same time. However, it is possible to accept (but also to reject) both a formula and its negation.

By means of the notions of denial and assertion, Mares characterizes two pragmatical versions of **DS**, which he calls respectively **PDS** and **PDS′**. The former states that when a disjunction is accepted, while one of its disjuncts is rejected, then the other disjunct should be accepted.

$$\text{[PDS]}\ \frac{\begin{array}{c} Acc(A \vee B) \\ Rej(A) \end{array}}{Acc(B)}$$

The latter states that when a conjunction is rejected, while one of its conjuncts is accepted, then the other conjunct should be rejected.

$$\text{[PDS']}\ \frac{\begin{array}{c} Rej(A \wedge B) \\ Acc(A) \end{array}}{Rej(B)}$$

Remark however that these pragmatic versions of **DS** do not yet reintroduce the non–pragmatic version of the inference rule **DS**, which would mean that the acceptance of a formula $A \vee B$ and the acceptance of the formula $\sim\! A$ would necessitate the acceptance of the formula B. This is because, according to Mares, there is no direct relation between the rejection of a formula and the negation of that formula. Such a relation is nevertheless necessary in order to reintroduce the non–pragmatic version of **DS**.

> The conclusion that we should draw, I think is that there isn't what we could call a 'deductive connection' between negation and rejection. There should, however, be some relationship between them, but what is it?[11]

The gap between the rejection of a formula and its negation is bridged by the human inclination to reject contradictions. Mares takes this inclination to be a human *default tendency*, expressed by the following two 'ceteris paribus'–laws:

(**RC**) All things being equal, reject contradictions.
(**AC**) All things being equal, modify your belief set in such a way as to retain the rejection of contradictions.

This kind of behavior, together with the pragmatic rules **PDS** and **PDS**′, indeed reintroduces the inference rule **DS**, as is shown by the example below:

1	$Acc(A \vee B)$	**PREM**
2	$Acc({\sim}A)$	**PREM**
3	$Rej(A \wedge {\sim}A)$	**RA**
4	$Rej(A)$	2,3;**PDS**′
5	$Acc(B)$	1,4;**PDS**

Finally, remark that as they are 'ceteris paribus'–laws, the above default assumptions are defeasible. Nevertheless, according to Mares, it is justified to use them, because of the fact that

> In the vast majority of cases, using (RC) and (RA) help us to find our way around in the world and manipulate things. They have been reliable rules. We have good inductive justification for thinking them to be reliable.[12]

Heuristics? In Mares' account, **DS** is explicitly interpreted as consequence of some default rules that are used for rationally changing an inconsistent belief set. As such, **DS** is considered as being part of the heuristics that pertain in reasoning and not as being part of the deductive core.

Although I think Mares quite nicely captures the dynamics present in reasoning contexts where people try to turn an inconsistent theory into a consistent one, I am not convinced that **DS** should be interpreted as a heuristic rule instead of as a deductive rule. The problem with Mares approach is that probably all other inference rule can be treated in a heuristic way as well. So, why specifically **DS** and only **DS**?

[11] See Mares [69, pp. 505–506].
[12] See Mares [69, p. 513].

2.5 Conclusion

In this chapter, I have presented some of the proposals to solve the **DS**–problem in relevant logics. Moreover, I have shown that none of them is able to cope with the problem in a satisfactory way.

Necessary Conditions. By considering the proposed solutions to the **DS**–problem, it has become clear which conditions a solution should satisfy in order to be considered as a satisfactory solution. I will recapitulate them below.

(1) No relevance between the disjuncts of a disjunction has to be presupposed in order to be able to apply **DS** (Anderson & Belnap).
(2) A solution to the **DS**–problem should not restrict **DS** to an unjustified extent (Mortensen).
(3) A solution to the **DS**–problem should not lead to irrelevant consequences (Read, Lavers).
(4) A solution to the **DS**–problem should not have to rely on extra–logical means (Mortensen, Lavers, Mares?).
(5) A solution to the **DS**–problem should not consider **DS** as a heuristic rule, but as a deductive rule, expressing a deductive connection (Mares).
(6) Finally, a condition that was not mentioned yet: a solution to the **DS**–problem should treat hypothetical and non–hypothetical reasoning on a par. This means that **DS** should also be applied within the scope of an implication, as in the following example:

- $A \rightarrow (B \vee (C \wedge {\sim}C)) \mapsto_{\mathbf{RL}} A \rightarrow B$.

Because implications can be considered as the result of hypothetical reasoning processes (which is most clearly shown in Fitch–style proof theories where they are the result of subproofs), and because rationality is not different for hypothetical reasoning, a solution to the **DS**–problem should not distinguish between hypothetical and non–hypothetical reasoning.

Part II

Logical Preliminaries

The Aim of Part II

In this part, I will introduce the logic background needed to understand the remaining of this dissertation. Although this might seem quite reproductive, this part also contains some new results. For example, a Fitch–style proof theory is presented for the paralogics presented in chapter 4, and a new semantic characterization is given for the standard relevant logics presented in chapter 5.

Overview of Part II

In chapter 3, the Adaptive Logics Programme (**ALP**) is introduced, which is the research tradition within which my research is to be situated. In chapters 4 and 5, I will present respectively the paralogics and the relevant logics that will be used in later parts of this dissertation.

Chapter 3

The Adaptive Logics Programme

3.1 Introduction

This first chapter of the logical preliminaries is intended as an introduction into the *Adaptive Logics Programme* (**ALP**). In short, the intention of the **ALP** is to restore formal precision in philosophy and in philosophy of science in particular. More specifically, it's central aim is the development of *adaptive logics* that can explicate those reasoning processes that are central in science (and in other interesting domains, such as for example ethics).

> Adaptive logic are intended to *explicate actual forms of reasoning* and only their dynamic proofs provide one with such an explication.[1]

In general, adaptive logics are very suitable tools to capture reasoning processes that display an internal and/or an external dynamics (usually triggered by the absence of a positive test). The *external dynamics* in fact comes down to non–monotonicity: if a premise set is extended, some consequences might not be derivable anymore.[2] The *internal dynamics* is a proof theoretical characteristic: growing insight in the premises, obtained by deriving new consequences from the premises, may lead to the withdrawal of earlier reached conclusions, or to the rehabilitation of earlier withdrawn conclusions.

The first adaptive logics were introduced by Diderik Batens in the early nineteen eighties with the intention to explicate reasoning processes based on inconsistent theories. By now, adaptive logics for handling reasoning in inconsistent contexts — the so–called inconsistency–adaptive logics — are no longer the only ones around. There are also adaptive logics for induction,

[1] See Batens [15, p. 47], sic.

[2] Formally: there are Γ, Δ and A such that $\Gamma \vdash_{\mathbf{AL}} A$ and $\Gamma \cup \Delta \nvdash_{\mathbf{AL}} A$.

abduction, compatibility, question evocation,...[3]

Despite the enormous diversity among adaptive logics, they all share a common structure.[4] This common structure, labeled "the standard format", was first described in [15], and was extended and refined in [26] and [33, p. 6–11].[5] As it is necessary for a proper understanding of this dissertation, I will present the standard format in the sections below. The sole purpose of this presentation is to safeguard the self–containment of this dissertation, which means that readers who are already well acquainted with adaptive logics can easily move on to the next chapter.

3.2 Flat Adaptive Logics

The dynamic behavior of an adaptive logic is generated by the interplay between the three constitutive elements shared by all adaptive logics:

(1) A *lower limit logic* (**LLL**): a reflexive, transitive, monotonic, and compact logic that has a characteristic semantics (with no trivial models).

(2) The *set of abnormalities* Ω of an **AL** is a set of formulas characterized by a (possibly restricted) logical form `F`. However, not only the abnormalities themselves are important for the characterization of **AL**, also (classical) disjunctions of abnormalities are. They are called *Dab*–formulas, and are usually referred to by means of $Dab(\Delta)$ ($\Delta \subseteq \Omega$).

(3) An *adaptive strategy* which determines how to interpret a premise set Γ "as normally as possible".

Intuitively, **AL** extend the **LLL**–consequence set of a premise set Γ by interpreting as false as many abnormal formulas (= elements of Ω) as possible. More specifically, whenever a formula $A \vee Dab(\Delta)$ is **LLL**–derivable from a premise set, the formula A will be considered an **AL**–consequence of the premise set, unless the adaptive strategy has determined that it is not safe to interpret some element(s) of Δ as false.

Which abnormalities an **AL** will in the end interpret as false, depends on how its adaptive strategy treats the *Dab*–consequences of a premise set. These are the *Dab*–formulas that are **LLL**–derivable from the premise set. Obviously, all disjuncts of a *Dab*–consequence can be interpreted as false, as this would lead to triviality.

[3]A nice introduction into the **ALP** and a more or less exhaustive list of the existing adaptive logics can be found at *http://logica.ugent.be/adlog*.

[4]To be honest, not all of them do, but as all **AL** in this dissertation are more or less standard, I do not consider this to be a problem.

[5]Although [33] was not intended to be about the standard format, it adds some extra elements to it, which are not mentioned in the other two papers, but which are nevertheless important for the adaptive logics I will present in this dissertation.

Self–evidently, every adaptive strategy will handle the *Dab*–consequences of a premise set in a different way. Below, I will only discuss two of them, the reliability strategy (**RS**) and the normal selections strategy (**NS**).[6]

The Upper Limit Logic. The *upper limit logic* (**ULL**) of an **AL** is the logic which by definition interprets all abnormalities as false. As a consequence, if $Neg(\Omega) = \{\neg A \mid A \in \Omega\}$ (with $\neg$ the classical negation), then **ULL**–derivability is characterized as follows:

Definition 3.1 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma \cup Neg(\Omega) \vdash_{\mathbf{LLL}} A$.

Remark that from this definition, it follows that the **LLL** and the **ULL** are related in the following way:

Theorem 3.1 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff there is a finite* $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$.

This theorem is called the *Derivability Adjustment Theorem* (**DAT**). It makes clear in which sense the **AL**–consequence set of a premise set is always situated in between the **LLL**–consequence set and the **ULL**–consequence set of that premise set:

Theorem 3.2 $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

How close the **AL**–consequence set of a premise set will come to the **ULL**–consequence set, depends on the *Dab*–consequences of the premise set. The lesser *Dab*–consequences derivable from it, the bigger its **AL**–consequence set will be. At the extreme, there are premise sets which have no *Dab*–consequences. They are called *normal premise sets*, and evidently, their **AL**–consequence set is equal to their **ULL**–consequence set:

Theorem 3.3 *When* $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta)$ *for all* $\Delta \subseteq \Omega$, *then* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

Of course, there are also premise sets that have an **AL**–consequence set which is equal to their **LLL**–consequence set. They are called *maximally abnormal premise sets*.

[6] There are still more adaptive strategies, most notably the minimal abnormality strategy, but as I don't need them in the remaining of my dissertation, I will not discuss them here.

3.2.1 Proof Theory

The proof theory of an adaptive logic consists of *deduction rules* and a *marking criterium.* While the former determine how new lines may be added to an **AL**–proof, the latter determines at every stage of an **AL**–proof which lines are considered as 'in' and which are considered as 'out' of the proof. When a line is considered as 'out' at a certain stage of the proof, its formula is not considered as derivable at that stage.

Deduction Rules. One of the specific features of **AL**–proofs is that their lines do not consist of three, but of four elements: a line number, a formula, a justification (the line numbers of the formulas from which the formula is derived and the rule by which the formula is derived), and an *adaptive condition.* The latter is a finite subset of Ω, and is taken to express that the formula of a line may be considered as derived, when the elements of its condition may be interpreted as false.

As the deduction rules determine how to add new lines to an **AL**–proof, they also determine how adaptive conditions are introduced.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\mathbf{LLL}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RC, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Remark that the deduction rules are fully determined by the **LLL**–rules and the set of abnormalities Ω. This makes clear that an adaptive proof is in fact a **LLL**–proof 'in disguise':

Theorem 3.4 *There is an* **AL**–*proof from* Γ *that contains a line on which* A *is derived on the condition* Δ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$.

This theorem shows how the proof theory of adaptive logics is related to the *Derivability Adjustment Theorem* from the foregoing section.

Marking Rules. A line i of an adaptive proof can be considered as 'in' at a certain stage of the proof, and considered as 'out' at an earlier or at a later stage. As stated above, this dynamic behavior is governed by the marking criterium, which is related to the adaptive strategy of an **AL**.

The Reliability Strategy. The marking rule related to the reliability strategy is based on the set $U_s(\Gamma)$, the set of *unreliable formulas* of Γ at stage s of the proof. It is determined by relying on the minimal *Dab*–consequences of Γ at stage s.

Definition 3.2 *$Dab(\Delta)$ is a minimal Dab–consequence at stage s of the proof iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also a Dab–consequence at stage s of the proof.*

Definition 3.3 *$U_s(\Gamma) = \Delta_1 \cup \Delta_2 \cup ...$, with $Dab(\Delta_1)$, $Dab(\Delta_2)$,... the minimal Dab–consequences of Γ at stage s of the proof.*

The reliability–marking rule is now defined as follows:

Definition 3.4 *Marking for* Reliability*: Line i is marked at stage s iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

The Normal Selections Strategy. Marking for the normal selections strategy is more straightforward.

Definition 3.5 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ has been derived at stage s on a line with condition $\emptyset$.*

Final Derivability. It should be clear by now that a formula will be considered as derivable from a premise set, when it occurs in a proof from that premise set as the second element of an unmarked line. However, because of the dynamical nature of adaptive proofs, this definition of derivability is rather problematic. Markings may change at every stage, so that for every new stage, it has to be reconsidered whether or not a formula is to be considered as derivable. Despite this stage–dependency of derivability, it still remains possible to define a stable notion of derivability. It is called *final derivability*, which is a very appropriate name, as for some formulas, derivability can only be decided at the final stage of a proof.

Definition 3.6 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 3.7 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is finally* **AL***–derivable from Γ) iff A is finally derived on a line of a proof from Γ.*

Notational Convention. Before discussing the semantics of adaptive logics, first consider the following notational convention:

Convention 3.1 *The strategy of an* **AL** *will be written as a superscript to the name of the* **AL**.

So, for example the adaptive logic $\mathbf{AL^r}$ is based on the reliability strategy, while $\mathbf{AL^s}$ is based on the normal selections strategy.

3.2.2 Semantics

Validity and semantic consequence for **AL** are always defined with respect to one or more subsets of the **LLL**–models of a premise set. They are called the *preferred sets* of **LLL**–models of a premise set. How the preferred sets are determined differs for each adaptive strategy.

In any case, whether or not a particular **LLL**–model will end up as the element of some preferred set, will depend on its abnormal part, which is the set of abnormalities that it verifies:[7]

Definition 3.8 *Where M is a* **LLL**–*model,* $Ab(M) = \{A \in \Omega \mid M \models A\}$.

Reliability. Semantic consequence for **AL** based on the reliability strategy, is defined with respect to the *reliable* **LLL**–models of a premise set Γ. The latter are those **LLL**–models of Γ that verify only a subset of the set $U(\Gamma)$, the set of all abnormalities that occur in the minimal *Dab*–consequences of Γ.

Definition 3.9 $Dab(\Delta)$ *is a minimal Dab–consequence of* Γ *iff* $\Gamma \models_{\mathbf{LLL}} Dab(\Delta)$ *and for all* $\Delta' \subset \Delta$, $\Gamma \not\models_{\mathbf{LLL}} Dab(\Delta')$.

If $Dab(\Delta_1), Dab(\Delta_2), ...$ are the minimal *Dab*–consequences of Γ, then $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup ...$

Definition 3.10 *A* **LLL**–*model M of Γ is reliable iff* $Ab(M) \subseteq U(\Gamma)$.

Definition 3.11 $\Gamma \models_{\mathbf{AL^r}} A$ *iff A is verified by all reliable models of* Γ.

Normal Selections. Semantic consequence for **AL** based on the normal selections strategy is defined with respect to the so–called *normal sets* of **LLL**–models of a premise set Γ. These are specific subsets of the set of minimally abnormal **LLL**–models of a premise set.

Definition 3.12 *A* **LLL**–*model M of Γ is minimally abnormal iff there is no* **LLL**–*model M' of Γ such that* $Ab(M') \subset Ab(M)$.

[7]Obviously, $M \models A$ (resp. $M \models \Gamma$) denotes that the model M verifies the formula A (resp. all members of Γ).

More specifically, a subset of the minimally abnormal **LLL**–models of Γ is a normal set of **LLL**–models of Γ, when all its elements verify the same abnormalities.

Definition 3.13 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 3.14 *A set* Σ *of* **LLL**–*models of* Γ *is a* normal set *iff for some* $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.

Definition 3.15 $\Gamma \models_{\mathbf{AL^s}} A$ *iff* A *is verified by all members of at least one normal set of* **LLL**–*models of* Γ.

3.2.3 Metatheory

Adaptive logics in standard format share a lot of metatheoretical characteristics. I will only mention soundness and completeness, while for the other characteristics, I refer the reader to [15, 26, 33].

Soundness and Completeness. Soundness and Completeness have been proven in [26] for all standard adaptive logics based on the reliability strategy, and in [33] for all standard adaptive logics based on the normal selections strategy.

Theorem 3.5 $\Gamma \vdash_{\mathbf{AL^r}} A$ *iff* $\Gamma \models_{\mathbf{AL^r}} A$.

Theorem 3.6 $\Gamma \vdash_{\mathbf{AL^s}} A$ *iff* $\Gamma \models_{\mathbf{AL^s}} A$.

Conclusion. The advantage of the standard format is obvious: for all adaptive logics that fall within the standard format, the proof theory, semantics, and a lot of metatheoretical properties are plainly given!

3.3 Combined Adaptive Logics

It is sometimes very useful to combine (adaptive) logics. Although this can be done in a number of ways, I will here limit myself to the case where the adaptive logics $\mathbf{AL_1}$,...,$\mathbf{AL_n}$ that are combined are based on the same **LLL** and the same adaptive strategy, but which have different sets of abnormalities $\Omega_1, ..., \Omega_n$. Their combination can result in a simple combined adaptive logic or in a prioritized adaptive logic.

3.3.1 Simple Combined Adaptive Logics

A simple combined adaptive logic **AL** adopts the **LLL** and the adaptive strategy of the adaptive logics $\mathbf{AL_1}$,...,$\mathbf{AL_n}$ on which it is based, but its set of abnormalities Ω is the union of their sets of abnormalities:

Definition 3.16 $\Omega = \Omega_1 \cup ... \cup \Omega_n$.

Both the proof theory and the semantics of simple combined adaptive logics are equal to those for flat adaptive logics.

3.3.2 Prioritized Adaptive Logics

Prioritized adaptive logics are a special kind of combined adaptive logics. They are well–studied in the literature — Most properly in [15, 34, 120] — and can also be characterized by means of the standard format. As such, a prioritized adaptive logic is also characterized by means of a lower limit logic **LLL**, a set of abnormalities Ω and an adaptive strategy. The difference with flat adaptive logics constitutes in the fact that the set of abnormalities Ω of a prioritized adaptive logic is a structurally ordered set of sets of abnormalities:

Definition 3.17 $\Omega = \Omega_1 < ... < \Omega_n$.

The order imposed on the set of abnormalities expresses a priority relation, which plays a decisive role in both the proof theory and the semantics of a prioritized adaptive logic. Intuitively, a premise set will first be interpreted "as normally as possible" with respect to the abnormalities of priority level 1, then with respect to abnormalities of priority level 2, etc.[8]

It is probably most convenient to interpret a prioritized adaptive logic as a superposition of adaptive logics that are based on the same **LLL** and on the same adaptive strategy, but which have different sets of abnormalities. As such, the consequence set of a prioritized adaptive logic **AL** can also be characterized as follows:

Definition 3.18 $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL_n}}(...(Cn_{\mathbf{AL_2}}(Cn_{\mathbf{AL_1}}(\Gamma)))...)$.

Proof Theory. The proof theory of prioritized adaptive logics comes quite close to the one for flat adaptive logics. The deduction rules PREM, RU and RC are as for flat adaptive logics, so that the only difference resides in the marking criterium.

[8]Normally the priorities are also expressed in the object language by means of the modal operator $\Diamond$ (see [34]). However, this is only necessary for the characterization of some prioritized adaptive logics, not for all of them. Hence, I will not express priorities in the object language.

In order to characterize the reliability–marking rule for prioritized adaptive logics (the only marking criterium I will consider here), first consider the minimal Dab^i–consequences of a premise set. When $Dab^i(\Delta)$ $(1 \leqslant i \leqslant n)$ is used to denote a *Dab*–formula for which $\Delta \subseteq \Omega_i$, $Dab^i(\Delta)$ is a *minimal Dab^i–consequence* of a premise set Γ at stage s of a proof, when (1) it occurs on an unmarked line at stage s, (2) all members of its adaptive condition belong to a Ω_j such that $j < i$, and (3) there is no $\Delta' \subset \Delta$ for which the same applies.

Next, for all priority levels i $(1 \leqslant i \leqslant n)$, the set $U^i_s(\Gamma)$ of *unreliable formulas* of Γ with priority i is defined.

Definition 3.19 $U^i_s(\Gamma) = \Delta^i_1 \cup \Delta^i_2 \cup ...$ *with* $Dab^i(\Delta_1)$*,* $Dab^i(\Delta_2)$*,... the minimal* Dab^i*–consequences of* Γ *at stage* s *of the proof.*

Finally, marking for reliability is defined as for flat adaptive logics.

Definition 3.20 *Marking for* Reliability*: Line* i *is marked at stage* s *iff, where* Δ *is its condition,* $\Delta \cap U^i_s(\Gamma) \neq \emptyset$*.*

The difference with the reliability–marking rule for flat adaptive logics consists in the fact that marking for prioritized adaptive logics *proceeds stepwise*: first for level 1, then for level 2,...

To conclude the proof theory, final derivability for prioritized adaptive logics is governed by definitions 3.6 and 3.7 from section 3.2.

Semantics. As the semantics for flat adaptive logics, the semantics for prioritized adaptive logics is also based on a selection (or on multiple selections) of **LLL**–models of a premise set. First of all, the abnormal parts of the **LLL**–models are determined. Each **LLL**–model has multiple abnormal parts, one for each priority level i.

Definition 3.21 *For every* **LLL***–model* M *and for every priority level* i*:* $Ab^i(M) = \{A \in \Omega_i \mid M \models A\}$*.*

Next, consider the set $\mathcal{M}_0$, the set of all **LLL**–models of a premise set.

Definition 3.22 $\mathcal{M}_0 =_{df} \{M \mid M \models \Gamma\}$*.*

For every priority level i, we can now define a set $\mathcal{M}_i$ which contains the selected **LLL**–models of priority level i. For the reliability strategy (the only strategy which I will consider), the set $\mathcal{M}_i$ is defined as follows:

Definition 3.23 $\mathcal{M}_i =_{df} \{M \in \mathcal{M}_{i-1} \mid Ab^i(M) \subseteq U^i(\Gamma)\}$*, where* $U^i(\Gamma) = \bigcup\{\Delta \mid Dab_i(\Delta)$ *a minimal Dab–consequence of* Γ*.*

Finally, the set of selected models of a premise set is defined as the section of all sets $\mathcal{M}_i$, and semantic consequence is defined by relying on those selected models. Consider below the definitions for the reliability strategy.

Definition 3.24 *M is a reliable model of Γ iff $M \in \mathcal{M}_1^r \cap \mathcal{M}_2^r \cap ...$*

Definition 3.25 *$\Gamma \models_{\mathbf{PAL^r}} A$ iff A is verified by all reliable models of Γ.*

Soundness and Completeness. Soundness and Completeness for prioritized adaptive logics follows in a rather straightforward way from the soundness and completeness of flat adaptive logics.

3.4 Conclusion

In this chapter, I have not only stated the main objectives of the Adaptive Logics Programme, I have also presented the standard format of adaptive logics. I found this necessary, as the logics I will present in later chapters, will all be adaptive logics.

Chapter 4

Introducing Paralogics

4.1 Paralogics

Classical Logic (**CL**) presupposes both consistency and completeness. However, other logics do not. These are called paralogics, as they are either paraconsistent (not presupposing consistency), paracomplete (not presupposing completeness), or both.

In this chapter, I will present those paralogics that will be used in later chapters. These are the basic paralogics **CLūNs**, **CLāNs** and **CLōNs** (see section 4.2), their full versions (see section 4.3), and their modal extensions (see section 4.4).

Preliminary Remark. In this dissertation, I limit myself to propositional logic, so that only the propositional fragments of the considered paralogics will be discussed.

4.2 Basic Paralogics

The semantics of classical negation expresses both the consistency and completeness presupposition inherent in **CL**:

CONSISTENCY	If $v(\neg A) = 1$ then $v(A) = 0$.
COMPLETENESS	If $v(\neg A) = 0$ then $v(A) = 1$.

If one (or both) of these semantical clauses is dropped from **CL**, we obtain the paralogics **CLuN** (by dropping the consistency requirement), **CLaN** (by dropping the completeness requirement), or **CLoN** (by dropping both requirements).

Although these paralogics contain the full positive part of **CL**, their negation has become extremely weak. Not only are double negation and the *De Morgan*–properties lost for it, within the scope of a negation the replacement of logically equivalent formulas is not even possible anymore, e.g. $\neg(A \wedge B) \not\models \neg(B \wedge A)$.[1]

It is however possible to allow all properties that drive the negation inwards, which results in the stronger paralogics **CLūNs**, **CLāNs** and **CLōNs**. These are the logics that I will present below, and from now on, it will be to them that I mean to refer when speaking about (basic) paralogics.

4.2.1 The Language Schema

The language $\mathcal{L}$ of **CLūNs**, **CLāNs** and **CLōNs** is the $\neg, \wedge, \vee, \supset$–fragment of the classical propositional language.[2] Consequently, the set of well–formed formulas $\mathcal{W}$ of the language $\mathcal{L}$, is made up as follows:

(i) $\mathcal{S} \subset \mathcal{W}$ for $\mathcal{S}$ the set of sentential letters,
(ii) If $A \in \mathcal{W}$ then $\neg A \in \mathcal{W}$,
(iii) If $A, B \in \mathcal{W}$ then $(A \wedge B), (A \vee B), (A \supset B) \in \mathcal{W}$.

Classes of Well–Formed Formulas. Based on an idea from [37, 19], I will subdivide the set of well–formed formulas into specific classes, named $\mathfrak{a}$–, $\mathfrak{b}$–formulas. $\mathfrak{a}$– and $\mathfrak{b}$–formulas are assigned two other formulas, in accordance with the following table:

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$		$A \vee B$	A	B
$\neg(A \supset B)$	A	$\neg B$		$A \supset B$	$\neg A$	B
$\neg\neg A$	A	A				

Table 4.1: $\mathfrak{a}$– and $\mathfrak{b}$–formulas for paralogics.

The formulas assigned to a particular $\mathfrak{a}/\mathfrak{b}$–formula, are called the constituting parts of that $\mathfrak{a}/\mathfrak{b}$–formula. By using the constituting parts of a formula in the semantics of paralogics, their non–truthfunctional semantics gets a more classical (read: truthfunctional) outlook.

4.2.2 Semantics for Basic Paralogics

I will first present the semantics of the logic **CLōNs**, which allows for b*o*th gluts and gaps for negation. In other words, it is both paraconsistent and

[1] For a thorough characterization of these logics and their extensions, see [7].

[2] For reasons not even completely clear to me, I leave out material equivalence. However, material equivalence can easily be reintroduced by defining it in terms of the material implication: $(A \equiv B) =_{df} (A \supset B) \wedge (B \supset A)$.

paracomplete. As such, it is the weakest paralogic of the three I will present.[3] Also remark that **CLōNs** is in fact equivalent to the logic **FDE** as presented in Priest [86, ch. 8].[4]

A **CLōNs**–model for the language $\mathcal{L}$, with $\mathcal{S}$ and $^{\neg}\mathcal{S}$ respectively the set of sentential letters and the set of negated sentential letters ($^{\neg}\mathcal{S} = \{\neg A \mid A \in \mathcal{S}\}$), is an assignment function v, characterized as follows:

AP1 $v: \mathcal{S} \mapsto \{0,1\}$.
AP2 $v: {}^{\neg}\mathcal{S} \mapsto \{0,1\}$.

The valuation function v_M determined by the model M is defined as follows:

SP1 $v_M : \mathcal{W} \mapsto \{0,1\}$.
SP2 For $A \in \mathcal{S}$: $v_M(A) = 1$ iff $v(A) = 1$.
SP3o For $A \in \mathcal{S}$: $v_M(\neg A) = 1$ iff $v(\neg A) = 1$.
SP4 $v_M(\mathfrak{a}) = 1$ iff $v_M(\mathfrak{a}_1) = 1$ and $v_M(\mathfrak{a}_2) = 1$.
SP5 $v_M(\mathfrak{b}) = 1$ iff $v_M(\mathfrak{b}_1) = 1$ or $v_M(\mathfrak{b}_2) = 1$.

Truth in a model, semantical consequence, and validity are defined as usual:

Definition 4.1 *A is true in a model M iff $v_M(A) = 1$.*

Definition 4.2 *$\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$ iff A is true in all models in which all elements of Γ are true.*

Definition 4.3 *$\models_{\mathbf{CL\bar{o}Ns}} A$ iff A is true in all models.*

Consistency and Completeness. It is of course possible to reintroduce the consistency and/or the completeness requirements into the logic **CLōNs**. First, the completeness requirement is reintroduced by replacing the clause **SP3o** in the above **CLōNs**–semantics by the semantical clause **SP3u**:

SP3u For $A \in \mathcal{S}$: $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$.

This gives us the paraconsistent logic **CLūNs** (Classical Logic with gl*u*ts for negation) which is in fact equivalent to the logic **LP** of Priest (see e.g. Priest [83], and Priest [85, ch. 7–8]).

Secondly, to reintroduce the consistency requirement, replace the semantical clause **SP3o** above by the following one:

SP3a For $A \in \mathcal{S}$: $v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $v(\neg A) = 1$.

[3] 'Weak' interpreted here as giving the smallest consequence set for a particular premise set.

[4] Moreover, the logic **FDE** characterizes tautological entailments, see [5].

This gives us the paracomplete logic **CLāNs** (Classical Logic with g*a*ps for negation) which is actually equivalent to Kleene's logic $\mathbf{K_3}$ (see Priest [85, ch. 7–8]).

Finally, it is obvious that if we add both presuppositions to **CLōNs**, we get **CL** again. This can also be done by replacing **SP3o** by **SP3**:

SP3 For $A \in \mathcal{S}$: $v_M(\neg A) = 1$ iff $v_M(A) = 0$.

Notational Convention. It is possible to construct logical systems that contain more than one negation, e.g. both the **CLāNs**– and the **CL**–negation. In order to keep them apart, I will use $\neg_!$ for classical negation, $\neg_u$ for the **CLūNs**–negation, $\neg_a$ for the **CLāNs**–negation, and $\neg_o$ for the **CLōNs**–negation. However, in contexts where there is only one kind of negation and no mix up is possible, I will always use $\neg$.

4.2.3 Proof Theory for Basic Paralogics

The proof theories I will present for the paralogics **CLūNs**, **CLāNs** and **CLōNs** are Fitch–style natural deduction systems.

Structural Rules. The proof theories of **CLūNs**, **CLāNs** and **CLōNs** all contain the same structural rules. These are quite standard, except for the rule **CSP**, which allows to introduce *pseudo–formulas* into the proofs. These are "formulas" of the form $S(A, B)$ with $A, B \in \mathcal{W}$, which express that there is a subproof with hypothesis A on its first line and formula B on its closing line.

PREM At any place in the proof, one may write down a premise.
HYP At any place in the proof, one may start a new subproof. This is done by introducing a new hypothesis, together with a new vertical line on its left.
CSP If the formula B is the formula on the last line of a subproof that started with the hypothesis A, one may conclude to the *pseudo–formula* $S(A, B)$. This of course also closes the subproof.
REP In the main proof and in subproofs, formulas may be repeated.
REIT In subproofs, one may reiterate formulas from lines in the main proof and from lines in unclosed subproofs.

Inference Rules. Also consider the inference rules for the logic **CLōNs**. They allow one to derive a formula from other formulas. The inference rules presented by means of ◀▶ allow for derivation in both directions.[5]

[5]Remark that the rules **DIL**,**CONT**, and **ASS** can be replaced by the following rule: $A \vee B$, $S(A, C)$, $S(B, C) \vdash C$. I opted for the former three rules, as this will come out handy later on in my dissertation.

CON	$A, B \blacktriangleright A \wedge B$
SIM	$A \wedge B \blacktriangleright A$, $A \wedge B \vdash B$
ADD	$A \vdash A \vee B$, $B \blacktriangleright A \vee B$
DIL	$A \vee B$, $S(A, C)$, $S(B, D) \blacktriangleright C \vee D$
CONT	$A \vee A \blacktriangleright A$
ASS	$A \vee (B \vee C) \blacktriangleleft\!\blacktriangleright (A \vee B) \vee C$
IMP	$A \sqsupset B \blacktriangleleft\!\blacktriangleright \neg A \vee B$
DN	$\neg\neg A \blacktriangleleft\!\blacktriangleright A$
NC	$\neg(A \wedge B) \blacktriangleleft\!\blacktriangleright \neg A \vee \neg B$
ND	$\neg(A \vee B) \blacktriangleleft\!\blacktriangleright \neg A \wedge \neg B$
NI	$\neg(A \sqsupset B) \blacktriangleleft\!\blacktriangleright \neg A \wedge B$

CL$\bar{\text{o}}$Ns–Derivability. A **CL$\bar{\text{o}}$Ns**–proof is defined as a finite sequence of wffs (and pseudo–wffs), each of which is either a premise or follows from wffs earlier in the list by means of a structural rule or a rule of inference. Moreover, in order for such a sequence to be a proof, all its subproofs should be closed.

Finally, **CL$\bar{\text{o}}$Ns**–derivability is defined as follows:

Definition 4.4 $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$ *(A is a* **CL$\bar{\text{o}}$Ns**–*consequence of* Γ*) iff there is a proof of the formula A from $B_1, ..., B_n \in \Gamma$ so that A has been derived on a line i of the main proof.*

Example. In order to make the proof theory more concrete, consider the following example for the logic **CL$\bar{\text{o}}$Ns**.

1	$p \vee q$	PREM	
2	$r \vee s$	PREM	$\triangle\ (p \vee r) \vee (q \wedge s)$
3	$\quad \mid p$	HYP	
4	$\quad \mid p$	3;REP	
5	$S(p, p)$	3,4;CSP	
6	$\quad \mid q$	HYP	
7	$\quad \mid r \vee s$	2;REIT	
8	$\quad \mid \mid r$	HYP	
9	$\quad \mid \mid r$	8;REP	
10	$\quad \mid S(r, r)$	8,9;CSP	
11	$\quad \mid \mid s$	HYP	
12	$\quad \mid \mid q$	6;REIT	
13	$\quad \mid \mid q \wedge s$	11,12;CON	
14	$\quad \mid S(s, q \wedge s)$	11,13;CSP	
15	$\quad \mid r \vee (q \wedge s)$	7,10,14;DIL	
16	$S(q, r \vee (q \wedge s))$	6,15;CSP	
17	$p \vee (r \vee (q \wedge s))$	1,5,16;DIL	
18	$(p \vee r) \vee (q \wedge s)$	17;ASS	

Extra Inference Rules. To obtain the proof theories of **CLūNs** and **CLāNs**, add respectively the inference rule **TH** or **DS** to the proof theory of **CLōNs**.

TH	For $C \in \mathcal{S}$: $\blacktriangleright A \vee \neg A$.
DS	For $C \in \mathcal{S}$: $A \vee B$, $S(B, C \wedge \neg C) \blacktriangleright A$

It is obvious that if both rules are added, we get a proof theory for **CL**.

4.2.4 Pseudo–Deduction Theorem for Paralogics

Although the usual deduction theorem is not valid for paralogics (**PL**), a variant of the usual deduction theorem is. Let's call it the *pseudo–deduction theorem.* Consider it below, together with its converse which is also valid.

Theorem 4.1 *$A_1, ..., A_n \vdash_{\mathbf{PL}} B$ iff $A_1, ..., A_{n-1} \vdash_{\mathbf{PL}} S(A_n, B)$.*

Proof.=> Suppose $A_1, ..., A_n \vdash_{\mathbf{PL}} B$. Hence, consider the following generic proof:

1	A_1	PREM
...	...	PREM
n-1	A_{n-1}	PREM
n	A_n	HYP
n+1	A_1	1;REIT
...	...	REIT
n+(n-1)	A_{n-1}	n-1;REIT
...	...	...
m	B	Supposition
m+1	$S(An, B)$	n,m;CSP

<= Suppose $A_1, ..., A_{n-1} \vdash_{\mathbf{PL}} S(A_n, B)$. Now, consider the following generic proof:

1	A_1	PREM
...	...	PREM
n-1	A_{n-1}	PREM
n	A_n	PREM
n+1	$A_n \vee A_n$	n;ADD
n+2	$S(A_n, B)$	1 till n-1;Supposition
n+3	$S(A_n, B)$	n+2;REP
n+4	$B \vee B$	n+1,n+2,n+3;DIL
n+5	B	n+4;CONT

■

The pseudo–deduction theorem will turn out to be very useful for proving soundness and completeness of the presented paralogics.

4.2.5 Soundness and Completeness

The soundness and completeness proofs of the basic paralogics **CLūNs**, **CLāNs** and **CLōNs**, are inspired by the soundness and completeness proofs I found in Roy [103] and Priest [85]. I shall start with soundness and completeness for **CLōNs**, as the other paralogics in a sense all contain it.

Theorem 4.2 (Soundness) *If* $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$ *then* $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$.

Before proving this theorem, first consider the following lemma:

Lemma 4.1 *If* $\Gamma \subseteq \Gamma'$ *and* $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$ *then* $\Gamma' \models_{\mathbf{CL\bar{o}Ns}} A$.

Proof. Suppose (1) $\Gamma \subseteq \Gamma'$, (2) $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$ and (3) $\Gamma' \not\models_{\mathbf{CL\bar{o}Ns}} A$. From (3), it follows that there is a **CLōNs**–model M such that $v_M(\Gamma') = 1$ and $v_M(A) = 0$.[6] As $\Gamma \subseteq \Gamma'$ (by (1)), it also follows that $v_M(\Gamma) = 1$. Consequently, as there is a model M for which $v_M(\Gamma) = 1$ and $v_M(A) = 0$, it follows that $\Gamma \not\models_{\mathbf{CL\bar{o}Ns}} A$, which is impossible because of (2). ■

Next, consider some terminological remarks. Let A_i express that the formula A is derived in a proof on line i, and let Γ_i stand for the set of all free premises (premises that do not occur in a closed subproof) and all free hypotheses (hypotheses of subproofs that are unclosed for the proof at line i) that occur on those lines j of the proof for which $j \leqslant i$.

Finally, consider the proof of theorem 4.2. It is an induction proof on the line numbers of a **CLōNs**–derivation.

Proof. Suppose $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$. This means that there is a proof of A from Γ. In order to proof that this also gives us $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$, I will proof by induction that for all lines i of the proof, it is the case that $\Gamma_i \models_{\mathbf{CL\bar{o}Ns}} B_i$. This will give us the desired result, as Γ_i (with i the line on which A occurs) cannot contain any hypotheses (because A is in the main proof), which means that $\Gamma_i \subseteq \Gamma$, so that by lemma 4.1, it follows that $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$.

First, consider the base case: B_0 is a premise or an assumption. This means that $B_0 \in \Gamma_0$ such that it is impossible that $v_M(\Gamma_0) = 1$ and $v_M(B_0) = 0$. Consequently, $\Gamma_0 \models_{\mathbf{CL\bar{o}Ns}} B_0$.

Next, consider the induction hypothesis:

Induction Hypothesis 4.1 *For any* i, $1 \leqslant i < k$: $\Gamma_i \models_{\mathbf{CL\bar{o}Ns}} B_i$.

[6] Obviously, $v_M(\Gamma') = 1$ means that for all formulas $B \in \Gamma'$: $v_M(B) = 1$.

It remains to be proven that $\Gamma_k \models_{\mathbf{CL\bar{o}Ns}} B_k$. B_k is either a premise, an assumption or is derived from previous lines by means of **REP**, **REIT**, **CON**, **SIM**, **ADD**, **DIL**, **CONT**, **ASS**, **IMP**, **DN**, **NC**, **ND** or **NI**. In case B_k is a premise or an assumption, the proof is analogous to the base case. Hence, $\Gamma_k \models_{\mathbf{CL\bar{o}Ns}} B_k$. So, we only need to show that this is also the case when B_k is derived by means of one of the above rules. In fact, as the proofs for most of them are trivial, I leave them to the reader. I will here only prove it for the rule **DIL**.

DIL Suppose that (1) $B_k = C \vee D$ and (2) that it has been derived from $B_i = A \vee B$, $S(A, C)$ (with C on line g) and $S(B, D)$ (with D on line h) by means of DIL.
Consequence 1. From (2), it follows that $\Gamma_i \models_{\mathbf{CL\bar{o}Ns}} A \vee B$ (by the induction hypothesis). Moreover, that $i < k$ gives us that $\Gamma_i \subseteq \Gamma_k$. From both the above, it follows that $\Gamma_k \models_{\mathbf{CL\bar{o}Ns}} A \vee B$ (by lemma 4.1).
Consequence 2. That $S(A, C)$, with C on line g (see (2)), gives us that $\Gamma_g \models_{\mathbf{CL\bar{o}Ns}} C$ (by the induction hypothesis). Moreover, that $g < k$ gives us that $\Gamma_g \subseteq \Gamma_k \cup \{A\}$. From both the above, it follows that $\Gamma_k \cup \{A\} \models_{\mathbf{CL\bar{o}Ns}} C$ (by lemma 4.1).
Consequence 3. That $S(B, D)$, with D on line h (see (2)), gives us that $\Gamma_h \models_{\mathbf{CL\bar{o}Ns}} D$ (by the induction hypothesis). Moreover, that $h < k$ gives us that $\Gamma_h \subseteq \Gamma_k \cup \{B\}$. From both the above, it follows that $\Gamma_k \cup \{B\} \models_{\mathbf{CL\bar{o}Ns}} D$ (by lemma 4.1).

Supposition. $\Gamma_k \not\models_{\mathbf{CL\bar{o}Ns}} B_k$. From this it follows that there is a **CLōNs**–model M such that $v_M(\Gamma_k) = 1$ and $v_M(B_k) = 0$. As $B_k = C \vee D$, this means that $v_M(C \vee D) = 0$. By **SP5**, it now follows that $v_M(C) = 0$ and $v_M(D) = 0$ (∗).
From the foregoing paragraph, it follows that for M: $v_M(\Gamma_k) = 1$. This means also that $v_M(A \vee B) = 1$ (by consequence 1), such that $v_M(A) = 1$ or $v_M(B) = 1$ (by **SP5**). This in fact means that $v_M(\Gamma_k \cup \{A\}) = 1$ or that $v_M(\Gamma_k \cup \{B\}) = 1$. But, the former leads to $v_M(C) = 1$ (by consequence 2) and the latter leads to $v_M(D) = 1$ (by consequence 3), which are both impossible because of (∗).

■

Theorem 4.3 (Completeness) *If* $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$ *then* $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$.

Proof. Suppose $\Gamma \nvdash_{\mathbf{CL\bar{o}Ns}} A$. Consider a sequence B_1,B_2,... that contains all wffs of the language $\mathcal{L}$. We then define:

$$\Delta_0 = Cn_{\mathbf{CL\bar{o}Ns}}(\Gamma)$$
$$\Delta_{i+1} = Cn_{\mathbf{CL\bar{o}Ns}}(\Delta_i \cup \{B_i\}) \text{ if } A \notin Cn_{\mathbf{CL\bar{o}Ns}}(\Delta_i \cup \{B_i\}), \text{ and}$$

$\Delta_{i+1} = \Delta_i$ otherwise.
$\Delta = \Delta_0 \cup \Delta_1 \cup ...$

Each of the following is provable:

(i) $\Gamma \subseteq \Delta$ (by the construction).
(ii) $A \notin \Delta$ (by the construction).
(iii) Δ is deductively closed (by the definition of Δ).
(iv) Δ is non–trivial (as $A \notin \Delta$).
(v) Δ is prime, i.e. if $C \vee D \in \Delta$, then $C \in \Delta$ or $D \in \Delta$.

Suppose that (1) $C \vee D \in \Delta$, but that (2) $C \notin \Delta$ and $D \notin \Delta$. From (2), it follows that there must be an m and n such that $\Delta_m \cup \{C\} \vdash_{\mathbf{CL\bar{o}Ns}} A$ and $\Delta_n \cup \{D\} \vdash_{\mathbf{CL\bar{o}Ns}} A$ (by the construction of Δ). From these, it follows that $\Delta_m \vdash_{\mathbf{CL\bar{o}Ns}} S(C, A)$ and $\Delta_n \vdash_{\mathbf{CL\bar{o}Ns}} S(D, A)$ (by theorem 4.1). But, this also means that $\Delta \vdash_{\mathbf{CL\bar{o}Ns}} S(C, A)$ and $\Delta \vdash_{\mathbf{CL\bar{o}Ns}} S(D, A)$ (by the construction of Δ and the syntactic version of lemma 4.1 which is left to the reader). From this, together with (1), it follows that $A \in \Delta$ (by the deductive closure of Δ), which contradicts (ii).

I now define a **CLōNs**–model M from Δ in the following way:

AP1 For all $C \in \mathcal{S}$, $v(C) = 1$ iff $C \in \Delta$.
AP2 For all $C \in {}^{\neg}\mathcal{S}$, $v(C) = 1$ iff $C \in \Delta$.

Finally, I show that for all wffs C of the language $\mathcal{L}$, $v_M(C) = 1$ iff $C \in \Delta$. This is done by a straightforward induction on the complexity of the wffs.

The Base Case. For primitive formulas, the proof is immediate because of **AP1**, **AP2**, **SP2** and **SP3**.

The Induction Cases. As this is all completely standard, I will only show how this is done for formulas of the form $\neg(A \wedge B)$. The remaining cases, I leave to the reader.

$\neg(A \wedge B) \in \Delta$ iff $\neg A \vee \neg B \in \Delta$ (as Δ is deductively closed)
iff $\neg A \in \Delta$ or $\neg B \in \Delta$ (as Δ is prime)
iff $v_M(\neg A) = 1$ or $v_M(\neg B) = 1$ (by the induction hypothesis)
iff $v_M(\neg(A \wedge B)) = 1$ (by **SP5**)

As $v_M(C) = 1$ iff $C \in \Delta$, (i) and (ii) give us that $v_M(\Gamma) = 1$ and $v_M(A) = 0$. Hence, $\Gamma \nvDash_{\mathbf{CL\bar{o}Ns}} A$. ■

Because of the two theorems above, we can conclude to the following corollary:

Corollary 4.1 $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$ *iff* $\Gamma \models_{\mathbf{CL\bar{o}Ns}} A$.

The Remaining Paralogics. Now that I have proven soundness and completeness for **CLōNs**, I can easily do so for **CLūNs** and **CLāNs** as well.

Theorem 4.4 $\Gamma \vdash_{\mathbf{CL\bar{u}Ns}} A$ *iff* $\Gamma \models_{\mathbf{CL\bar{u}Ns}} A$.

Proof. => The soundness proof for **CLūNs** is completely equivalent to the one for **CLōNs**. I only need to prove one extra induction case, namely for the rule **TH**. As this is a trivial case, I leave it to the reader.

<= The completeness proof for **CLūNs** also comes very close to the one for **CLōNs**. The only difference with the proof for **CLōNs** concerns the case for negations of sentential letters: $v_M(\neg A) = 1$ iff $\neg A \in \Delta$ (for $A \in \mathcal{S}$). This case is proven as follows:

Right–Left Suppose for $A \in \mathcal{S}$, $\neg A \in \Delta$. By **AP2**, it follows that $v(\neg A) = 1$, so that also $v_M(\neg A) = 1$ by **SP3u**.

Left–Right Suppose for $A \in \mathcal{S}$, $v_M(\neg A) = 1$. By **SP3u**, it follows that (1) $v_M(A) = 0$ or (2) $v(\neg A) = 1$. From (2), it follows immediately that $\neg A \in \Delta$ (by **AP2**), and from (1), it follows that $A \notin \Delta$ (also by **AP2**). From the latter, it also follows that $\neg A \in \Delta$, as $A \vee \neg A \in \Delta$ (by deductive closure of Δ) and Δ is a prime theory.

■

Theorem 4.5 $\Gamma \vdash_{\mathbf{CL\bar{a}Ns}} A$ *iff* $\Gamma \models_{\mathbf{CL\bar{a}Ns}} A$.

Proof. => The soundness proof for **CLāNs** is also equivalent to the one for **CLōNs**, except for one extra induction case, namely for the rule **DS**.

DS Suppose that (1) $A_k = A$ and (2) that it has been derived from $A_i = A \vee B$ and $S(B, C \wedge \neg C)$ (with $C \wedge \neg C$ on line j) by means of DS.
Consequence 1. From (2), it follows that $\Gamma_i \models_{\mathbf{CL\bar{a}Ns}} A \vee B$ (by the induction hypothesis). Moreover, that $i < k$ gives us that $\Gamma_i \subseteq \Gamma_k$. From both the above, it follows that $\Gamma_k \models_{\mathbf{CL\bar{a}Ns}} A \vee B$ (by lemma 4.1).
Consequence 2. That $S(B, C \wedge \neg C)$, with C on line j (see (2)), gives us that $\Gamma_j \models_{\mathbf{CL\bar{a}Ns}} C \wedge \neg C$ (by the induction hypothesis).

Moreover, that $j < k$ gives us that $\Gamma_j \subseteq \Gamma_k \cup \{B\}$. From both the above, it follows that $\Gamma_k \cup \{B\} \models_{\mathbf{CL\bar{a}Ns}} C \wedge \neg C$ (by lemma 4.1).
Supposition. $\Gamma_k \not\models_{\mathbf{CL\bar{a}Ns}} A_k$. From this it follows that there is a **CLāNs**–model M such that $v_M(\Gamma_k) = 1$ and $v_M(A_k) = 0$. As $A_k = A$, this means that $v_M(A) = 0$ $(*)$.
From the foregoing paragraph, it follows that for M: $v_M(\Gamma_k) = 1$. This means also that $v_M(A \vee B) = 1$ (by consequence 1), such that $v_M(A) = 1$ or $v_M(B) = 1$ (by **SP5**). This in fact means that $v_M(\Gamma_k \cup \{A\}) = 1$ or that $v_M(\Gamma_k \cup \{B\}) = 1$. But, as the latter leads to $v_M(C) = 1$ and $v_M(C) = 0$ (by consequence 2, **SP4** and **SP3a**), it follows that $v_M(\Gamma_k \cup \{A\}) = 1$ is the case. This however contradicts with $(*)$.

<= Also the completeness proof for **CLāNs** differs only from the one for **CLōNs** concerning the induction case for negations of sentential letters.

Right–Left Suppose for $A \in \mathcal{S}$, $\neg A \in \Delta$. It follows that $v(\neg A) = 1$ (by **AP2**), and it also follows that $A \notin \Delta$ (otherwise Δ would be trivial, which it is not). Hence, $v(A) = 0$ (by **AP1**), which means that also $v_M(A) = 0$ (by **SP2**). From the foregoing, it follows that $v_M(\neg A) = 1$ (by **SP3a**).

Left–Right Suppose for $A \in \mathcal{S}$, $v_M(\neg A) = 1$. By **SP3a**, it follows that $v(\neg A) = 1$, which means that $\neg A \in \Delta$ (by **AP2**).

■

Classical Logic Again. Soundness and completeness for **CL** follows from the soundness and completeness proofs for **CLūNs** and **CLāNs**.

Theorem 4.6 $\Gamma \vdash_{\mathbf{CL}} A$ *iff* $\Gamma \models_{\mathbf{CL}} A$.

4.2.6 Some interesting Relations Between the Logics

The following theorems show us a very interesting relation between the logics **CLūNs** and **CL**, and between the logics **CLōNs** and **CLāNs**. As these relations will play an important role later on in this dissertation, I now already state them.

Theorem 4.7 *For* $B_1, ..., B_n \in \mathcal{S}$: $\Gamma \vdash_{\mathbf{CL}} A$ *iff* $\Gamma \vdash_{\mathbf{CL\bar{u}Ns}} A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$.

Theorem 4.8 *For* $B_1, ..., B_n \in \mathcal{S}$: $\Gamma \vdash_{\mathbf{CL\bar{a}Ns}} A$ *iff* $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$.

The proof of both theorems is completely equivalent, so that I will give it only once. But first, consider the following lemma:

Lemma 4.2 *$A \vee B$, $S(B, C) \vdash A \vee C$ is a derived inference rule in the proof theory of the presented paralogics.*

Proof. Suppose $A \vee B$, $S(B, C)$. Because it is obvious that $S(A, A)$ is derivable, it also follows that $A \vee C$ by means of DIL. ■

In the proof of the theorems 4.7 and 4.8, I will make use of the above derived rule. I will call it **DIL′**.

Proof.$\Rightarrow$ When $A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$ $(B_1, ..., B_n \in \mathcal{S})$ is derivable on a line in a **CLūNs**–proof (resp. **CLōNs**–proof), it is also derivable in a **CL**–proof (resp. **CLāNs**–proof), as all rules for **CLūNs** (resp. **CLōNs**) are also rules for **CL** (resp. **CLāNs**). Moreover, $S(B_i \wedge \neg B_i, B_i \wedge \neg B_i)$ is derivable for all $i \in \{1, n\}$, so that A will be derivable too in the **CL**–proof (resp. **CLāNs**–proof) because of the extra rule **DS**.

$\Leftarrow$ Consider an arbitrary **CL**–proof (resp. **CLāNs**–proof) of A from Γ. In order to prove there is also a **CLūNs**–proof (resp. **CLōNs**–proof) of $A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$, I will show how we can transform the original **CL**–proof (resp. **CLāNs**–proof) of A into a **CLūNs**–proof (resp. **CLōNs**–proof) of $A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$.

Remind that the only **CL**–rule (resp. **CLāNs**–rule) that is not valid for **CLūNs** (resp. **CLōNs**) is the rule **DS**. In order to get the transformation, we will proceed as follows: we start the proof and proceed as the original proof, until we arrive at a line i on which **DS** has been applied in the original proof. This means that we have a formula $A_1 \vee C_1$ and a subproof $S(C_1, B_1 \wedge B_1)$. In the original proof this leads to a line j with formula A_1. In the **CLūNs**–proof (resp. **CLōNs**–proof), we first apply **DIL′**, so that we arrive at $A_1 \vee (B_1 \wedge B_1)$. Next, we start a subproof with hypothesis A_1 and proceed within the subproof as we did in the original proof after the application of **DS**. This procedure is repeated any time we arrive at a line on which **DS** is applied in the original proof. At some point in the proof, we will be able to derive A on the closing line of a subproof.

1	P_1	PREM
...	...	PREM
g	P_n	PREM
...	...	**PL**–rules
h	$S(C_1, B_1 \wedge B_1)$	**PL**–rules
...	...	**PL**–rules
i	$A_1 \vee C_1$	**PL**–rules
i+1	$A_1 \vee (B_1 \wedge B_1)$	h,i;**DIL′**
i+2	$\mid A_1$	HYP

...	...	**PL**–rules
m	$A_2 \vee (B_2 \wedge \neg B_2)$	**DIL′**
m+1	$\mid A_2$	HYP
...	$\mid$ ...	...
n	$\mid \ldots \mid A_n$	HYP
...	$\mid \ldots \mid \ldots$	**PL**–rules
k	$\mid \ldots \mid A$	**PL**–rules

Once A has been derived in this way, we apply the rule **DIL′** n times, which gives us the formula $A \vee (B_1 \wedge \neg B_1) \vee ... \vee (B_n \wedge \neg B_n)$. ■

4.3 Full Paralogics

In the paralogics presented above, the implication is treated as a defined connective ($A \sqsupset B =_{df} \neg A \vee B$). As a consequence, in **CLūNs** and **CLōNs**, modus ponens is not a valid inference rule. It is nevertheless possible to add a stronger implication to those logics, an implication for which modus ponens is valid.

Actually, the logics that are obtained by adding this stronger implication to the basic paralogics (and by leaving out the weaker one), are the logics that are usually called **CLuNs**, **CLaNs** and **CLoNs** in the literature (see e.g. Batens et al. [7, 28, 29]. I will also refer to them as to the *full* versions of **CLūNs**, **CLāNs** and **CLōNs**.

4.3.1 Language Schema for Full Paralogics

The language $\mathcal{L}^f$ of full paralogics is the language $\mathcal{L}$ of basic paralogics, extended with the implication symbol $\supset$. Consequently, the set of wffs $\mathcal{W}^f$ of $\mathcal{L}^f$ is constructed as follows:

(i) $\mathcal{S} \subset \mathcal{W}^f$ for $\mathcal{S}$ the set of sentential letters,
(ii) When $A \in \mathcal{W}^f$ then $\neg A \in \mathcal{W}^f$,
(iii) When $A, B \in \mathcal{W}^f$ then $(A \wedge B), (A \vee B), (A \sqsupset B), (A \supset B) \in \mathcal{W}^f$.

Classes of Well–Formed Formulas. The formulas of full paralogics are also subdivided into $\mathfrak{a}$– and $\mathfrak{b}$–formulas. This is done as for basic paralogics (see table 4.1), although extended with some extra $\mathfrak{a}$–formulas:

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$
$\neg(A \supset B)$	A	$\neg B$

Table 4.2: Extra $\mathfrak{a}$–formulas for full paralogics.

4.3.2 Semantics for Full Paralogics

To obtain a semantics for their full versions, we simply add the following semantical clause to the semantics of respectively **CLūNs**, **CLāNs** and **CLōNs**.

SP6 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$.

In fact, this interprets the implication in a classical way. In other words, this stronger implication is nothing more nor less than good old material implication.

Negated Implications. Remark that as negations of implications are treated as $\mathfrak{a}$–formulas, they are semantically governed by clause **SP4**.

4.3.3 Proof Theory for Full Paralogics

Also the proof theory for full paralogics is easily obtained. It suffices to add the inference rules below to the proof theory of **CLūNs**, **CLāNs** or **CLōNs**.

CP $S(A, B) \blacktriangleright A \supset B$
MP A, $A \supset B \blacktriangleright B$
PC $(A \supset B) \supset A \blacktriangleright A$
NMI $\neg(A \supset B) \blacktriangleleft\blacktriangleright A \wedge \neg B$

CLuNs–, **CLaNs**– and **CLoNs**–derivability are defined as for **CLōNs** (see section 4.2.3).

Deduction Theorem. Remark that in contradistinction with the basic paralogics (see section 4.2.4), full paralogics have a traditional deduction theorem.

Theorem 4.9 $A_1, ..., A_n \vdash_{\mathbf{PL}} B$ *iff* $A_1, ..., A_{n-1} \vdash_{\mathbf{PL}} A_n \supset B$.

The proof of theorem 4.9 is standard, so that it is not necessary to present it here.

4.3.4 Soundness and Completeness

Soundness and completeness for full paralogics is easily proven. I will do it here only for **CLoNs**, but the method is completely equivalent for **CLuNs** and **CLaNs**.

Theorem 4.10 $\Gamma \vdash_{\mathbf{CLoNs}} A$ *iff* $\Gamma \models_{\mathbf{CLoNs}} A$.

Theorem 4.11 $\Gamma \vdash_{\mathbf{CLuNs}} A$ *iff* $\Gamma \models_{\mathbf{CLuNs}} A$.

Theorem 4.12 $\Gamma \vdash_{\mathbf{CLaNs}} A$ *iff* $\Gamma \models_{\mathbf{CLaNs}} A$.

Proof.=> Soundness for **CLoNs** is obtained by adding the induction cases for the extra inference rules to the soundness proof for **CLōNs**. As the ones for **MP**, **PC** and **NMI** are trivial, I will only prove the case for **CP**.

CP Suppose that (1) $A_k = A \supset B$ and (2) that it has been derived from $S(A, B)$ (with B on line i) by means of CP.
Consequence 1. From (2), it follows that $\Gamma_i \models_{\mathbf{CLoNs}} B$ (by the induction hypothesis). Moreover, that $i < k$ gives us that $\Gamma_i \subseteq \Gamma_k \cup \{A\}$. From both the above, it follows that $\Gamma_k \cup \{A\} \models_{\mathbf{CLoNs}} B$ (by lemma 4.1).
Supposition. $\Gamma_k \not\models_{\mathbf{CLoNs}} A_k$. From this, it follows that there is an **CLoNs**–model M such that $v_M(\Gamma_k) = 1$ and $v_M(A_k) = 0$. As $A_k = A \supset B$, this means that $v_M(A \supset B) = 0$, from which it follows that $v_M(A) = 1$ and $v_M(B) = 0$ $(*)$ (by **SP6**). But, as $v_M(\Gamma_k) = 1$ and $v_M(A) = 1$, also $v_M(\Gamma_k \cup \{A\}) = 1$ such that $v_M(B) = 1$ (by consequence 1). The latter however contradicts with $(*)$.

<= Completeness for **CLoNs** is proven in the same way as for **CLōNs**. Only the induction cases for implications and negations of implications should be added. As the case for negations of implicational formulas is trivial, I will only prove it for implicational wffs.

First, I need to show that the constructed set Δ that is used to define a **CLoNs**–model, is not only non–trivial (see the completeness proof for **CLōNs**), but that it is maximally so (meaning that any extension would lead to the trivial set). In order to prove this, I will first show that $A \supset C \in \Delta$, with A the formula we deliberately kept out of Δ and with C an arbitrary formula (representing triviality). If this were not the case, then $\Delta \cup \{A \supset C\} \vdash_{\mathbf{CLoNs}} A$ (by the construction for Δ), and also $\Delta \vdash_{\mathbf{CLoNs}} (A \supset C) \supset A$ (by theorem 4.9). But this would mean that $\Delta \vdash_{\mathbf{CLoNs}} A$ (by the deductive closure of Δ), which is impossible.

Secondly, take an arbitrary formula D for which $D \notin \Delta$. This gives us $\Delta \cup \{D\} \vdash_{\mathbf{CLoNs}} A$ (by the construction for Δ). So, adding D to Δ would give us A, but as $A \supset C \in \Delta$ for all C, adding D to Δ would also give us all C (by the deductive closure of Δ). Conclusion: Δ is max. non–trivial.

Knowing this, I can now also prove that $v_M(A \supset B) = 1$ iff $A \supset B \in \Delta$. For the right–left direction, suppose $A \supset B \in \Delta$. This means that $\Delta \vdash_{\mathbf{CLoNs}} A \supset B$, from which it follows that $\Delta \cup \{A\} \vdash_{\mathbf{CLoNs}} B$ (by theorem 4.9). There are two possibilities concerning the formula A: $A \notin \Delta$ or $A \in \Delta$. The latter would mean that $B \in \Delta$ (by the deductive closure of Δ). From the foregoing, it now follows that $v_M(A) = 0$ or $v_M(B) = 1$ (by the induction hypothesis). This gives us $v_M(A \supset B) = 1$ (by **SP6**).

For the left–right direction, suppose $v_M(A \supset B) = 1$. By **SP6**, it follows that $v_M(A) = 0$ or $v_M(B) = 1$, which also means that $A \notin \Delta$ or $B \in \Delta$ (by the induction hypothesis). Both of these lead to $A \supset B \in \Delta$. The latter by the deductive closure of Δ, and the former because Δ is max. non–trivial (As $A \notin \Delta$, adding A to Δ would lead to triviality. From this follows that $\Delta \cup \{A\} \vdash_{\mathbf{CLoNs}} B$, which gives us the desired result by means of theorem 4.9). ■

4.4 Modal Paralogics

Modal versions are possible for all paralogics presented in the foregoing sections. Fot these *modal paralogics*, worlds are not consistent and complete, but paraconsistent and/or paracomplete. As such, it is probably a good idea not to call them worlds, but to call them set–ups, as is widely done in the literature. Nevertheless, I will keep referring to them as to worlds.

Although there are a lot of modal paralogics, only the logic **KōNs** will be discussed in full. The discussion of the other modal paralogics will be restricted to side remarks.

4.4.1 The Modal Language Schema

The language $\mathcal{L}^{\mathcal{M}}$ of modal paralogics is obtained by extending the language $\mathcal{L}$ of basic paralogics with the usual modal operators $\Box$ and $\Diamond$. Consequently, the set of well–formed formulas $\mathcal{W}^{\mathcal{M}}$ of $\mathcal{L}^{\mathcal{M}}$ is constructed as follows:

(i) $\mathcal{S} \subset \mathcal{W}^{\mathcal{M}}$ for $\mathcal{S}$ the set of sentential letters,
(ii) When $A \in \mathcal{W}^{\mathcal{M}}$ then $\neg A, \Box A, \Diamond A \in \mathcal{W}^{\mathcal{M}}$,
(iii) When $A, B \in \mathcal{W}^{\mathcal{M}}$ then $(A \wedge B), (A \vee B), (A \sqsupset B) \in \mathcal{W}^{\mathcal{M}}$.

Classes of Well–Formed Formulas. The $\mathfrak{a}$– and $\mathfrak{b}$–formulas of modal paralogics are the same as for basic paralogics (see table 4.1), with the following $\mathfrak{a}$–formulas added:

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$
$\neg\Box A$	$\Diamond\neg A$	$\Diamond\neg A$
$\neg\Diamond A$	$\Box\neg A$	$\Box\neg A$

Table 4.3: Extra $\mathfrak{a}$–formulas for modal paralogics.

4.4.2 Semantics for Modal Paralogics

A **KōNs**–model for the language $\mathcal{L}^{\mathcal{M}}$, with $\mathcal{S}$ and $^{\neg}\mathcal{S}$ respectively the set of sentential letters and the set of negated sentential letters ($^{\neg}\mathcal{S} = \{\neg A \mid$

$A \in \mathcal{S}\}$), is a quadruple $< W, g, R, v >$, with W a set of worlds, g the base world, R a binary relation on W, and v a valuation function, characterized as follows:

AP1 $v:\ \mathcal{S} \times W \mapsto \{0, 1\}$.
AP2 $v:\ {}^{\neg}\mathcal{S} \times W \mapsto \{0, 1\}$.

The valuation function v_M determined by the model M is defined as follows:

SP1 $v_M : \mathcal{W}^{\mathcal{M}} \times W \mapsto \{0, 1\}$.
SP2 For $A \in \mathcal{S}$: $v_M(A, a) = 1$ iff $v(A, a) = 1$.
SP3o For $A \in \mathcal{S}$: $v_M(\neg A, a) = 1$ iff $v(\neg A, a) = 1$.
SP4 $v_M(\mathfrak{a}, a)$= 1 iff $v_M(\mathfrak{a}_1, a) = 1$ and $v_M(\mathfrak{a}_2, a) = 1$.
SP5 $v_M(\mathfrak{b}, a)$= 1 iff $v_M(\mathfrak{b}_1, a) = 1$ or $v_M(\mathfrak{b}_2, a) = 1$.
SP7 $v_M(\Box A, a) = 1$ iff for all $b \in W$: if Rab then $v_M(A, b) = 1$.
SP8 $v_M(\Diamond A, a) = 1$ iff there is at least one $b \in W$: Rab and $v_M(A, b) = 1$.

Truth in a model, semantical consequence, and validity are defined as usual:

Definition 4.5 *A is true in a model M iff $v_M(A, g) = 1$.*

Definition 4.6 *$\Gamma \models_{\mathbf{K\bar{o}Ns}} A$ iff A is true in all models in which all elements of Γ are true.*

Definition 4.7 *$\models_{\mathbf{K\bar{o}Ns}} A$ iff A is true in all models.*

Features of the Accessibility Relation. The accessibility relation of the logic **Kōns** has no special characteristics. It is nevertheless possible to strengthen the relation R in such a way that it becomes reflexive, symmetrical and/or transitive.[7] We then obtain the following logics:

ToNs R is reflexive.
BoNs R is reflexive and symmetrical.
S4oNs R is reflexive and transitive.
S5oNs R is reflexive, symmetrical and transitive.

More Modal Paralogics. As stated above, **KōNs**–worlds are both paraconsistent and paracomplete. But, it is also possible to take worlds to be only paraconsistent or only paracomplete. This is done by replacing the semantical clause **SP3o** by respectively **SP3u** or **SP3a** (see section 4.2.2), which gives us the modal versions of respectively **CLūNs** and **CLāNs**.

[7] Obviously, other characteristics are also possible, but I limit myself to the best–known ones.

4.4.3 Proof Theory for Modal Paralogics

Although the proof theory for the modal paralogic **KōNs** is quite resemblant to the one for **CLōNs**, there are nevertheless some striking differences. First, consider the structural rules.

HYP At any place in the proof, one may start a new subproof. This is done by introducing a new hypothesis, together with a new vertical line on its left.

♦HYP At any place in the proof, one may start a new modal subproof. This is done by introducing a new hypothesis, together with a new vertical line on its left. Modal subproofs will be differentiated from non–modal subproofs by writing a ♦–symbol next to their vertical line.

i	... ♦	A	♦HYP
i+1	...	...	

These subproofs show us what would follow from the hypothesis if it were true in an arbitrary world.

CSP If the formula B is the formula on the last line of a subproof that started with the hypothesis A, one may conclude to the pseudo–formula $S(A, B)$.

MCSP If the formula B is the formula on the last line of a modal subproof that started with the hypothesis A, one may conclude to the *pseudo–formula* $\blacklozenge S(A, B)$.[8]

PREM Premises may be written down at any place in the proof, except in modal subproofs. The reason is obvious: the premises are only taken to be true at the base world g, and are not necessarily also true in an arbitrary world.

REP In the main proof and in both modal and non–modal subproofs, formulas may be repeated.

REIT Reiteration is restricted to non–modal subproofs, which means that formulas can only be reiterated into unclosed non–modal subproofs.

Inference Rules. The inference rules for **KōNs** are obtained by adding the rules below to the inference rules for **CLōNs**.

$\Box$I	$\Box A$, $\blacklozenge S(A, B) \blacktriangleright \Box B$
$\Diamond$I	$\Diamond A$, $\blacklozenge S(A, B) \blacktriangleright \Diamond B$
$\Box\vee$E	$\Box(A \vee B) \blacktriangleright \Box A \vee \Diamond B$

[8] This means that in a modal proof, a modal subproof is represented by means of $\blacklozenge S(A, B)$, which is completely in accordance with the practice of representing a non–modal subproof by means of $S(A, B)$.

$\Diamond\vee$E	$\Diamond(A \vee B) \blacktriangleright \Diamond A \vee \Diamond B$
$\Diamond\wedge$I	$\Box A \wedge \Diamond B \blacktriangleright \Diamond(A \wedge B)$
$\Box\wedge$I	$\Box A \wedge \Box B \blacktriangleright \Box(A \wedge B)$
$\neg\Box$	$\neg\Box A \blacktriangleleft\blacktriangleright \Diamond\neg A$
$\neg\Diamond$	$\neg\Diamond A \blacktriangleleft\blacktriangleright \Box\neg A$

KōNs–Derivability. A **KōNs**–proof is defined in the same way as a **CLōNs**–proof, and also the definition of **KōNs**–derivability is equivalent to the one for **CLōNs**.

Example. To make the **KōNs**–proof theory more concrete, consider the following example:

1	$\Box(p \wedge q)$	PREM	
2	$\Diamond(r \vee s)$	PREM	$\triangle\ \Diamond(r \vee (q \wedge s))$
3	$\blacklozenge\mid p \wedge q$	$\blacklozenge$HYP	
4	$\mid q$	3;SIM	
5	$\blacklozenge S(p \wedge q, q)$	3,4;MCSP	
6	$\Box q$	1,5;$\Box$I	
7	$(\Box q) \wedge (\Diamond(r \vee s))$	2,6;CON	
8	$\Diamond(q \wedge (r \vee s))$	7;$\Diamond\wedge$I	
9	$\blacklozenge\mid q \wedge (r \vee s)$	$\blacklozenge$HYP	
10	$\mid r \vee s$	9;SIM	
11	$\mid\ \mid r$	HYP	
12	$\mid\ \mid r$	11;REP	
13	$\mid S(r, r)$	11,12;CSP	
14	$\mid\ \mid s$	HYP	
15	$\mid\ \mid q$	9;SIM	
16	$\mid\ \mid q \wedge s$	14,15;CON	
17	$\mid S(s, q \wedge s)$	14,16;CSP	
18	$\mid r \vee (q \wedge s)$	10,13,17;DIL	
19	$\blacklozenge S(q \wedge (r \vee s), r \vee (q \wedge s))$	9,18;MCSP	
20	$\Diamond(r \vee (q \wedge s))$	8,19;$\Diamond$I	

Strengthening the Accessibility Relation. Modal paralogics with a stronger accessibility relation have extra inference rules, rules that express the specific features of the accessibility relation. When the relation R is reflexive, the following inference rules should be added:

Refl$\Box$	$\Box A \blacktriangleright A$
Refl$\Diamond$	$A \blacktriangleright \Diamond A$

For a symmetrical accessibility relation, the extra inference rules are the following:

Sym$\Box$	$A \blacktriangleright \Box\Diamond A$
Sym$\Diamond$	$\Diamond\Box A \blacktriangleright A$

Finally, when the accessibility relation is considered transitive, the following inference rules should be added:

Tran$\Box$	$\Box A \blacktriangleright \Box\Box A$
Tran$\Diamond$	$\Diamond\Diamond A \blacktriangleright \Diamond A$

To show that these rules give some adequate results, consider the example below for the logic **S5ōNs**:

1	$\Diamond p$	PREM	$\triangle\ \Box\Diamond p$
2	$\Box\Diamond\Diamond p$	1;Sym$\Box$	
3	$\blacklozenge\mid \Diamond\Diamond p$	$\blacklozenge$HYP	
4	$\mid \Diamond p$	3;Tran$\Diamond$	
5	$\blacklozenge S(\Diamond\Diamond p, \Diamond p)$	3,4;MCSP	
6	$\Box\Diamond p$	2,5;$\Box$I	

More Modal Paralogics. Some extra inference rules are also required to obtain the modal versions of **CLūNs** and **CLāNs**. To obtain modal **CLūNs**–paralogics, add the rules **TH** and **NEC**, and to obtain modal **CLāNs**–paralogics, add the rules **DS** and **MDS**.

TH	For $C \in \mathcal{S}$: $\blacktriangleright A \vee \neg A$
NEC	$\vdash A \blacktriangleright \vdash \Box A$
DS	For $C \in \mathcal{S}$: $A \vee B$, $S(B, C \wedge \neg C) \blacktriangleright A$
MDS	For $C \in \mathcal{S}$: $A \vee \Diamond ... \Diamond B$, $\blacklozenge S(B, C \wedge \neg C) \blacktriangleright A$

I think it is also clear by now that the usual normal modal logics are reached when all these rules are added to the modal **CLōNs**–paralogics.

4.4.4 Soundness and Completeness

In this section, I will prove soundness and completeness for the logic modal paralogic **KōNs**. First, consider soundness.

Theorem 4.13 (Soundness) *If* $\Gamma \vdash_{\mathbf{K\bar{o}Ns}} A$ *then* $\Gamma \models_{\mathbf{K\bar{o}Ns}} A$.

Before proving theorem 4.13, first consider the lemma below:

Lemma 4.3 *If* $\Gamma \subseteq \Gamma'$ *and if for all* **KōNs**–*models* M *it is the case that for all* $w \in W$, *if* $v_M(\Gamma, w) = 1$ *then* $v_M(A, w) = 1$, *then it follows for all* **KōNs**–*models* M *that it is the case that for all* $w \in W$, *if* $v_M(\Gamma', w) = 1$ *then* $v_M(A, w) = 1$.

As the proof of lemma 4.3 is straightforward, it is left to the reader.

Next, consider some terminological remarks. Let A_i express that the formula A is derived in a proof on line i, and let Γ_i be the set of all premises and all hypotheses that have the formula on line i in their scope, where a formula A on line i is *in the scope of* a formula B on line j whenever (1) $j \leqslant i$, and (2) B can be reiterated into the subproof were A is in.

Finally, consider the proof of theorem 4.13. It is an induction proof on the line numbers of a **Kō̄Ns**–derivation.

Proof. Suppose $\Gamma \vdash_{\mathbf{K\bar{o}Ns}} A$. This means that there is a proof of A from Γ. To proof that this also gives us $\Gamma \models_{\mathbf{K\bar{o}Ns}} A$, I will proof by induction that for all lines i of the proof, it is the case for all **Kō̄Ns**–models that $\forall w \in W$, if $v_M(\Gamma_i, w) = 1$ then $v_M(A_i, w) = 1$. As Γ_i (with i the line on which A occurs) cannot contain any hypotheses (because A is only derivable when it occurs in the main proof), Γ_i will be a subset of Γ so that by lemma 4.3, it will follow for all **Kō̄Ns**–models that $\forall w \in W$, if $v_M(\Gamma, w) = 1$ then $v_M(A, w) = 1$. This gives us $\Gamma \models_{\mathbf{K\bar{o}Ns}} A$ (because $g \in W$).

First, consider the base case: A_0 is necessarily a premise or a hypothesis, which means that $A_0 \in \Gamma_0$ so that for all **Kō̄Ns**–models, it is impossible that $\exists w \in W$ such that $v_M(\Gamma_0, w) = 1$ and $v_M(A_0, w) = 0$.

Next, consider the induction hypothesis:

Induction Hypothesis 4.2 *For any i, $1 \leqslant i < k$: For all* **Kō̄Ns***–models M, $\forall w \in W$, if $v_M(\Gamma_i, w) = 1$ then $v_M(A_i, w) = 0$.*

It remains to be proven that $\forall M\ \forall w \in W$, if $v_M(\Gamma_k, w) = 1$ then $v_M(A_k, w) = 1$. A_k is either a premise, an assumption or is derived from previous lines by means of the inference rules. When A_k is a premise or an assumption, the proof is analogous to the base case. Hence, $\forall M\ \forall w \in W$, if $v_M(\Gamma_k, w) = 1$ then $v_M(A_k, w) = 0$. So, we only need to show that this is also the case when A_k is derived by means of one of the inference rules. As the proofs for most of them are fairly easy, I leave most of them to the reader, and will only prove the case for the inference rule $\Box$**I**.

$\Box$I Suppose (1) that $A_k = \Box B$ and (2) that it has been derived from $A_i = \Box A$ and $\blacklozenge S(A, B)$ (with B on line j) by means of $\Box$**I**.
Consequence 1. From (2), it follows that $\forall M\ \forall w \in W$, if $v_M(\Gamma_i, w) = 1$ then $v_M(\Box A, w) = 1$ (by the induction hypothesis). Moreover, that $i < k$ and that they both belong to the same subproof, gives us that $\Gamma_i \subseteq \Gamma_k$. From both the above, it follows that $\forall M\ \forall w \in W$, if $v_M(\Gamma_i, w) = 1$ then $v_M(\Box A, w) = 1$ (by lemma 4.3).
Consequence 2. That $\blacklozenge S(A, B)$, with B on line j (see (2)), gives us that $\forall M\ \forall w \in W$, if $v_M(\Gamma_j, w) = 1$ then $v_M(B, w) = 1$ (by the induction hypothesis). Moreover, because no premises can be introduced in a modal subproof, and because reiteration is not possible

for modal subproof, it follows that $\Gamma_j = \{A\}$, which means that $\forall M\ \forall w \in W$, if $v_M(A, w) = 1$ then $v_M(B, w) = 1$.
Supposition. Suppose $\exists a \in W$, $v_M(\Gamma_k, a) = 1$ and $v_M(A_k, a) = 0$. As $A_k = \Box B$, this means that $v_M(\Box B, a) = 0$, from which it follows that there is a world $b \in W$ such that Rab and $v_M(B, b) = 0$ $(*)$ (by **SP7**).
From the foregoing paragraph, it follows that $v_M(\Gamma_k, a) = 1$, which means that also $v_M(\Box A, a) = 1$ (by consequence 1). From the latter, it follows that $v_M(A, b) = 1$ (by **SP7**), which gives us that $v_M(B, b) = 1$ (by consequence 2). This contradicts $(*)$.

■

Next, also consider completeness for **KōNs**.

Theorem 4.14 (Completeness) *If* $\Gamma \models_{\mathbf{K\bar{o}Ns}} A$ *then* $\Gamma \vdash_{\mathbf{K\bar{o}Ns}} A$.

Proof. Suppose $\Gamma \nvdash_{\mathbf{K\bar{o}Ns}} A$. It is now possible to extend Γ to a set Π such that

(i) $\Gamma \subseteq \Pi$,
(ii) $A \notin \Pi$,
(iii) Π is deductively closed,
(iv) Π is prime.

This is done in the same way as in the completeness proof for **CLōNs** (see section 4.2.5).

A **KōNs**–model is now defined as the 4–tuple $< W, \Pi, R, v >$, with W the set of all prime deductively closed theories, R a binary relation on W such that $R_{\Gamma\Delta}$ iff for all $B \in \mathcal{W}^{\mathcal{M}}$:

RP1 if $\Box B \in \Gamma$ then $B \in \Delta$, and
RP2 if $B \in \Delta$ then $\Diamond B \in \Gamma$.

and v an assignment function for which

AP1 For $A \in \mathcal{S}$ and $\Sigma \in W$, $v(A, \Sigma) = 1$ iff $A \in \Sigma$.
AP2 For $A \in \mathcal{S}$ and $\Sigma \in W$, $v(\neg A, \Sigma) = 1$ iff $\neg A \in \Sigma$.

Notice that $\Pi \in W$ (because of (iii) and (iv)). By induction on the complexity of wffs, it can be now shown that for all wffs $C \in \mathcal{W}^{\mathcal{M}}$, $v_M(C, \Sigma) = 1$ iff $C \in \Sigma$. As the base case and most induction cases are quite straightforward, I will only prove the induction cases for formulas of the form $\Box A$ and $\Diamond A$, as these require some extra work. First, consider the following two lemmas:

Lemma 4.4 $\Box B \in \Sigma$ *iff* $\forall \Theta \in W$*, if* $\mathrm{R}_{\Sigma\Theta}$ *then* $B \in \Theta$*.*

Proof.=> Suppose (1) $\Box B \in \Sigma$, (2) $R_{\Sigma\Theta}$ and (3) $B \notin \Theta$. From (1) and (2), it follows that $B \in \Theta$ (by **RP1**), which contradicts with (3).

<= Suppose $\Box B \notin \Sigma$. Construct $\Sigma_{\Box} = \{C \mid \Box C \in \Sigma\}$ and $\Sigma_{\Diamond} = \{D \mid \Diamond D \notin \Sigma\}$. It follows that for all $C_1, ..., C_n \in \Sigma_{\Box}$ and $D_1, ..., D_m \in \Sigma_{\Diamond}$, $C_1 \wedge ... \wedge C_n \not\vdash_{\mathbf{K\bar{o}Ns}} B \vee D_1 \vee ... \vee D_m$ (otherwise, because of the deductive closure of Σ, $\Box B \vee \Diamond D_1 \vee ... \vee \Diamond D_m \in \Sigma$, with $D_1, ..., D_m \in \Sigma_{\Diamond}$. This would mean that $\Box B \in \Sigma$ or $\Diamond D_1 \in \Sigma$ or ... or $\Diamond D_n \in \Sigma$, as Σ is prime. The latter contradicts with our supposition and with the construction of $\Sigma_{\Diamond}$). $\Sigma_{\Box}$ can be extended to a deductively closed, prime theory Θ such that $R_{\Sigma\Theta}$ and $B \notin \Theta$. ■

Lemma 4.5 $\Diamond B \in \Sigma$ *iff* $\exists \Theta \in W$, $\mathrm{R}_{\Sigma\Theta}$ *and* $B \in \Theta$.

Proof.=> Suppose $\Diamond B \in \Sigma$. Construct $\Sigma_{\Box} = \{C \mid \Box C \in \Sigma\}$ and $\Sigma_{\Diamond} = \{D \mid \Diamond D \notin \Sigma\}$. It follows that for all $C_1, ..., C_n \in \Sigma_{\Box}$ and $D_1, ..., D_m \in \Sigma_{\Diamond}$, $C_1 \wedge ... \wedge C_n \wedge B \not\vdash_{\mathbf{K\bar{o}Ns}} D_1 \vee ... \vee D_m$ (otherwise, because of the deductive closure of Σ, $\Diamond D_1 \vee ... \vee \Diamond D_m \in \Sigma$, with $D_1, ..., D_m \in \Sigma_{\Diamond}$. This would mean that $\Diamond D_1 \in \Sigma$ or ... or $\Diamond D_n \in \Sigma$, as Σ is prime. The latter contradicts with the construction of $\Sigma_{\Diamond}$). $\Sigma_{\Box}$ can be extended to a deductively closed, prime theory Θ such that $R_{\Sigma\Theta}$ and $B \in \Theta$

<= Suppose $R_{\Sigma\Theta}$ and $B \in \Theta$. From these, it follows that $\Diamond B \in \Sigma$ (by **RP2**). ■

Next, consider the induction cases for formulas of the form $\Box A$ and $\Diamond A$.

$\Box A \in \Sigma$	iff	$\forall \Theta \in W$, if $R_{\Sigma\Theta}$ then $A \in \Theta$ (by lemma 4.4).
	iff	$\forall \Theta \in W$, if $R_{\Sigma\Theta}$ then $v_M(A, \Theta) = 1$ (by the induction hypothesis).
	iff	$v_M(\Box A, \Sigma) = 1$ (by **SP7**).
$\Diamond A \in \Sigma$	iff	$\exists \Theta \in W$, $R_{\Sigma\Theta}$ and $A \in \Theta$ (by lemma 4.5).
	iff	$\exists \Theta \in W$, $R_{\Sigma\Theta}$ and $v_M(A, \Theta)$ (by the induction hypothesis).
	iff	$v_M(\Diamond A, \Sigma) = 1$ (by **SP8**).

Finally, from the induction proof, together with (i) and (ii), it follows that $v_M(\Gamma, \Pi) = 1$ and $v_M(A, \Pi) = 0$ such that $\Gamma \not\models_{\mathbf{K\bar{o}Ns}} A$. ■

The corollary below now follows immediately from theorem 4.13 and theorem 4.14.

Corollary 4.2 $\Gamma \vdash_{\mathbf{K\bar{o}Ns}} A$ *iff* $\Gamma \models_{\mathbf{K\bar{o}Ns}} A$.

Other Modal Paralogics. Soundness and the completeness proofs for other modal paralogics are simple adaptations of the proofs above. As this is not the topic of this dissertation, I leave it to the reader to work out the details.

4.5 Conclusion

In this chapter, I have presented those paralogics that will be needed in later chapters of this dissertation. More specifically, I have presented the basic paralogics **CLūNs**, **CLāNs** and **CLōNs**, the paralogics **CLuNs**, **CLaNs** and **CLoNs**, and some modal paralogics based on **CLūNs**–, **CLāNs**– and **CLōNs**–worlds.

Chapter 5

Introducing Relevant Logics

5.1 Introduction

In this chapter, I will present a non–truthfunctional semantics for standard relevant logics (**RL**). This will characterize **RL**–negation along the lines of chapter 4. Special attention will be given to the semantic characterization of the basic relevant logic **BD** and the well–known relevant logic **R**. Moreover, not only classical derivability will be characterized, but also relevant derivability.

5.2 Relevant Logics

As made clear in chapter 1, Relevance Logic started off in an attempt to avoid the paradoxes of the material and the strict implication. This was done by constructing relevant logics (**RL**) for which only relevant entailments[1] are valid.[2]

Anderson and Belnap were the first to systematically discuss the topic in their Entailment [5], in which they gave a full proof theoretical characterization of the first relevant logics **T**, **R** and **E**. Only later a semantics was devised for these logics. The best known semantical characterizations are those of Urquhart [115], Fine [55], Dunn [52], and Routley & Meyer [97, 98, 99, 100]. Of these, the Routley–Meyer semantics is without any doubt the most popular one. Shortly put, it is a modal semantics that makes use of a ternary accessibility relation between worlds to characterize relevant implication:

Definition 5.1 *$A \to B$ is true at a world a iff for all worlds b and c such that $\mathrm{R}abc$, if A is true at b then B is true at c.*

[1] Remember that entailments refer to logical truths in implicational form, see ch. 1.

[2] A very nice and short introduction into standard Relevance Logic is the one by Mares [68] in the *Standford Encyclopedia of Philosophy*. Also Dunn & Restall [54] is nice, but more tedious.

In order to reach the exact semantical characterization of the logics **T**, **R** and **E**, certain constraints have to be put upon this ternary accessibility relation R. Of course, it is possible to alter these constraints in numerous ways. This has lead to a whole bunch of relevant logics (see e.g. Routley et al. [101]). These can be considered the "standard relevant logics".

The ternary accessibility relation is not the only non–standard element in the semantics of **RL**. As contradictions don't imply everything, it is not surprising that their semantics also makes use of paraconsistent and paracomplete worlds.[3] This immediately implies that negation in **RL** is not characterized as in **CL**. In the literature, two ways to treat the **RL**–negation have been proposed. The first one is called the *Australian Plan* (**AUP**) and is originally due to Routley & Routley [102]. The second one is called the *American Plan* (**AMP**) and is originally due to Dunn [52], but completed in different ways by Routley [96] and Restall [94].

In this chapter, I will show that it is also possible to characterize the **RL**–negation in a non–truthfunctional way, as it was done for paralogics in chapter 4. I will call this the *Ghent Plan* (**GP**). Special attention will be given to the characterization of the logic **R**, as it still has a role to play in the remaining of this dissertation.

5.3 The Ghent Plan Completed

Before I start presenting the proof theory and semantics of **RL**, I feel obliged to first give some extra reasons as to why I consider it necessary to develop another semantical characterization of the **RL**–negation. I do not intend to give a complete discussion of the topic, which is left for further research. For the moment, some short remarks will suffice.

First of all, that all logics in this dissertation are characterized in a uniform way, would be more elegant. This enables one to see the existing relations between them more easily. Secondly, it can be argued that the non–truthfunctional semantics avoids some nasty disadvantages of the other two approaches.

Firstly, in the **AUP**, negation is handled by means of the so–called "Routley Star", which is a binary operator on worlds. Each world a is given a star–world a^*. Negation is then characterized by the following definition:

Definition 5.2 (AUP) *$\sim A$ is true at world a iff A is false at a^*.*

Obviously, some extra constraints can be placed upon the star operator. This is also necessary to characterize the logic **R**. The biggest problem with the **AUP** is that, despite numerous attempts (most notably the one of Dunn [53], who interprets the star–world as the world that is maximally

[3] Remember that these are usually called set–ups.

compatible with its non–starred counterpart), no satisfactory philosophical interpretation has yet been found for the Routley Star. After all, it remains quite strange to treat negation by relying on a relation between worlds. This strange feature of the **AUP** is avoided in the **GP**, as the latter doesn't make use of the Routley Star, nor of any other relation on the set of worlds. Negation is treated as a simple, one–world connective.

Secondly, in the **AMP**, negation is handled by extending the number of truth values from two (*True* and *False*) to four (*True*, *False*, *Neither*, and *Both*), which explains why the **AMP** is also called the "four–valued"–approach. Negation is here characterized as follows:

Definition 5.3 (AMP) *"True" is a truth value of* $\sim A$ *at world* a *iff "False" is a truth value of* A *at* a*.*

It should be immediately clear that the **AMP**–approach avoids the problems related to the Routley Star.[4] Nevertheless, it has some problems of its own. The first one being the enormous increase in technicality (see e.g. Routley [96]), and the second one being that it has not been proved possible to characterize all **RL** by means of a four–valued semantics. Especially the richer relevant logics have caused some problems, most notably the logic **R**.[5] Moreover, and this is the final objection towards **AMP**, it is quite counter–intuitive to allow formulas to be both True and False. In reasoning contexts, the contextual premises are considered the best ones available for solving the problem of the context.[6] As such, they are considered contextually True, as are their consequences.[7] In other contexts, they might be considered contextually False, but it is never the case that they are considered both True and False.[8] If the contextual model is a good model to capture how people reason, it is clear that a two valued approach has to be preferred over a four valued one. It goes without saying that the semantics I will present below is two valued. Moreover, its technicality is kept within boundaries and it can be used to characterize a wide range of relevant logics, the richer ones included.

[4] Remark that this is not always the case. Restall [94] also used the Routley Star in his four–valued characterization of **RL**. This of course immediately raises the question as to why we should use four values if two are enough.

[5] It is possible that the recent four-valued approach due to Mares [70] is able to avoid this problem. For the moment, I am however not enough acquainted with this approach to judge this correctly.

[6] I here adhere to the contextual approach in reasoning, proposed by Batens in [21, 9, 8]. The approach takes reasoning to proceed goal–directed, with the intention to solve problems.

[7] True, it sometimes happens that they are considered both, but this is usually due to an ambiguous interpretation of those formulas, which is not what is referred to here.

[8] Remark that truth is not taken as correspondence with the world, but as an instrumental, pragmatical notion.

Does it now automatically follow that the non–truthfunctional semantics is better off than its opponents? To be honest, I'm not sure, but the least one can say is that it definitely is a valuable alternative to the existing approaches and in a sense that should be enough to grant it its right of existence.

5.4 Characterizing Standard Relevant Logics

In this section, I will give both the proof–theoretic and semantic characterization of a lot of standard relevant logics. Before I can get started, it is necessary to give two final preliminary remarks.

Firstly, the derivability relation of the **RL** characterized in this section, is the classical derivability relation (see chapter 1). The relevant derivability relation will be discussed in section 5.5.

Secondly, the non–truthfunctonal semantics that will be presented in this section is a modification of the simplified Routley–Meyer semantics for **RL**, due to Priest & Sylvan [88], Restall [93], and Restall & Roy [95]. Besides the fact that I owe a great deal to those papers (especially concerning the soundness and completeness proofs), this also has two important consequences. The first one being that the semantics of the positive fragments of **RL** I will present, is equal to the one presented in those papers. The second one being that the semantical characterization I will present below is only useful to model *disjunctive systems*.[9] These are systems for which the following theorem is valid:

Theorem 5.1 *If* $A \vdash B$ *then* $C \vee A \vdash C \vee B$.

It is important to notify this, as not all relevant logics are disjunctive systems. The most notable one being without any doubt the logic **E**, as it well–known that $\sim A \vee A \nvdash_{\mathbf{E}} \sim A \vee ((A \to B) \to B)$ is not valid, despite the fact that $A \vdash_{\mathbf{E}} (A \to B) \to B$ is valid.

5.4.1 The Language Schema of RL

The language $\mathcal{L}$ of **RL** is the classical $\wedge, \vee$–fragment of standard propositional language,[10] upgraded with a relevant negation and implication symbol $\sim$ and $\to$. Consequently, the set of well–formed formulas $\mathcal{W}$ of the language $\mathcal{L}$, is made up as follows:

(i) $\mathcal{S} \subset \mathcal{W}$ for $\mathcal{S}$ the set of sentential letters,
(ii) When $A \in \mathcal{W}$ then $\sim A \in \mathcal{W}$,
(iii) When $A, B \in \mathcal{W}$ then $(A \wedge B), (A \vee B), (A \to B) \in \mathcal{W}$.

[9] See Brady [40] for an introduction to the notion of disjunctive systems.

[10] I will not discuss predicative **RL**. How these should be treated by the semantics I will propose in this chapter, is left for further research.

Classes of Well–Formed Formulas. As in chapter 4, I will make use of $\mathfrak{a}$– and $\mathfrak{b}$–formulas, which represent classes of well–formed formulas.

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\sim(A \wedge B)$	$\sim A$	$\sim B$
$\sim(A \vee B)$	$\sim A$	$\sim B$		$A \vee B$	A	B
$\sim\sim A$	A	A				

Table 5.1: $\mathfrak{a}$– and $\mathfrak{b}$–formulas for relevant logics

Remark that the classes of well-formed formulas for **RL** are the same as those for basic paralogics. This will give us a good view on the relation(s) between both kinds of logics.

5.4.2 The Basic Relevant Logic

In Routley et al. [101], the relevant logic **B** is put forth as the weakest, most basic relevant logic (see table 5.5). The other relevant logics are obtained by adding extra conditions on the ternary accessibility relation. Moreover, as in the simplified Routley–Meyer semantics, no constraints are put upon the ternary accessibility relation, it can be claimed that the relevant logic **B** is for relevant logicians what the logic **K** is for modal logicians.[11]

Although the logic **B** is usually considered the weakest relevant logic, this is not completely true. When also all constraints on the star operator are removed, the logic **BM** is obtained (see table 5.5). Consequently, people adhering to the Routley–Meyer semantics, might also claim this to be the basic logic.

Because of the absence of the star–operator, the basic relevant logic in the **GP** is not the logic **B**, nor the logic **BM**, but the logic **BD**, which is also considered the basic relevant logic by adherents of the four–valued approach. As such, I will first give a complete proof–theoretic and semantic characterization of the logic **BD**, before doing the same for stronger relevant logics.

The Basic Logic? Despite the fact that in this section, the logic **BD** will be considered as the basic relevant logic, I will show in section 5.4.5 that there are relevant logics that are even more basic than **BD**.

A. Proof Theory and Semantics

First, consider the proof theory of the logic **BD**. It is the axiom system given in table 5.2 below.

[11] This is not the case in the original Routley–Meyer semantics, where there are several conditions on the accessibility relation.

A1	$A \to A$
A2	$(A \wedge B) \to A$, $(A \wedge B) \to B$
A3	$A \to (A \vee B)$, $B \to (A \vee B)$
A4	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
A5	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
A6	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
NA1	$\sim\sim A \to A$, $A \to \sim\sim A$
NA2	$\sim(A \vee B) \to (\sim A \wedge \sim B)$, $(\sim A \wedge \sim B) \to \sim(A \vee B)$
NA3	$\sim(A \wedge B) \to (\sim A \vee \sim B)$, $(\sim A \vee \sim B) \to \sim(A \wedge B)$
R1	$A, A \to B \blacktriangleright B$
R2	$A, B \blacktriangleright A \wedge B$
R3	$A \to B, C \to D \blacktriangleright (B \to C) \to (A \to D)$
MR1	If $A \blacktriangleright B$ then $C \vee A \blacktriangleright C \vee B$

Table 5.2: Axiom system of **BD**

A proof for the logic **BD** is defined in the standard way, as a sequence of wffs each of which is either a premise, an axiom or follows from those earlier in the list by a rule of inference.

Definition 5.4 $\Gamma \vdash_{\mathbf{BD}} A$ *iff there are* $B_1, ..., B_n \in \Gamma$ *such that there is a* **BD***–proof of* A *from* $B_1, ..., B_n$.[12]

Next, consider the semantics of the logic **BD**. Let $\mathcal{L}$ be the standard language of relevant logics, with $\mathcal{S}$ the set of sentential letters, ${}^{\sim}\mathcal{S} = \{\sim A \mid A \in \mathcal{S}\}$ the set of negated sentential letters, and ${}^{\sim}\mathcal{I} = \{\sim(A \to B) \mid (A \to B) \in \mathcal{W}\}$ the set of negated implicational formulas.

A **BD**–model for the language $\mathcal{L}$ is than a 4–tuple $< g, W, R, v >$, where W is a set of worlds, $g \in W$ the base world, R a ternary relation on W, satisfying

FP0 For all $a, b \in W$: $Rgab$ iff $a = b$.

and v an assignment function such that:

AP1 v: $\mathcal{S} \times W \mapsto \{0, 1\}$.
AP2 v: ${}^{\sim}\mathcal{S} \times W \mapsto \{0, 1\}$.
AP3 v: ${}^{\sim}\mathcal{I} \times W \mapsto \{0, 1\}$.

The valuation function v_M determined by the model **M** is defined as follows:

[12] Usually, when **RL** are characterized by means of an axiom system, the definitions of a *classical* **RL**–proof and *classical* **RL**–derivability are not mentioned explicitly. However, I am convinced that, as the definitions I have given are standard, they express what is implicitly assumed.

SP0 v_M: $\mathcal{W} \times W \mapsto \{0, 1\}$.
SP1 For $A \in \mathcal{S}$: $v_M(A, a) = 1$ iff $v(A, a) = 1$.
SP2 For $A \in \mathcal{S}$: $v_M(\sim A, a) = 1$ iff $v(\sim A, a) = 1$.
SP3 $v_M(\mathfrak{a}, a) = 1$ iff $v_M(\mathfrak{a}_1, a) = 1$ and $v_M(\mathfrak{a}_2, a) = 1$.
SP4 $v_M(\mathfrak{b}, a) = 1$ iff $v_M(\mathfrak{b}_1, a) = 1$ or $v_M(\mathfrak{b}_2, a) = 1$.
SP5 $v_M(A \to B, g) = 1$ iff for all $a, b \in W$: if $Rgab$ then $v_M(A, a) = 0$ or $v_M(B, b) = 1$.
SP6 $v_M(A \to B, a) = 1$ $(a \neq g)$ iff for all $b, c \in W$: if $Rabc$ then $v_M(A, b) = 0$ or $v_M(B, c) = 1$.
NP1 $v_M(\sim(A \to B), a) = 1$ iff $v(\sim(A \to B), a) = 1$.

The definitions for semantical validity and semantical consequence are the following:

Definition 5.5 *A valuation function v_M verifies A iff $v_M(A, g) = 1$, and falsifies A iff $v_M(A, g) = 0$.*

Definition 5.6 *A valuation function v_M is a model of Γ iff it verifies all $A \in \Gamma$.*

Definition 5.7 $\Gamma \models_{\mathbf{BD}} A$ *(A is a* **BD***–consequence of Γ) iff no model of Γ falsifies A.*

Definition 5.8 $\models_{\mathbf{BD}} A$ *(A is a* **BD***–theorem) iff no* **BD***–model falsifies A.*

B. Soundness and Completeness

Although relevant logics are usually only studied for their theorems, I will prove soundness and completeness in full, thus not only for theoremhood, but also for logical consequence.

Theorem 5.2 (Soundness) *If $\Gamma \vdash_{\mathbf{BD}} A$ then $\Gamma \models_{\mathbf{BD}} A$*

Proof. Soundness for **BD** is proven in the standard way, namely by proving the semantical validity of all **BD**–axioms and –rules.

A1 Suppose $v_M(A \to A, g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, $v_M(A, a) = 1$ and $v_M(A, a) = 0$ (by **SP5** and **FP0**), which is impossible.
A2 Suppose $v_M((A \wedge B) \to A, g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(A \wedge B, a) = 1$ and (2) $v_M(A, a) = 0$ (by **SP5** and **FP0**). From (2), it follows by **SP3** that $v_M(A, a) = 1$, which contradicts (2).
A3 Suppose $v_M(A \to (A \vee B), g) = 0$. Then, for at least one $a \in W$: $Rgaa$, (1) $v_M(A, a) = 1$ and (2) $v_M(A \vee B, a) = 0$ (by **SP5** and **FP0**). From (2), it follows by **SP4** that $v_M(A, a) = 0$, which contradicts (1).

A4 Suppose $v_M((A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C)), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A \wedge (B \vee C), a) = 1$ and (2) $v_M((A \wedge B) \vee (A \wedge C), a) = 0$ (by **SP5** and **FP0**). From (1), it follows by **SP3** and **SP4** that (1a) $v_M(A, a) = 1$ and $v_M(B, a) = 1$, or that (1b) $v_M(A, a)$ =1 and $v_M(C, a) = 1$. From (2), it follows by **SP3** and **SP4** that (2a) $v_M(A, a) = 0$, (2b) $v_M(A, a) = v_M(C, a) = 0$, (2c) $v_M(A, a) = v_M(B, a) = 0$, or (2d) $v_M(B, a) = v_M(C, a) = 0$, which leads to a contradiction in all cases.

A5 Suppose $v_M(((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A \to B, a) = 1$, (2) $v_M(A \to C, a) = 1$, and (3) $v_M(A \to (B \wedge C), a) = 0$ (by **SP5**,**SP3** and **FP0**).

i. Suppose $a = g$
From (3), it follows that for at least one $b \in W$: *Rgbb* so that (3a) $v_M(A, b) = 1$ and $v_M(B, b) = 0$, or (3b) $v_M(A, b) = 1$ and $v_M(C, b) = 0$ (because of **SP5**, **FP0** and **SP3**).
From (1), together with *Rgbb*, it follows by **SP5** that $v_M(A, b) = 0$ or $v_M(B, b) = 1$, which contradicts (3a) in both cases.
From (2), together with *Rgbb*, it follows by **SP5** that $v_M(A, b) = 0$ or $v_M(C, b) = 1$, which contradicts (3b) in both cases.

ii. Suppose $a \neq g$
From (3), it follows for at least one $b, c \in W$: *Rabc*, so that (3a) $v_M(A, b) = 1$ and $v_M(B, c) = 0$, or (3b) $v_M(A, b) = 1$ and $v_M(C, c) = 0$ (because of **SP6**, **FP0** and **SP3**).
From (1), together with *Rabc*, it follows by **SP6** that $v_M(A, b) = 0$ or $v_M(B, c) = 1$, which contradicts (3a) in both cases.
From (2), together with *Rabc*, it follows by **SP6** that $v_M(A, b) = 0$ or $v_M(C, c) = 1$, which contradicts (3b) in both cases.

A6 Suppose $v_M(((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A \to C, a) = 1$, (2) $v_M(B \to C, a) = 1$, and (3) $v_M((A \vee B) \to C, a) = 0$ (by **SP5**,**SP4** and **FP0**).

i. Suppose $a = g$
From (3), it follows that for at least one $b \in W$: *Rgbb* so that (3a) $v_M(A, b) = 1$ and $v_M(C, b) = 0$, or (3b) $v_M(B, b) = 1$ and $v_M(C, b) = 0$ (because of **SP5**, **FP0** and **SP4**).
From (1), it also follows by **SP5** that $v_M(A, b) = 0$ or $v_M(C, b) = 1$, which contradicts (3a) in both cases.
From (2), it follows by **SP5** that $v_M(B, b) = 0$ or $v_M(C, b) = 1$, which contradicts (3b) in both cases.

ii. Suppose $a \neq g$
From (3), it follows that for at least one $b, c \in W$: *Rabc*, so that (3a) $v_M(A, b) = 1$ and $v_M(C, c) = 0$, or (3b) $v_M(B, b) = 1$ and

$v_M(C, c) = 0$ (because of **SP6**, **FP0** and **SP4**).
From (1), together with $Rabc$, it also follows by **SP6** that $v_M(A, b) = 0$ or $v_M(C, c) = 1$, which contradicts (3a) in both cases.
From (2), together with $Rabc$, it follows by **SP6** that $v_M(B, b) = 0$ or $v_M(C, c) = 1$, which contradicts (3b) in both cases.

NA1 $\Leftarrow$ Suppose $v(\sim\sim A \rightarrow A, g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(\sim\sim A, a) = 1$ and (2) $v_M(A, a) = 0$ (by **SP5** and **FP0**). From (1), it follows by **SP3** that $v_M(A, a) = 1$, which contradicts (2).
$\Rightarrow$ Suppose $v_M(A \rightarrow \sim\sim A, g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(A, a) = 1$ and (2) $v_M(\sim\sim A, a) = 0$ (by **SP5** and **FP0**). From (2), it follows by **SP3** that $v_M(A, a) = 0$, which contradicts (1).

NA2 $\Leftarrow$ Suppose $v_M(\sim(A \vee B) \rightarrow (\sim A \wedge \sim B), g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(\sim(A \vee B), a) = 1$ and (2) $v_M(\sim A \wedge \sim B, a) = 0$ (by **SP5** and **FP0**). From (1), it follows by **SP3** that $v_M(\sim A \wedge \sim B, a) = 1$, which contradicts (2).
$\Rightarrow$ Suppose $v_M((\sim A \wedge \sim B) \rightarrow \sim(A \vee B), g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(\sim A \wedge \sim B, a) = 1$ and (2) $v_M(\sim(A \vee B), a) = 0$ (by **SP5** and **FP0**). From (2), it follows by **SP3** that $v_M(\sim A \wedge \sim B, a) = 0$, which contradicts (1).

NA3 $\Leftarrow$ Suppose $v_M(\sim(A \wedge B) \rightarrow (\sim A \vee \sim B), g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(\sim(A \wedge B), a) = 1$ and (2) $v_M(\sim A \vee \sim B, a) = 0$ (by **SP5** and **FP0**). From (1), it follows by **SP4** that $v_M(\sim A \vee \sim B, a) = 1$, which contradicts (2).
$\Rightarrow$ Suppose $v_M((\sim A \vee \sim B) \rightarrow \sim(A \wedge B), g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(\sim A \vee \sim B, a) = 1$ and (2) $v_M(\sim(A \wedge B), a) = 0$ (by **SP5** and **FP0**). From (2), it follows by **SP4** that $v_M(\sim A \vee \sim B, a) = 0$, which contradicts (1).

R1 Suppose (1) $v_M(A, g) = 1$, (2) $v_M(A \rightarrow B, g) = 1$. From (2), together with $Rggg$ (which follows from **FP0**), it follows that $v_M(A, g) = 0$ or $v_M(B, g) = 1$ (by **SP5**). As the former is impossible (because of (1)), it follows that $v_M(B, g) = 1$.

R2 Suppose (1) $v_M(A, g) = 1$, (2) $v_M(B, g) = 1$ and (3) $v_M(A \wedge B, g) = 0$. From (3), it follows by **SP3** that $v_M(A, g) = 0$ or $v_M(B, g) = 0$, which is impossible in both cases.

R3 Suppose (1) $v_M(A \rightarrow B, g) = 1$, (2) $v_M(C \rightarrow D, g) = 1$, and (3) $v((B \rightarrow C) \rightarrow (A \rightarrow D), g) = 0$. From (3), it follows that there is at least one $a \in W$: $Rgaa$, (3a) $v_M(B \rightarrow C, a) = 1$ and (3b) $v_M(A \rightarrow D, a) = 0$ (by **SP5** and **FP0**).

i. Suppose $a = g$
From (3a) and (3b), it follows that there is at least one $b \in W$

such that $Rgbb$, (a) $v_M(A,b) = 1$ and $v_M(B,b) = v_M(D,b) = 0$, or (b) $v_M(A,b) = v_M(C,b) = 1$ and $v_M(D,b) = 0$ (by **SP5** and **FP0**).
From (1), together with $Rgbb$, it follows by **SP5** that $v_M(A,b) = 0$ or $v_M(B,b) = 1$, which contradicts (a).
From (2), together with $Rgbb$, it follows by **SP5** that $v_M(C,b) = 0$ or $v_M(D,b) = 1$, which contradicts (b).

ii. Suppose $a \neq g$
From (3a) and (3b), it follows that there is at least one $b, c \in W$ such that $Rabc$, (a) $v_M(A,b) = 1$ and $v_M(B,b) = v_M(D,c) = 0$, or (b) $v_M(A,b) = v_M(C,c) = 1$ and $v_M(D,c) = 0$ (by **SP6**).
From (1), together with $Rgbb$, it follows by **SP6** that $v_M(A,b) = 0$ or $v_M(B,b) = 1$, which contradicts (a)
From (2), together with $Rgcc$, it follows by **SP6** that $v_M(C,c) = 0$ or $v_M(D,c) = 1$, which contradicts (b).

MR1 Suppose $A \models B$ and $C \vee A \not\models C \vee B$. From these, it follows that (1) for all **B**–models M, if $v_M(A,g) = 1$ then $v_M(B,g) = 1$, and (2) there is at least one **B**–model M such that if $v_M(C \vee A, g) = 1$ then $v_M(C \vee B, g) = 0$. From (2), it follows by **SP4** that (2a) if $v_M(C,g) = 1$ then $v_M(C,g) = v_M(B,g) = 0$ (which is impossible), or (2b) if $v_M(A,g) = 1$ then $v_M(C,g) = v_M(B,g) = 0$ (which is also impossible because of (1)).

■

Theorem 5.3 (Strong Completeness) *If* $\Gamma \models_{\mathbf{BD}} A$ *then* $\Gamma \vdash_{\mathbf{BD}} A$

Completeness will be proven for **BD** by relying on the completeness proof from Priest & Sylvan [88]. First, consider the key notions:[13]

(i) If Π is a set of $\mathcal{L}$–sentences, let $\Pi_{\rightarrow}$ be the set of all members of Π of the form $A \rightarrow B$.

(ii) $\Sigma \vdash_\pi A$ iff $\Sigma \cup \Pi_{\rightarrow} \vdash A$.

(iii) Σ is a Π–*theory* iff:

   (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$,
   (b) if $\vdash_\pi A \rightarrow B$ then (if $A \in \Sigma$ then $B \in \Sigma$).

(iv) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).

(v) If X is any set of sets of formulas, the ternary relation R on X is defined thus:

$$R_{\Pi\Gamma\Delta} \text{ iff } \Gamma = \Delta.$$

[13] I have slightly altered some of the definitions in view of what is to come. Nevertheless, basically nothing changes.

$R_{\Sigma\Gamma\Delta}$ $(\Sigma \neq \Pi)$ iff (if $A \rightarrow B \in \Sigma$ then (if $A \in \Gamma$ then $B \in \Delta$)).

(vi) $\Sigma \vdash_\pi \Delta$ iff for some $B_1, ..., B_n \in \Delta$: $\Sigma \vdash_\pi B_1 \vee ... \vee B_n$.
(vii) $\vdash_\pi \Sigma \rightarrow \Delta$ iff for some $A_1, ..., A_n \in \Sigma$ and $B_1, ..., B_m \in \Delta$: $\vdash_\pi A_1 \wedge ... \wedge A_n \rightarrow B_1 \vee ... \vee B_m$.
(viii) Σ is Π–*deductively closed* iff (if $\Sigma \vdash_\pi A$ then $A \in \Sigma$).
(ix) If Θ is the set of all formulas, $< \Sigma, \Delta >$ is a Π–partition iff:

(a) $\Sigma \cup \Delta = \Theta$,
(b) $\not\vdash_\pi \Sigma \rightarrow \Delta$.

Based on the **BD**–proof theory and the foregoing definitions, Priest & Sylvan [88, pp. 222–226] have proven the following lemmas:[14]

Lemma 5.1 *If $A \vdash B$ then $C \vee A \vdash C \vee B$.*

Lemma 5.2 *If $< \Sigma, \Delta >$ is a Π–partition then Σ is a prime Π–theory.*

Lemma 5.3 *If $\not\vdash_\pi \Sigma \rightarrow \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $< \Sigma', \Delta' >$ is a Π–partition.*

Corollary 5.1 *Let Σ be a Π–theory, Δ be closed under disjunction, and $\Sigma \cap \Delta = \emptyset$. Then there is a $\Sigma' \supset \Sigma$ such that $\Sigma' \cap \Delta = \emptyset$ and Σ' is a prime Π–theory.*

Lemma 5.4 *If $\Sigma \not\vdash \Delta$ then there is $\Sigma' \supseteq \Sigma$, $\Delta' \supseteq \Delta$ such that $< \Sigma', \Delta' >$ is a partition and Σ' is deductively closed.*

Corollary 5.2 *If $\Sigma \not\vdash A$ then there is a $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, Π is a prime Π–theory and Π is Π–deductively closed.*

Lemma 5.5 *If Π is a prime Π–theory, is Π–deductively closed and $A \rightarrow B \notin \Pi$, then there is a prime Π–theory, Γ, such that $A \in \Gamma$ and $B \notin \Gamma$.*

Lemma 5.6 *If Σ, Γ, Δ are Π–theories ($\Sigma \neq \Pi$), $\mathrm{R}_{\Sigma\Gamma\Delta}$ and $A \notin \Delta$ then there are prime Π–theories, Γ', Δ', such that $\Gamma' \supseteq \Gamma$, $A \notin \Delta'$ and $\mathrm{R}_{\Sigma\Gamma'\Delta'}$.*

Lemma 5.7 *Let Σ be a prime Π–theory ($\Sigma \neq \Pi$) and $A \rightarrow B \notin \Sigma$. Then there are prime Π–theories, Γ', Δ' such that $\mathrm{R}_{\Sigma\Gamma'\Delta'}$, $A \in \Gamma'$, $B \notin \Delta'$.*

[14] In view of the slight alterations I've made in the definitions above, also here, I had to make some alterations in the extension lemmas. They are a bit more restrictive, which means that they nevertheless still remain provable. I will not give the proofs here as the changes are straightforward.

By means of the above lemmas and corollaries, I'm now able to prove completeness for the logic **BD**.

Proof. Suppose $\Theta \not\vdash A$. By corollary 5.2 there is a $\Pi \supseteq \Theta$ such that $A \notin \Pi$, Π is a prime Π–theory and Π is Π–deductively closed. I can now define the **BD**–model $M = <\Pi, X, R, v>$, where X is the set of all prime Π–theories (= the set of worlds with Π the base world), R a ternary relation on X satisfying the following constraints:

FC1 For all $\Gamma, \Delta \in X$: $R_{\Pi\Gamma\Delta}$ iff $\Gamma = \Delta$
FC2 For all $\Sigma, \Gamma, \Delta \in X$ $(\Sigma \neq \Pi)$: $R_{\Sigma\Gamma\Delta}$ iff (if $A \rightarrow B \in \Sigma$ then (if $A \in \Gamma$ then $B \in \Delta$)),

and v an assignment function defined as follows:

AC1 For all $\Sigma \in X$ and for all $A \in \mathcal{S}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.
AC2 For all $\Sigma \in X$ and for all $A \in {}^{\sim}\mathcal{S}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.
AC3 For all $\Sigma \in X$ and for all $A \in {}^{\sim}\mathcal{I}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.

Finally, we can now define a valuation function v_M based on the model M:

$(*)$ For all $\Sigma \in X$ and for all $A \in \mathcal{W}$: $v_M(A, \Sigma) = 1$ iff $A \in \Sigma$.

Finally, as $\Theta \subseteq \Pi$ and $A \notin \Pi$, it follows that for all formulas $B \in \Theta$, $v_M(B, \Pi) = 1$ and $v_M(A, \Pi) = 0$, which means that $\Theta \not\models A$. ■

Remark that I still have to prove $(*)$. In other words, I still have to prove that the function v_M really is a valuation function of the logic **BD**. This will be done by proving that the semantical clauses **SP1**–**SP6** and **NP1** are valid for v_M.

SP1 For $A \in \mathcal{S}$, $v_M(A, \Sigma) = 1$ iff $v(A, \Sigma) = 1$.

Proof. $\Rightarrow$ Suppose $v_M(A, \Sigma) = 1$ for $A \in \mathcal{S}$. Hence, it follows that $A \in \Sigma$ (by $*$), so that also $v(A, \Sigma) = 1$ (by **AP1**).
$\Leftarrow$ Suppose $v(A, \Sigma) = 1$ for $A \in \mathcal{S}$. Hence, it follows that $A \in \Sigma$ (by **AP1**), so that also $v_M(A, \Sigma) = 1$ (by $*$). ■

SP2 For $A \in \mathcal{S}$, $v_M({\sim}A, \Sigma) = 1$ iff $v({\sim}A, \Sigma) = 1$.

Proof. $\Rightarrow$ Suppose $v_M({\sim}A, \Sigma) = 1$ for $A \in \mathcal{S}$. Hence, it follows that ${\sim}A \in \Sigma$ (by $*$), so that also $v({\sim}A, \Sigma) = 1$ (by **AP2**).
$\Leftarrow$ Suppose $v({\sim}A, \Sigma) = 1$ for $A \in \mathcal{S}$. Hence, it follows that ${\sim}A \in \Sigma$ (by **AP2**), so that also $v_M({\sim}A, \Sigma) = 1$ (by $*$). ■

SP3 $v_M(\mathfrak{a}, \Sigma) = 1$ iff $v_M(\mathfrak{a}_1, \Sigma) = 1$ and $v_M(\mathfrak{a}_2, \Sigma) = 1$.

Proof. $\Rightarrow$ Suppose $v_M(\mathfrak{a}, \Sigma) = 1$. Hence, it follows that $\mathfrak{a} \in \Sigma$ (by $*$), so that also $\mathfrak{a}_1 \in \Sigma$ and $\mathfrak{a}_2 \in \Sigma$ (by **A2**, **NA1**, **NA2** and the fact that Σ is a Π–theory). It now follows that $v_M(\mathfrak{a}_1, \Sigma) = 1$ and $v_M(\mathfrak{a}_2, \Sigma) = 1$ (by $*$).
$\Leftarrow$ Suppose $v_M(\mathfrak{a}_1, \Sigma) = 1$ and $v_M(\mathfrak{a}_2, \Sigma) = 1$. Hence, it follows that $\mathfrak{a}_1, \mathfrak{a}_2 \in \Sigma$ (by $*$), so that also $\mathfrak{a} \in \Sigma$ (by **R2**, **NA1**, **NA2** and the fact that Σ is a Π–theory). It now follows that $v_M(\mathfrak{a}, \Sigma) = 1$ (by $*$). ■

SP4 $v_M(\mathfrak{b}, \Sigma) = 1$ iff $v_M(\mathfrak{b}_1, \Sigma) = 1$ or $v_M(\mathfrak{b}_2, \Sigma) = 1$.

Proof. $\Rightarrow$ Suppose $v_M(\mathfrak{b}, \Sigma) = 1$. Hence, it follows that $\mathfrak{b} \in \Sigma$ (by $*$), so that also $\mathfrak{b}_1 \in \Sigma$ or $\mathfrak{b}_2 \in \Sigma$ (by **NA3** and the fact that Σ is a prime Π–theory). It now follows that $v_M(\mathfrak{b}_1, \Sigma) = 1$ or $v_M(\mathfrak{b}_2, \Sigma) = 1$ (by $*$).
$\Leftarrow$ Suppose $v_M(\mathfrak{b}_1, \Sigma) = 1$ or $v_M(\mathfrak{b}_2, \Sigma) = 1$. Hence, it follows that $\mathfrak{b}_1 \in \Sigma$ or $\mathfrak{b}_2 \in \Sigma$ (by $*$), so that also $\mathfrak{b} \in \Sigma$ (by **A3**, **NA3** and the fact that Σ is a Π–theory). It now follows that $v_M(\mathfrak{b}, \Sigma) = 1$ (by $*$). ■

SP5 $v_M(A \to B, \Pi) = 1$ iff for all $\Gamma, \Delta \in X$: if $R_{\Pi\Gamma\Delta}$ then $v_M(A, \Gamma) = 0$ or $v_M(B, \Delta) = 1$.

Proof. $\Rightarrow$ Suppose (1) $v_M(A \to B, \Pi) = 1$ and suppose (2) that there is a $\Gamma, \Delta \in X$ such that (2a) $R_{\Pi\Gamma\Delta}$, (2b) $v_M(A, \Gamma) = 1$ and (2c) $v_M(B, \Delta) = 0$. From (1), it follows that $A \to B \in \Pi$ (by $*$), and, from (2a), it follows that $\Gamma = \Delta$ (by **FP0**). From the foregoing, it follows that (if $A \in \Gamma$ then $B \in \Delta$) (as Γ is a Π–theory), or, which comes to the same (because of $*$): (if $v_M(A, \Gamma) = 1$ then $v_M(B, \Delta)$ $= 1$). From the latter, together with (2c), it follows that $v_M(A, \Gamma)$ $= 0$, which contradicts (2b).
$\Leftarrow$ Suppose that (1) for all $\Gamma, \Delta \in X$: if $R_{\Pi\Gamma\Delta}$ then $v_M(A, \Gamma) = 0$ or $v_M(B, \Delta) = 1$, and (2) $v_M(A \to B, \Pi) = 0$. From (2), it follows that $A \to B \notin \Pi$ (by $*$), so that there are prime Π–theories Γ' and Δ' for which (2a) $R_{\Pi\Gamma'\Delta'}$, (2b) $A \in \Gamma'$ and (2c) $B \notin \Delta'$ (by lemma 5.5). From (2a), together with (1) and ($*$), it follows that (3a) $A \notin \Gamma'$ or (3b) $B \in \Delta'$, which contradict respectively (2b) and (2c). ■

SP6 $v_M(A \to B, \Sigma) = 1$ ($\Sigma \neq \Pi$) iff for all $\Gamma, \Delta \in X$: if $R_{\Sigma\Gamma\Delta}$ then $v_M(A, \Gamma) = 0$ or $v_M(B, \Delta) = 1$.

Proof. $\Rightarrow$ Suppose (1) $v_M(A \to B, \Sigma) = 1$ ($\Sigma \neq \Pi$) and suppose (2)

that there is a $\Gamma, \Delta \in X$ such that (2a) $R_{\Sigma\Gamma\Delta}$, (2b) $v_M(A, \Gamma) = 1$ and (2c) $v_M(B, \Delta) = 0$. From (1), it follows that $A \rightarrow B \in \Sigma$ (by $*$), and, from (2a), it follows that (if $A \rightarrow B \in \Sigma$ then (if $A \in \Gamma$ then $B \in \Delta$)). From these, it follows that (if $A \in \Gamma$ then $B \in \Delta$), or which comes to the same (because of $*$): (if $v_M(A, \Gamma) = 1$ then $v_M(B, \Delta) = 1$). From the latter, together with (2c), it follows that $v_M(A, \Gamma) = 0$, which contradicts (2b).
$\Leftarrow$ Suppose that (1) for all $\Gamma, \Delta \in X$: if $R_{\Sigma\Gamma\Delta}$ then $v_M(A, \Gamma) = 0$ or $v_M(B, \Delta) = 1$, and (2) $v_M(A \rightarrow B, \Sigma) = 0$ ($\Sigma \neq \Pi$). From (2), it follows that $A \rightarrow B \notin \Sigma$ (by $*$). So that there are prime Π–theories Γ' and Δ' such that (2a) $R_{\Sigma\Gamma'\Delta'}$, (2b) $A \in \Gamma'$ and (2c) $B \notin \Delta'$ (by lemma 5.7). From (2a), together with (1) and ($*$), it follows that (3a) $A \notin \Gamma'$ or (3b) $B \in \Delta'$, which contradict respectively (2b) and (2c). ■

NP1 $v_M({\sim}(A \rightarrow B), \Sigma) = 1$ iff $v({\sim}(A \rightarrow B), \Sigma) = 1$.

Proof. $\Rightarrow$ Suppose $v_M({\sim}(A \rightarrow B), \Sigma) = 1$. Hence, it follows that ${\sim}(A \rightarrow B) \in \Sigma$ (by $*$), so that also $v({\sim}(A \rightarrow B), \Sigma) = 1$ (by **AP3**).
$\Leftarrow$ Suppose $v({\sim}(A \rightarrow B), \Sigma) = 1$. Hence, it follows that ${\sim}(A \rightarrow B) \in \Sigma$ (by **AP3**), so that also $v_M({\sim}(A \rightarrow B), \Sigma) = 1$ (by $*$). ■

Now that both soundness and completeness have been proven, the following corollary follows immediately:

Corollary 5.3 $\Gamma \vdash_{\mathbf{BD}} A$ *iff* $\Gamma \models_{\mathbf{BD}} A$.

5.4.3 Relevant Logics Extending BD

Being the basic relevant logic, **BD** can be strengthened in numerous ways in order to obtain stronger relevant logics. In this section, I will show how this can be done, both proof theoretically and semantically.

Proof Theory. Proof theoretically, the basic relevant logic **BD** is strengthened by adding certain axioms and/or rules to its axiom system (see section 5.4.2). Consider for example the axioms and rules that are stated in table 5.3.[15]

[15]The axioms and rules presented in table 5.3 are obtained from Restall [93], Restall & Roy [95] and Brady [42]. More axioms and rules can be found in Routley et al. [101].

A7	$(A \to B) \to ((B \to C) \to (A \to C))$
A8	$(A \to B) \to ((C \to B) \to (C \to A))$
A9	$(A \to (A \to B)) \to (A \to B)$
A10	$A \to ((A \to B) \to B)$
A11	$((A \to B) \wedge (B \to C)) \to (A \to C)$
A12	$(A \wedge (A \to B)) \to B$
A13	$(A \to (B \to C)) \to (B \to (A \to C))$
A14	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
A15	$(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
A16	$A \to (B \to B)$
A17	$B \to (A \to B)$
A18	$A \to (B \to (C \to A))$
A19	$A \to (B \to (A \wedge B))$
A20	$((A \wedge B) \to C) \to (A \to (B \to C))$
A21	$(A \to B) \vee (B \to A)$
A22	$A \to (A \to A)$
A23	$((A \wedge B) \to C) \to ((A \to C) \vee (B \to C))$
R4	$A \blacktriangleright (A \to B) \to B$
NA4	$A \vee {\sim}A$
NA5	$({\sim}A \wedge (A \vee B)) \to B$
NA6	$(A \wedge {\sim}B) \to {\sim}(A \to B)$
NR2	${\sim}A, A \vee B \blacktriangleright B$
NR3	$A, {\sim}B \blacktriangleright {\sim}(A \to B)$
NA6	$(A \to {\sim}B) \to (B \to {\sim}A)$
NA7	$(A \to {\sim}A) \to {\sim}A$
NRP1	$A \to B \blacktriangleright {\sim}B \to {\sim}A$

Table 5.3: Axioms and rules for relevant logics

For all **RL** extending **BD** by adding some of the axioms and/or rules from table 5.3, the proof theoretic characterization is equivalent to the one for **BD**. First of all, an **RL**–proof is a sequence of wffs each of which is either a premise, an axiom or follows from those earlier in the list by a rule of inference. Secondly, **RL**–derivability is defined as follows:

Definition 5.9 $\Gamma \vdash_{\mathbf{RL}} A$ *iff there are* $B_1, ..., B_n \in \Gamma$ *such that there is a* **RL**–*proof of* A *from* $B_1, ..., B_n$.

Semantics. The semantic characterization of the logics extending **B** in the above specified way, is quite standard. At least, for all axioms and rules of table 5.3, with exception of the last three of them (**NA6**,**NA7**,**NRP1**). In this section, I will only consider the standard extensions. They are semantically obtained by putting certain constraints upon the ternary accessibility relation R, or by some extra semantic clause. Table 5.4 gives an overview of those constraints and clauses that correspond to the axioms and rules from table 5.3. In order to understand all of them, first consider the following definitions:

Definition 5.10 $\mathrm{R}^2abcd = (\exists x)(\mathrm{R}abx \wedge \mathrm{R}xcd)$

Definition 5.11 $\mathrm{R}^2a(bc)d = (\exists x)(\mathrm{R}bcx \wedge \mathrm{R}axd)$

Definition 5.12 $\mathrm{R}^3ab(cd)e = (\exists x)(\mathrm{R}^2abxe \wedge \mathrm{R}cdx)$

Definition 5.13 *If* $a \leqslant b$ *then:*

- *if* $v(A,a) = 1$ *then* $v(A,b) = 1$,
- *if* $a \neq g$ *and* $\mathrm{R}bcd$ *then* $\mathrm{R}acd$, *and*
- *if* $a = g$ *and* $\mathrm{R}bcd$ *then* $c \leqslant d$.

SP7	For all $a,b,c,d \in W$: if R^2abcd then $R^2b(ac)d$.
SP8	For all $a,b,c,d \in W$: if R^2abcd then $R^2a(bc)d$.
SP9	For all $a,b,c \in W$: if $Rabc$ then R^2abbc.
SP10	For all $a,b,c \in W$: if $Rabc$ then $\exists x \in W$: $a \leqslant x$ and $Rbxc$.
SP11	For all $a,b,c \in W$: if $Rabc$ then $R^2a(ab)c$.
SP12	For all $a \in W$: $Raaa$.
SP13	For all $a,b,c,d \in W$: if R^2abcd then $\exists x,y \in W$: $b \leqslant x$, $c \leqslant y$ and R^2ayxd.
SP14	For all $a,b,c,d \in W$: if R^2abcd then $R^3ac(bc)d$.
SP15	For all $a,b,c,d \in W$: if R^2abcd then $R^3bc(ac)d$.
SP16	For all $a,b,c \in W$: if $Rabc$ then $b \leqslant c$.
SP17	For all $a,b,c,d \in W$: if $Rabc$ then $a \leqslant c$.
SP18	For all $a,b,c,d \in W$: if R^2abcd then $a \leqslant d$.
SP19	For all $a,b,c \in W$: if $Rabc$ then $a \leqslant c$ and $b \leqslant c$.
SP20	For all $a,b,c,d \in W$: if R^2abcd then $\exists x \in W$, $b \leqslant x$, $c \leqslant x$, and $Raxd$.
SP21	For all $a,b \in W$: $a \leqslant b$ or $b \leqslant a$.
SP22	For all $a,b,c \in W$: if $Rabc$ then $a \leqslant b$ or $b \leqslant c$.
SP23	For all $a,b,c,d,e \in W$: if $Rabc$ and $Rade$ then $\exists x \in W$: $b \leqslant x$, $d \leqslant x$, and ($Raxc$ or $Raxe$).
RP4	For all $a \in W$: $Raga$.
NP4	$v_M({\sim}A,g) = 1$ iff $v_M(A,g) = 1$ or $v({\sim}A,g) = 1$.
NP5	$v_M({\sim}A,a) = 1$ iff $v_M(A,a) = 0$ and $v({\sim}A,a) = 1$.
NP6	$v_M({\sim}(A \to B),a) = 1$ iff $v_M(A,a) = 1$ and $v_M({\sim}B,a) = 1$, or $v({\sim}(A \to B),a) = 1$.
NRP2	$v_M({\sim}A,g) = 1$ iff $v_M(A,g) = 0$ and $v({\sim}A,g) = 1$.
NRP3	$v_M({\sim}(A \to B),g) = 1$ iff $v_M(A,g) = 1$ and $v_M({\sim}B,g) = 1$, or $v({\sim}(A \to B),g) = 1$.

Table 5.4: Semantic Postulates corresponding to axioms and rules

The soundness and completeness proofs for the straightforward extensions of the logic **BD** are easy adaptations of the proofs for **BD**. I will not give them here, as most of them are given in Priest & Sylvan [88], Restall [93] and Restall & Roy [95], namely those for extensions based on **SP7**–**SP23** and **RP4**. The others are straightforward in view of chapter 4, and are left to the reader.

Some Known Relevant Logics. Some of the extensions of the logic **BD** are better known than others. In table 5.5, an overview is given of some of the better known ones. Remark that the axiom systems of most of them include **NRP1** or **NA6**, which means that their semantics cannot be characterized by simply adding the corresponding semantic clauses from table 5.4 to the semantics of **BD**. In section 5.4.4, I will however show how to characterize one of them, namely the logic **R**.

BM	**BD** - NA1 + NR1
B	**BD** + NR1
BX	**B** + NA6
BC	**BX** + NR3
DW	**B** + NA4
DWX	**DW** + NA6
DWC	**DWX** + NR2
DJ	**DW** + A11
DK	**DJ** + NA6
DL	**DK** + NA5 - NA6
D	**DL** + A9 + A12
DC	**D** + NR2
RBC	**DC** + A10
TW	**DW** + A7 + A8 - R3
TWX	**TW** + NA6
TWC	**TWX** + NR2
RW	**TW** + A10 - A8
RWK	**RW** + A17
EW	**TW** + R4
T	**TW** + A9 + NA5
E	**T** + R4
R	**T** + A10

Table 5.5: Some relevant logics

5.4.4 The relevant logic R

In this section, I will give a semantic characterization of the logic **R**. There are two reasons for doing so. The first one is that this will prove that the non–truthfunctional approach can also characterize relevant logics that have **NA6**, **NA7** and **NRP1** in their axiom system. The second one is that the adaptive logics that I will present in chapters 10 and 11 are all based on the relevant logic **R**.

A. Proof Theory

First, consider the proof theory of the logic **R**. It is the axiom system given in table 5.6 below.[16]

A1	$A \to A$
A2	$(A \wedge B) \to A$, $(A \wedge B) \to B$
A3	$A \to (A \vee B)$, $B \to (A \vee B)$
A4	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
A5	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
A6	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
A7	$(A \to B) \to ((B \to C) \to (A \to C))$
A8	$(A \to B) \to ((C \to A) \to (C \to B))$
A9	$(A \to (A \to B)) \to (A \to B)$
A10	$A \to ((A \to B) \to B)$
NA1	$\sim\sim A \to A$, $A \to \sim\sim A$
NA2	$\sim(A \vee B) \to (\sim A \wedge \sim B)$, $(\sim A \wedge \sim B) \to \sim(A \vee B)$
NA3	$\sim(A \wedge B) \to (\sim A \vee \sim B)$, $(\sim A \vee \sim B) \to \sim(A \wedge B)$
NA6	$(A \to B) \to (\sim B \to \sim A)$
NA7	$(A \to \sim A) \to \sim A$
R1	$A, A \to B \blacktriangleright B$
R2	$A, B \blacktriangleright A \wedge B$

Table 5.6: Axiom system of **R**

The definition of an **R**–proof and of **R**–derivability are as for all relevant logics, and will not be repeated here.

B. Semantics

The semantics of the relevant logic **R** that I will present below, is a rather unusual kind of semantics. It is an adaptive logics–semantics,[17] which means that the set of **R**–models will be a specific subset of some set $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$ of models of the premise set (the set of **LLL**–models, see chapter 3). I will call the models that belong to $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$, the $\mathbf{R}^{\mathbf{LLL}}$–models of the premise set Γ, and I will characterize them first.

Let $\mathcal{L}$ be the standard language of relevant logics, with $\mathcal{S}$ the set of sentential letters, ${}^{\sim}\mathcal{S} = \{\sim A \mid A \in \mathcal{S}\}$ the set of negated sentential letters, and ${}^{\sim}\mathcal{I} = \{\sim(A \to B) \mid A, B \in \mathcal{W}\}$ the set of negated implicational formulas.

[16] Remark that some of the axioms are superfluous.

[17] By now, I have also found a way to characterize the **R**–semantics in a non–adaptive way. It is done by introducing a second a new kind of relation between world $\cong$, which in a sense resembles $\leqslant$ and which is used to put some extra constraints on the accessibility relation. However, I did not find the time yet to prove completeness, so I have left it out of this dissertation.

An $\mathbf{R^{LLL}}$–model for the language $\mathcal{L}$ is a 5–tuple $< g, W, R, \leqslant, v >$, where W is a set of worlds, with $g \in W$ the base world, R a ternary relation on W, satisfying

FP0 For all $a, b \in W$: $Rgab$ iff $a = b$,
FP7 For all $a, b, c, d \in W$: if R^2abcd then $R^2b(ac)d$.
FP8 For all $a, b, c, d \in W$: if R^2abcd then $R^2a(bc)d$.
FP9 For all $a, b, c \in W$: if $Rabc$ then R^2abbc.
FP10 For all $a, b, c \in W$: if $Rabc$ then $\exists x \in W$: $a \leqslant x$, $x \leqslant a$ and $Rbxc$.

$\leqslant$ a reflexive and transitive binary relation on W (usually called a containment relation), satisfying:

C1 For all $a, b \in W$ such that $a \leqslant b$: if $v(A, a) = 1$ then $v(A, b) = 1$.
C2 For all $a, b, c, d \in W$ $(a \neq g)$ such that $a \leqslant b$: if $Rbcd$ then $Racd$.
C3 For all $a, b, c \in W$ such that $g \leqslant a$: if $Rabc$ then $b \leqslant c$.

and v an assignment function such that:

AP1 v: $\mathcal{S} \times W \mapsto \{0, 1\}$.
AP2 v: ${}^{\sim}\mathcal{S} \times W \mapsto \{0, 1\}$.
AP3 v: ${}^{\sim}\mathcal{I} \times W \mapsto \{0, 1\}$.

The valuation function v_M based on the interpretation $\mathbf{M}$ is characterized as follows:

SP0 v_M: $\mathcal{W} \times W \mapsto \{0, 1\}$.
SP1 For $A \in \mathcal{S}$: $v_M(A, a) = 1$ iff $v(A, a) = 1$.
SP2 For $A \in \mathcal{S}$: $v_M({\sim}A, a) = 1$ $(a \neq g)$ iff $v({\sim}A, a) = 1$.
SP3 $v_M(\mathfrak{a}, a) = 1$ iff $v_M(\mathfrak{a}_1, a) = 1$ and $v_M(\mathfrak{a}_2, a) = 1$.
SP4 $v_M(\mathfrak{b}, a) = 1$ iff $v_M(\mathfrak{b}_1, a) = 1$ or $v_M(\mathfrak{b}_2, a) = 1$.
SP5 $v_M(A \to B, a) = 1$ iff for all $b, c \in W$: if $Rabc$ then $v_M(A, b) = 0$ or $v_M(B, c) = 1$.
NP1 $v_M({\sim}(A \to B), a) = 1$ iff $v({\sim}(A \to B), a) = 1$.

As usual, a valuation function is called a model if it verifies all elements of the premise set Γ.

Definition 5.14 *A valuation function v_M verifies A iff $v_M(A, g) = 1$, and falsifies A iff $v_M(A, g) = 0$.*

Definition 5.15 *A valuation function v_M is an $\mathbf{R^{LLL}}$–model of Γ iff it verifies all $A \in \Gamma$.*

Now, in order to distinguish the actual $\mathbf{R}$–models from the $\mathbf{R^{LLL}}$–models, it is necessary to define a set of abnormalities Ω. In this case, Ω is the union of the following two sets:

a) $\Omega_1 = \{(A \to {\sim}B) \to (B \to {\sim}A) \mid A, B \in \mathcal{W}\}$
b) $\Omega_2 = \{(A \to {\sim}A) \to {\sim}A \mid A \in \mathcal{W}\}$

Remark that the abnormalities are the instances of the axiom schemas **NA6** and **NA7**. But, in the logic **R**, those instances are all true! In fact, this means that only those $\mathbf{R}^{\mathbf{LLL}}$–models of a premise that verify all elements of Ω can be considered as **R**–models of that premise set. Hence, the abnormal part of an $\mathbf{R}^{\mathbf{LLL}}$–model should be defined not by reference to which abnormalities it verifies, but to which it falsifies.

Definition 5.16 *For each* $\mathbf{R}^{\mathbf{LLL}}$*–model* M*,* $Ab(M) = \{A \in \Omega \mid M \not\models A\}$*.*

As a consequence, an **R**–model of a premise set can now be defined as an $\mathbf{R}^{\mathbf{LLL}}$–model with an empty set of abnormalities.[18]

Definition 5.17 *An* $\mathbf{R}^{\mathbf{LLL}}$*–model* M *of* Γ *is an* **R***–model of* Γ *iff* $Ab(M) = \emptyset$*.*

Finally, semantic consequence is defined by relying on the **R**–models of the premise set.

Definition 5.18 $\Gamma \models_{\mathbf{R}} A$ *iff* A *is verified by all* **R***–models of* Γ*.*

Hereditariness Lemma. Remark that the so–called *hereditariness lemma* can be proven by an easy induction over the complexity of formulas (see e.g. Restall [93, p. 498]).

Lemma 5.8 *For all* $A \in \mathcal{W}$*, if* $a \leqslant b$ *and* $v_M(A, a) = 1$ *then* $v_M(A, b) = 1$*.*

Soundness and Completeness for R. The soundness and completeness proofs for **R** are very resemblant to the ones for **BD**. As a consequence, the proofs will not be given in full detail. Only the extra elements will be given.

Theorem 5.4 (Soundness) *If* $\Gamma \vdash_{\mathbf{R}} A$ *then* $\Gamma \models_{\mathbf{R}} A$*.*

Proof. Soundness is proven for **R** by proving the semantical validity of all its axioms and rules.

A7 Suppose $v_M((A \to B) \to ((B \to C) \to (A \to C)), g) = 0$. Hence, for at least one $a \in W$: $Rgaa$, (1) $v_M(A \to B, a) = 1$, and (2) $v_M((B \to C) \to (A \to C), a) = 0$ (by **SP5** and **FP0**).
From (2), it follows by **SP5** that there is at least one $b, c \in W$: $Rabc$, (2a) $v_M(B \to C, b) = 1$ and (2b) $v_M(A \to C, c) = 0$.

[18] In adaptive logics–terminology, this strategy is called the *Blindness Strategy*, as it remains blind to any abnormalities derivable from the premise set (see Batens [14]).

From (2b), it follows by **SP5** that there is at least one $d, e \in W$: *Rcde*, (2b$_1$) $v_M(A, d) = 1$ and (2b$_2$) $v_M(C, e) = 0$.
From *Rabc* and *Rcde*, it follows that there is at least one $f \in W$ such that *Radf* and $Rbfe$ (by **FP7**). It now also follows that $v_M(A, d) = 0$ or $v_M(B, f) = 1$, and $v_M(B, f) = 0$ or $v_M(C, e) = 1$ (by (1),(2a), and **SP5**), which gives a contradiction in all cases.

A8 Suppose $v_M((A \to B) \to ((C \to A) \to (C \to B)), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A \to B, a) = 1$, and (2) $v_M((C \to A) \to (C \to B), a) = 0$ (by **SP5** and **FP0**).
From (2), it follows by **SP5** that there is at least one $b, c \in W$: *Rabc*, (2a) $v_M(C \to A, b) = 1$ and (2b) $v_M(C \to B, c) = 0$.
From (2b), it follows by **SP5** that there is at least one $d, e \in W$: *Rcde*, (2b$_1$) $v_M(C, d) = 1$ and (2b$_2$) $v_M(B, e) = 0$.
From *Rabc* and $Rdef$, it follows that there is at least one $f \in W$ such that *Rbdf* and $Rafe$ (by **FP8**). It now also follows that $v_M(A, f) = 0$ or $v_M(B, e) = 1$, and $v_M(C, d) = 0$ or $v_M(A, f) = 1$ (by (1), (2a), and **SP5**), which gives a contradiction in all cases.

A9 Suppose $v_M((A \to (A \to B)) \to (A \to B), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A \to (A \to B), a) = 1$ and (2) $v_M(A \to B, a) = 0$ (by **SP5** and **FP0**).
From (2), it follows by **SP5** that there is at least one $b, c \in W$: *Rabc*, (2a) $v_M(A, b) = 1$ and (2b) $v_M(B, c) = 0$.
From *Rabc*, it follows that there is at least one $d \in W$ such that *Rabd* and *Rdbc* (by **FP9**). It now also follows that (1a) $v_M(A, b) = 0$ or (1b) $v_M(A \to B, d) = 1$ (by (1) and **SP5**). As (1a) is impossible because of (2a), only (1b) remains.
From (1b), it follows that $v_M(A, b) = 0$ or $v_M(B, c) = 1$ (by **SP5**), which contradict respectively (2a) and (2b).

A10 Suppose $v_M(A \to ((A \to B) \to B), g) = 0$. Hence, for at least one $a \in W$: *Rgaa*, (1) $v_M(A, a) = 1$ and (2) $v_M((A \to B) \to B, a) = 0$ (by **SP5** and **FP0**).
From (2), it follows by **SP5** that there is at least one $b, c \in W$: *Rabc*, (2a) $v_M(A \to B, b) = 1$ and (2b) $v_M(B, c) = 0$.
From *Rabc*, it follows that there is at least one $d \in W$ such that $a \leqslant d$ and *Rbdc* (by **FP10**). It now follows that $v_M(A, d) = 1$ and $v_M(A, d) = 0$ or $v_M(B, c) = 1$ (by (1), (2a), **C1** and **SP5**), which gives a contradiction in all cases.

NA6 Suppose $v_M((A \to {\sim}B) \to (B \to {\sim}A, g) = 0$. This is impossible, as it would mean that $Ab(M) \neq \emptyset$.

NA7 Suppose $v_M((A \to {\sim}A) \to {\sim}A, g) = 0$. This is impossible, as it would mean that $Ab(M) \neq \emptyset$.

■

Theorem 5.5 (Strong Completeness) *If* $\Gamma \models_{\mathbf{R}} A$ *then* $\Gamma \vdash_{\mathbf{R}} A$.

In order to prove completeness for **R**, I will rely on the completeness proof given in Restall [93] and Restall & Roy [95].[19]

Proof. Suppose $\Theta \not\vdash A$. By corollary 5.2 there is a $\Pi \supseteq \Theta$ such that $A \notin \Pi$, Π is a prime Π–theory and Π is Π–deductively closed. We can now define the $\mathbf{R}^{\mathbf{LLL}}$–model $M = <\Pi, X, R, \subseteq, v>$, where X is the set of all prime Π–theories (= the set of worlds with Π the base world), R a ternary relation on X satisfying the following constraints:

FC0 For all $\Gamma, \Delta \in X$: $R_{\Pi\Gamma\Delta}$ iff $\Gamma = \Delta$
FC1 For all $\Sigma, \Gamma, \Delta \in X$ ($\Sigma \neq \Pi$): $R_{\Sigma\Gamma\Delta}$ iff (if $A \to B \in \Sigma$ then (if $A \in \Gamma$ then $B \in \Delta$)),

$\subseteq$ a subset relation between elements of X (corresponding to the containment relation in the semantics), and v an assignment function defined as follows:

AC1 For all $\Sigma \in X$ and all $A \in \mathcal{S}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.
AC2 For all $\Sigma \in X$ and all $A \in {}^{\sim}\mathcal{S}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.
AC3 For all $\Sigma \in X$ and all $A \in {}^{\sim}\mathcal{I}$: $v(A, \Sigma) = 1$ iff $A \in \Sigma$.

A remark has to be made concerning the set of all prime Π–theories. It not only contains the theory Π (corresponding to the base world), but also Π_p which has the following characteristics:[20]

PW1 For all $\Gamma \in X$: $\Pi \subseteq \Gamma$ iff $\Pi_p \subseteq \Gamma$,
PW2 For all $\Gamma \in X$: $\Gamma \subseteq \Pi$ iff $\Gamma \subseteq \Pi_p$,

Finally, we can now define a valuation function v_M, based on the model M:

(∗) For all $\Sigma \in X$ and for all $A \in \mathcal{W}$: $v_M(A, \Sigma) = 1$ iff $A \in \Sigma$.

As $\Theta \subseteq \Pi$ and $A \notin \Pi$, it follows that for all formula $B \in \Theta$, $v_M(B, \Pi) = 1$ and $v_M(A, \Pi) = 0$. Moreover, as Π is Π–deductively closed, for all $C \in \Omega$, $C \in \Phi$. Hence, for all $C \in \Omega$, $v_M(C, \Pi) = 1$, so that $Ab(v_M) = \emptyset$. As a consequence, v_M is an **R**–model, which means that $\Theta \not\models A$. ■

Remark that I still have to prove that (1) the valuation function v_M really is a valuation function of the logic **R**, (2) the subset relation $\subseteq$ really

[19] It is interesting to remark that Tony Roy has found a mistake in Restall's completeness proof of [93]. It is a mistake that only has effect on rich relevant logics, e.g. the logic **R**. For them, Restall's semantics overgenerate. A solution has been proposed in Restall & Roy [95].

[20] Restall has introduced such a theory in [93]. It corresponds to a world which is different from the base world, but which has the same elements. Restall called it the pseudo–base world.

corresponds to the containment relation $\leqslant$, and (3) the ternary relation R satisfies the **R**–conditions **FP7**–**FP10**.

First of all, I need to prove that the valuation function v_M really is a valuation function of the logic **R**. This can be done by proving that it has the features an **R**–valuation function should have. However, as **SP1**–**SP5** and **NP1** remain as for **BD**, I consider this done.

Next, I will show that $\subseteq$ validates the conditions **C1**–**C3**. This suffices to prove the correspondence between $\subseteq$ and $\leqslant$, as $\subseteq$ is both reflexive and transitive.

C1 For all $\Sigma, \Gamma \in X$ such that $\Sigma \subseteq \Gamma$: if $v(A, \Sigma) = 1$ then $v(A, \Gamma) = 1$.

Proof. Suppose (1) $\Sigma \subseteq \Gamma$, (2) $v_M(A, \Sigma) = 1$ and (3) $v_M(A, \Gamma) = 0$. From (2) and (3), it follows by $(*)$ that $A \in \Sigma$ and $A \notin \Gamma$, which contradicts (1). ■

C2 For all $\Theta, \Sigma, \Gamma, \Delta \in X$ such that $\Theta \subseteq \Sigma$ $(\Theta \neq \Pi)$: if $R_{\Sigma\Gamma\Delta}$ then $R_{\Theta\Gamma\Delta}$.

Proof. Suppose (1) $\Theta \subseteq \Sigma$ $(\Theta \neq \Pi)$ and (2) $R_{\Sigma\Gamma\Delta}$. In order to show that $R_{\Theta\Gamma\Delta}$, consider an arbitrary formula $A \to B \in \Theta$ and $A \in \Gamma$. From the former, together with (1), it follows that $A \to B \in \Sigma$, which gives us $B \in \Delta$ because of (2) and $A \in \Gamma$. ■

C3 For all $\Sigma, \Gamma, \Delta \in X$ such that $\Pi \subseteq \Sigma$: if $R_{\Sigma\Gamma\Delta}$ then $\Gamma \subseteq \Delta$.

Proof. Suppose (1) $\Pi \subseteq \Sigma$ and (2) $R_{\Sigma\Gamma\Delta}$. From (1), it follows that all formulas of the form $A \to A \in \Sigma$, which gives us, together with (2) the certainty that all formulas that are in Γ are also in Δ. Differently put, $\Gamma \subseteq \Delta$. ■

Finally, I still have to prove that the relation R satisfies the constraints that are put upon it. As **FP0** follows immediately from the construction above, I only need to prove **FP7**–**FP10**. In order to do so, I will make use of the following priming lemmas:

Lemma 5.9 *In [93], Restall has proven the following priming lemmas:*[21]

1. *If* Σ, Γ, Δ *are* Π*–theories (*$\Sigma \neq \Pi$*), such that* $\mathrm{R}_{\Sigma\Gamma\Delta}$ *and* Δ *is prime, then there is a prime* $\Gamma' \supseteq \Gamma$ *where* $\mathrm{R}_{\Sigma\Gamma'\Delta}$*.*
2. *If* Σ, Γ, Δ *are* Π*–theories (*$\Sigma \neq \Pi$*), such that* $\mathrm{R}_{\Sigma\Gamma\Delta}$ *and* Δ *is prime, then there is a prime* $\Sigma' \supseteq \Sigma$ *where* $\mathrm{R}_{\Sigma'\Gamma\Delta}$*.*

[21] Restall needed to prove these lemmas because his completeness proof doesn't make use of the canonical model, but of (as he calls it) the quasi–canonical model. It only differs from the canonical model by the presence of the pseudo–base world.

3.. If Σ, Γ, Δ are Π–theories ($\Sigma \neq \Pi$), such that $\mathrm{R}_{\Sigma\Gamma\Delta}$ *and Δ is prime, then there are prime $\Sigma' \supseteq \Sigma$ and $\Gamma' \supseteq \Gamma$ where* $\mathrm{R}_{\Sigma'\Gamma'\Delta}$.

4. If Σ, Γ, Δ are Π–theories ($\Sigma \neq \Pi$), such that $\mathrm{R}_{\Sigma\Gamma\Delta}$ *and $A \notin \Delta$, then there are prime Π–theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ where* $\mathrm{R}_{\Sigma\Gamma'\Delta'}$ *and $A \notin \Delta'$.*

FP7 For $\Sigma, \Gamma, \Xi, \Theta$ and Δ arbitrary prime Π–theories: if $R_{\Sigma\Gamma\Xi}$ and $R_{\Xi\Theta\Delta}$, then there is a prime Π–theory Ω' such that $R_{\Sigma\Theta\Omega'}$ and $R_{\Gamma\Omega'\Delta}$.

Proof. Suppose (1) $R_{\Sigma\Gamma\Xi}$ and (2) $R_{\Xi\Theta\Delta}$:

(i) $\Gamma \neq \Pi$ and $\Sigma \neq \Pi$
First, construct Ω as follows: $\{B \mid (\exists A) A \rightarrow B \in \Sigma$ and $A \in \Theta\}$. This is a Π–theory and $R_{\Sigma\Theta\Omega}$ follows from the construction.
Secondly, in order to show that also $R_{\Gamma\Omega\Delta}$ is the case, consider an arbitrary $A \rightarrow B \in \Gamma$ and $A \in \Omega$. It follows that there is a formula C such that $C \rightarrow A \in \Sigma$ and $C \in \Theta$ (by the construction of Ω). Because of **A7** and the fact that Σ is a Π–theory, it follows that $(A \rightarrow B) \rightarrow (C \rightarrow B) \in \Sigma$. From this, together with $R_{\Sigma\Gamma\Xi}$, it follows that $C \rightarrow B \in \Xi$, so that $R_{\Xi\Theta\Delta}$ gives us $B \in \Delta$. From this, it follows that $R_{\Gamma\Omega\Delta}$.
Finally, from the foregoing, together with lemma 5.9, it follows that there is a prime Π–theory $\Omega' \supseteq \Omega$ such that $R_{\Gamma\Omega'\Delta}$ and $R_{\Sigma\Theta\Omega'}$.

(ii) $\Sigma = \Pi$ and $\Gamma \neq \Pi$
First, set $\Omega = \Theta$. From this, it follows immediately that $R_{\Sigma\Theta\Omega}$.
Secondly, as $\Gamma = \Xi$ (because of (1) and **FP0**), $R_{\Gamma\Omega\Delta}$ follows immediately from (2).

(iii) $\Gamma = \Pi$ and $\Sigma \neq \Pi$
First, set $\Omega = \Delta$. From this, it follows immediately that $R_{\Gamma\Omega\Delta}$.
Secondly, in order to show that also $R_{\Sigma\Theta\Omega}$ is the case, consider an arbitrary $A \rightarrow B \in \Sigma$ and $A \in \Omega$. From the former, together with **A7** and the fact that Σ is a Π–theory, it follows that $(B \rightarrow B) \rightarrow (A \rightarrow B) \in \Sigma$. As $B \rightarrow B \in \Gamma$ ($\Gamma = \Pi$) and as $R_{\Sigma\Gamma\Xi}$ holds, $A \rightarrow B \in \Xi$. This, together with $R_{\Xi\Theta\Delta}$ and $A \in \Theta$, gives us $B \in \Omega$ (as $\Delta = \Omega$).

(iv) $\Sigma = \Pi$ and $\Gamma = \Pi$
Set $\Omega = \Delta$. Both $R_{\Sigma\Theta\Omega}$ and $R_{\Gamma\Omega\Delta}$ follow immediately.

■

FP8 For $\Sigma, \Gamma, \Xi, \Theta$ and Δ arbitrary prime Π–theories: if $R_{\Sigma\Gamma\Xi}$ and $R_{\Xi\Theta\Delta}$, then there is a prime Π–theory Ω' such that $R_{\Gamma\Theta\Omega'}$ and $R_{\Sigma\Omega'\Delta}$.

Proof. Suppose (1) $R_{\Sigma\Gamma\Xi}$ and (2) $R_{\Xi\Theta\Delta}$:

(i) $\Gamma \neq \Pi$ and $\Sigma \neq \Pi$.
First, construct Ω as follows: $\{B \mid (\exists A)A \to B \in \Gamma$ and $A \in \Theta\}$. This is a Π–theory and $R_{\Gamma\Theta\Omega}$ follows from the construction.
Secondly, in order to show that also $R_{\Sigma\Omega\Delta}$ is the case, consider an arbitrary $A \to B \in \Sigma$ and $A \in \Omega$. From this, it follows that there is a formula C such that $C \to A \in \Gamma$ and $C \in \Theta$ (by the construction of Ω). Now, because of **A8** and the fact that Σ is a Π–theory, it follows that $(C \to A) \to (C \to B) \in \Sigma$. From this, together with $R_{\Sigma\Gamma\Xi}$ and $C \to A \in \Gamma$, it follows that $C \to B \in \Xi$, so that $R_{\Xi\Theta\Delta}$, together with $C \in \Theta$, gives us $B \in \Delta$. From this, it follows that $R_{\Sigma\Omega\Delta}$.
Finally, from the foregoing, together with lemma 5.9, it follows that there is a prime Π–theory $\Omega' \supseteq \Omega$ such that that $R_{\Sigma\Omega'\Delta}$ and $R_{\Gamma\Theta\Omega'}$.

(ii) $\Gamma = \Pi$ and $\Sigma \neq \Pi$
First, set $\Omega = \Theta$. From this, it follows immediately that $R_{\Gamma\Theta\Omega}$. Secondly, in order to show that also $R_{\Sigma\Omega\Delta}$ is the case, consider an arbitrary $A \to B \in \Sigma$ and $A \in \Omega$. From the former, together with **A8** and the fact that Σ is a Π–theory, it follows that $(A \to A) \to (A \to B) \in \Sigma$. As $A \to A \in \Gamma$ ($\Gamma = \Pi$) and as $R_{\Sigma\Gamma\Xi}$ holds, $A \to B \in \Xi$. This, together with $R_{\Xi\Theta\Delta}$ and $A \in \Theta$ (because $\Theta = \Omega$), gives us $B \in \Delta$.

(iii) $\Sigma = \Pi$ and $\Gamma \neq \Pi$
First, set $\Omega = \Delta$. From this, it follows immediately that $R_{\Sigma\Omega\Delta}$. Secondly, as $\Gamma = \Xi$ (because of (1) and **FP0**), $R_{\Gamma\Theta\Omega}$ follows immediately from (2).

(iv) $\Gamma = \Pi$ and $\Sigma = \Pi$
Set $\Omega = \Delta$. Both $R_{\Sigma\Theta\Omega}$ and $R_{\Gamma\Omega\Delta}$ follow immediately.

■

FP9 For all Σ, Γ, Δ arbitrary prime Π–theories: if $R_{\Sigma\Gamma\Delta}$ then there is a prime Π–theory Ω' such that $R_{\Sigma\Gamma\Omega'}$ and $R_{\Omega'\Gamma\Delta}$.

Proof. First, notice that (T1) $\vdash_\pi (C \to (A \to B)) \to ((A \wedge C) \to ((A \wedge C) \to B))$ and (T2) $\vdash_\pi (C \to (A \to B)) \to ((A \wedge C) \to ((A \wedge C) \to B))$ are valid in **B** + **A9**:

1	$\vdash_\pi (A \wedge C) \to A$	**A2**
2	$\vdash_\pi B \to B$	**A1**
3	$\vdash_\pi (A \to B) \to ((A \wedge C) \to B)$	1,2;**R3**
4	$\vdash_\pi C \to C$	**A1**
5	$\vdash_\pi (C \to (A \to B)) \to (C \to ((A \wedge C) \to B))$	3,4;**R3**
6	$\vdash_\pi ((A \wedge C) \to B) \to ((A \wedge C) \to B)$	**A1**
7	$\vdash_\pi (A \wedge C) \to C$	**A2**

8	$\vdash_\pi (C \to ((A \wedge C) \to B)) \to ((A \wedge C) \to ((A \wedge C) \to B))$	6,7;**R3**
9	$\vdash_\pi (C \to (A \to B)) \to ((A \wedge C) \to ((A \wedge C) \to B))$	5,8;**Trans**

1	$\vdash_\pi (A \wedge (A \to B)) \to A$	**A2**
2	$\vdash_\pi B \to B$	**A1**
3	$\vdash_\pi (A \to B) \to ((A \wedge (A \to B)) \to B)$	1,2;**R3**
4	$\vdash_\pi (A \wedge (A \to B)) \to (A \to B)$	**A2**
5	$\vdash_\pi (A \wedge (A \to B)) \to ((A \wedge (A \to B)) \to B)$	3,4;**Trans**
6	$\vdash_\pi ((A \wedge (A \to B)) \to ((A \wedge (A \to B)) \to B)) \to ((A \wedge (A \to B)) \to B)$	**A9**
7	$\vdash_\pi ((A \wedge (A \to B)) \to B)$	5,6;**MP**

Next, suppose (1) $R_{\Sigma\Gamma\Delta}$:

(i) $\Sigma \neq \Pi$

First, construct Ω as follows: $\{B \mid (\exists A)A \to B \in \Sigma$ and $A \in \Gamma\}$. This is a Π–theory and $R_{\Sigma\Gamma\Omega}$ follows from the construction.

Secondly, in order to show that also $R_{\Omega\Gamma\Delta}$ is the case, consider an arbitrary $A \to B \in \Omega$ and $A \in \Gamma$. From this, it follows that there is a formula C such that $C \to (A \to B) \in \Sigma$ and $C \in \Gamma$ (by the construction of Ω). It now follows that $(A \wedge C) \to ((A \wedge C) \to B) \in \Sigma$ (by **T1** and the fact that Σ is a Π–theory). The latter gives us $(A \wedge C) \to B \in \Sigma$ (by **A9**), from which it follows that $B \in \Delta$ (because $R_{\Sigma\Gamma\Delta}$, $A, C \in \Gamma$ and because Γ is a Π–theory).

Finally, from the foregoing, together with lemma 5.9, it follows that there is a prime Π–theory $\Omega' \supseteq \Omega$ such that that $R_{\Sigma\Gamma\Omega'}$ and $R_{\Omega'\Gamma\Delta}$. This will even be the case when $\Omega' = \Pi$ (meaning that it are equal sets), because in that case, we can take Ω' to be Π_p.[22]

(ii) $\Sigma = \Pi$

First, set $\Omega = \Gamma$. From this, it follows immediately that $R_{\Sigma\Gamma\Omega}$.

Secondly, in order to show that also $R_{\Omega\Gamma\Delta}$ is the case (with $\Omega = \Gamma = \Delta$, as $\Sigma = \Pi$), consider an arbitrary $A \to B \in \Omega$ and $A \in \Gamma$. From this, it follows that $A \wedge (A \to B) \in \Delta$ (because $\Gamma = \Delta$, and because Δ is a Π–theory). It now also follows that $B \in \Delta$ (by **T2** and the fact that Δ is a Π–theory).

■

FP10 For all Σ, Γ, Δ arbitrary Π–theories: if $R_{\Sigma\Gamma\Delta}$ then there is a prime Π–theory Ω' such that $\Sigma \subseteq \Omega'$, $\Omega' \subseteq \Sigma$ and $R_{\Gamma\Omega'\Delta}$.

[22] This shows the necessity of the pseudo–base world, because if it would not be there, it would not be guaranteed that $R_{\Pi\Gamma\Delta}$ is the case, as it cannot be proven that $\Gamma = \Delta$.

Proof. Suppose (1) $R_{\Sigma\Gamma\Delta}$:

(i) $\Sigma \neq \Pi$ and $\Gamma \neq \Pi$
First, set $\Omega = \Sigma$. From this, it follows immediately that $\Sigma \subseteq \Omega$ and $\Omega \subseteq \Sigma$. Secondly, in order to show that also $R_{\Gamma\Omega\Delta}$ is the case, consider an arbitrary $A \rightarrow B \in \Gamma$ and $A \in \Omega$. From the latter, together with **A10** and the fact that Ω is a Π–theory, it follows that $(A \rightarrow B) \rightarrow B \in \Omega$. This, together with $A \rightarrow B \in \Gamma$ and (1), gives us $B \in \Delta$.

(ii) $\Sigma = \Pi$ and $\Gamma \neq \Pi$
First, set $\Omega = \Pi_p$. From this, it follows immediately that $\Sigma \subseteq \Omega$ and $\Omega \subseteq \Sigma$ (as $\Sigma = \Pi$ and $\Omega = \Pi_p$). Secondly, in in order to show that also $R_{\Gamma\Omega\Delta}$ is the case (with $\Gamma = \Delta$, as $\Sigma = \Pi$), consider an arbitrary $A \rightarrow B \in \Gamma$ and $A \in \Omega$. From the latter, it follows that also $A \in \Sigma$ (as $\Pi_p \subseteq \Pi$). Now, from this, together with **A10** and the fact that Σ is a Π–theory, it follows that $(A \rightarrow B) \rightarrow B \in \Omega$. This, together with $A \rightarrow B \in \Gamma$, gives us $B \in \Delta$.

(iii) $\Sigma = \Pi$ and $\Gamma = \Pi$
Set $\Omega = \Pi$, which gives the required results.

■

Now that both soundness and completeness have been proven, the following corollary follows immediately:

Corollary 5.4 $\Gamma \vdash_{\mathbf{R}} A$ *iff* $\Gamma \models_{\mathbf{R}} A$.

5.4.5 Relation with Paralogics

From the semantical characterization above, it is immediately clear that relevant logics are intimately related to the paralogics from chapter 4. In fact, the semantics of relevant logics makes use of paraworlds, which are worlds in which negation, conjunction and disjunction are treated as for the logics **CLūNs**, **CLāNs** or **CLōNs**. The worlds that are most frequently used for **RL** are obviously **CLōNs**–worlds, because they are both paraconsistent and paracomplete, which are necessary features for relevance (see chapter 1).

Remark however that for some relevant logics, most notably the logic **R**, not all worlds treat negation, conjunction and disjunction on a par. For example, although **R** makes use of **CLōNs**–worlds, its base world is a **CLūNs**–world. This is obvious from the fact that all classical theorems are valid for **R**, which is a well–known feature of this relevant logic.

Last Call for the Basic Relevant Logic. Because of the stated relation between relevant logics and paralogics, it is also immediately clear that a lot of relevant logics can be constructed that are even weaker than **BM**, **BD** and **B**. They are not based on **CLūNs**–, **CLāNs**– or **CLōNs**–worlds, but on **CLuN**–, **CLaN**– or **CLoN**–worlds (see chapter 4).

Fitch–Style Proofs. In section 5.4.4, I presented an axiom system for the logic **R**. There is however also a Fitch–style proof theory for **R**. It was given in Anderson & Belnap [5]. I will here give the version from Brady [40], but with some slight alterations in the light of the proof theories of **CLūNs** and **CLōNs** presented in chapter 4. The alterations are straightforward, because of the relation between those paralogics and the logic **R**. Consequently, I will not explicitly prove the equivalence of the proof theory presented here with the original one. I simply state that they are equivalent.[23]

Before I start presenting the proof rules, I have to make a final preliminary remark: formulas occurring in the proof will have sets of numerals attached to them. In accordance with Brady [40], I will these "index sets".

As for the actual proof theory, first consider the structural rules below.

PREM Premises may be written down at any place in the proof, with the index set $\emptyset$.

HYP At any place in the proof, one may start a new subproof. This is done by introducing a new hypothesis A with an arbitrary index set Δ attached to it. That a new subproof has been started will be denoted by a new vertical line to the left of the formula A.

RHYP At any place in the proof, one may start a new *relevant subproof.* This is done by introducing a new hypothesis A with an index set $\{k\}$ attached to it. That a new relevant subproof has been started will be denoted by a new vertical line to the left of the formula A. A relevant subproof will be differentiated from an ordinary subproof by writing an R–symbol next to this vertical line. Such lines will be called R–lines.

i	... R	$A_{\{k\}}$	RHYP
i+1	...	...	

The index set of a relevant hypothesis is not an arbitrary set, but a singleton $\{k\}$, with k the rank of the new relevant subproof. The rank of a relevant subproof is the number of vertical R–lines to the left of the formula.

REP A formula A may be repeated in the same subproof, retaining its index set Δ.

REIT A formula A may be reiterated in an unclosed subproof, retaining its index set Δ.

[23] Obviously, critical readers can always try to prove the equivalence themselves.

Next, consider the inference rules. As for the proof theory of paralogics (see chapter 4), pseudo–formulas are introduced. Remark however that in all inference rules below, A and B always represent formulas, never pseudo–formulas.

CSP	If the formula B_Δ is the formula on the last line of an ordinary subproof that started with the hypothesis A_Δ, one may add a new line to the proof with the pseudo–formula $S(A,B)_\Delta$.
CON	A_Δ, B_Δ ► $(A \wedge B)_\Delta$
SIM	$(A \wedge B)_\Delta$ ► A_Δ; $(A \wedge B)_\Delta$ ► B_Δ
ADD	A_Δ ► $(A \vee B)_\Delta$; B_Δ ► $(A \vee B)_\Delta$
DIL	$(A \vee B)_\Delta$, $S(A,C)_\Delta$, $S(B,D)_\Delta$ ► $(C \vee D)_\Delta$
CONT	$(A \vee A)_\Delta$ ► A_Δ
ASS	$(A \vee (B \vee C))_\Delta$ ◄► $((A \vee B) \vee C)_\Delta$
DN	$(\neg\neg A)_\Delta$ ◄► A_Δ
NC	$(\neg(A \wedge B))_\Delta$ ◄► $(\neg A \vee \neg B)_\Delta$
ND	$(\neg(A \vee B))_\Delta$ ◄► $(\neg A \wedge \neg B)_\Delta$
EM	$S(A,B)_\emptyset$, $S({\sim}A,B)_\emptyset$ ► $B_\emptyset$
RCP	If the formula B_Δ is the formula on the last line of a relevant subproof that started with the hypothesis $A_{\{k\}}$, one may add a new line to the proof with formula $A \rightarrow B_{\Delta-\{k\}}$, provided that $k \in \Delta$.
RMP	A_Δ, $(A \rightarrow B)_\Theta$ ► $B_{\Delta\cup\Theta}$
RMT	$({\sim}B)_\Delta$, $(A \rightarrow B)_\Theta$ ► $({\sim}A)_{\Delta\cup\Theta}$
RDIL	$(A \vee B)_\Delta$, $(A \rightarrow C)_\Theta$, $(B \rightarrow C)_\Theta$ ► $C_{\Delta\cup\Theta}$
RRAA	$(A \rightarrow {\sim}A)_\Delta$ ► $({\sim}A)_\Delta$

The definition of an **R**–proof is now slightly different than before. A proof is now a sequence of wffs each of which is either a premise or follows from those earlier in the list by a stuctural rule or a rule of inference. Moreover, in order for such a sequence to be a proof, all its subproofs should be closed.

Also the definition of **R**–derivability slightly differs from the one given above:

Definition 5.19 $\Gamma \vdash_{\mathbf{R}} A$ *(A is an* **R***–consequence of* Γ*) iff there is a proof of the formula* $A_\emptyset$ *from* $B_1, ..., B_n \in \Gamma$ *so that* $A_\emptyset$ *has been derived on a line* i *of the main proof.*

Finally, consider the proof of the theorem $(p \rightarrow (p \rightarrow (s \wedge t))) \rightarrow (p \rightarrow s)$.

1	R| $p \rightarrow (p \rightarrow (s \wedge t))_{\{1\}}$	RHYP
2	| R| $p_{\{2\}}$	RHYP
3	| | $p \rightarrow (p \rightarrow (s \wedge t))_{\{1\}}$	1;RREIT
4	| | $p \rightarrow (s \wedge t))_{\{1,2\}}$	2,3;RMP
5	| | $s \wedge t_{\{1,2\}}$	2,4;RMP
6	| | $s_{\{1,2\}}$	5;RSIM

7	$\quad\vert\; p \rightarrow s_{\{1\}}$	2,6;RCP
8	$(p \rightarrow (p \rightarrow (s \wedge t))) \rightarrow (p \rightarrow s)_{\emptyset}$	1,7;RCP

Reintroducing Material Implication. Even for relevant logics, which make use of a relevant implication, it is possible to reintroduce the irrelevant material implication. This is done in the same way as for paralogics. Semantically, the following semantic clauses are added:[24]

AP $\quad v$: ${}^{\sim}\mathcal{MI} \times W \mapsto \{0,1\}$
SP $\quad v_M(A \supset B, a) = 1$ iff $v_M(A, a) = 0$ or $v_M(B, a) = 1$.
NP $\quad v_M(A \supset B, a) = 1$ iff $v(A \supset B, a) = 1$.

Proof theoretically, the following inference rules are added to the Fitch–style proof theory:

CP $\quad S(A_\Delta, B_\Delta) \blacktriangleright (A \supset B)_\Delta$
MP $\quad A_\Delta,\ (A \supset B)_\Delta \blacktriangleright B_\Delta$
PC $\quad ((A \supset B) \supset A)_\Delta \blacktriangleright A_\Delta$

5.5 Characterizing Relevant Derivability

In chapter 1, I already mentioned that in Relevance Logic, the notion of relevant derivability is dependent upon the notion of classical derivability. In this section, I will discuss relevant derivability. This is important, as in the remaining of this dissertation, I will solely be concerned with derivability in the sense here discussed.

First, consider again the definition of relevant deduction, usually given in the literature, e.g. by Brady in [43, pp. 302–308]:[25]

Definition 5.20 $\Gamma \mid\!\!\rightarrow_{\mathbf{RL}} A$ *is a relevant deduction of* A *from* Γ *iff there are* $B_1, ..., B_n \in \Gamma$ *such that* $\vdash_{\mathbf{RL}} (B_1 \wedge ... \wedge B_n) \rightarrow A$.

Below, I will show how this can be captured proof theoretically and even semantically. This will however only be done for the logic **R**, but the results can easily be extended to other relevant logics.

5.5.1 Proof Theories for Relevant Derivability

In the foregoing sections, I have given both an axiomatic and a Fitch–style proof theory for classical **R**–derivability. Consequently, I will here also consider both an axiomatic and a Fitch–style proof theory for relevant **R**–derivability.

[24] The set ${}^{\sim}\mathcal{MI} = \{{\sim}(A \supset B) \mid A, B \in \mathcal{W}\}$ is the set of negated material implications.
[25] An alternative one is given in Batens & Van Bendegem [27].

Axiomatic Proof Theory. The proof theory for relevant **R**–derivability is captured by two structural rules and some inference rules. Remark that these are dependent upon the proof theory of classical **R**–derivability. First, consider the structural rules.

PREM Premises may be written down at any place in the proof, always with a star attached to them.
AX **R**–axioms (see table 5.6) may be written down at any place in the proof, never with a star attached to them.

Next, consider the inference rules.

MP $A \to B,\, A \blacktriangleright B$
MP∗ $A \to B,\, A^* \blacktriangleright B^*$
CON $A,\, B \blacktriangleright A \wedge B$
CON∗ $A^*,\, B^* \blacktriangleright (A \wedge B)^*$

A relevant **R**–proof is now defined as a sequence of wffs each of which is either a starred premise, a non–starred **R**–axiom or derived from those earlier in the list by a rule of inference.

Relevant **R**–derivability is now defined as follows:

Definition 5.21 $\Gamma \models\!\!\!\rightarrow_{\mathbf{R}} A$ *iff there are* $B_1, ..., B_n \in \Gamma$ *such that there is a relevant* **R***–proof of the formula* A^* *from* $B_1, ..., B_n$.

In order to show that the definition of relevant derivability given here is equivalent with the one given in definition 5.20, consider the following proof:

Proof. => Suppose there are $B_1, ..., B_n \in \Gamma$ such that $\vdash_{\mathbf{R}} (B_1 \wedge ... \wedge B_n) \to A$. It is now plainly obvious that there will be a proof of A^* from Γ, because of the fact that $B_1, ..., B_n \in \Gamma$, and the rules **PREM**, **AX**, **CON**∗ and **MP**∗.

<= Suppose there here are $B_1, ..., B_n \in \Gamma$ such that there is a relevant **R**–proof of the formula A^* from $B_1, ..., B_n$. Suppose that $\langle C_1, ..., C_m \rangle$ is such a proof. It is now possible to transform this sequence into a different one by placing "$(B_1 \wedge ... \wedge B_n) \to$" before its starred members. Moreover, in the new sequence, the stars are removed so that all formulas in it are now non–starred. It can now be shown that this new sequence is in fact a classical **R**–proof.

This will be done as follows. First take notice of the fact that, in the original proof, each C_i ($1 \leqslant i \leqslant m$) is either a starred premise, a non–starred **R**–axiom, or derived from earlier members by means of a rule of inference. Let's consider what happens with each of those cases in the new proof:

No Star For C_i a non–starred formula in the original proof, nothing changes in the new proof. Moreover, it is obvious that all C_i remain derivable.

PREM For C_i a starred premise in the original proof (= an element of $\{B_1, ..., B_n\}$), it is obvious that $(B_1 \wedge ... \wedge B_n) \to C_i$ can be derived in the new proof (because of **R**–axiom **A2**).

CON$*$ For C_i a starred formula derived in the original proof from C_j and C_k by means of **CON**$*$, it is obvious that $(B_1 \wedge ... \wedge B_n) \to C_i$ can be derived from $(B_1 \wedge ... \wedge B_n) \to C_j$ and $(B_1 \wedge ... \wedge B_n) \to C_k$ in the new proof (because of **R**–axiom **A5**).

MP$*$ For C_i a starred formula derived in the original proof from the non–starred formula $C_k \to C_i$ and the starred formula C_k by means of **MP**$*$, it is obvious that $(B_1 \wedge ... \wedge B_n) \to C_i$ can be derived from $C_k \to C_i$ and $(B_1 \wedge ... \wedge B_n) \to C_k$ in the new proof (because transitivity is valid).

As $C_m = A^*$, and as it has been proven that the new sequence is a classical **R**–proof, it follows that $\vdash_{\mathbf{R}} (B_1 \wedge ... \wedge B_n) \to A$. ■

Fitch–Style Proofs. The Fitch–style proof theory from section 5.4.5 can also be used to characterize relevant derivability. In fact, it is reached by changing the **PREM**–rule and the **REIT**–rule in the Fitch style–proof theory for classical derivability. This should be done in the following way:

PREM Premises may be written down at any place in the main proof, with the index set $\{0\}$.

REIT A formula A may be reiterated into a subproof retaining its index set Δ, provided $0 \notin \Delta$.

A relevant **R**–proof is here defined as a sequence of wffs, each of which is a premise or follows from those earlier in the list by a stuctural rule or a rule of inference. Moreover, in order for such a sequence to be a proof, all its subproofs should be closed.

The definition of relevant **R**–derivability is here the following one:

Definition 5.22 $\Gamma \vdash_{\mathbf{R}} A$ *(A is an* **R***–consequence of* Γ*) iff there is a proof of the formula* $A_{\{0\}}$ *from* $B_1, ..., B_n \in \Gamma$ *so that* $A_{\{0\}}$ *has been derived on a line* i *of the main proof.*

I will not prove that this definition of relevant derivability is equivalent with the foregoing ones, I merely state it.[26]

5.5.2 Semantics for Relevant Derivability

Relevant **R**–derivability has not only been characterized proof theoretically, but also semantically, e.g. by Routley et al. [101, ch. 2] and by Read [92, ch. 5]. In both of them, relevant **R**–derivability has been defined as follows:

[26] Again, critical readers may prove this equivalence themselves.

Definition 5.23 $\Gamma \models\!\!\!\!\!\!\rightarrow_{\mathbf{R}} A$ *iff for all* **R***–models* M, $\forall a \in W$*: if* $v_M(\Gamma, a) =$ *1 then* $v_M(A, a) = 1$*.*

Soundness and Completeness. Also for relevant derivability, it is necessary to prove soundness and completeness.

Theorem 5.6 $\Gamma \vdash\!\!\!\rightarrow_{\mathbf{R}} A$ *iff* $\Gamma \models\!\!\!\!\!\!\rightarrow_{\mathbf{R}} A$*.*

Proof. => Suppose (1) $\Gamma \vdash\!\!\!\rightarrow_{\mathbf{R}} A$ and (2) $\Gamma \not\models\!\!\!\!\!\!\rightarrow_{\mathbf{R}} A$. From (1), it follows that there are $B_1, ..., B_n \in \Gamma$ such that $\vdash_{\mathbf{R}} (B_1 \wedge ... \wedge B_n) \rightarrow A$ (because of definition 5.20). From the latter, it follows that $\models_{\mathbf{R}} (B_1 \wedge ... \wedge B_n) \rightarrow A$ (by corollary 5.4), which means that for all **R**–models M, $v_M((B_1 \wedge ... \wedge B_n) \rightarrow A, g) = 1$. This also means that for all worlds $a \in W$, $v_M(B_1, a) = 0$ or ... or $v_M(B_1, a) = 0$ or $v_M(A, a) = 1$. However, from (2), it follows there is an **R**–model M such that $\exists a \in W$, $v_M(B_1, a) = 1$ and ... and $v_M(B_n, a) = 1$ and $v_M(A, a) = 0$ (by definition 5.23), which contradicts with the foregoing.

<= Suppose $\Gamma \not\vdash\!\!\!\rightarrow_{\mathbf{R}} A$. From definition 5.20, it follows that the consequence set Σ of Γ is a Π–theory (see section 5.4.2). As $\Sigma \cap \{A\} = \emptyset$, it follows by corollary 5.1, that there is a $\Sigma' \supseteq \Sigma$ such that Σ' is a prime Π–theory and $A \notin \Sigma'$. An **R**–model M can now be defined in the same way as it was done in section 5.4.4, for which it is the case that $\Sigma' \in X$ and $\Sigma' \neq \Pi$. From this, it follows that there is at least one **R**–model M for which there is a world, namely Σ' such that $v_M(\Gamma, \Sigma') = 1$ and $v_M(A, \Sigma') = 0$. This however means that $\Gamma \not\models\!\!\!\!\!\!\rightarrow_{\mathbf{R}} A$ (by definition 5.23). ■

The Deductive World

In view of what is to come, I will slightly alter the definition of relevant semantic consequence, as to make it useful in later chapters.

First, remember that in section 5.4.4, an **R**–model was characterized as a 5–tuple $\langle g, W, R, \leqslant, v \rangle$. Next, I would like to add an extra element, namely a *deductive world*. Consequently, an **R**–model becomes a 6–tuple $\langle g, d, W, R, \leqslant, v \rangle$, with $d \in W$ the deductive world. Moreover, take it that $d \neq g$! Obviously, this will not change anything to the soundness and the completeness proofs, but it does make it easier to define relevant derivability. This can now be done in the following way: first, the $\mathbf{R^{LLL}}$-models of a premise set are characterized. This is done by reference to the deductive world d.

Definition 5.24 *A valuation function* v_M *d–verifies* A *iff* $v_M(A, d) = 1$*, and d–falsifies* A *iff* $v_M(A, d) = 0$*.*

Definition 5.25 *A valuation function* v_M *is an* $\mathbf{R^{LLL}}$*–model of* Γ *iff it d–verifies all* $A \in \Gamma$*.*

Next, the **R**–models are characterized as those $\mathbf{R}^{\mathbf{LLL}}$–models that do not falsify any abnormalities. This is of course not done by reference to the deductive world d, but by reference to the base world g.

Definition 5.26 *A valuation function v_M g–verifies A iff $v_M(A, g) = 1$, and g–falsifies A iff $v_M(A, g) = 0$.*

Definition 5.27 *For each $\mathbf{R}^{\mathbf{LLL}}$–model M, $Ab(M) = \{A \in \Omega \mid M$ g–falsifies $A\}$.*

Definition 5.28 *An $\mathbf{R}^{\mathbf{LLL}}$–model of Γ is an **R**–model of Γ iff $Ab(M) = \emptyset$.*

Finally, semantic consequence is defined by means of the **R**–models of a premise set.

Definition 5.29 *$\Gamma \models\!\!\!\!\rightarrow_{\mathbf{R}} A$ iff A is d–verified by all **R**–models of Γ.*

It is easily conceived that this way of characterizing relevant semantic consequence is equivalent to the one stated above. As a consequence:

Theorem 5.7 *$\Gamma \vdash\!\!\!\!\rightarrow_{\mathbf{R}} A$ iff $\Gamma \models\!\!\!\!\rightarrow_{\mathbf{R}} A$.*

Notational Convention. As both the semantics and the proof theory for classical **R**–derivability (see section 5.4.4) and relevant **R**–derivability (see section 5.5.2) differ, it seems odd to keep on talking as if it only concerns one logic, namely the logic **R**. Obviously, it doesn't concern only one, but two logics! Consequently, in order to avoid confusion, I will give them different names. From now on, the logic **R** will refer to the logic characterizing classical derivability, while I will use $\mathbf{R_d}$ in order to refer to the logic characterizing relevant derivability.

Definition 5.30 *$\Gamma \vdash_{\mathbf{R_d}} A$ iff $\Gamma \vdash\!\!\!\!\rightarrow_{\mathbf{R}} A$.*

Finally, in the remaining of this dissertation, I will only be concerned with the logic $\mathbf{R_d}$, as I still aim to characterize relevant deduction.

5.6 Conclusion

In this chapter, several objectives were met. First of all, I provided a new semantic characterization of standard **RL** (see section 5.4). It treats **RL**–negation in a non–truthfunctional way, and is meant as an alternative to the Routley–star–characterization and the four valued–characterization of **RL**–negation (see sections 5.2 and 5.3). Next, I also pointed to the striking relations between the standard **RL** and the paralogics from chapter 4 (see section 5.4.5). Finally, I also provided a new semantic characterization of relevant derivability (see section 5.5.2).

Part III

First Degree Relevance

The Aim of Part III

In chapter 1, I stated that the main relevance criterium of standard Relevance Logic (the use–criterium), is only appropriate to cope with the paradoxes of the material implication. Moreover, it only gives nice results for $\mathbf{R}_{\rightarrow}$, the pure implicational fragment of **R**. Once the other connectives are added to $\mathbf{R}_{\rightarrow}$, the standard account of relevance breaks down. This became most obvious from the blunt rejection of the entailment **EDS**.

$$\nvdash_{\mathbf{R}} (A \wedge ({\sim}A \vee B)) \rightarrow B$$

It was not rejected on grounds of any relevance criterium, but solely because it reintroduces some of the paradoxes of the material implication. Moreover, the choice to reject **EDS** was completely arbitrary. Rejecting some other entailment might have lead to equally good results.

In $\mathbf{R_d}$, the deductive counterpart of **R**, the entailment **EDS** corresponds to the inference rule disjunctive syllogism (**DS**). As such, the latter is obviously not a valid rule of inference in $\mathbf{R_d}$.

$$A, {\sim}A \vee B \nvdash_{\mathbf{R_d}} B$$

This is particularly damaging, as **DS** is a frequently used inference rule in both scientific and common sense reasoning. Moreover, it usually doesn't lead to the derivation of irrelevant consequences, so that its rejection unnecessarily limits the deductive strength of relevant logics.

Consequently, I have claimed that standard Relevance Logic doesn't capture relevant deduction in an adequate way. It only succeeds in giving a more or less nice account of relevant implication. In order to capture relevant deduction in full, it is also necessary to adequately cope with *first degree relevance*. As there is no implication at the first degree, the only fallacies of relevance that should be taken into consideration are the ones related to the **EQV**– and **EFQ**–paradoxes. As a consequence, to capture *first degree relevance*, it is necessary to avoid the **EQV**– and **EFQ**–paradoxes without unnecessary limiting deductive strength.

In this part of my dissertation, I will look for a way to capture relevance at the first degree. In part IV, the resulting theory of first degree relevance will be combined with the relevant implication from standard Relevance Logic, to obtain a complete and adequate theory of relevant deduction.

Overview of Part III

In the first chapter of this part (ch. 6), I will present a theory of first degree relevance. In the next three chapters (ch. 7–9), I will show how the given account of first degree relevance can be captured by means in a logically stringent way.

Chapter 6

Theory of First Degree Relevance

6.1 First Degree Relevance

As said in chapter 1, Relevance Logic basically investigates deductive connection.[1] What is sought after, is a theory of deduction that explicates the *substantial* connection between premises and conclusions in real deductive proofs. This substantiality is definitely not captured by the use–criterium, which is made clear by C.I. Lewis' *Independent Proof*:

1	p	**PREM**
2	$\neg p$	**PREM**
3	$p \vee q$	1;**ADD**
4	q	2,3;**DS**

Both premises were used in the derivation of the formula on line 4. Nevertheless, that formula is (and should be!) considered an irrelevant consequence of the premise set. The question thus remains how it is possible to explicate substantiality in an adequate way.

Preliminary Remark. Before I answer the question above, remark that I will take *Classical Logic* as my starting point. Not all logicians (e.g. intuitionists) might agree on this. I nevertheless do not consider this to be that problematic, as I am quite confident that the approach below can be adapted to fit their philosophical standpoints as well.

[1] Consequently, its first concern is not with relevance "an sich". Relevance is merely considered an epiphenomenon of any good derivability relation. This is most clearly stated in Routley et al. [101, p. x].

6.1.1 Transfer of Deductive Weight

As there should be something substantial to a relevant deductive connection, let's suppose the premises carry some real (deductive) *weight*. More specifically, the weight is carried by the sentential letters occurring in the premises, and through them, it is passed over to their consequences. Consequently, relevant deduction can be interpreted as the *transfer of deductive weight* from premises to conclusions.

It is now possible to state that a formula carries *full deductive weight* when all its sentential letters carry some weight (meaning that they were part of the premises). Moreover, an inference rule *transfers full deductive weight* from its premises to its conclusions when all sentential letters in the latter still carry some deductive weight. This is not the case for all inference rules. Consider for example the inference rule *addition* (**ADD**).

[**ADD**] $A \blacktriangleright A \vee B$

Obviously, **ADD** allows to introduce sentential letters (those occurring in B) that do not carry any deductive weight, as they were not physically present in the premises (which is the only way to get some weight). This of course doesn't mean that **ADD** leads to irrelevant consequences. Its consequences might still carry some weight (bundled in A), so that there can be a substantial connection between them and the premises. Problems only arise when some other inference rule, as for example **DS**, is applied to formulas that do not carry full deductive weight anymore. This might lead to formulas in which there do not occur any weighty sentential letters, meaning that there is no substantial connection between them and the premises. It are those consequences that are the real irrelevant consequences of a premise set. In order to make this more concrete, let's consider the example below.

Example. Take the premise set $\Gamma = \{p, p \vee q, \neg p\}$. The sentential letters occurring in this premise set all carry some deductive weight, which will be denoted by placing them between brackets. When we now consider a **CL**–proof from this premise set, it is possible to see whether or not its conclusion also carries some weight.

1	$[p]$	**PREM**
2	$\neg[p]$	**PREM**
3	$[p] \vee [q]$	**PREM**
4	$[p] \vee q$	1;**ADD**
5	$\mid [p]$	**HYP**
6	$\mid \neg[p]$	2;**REIT**
7	$\mid [p] \wedge \neg[p]$	5,6;**CON**
8	$S([p], [p] \wedge \neg[p])$	5,7;**CON**
9	$[q]$	3,8;**DS**

10 q 4,8;**DS**

This example clearly illustrates what was stated above. On line 4, a formula is derived by means of the inference rule **ADD**. Although it doesn't carry full deductive weight anymore, it is a relevantly obtained consequence of the premise set (as it still carries some weight). This cannot be said of the formula on line 10, which doesn't carry any deductive weight anymore. It has been derived from lines 4 and 8 by means of the inference rule **DS**. This however doesn't mean that all applications of **DS** lead to irrelevant consequences, which is made clear by the formula on line 9. It has been derived by means of **DS** and it definitely does carry deductive weight.

In fact, the example above makes clear that the fallacies of relevance are not caused by some specific rule(s) of inference, as for example **ADD** or **DS**. All of them can pass on deductive weight, so that none of them can be blamed in particular. The problem really lies with their combined *use*.

Relevant Proofs. After all, it seems now that *use* indeed has something to do with relevance. But, it is important to notify that what matters is not the use that has been made of the premises (as the standard use–criterium has it), but the use that has been made of the inference rules. As a consequence, I consider relevance not in the first place a property applicable to formulas (premises or consequences), but to proofs.[2] A proof is considered relevant when the inference rules have been used in such a way that its conclusion still carries some deductive weight.[3]

The Classical Inference Rules. In order to be able to decide whether or not the conclusion of a **CL**–proof still carries some deductive weight, it is necessary to investigate in which sense the **CL**–inference rules are capable of transferring it. Remark that not only the main rules, but also the most important derived rules should be taken into consideration.

I will distinguish between four kinds of **CL**–inference rules, based on the subdivision made by Meheus in [72]: analyzing rules, constructive rules, adjunctive rules and transformation rules. First, consider the transformation rules. They allow to replace a formula with a formula that is equivalent to it.

(1) $A \vee A \blacktriangleleft\!\blacktriangleright A$
(2) $A \vee B \blacktriangleleft\!\blacktriangleright B \vee A$

[2]As far as I know, only Neil Tennant [107, 108] also attributed relevance to proofs. There is a very interesting relation between his approach and the one described here, which I will discuss in chapter 7.

[3]I will call the consequences of a premise set for which there exists a relevant proof, the relevant consequences of that premise set.

(3) $A \vee (B \vee C)$ ◀▶ $(A \vee B) \vee C$
(4) $A \wedge B$ ◀▶ $B \wedge A$
(5) $A \wedge (B \wedge C)$ ◀▶ $(A \wedge B) \wedge C$
(6) $\neg\neg A$ ◀▶ A
(7) $\neg(A \vee B)$ ◀▶ $\neg A \wedge \neg B$
(8) $\neg(A \wedge B)$ ◀▶ $\neg A \vee \neg B$
(9) $A \vee (B \wedge C)$ ◀▶ $(A \vee B) \wedge (A \vee C)$
(10) $A \wedge (B \vee C)$ ◀▶ $(A \wedge B) \vee (A \wedge C)$

As the sentential letters in the premises of a transformation rule are the same as those in its conclusion, applying it will not change the deductive weight of a formula. So, if their premises still carry full deductive weight, then also their conclusions will do so. But, on the contrary, if their premises contain some weightless sentential letters, then also their conclusions will do so. As a consequence, there are no objections towards the use of those rules in relevant proofs.

It is easily verified that the same situation also applies for the adjunctive rules, which are the inference rules related to the conjunction connective.

(11) A, B ▶ $A \wedge B$
(12) $A \wedge B$ ▶ A; $A \wedge B$ ▶ B

Next, consider the constructive rules, which allow to derive formulas that are more complex than their premises.

(13) A ▶ $A \vee B$
(14) For $A \in \mathcal{S}$: ▶ $A \vee \neg A$

As these rules introduce sentential letters that do not carry any deductive weight, they need to be handled with care. Moreover, as relevant consequences still carry some weight, these rules should not lead to the introduction of weightless formulas. Because, if they do so, it is impossible for them to be used in a relevant proof. As such, they should be dropped from any proof theory trying to capture first degree relevance.

Normative Statement 6.1 *Inference rules that allow to introduce formulas which do not carry any deductive weight should be dropped altogether.*

It is now clear that the second constructive rule (rule (14)) should be dropped altogether. The first one (rule (13)) can be kept, because although it does some weightless sentential letters, it also transfers the deductive weight of its premises.

Finally, also consider the analyzing rules. Actually, there is only one, namely **DS**. It is called an analyzing rule because it allows to derive conclusions which are less complex than its premises.

(15) For $C \in \mathcal{S}$: $A \vee B$, $S(B, C \wedge \neg C) \blacktriangleright A$

It is obvious that there are no objections to its use, as long as one can be sure that its conclusion still carries some weight. As this will always be the case when it is applied to formulas that were derived without the use of a constructive rule, it is possible to claim that:

Normative Statement 6.2 *Analyzing rules should only be applied to formulas that still carry full deductive weight.*

Because, as long as no constructive rule has been applied, this will obviously be the case.

First Degree Relevance. Relevant deduction is intuitively captured by the normative statements 6.1 and 6.2. It is easily verified that the former rules out the **EQV**–paradoxes, while the latter prevents the **EFQ**–paradoxes from occurring.

6.1.2 Classical Relevance

It is fairly straightforward to incorporate the normative statements 6.1 and 6.2 into the **CL**–proof theory that was presented in chapter 4. In this section, I will however only do so for normative statement 6.2. The resulting logic, called **CL***, does not capture first degree relevance yet, but it does capture what I've called *classical relevance*, which is the avoidance of the **EFQ**–paradoxes without unnecessarily limiting deductive strength.

Proof Theory. The proof theory of **CL*** is obtained by changing the **CL**–proof theory presented in chapter 4 in a straightforward way. First of all, in order to denote that a formula carries full deductive weight, a star is attached to it. Secondly, a distinction is made between those classical inference rules that transfer the star from their premises to their conclusions, and those that do not. Finally, the application of the analyzing rule **DS** is restricted to formulas that have stars attached to them.

In order to make this intuitive description more clear, I will present the **CL***–proof theory in full detail. First, consider the structural rules.

PREM Premises may be written down at any place in the proof. Moreover, as a premise carries full deductive weight, it is always introduced carrying a star.

HYP At any place in the proof, one may start a new subproof. This is done by introducing a new hypothesis, together with a new vertical line on its left. Hypotheses are also introduced carrying a star. This might seem strange at first, as they are not part of the premises. Nevertheless, it makes perfectly good sense to do so. If they aren't supposed to carry some weight, why introduce them in the first place?

REP In the main proof and in subproofs, formulas may be repeated. Obviously, repeated formulas retain their star.

REIT In subproofs, one may reiterate formulas from lines in the main proof and from lines in unclosed subproofs. However, reiterated formulas do not retain their star![4]

Secondly, consider the inference rules that allow to pass on stars from their premises to their conclusions. It is easily verified that they correspond to the transformation, the adjunctive and the analyzing rules.

CSP* If the formula B^* is the formula on the last line of a subproof that started with the hypothesis A^*, one may conclude to the pseudo–formula $S(A, B)^*$. This of course also closes the subproof.

CON* A^*, $B^* \blacktriangleright A \wedge B^*$

SIM* $A \wedge B^* \blacktriangleright A^*$; $A \wedge B^* \blacktriangleright B^*$

DIL* $A \vee B^*$, $S(A, C)^*$, $S(B, D)^* \blacktriangleright C \vee D^*$

CONT* $A \vee A^* \blacktriangleright A^*$

ASS* $A \vee (B \vee C)^* \blacktriangleleft\!\blacktriangleright (A \vee B) \vee C^*$

IMP* $A \sqsupset B^* \blacktriangleleft\!\blacktriangleright \neg A \vee B^*$

DN* $\neg\neg A^* \blacktriangleleft\!\blacktriangleright A^*$

NC* $\neg(A \wedge B)^* \blacktriangleleft\!\blacktriangleright \neg A \vee \neg B^*$

ND* $\neg(A \vee B)^* \blacktriangleleft\!\blacktriangleright \neg A \wedge \neg B^*$

NI* $\neg(A \sqsupset B)^* \blacktriangleleft\!\blacktriangleright \neg A \wedge B^*$

DS* For $C \in \mathcal{S}$: $A \vee C^*$, $\neg C^* \blacktriangleright A^*$; $A \vee C^*$, $B \vee \neg C^* \blacktriangleright A \vee B^*$

Finally, also consider those inference rules that do not allow to pass on stars to their consequences, and those inference rules that can be applied to formulas without a star. They obviously correspond to the transformation and the constructive rules.

CSP If the formula B is the formula on the last line of a subproof that started with the hypothesis A^*, one may conclude to the pseudo–formula $S(A, B)$. This evidently closes the subproof.

CON $A^{(*)}$, $B^{(*)} \blacktriangleright A \wedge B$

SIM $A \wedge B \blacktriangleright A$, $A \wedge B \vdash B$

ADD $A^{(*)} \vdash A \vee B$, $B^{(*)} \blacktriangleright A \vee B$

DIL $A \vee B^{(*)}$, $S(A, C)^{(*)}$, $S(B, D)^{(*)} \blacktriangleright C \vee D$

CONT $A \vee A \blacktriangleright A$

[4] If reiterated formulas were allowed to retain their star, a restricted kind of **ADD** would be able to transfer stars (and as such, full deductive weight). For example, from the formulas $p \vee q^*$,$\neg r^*$ and r^*, the formula $r \vee p^*$ would be derivable, which would lead eventually to p. But, this is obviously an irrelevant consequence, and as such, it should not be derivable.

ASS	$A \vee (B \vee C) \blacktriangleleft\!\blacktriangleright (A \vee B) \vee C$
IMP	$A \sqsupset B \blacktriangleleft\!\blacktriangleright \neg A \vee B$
DN	$\neg\neg A \blacktriangleleft\!\blacktriangleright A$
NC	$\neg(A \wedge B) \blacktriangleleft\!\blacktriangleright \neg A \vee \neg B$
ND	$\neg(A \vee B) \blacktriangleleft\!\blacktriangleright \neg A \wedge \neg B$
NI	$\neg(A \sqsupset B) \blacktriangleleft\!\blacktriangleright \neg A \wedge B$
TH	For $C \in \mathcal{S}$: $\blacktriangleright A \vee \neg A$.

A **CL***–proof is defined as a sequence of wffs, each of which is a premise, a hypothesis or a formula that follows from earlier ones in the list by a rule of inference. Moreover, in order for such a sequence to be a proof, all its subproofs should be closed.

Based on the above definition, **CL***–derivability can now be defined as follows:

Definition 6.1 $\Gamma \vdash_{\mathbf{CL}^*} A$ *(A is an* **CL****–consequence of* Γ*) iff there is a proof of the formula A from $B_1, ..., B_n \in \Gamma$ such that A occurs on a line of the main proof. It is not necessary for A to be starred.*

Examples. Consider the premise set $\Gamma = \{p, p \vee (\neg p \vee q), \neg p, \neg(p \wedge r), r\}$. I will now show that it is possible to **CL***–derive the formulas $q \vee s$ and $\neg p \wedge \neg r$ from Γ. First, consider the **CL***–proof for $q \vee s$.

1	$p \vee (\neg p \vee q)^*$	**PREM**	
2	p^*	**PREM**	
3	$\neg p^*$	**PREM**	$\triangle\ q \vee s$
4	$\neg p \vee q^*$	1,3;**DS***	
5	q^*	3,4;**DS***	
6	$q \vee s$	5;**ADD**	

Next, consider the **CL***–proof for $\neg p \wedge \neg r$.

1	$\neg(p \wedge r)^*$	**PREM**	
2	p^*	**PREM**	
3	r^*	**PREM**	$\triangle\ \neg p \wedge \neg r$
4	$\neg p \vee \neg r^*$	1;**NC**	
5	$\neg p^*$	3,4;**DS***	
6	$\neg r^*$	2,4**DS***	
7	$\neg p \wedge \neg r^*$	5,6;**CON***	

No Implication? Although this chapter is about first degree relevance, the $\mathbf{CL}^*$–proof theory described above does contain inference rules related to the implication. Moreover, it is an irrelevant implication, as it is defined in the usual way, by means of the disjunction:

$$A \supset B =_{df} \neg A \vee B$$

Consequently, if one would ask for its meaning, I would state that it is not to be interpreted as a real implication, but merely as a disjunction "in disguise". As such, I consider it harmless to incorporate it in a theory of first degree relevance.

6.1.3 Relevant Deduction

In order to capture first degree relevance in full, it is not enough only to incorporate normative statement 6.2 into the classical proof theory. Also normative statement 6.1 should be incorporated. Obviously, this can be done by dropping the inference rule **TH** from the $\mathbf{CL}^*$–proof theory, as it is the only inference rule that can introduce formulas lacking deductive weight. The result is a proof theory which adequately captures relevant deduction at the first degree.

Remark that the obtained proof theory is in fact the proof theory of the paralogic **CLāNs** (see chapter 4) into which normative statement 6.2 has been incorporated. As such, it is appropriate to refer to the resulting relevant logic as to the logic $\mathbf{CL\bar{a}Ns}^*$.

6.1.4 Weak Relevance Criteria

The above theory of first degree relevance is based on weak relevance criteria. The existence of a weak deductive connection, denoted by the presence of *some* deductive weight, is sufficient to allow for derivation. Some people might wonder whether these criteria are not too weak. They might claim that in some reasoning contexts, it is more appropriate to opt for stronger relevance criteria. In those contexts, they might claim, relevant proofs should only lead to conclusions that still carry full deductive weight, which would lead to a more restrictive attitude towards the inference rule **ADD**.[5]

Although those people might be right, I here restrict myself to the weak relevance criteria stated above and leave it open for discussion whether it is really necessary to strengthen them in some reasoning contexts, and, if so, in which sense this should be done.

[5]Based on previous writings, some possible candidates are Parry [81], Verhoeven [122, 121], Meheus [72],...

6.1.5 Relation with Other Logics

Both **CL*** and **CLāNs*** bear a lot of relations with other logics. In this section, I will state some of those relations, but remark that none of the claims I will make are proven yet (this is left for further research), except when explicitly stated.

First of all, the logic **CL*** is equivalent to Tennant's **CR** from [108, 109, 114]. This will be proven in chapter 7. Moreover, if in the **CL***–proof theory, the inference rule **ADD** is replaced by the inference rule **ADD**$'$ stated below, then a proof theory is obtained for the logic **AN** from Meheus [72, 73].

ADD$'$ $A^{(*)} \blacktriangleright A \vee A$.

Moreover, if the inference rule **REIT** is also allowed to transfer stars (which is now not the case), then a proof theory is obtained for the logic **DAI** from Dunn [51] (see also Oller [80]).

Concerning **CLāNs***, I suppose that it is equivalent to Besnard and Hunter's *Quasi–classical Logic* (**QCL**) that was presented in Besnard and Hunter [38] and Hunter [59, 60].

6.2 Some Metatheory

In this section, I will state some metatheoretical properties of **CL***. These will provide a better understanding of the system, and moreover, they will come out very handy in the next chapter(s).

The CL*–Consequence Set. The first important metatheoretical characteristic states that the **CL***–consequence set of a premise set is always situated somewhere in between the **CLūNs**– and the **CL**–consequence set of that premise set.

Theorem 6.1 *For a premise set* Γ, $Cn_{\mathbf{CL\bar{u}Ns}}(\Gamma) \subseteq Cn_{\mathbf{CL}^*}(\Gamma) \subseteq Cn_{\mathbf{CL}}(\Gamma)$.

Proof. Obvious from the fact that the **CLūNs**–rules are all admitted rules in **CL***, and from the fact that the **CL***–rules are all admitted rules in **CL**. ■

Transitivity of CL*–Deduction. It is easily verified that transitivity is not in general valid for the **CL***–derivability relation. Consider the example below:

Example 6.1 *Although* $\{p, \neg p\} \vdash_{\mathbf{CL}^*} (p \vee q) \wedge \neg p$ *and* $\{(p \vee q) \wedge \neg p\} \vdash_{\mathbf{CL}^*} q$, *it is not the case that* $\{p, \neg p\} \vdash_{\mathbf{CL}^*} q$.

Nevertheless, a restricted kind of transitivity is valid for $\mathbf{CL}^*$. It is stated as follows:[6]

Theorem 6.2 *If* $\Gamma \vdash_{\mathbf{CL}^*} A^*$ *and* $\Gamma' \cup \{A\} \vdash_{\mathbf{CL}^*} B^{(*)}$*, then* $\Gamma \cup \Gamma' \vdash_{\mathbf{CL}^*} B^{(*)}$.

Proof. Suppose (1) $\Gamma \vdash_{\mathbf{CL}^*} A^*$ and (2) $\Gamma' \cup \{A\} \vdash_{\mathbf{CL}^*} B$. From (2), it follows that there is a $\mathbf{CL}^*$–proof of B from $\Gamma' \cup \{A\}$. This can easily be turned into a $\mathbf{CL}^*$–proof of B from $\Gamma' \cup \Gamma$. It is done by replacing the line where A^* was introduced as a premise, by the $\mathbf{CL}^*$–proof for the formula A^* from Γ. This is possible because of (1). ■

Relation with CLūNs and CL. The logic $\mathbf{CL}^*$ bears some interesting relations with the logics **CLūNs** and **CL**. In order to spell them out, I first need to introduce some extra terminology.

Conjunctive Normal Forms. A formula is in conjunctive normal form (**CNF**) when it is a conjunction of disjunctions of primitive formulas (sentential letters or negated sentential letters).

Definition 6.2 $CNF(A)$ *refers to the conjunctive normal form of the formula* A.

Any formula can be transformed into a formula in **CNF**. This is done in a 3–step manner:

1. Replace all implications $A \supset B$ in the original formula by disjunctions $\neg A \vee B$.
2. Drive the negation inwards by means of the De Morgan laws and double negation.
3. Use the distributive laws to reach a formula in **CNF**.

It is a well–known fact that in **CL** a formula is completely equivalent with its **CNF**.

Fact 6.1 $A \vdash_{\mathbf{CL}} CNF(A)$ *and* $CNF(A) \vdash_{\mathbf{CL}} A$.

This is not only a fact for **CL**. It can easily be verified that it is also valid for the paralogics **CLōNs**, **CLūNs** and **CLāNs**. Moreover, it is also valid for $\mathbf{CL}^*$.[7]

Fact 6.2 $A \vdash_{\mathbf{CL\bar{x}Ns}} CNF(A)$ *and* $CNF(A) \vdash_{\mathbf{CL\bar{x}Ns}} A$.

[6] Obviously, $\Gamma \vdash_{\mathbf{CL}^*} A^*$ means that there is a $\mathbf{CL}^*$–proof of A from Γ such that A occurs on a line of the main proof and A is starred.

[7] Proofs are straightforward and left to the reader.

Fact 6.3 $A \vdash_{\mathbf{CL}^*} CNF(A)^*$.

Fact 6.4 $A \vdash_{\mathbf{CL}^*} CNF(A)$ *and* $CNF(A) \vdash_{\mathbf{CL}^*} A$.

Clauses. A clause is a finite disjunction of primitive formulas, which means that a formula in **CNF** is a conjunction of clauses. Consequently, it is possible to introduce the notion of a clause set, defined as follows:

Definition 6.3 *$CNF^\circ(A)$ is the clause set of the formula A. It is the set of clauses making up the conjunctive normal form of the formula A.*

Below, clauses might be represented by making use of the following definition:

Definition 6.4 *$\bigvee(\Delta)$ refers to the disjunction of the elements of Δ.*

Resolvents. Consider also the definition of the set $Res_{\mathbf{CL}}(\Gamma)$, the set of **CL**–*resolvents* of the premise set Γ:

Definition 6.5 *$Res_{\mathbf{CL}}(\Gamma)$ is the set of clauses that are characterized as follows:*

1. *For all $A \in \Gamma$, if $B \in CNF^\circ(A)$ then $B \in Res_{\mathbf{CL}}(\Gamma)$.*
2. *If $\bigvee(\Delta \cup \{A\}) \in Res_{\mathbf{CL}}(\Gamma)$, $\bigvee(\Delta' \cup \{\neg A\}) \in Res_{\mathbf{CL}}(\Gamma)$ and $\Delta \cup \Delta' \neq \emptyset$, then also $\bigvee(\Delta \cup \Delta') \in Res_{\mathbf{CL}}(\Gamma)$.*

It can easily be proven that the set of **CL**–resolvents of a premise set Γ leads to the same **CL**–consequences as the premise set itself.

Theorem 6.3 $\Gamma \vdash_{\mathbf{CL}} A$ *iff* $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}} A$.

Proof. => Suppose $\Gamma \vdash_{\mathbf{CL}} A$. Moreover, it is obvious that for all $B \in \Gamma$, $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}} B$ (by definition 6.5, fact 6.1 and the **CL**–proof theory). By the transitivity of **CL**–deduction, it now follows that $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}} A$.

<= Suppose $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}} A$. Moreover, it is obvious that for all $B \in Res_{\mathbf{CL}}(\Gamma)$, $\Gamma \vdash_{\mathbf{CL}} B$ (by definition 6.5, fact 6.1 and the **CL**–proof theory). By the transitivity of **CL**–deduction, it now follows that $\Gamma \vdash_{\mathbf{CL}} A$. ■

Besides the set of **CL**–resolvents, a premise set Γ also has a set of **CL***–resolvents. It is characterized as follows:

Definition 6.6 $Res_{\mathbf{CL}^*}(\Gamma) = \{\bigvee(\Delta) \mid \bigvee(\Delta)$ *a clause and* $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta)^*\}$.

It can now be shown that both sets of resolvents are completely equivalent to one another. First, consider two preliminary lemmas.

Lemma 6.1 *For all $A \in Res_{\mathbf{CL}}(\Gamma)$, $\Gamma \vdash_{\mathbf{CL}^*} A^*$.*

Proof. First of all, for all $B \in \Gamma$, $\Gamma \vdash_{\mathbf{CL}^*} CNF(B)^*$ (by fact 6.3). From this, it follows that for all $A \in Res_{\mathbf{CL}}(\Gamma)$, $\Gamma \vdash_{\mathbf{CL}^*} A^*$, which is verified by checking that all inference rules needed to derive the elements of $Res_{\mathbf{CL}}(\Gamma)$ from $CNF(\Gamma)$[8] (most notably $\mathbf{DS}^*$) also transfer stars to their conclusions. ■

Lemma 6.2 *$A \in Res_{\mathbf{CL}^*}(\Gamma)$ iff $A \in Res_{\mathbf{CL}}(\Gamma)$.*

Proof. $\Rightarrow$ Suppose $\bigvee(\Delta) \in Res_{\mathbf{CL}^*}(\Gamma)$. Hence, $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta)^*$ (by definition 6.6). But, it is easily verified that this can only be true for $\bigvee(\Delta)$ when:

1. $\bigvee(\Delta) \in CNF^\circ(A)$, for $A \in \Gamma$.
2. $\bigvee(\Delta)$ is derived from $CNF^\circ(\Gamma)$[9] by applying $\mathbf{DS}^*$ (possibly multiple times). This means that $\bigvee(\Delta)$ is derived from two starred clauses in $Res_{\mathbf{CL}}(\Gamma)$, $\bigvee(\Theta \cup \{C\})$ and $\bigvee(\Theta' \cup \{\neg C\})$ for which $\Theta \cap \Theta' \neq \emptyset$.

As a consequence, $A \in Res_{\mathbf{CL}}(\Gamma)$ (by definition 6.5).

$\Leftarrow$ Suppose $A \in Res_{\mathbf{CL}}(\Gamma)$. Hence, $A \in Res_{\mathbf{CL}^*}(\Gamma)$ (by lemma 6.1). ■

From lemma 6.2 now immediately follows that the set of $\mathbf{CL}^*$–resolvents is equal to the set of $\mathbf{CL}$–resolvents.

Theorem 6.4 *$Res_{\mathbf{CL}^*}(\Gamma) = Res_{\mathbf{CL}}(\Gamma)$.*

Consistent Premise Sets. By means of the above definitions and their consequences, some interesting properties of $\mathbf{CL}^*$ can be explicated. The first one is that consistent premise sets yield exactly the same consequence sets as they would for **CL**. This is stated in theorem 6.5 below. To prove this theorem, first consider the lemmas 6.3 and 6.4.

Lemma 6.3 *If Γ is consistent and $\Gamma \vdash_{\mathbf{CL}} A$, then $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$.*

Proof. Suppose (1) Γ is consistent, (2) $\Gamma \vdash_{\mathbf{CL}} A$ and (3) $Res_{\mathbf{CL}}(\Gamma) \nvdash_{\mathbf{CL\bar{u}Ns}} A$.

Consequence 1. From (1) and (2), it follows that $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}} CNF(A)$

[8] By $CNF(\Gamma)$, I obviously mean the set $\{CNF(B) \mid B \in \Gamma\}$.

[9] By $CNF^\circ(\Gamma)$, I obviously mean the union of all sets $CNF^\circ(A)$ with $A \in \Gamma$.

(by theorem 6.3, fact 6.1, and the transitivity of **CL**–deduction). From this, it follows that for all $\bigvee(\Delta) \in CNF^{\circ}(A)$ there is a $\bigvee(\Delta')$ with $\Delta' \subseteq \Delta$ such that $\bigvee(\Delta') \in Res_{\mathbf{CL}}(\Gamma)$ (otherwise, as Γ is consistent, $Res_{\mathbf{CL}}(\Gamma) \nvdash_{\mathbf{CL}} CNF(A)$).
Consequence 2. From (3), it follows that $Res_{\mathbf{CL}}(\Gamma) \nvdash_{\mathbf{CL\bar{u}Ns}} CNF(A)$ (by fact 6.2). From the latter, it follows that there is a $\bigvee(\Delta) \in CNF^{\circ}(A)$ such that for all $\bigvee(\Delta')$ with $\Delta' \subseteq \Delta$, it is the case that $\bigvee(\Delta') \notin Res_{\mathbf{CL}}(\Gamma)$ (otherwise $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} CNF(A)$ because of the inference rule **ADD**).

As consequence 1 and consequence 2 are contradictory, it follows that $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$. ■

Lemma 6.4 *If* Γ *is consistent, then* $\Gamma \vdash_{\mathbf{CL}^*} A$ *iff* $\Gamma \vdash_{\mathbf{CL}} A$.

Proof. The left–right direction is obvious from theorem 6.1. The right–left direction is a bit harder to prove. Suppose Γ is consistent and $\Gamma \vdash_{\mathbf{CL}} A$. From this, it follows that:

(1) for all $B \in Res_{\mathbf{CL}}(\Gamma)$, $\Gamma \vdash_{\mathbf{CL}^*} B^*$ (by lemma 6.1).
(2) $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$ (by lemma 6.3). This gives us $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}^*} A$ (by theorem 6.1).

From (1) and (2), it follows that $\Gamma \vdash_{\mathbf{CL}^*} A$ (by theorem 6.2). ■

From lemma 6.4, it immediately follows that for consistent premise sets, $\mathbf{CL}^*$ yields the same consequence set as **CL**.

Theorem 6.5 *If* Γ *is consistent, then* $Cn_{\mathbf{CL}^*}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$.

In Between **CL** *and* **CLūNs**. Theorem 6.1 already showed that the logic $\mathbf{CL}^*$ is situated somewhere in between the logics **CLūNs** and **CL**. It is nevertheless possible to pin down its exact position. This is done by theorem 6.6 below. As such, this theorem shows us how $\mathbf{CL}^*$ is related to both **CLūNs** and **CL**.

Lemma 6.5 *For* $\bigvee(\Delta)$ *a clause, if* $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta)$ *and there is no* $\Delta' \subset \Delta$ *such that* $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta')$, *then* $\bigvee(\Delta) \in Res_{\mathbf{CL}^*}(\Gamma)$ *or* $\vdash_{\mathbf{CL}^*} \bigvee(\Delta)$.

Proof. Suppose (1) the antecedent is true, (2) $\bigvee(\Delta) \notin Res_{\mathbf{CL}^*}(\Gamma)$ and (3) $\nvdash_{\mathbf{CL}^*} \bigvee(\Delta')$. From (2), it follows that $\Gamma \nvdash_{\mathbf{CL}^*} \bigvee(\Delta)^*$ (by definition 6.6). Hence, there is no $\mathbf{CL}^*$–proof of $\bigvee(\Delta)^*$ from Γ. As a consequence, it is necessary to use **ADD** or **TH** in order to prove $\bigvee(\Delta)$, which means that $\vdash_{\mathbf{CL}^*} \bigvee(\Delta)$ (which contradicts (3)), or that there is a $\Delta' \subset \Delta$ such that $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta')$ (which contradicts (1)). Consequently, $\bigvee(\Delta) \in Res_{\mathbf{CL}^*}(\Gamma)$ or $\vdash_{\mathbf{CL}^*} \bigvee(\Delta)$. ■

Theorem 6.6 $\Gamma \vdash_{\mathbf{CL}^*} A$ *iff* $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$.

Proof. $\Rightarrow$ Suppose $\Gamma \vdash_{\mathbf{CL}^*} A$. From this, it follows that $\Gamma \vdash_{\mathbf{CL}^*} CNF(A)$ (by fact 6.4). Hence, for all $\bigvee(\Delta) \in CNF^{\circ}(A)$, $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta)$. As a consequence, for all those $\bigvee(\Delta)$, there will a $\Delta' \subseteq \Delta$, $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta')$ such that there is no $\Delta'' \subset \Delta'$, $\Gamma \vdash_{\mathbf{CL}^*} \bigvee(\Delta'')$. As a consequence, $\bigvee(\Delta') \in Res_{\mathbf{CL}^*}(\Gamma)$ or $\vdash_{\mathbf{CL}^*} \bigvee(\Delta')$ (by lemma 6.5). From this, it follows that $Res_{\mathbf{CL}^*}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$ (by the **CLūNs**–inference rules **TH** and **ADD** and fact 6.2), which also means that $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$ (by theorem 6.4).

$\Leftarrow$ Suppose $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$. First of all, from this, it follows that $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL}^*} A$ (by theorem 6.1). Secondly, for all $B \in Res_{\mathbf{CL}}(\Gamma)$, $\Gamma \vdash_{\mathbf{CL}^*} B^*$ (by lemma 6.1). From both, it now follows that $\Gamma \vdash_{\mathbf{CL}^*} A$ (by theorem 6.2). ■

Relevant Deduction. It goes without saying that similar metatheoretical properties are also valid for **CLāNs***. I will not prove them anymore, as the proofs are analogous to the ones for **CL***. Nevertheless, the most important theorems are stated below.

Theorem 6.7 *For a premise set* Γ, $Cn_{\mathbf{CL\bar{o}Ns}}(\Gamma) \subseteq Cn_{\mathbf{CL\bar{a}Ns}^*}(\Gamma) \subseteq Cn_{\mathbf{CL\bar{a}Ns}}(\Gamma)$.

Theorem 6.8 *If* $\Gamma \vdash_{\mathbf{CL\bar{a}Ns}^*} A^*$ *and* $\Gamma' \cup \{A\} \vdash_{\mathbf{CL\bar{a}Ns}^*} B^{(*)}$, *then* $\Gamma \cup \Gamma' \vdash_{\mathbf{CL\bar{a}Ns}^*} B^{(*)}$.

Theorem 6.9 *If* Γ *is consistent, then* $Cn_{\mathbf{CL\bar{a}Ns}^*}(\Gamma) = Cn_{\mathbf{CL\bar{a}Ns}}(\Gamma)$.

Theorem 6.10 $\Gamma \vdash_{\mathbf{CL\bar{a}Ns}^*} A$ *iff* $Res_{\mathbf{CL}}(\Gamma) \vdash_{\mathbf{CL\bar{o}Ns}} A$.

6.3 Conclusion

In this chapter, I have presented an intuitive theory of first degree relevance, based on the transfer of deductive weight. But, although this also resulted in a nice proof theoretical characterization of first degree relevance, I still need to characterize the logic behind this proof theory, which will be done in the next chapters.

Chapter 7

Classical Relevance: Part 1

7.1 Introduction

In the previous chapter, I presented the theory of classical relevance. Moreover, I also showed how it could be captured proof theoretically. But, a proof theory alone does not make for a logic yet. One needs a semantics too. Hence, in this chapter, I will present the adaptive logic $\exists\mathbf{CL^s}$ of which it will be proven that it is equivalent to the logic $\mathbf{CL}^*$. As such, it can be claimed that the logic $\exists\mathbf{CL^s}$ is the logic behind the $\mathbf{CL}^*$–proof theory.

7.2 The General Idea

The logic $\exists\mathbf{CL^s}$ belongs to the class of the so–called ambiguity–adaptive logics (**AAL**). These were first introduced by Vanackere in [117] and were elaborated on in Vanackere [118, 119, 116] and Batens [17]. The basic idea behind **AAL** is quite simple: as it is possible to interpret a word or sentence in different ways, inconsistencies in a (scientific or common sense) theory might be due to the (semantical) ambiguity of its non–logical constants. To be able to distinguish between those different meanings, indices are attached to them, so that the same sentential letter with two different indices is then treated as two distinct letters.

[AMB] For all $i, j \in \mathbb{N}$ and For all $A \in \mathcal{S}$: if $i \neq j$ then $A^i \neq A^j$.

Ambiguity–adaptive logics were originally devised to interpret an ambiguous premise set as unambiguously as possible. This means that two occurrences of the same sentence letter with different indices, are supposed to mean the same, until or unless it turns out otherwise.

Although the logic $\exists\mathbf{CL^s}$ clearly belongs to the **AAL**, it has nothing do do with ambiguity. It merely makes use of the techniques from **AAL**,[1] and

[1] As such, it could be stated that it comes much closer to the intentions of Brown [44, 45], despite the fact that he was merely interested in paraconsistency.

was devised to capture classical relevance in a decent way. It is related to the logic $\mathbf{CL}^*$ from the foregoing chapter in the following way:

Definition 7.1 $\Gamma \vdash_{\mathbf{CL}^*} A$ *iff* $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$.

In order to understand definition 7.1, some extra definitions are necessary. First, consider the one that shows how the premise set $\Gamma^{\exists \mathbf{i}}$ is constructed:

Definition 7.2 $\Gamma^{\exists \boldsymbol{i}} = \{(\exists i)B^{(i)} \mid B \in \Gamma\}$

Remark that the existential quantifiers mentioned in both definitions above are quantifiers over the indices, not over individual constants.[2] The language of $\exists\mathbf{CL^s}$ remains the standard propositional language $\mathcal{W}$, and is only extended with indices (attached to sentential letters) and quantifiers over those indices.

Next, also consider the definition that shows how to interpret the existential formulas in both definitions above:

Definition 7.3 *When* $A(\xi_1, ..., \xi_n)$ *means that* $\xi_1, ..., \xi_n \in \mathcal{S}$ *are the sentential letters that occur in* $A \in \mathcal{W}$, $(\exists i)A^{(i)}$ *is taken to stand for* $(\exists i_1)...(\exists i_n)$ $A(\xi_1^{i_1}, ..., \xi_n^{i_n})$.

In words, $(\exists i)A^{(i)}$ expresses that all sentential letters in the formula A $(\in \mathcal{W})$ have different variables as their index. Moreover, those variables are all bound by an existential quantifier at the front of the formula. In order to make this more concrete, consider the following example:

Example 7.1 $(\exists i)((p \wedge q) \vee r)^{(i)} =_{df} (\exists i_1)(\exists i_2)(\exists i_3)((p^{i_1} \wedge q^{i_2}) \vee r^{i_3})$

The logic $\exists\mathbf{CL^s}$ is an adaptive logic, which means that it is built on three main components (remember chapter 3): a lower limit logic (**LLL**), a set of abnormalities and an adaptive strategy. As a lot of metatheoretical properties depend on the **LLL**, I will first describe the **LLL** of $\exists\mathbf{CL^s}$.

7.3 The Lower Limit Logic

The adaptive logic $\exists\mathbf{CL^s}$ is based on the lower limit logic $\exists\mathbf{CL}$. As its name might have lead one to suspect, it is very resemblant of first order classical logic. Moreover, as it can be shown that $\exists\mathbf{CL}$ is monotonic, transitive, reflexive and compact, the adaptive logic $\exists\mathbf{CL^s}$ will be a standard adaptive logic.

[2]The language of the first **AAL** did not contain quantifiers. They were introduced in **AAL** by Batens in [17].

7.3.1 Language Schema

Let $\mathcal{L}^{\exists i}$ be the language of $\exists$**CL**. It is defined from $\langle \mathcal{S}^{\mathcal{I}}, \mathcal{V} \rangle$, where $\mathcal{S}^{\mathcal{I}} = \{A^i \mid A \in \mathcal{S} \text{ and } i \in \mathbb{N}\}$ is the set of indexed (sentential) letters, and $\mathcal{V}$ the set of variables. The set of well–formed formulas $\mathcal{W}^{\exists i}$ is the set of all closed formulas of the language $\mathcal{L}^{\exists i}$. It is defined as follows:

(i) $\mathcal{S}^{\mathcal{I}} \subset \mathcal{W}^{\exists i}$.
(ii) When $A \in \mathcal{W}^{\exists i}$ then $\neg A \in \mathcal{W}^{\exists i}$.
(iii) When $A, B \in \mathcal{W}^{\exists i}$ then $(A \wedge B), (A \vee B), (A \sqsupset B) \in \mathcal{W}^{\exists i}$.
(iv) When $A \in \mathcal{W}^{\exists i}$ and $i \in \mathcal{V}$ then $(\exists i)A[i]$, $(\forall i)A[i] \in \mathcal{W}^{\exists i}$.[3]

Classes of Well–formed Formulas. In the characterization of $\exists$**CL**, I will also make use of $\mathfrak{a}$– and $\mathfrak{b}$–formulas. I've put them in table 7.1 below.

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$		$A \vee B$	A	B
$\neg(A \sqsupset B)$	A	$\neg B$		$A \sqsupset B$	$\neg A$	B
$\neg\neg A$	A	A				
$\neg(\exists i)A$	$(\forall i)\neg A$	$(\forall i)\neg A$				
$\neg(\forall i)A$	$(\exists i)\neg A$	$(\exists i)\neg A$				

Table 7.1: $\mathfrak{a}$– and $\mathfrak{b}$–formulas for $\exists$**CL**.

Indexed Wffs. In view of what is to come, I will also introduce some extra terminology. First, the set of *indexed wffs* $\mathcal{W}^{\mathcal{I}} \subset \mathcal{W}^{\exists i}$ is the set of wffs defined as follows:

(i) $\mathcal{S}^{\mathcal{I}} \subset \mathcal{W}^{\mathcal{I}}$.
(ii) When $A \in \mathcal{W}^{\mathcal{I}}$ then $\neg A \in \mathcal{W}^{\mathcal{I}}$.
(iii) When $A, B \in \mathcal{W}^{\mathcal{I}}$ then $(A \wedge B), (A \vee B), (A \sqsupset B) \in \mathcal{W}^{\mathcal{I}}$.

Next, the set of the set of *interpretations* of a formula $A \in \mathcal{W}$ will be denoted by $\mathcal{I}(A)$. A formula $A^{\mathcal{I}}$ is an element of $\mathcal{I}(A)$ iff

i. $A^{\mathcal{I}} \in \mathcal{W}^{\mathcal{I}}$, and
ii. when we drop the indices from $A^{\mathcal{I}}$, we get the formula $A \in \mathcal{W}$.

7.3.2 Proof Theory

The proof theory of $\exists$**CL** is very resemblant to the one for predicative **CL** without identity. It is obtained by adding the inference rules below to the

[3] Obviously, $A[i]$ means that i occurs in A.

proof theory for propositional **CL** that was proposed in chapter 4 (section 4.2.3).

NU	$\neg(\forall i)A[i] \blacktriangleright (\exists i)\neg A[i]$
NE	$\neg(\exists i)A[i] \blacktriangleright (\forall i)\neg A[i]$
UI	$(\forall i)A[i] \blacktriangleright A[i/j]$ $(j \in \mathbb{N})$
EG	$A[i/j] \blacktriangleright (\exists i)A[i]$ $(j \in \mathbb{N})$
UG	$A[i/j] \blacktriangleright (\forall i)A[i]$, provided $j \in \mathbb{N}$ doesn't occur in A, in a premise, or in the hypothesis of an unclosed subproof.
MPE	$(\exists i)A[i]$, $S(A[i/j], B) \blacktriangleright B$, provided $j \in \mathbb{N}$ doesn't occur in A, in B, in a premise, or in the hypothesis of an unclosed subproof.

A $\exists$**CL**–proof is a sequence of wffs each of which is either a premise, a hypothesis, or follows from those earlier in the list by a rule of inference. $\exists$**CL**–derivability can now be defined as follows:

Definition 7.4 $\Gamma \vdash_{\exists\mathbf{CL}} A$ *iff there are* $B_1, ..., B_n \in \Gamma$*, such that there is a* $\exists$**CL**–*proof of* A *from* $B_1, ..., B_n$ *so that* A *has been derived on a line* i *of the main proof.*

The compactness and (pseudo–)deduction theorem are obviously valid, and can be proven in the standard way.

Theorem 7.1 (Compactness Theorem) $\Gamma \vdash_{\exists\mathbf{CL}} A$ *iff there is a finite* $\Delta \subseteq \Gamma$ *such that* $\Gamma \vdash_{\exists\mathbf{CL}} A$.

Theorem 7.2 (Deduction Theorem) *If* $A_1, ..., A_n \vdash_{\exists\mathbf{CL}} B$ *then* $A_1, ...,$ $A_{n-1} \vdash_{\exists\mathbf{CL}} S(A_n, B)$.[4]

7.3.3 Semantics

The semantics of $\exists$**CL** is not characterized w.r.t. the language $\mathcal{L}^{\exists i}$, but w.r.t. the pseudo–language $\mathcal{L}^{\exists i}_{+}$. This might seem rather strange at first, but it is a technique used quite regularly (and successfully) by people from the Ghent Group, see for example [10, 28, ...].

Let $\mathbb{N}'$ be a denumerable set of pseudo–indices, e.g. $1', 2', 3', ...$ The pseudo–language $\mathcal{L}^{\exists i}_{+}$ is now defined from $\langle \mathcal{S}^{\mathcal{I}}_{+}, \mathcal{V} \rangle$, with $\mathcal{S}^{\mathcal{I}}_{+} = \{A^j \mid A \in \mathcal{S}$ and $j \in \mathbb{N} \cup \mathbb{N}'\}$ the extended set of indexed letters, and $\mathcal{V}$ the set of variables. The set of well–formed formulas $\mathcal{W}^{\exists i}_{+}$ is defined for $\mathcal{L}^{\exists i}_{+}$ in the same way as $\mathcal{W}^{\exists i}$ is defined for $\mathcal{L}^{\exists i}$.[5]

An $\exists$**CL**–model for the language $\mathcal{L}^{\exists i}_{+}$ is an assignment function v, characterized as follows:

[4]Remark that for $\exists$**CL** also the usual deduction theorem will be valid.

[5]The pseudo–indices are introduced to safeguard the compactness of the logic $\exists$**CL**. Compactness is necessary, as (standard) adaptive logics are based on a compact **LLL** (see chapter 3).

AP1 $v : \mathcal{S}_{+}^{\mathcal{I}} \mapsto \{0, 1\}$.

The valuation function v_M determined by the model M is defined as follows:

SP1 $v_M : \mathcal{W}_{+}^{\exists i} \mapsto \{0, 1\}$.
SP2 For $A \in \mathcal{S}_{+}^{\mathcal{I}}$, $v_M(A) = 1$ iff $v(A) = 1$.
SP3 For $A \in \mathcal{S}_{+}^{\mathcal{I}}$, $v_M(\neg A) = 1$ iff $v_M(A) = 0$.
SP4 $v_M(\mathfrak{a})= 1$ iff $v_M(\mathfrak{a}_1) = 1$ and $v_M(\mathfrak{a}_2) = 1$.
SP5 $v_M(\mathfrak{b})= 1$ iff $v_M(\mathfrak{b}_2) = 1$ or $v_M(\mathfrak{b}_2) = 1$.
SP6 For $\xi \in \mathcal{S}$: $v_M((\exists i)A[\xi^j]) = 1$ iff $v_M(A[\xi^i]) = 1$ for at least one $j \in \mathbb{N} \cup \mathbb{N}'$.
SP7 For $\xi \in \mathcal{S}$: $v_M((\forall i)A[\xi^i]) = 1$ iff $v_M(A[\xi^j]) = 1$ for all $j \in \mathbb{N} \cup \mathbb{N}'$.

Truth in a model, semantical consequence and validity are defined as usual:

Definition 7.5 *A is true in an $\exists$**CL**–model M iff $v_M(A) = 1$.*

Definition 7.6 *$\Gamma \models_{\exists\mathbf{CL}} A$ iff A is true in all $\exists$**CL**–models in which all elements of Γ are true.*

Definition 7.7 *$\models_{\exists\mathbf{CL}} A$ iff A is true in all $\exists$**CL**–models.*

Soundness and Completeness. It is now quite easy to prove soundness and completeness for $\exists$**CL**. Remark that for completeness to be valid, all $B \in \Gamma$ and A should be restricted to elements of $\mathcal{L}^{\exists i}$.

Theorem 7.3 (Soundness) *If $\Gamma \vdash_{\exists\mathbf{CL}} A$ than $\Gamma \models_{\exists\mathbf{CL}} A$.*

Proof. The soundness proof is an easy extension of the one for **CL** in chapter 4. Hence, it is left to the reader. ■

Theorem 7.4 (Strong Completeness) *If $\Gamma \models_{\exists\mathbf{CL}} A$ than $\Gamma \vdash_{\exists\mathbf{CL}} A$.*

In order to prove the completeness theorem, first consider the following lemma:

Lemma 7.1 *If $\Gamma \supset \Gamma'$ and $\Gamma' \vdash_{\exists\mathbf{CL}} A$, then $\Gamma \vdash_{\exists\mathbf{CL}} A$.*

The proof of this lemma is obvious (because of definition 7.4) and left to the reader. Next, consider the proof for the completeness theorem of $\exists$**CL**.

Proof. Suppose that $\Gamma \nvdash_{\exists\mathbf{CL}} A$. Consider a sequence B_1,B_2,... that contains all wffs of the language $\mathcal{L}_{+}^{\exists i}$ and in which each wff of the form $(\exists i)A[i]$ is followed immediately by an instance with an index that does not occur in Γ, in A, or in any previous member of the sequence. We then define:

$$\begin{array}{ll}\Delta_0 & = Cn_{\exists\mathbf{CL}}(\Gamma)\\ \Delta_{i+1} & = Cn_{\exists\mathbf{CL}}(\Delta_i \cup \{B_i\}) \text{ if } A \notin Cn_{\exists\mathbf{CL}}(\Delta_i \cup \{B_i\}), \text{ and}\\ \Delta_{i+1} & = \Delta_i \text{ otherwise.}\\ \Delta & = \Delta_0 \cup \Delta_1 \cup ...\end{array}$$

Each of the following is provable:

(i) $\Gamma \subseteq \Delta$ (by the construction).
(ii) $A \notin \Delta$ (by the construction).
(iii) Δ is deductively closed (by the definition of Δ).
(iv) Δ is non–trivial (as $A \notin \Delta$).
(v) Δ is prime, i.e. if $C \vee D \in \Delta$, then $C \in \Delta$ or $D \in \Delta$.

Suppose that (1) $C \vee D \in \Delta$, but that (2) $C \notin \Delta$ and $D \notin \Delta$. From (2), it follows that there must be an m and n such that $\Delta_m \cup \{C\} \vdash_{\exists\mathbf{CL}} A$ and $\Delta_n \cup \{D\} \vdash_{\exists\mathbf{CL}} A$ (by the construction of Δ). From these, it follows that $\Delta_m \vdash_{\exists\mathbf{CL}} S(C, A)$ and $\Delta_n \vdash_{\exists\mathbf{CL}} S(D, A)$ (by theorem 7.2). But, this also means that $\Delta \vdash_{\exists\mathbf{CL}} S(C, A)$ and $\Delta \vdash_{\exists\mathbf{CL}} S(D, A)$ (by the construction of Δ and lemma 7.1). From this, together with (1), it follows that $A \in \Delta$ (by the deductive closure of Δ), which is impossible (because of the construction of Δ).

(vi) Δ is ω–complete with respect to $\mathcal{L}_+^{\exists i}$.[6]

Suppose that Δ is not ω–complete with respect to $\mathcal{L}_+^{\exists i}$. Hence, there is a formula $(\exists i)C[i] \in \Delta$ for which there is no $j \in \mathbb{N} \cup \mathbb{N}'$ such that $C[i/j] \in \Delta$. Now, take the formula $(\exists i)C[i]$ to be the k–th formula in the sequence $B_1, B_2, ...$ of $\mathcal{L}_+^{\exists i}$–wffs. Hence, its successor is a formula $C[i/j]$ with j not occurring in Γ, in A, or in a previous member of the sequence $B_1, B_2, ...$ (because of the construction of Δ). Moreover, $C[i/j] \notin \Delta$ (because of our supposition). From the latter, it follows that $\Delta_i \cup \{C[i/j]\} \vdash A$ (by the construction of Δ), so that also $\Delta_i \vdash S(C[i/j], A)$ (by theorem 7.2). But, this means that there are $D_1, ..., D_n \in \Gamma$ such that $D_1, ..., D_n, B_1, ..., B_k \vdash S(C[i/j], A)$ (by definition 7.4). This means that also $\Delta_k \vdash S(C[i/j], A)$ (by lemma 7.1), so that $A \in \Delta_k$ (because $(\exists i)C[i] \in \Delta_k$ and the deductive closure of Δ_k), which is impossible.

I now define an $\exists$**CL**–model M from Δ in the following way:

AP1 For all $C \in \mathcal{S}_+^{\mathcal{I}}$, $v(C) = 1$ iff $C \in \Delta$.

Finally, I show that for all wffs C of the language $\mathcal{L}$, $v_M(C) = 1$ iff $C \in \Delta$. This is done by a straightforward induction on the complexity of the wffs.

[6] Δ is ω–complete iff, if $(\exists i)A(i) \in \Delta$ than $A(j) \in \Delta$ for some $j \in \mathbb{N} \cup \mathbb{N}'$.

The Base Case. For primitive formulas, the proof is immediate because of **AP1** and **SP2**.

The Induction Cases. As this is all completely standard, I will only show how this is done for formulas of the form $(\exists i)C[i]$. The remaining cases, I leave to the reader.

$(\exists i)C\lceil i\rceil \in \Delta$ iff $C[j] \in \Delta$, with $j \in \mathbb{N} \cup \mathbb{N}'$ (as Δ is ω–complete w.r.t. $\mathcal{L}_{+}^{\exists i}$)
iff $v_M(C[j]) = 1$, with $j \in \mathbb{N} \cup \mathbb{N}'$ (by the induction hypothesis)
iff $v_M((\exists i)C[i]) = 1$ (by **SP6**)

As $v_M(C) = 1$ iff $C \in \Delta$, (i) and (ii) give us that $v_M(\Gamma) = 1$ and $v_M(A) = 0$. Hence, $\Gamma \nvDash_{\exists\mathbf{CL}} A$. ■

Now that both soundness and completeness have been proven, the corollary below follows immediately.

Corollary 7.1 $\Gamma \vdash_{\exists\mathbf{CL}} A$ *iff* $\Gamma \vDash_{\exists\mathbf{CL}} A$.

7.3.4 The LLL of $\exists$CL^{s}

From the foregoing sections, it is clear that $\exists$**CL** really is monotonic, transitive, reflexive and compact. But, $\exists$**CL** still has another very interesting characteristic. In relation with the premise set $\Gamma^{\exists \mathbf{i}}$, it is equivalent to the paralogic **CLūNs**, which is made clear by the following theorem:[7]

Theorem 7.5 $\Gamma^{\exists i} \vDash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ *iff* $\Gamma \vDash_{\mathbf{CL\bar{u}Ns}} A$.

Proof. <= Suppose $\Gamma^{\exists i} \nvDash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ and suppose that M (a valuation function v) is an $\exists$**CL**–model that verifies $\Gamma^{\exists i}$ and falsifies $(\exists i)A^{(i)}$. From M, I can now define a **CLūNs**–model M' (a valuation function v') such that M' verifies Γ and falsifies A, which gives us the desired result.

In order to prove the equivalence between both models, define M' in the following way:

(1) Where $A \in \mathcal{S}$, $v'(A) = 1$ iff $v(A^{j}) = 1$ for some $j \in \mathbb{N} \cup \mathbb{N}'$.
(2) Where $A \in \mathcal{S}$, $v'(\neg A) = 1$ iff $v(A^{j}) = 0$ for some $j \in \mathbb{N} \cup \mathbb{N}'$.

Consider a formula $A \in \mathcal{S}$ (the base case). Clearly, $v_{M'}(A) = 1$ iff $v_M((\exists i)(A)^{(i)}) = 1$, and $v_{M'}(\neg A) = 1$ iff $v_M((\exists i)(\neg A)^{(i)}) = 1$.

[7] The proof of theorem 8.4 is based on the proof of theorem 3 of Batens [17].

This result is now easily generalized to all **CLūNs**–wffs, by a straightforward induction over the complexity of formulas. Consider for example the induction case for the conjunction.

$v_{M'}(A \wedge B) = 1$ iff $v_{M'}(A) = 1$ and $v_{M'}(B) = 1$ (by the **CLūNs**–semantics)
iff $v_M((\exists i)A^{(i)}) = 1$ and $v_M((\exists i)B^{(i)}) = 1$ (by the induction hypothesis)
iff $v_M((\exists i)(A \wedge B)^{(i)}) = 1$ (Obvious from the $\exists$**CL**–semantics and definition 7.3)

=> Suppose $\Gamma \nvDash_{\mathbf{CL\bar{u}Ns}} A$ and suppose that M' (a valuation function v') is an **CLūNs**–model that verifies Γ and falsifies A. From M', I can now define a $\exists$**CL**–model M (a valuation function v) such that M verifies $\Gamma^{\exists\mathbf{i}}$and falsifies $(\exists i)A^{(i)}$, which gives us the desired result.

In order to prove the equivalence between both models, define M in the following way:

(1) Where $A \in \mathcal{S}$, $v(A^j) = 1$ for some $j \in \mathbb{N} \cup \mathbb{N}'$ iff $v'(A) = 1$.
(2) Where $A \in \mathcal{S}$, $v(A^j) = 0$ for some $j \in \mathbb{N} \cup \mathbb{N}'$ iff $v'(\neg A) = 1$.

Consider a formula $A \in \mathcal{S}$ (the base case). Clearly, $v_M((\exists i)(A)^{(i)}) = 1$ iff $v_{M'}(A) = 1$, and $v_M((\exists i)(\neg A)^{(i)}) = 1$ iff $v_{M'}(\neg A) = 1$.

This result is now easily generalized to all **CLūNs**–wffs, by a straightforward induction over the complexity of formulas. Consider again the induction case for the conjunction.

$v_M((\exists i)(A \wedge B)^{(i)}) = 1$ iff $v_M((\exists i)A^{(i)}) = 1$ and $v_M((\exists i)B^{(i)}) = 1$ (by the $\exists$**CL**–semantics and definition 7.3)
iff $v_{M'}(A) = 1$ and $v_{M'}(B) = 1$ (by the induction hypothesis)
iff $v_{M'}(A \wedge B) = 1$ (by the **CLūNs**–semantics)

■

7.4 The Adaptive Logic $\exists$CL$^{\mathbf{s}}$

The logic $\exists$**CL**$^{\mathbf{s}}$ is a standard (flat) adaptive logic, so that it is characterized by three components:

(1) Its **LLL** is the logic $\exists$**CL**.
(2) Its set of abnormalities Ω is defined as follows:

Definition 7.8 $\Omega = \{(\exists i)A^{(i)} \wedge \neg A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists i}$ *and* $A^{\mathcal{I}} \in \mathcal{I}(A)\}$.

(3) Its adaptive strategy is the normal selections strategy.

As the logic $\exists\mathbf{CL^s}$ is meant to be used in relation with a premise set $\Gamma^{\exists\mathbf{i}}$ it is possible to add this premise set as the fourth element on which this adaptive logic is built.

(4) A premise set $\Gamma^{\exists\mathbf{i}}$, defined as in definition 7.2.

This completes the general characterization of $\exists\mathbf{CL^s}$, so that we can now have a closer look at its proof theory and semantics.

7.4.1 Proof Theory of $\exists$CL$^{\mathrm{s}}$

The $\exists\mathbf{CL^s}$–proof theory is standard adaptive logics based on the normal selections strategy. First, let's have a look at the deduction rules.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists\mathbf{CL}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\exists\mathbf{CL}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Next, the marking criterium is characterized as follows:

Definition 7.9 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ has been derived at stage s on a line with condition $\emptyset$.*

Finally, also the definitions for final derivability are completely standard.

Definition 7.10 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 7.11 $\Gamma \vdash_{\exists\mathbf{CL^s}} A$ *(A is finally $\exists\mathbf{CL^s}$–derivable from Γ) iff A is finally derived on a line of a proof from Γ.*

Example. The example below clearly illustrates the proof theory of the logic $\exists\mathbf{CL^s}$. Remark that I use Ω_i to refer to the adaptive condition on line i, and that $\bigvee(\Delta)$ is used to refer to the disjunction of the members of Δ (see also definition 6.4).

Consider the premise set $\Gamma^{\exists i} = \{(\exists i)(p)^{(i)}, (\exists i)(\neg p)^{(i)}, (\exists i)(\neg q)^{(i)}, (\exists i)(p \vee q)^{(i)}, (\exists i)(p \vee q \vee r)^{(i)}, (\exists i)(p \vee \neg p \vee s)^{(i)}\}$.

1	$(\exists i)(\neg p)^{(i)}$	PREM	$\emptyset$
2	$(\exists i)(\neg q)^{(i)}$	PREM	$\emptyset$
3	$(\exists i)(p \vee q \vee r)^{(i)}$	PREM	$\emptyset$
4	$p^1 \vee q^2 \vee r^1$	3;RC	$\{(\exists i)(p \vee q \vee r)^{(i)} \wedge \neg(p^1 \vee q^2 \vee r^1)\}$
5	$\neg p^1$	1;RC	$\{(\exists i)(\neg p)^{(i)} \wedge \neg(\neg p^1)\}$
6	$\neg q^2$	2;RC	$\{(\exists i)(\neg q)^{(i)} \wedge \neg(\neg q^2)\}$
7	$(\exists i)(r)^{(i)}$	4,5,6;RU	$\Omega_4 \cup \Omega_5 \cup \Omega_6$
8	$(\exists i)(p)^{(i)}$	PREM	$\emptyset$
9	$(\exists i)(p \vee \neg p \vee s)^{(i)}$	PREM	$\emptyset$
10	p^2	8;RC	$\{(\exists i)(p)^{(i)} \wedge \neg p^2\}$
11	$p^1 \vee \neg p^2 \vee s^1$	9;RC	$\{(\exists i)(p \vee \neg p \vee s)^{(i)} \wedge \neg(p^1 \vee \neg p^2 \vee s^1)\}$
12	$(\exists i)(s)^{(i)}$	5,10,11;RU	$\Omega_5 \cup \Omega_{10} \cup \Omega_{11}$
13	$(\exists i)(p \vee q)^{(i)}$	PREM	$\emptyset$
14	$p^1 \vee q^2$	13;RC	$\{(\exists i)(p \vee q)^{(i)} \wedge \neg(p^1 \vee q^2)\}$
15	$(\exists i)(p \wedge q)^{(i)}$	5,6,14;RU	$\Omega_5 \cup \Omega_6 \cup \Omega_{14}$
16	$(\exists i)(t)^{(i)}$	5,6,14;RU	$\Omega_5 \cup \Omega_6 \cup \Omega_{14}$

Until now, no *Dab*–formulas have been derived, so that no lines are marked yet. This also means that all formulas derived on lines 1–16 are still considered as derivable from the premise set $\Gamma^{\exists \mathbf{i}}$. This however changes once we add line 17 to the proof.

15	$(\exists i)(p \wedge q)^{(i)}$	5,6,14;RU	$\Omega_5 \cup \Omega_6 \cup \Omega_{14}$	✓
16	$(\exists i)(t)^{(i)}$	5,6,14;RU	$\Omega_5 \cup \Omega_6 \cup \Omega_{14}$	✓
17	$\bigvee(\Omega_5, \Omega_6, \Omega_{14})$	1,2,13;RU	$\emptyset$	

Line 17 obviously leads to the marking of lines 15 and 16. All other lines remain derivable.

Remark that the formula on line 15 is ultimately derived from the premises introduced on lines 1,2 and 13. Although that formula is marked on line 15, it is nevertheless possible to derive it from those same premises. This is done by extending the proof in the following way:

18	$p^3 \vee q^4$	13;RC	$\{(\exists i)(p \vee q)^{(i)} \wedge \neg(p^3 \vee q^4)\}$

19	$\neg q^4$	2;RC	$\{(\exists i)(\neg q)^{(i)} \wedge \neg(\neg q^4)\}$
20	$(\exists i)(p \wedge q)^{(i)}$	5,14,18,19;RU	$\Omega_5 \cup \Omega_{14} \cup \Omega_{18} \cup \Omega_{19}$

It can easily be checked that the formula on line 20 is finally derivable from the premise set $\Gamma^{\exists\mathbf{i}}$.

Some Metatheoretical Properties of $\exists\mathbf{CL^s}$

A lot of very interesting metatheoretical properties are provable from the proof theoretical description of $\exists\mathbf{CL^s}$. The first one is given by the theorem below.

Theorem 7.6 *$\Gamma^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$ iff there is a finite $\Delta \subset \Omega$ such that $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL}} (\exists i)A^{(i)} \vee Dab(\Delta)$ and $\Gamma^{\exists i} \nvdash_{\exists\mathbf{CL}} Dab(\Delta)$.*

This theorem is valid for $\exists\mathbf{CL^s}$, because it is an instantiation of theorem 11 from Batens et al. [33]. Consequently, the proof is not given here, and I will concentrate on some other properties.

The Nature of Dab–Formulas. In the example above, the formula $(\exists i)(p \wedge q)^{(i)}$ is derived on both line 15 and line 20. Moreover, in both cases, it is derived from exactly the same premises. But, only line 15 gets marked. Although this might seem quite odd at first, it is a straightforward consequence of (1) the logical form of the $\exists\mathbf{CL^s}$–abnormalities, and (2) the construction of the premise set.

First, consider the set of abnormalities Ω. It only contains conjunctions that consist of an existentially quantified formula and the negation of a particular instantiation of that formula. Now, as the existential formulas are taken to be a members of the premise set (which is clearly specified in definition 7.8), they will always be derivable. This however also means that it will solely depend on the instantiations of the existential formulas, whether or not a *Dab*–consequence will be derivable from the premise set. This is made clear by the following theorem:[8]

Theorem 7.7 *For $A \in \Gamma$, $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL}} ((\exists i)A_1^{(i)} \wedge \neg A_1^{\mathcal{I}}) \vee ... \vee ((\exists i)A_n^{(i)} \wedge \neg A_n^{\mathcal{I}})$ iff $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL}} \neg A_1^{\mathcal{I}} \vee ... \vee \neg A_n^{\mathcal{I}}$.*

Secondly, also take the premise set $\Gamma^{\exists\mathbf{i}}$ into account. As it cannot contain indexed wffs (elements of $\mathcal{W}^{\mathcal{I}}$), it can only lead to *Dab*–formulas when the required disjunction of negated instantiations is a $\exists\mathbf{CL}$–theorem. As such, theorem 7.7 can be rewritten as follows:

[8] The proofs of theorems 7.7, 7.8, 7.9, 7.10 and 7.11 are obvious and left to the reader.

Theorem 7.8 *For* $A \in \Gamma$, $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}} ((\exists i)A_1^{(i)} \wedge \neg A_1^{\mathcal{I}}) \vee ... \vee ((\exists i)A_n^{(i)} \wedge \neg A_n^{\mathcal{I}})$ *iff* $\vdash_{\exists \mathbf{CL}} \neg A_1^{\mathcal{I}} \vee ... \vee \neg A_n^{\mathcal{I}}$.

Because of the relation between $\exists$**CL** and **CL**, such a disjunction will only be derivable when the set $\{A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}\}$ is inconsistent. As a consequence, theorem 7.8 is equivalent to the theorem below:

Theorem 7.9 *For* $A \in \Gamma$, $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}} ((\exists i)A_1^{(i)} \wedge \neg A_1^{\mathcal{I}}) \vee ... \vee ((\exists i)A_n^{(i)} \wedge \neg A_n^{\mathcal{I}})$ *iff* $\{A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}\}$ *is inconsistent.*

From the above, and given the fact that we define $Dab[A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}]$ as in definition 7.12, both theorem 7.10 and 7.11 below can be proven in a straightforward way.

Definition 7.12 $Dab[A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}] =_{df} ((\exists i)A_1^{(i)} \wedge \neg A_1^{\mathcal{I}}) \vee ... \vee ((\exists i)A_n^{(i)} \wedge \neg A_n^{\mathcal{I}})$.

Theorem 7.10 $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$ *iff there is a finite* $\{B_1^{\mathcal{I}}, .., B_n^{\mathcal{I}}\} \subset \mathcal{W}^{\mathcal{I}}$ *such that* $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$ *and* $\nvdash_{\exists \mathbf{CL}} \neg B_1^{\mathcal{I}} \vee ... \vee \neg B_n^{\mathcal{I}}$.

Theorem 7.11 $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$ *iff there is a finite* $\{B_1^{\mathcal{I}}, .., B_n^{\mathcal{I}}\} \subset \mathcal{W}^{\mathcal{I}}$ *such that* $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$ *and* $\{B_1^{\mathcal{I}}, .., B_n^{\mathcal{I}}\}$ *is consistent.*

If we now again turn to the example, it has become clear why line 15 gets marked and line 20 doesn't, even though both have the same formula as their second element ($(\exists i)(p \wedge q)^{(i)}$) and are obtained by means of exactly the same premises. On line 15, the formula was derived by presupposing the falsity of a *Dab*–formula based on an inconsistent set of instantiations of the premises. On the other hand, on line 20, it was derived by presupposing the falsity of a *Dab*–formula based on a consistent set.

Monotonicity. From theorem 7.10, it follows that $\exists$**CL**$^{\mathbf{s}}$–derivability solely depends on the set of $\exists$**CL**–theorems. Consequently, the logic $\exists$**CL**$^{\mathbf{s}}$ will be monotonic, as it is obvious that enlarging a premise set will not change the set of $\exists$**CL**–theorems.

Theorem 7.12 *If* $\Gamma_1^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$ *then* $(\Gamma_1 \cup \Gamma_2)^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$.

Proof. Suppose $\Gamma_1^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$. From this, it follows that there is a finite $\{B_1^{\mathcal{I}}, .., B_n^{\mathcal{I}}\} \subset \mathcal{W}^{\mathcal{I}}$ such that $\Gamma_1^{\exists i} \vdash_{\exists \mathbf{CL}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$ and $\nvdash_{\exists \mathbf{CL}} \neg B_1^{\mathcal{I}} \vee ... \vee \neg B_n^{\mathcal{I}}$ (by theorem 7.10). It is obvious that this formula will not become derivable by extending the premise set. As a consequence, $(\Gamma_1 \cup \Gamma_2)^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$. ■

Transitivity. The logic $\exists\mathbf{CL^s}$ is obviously not transitive in general, as the following example makes clear:

Example 7.2 *Although* $\{(\exists i)(p)^{(i)}, (\exists i)(\neg p)^{(i)}\} \vdash_{\exists\mathbf{CL^s}} (\exists i)((p \vee q) \wedge \neg p)^{(i)}$ *and* $\{(\exists i)((p \vee q) \wedge \neg p)^{(i)}\} \vdash_{\exists\mathbf{CL^s}} (\exists i)(q)^{(i)}$*, it is not the case that* $\{(\exists i)(p)^{(i)}, (\exists i)(\neg p)^{(i)}\} \vdash_{\exists\mathbf{CL^s}} (\exists i)(q)^{(i)}$.

Nevertheless, a restricted kind of transitivity is valid for $\exists\mathbf{CL^s}$, as is stated in theorem 7.13 below.

Theorem 7.13 *If for all* $(\exists i)A^{(i)} \in \Gamma_2^{\exists i}$, $\Gamma_1^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$ *and* $\Gamma_2^{\exists i} \vdash_{\exists\mathbf{CL}} (\exists i)B^{(i)}$ *then* $\Gamma_1^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)B^{(i)}$.

Proof. Suppose (1) for all $(\exists i)A^{(i)} \in \Gamma_2^{\exists\mathbf{i}}$, $\Gamma_1^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$, and (2) $\Gamma_2^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}} (\exists i)B^{(i)}$. From (2), it follows that there is a finite $\{(\exists i)A_1^{(i)}, ..., (\exists i)A_n^{(i)}\} \subseteq \Gamma_2^{\exists\mathbf{i}}$ such that $(\exists i)A_1^{(i)}, ..., (\exists i)A_n^{(i)} \vdash_{\exists\mathbf{CL}} (\exists i)B^{(i)}$ (†) (by theorem 7.1). From this, together with (1), it follows that for all $(\exists i)A_j^{(i)}$ $(1 \leqslant j \leqslant n)$, $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}} (\exists i)A_j^{(i)} \vee Dab[B_{1_j}^{\mathcal{I}}, ..., B_{m_j}^{\mathcal{I}}]$ and $\{B_{1_j}^{\mathcal{I}}, ..., B_{m_j}^{\mathcal{I}}\}$ is consistent (††) (by theorem 7.11). From (†) and (††), it follows that $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}} (\exists i)B^{(i)} \vee Dab[B_{1_1}^{\mathcal{I}}, ..., B_{m_1}^{\mathcal{I}}] \vee ... \vee Dab[B_{1_n}^{\mathcal{I}}, ..., B_{m_n}^{\mathcal{I}}]$, and $\{B_{1_1}^{\mathcal{I}}, ..., B_{m_1}^{\mathcal{I}}\}$ is consistent,..., and $\{B_{1_n}^{\mathcal{I}}, ..., B_{m_n}^{\mathcal{I}}\}$ is consistent (by theorem 7.2 and the transitivity of $\exists\mathbf{CL}$). This would give us the desired result (by theorem 7.11) if it could be guaranteed that the set $\{B_{1_1}^{\mathcal{I}}, ..., B_{m_1}^{\mathcal{I}}\} \cup ... \cup \{B_{1_n}^{\mathcal{I}}, ..., B_{m_n}^{\mathcal{I}}\}$ is also consistent. This however can not be guaranteed.

It can nevertheless be guaranteed that there is an equivalent set which does give us the desired result. It is constructed by mapping every set $\{B_{1_j}^{\mathcal{I}}, ..., B_{m_j}^{\mathcal{I}}\}$ $((1 \leqslant j \leqslant n))$ to an equivalent set $\{C_{1_j}^{\mathcal{I}}, ..., C_{m_j}^{\mathcal{I}}\}$ by substituting some or all indexed letters by different ones. This should be done in such a way that (1) all sentential letters that had the same index in the original set, still have the same index, and that (2) no index occurs in more than one set. It is now easily verified that

1. all these sets $\{C_{1_j}^{\mathcal{I}}, ..., C_{m_j}^{\mathcal{I}}\}$ are consistent (if not, also the original set should be inconsistent, which it is not). Moreover, as none of them have indexed letters in common, also their union will be consistent.
2. for all $(\exists i)A_j^{(i)}$ $(1 \leqslant i \leqslant n)$, it will be the case that $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}} (\exists i)A_j^{(i)} \vee Dab[C_{1_j}^{\mathcal{I}}, ..., C_{m_j}^{\mathcal{I}}]$ (This result is reached by simply changing the indices in the proof based on the original set. Remark that this only yields the desired result because no indices occur in $(\exists i)A_j^{(i)}$).

As a consequence, $\Gamma_1^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL^s}} (\exists i)B^{(i)}$ (by theorem 7.11). ∎

7.4.2 Semantics of $\exists$CL^s

The semantics of $\exists\mathbf{CL^s}$ is a standard semantics for adaptive logics based on the normal selections strategy. This means that it first selects those **LLL**–models of the premise set that have minimal abnormal parts. They are called the minimally abnormal models of the premise set.

Definition 7.13 *Where M is a $\exists\mathbf{CL}$–model: its abnormal part is the set $Ab(M) = \{A \in \Omega \mid M \models A\}$.*

Definition 7.14 *A $\exists\mathbf{CL}$–model M of Γ is a minimal abnormal model iff there is no $\exists\mathbf{CL}$–model M' of Γ for which $Ab(M') \subset Ab(M)$.*

Next, all minimal abnormal models that have equal abnormal parts are grouped together in a normal set.

Definition 7.15 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 7.16 *A set Σ of $\exists\mathbf{CL}$–models of Γ is a normal set iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.*

Finally, semantical consequence is defined by relying on the normal sets of a premise set.

Definition 7.17 $\Gamma \models_{\exists\mathbf{CL^s}} A$ *iff A is verified by all members of at least one normal set of $\exists\mathbf{CL}$–models of Γ.*

7.5 Does $\exists$CL^s Capture Classical Relevance?

It can now be shown that the adaptive logic $\exists\mathbf{CL^s}$ really captures classical relevance by proving its equivalence with $\mathbf{CL}^*$. First, consider some terminological remarks.

Terminological Remarks. In the lemmas below, I will make use of a special kind of premise set $\Gamma^{\mathcal{I}}$, denoting the set of all possible interpretations of a premise set Γ. More specifically, $\Gamma^{\mathcal{I}}$ is defined as follows:

Definition 7.18 $\Gamma^{\mathcal{I}} = \{A^{\mathcal{I}} \in \mathcal{I}(A) \mid A \in \Gamma\}$.

Moreover, I will also make use of the set $\mathcal{I}'(A)$, the set of *interpretations* and *pseudo–interpretations* of a formula $A \in \mathcal{W}$. It will be denoted by $\mathcal{I}'(A)$. A formula $A^{\mathcal{I}}$ is an element of $\mathcal{I}'(A)$ iff

i. $A^{\mathcal{I}} \in \mathcal{W}^{\mathcal{I}}_{+}$ (the set of indexed wffs of the language $\mathcal{L}^{\exists i}_{+}$), and

ii. when we drop the indices and pseudo–indices from $A^{\mathcal{I}}$, we get the formula $A \in \mathcal{W}$.

Finally, also consider the following two definitions:

Definition 7.19 $\Omega^* = \{A^i \wedge \neg A^j \mid A \in \mathcal{S}$ *and* $i, j \in \mathbb{N}\}$.

Definition 7.20 Γ^{amb} *is a maximally ambiguous interpretation of* Γ*, which means that all indexed letters occur maximally once in it.*

Equivalence of CL* and $\exists$CL^{s}. In order to prove the equivalence of **CL*** and $\exists$**CL**^{s}, it is necessary to first prove the lemmas below.

Lemma 7.2 $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}^{s}} (\exists i)(A)^{(i)}$ *iff there is a consistent* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ *such that* $\Gamma^{\exists i} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)}$.

Proof. $\Rightarrow$ Suppose $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}^{s}} (\exists i)(A)^{(i)}$. From this, it follows that there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$ (by theorem 7.11). From the latter, it follows that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)} \vee \neg B_1^{\mathcal{I}} \vee ... \vee \neg B_n^{\mathcal{I}}$. This also gives us $\Gamma^{\exists \mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)}$.

$\Leftarrow$ Suppose there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\Gamma^{\exists \mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)}$. From this, it follows that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)} \vee \neg B_1^{\mathcal{I}} \vee ... \vee \neg B_n^{\mathcal{I}}$ (by theorem 7.2 and the $\exists$**CL**–proof theory), which gives us also $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$. As $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ is consistent, it follows from the latter that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)(A)^{(i)}$ (by theorem 7.11). ■

Lemma 7.3 $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL}^{s}} (\exists i)(A)^{(i)}$ *iff for some* $A^{\mathcal{I}} \in \mathcal{I}'(A)$*, there is a consistent* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ *such that* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} A^{\mathcal{I}}$.

Proof. $\Leftarrow$ Suppose for some $A^{\mathcal{I}} \in \mathcal{I}'(A)$, there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} A^{\mathcal{I}}$. From this, it follows that $\vdash_{\exists \mathbf{CL}} (\exists i)A^{(i)} \vee \neg B_1^{\mathcal{I}} \vee ... \vee \neg B_n^{\mathcal{I}}$ (by theorem 7.2 and the $\exists$**CL**–proof theory). This now gives us that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}]$ (because $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$). As $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ is consistent, it follows that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}^{s}} (\exists i)A^{(i)}$ (by theorem 7.11).

$\Rightarrow$ Suppose that (1) for all $A^{\mathcal{I}} \in \mathcal{I}'(A)$ and for all consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$, it is the case that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists \mathbf{CL}} A^{\mathcal{I}}$, and (2) $\Gamma^{\exists \mathbf{i}} \vdash_{\exists \mathbf{CL}^{s}} (\exists i)(A)^{(i)}$.

From (1), it follows that for all $A^{\mathcal{I}} \in \mathcal{I}'(A)$ and for all $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$, it is also the case that $\Gamma^{\exists \mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists \mathbf{CL}} A^{\mathcal{I}}$ (as existential formulas cannot lead to non–existential ones, adding $\Gamma^{\exists \mathbf{i}}$ will not make some $A^{\mathcal{I}} \in \mathcal{I}'(A)$ derivable). As a consequence, also $\Gamma^{\exists \mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvDash_{\exists \mathbf{CL}} A^{\mathcal{I}}$ (by corollary 7.1), which means that not all $\exists$**CL**–models that verify $\Gamma^{\exists \mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ also verify some $A^{\mathcal{I}} \in \mathcal{I}'(A)$ (†) (by definition 7.6).

From (2), it follows that there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\Gamma^{\exists\mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ (by lemma 7.2). This means that also $\Gamma^{\exists\mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \models_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ (by corollary 7.1). This means that all $\exists$**CL**–models that verify $\Gamma^{\exists\mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$, also verify $(\exists i)A^{(i)}$ (by definition 7.6). From the latter, it follows that all models that verify $\Gamma^{\exists\mathbf{i}} \cup \{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$, also verify some $A^{\mathcal{I}} \in \mathcal{I}'(A)$ (††) (by the $\exists$**CL**–semantics).

As (†) and (††) are contradictory, it follows that $\Gamma^{\exists\mathbf{i}} \nvdash_{\exists\mathbf{CL^s}} (\exists i)(A)^{(i)}$.
■

Lemma 7.4 *If for some* $A^{\mathcal{I}} \in \mathcal{I}(A)$, $\Gamma^{amb} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}} \vee \bigvee(\Delta)$ *(*$\Delta \subset \Omega^*$*) and* $\Gamma^{amb} \nvdash_{\exists\mathbf{CL}} \bigvee(\Delta)$ *then there is a consistent* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ *such that* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}}$.

Proof. Suppose that for some $A^{\mathcal{I}} \in \mathcal{I}(A)$, $\Gamma^{amb} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}} \vee \bigvee(\Delta)$ $(\Delta \subset \Omega^*)$ and $\Gamma^{amb} \nvdash_{\exists\mathbf{CL}} \bigvee(\Delta)$. From this, it follows that there is some $\Theta = \{\neg(A^i \equiv A^j) \mid A \in \mathcal{S}$, and $i, j \in \mathbb{N}\}$, such that $\Gamma^{amb} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}} \vee \bigvee(\Theta)$ and $\Gamma^{amb} \nvdash_{\exists\mathbf{CL}} \bigvee(\Theta)$ (because in **CL**, $\neg(A^i \equiv A^j)$ is equivalent to $(A^i \wedge \neg A^j) \vee (A^j \wedge \neg A^i)$). Hence, for $\Theta' = \{A \mid \neg A \in \Theta\}$, it follows that $\Gamma^{amb} \cup \Theta' \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}}$ (by the metatheoretical characterization of $\exists$**CL**). Moreover, as $\Gamma^{amb} \nvdash_{\exists\mathbf{CL}} \bigvee(\Theta)$, it follows that $\Gamma^{amb} \cup \Theta'$ will remain consistent. If all indices of those indexed letters in Γ^{amb} that are identified in Θ', are replaced by new indices (not occurring in Γ^{amb}), a set $\Gamma^{amb'}$ is obtained that is still consistent and for which it is the case that $\Gamma^{amb'} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}}$, for some $A^{\mathcal{I}} \in \mathcal{I}(A)$. This means that there is a consistent set $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}}$ (by the compactness of $\exists$**CL**).
■

Lemma 7.5 *If* $A \in Res_{\mathbf{CL}^*}(\Gamma)$*, then* $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)(A)^{(i)}$.

Proof. Suppose $A \in Res_{\mathbf{CL}^*}(\Gamma)$ Hence, it follows that $\Gamma \vdash_{\mathbf{CL}^*} A^*$ (by definition 6.6), which means that there is a **CL***–proof Φ of A^* from Γ (by definition 6.1). Remark that it is possible to transform the **CL***–proof Φ of A^* from Γ into an $\exists$**CL**–proof Φ' of $B^{\mathcal{I}} \vee \bigvee(\Delta)$ from Γ^{amb} (†), so that A is **CL***–derivable from B by means of the rule **CONT** (‡). The transformation proceeds as in the second part of the proof of theorem 4.8 (see chapter 4), which is possible because the indices only block the inference rules **DS*** and **CONT**. Moreover, as Φ' is a mirror proof of Φ, the rules **ADD** and **TH** are not used in it. As a consequence, none of the indexed letters in $B^{\mathcal{I}}$ occurs in $\bigvee(\Delta)$. From this, it follows that $\Gamma^{amb} \nvdash_{\exists\mathbf{CL}} \bigvee(\Delta)$ (††).

From (†) and (††), it now follows that there is a consistent $\{C_1^{\mathcal{I}}, ..., C_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{C_1^{\mathcal{I}}, ..., C_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} B^{\mathcal{I}}$. (by lemma 7.4). From the latter, it follows that $\Gamma^{\exists\mathbf{i}} \cup \{C_1^{\mathcal{I}}, ..., C_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ (by ‡ and the $\exists$**CL**–proof theory), which also gives us $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL^s}} (\exists i)(A)^{(i)}$ (by lemma 7.2). ■

Finally, it is possible to prove the equivalence of $\exists$**CL**$^{\mathbf{s}}$ and **CL***.

Theorem 7.14 $\Gamma \vdash_{\mathbf{CL}^*} A$ *iff* $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL}^{\mathbf{s}}} (\exists i)A^{(i)}$.

Proof.$\Rightarrow$ Suppose $\Gamma \vdash_{\mathbf{CL}^*} A$. From this, it follows that $Res_{\mathbf{CL}^*}(\Gamma) \vdash_{\mathbf{CL\bar{u}Ns}} A$ (by theorems 6.4 and 6.6), which gives us $(Res_{\mathbf{CL}^*}(\Gamma))^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ (†) (by theorem 7.5). From lemma 7.5, it also follows that for all $B \in Res_{\mathbf{CL}^*}(\Gamma)$, $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}^{\mathbf{s}}} (\exists i)B^{(i)}$ (††). From (†) and (††), it now follows that $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}^{\mathbf{s}}} (\exists i)A^{(i)}$ (by theorem 7.13).

$\Leftarrow$ Suppose $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL}^{\mathbf{s}}} (\exists i)A^{(i)}$. From this, it follows that for some $A^{\mathcal{I}} \in \mathcal{I}(A)$, there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} A^{\mathcal{I}}$ (by lemma 7.3). However, this gives us also $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\mathbf{CL}^*} A^{\mathcal{I}}$ (by lemma 6.4), which means that there is a $\mathbf{CL}^*$–proof of $A^{\mathcal{I}}$ from $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ (by definition 6.1). But, if we now drop the indices, we get a $\mathbf{CL}^*$–proof of A from $\{B_1, ..., B_n\}$ (what can be derived with the indices, can obviously also be derived without them). As a consequence, $\{B_1, ..., B_n\} \vdash_{\mathbf{CL}^*} A$ (by definition 6.1), which means that also $\Gamma \vdash_{\mathbf{CL}^*} A$ (as $\{B_1, ..., B_n\} \subseteq \Gamma$). ■

7.6 Compassionate Relevantism

In multiple publications, Neil Tennant has presented a logical system which he called **CR**, standing for *Compassionate Relevantism*, see e.g. Tennant [107, 108, 109, 111, 113, 114].[9] In this section, I will show that his **CR** is in some way equivalent to the logic $\mathbf{CL}^*$.

Relevant Classical Proofs. In [108, 109], Neil Tennant split up **CL**–proofs into *explosive classical proofs* and *relevant classical proofs*.[10] Conclusions of explosive **CL**–proofs are only derivable because of the inconsistency of the premise set:

> A 'follows' from Γ by dint of Γ's inconsistency, rather than by dint of any genuine deductive connection between Γ and A.[11]

On the contrary, conclusions of relevant **CL**–proofs follow from the premises by "relevant use" of the classical derivation rules. In other words, relevant classical proofs give us all and only those classical consequences of a premise set that are somehow "in" the premises. Those are the relevant consequences of that premise set.

It is immediately obvious that Tennant's approach towards relevance bears a lot of similarities with the approach I presented in chapter 6. As such, it is not surprising that the resulting logical systems are equivalent.

[9] He also developed an intuitionistic variant, see e.g. Tennant [110, 112].
[10] My terminology!
[11] See [114, p. 706]. Remark that here and in the remaining of this section, I have adapted Tennant's logical notation to mine in order to preserve overall coherence.

Semantics of CR. Although Tennant attached more importance to the proof theoretical characterization of **CR** (stated in sequent calculus, see e.g. [109]), I will only present the semantical characterization.

First, remark that Tennant makes use of *set sequents* to characterize **CR**. These are "formulas" of the form Γ:Δ with Γ and Δ sets of **CL**–wffs in which the order and the repetition of elements are considered irrelevant. In the following, I will always restrict the succedent set Δ to a singleton. This doesn't lead to a change in the logic, and will make it easier to prove the equivalence between **CR** and **CL***.

Next, in order to capture the notion of relevant classical proof in a semantical way, Tennant makes use of the notion *entailment*:[12]

Definition 7.21 $\Gamma \models_{\mathbf{CR}} A$ *iff* Γ:Δ *is a* **CR**–*entailment.*

Whether or not a set sequent is an *entailment*, depends on the following definitions:[13]

Definition 7.22 *A **valid sequent*** Γ:A *is a sequent for which there exists no* **CL**–*model that makes all elements of* Γ *true and* A *false.*

Definition 7.23 *A **perfectly valid sequent*** Γ:A *is a valid sequent that has no valid proper subsequents.*

Definition 7.24 *A **proper subsequent** of a sequent* Γ:A *is*

(a) *a valid sequent* Γ':A *such that* $\Gamma' \subset \Gamma$ *(meaning that not all elements of* Γ *are needed in order to derive* A*), or*
(b) *the sequent* Γ:$\emptyset$ *(meaning that* Γ *is inconsistent).*

Definition 7.25 *A sequent* Γ':A' *is a **suprasequent** of* Γ:A*, iff there is a function* s *which replaces each sentential letter from* Γ:A *by a (possibly complex) formula, so that* $s(\Gamma{:}A) = \Gamma'{:}A'$.

Definition 7.26 *A sequent* Γ:A *is an **entailment** iff* Γ:A *has a perfectly valid suprasequent.*

The semantic characterization of **CR** can now be clearly illustrated by some examples. First, consider the following example.

Example 7.3 $\{p, \neg p, p \vee \neg p \vee q\} : q$ *is an entailment, because* $\{r, \neg p, p \vee \neg r \vee q\} : q$ *is a perfectly valid suprasequent of* $\{p, \neg p, p \vee \neg p \vee q\} : q$.

[12] Remark that Tennant uses the notion "entailment" slightly different than in standard Relevance Logic. Nevertheless, he still refers to some kind of "theorem", as in sequent calculus, it is impossible to introduce **CL**–wffs. Only sequents can be introduced.

[13] See Tennant [109, pp. 184–187].

This example is very straightforward. There are also examples that are more demanding. Consider for example the one below.

Example 7.4 $\{p, q, \neg(p \wedge q)\} : \neg p \wedge \neg q$ (†) *is an entailment, because* $\{r, \neg(p \wedge r), s, \neg(q \wedge s)\} : q$ (††) *is a perfectly valid suprasequent of* $\{p, q, \neg(p \wedge q)\} : \neg p \wedge \neg q$.

It seems quite odd to consider (††) as a valid suprasequent of (†). Nevertheless, it becomes obvious when we take into account that Tennant has made use of set sequents. For those, the order and repetition of the elements do not matter, so that the set sequents $\{p, q, \neg(p \wedge q)\}{:}\neg p \wedge \neg q$ and $\{p, q, \neg(p \wedge q), \neg(p \wedge q)\}{:}\neg p \wedge \neg q$ are equivalent.[14]

Relevant Deduction? Definition 7.24 shows that entailments always have minimal antecedent sets. They contain only formulas that are really needed for deriving the formula in the succedent part. Although this might capture relevant classical *proofs* (which was Tennant's objective), it certainly does not capture relevant classical *deduction*. The following example shows us why:

Example 7.5 *Because* $\{p, {\sim}p \vee q\}{:}q$ *is an entailment,* $\{{\sim}p, p, {\sim}p \vee q\}{:}q$ *is not an entailment, even though* $\{p \wedge {\sim}p, {\sim}p \vee q\}{:}q$*, and* $\{p, {\sim}p \wedge ({\sim}p \vee q)\}{:}q$ *are entailments.*

Does this mean that there is no relevant proof of q from $\{{\sim}p, p, {\sim}p \vee q\}$? In a sense, it does, as the antecedent set contains more elements than we really need in order to derive q. But, as there is a relevant proof from a subset of Γ, this obviously shouldn't also mean that q cannot be relevantly *deduced* from the premise set $\Gamma = \{{\sim}p, p, {\sim}p \vee q\}$.

From the above, it is obvious that **CR** can be used in order to capture relevant classical deduction: there is a relevant classical deduction of A from Γ whenever there is a $\Gamma' \subseteq \Gamma$ such that $\Gamma'{:}A$ is an *entailment*. I will call this *deductive Compassionate Relevantism* (**dCR**).

Definition 7.27 $\Gamma \models_{\mathbf{dCR}}$ *iff there is a* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma'{:}A$ *is an* entailment.

Relation with CL*. It can now easily be shown that **dCR** is equivalent to **CL***. First, consider the lemma below.

Lemma 7.6 $\Gamma \models_{\mathbf{dCR}} A$ *iff* $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$.

[14] This is a subtlety in Tennant's approach that is easily overlooked. At least, I did so in [65, 66]. As a consequence, the adaptive logic **AAL**$^{\mathbf{ns}}$ presented in those papers, is not equivalent to **CR**, but to some weaker variant.

Proof. $\Rightarrow$ Suppose $\Gamma \models_{\mathbf{dCR}} A$. From this, it follows that there is a $\Gamma' \subseteq \Gamma$ such that Γ':A is an *entailment* (by definition 7.27), which means that Γ':A has a perfectly valid suprasequent (by definition 7.26). From this, it follows that there is a $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma'^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$:$A^{\mathcal{I}}$ for some $A^{\mathcal{I}} \in \mathcal{I}(A)$ (by the compactness of **CL** and the fact that indexed wffs can be considered substitution instances of the ordinary sentential letters). Moreover, $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ has to be consistent, otherwise $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$:$A^{\mathcal{I}}$ is not a perfectly valid suprasequent of Γ':A (by definitions 7.23 and 7.24). From this, it follows that $\Gamma'^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$ (by theorem 7.3), which also gives us $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$ (by theorem 7.12).

$\Leftarrow$ Suppose $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL^s}} (\exists i)A^{(i)}$. From this, it follows that there is a consistent and minimal $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists \mathbf{CL}} A^{\mathcal{I}}$ for some $A^{\mathcal{I}} \in \mathcal{I}'(A)$ (by lemma 7.3). This means that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$:$A^{\mathcal{I}}$ is a perfectly valid suprasequent of some $\Gamma' \subset \Gamma$ (by corollary 7.1 and definitions 7.23 and 7.24), such that $\{B_1, ..., B_n\}$:A will be an entailment (by definition 7.26). As $\{B_1, ..., B_n\}$ has to be a subset of Γ (because $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ and definition 7.18), it now follows that $\Gamma \models_{\mathbf{dCR}} A$ (by definition 7.27). ■

In view of lemma 7.6, the theorem below now follows immediately.

Theorem 7.15 $\Gamma \vdash_{\mathbf{dCR}} A$ *iff* $\Gamma \vdash_{\mathbf{CL}^*} A$.

7.7 Conclusion and Further Research

In this chapter, I have presented the adaptive logic $\exists\mathbf{CL^s}$, which is equivalent to the logical system $\mathbf{CL}^*$. As such, it captures classical relevance in a logically stringent way. Moreover, I have also shown that the logic $\exists\mathbf{CL^s}$ is also equivalent to the logic **dCR**, the deductive variant of Tennant's **CR**.

Further Research. There are two points of further research concerning the material of this chapter. First of all, the logic $\exists\mathbf{CL^s}$ should be extended to the predicative level. Secondly, also the relation between $\exists\mathbf{CL^s}$ and the goal directed proof procedure developed by Batens and Provijn in [37], should be investigated. As said in chapter 1, this procedure pushes some of the heuristics into the proof theory. If the (artificial) rule *ex falso quodlibet* (**EFQ**) is dropped from the goal directed proof procedure for **CL**, one reaches the proof procedure for an extremely rich paraconsistent logic, called $\mathbf{CL}^-$ in Batens [22, 24]. But, $\mathbf{CL}^-$ seems to be equivalent to $\exists\mathbf{CL^s}$. If this would be the case, then $\exists\mathbf{CL^s}$ can not only be considered the logic behind the goal directed proof procedure, it would also prove that deductive relevance and heuristic relevance are related to one another in a very fundamental way.

Chapter 8

Classical Relevance: Part 2

8.1 Introduction

The adaptive logic $\exists\mathbf{CL^s}$ from the foregoing chapter, was obtained by means of a two–step procedure. The first step consisted in the *ambiguization* of **CL** (resulting in the ambiguity logic $\exists$**CL**), the second step in the *adaptive treatment* of this ambiguization (resulting in $\exists\mathbf{CL^s}$). As it was shown that $\exists\mathbf{CL^s}$ nicely captures classical relevance, this two–step procedure in a sense *relevantized* the non–relevant logic **CL**.

At first sight, it might appear that other irrelevant logics can be relevantized in the same way. This would give us a general (or universal) logic method for handling first degree (classical) relevance (**FDR**). Unfortenately, it is not that straightforward, as is most easily shown be means of an example.

Example 8.1 *Suppose the logic* $\exists$**K** *is the ambiguity logic based on the modal logic* **K** *(see chapter 4). If we now consider the adaptive logic* $\exists\mathbf{K^s}$*, constructed in the same way as* $\exists\mathbf{CL^s}$*, then we get the following results:*

From the premise set $\Gamma = \{(\exists i)(\Box p)^{(i)}, (\exists i)(\Box\neg p)^{(i)}\}$*, it is possible to* $\exists$**K**–*derive the formula* $(\exists i)(\Box q)^{(i)} \vee \neg\Box p^1 \vee \neg\Box\neg p^1$*, while it is not possible to* $\exists$**K**–*derive the formula* $\neg\Box p^1 \vee \neg\Box\neg p^1$*. As a consequence, the formula* $(\exists i)(\Box q)^{(i)}$ *is an* $\exists\mathbf{K^s}$*–consequence of the premise set* Γ*. But, as this is clearly an irrelevant consequence, the logic* $\exists\mathbf{K^s}$ *cannot be considered a relevant modal logic.*

In this chapter, I will show that in order to reach a general method for handling **FDR**, it is not necessary to profoundly alter the relevantizing procedure from above (some minor changes will be necessary though). What is demanded, is a change of perspective. Remember that the logic $\mathbf{CL}^*$ (the logic expressing classical relevance) is situated somewhere in between the logics **CL** and **CLūNs** (see chapter 6). The logic $\exists\mathbf{CL^s}$ captures classical relevance by weakening the logic **CL**. But, and here's the change of perspective, it is also possible to do so by strengthening the logic **CLūNs**. The latter is done by the ambiguity–adaptive logic ∃**CLūNs**$^{\mathbf{s}}$ that I will present below. In the subsequent chapters, it will be shown how this approach can be extended to relevantize a lot of other logics. As such, it can be considered a general method to approach first degree relevance.

8.2 The Lower Limit Logic

The **LLL** of the adaptive logic ∃**CLūNs**$^{\mathbf{s}}$ is the ambiguity logic ∃**CLūNs**, based on the logic **CLūNs**. Its characterization bears a lot of similarities with the one for ∃**CL** (see chapter 7, section 7.3), but there are nevertheless some important differences. First, consider the language schema of ∃**CLūNs**.

8.2.1 Language Schema

The ∃**CLūNs**–language schema is obtained from the one for ∃**CL** (see chapter 7, section 7.3.1) by extending the ∃**CL**–language $\mathcal{L}^{\exists i}$ with an extra negation symbol. As a consequence, the ∃**CLūNs**–language $\mathcal{L}_\circ^{\exists i}$ contains two negation symbols, the **CLūNs**–negation and the **CL**–negation. The former will be denoted as usual (without a subscript), while the latter will be denoted by means of "$\neg_!$". The set of ∃**CLūNs**–wffs $\mathcal{W}_\circ^{\exists i}$ is now constructed in the usual way:

(i) $\mathcal{S}^{\mathcal{I}} \subset \mathcal{W}_\circ^{\exists i}$.
(ii) When $A \in \mathcal{W}_\circ^{\exists i}$ then $\neg A$, $\neg_! A \in \mathcal{W}_\circ^{\exists i}$.
(iii) When $A, B \in \mathcal{W}_\circ^{\exists i}$ then $(A \wedge B), (A \vee B), (A \sqsupset B) \in \mathcal{W}_\circ^{\exists i}$.
(iv) When $A \in \mathcal{W}_\circ^{\exists i}$ and $i \in \mathcal{V}$ then $(\exists i)A[i]$, $(\forall i)A[i] \in \mathcal{W}_\circ^{\exists i}$.

Overview. Consider table 8.1. It clearly states the relations between the languages $\mathcal{L}$ of **CL***, $\mathcal{L}^{\exists i}$ of ∃**CL**, and $\mathcal{L}_\circ^{\exists i}$ of ∃**CLūNs**.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\neg, \wedge, \vee, \sqsupset$	$\mathcal{W}$
$\mathcal{L}^{\exists i}$	$\mathcal{S}^{\mathcal{I}}$	$\neg, \wedge, \vee, \sqsupset, \exists, \forall$	$\mathcal{W}^{\exists i}$
$\mathcal{L}_\circ^{\exists i}$	$\mathcal{S}^{\mathcal{I}}$	$\neg_!, \sim, \wedge, \vee, \sqsupset, \exists, \forall$	$\mathcal{W}_\circ^{\exists i}$

Table 8.1: Relations between $\mathcal{L}$, $\mathcal{L}^{\exists i}$ and $\mathcal{L}_\circ^{\exists i}$.

8.2.2 Proof Theory and Semantics

First of all, the language schema of ∃**CLūNs** is the same as the one for ∃**CL** (see chapter 7, section 7.3.1). Secondly, also the ∃**CLūNs**–proof theory and

–semantics are very similar to those of $\exists$**CL**.

Proof Theory of $\exists$CLūNs. The proof theory of $\exists$**CLūNs** is obtained by adding the inference rules below to the proof theory of **CLūNs**, presented in chapter 4.

NU	$\neg(\forall i)A[i] \blacktriangleright (\exists i)\neg A[i]$
NE	$\neg(\exists i)A[i] \blacktriangleright (\forall i)\neg A[i]$
UI	$(\forall i)A[i] \blacktriangleright A[i/j]$ $(j \in \mathbb{N})$
EG	$A[i/j] \blacktriangleright (\exists i)A[i]$ $(j \in \mathbb{N})$
UG	$A[i/j] \blacktriangleright (\forall i)A[i]$, provided $j \in \mathbb{N}$ doesn't occur in A, in a premise, or in the hypothesis of an unclosed subproof.
MPE	$(\exists i)A[i]$, $S(A[i/j], B) \blacktriangleright B$, provided $j \in \mathbb{N}$ doesn't occur in A, in B, in a premise, or in the hypothesis of an unclosed subproof.
EFQ	A, $\neg_! A \blacktriangleright B$.
EM	$\blacktriangleright A \vee \neg_! A$.

An $\exists$**CLūNs**–proof is a sequence of wffs each of which is either a premise, a hypothesis, or follows from those earlier in the list by a rule of inference. $\exists$**CLūNs**–derivability can now be defined as follows:

Definition 8.1 $\Gamma \vdash_{\exists \mathbf{CL\bar{u}Ns}} A$ *iff there is an* $\exists$**CLūNs**–*proof of* A *from* $B_1, ..., B_n \in \Gamma$ *such that* A *has been derived on a line* i *of the main proof.*

The compactness and (pseudo–)deduction theorem are obviously valid, and can be proven in the standard way.

Theorem 8.1 (Compactness Theorem) $\Gamma \vdash_{\exists \mathbf{CL\bar{u}Ns}} A$ *iff there is a finite* $\Delta \subseteq \Gamma$ *such that* $\Delta \vdash_{\exists \mathbf{CL\bar{u}Ns}} A$.

Theorem 8.2 (Deduction Theorem) *If* $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{u}Ns}} B$ *then* $A_1, ...,$ $A_{n-1} \vdash_{\exists \mathbf{CL\bar{u}Ns}} S(A_n, B)$.

Semantics of $\exists$CLūNs. As for $\exists$**CL**, the $\exists$**CLūNs**–semantics is characterized w.r.t. the pseudo–language $\mathcal{L}^{\exists i}_{\circ +}$ (see chapter 7, section 7.3.3).

An $\exists$**CLūNs**–model M is an assignment function v defined in the following way:

AP1 $v : \mathcal{S}^{\mathcal{I}}_{+} \mapsto \{0, 1\}$.
AP2 $v :\ ^{\neg}\mathcal{S}^{\mathcal{I}}_{+} \mapsto \{0, 1\}$.

The valuation function v_M determined by the model M is characterized as follows:

SP1 $v_M : \mathcal{W}^{\exists i}_{\circ +} \mapsto \{0, 1\}$.

SP2 For $A \in \mathcal{S}^{\mathcal{I}}_{+}$: $v_M(A) = 1$ iff $v(A) = 1$.
SP3u For $A \in \mathcal{S}^{\mathcal{I}}_{+}$: $v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$.
SP3! $v_M(\neg_! A) = 1$ iff $v_M(A) = 0$.
SP4 $v_M(\mathfrak{a}) = 1$ iff $v_M(\mathfrak{a}_1) = 1$ and $v_M(\mathfrak{a}_2) = 1$.
SP5 $v_M(\mathfrak{b}) = 1$ iff $v_M(\mathfrak{b}_1) = 1$ or $v_M(\mathfrak{b}_2) = 1$.
SP6 For $\xi \in \mathcal{S}$: $v_M((\exists i)A[\xi^i]) = 1$ iff $v_M(A[\xi^j]) = 1$ for at least one $j \in \mathbb{N} \cup \mathbb{N}'$.
SP7 For $\xi \in \mathcal{S}$: $v_M((\forall i)A[\xi^i]) = 1$ iff $v_M(A[\xi^j]) = 1$ for all $j \in \mathbb{N} \cup \mathbb{N}'$.

Truth in a model, semantical consequence and validity are defined as for ∃**CL** from chapter 7.

Soundness and Completeness. The logic ∃**CLūNs** is sound and complete with respect to its semantics.

Theorem 8.3 $\Gamma \vdash_{\exists \mathbf{CL\bar{u}Ns}} A$ *iff* $\Gamma \models_{\exists \mathbf{CL\bar{u}Ns}} A$.

The soundness and completeness proofs for ∃**CLūNs** are easily obtained from those for ∃**CL** (see chapter 7, theorem 7.3 and 7.4).

8.2.3 The LLL of ∃CLūNs$^{\mathbf{s}}$

As the logic ∃**CLūNs** is reflexive, transitive, monotonic and compact, it is capable to serve as the **LLL** of an adaptive logic.

Relation with CLūNs. If the ∃**CLūNs**–language is restricted to $\mathcal{L}^{\exists i}$ (the language without the **CL**–negation), then ∃**CLūNs** captures **CLūNs**–derivability:[1]

Theorem 8.4 $\Gamma^{\exists i} \models_{\exists \mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)}$ *iff* $\Gamma \vdash_{\mathbf{CL\bar{u}Ns}} A$.

Proof. <= Suppose $\Gamma^{\exists i} \not\models_{\exists \mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)}$ and suppose that M (a valuation function v) is an ∃**CLūNs**–model that verifies $\Gamma^{\exists i}$ and falsifies $(\exists i)A^{(i)}$. From M, define a **CLūNs**–model M' (a valuation function v') that verifies Γ and falsifies A. This gives us the desired result.

In order to prove the equivalence between both models, define M' in the following way:

(1) Where $A \in \mathcal{S}$, $v'(A) = 1$ iff $v(A^j) = 1$ for some $j \in \mathbb{N} \cup \mathbb{N}'$.
(2) Where $A \in \mathcal{S}$, $v'(\neg A) = 1$ iff $v(A^j) = 0$ or $v(\neg A^j) = 1$, for some $j \in \mathbb{N} \cup \mathbb{N}'$.

[1] The proof of theorem 8.4 is based on the proof of theorem 3 of Batens [17].

Consider a formula $A \in \mathcal{S}$ (the base case). Clearly, $v_{M'}(A) = 1$ iff $v_M((\exists i)(A)^{(i)}) = 1$, and $v_{M'}(\neg A) = 1$ iff $v_M((\exists i)(\neg A)^{(i)}) = 1$.

This result is now easily generalized to all **CLūNs**–wffs, by a straightforward induction over the complexity of formulas. Consider for example the induction case for the conjunction.

$v_{M'}(A \wedge B) = 1$ iff $v_{M'}(A) = 1$ and $v_{M'}(B) = 1$ (by the **CLūNs**–semantics)
iff $v_M((\exists i)A^{(i)}) = 1$ and $v_M((\exists i)B^{(i)}) = 1$ (by the induction hypothesis)
iff $v_M((\exists i)(A \wedge B)^{(i)}) = 1$ (Obvious from the $\exists$**CLūNs**–semantics and definition 7.3)

=> Suppose $\Gamma \nvDash_{\mathbf{CL\bar{u}Ns}} A$ and suppose that M' (a valuation function v') is an **CLūNs**–model that verifies Γ and falsifies A. From M', I can now define a $\exists$**CLūNs**–model M (a valuation function v) such that M verifies $\Gamma^{\exists \mathbf{i}}$and falsifies $(\exists i)A^{(i)}$, which gives us the desired result.

In order to prove the equivalence between both models, define M in the following way:

(1) Where $A \in \mathcal{S}$, $v(A^j) = 1$ for some $j \in \mathbb{N} \cup \mathbb{N}'$ iff $v'(A) = 1$.
(2) Where $A \in \mathcal{S}$, $v(A^j) = 0$ or $v(\neg A^j) = 1$ for some $j \in \mathbb{N} \cup \mathbb{N}'$ iff $v'(\neg A) = 1$.

Consider a formula $A \in \mathcal{S}$ (the base case). Clearly, $v_M((\exists i)(A)^{(i)}) = 1$ iff $v_{M'}(A) = 1$, and $v_M((\exists i)(\neg A)^{(i)}) = 1$ iff $v_{M'}(\neg A) = 1$.

This result is now easily generalized to all **CLūNs**–wffs, by a straightforward induction over the complexity of formulas. Consider again the induction case for the conjunction.

$v_M((\exists i)(A \wedge B)^{(i)}) = 1$ iff $v_M((\exists i)A^{(i)}) = 1$ and $v_M((\exists i)B^{(i)}) = 1$ (by the $\exists$**CLūNs**–semantics and definition 7.3)
iff $v_{M'}(A) = 1$ and $v_{M'}(B) = 1$ (by the induction hypothesis)
iff $v_{M'}(A \wedge B) = 1$ (by the **CLūNs**–semantics)

■

8.3 The Adaptive Logic $\exists$CLūNs$^{\mathbf{s}}$

The adaptive logic $\exists$**CLūNs**$^{\mathbf{s}}$ is not a flat adaptive logic, but a simple combined adaptive logic. As such, it is characterized by the following three components:

(1) Its **LLL** is the logic $\exists$**CLūNs**.
(2) Its set of abnormalities Ω is the union of the sets Ω_1 and Ω_2 below:

a) $\Omega_1 = \{(\exists i)A^{(i)} \wedge \neg_! A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}} \text{ and } A^{\mathcal{I}} \in \mathcal{I}(A)\}$.
b) $\Omega_2 = \{A^i \wedge \neg A^j \mid A \in \mathcal{S} \text{ and } i, j \in \mathbb{N}\}$.

(3) Its adaptive strategy is the normal selections strategy.

8.3.1 Proof Theory of $\exists$CLūNs$^{\mathbf{s}}$

As $\exists$**CLūNs**$^{\mathbf{s}}$ is a simple combined **AL**, its proof theory is the standard proof theory of flat **AL**. First, consider the deduction rules.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{u}Ns}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{u}Ns}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Next, consider the normal selections–marking criterium:

Definition 8.2 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ has been derived at stage s on a line i with condition $\emptyset$.*

Finally, also consider the definitions for final derivability:

Definition 8.3 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 8.4 $\Gamma \vdash_{\exists \mathbf{CL\bar{u}Ns^s}} A$ *(A is finally* $\exists$**CLūNs**$^{\mathbf{s}}$*–derivable from Γ) iff A is finally derived on a line of a proof from Γ.*

As for all adaptive logics based on the normal selections strategy, the core of the proof theory of $\exists$**CLūNs**$^{\mathbf{s}}$ is captured by the following theorem:[2]

Theorem 8.5 $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$ *iff there are finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$ such that $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab(\Delta \cup \Theta)$ and $\Gamma^{\exists i} \nvdash_{\exists \mathbf{CL\bar{u}Ns}} Dab(\Delta \cup \Theta)$.*

[2]This is proven for all **AL** in Batens et al. [33, pp. 10–11].

Example. In order to make it easier to compare $\exists$**CLūNs**$^{\mathbf{s}}$ with $\exists\mathbf{CL^s}$, I'll give the same example as I gave for $\exists\mathbf{CL^s}$ in section 7.4.1. First, consider again the premise set $\Gamma^{\exists\mathbf{i}}$:

$$\Gamma^{\exists i} = \{(\exists i)(p)^{(i)}, (\exists i)(\neg p)^{(i)}, (\exists i)(\neg q)^{(i)}, (\exists i)(p\vee q)^{(i)}, (\exists i)(p\vee q\vee r)^{(i)}, (\exists i)(p\vee \neg p\vee s)^{(i)}\}.$$

Next, consider the $\exists$**CLūNs**$^{\mathbf{s}}$–proof below:

1	$(\exists i)(\neg p)^{(i)}$	PREM	$\emptyset$	
2	$(\exists i)(\neg q)^{(i)}$	PREM	$\emptyset$	
3	$(\exists i)(p\vee q\vee r)^{(i)}$	PREM	$\emptyset$	
4	$p^1\vee q^2\vee r^1$	3;RC	$\{(\exists i)(p\vee q\vee r)^{(i)}\wedge\neg_!(p^1\vee q^2\vee r^1)\}$	
5	$\neg p^1$	1;RC	$\{(\exists i)(\neg p)^{(i)}\wedge\neg_!(\neg p^1)\}$	
6	$\neg q^2$	2;RC	$\{(\exists i)(\neg q)^{(i)}\wedge\neg_!(\neg q^2)\}$	
7	$(\exists i)(r)^{(i)}$	4,5,6;RU	$\Omega_4\cup\Omega_5\cup\Omega_6\cup\{p^1\wedge\neg p^1, q^2\wedge\neg q^2\}$	
8	$(\exists i)(p)^{(i)}$	PREM	$\emptyset$	
9	$(\exists i)(p\vee\neg p\vee s)^{(i)}$	PREM	$\emptyset$	
10	p^3	8;RC	$\{(\exists i)(p)^{(i)}\wedge\neg_! p^5\}$	
11	$p^1\vee\neg p^3\vee s^1$	9;RC	$\{(\exists i)(p\vee\neg p\vee s)^{(i)}\wedge\neg_!(p^1\vee\neg p^3\vee s^1)\}$	
12	$(\exists i)(s)^{(i)}$	5,10,11;RU	$\Omega_5\cup\Omega_{10}\cup\Omega_{11}\cup\{p^1\wedge\neg p^1, p^3\wedge\neg p^3\}$	
13	$(\exists i)(p\vee q)^{(i)}$	PREM	$\emptyset$	
14	$p^1\vee q^2$	13;RC	$\{(\exists i)(p\vee q)^{(i)}\wedge\neg_!(p^1\vee q^2)\}$	
15	$(\exists i)(p\wedge q)^{(i)}$	5,6,14;RU	$\Omega_5\cup\Omega_6\cup\Omega_{14}\cup\{p^1\wedge\neg p^1, q^2\wedge\neg q^2\}$	✓
16	$(\exists i)(t)^{(i)}$	5,6,14;RU	$\Omega_5\cup\Omega_6\cup\Omega_{14}\cup\{p^1\wedge\neg p^1, q^2\wedge\neg q^2\}$	✓
17	$\bigvee(\Omega_5,\Omega_6,\Omega_{14}\cup\{p^1\wedge\neg p^1, q^2\wedge\neg q^2\})$	1,2,13;RU	$\emptyset$	
18	$p^3\vee q^4$	13;RC	$\{(\exists i)(p\vee q)^{(i)}\wedge\neg_!(p^3\vee q^4)\}$	
19	$\neg q^4$	2;RC	$\{(\exists i)(\neg q)^{(i)}\wedge\neg_!(\neg q^4)\}$	
20	$(\exists i)(p\wedge q)^{(i)}$	5,14,18,19;RU	$\Omega_5\cup\Omega_{14}\cup\Omega_{18}\cup\Omega_{19}\cup\{p^1\wedge\neg p^1, q^4\wedge\neg q^4\}$	

8.3.2 Semantics of $\exists$CLūNs$^{\mathrm{s}}$

The semantics of $\exists$**CLūNs**$^{\mathbf{s}}$ is a standard **AL**–semantics for **AL** based on the normal selections strategy. This means that it first selects the minimal abnormal **LLL**–models of the premise set.

Definition 8.5 *Where M is a $\exists$**CLūNs**–model: its abnormal part is the set $Ab(M) = \{A\in\Omega \mid M\models A\}$.*

Definition 8.6 *A $\exists$**CLūNs**–model M of Γ is a minimal abnormal model iff there is no $\exists$**CLūNs**–model M' of Γ for which $Ab(M')\subset Ab(M)$.*

Next, all minimal abnormal models that have equal abnormal parts are grouped together in normal sets.

Definition 8.7 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 8.8 *A set* Σ *of* $\exists$**CLūNs**–*models of* Γ *is a normal set iff for some* $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.

Finally, semantic consequence is defined by relying on the normal sets of a premise set.

Definition 8.9 $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns^s}} A$ *iff* A *is verified by all members of at least one normal set of* $\exists$**CLūNs**–*models of* Γ.

Soundness and Completeness. As $\exists\mathbf{CL\bar{u}Ns^s}$ is a standard adaptive logic, soundness and completeness follow immediately.

Theorem 8.6 $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} A$ *iff* $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns^s}} A$.

8.4 Equivalence of ∃CL$^\mathbf{s}$ and ∃CLūNs$^\mathbf{s}$

The equivalence of $\exists\mathbf{CL\bar{u}Ns^s}$ and $\exists\mathbf{CL^s}$ is stated by theorem 8.7 below. In order to prove the correctness of that theorem, it is necessary to first consider definition 8.10 and lemma 8.1.

Definition 8.10 $Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] =_{df} Dab(\{(\exists i)B_1^{(i)} \wedge \neg_! B_1^{\mathcal{I}}, ..., (\exists i)B_n^{(i)} \wedge \neg_! B_n^{\mathcal{I}}\})$.

Lemma 8.1 $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$ *iff there is a* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ *such that* $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] \vee Dab(\Theta)$ $(\Delta \subset \Omega_2)$ *and* $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$.

Proof. $\Rightarrow$ Suppose $\Gamma^{\exists \mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$. Hence, there are finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$ such that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab(\Delta \cup \Theta)$ and $\Gamma^{\exists \mathbf{i}} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta \cup \Theta)$. (by theorem 8.5), which also means that there is a $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] \vee Dab(\Theta)$ and $\Gamma^{\exists \mathbf{i}} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] \vee Dab(\Theta)$ (by definition 8.10). As existential formulas can never lead to non–existential formulas, it immediately follows that $\nvdash_{\exists\mathbf{CL\bar{u}Ns}} \neg_! B_1^{\mathcal{I}} \vee ... \vee \neg_! B_n^{\mathcal{I}} \vee Dab(\Theta)$. Consequently, $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Theta)$.

$\Leftarrow$ Suppose there is a $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that (1) $\Gamma^{\exists \mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] \vee Dab(\Theta)$ $(\Theta \subset \Omega_2)$ and (2) $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$. From (1), it follows that there are finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$ such that $\Gamma^{\exists \mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab(\Delta \cup \Theta)$ (by definition 8.10). Moreover, from (2), it follows that $\nvdash_{\exists\mathbf{CL\bar{u}Ns}} \neg_! B_1^{\mathcal{I}} \vee ... \vee \neg_! B_n^{\mathcal{I}} \vee Dab(\Theta)$. As existential formulas can never lead to non–existential formulas, it immediately follows that $\Gamma^{\exists \mathbf{i}} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} \neg_! B_1^{\mathcal{I}} \vee ... \vee \neg_! B_n^{\mathcal{I}} \vee Dab(\Theta)$, which means that $\Gamma^{\exists \mathbf{i}} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta \cup \Theta)$. ■

Theorem 8.7 $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$ *iff* $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$.

Proof. $\Rightarrow$ I have to admit that I have not found the left–right part of this proof yet. As such, I leave it open for further research, and I will only give the proof of the right–left side.

$\Leftarrow$ Suppose $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL^s}} (\exists i)A^{(i)}$. Hence, it follows that there is a consistent $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \subset \Gamma^{\mathcal{I}}$ such that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL}} (\exists i)A^{(i)}$ (by lemma 7.3 and the $\exists$**CL**–proof theory). From this, it follows that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee (C_1^{i_1} \wedge \neg C_1^{i_1}) \vee ... \vee (C_m^{i_m} \wedge \neg C_m^{i_m})$ (†) (by theorem 4.7).

Consequence 1. From (†), if follows that $\vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee \neg_! B_1^{\mathcal{I}} \vee ... \vee \neg_! B_n^{\mathcal{I}} \vee (C_1^{i_1} \wedge \neg C_1^{i_1}) \vee ... \vee (C_m^{i_m} \wedge \neg C_m^{i_m})$. As a consequence, $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} (\exists i)A^{(i)} \vee Dab[B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}] \vee Dab(\Delta)$, with $\Delta \subset \Omega_2$ (by definition 8.10 and the characterization of Ω_2).

Consequence 2. As $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\}$ is consistent, it follows that $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} (C_1^{i_1} \wedge \neg C_1^{i_1}) \vee ... \vee (C_m^{i_m} \wedge \neg C_m^{i_m})$. Hence, $\{B_1^{\mathcal{I}}, ..., B_n^{\mathcal{I}}\} \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$.

Result. From consequence 1 and 2, it now follows that $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$ (by lemma 8.1). ■

8.5 Conclusion

From theorem 8.7, it immediately follows that the logic $\exists$**CLūNs**$^{\mathbf{s}}$ also captures classical relevance, which is stated by the theorem below.

Theorem 8.8 *$\Gamma \vdash_{\mathbf{CL}^*} A$ iff $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns^s}} (\exists i)A^{(i)}$.*

Moreover, the logic $\exists$**CLūNs**$^{\mathbf{s}}$ has cleared the way for a general logical theory of first degree relevance, as it can now be shown that the relevantizing procedure used to obtain $\exists$**CLūNs**$^{\mathbf{s}}$, can be used to construct a large set of relevant logics. This will be done in the subsequent chapters.

Chapter 9

First Degree Relevance

9.1 Introduction

In the previous two chapters, I was solely concerned with classical relevance, mostly because of its simplicity. In this chapter, I finally turn to real *first degree relevance* (**FDR**), the actual aim of this dissertation. More specifically, I will present the logic $\exists\mathbf{CL\bar{o}Ns^s}$ which nicely explicates **FDR**, because of its equivalence with the logic $\mathbf{CL\bar{a}Ns}^*$ from chapter 6.

Relevantizing Procedure. The logic $\exists\mathbf{CL\bar{o}Ns^s}$ is based on the relevantizing procedure from the foregoing chapter. As such, it captures **FDR** in the same way as $\exists\mathbf{CL\bar{u}Ns^s}$ captured classical relevance: by means of an ambiguity logic (the logic $\exists\mathbf{CL\bar{o}Ns}$), which is used as the **LLL** of an ambiguity–adaptive logic (the logic $\exists\mathbf{CL\bar{o}Ns^s}$).

9.2 The Lower Limit Logic

The **LLL** of the adaptive logic $\exists\mathbf{CL\bar{o}Ns^s}$ is the ambiguity logic $\exists\mathbf{CL\bar{o}Ns}$. As $\exists\mathbf{CL\bar{o}Ns}$ resembles $\exists\mathbf{CL\bar{u}Ns}$ (see chapter 8) in numerous ways, it is possible to present it rather quickly.

9.2.1 Proof Theory and Semantics

First of all, the language schema of $\exists\mathbf{CL\bar{o}Ns}$ is the same as the one for $\exists\mathbf{CL\bar{u}Ns}$ (see chapter 8, section 8.2.1). Secondly, also the $\exists\mathbf{CL\bar{o}Ns}$–proof theory and –semantics are very similar to those of $\exists\mathbf{CL\bar{u}Ns}$.

Proof Theory of $\exists\mathbf{CL\bar{o}Ns}$. The proof theory of $\exists\mathbf{CL\bar{o}Ns}$ is obtained by adding the inference rules below to the proof theory of the logic $\mathbf{CL\bar{o}Ns}$.

NU $\quad \neg(\forall i)A[i] \blacktriangleright (\exists i)\neg A[i]$

NE $\neg(\exists i)A[i] \blacktriangleright (\forall i)\neg A[i]$
UI $(\forall i)A[i] \blacktriangleright A[i/j]$ $(j \in \mathbb{N})$
EG $A[i/j] \blacktriangleright (\exists i)A[i]$ $(j \in \mathbb{N})$
UG $A[i/j] \blacktriangleright (\forall i)A[i]$, provided $j \in \mathbb{N}$ doesn't occur in A, in a premise, or in the hypothesis of an unclosed subproof.
MPE $(\exists i)A[i]$, $S(A[i/j], B) \blacktriangleright B$, provided $j \in \mathbb{N}$ doesn't occur in A, in B, in a premise, or in the hypothesis of an unclosed subproof.
EFQ A, $\neg_! A \blacktriangleright B$.
EM $\blacktriangleright A \vee \neg_! A$.

An $\exists$**CLōNs**–proof is a sequence of wffs each of which is either a premise, a hypothesis, or follows from those earlier in the list by a rule of inference. $\exists$**CLōNs**–derivability is now defined as follows:

Definition 9.1 $\Gamma \vdash_{\exists \mathbf{CL\bar{o}Ns}} A$ *iff there is an* $\exists$**CL*ō*Ns**–*proof of* A *from* $B_1, ..., B_n \in \Gamma$ *such that* A *has been derived on a line* i *of the main proof.*

The compactness and (pseudo–)deduction theorem are also valid, and can be proven in the standard way.

Theorem 9.1 (Compactness Theorem) $\Gamma \vdash_{\exists \mathbf{CL\bar{o}Ns}} A$ *iff there is a finite* $\Delta \subseteq \Gamma$ *such that* $\Gamma \vdash_{\exists \mathbf{CL\bar{o}Ns}} A$.

Theorem 9.2 (Deduction Theorem) *If* $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{o}Ns}} B$ *then* $A_1, ..., A_{n-1} \vdash_{\exists \mathbf{CL\bar{o}Ns}} S(A_n, B)$.

Semantics of $\exists$CLōNs. The $\exists$**CLōNs**–semantics is characterized with respect to the pseudo–language $\mathcal{L}^{\exists i}_{\circ +}$. It is obtained from $\mathcal{L}^{\exists i}_{\circ}$ in the same way the pseudo–language $\mathcal{L}^{\exists i}_{+}$ is obtained from $\mathcal{L}^{\exists i}$ (see chapter 7, section 7.3.3). Let $\mathcal{W}^{\exists i}_{\circ +}$ be the set of wffs of $\mathcal{L}^{\exists i}_{\circ +}$.

An $\exists$**CLōNs**–model M is now characterized by an assignment function v, defined as follows:

AP1 $v : \mathcal{S}^{\mathcal{I}}_{+} \mapsto \{0, 1\}$.
AP2 $v : \ {}^{\neg}\mathcal{S}^{\mathcal{I}}_{+} \mapsto \{0, 1\}$.

The valuation function v_M determined by the model M is characterized as follows:[1]

SP1 $v_M : \mathcal{W}^{\exists i}_{\circ +} \mapsto \{0, 1\}$.
SP2 For $A \in \mathcal{S}^{\mathcal{I}}_{+}$: $v_M(A) = 1$ iff $v(A) = 1$.
SP3o For $A \in \mathcal{S}^{\mathcal{I}}_{+}$: $v_M(\neg A) = 1$ iff $v(\neg A) = 1$.
SP3! $v_M(\neg_! A) = 1$ iff $v_M(A) = 0$.

[1]Remark that the classes of wffs in $\exists$**CLōNs** are the same as for $\exists$**CL**, which means that there are no $\mathfrak{a}$– or $\mathfrak{b}$–formulas of the form $\neg_! A$.

SP2 $v_M(\mathfrak{a})= 1$ iff $v_M(\mathfrak{a}_1) = 1$ and $v_M(\mathfrak{a}_2) = 1$.
SP3 $v_M(\mathfrak{b})= 1$ iff $v_M(\mathfrak{b}_1) = 1$ or $v_M(\mathfrak{b}_2) = 1$.
SP5 For $\xi \in \mathcal{S}$: $v_M((\exists i)A[\xi^i]) = 1$ iff $v_M(A[\xi^j]) = 1$ for at least one $j \in \mathbb{N} \cup \mathbb{N}'$.
SP6 For $\xi \in \mathcal{S}$: $v_M((\forall i)A[\xi^i]) = 1$ iff $v_M(A[\xi^j]) = 1$ for all $j \in \mathbb{N} \cup \mathbb{N}'$.

Truth in a model, semantical consequence and validity are defined as for $\exists$**CL** (see chapter 7).

Soundness and Completeness. It is easy to prove soundness and completeness for the logic $\exists$**CL$\bar{\text{o}}$Ns**. Both proofs are obtained from those for $\exists$**CL** in a straightforward way.

Theorem 9.3 *$\Gamma \vdash_{\exists\mathbf{CL\bar{o}Ns}} A$ iff $\Gamma \vDash_{\exists\mathbf{CL\bar{o}Ns}} A$.*

$\exists$CL$\bar{\text{o}}$Ns–Theorems. A lot of theorems are valid in $\exists$**CL$\bar{\text{o}}$Ns**. Consider for example the theorems below:

- $\vdash_{\exists\mathbf{CL\bar{o}Ns}} \neg!(p^1 \vee \neg q^2) \vee \neg!(\neg p^3) \vee \neg!(r^2 \wedge q^4) \vee (p^1 \wedge \neg p^3) \vee (q^4 \wedge \neg q^2)$.
- $\vdash_{\exists\mathbf{CL\bar{o}Ns}} \neg!(p^1 \wedge \neg p^1) \vee (p^1 \wedge \neg p^1)$.
- $\vdash_{\exists\mathbf{CL\bar{o}Ns}} \neg!(p^1) \vee \neg(r^2 \wedge \neg p^1)$.

As the paralogic **CL$\bar{\text{o}}$Ns** on which $\exists$**CL$\bar{\text{o}}$Ns** is based, doesn't validate any theorems (see chapter 4), all theorems are the result of the fact that the language was extended with the classical negation. Hence, it is the classical negation that makes the adaptive treatment of an ambiguous premise set possible in $\exists$**CL$\bar{\text{o}}$Ns$^{\text{s}}$**, as in the foregoing chapters it became clear that this adaptive treatment is heavily dependent on the derivability of theorems (see chapters 7 and 8).

Relation with CL$\bar{\text{o}}$Ns. If the $\exists$**CL$\bar{\text{o}}$Ns**–language is restricted to the language $\mathcal{L}^{\exists i}$ (the language without the **CL**–negation), then it can be shown that the logic $\exists$**CL$\bar{\text{o}}$Ns** is in a certain way equivalent to the logic **CL$\bar{\text{o}}$Ns**.

Theorem 9.4 *$\Gamma^{\exists i} \vDash_{\exists\mathbf{CL\bar{o}Ns}} (\exists i)A^{(i)}$ iff $\Gamma \vdash_{\mathbf{CL\bar{o}Ns}} A$.*

The proof of this theorem is completely equivalent to the proof of theorem 8.4 (see chapter 8, section 8.2.3).

9.3 The Adaptive Logic $\exists$CL$\bar{\text{o}}$Ns$^{\text{s}}$

The adaptive logic $\exists$**CL$\bar{\text{o}}$Ns$^{\text{s}}$** is a simple combined ambiguity–adaptive logic. As such, it is characterized by means of the following four components:

(1) Its **LLL** is the reflexive, transitive, monotonic and compact logic $\exists$**CL$\bar{\text{o}}$Ns**.

(2) Its set of abnormalities $\Omega = \Omega_1 \cup \Omega_2$, with

a) $\Omega_1 = \{(\exists i)A^{(i)} \wedge \neg_! A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}}\}$.
b) $\Omega_2 = \{A^i \wedge \neg A^j \mid A \in \mathcal{S} \text{ and } i, j \in \mathbb{N}\}$

(3) The adaptive strategy is the normal selections strategy.
(4) The premise set $\Gamma^{\exists \mathbf{i}} = \{(\exists i)A^{(i)} \in \mathcal{L}^{\exists i} \mid A \in \Gamma\}$.

Proof Theory and Semantics of $\exists$CLōNs$^{\mathbf{s}}$. The proof theory and semantics of $\exists$**CLōNs**$^{\mathbf{s}}$ are completely equivalent to those of $\exists$**CLūNs**$^{\mathbf{s}}$ (see chapter 8, sections 8.3.1 and 8.3.2). As such, I will not present them anymore.

First Degree Relevance. The logic $\exists$**CLōNs**$^{\mathbf{s}}$ now captures first degree relevance in the following way:[2]

Theorem 9.5 *$\Gamma \vdash_{\mathbf{CL\bar{a}Ns}^*} A$ iff $\Gamma^{\exists i} \vdash_{\exists \mathbf{CL\bar{o}Ns^s}} (\exists i)A^{(i)}$, for $\Gamma \subset \mathcal{W}$ and $A \in \mathcal{W}$.*

Remember that $\mathcal{W}$ is the set of wffs of the **CLōNs**–language $\mathcal{L}$ (see ch. 4).

Intuitively, it is immediately clear that the above theorem should be true, because of the fact that the only difference between the adaptive logics $\exists$**CLūNs**$^{\mathbf{s}}$ and $\exists$**CLōNs**$^{\mathbf{s}}$ consists in the fact that the **LLL** of the former allows the derivation of $\mathcal{L}$–theorems (theorems in which no classical negation occurs), while the **LLL** of the latter doesn't. As a consequence, the only difference between the adaptive logics is also that the former allows the derivation of $\mathcal{L}$–theorems, while the latter doesn't. As this is exactly the only difference between **CL*** and **CLāNs*** (see chapter 6), and the logic $\exists$**CLūNs**$^{\mathbf{s}}$ is equivalent to the logic **CL***, the logic $\exists$**CLōNs**$^{\mathbf{s}}$ should be equivalent to the logic **CLāNs***.

Example. The example below clearly illustrates how first degree relevance is captured by the adaptive logic $\exists$**CLōNs**$^{\mathbf{s}}$. First, consider the premise set $\Gamma = \{p \vee q, \neg p, p, \neg s, s \vee r, \neg r\}$. Next, consider the ambiguous premise set $\Gamma^{\exists \mathbf{i}}$, based on Γ:

$$\Gamma^{\exists \mathbf{i}} = \{(\exists i)(p \vee q)^{(i)}, (\exists i)(\neg p)^{(i)}, (\exists i)(p)^{(i)}, (\exists i)(\neg s)^{(i)}, (\exists i)(s \vee r)^{(i)}, (\exists i)(\neg r)^{(i)}\}.$$

It can now be shown that the formulas $(\exists i)(q)^{(i)}$ and $(\exists i)(r \wedge s)^{(i)}$ are $\exists$**CLōNs**$^{\mathbf{s}}$–derivable from $\Gamma^{\exists \mathbf{i}}$.

[2] In this chapter, I will not prove any metatheory anymore, as all metatheoretical proofs are straightforwardly obtained by equivalence to the proofs for $\exists$**CL**$^{\mathbf{s}}$ and $\exists$**CLūNs**$^{\mathbf{s}}$.

1	$(\exists i)(p \vee q)^{(i)}$	—;PREM	$\emptyset$
2	$(\exists i)(\neg p)^{(i)}$	—;PREM	$\emptyset$
3	$(\exists i)(p)^{(i)}$	—;PREM	$\emptyset$
4	$(\exists i)(s \vee r)^{(i)}$	—;PREM	$\emptyset$
5	$(\exists i)(\neg s)^{(i)}$	—;PREM	$\emptyset$
6	$(\exists i)(\neg r)^{(i)}$	—;PREM	$\emptyset$
7	$p^1 \vee q^1$	1;RC	$\{(\exists i)(p \vee q)^{(i)} \wedge \neg_!(p^1 \vee q^1)\}$
8	$\neg p^2$	2;RC	$\{(\exists i)(\neg p)^{(i)} \wedge \neg_!(\neg p^2)\}$
9	p^1	3;RC	$\{(\exists i)(p)^{(i)} \wedge \neg_!(p^1)\}$
10	$(\exists i)(q)^{(i)}$	7,8;RU	$\Omega_7 \cup \Omega_8 \cup \{p^1 \wedge \neg p^2\}$
11	$\bigvee(\Omega_8 \cup \Omega_9 \cup \{p^1 \wedge \neg p^2\})$	2,3;RU	$\emptyset$

It is easily verified that the formula on line 10 is finally derived, despite the fact that it is possible to derive *Dab*–formulas by means of the premises on which that line was derived. Remark that this is also the case for the formula on line 16 below.

12	$s^1 \vee r^1$	4;RC	$\{(\exists i)(s \vee r)^{(i)} \wedge \neg_!(s^1 \vee r^1)\}$
13	$\neg s^2$	5;RC	$\{(\exists i)(\neg s)^{(i)} \wedge \neg_!(\neg s^2)\}$
14	$\neg r^3$	6;RC	$\{(\exists i)(\neg r)^{(i)} \wedge \neg_!(\neg r^3)\}$
15	$s^4 \vee r^4$	4;RC	$\{(\exists i)(s \vee r)^{(i)} \wedge \neg_!(s^4 \vee r^4)\}$
16	$(\exists i)(r \wedge s)^{(i)}$	12,13,14,15;RU	$\Omega_{12} \cup \Omega_{13} \cup \Omega_{14} \cup \Omega_{15} \cup \{s^1 \wedge \neg s^2, r^4 \wedge \neg r^3\}$
17	$\bigvee(\Omega_{12}, \Omega_{13}, \Omega_{14} \cup \{s^1 \wedge \neg s^2, r^1 \wedge \neg r^3\})$	4,5,6;RU	$\emptyset$
18	$\bigvee(\Omega_{13}, \Omega_{14}, \Omega_{15} \cup \{s^4 \wedge \neg s^2, r^4 \wedge \neg r^3\})$	4,5,6;RU	$\emptyset$

9.4 Conclusion

In this chapter, I presented the ambiguity–adaptive logic $\exists$**CLōNs**$^{\mathbf{s}}$. It nicely explicates deductive relevance at the first degree, as it is equivalent with the logic **CLāNs*** from chapter 6.

Further Research. The logic $\exists$**CLōNs**$^{\mathbf{s}}$ only explicates deductive relevance. This is however not the only kind of relevance in reasoning. Also heuristical relevance is important, which is logically captured by the goal directed proof procedure from Batens and Provijn [36] (see chapter 1). As a consequence, it is necessary to investigate whether it is possible to devise a goal directed proof procedure for the logic $\exists$**CLōNs**$^{\mathbf{s}}$. This would then combine both deductive and heuristic relevance into a general formal theory of relevance in reasoning.

Part IV

Relevant Deduction: the return of the disjunctive syllogism

The Aim of Part IV

In the foregoing part, I presented a philosophically motivated account of first degree relevant deduction based on the transfer of deductive weight. Moreover, I also showed how this transfer of deductive weight is captured by the adaptive logic $\exists\mathbf{CL\bar{o}Ns^s}$. Now, the aim of this part is to combine the insights from part III with the relevant implication from standard Relevance Logic. This will result in relevant logics combining a satisfactory notion of relevance at the first degree with a relevant implication.

Overview of Part IV

However, before this combination can be made, I will first show in chapter 10 that it is possible to construct inconsistency–adaptive relevant logics (**IARL**), adaptive logics that combine the insights from inconsistency–adaptive logics with a relevant implication. But, as they already integrate some heuristical elements of reasoning (remember chapter **??**, section **??**), they do not completely capture the dependence relation between premises and conclusion expressed in real deductive reasoning. Nevertheless, they will turn out to be a useful intermediate step towards the adaptive logics that do capture this dependence relation, and that will be presented in chapter 11.

Chapter 10

Inconsistency–Adaptive Relevant Logics

10.1 Introduction

In this chapter, the insights from inconsistency–adaptive logics (**IAL**)[1] will be put to work for standard relevant logics (**RL**) (see chapter 5). This will result in inconsistency–adaptive relevant logics (**IARL**), relevant logics in which the inference rule **DS** has been reintroduced in a conditional way.

However, despite the fact that **IARL** reintroduce **DS** in relevant logics, they do not capture (first degree) deductive relevance yet, as they still unnecessarily limit the deductive strength of those logics. In this, they completely resemble the usual **IAL** . Nevertheless, as the latter, **IARL** might be the right logics to capture the heuristic behavior people display when trying to turn an inconsistent theory into a consistent one (see chapter 2).

Inconsistency–Adaptive Logics. Although **IAL** are well–known, I will nevertheless shortly summarize their most important characteristics. First of all, **IAL** are based on a paraconsistent lower limit logic, a set of abnormalities whose elements are inconsistencies (formulas of the form $A \wedge \neg A$),[2] and an adaptive strategy.

Proof theoretically, **IAL** allow the application of the inference rule **DS** in a conditional way: if a formula of the form $A \vee (B \wedge \neg B)$ has been derived on a line of an **IAL**–proof, the formula A may be derived on a new line, under the condition that $(B \wedge \neg B)$ is not proven problematic.[3] As a consequence, **IAL**

[1] Inconsistency–adaptive logics are the oldest adaptive logics around and are very well–studied, see e.g. Batens [12, 13] and Batens and Meheus [32].

[2] Sometimes the set of abnormalities has to be restricted to a specific subset of the set of inconsistencies. This is necessary in order to avoid flip–flop logics (see e.g. Batens [25]).

[3] Under which conditions a particular inconsistency is considered as problematic depends on the adaptive strategy.

do not invalidate the inference rule **DS** in general. They merely invalidate those applications of **DS** that are (heuristically) problematic.

Two Possibilities. It is important to notice that there are two possible ways to interpret the inference rule **DS**. First of all, it is possible to consider **DS** as an inference rule which is only valid outside the scope of an implication. I will call this *rule disjunctive syllogism* (**RDS**).[4] It is not to be confused with *hypothetical disjunctive syllogism* (**HDS**) which also allows for **DS** within the scope of an implication.

RDS $A \vee (B \wedge {\sim}B) \vdash A$.
HDS $A_1 \rightarrow \ldots \rightarrow A_n \rightarrow (B \vee (C \wedge {\sim}C)) \vdash A \rightarrow \ldots \rightarrow A_n \rightarrow B$.

As implications can be interpreted as the result of hypothetical reasoning processes (usually represented by means of subproofs), the restriction of **DS** to **RDS** would imply that other "laws" obtain in hypothetical and non–hypothetical reasoning. This however seems quite unlikely, as it is clearly in contradiction with human practice. Hence, in order to obtain interesting **IARL** also **HDS** should be allowed.

Overview. In this chapter, I will present the inconsistency–adaptive relevant logics $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ and $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$, respectively in section 10.2 and 10.3. The former only captures **RDS**, while the latter also captures **HDS**. As such, both these logics succeed in reintroducing **DS** in relevant logics.

Remark that I will only present **IARL** based on the relevant logic $\mathbf{R_d}$. But, **IARL** based on alternative **RL**, are obtained along the same lines, so that it is not necessary to also present them.

10.2 The Adaptive logic $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$

The adaptive logic $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ is a standard (flat) adaptive logic. As such, it can be characterized by the usual three components: a lower limit logic, a set of abnormalities and an adaptive strategy. For $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$, these are the following:

(1) Its **LLL** is the relevant logic $\mathbf{R_d}$ (see chapter 5, section 5.5.2).
(2) Its set of abnormalities $\Omega = \{A \wedge {\sim}A \mid A \in \mathcal{S}\}$.
(3) Its adaptive strategy is the normal selections strategy.[5]

Remark that by taking the logic $\mathbf{R_d}$ as the **LLL**, I only aim at strengthening relevant derivability. This also means that the adaptive logic will not change the set of entailments of the logic **R**.

[4] Remark that this resembles Ackermann's treatment of **DS**. In [1], he introduced **DS** as an independent rule, which he called γ.

[5] It is also possible to use one of the other adaptive strategies. I however prefer the normal selections strategy.

10.2.1 Proof Theory and Semantics of $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$

Both the proof theory and the semantics of $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ are completely standard. First, consider the proof theory.

Proof Theory. The $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$–proof theory consists of the usual deduction rules, and the standard marking rule for the normal selections strategy.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\mathbf{R_d}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\mathbf{R_d}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Definition 10.1 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ has been derived at stage s on a line with condition $\emptyset$.*

As the definitions for final $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$–derivability are also standard, I will not mention them anymore.

Example. Consider the example below, it is based on the premise set $\Gamma = \{p \to (r \vee s), p \to {\sim}r, {\sim}p, p \vee q\}$, and clearly illustrates the proof theory of $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$.

1	$p \to (r \vee s)$	—;PREM	$\emptyset$
2	$p \to {\sim}r$	—;PREM	$\emptyset$
3	${\sim}p$	—;PREM	$\emptyset$
4	$p \vee q$	—;PREM	$\emptyset$
5	q	3,4;RC	$\{p \wedge {\sim}p\}$
6	${\sim}p \vee (r \vee s)$	1;RU	$\emptyset$
7	${\sim}p \vee {\sim}s$	2;RU	$\emptyset$
8	${\sim}p \vee r$	6,7;RC	$\{s \wedge {\sim}s\}$

Remark that all lines derived until now are also finally derived, at least if the premise set is not extended. Moreover, also notice that the inference rule **HDS** can never be applied, so that it is for example not possible to derive $p \to s$ from the formulas on lines 1 and 2.

Semantics. As $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ is based on the normal selections strategy, semantic consequence for $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ is defined by relying on the normal sets of $\mathbf{R_d}$–models of a premise set. I will content in only given the relevant definitions.

Definition 10.2 *Where M is an $\mathbf{R_d}$–model: its abnormal part is the set $Ab(M) = \{A \in \Omega \mid M \models A\}$.*[6]

Definition 10.3 *An $\mathbf{R_d}$–model M of Γ is a minimal abnormal model iff there is no $\mathbf{R_d}$–model M' of Γ for which $Ab(M') \subset Ab(M)$.*

Definition 10.4 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 10.5 *A set Σ of $\mathbf{R_d}$–models of Γ is a normal set iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.*

Definition 10.6 $\Gamma \models_{\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}} A$ *iff A is verified by all members of at least one normal set of $\mathbf{R_d}$–models of Γ.*

Soundness and Completeness. As $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ is a standard adaptive logic, based on the normal selections strategy, soundness and completeness is guaranteed.

Theorem 10.1 $\Gamma \vdash_{\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}} A$ *iff* $\Gamma \models_{\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}} A$.

10.3 The Adaptive Logic $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$

The adaptive logic $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$ from the foregoing section, only reintroduces **RDS** in $\mathbf{R_d}$, which means that in $\mathbf{R}^{\mathbf{r}}_{\mathbf{d}}$–proofs, the inference rule **DS** is only applicable outside the scope of an implication. The logic $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$ that will be presented in this section, also reintroduces **HDS** in $\mathbf{R_d}$, so that in $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$–proofs, the inference rule **DS** will also be applicable inside the scope of an implication. As such, the logic $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$ is also able to capture hypothetical reasoning processes that makes use of the inference rule **DS**.

[6] Remember section 5.5.2, where I introduced the deductive world in order to characterize $\mathbf{R_d}$–derivability (there it is still called relevant $\mathbf{R}$–derivability). By now, it should be clear why it was necessary to do so. If $\mathbf{R_d}$–derivability is defined in the usual way (by reference to all worlds of an $\mathbf{R_d}$–model), then it is not possible to define the abnormal set of an $\mathbf{R}$–model. By introducing a deductive world d, this problem has been solved, as for all $\mathbf{R_d}$–models $M \models A$ just comes down to $v_M(A, d) = 1$.

General Characterization. The logic $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$ is characterized by means of the adaptive logic $\mathbf{AR}^{\Diamond}_{\mathbf{d}}$. More specifically, where $\mathcal{L}$ is the language of $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$, and $\mathcal{W}$ the set of wffs of $\mathcal{L}$ (see section 10.3.1), $\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}$ is defined as follows:

Definition 10.7 $\Gamma \vdash_{\mathbf{R}^{\mathbf{ia}}_{\mathbf{d}}} A$ *iff* $\Gamma \vdash_{\mathbf{AR}^{\Diamond}_{\mathbf{d}}} A$*, for* $\Gamma \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$*.*

In the remaining of this section, I will present the logic $\mathbf{AR}^{\Diamond}_{\mathbf{d}}$, starting with its lower limit logic.

10.3.1 The Lower Limit Logic

The lower limit logic of the logic $\mathbf{AR}^{\Diamond}_{\mathbf{d}}$ is the logic $\mathbf{R}^{\Diamond}_{\mathbf{d}}$, which is in fact the logic $\mathbf{R_d}$, extended with some extra connectives.

Language Schema. The logic $\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is based on the language $\mathcal{L}^{\Diamond}$, which is the standard language $\mathcal{L}$ of relevant logics (see chapter 5, section 5.4.1), extended with the classical negation "$\neg_!$" and a new implication symbol "$\Diamond\!\!\rightarrow$". The set of well–formed formulas $\mathcal{W}^{\Diamond}$ of the language $\mathcal{L}^{\Diamond}$ is constructed as usual.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\sim, \wedge, \vee, \rightarrow$	$\mathcal{W}$
$\mathcal{L}^{\Diamond}$	$\mathcal{S}$	$\neg_!, \sim, \wedge, \vee, \rightarrow, \Diamond\!\!\rightarrow$	$\mathcal{W}^{\Diamond}$

Table 10.1: Relations between $\mathcal{L}$ and $\mathcal{L}^{\Diamond}$.

Some words on the new implication symbol $\Diamond\!\!\rightarrow$. It is a define connective, which means that its meaning is completely determined by the meaning of the other connectives. More specifically, the implication $\Diamond\!\!\rightarrow$ is defined as follows:

Definition 10.8 $A \Diamond\!\!\rightarrow B =_{df} \neg_!(A \rightarrow \neg_! B) \vee (\neg_! A \vee B)$.

Boolean Relevant Logic. Because the language $\mathcal{L}^{\Diamond}$ contains classical negation, the logic $\mathbf{R}^{\Diamond}_{\mathbf{d}}$ will be a so–called *Boolean relevant logic*. Boolean **RL** are well–known and were studied most extensively by Bob Meyer, see for example Meyer & Routley [77] and Giambrone & Meyer [58].[7]

Moreover, as the implication $\Diamond\!\!\rightarrow$ is a defined connective, the logic $\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is in fact equivalent to the Boolean relevant logic $\mathbf{R}^{\neg}$ from Giambrone & Meyer [58].

[7] In [93], Restall also made some interesting comments concerning Boolean **RL**.

Semantics. The semantics of the logic $\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is obtained by adding the following clauses to the $\mathbf{R_d}$–semantics (see chapter 5, section 5.4.4):[8]

AP4	$v\colon \mathcal{N} \mapsto \{0,1\}$.
BP1	$v_M(\neg_{!}A, a) = 1$ iff $v_M(A, a) = 0$.
BP2	$v_M({\sim}\neg_{!}A, a) = 1$ iff $v({\sim}\neg_{!}A, a) = 1$.
BP3	$v_M(A \Diamond\!\!\rightarrow B, a) = 1$ iff $v_M(\neg_{!}(A \rightarrow \neg_{!}B) \vee (\neg_{!}A \vee B), a) = 1$.

The definitions of semantical consequence are the same as for $\mathbf{R_d}$.

Definition 10.9 *A valuation function v_M verifies A iff $v_M(A, d) = 1$, and falsifies A iff $v_M(A, d) = 0$.*

Definition 10.10 *A valuation function v_M is a model of Γ iff it verifies all $A \in \Gamma$.*

Definition 10.11 $\Gamma \models_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} A$ *iff no model of Γ falsifies A.*

Proof Theory. The proof theory of $\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is obtained by adding the axiom schemas from table 10.2 to the axiom system of $\mathbf{R_d}$ (see chapter 5, section 5.4.4). The definitions of an $\mathbf{R}^{\Diamond}_{\mathbf{d}}$–proof and of $\mathbf{R}^{\Diamond}_{\mathbf{d}}$–derivability remain as for $\mathbf{R_d}$.

BA1	$A \rightarrow (B \rightarrow (C \vee \neg_{!}C))$
BA2	$\neg_{!}(A \rightarrow B) \vee (\neg_{!}A \vee B)$
BA3	$(A \wedge \neg_{!}A) \rightarrow B$
BA4	$(A \Diamond\!\!\rightarrow B) \rightarrow (\neg_{!}(A \rightarrow \neg_{!}B) \vee (\neg_{!}A \vee B))$
BA5	$(\neg_{!}(A \rightarrow \neg_{!}B) \vee (\neg_{!}A \vee B)) \rightarrow (A \Diamond\!\!\rightarrow B)$
BR1	$(A \wedge B) \rightarrow C \blacktriangleright (A \wedge \neg_{!}C) \rightarrow \neg_{!}B$

Table 10.2: Axioms for Boolean Relevant Logics.

In order to get a better grasp on the behavior of the implication $\Diamond\!\!\rightarrow$, consider the following examples:

- $A \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} B \Diamond\!\!\rightarrow A$
- $A \rightarrow B \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} A \Diamond\!\!\rightarrow B$
- $A \rightarrow (B \vee C) \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} (A \rightarrow B) \vee (A \Diamond\!\!\rightarrow C)$
- $\neg_{!}A \vee B \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} A \Diamond\!\!\rightarrow B$

[8] The set $\mathcal{N} = \{{\sim}\neg_{!}A \mid A \in \mathcal{W}^{\Diamond}\}$.

Soundness and Completeness. Soundness and completeness proofs for $\mathbf{R}_{\mathbf{d}}^{\Diamond}$ are easily obtained from the soundness and completeness proofs for $\mathbf{R}_{\mathbf{d}}$, and will not be given here.

Theorem 10.2 $\Gamma \vdash_{\mathbf{R}_{\mathbf{d}}^{\Diamond}} A$ *iff* $\Gamma \models_{\mathbf{R}_{\mathbf{d}}^{\Diamond}} A$.

10.3.2 Abnormal Formulas

In the foregoing section, I presented the **LLL** of the logic $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$. In this section, I will present its set of abnormalities Ω. It is the union of two sets, Ω^* and $\Omega^{\blacklozenge}$. The former set of abnormalities is particularly easy to characterize, as it is the set of abnormalities of the adaptive logic $\mathbf{R}_{\mathbf{d}}^{\mathbf{r}}$ (see section 10.2):

Definition 10.12 $\Omega^* = \{A \wedge {\sim}A \mid A \in \mathcal{S}\}$.

The latter set of abnormalities is not that easily characterized. In order to do so, first consider $\mathcal{W}^{\Omega} \subset \mathcal{W}$, the set of all formulas of the following form:

$$[B_1\vee]A_1 \to ([B_2\vee]A_2 \to (...([B_{n-1}\vee]A_n \to (B_n \vee \bigvee(\Delta)))...)),$$

with $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{W}$, $1 \leqslant n$ and $\Delta \subset \Omega^*$. That some parts of the above (generic) formula are placed between brackets, means that those parts are not necessary in order for a formula to belong to $\mathcal{W}^{\Omega}$. As this will become clearer by considering some examples, consider the formulas below. It is easily conceived that these formulas all belong to $\mathcal{W}^{\Omega}$.

- $p \to (q \vee (r \wedge {\sim}r))$
- $r \vee p \to (q \vee (r \wedge {\sim}r))$
- $p \to (q \to (s \vee (r \wedge {\sim}r)))$
- $p \to ((t \to s) \vee (q \to (s \vee (r \wedge {\sim}r))))$

Next, for all formulas $A \in \mathcal{W}^{\Omega}$, there is a formula $A^{\blacklozenge}$, which is obtained from A in the following way:

1. replace in A the implication symbols outside $A_1, ..., A_n, B_1, ..., B_n$ by an implication symbol "$\Diamond\!\!\to$", and
2. replace in A the subformula $B_n \vee \bigvee(\Delta)$ by the formula $\bigvee(\Delta)$.

As a consequence, a formula $A^{\blacklozenge}$ will always be of the following form:[9]

$$[B_1\vee]A_1 \Diamond\!\!\to ([B_2\vee]A_2 \Diamond\!\!\to (...([B_{n-1}\vee]A_n \Diamond\!\!\to \bigvee(\Delta))...)),$$

[9] $\bigvee(\Delta)$ refers to the disjunction of the elements of Δ.

with $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{W}$ and $\Delta \subset \Omega^*$.[10] For example, consider again the examples from above:

- $p \Diamond\!\!\rightarrow (q \vee (r \wedge {\sim}r))$
- $r \vee p \Diamond\!\!\rightarrow (q \vee (r \wedge {\sim}r))$
- $p \Diamond\!\!\rightarrow (q \Diamond\!\!\rightarrow (s \vee (r \wedge {\sim}r)))$
- $p \Diamond\!\!\rightarrow ((t \rightarrow s) \vee (q \Diamond\!\!\rightarrow (s \vee (r \wedge {\sim}r))))$

Finally, the second set of abnormal formulas, $\Omega^{\blacklozenge}$, can now be defined:

Definition 10.14 $\Omega^{\blacklozenge} = \{A \wedge A^{\blacklozenge} \mid A \in \mathcal{W}^{\Omega}$*, and* $A \in Cn_{\mathbf{R}_{\mathbf{d}}^{\Diamond}}(\Gamma)\}$.

Remark that the elements of $\Omega^{\blacklozenge}$ are non–standard abnormalities, as they are not solely characterized by means of their logical form, but also by reference to the $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–consequence set (see chapter 3, section 3.2). Anyway, despite the dodgy abnormalities, the adaptive logic $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$ remains quite standard, as will become clear below.

10.3.3 The Adaptive Logic $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$

The adaptive logic $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$ is a simple combined adaptive logic (see chapter 3, section 3.3.1). As such, it can be characterized by the following three components:

(1) The **LLL** is the logic $\mathbf{R}_{\mathbf{d}}^{\Diamond}$ from section 10.3.1.
(2) The set of abnormalities $\Omega = \Omega^* \cup \Omega^{\blacklozenge}$, with Ω^* and $\Omega^{\blacklozenge}$ defined as in section 10.3.2.
(3) The adaptive strategy is the normal selections strategy.[11]

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[10]For relevant logics with a non–reflexive accessibility relation — which means that not for all worlds $a \in W$, it is the case that $Raaa$ — $A \Diamond\!\!\rightarrow B$ does not follow from $A \rightarrow B$. Consequently, for those **RL**, it is necessary to define $A^{\blacklozenge}$ in a slightly different way.

First, a new connective is defined, namely "$\blacklozenge\!\!\rightarrow$":

Definition 10.13 $A \blacklozenge\!\!\rightarrow B =_{df} (A \rightarrow B) \vee (A \Diamond\!\!\rightarrow B)$.

Next, $A^{\blacklozenge}$ is now obtained from the formula $A \in \mathcal{W}^{\Omega}$ in the following way:

1. replace in A the implication symbols outside $A_1, ..., A_n, B_1, ..., B_n$ by an implication symbol "$\blacklozenge\!\!\rightarrow$", and
2. replace in A the subformula $B_n \vee \bigvee(\Delta)$ by the formula $\bigvee(\Delta)$.

As a consequence, a formula $A^{\blacklozenge}$ will always be of the following form:

$$[B_1\vee]A_1 \blacklozenge\!\!\rightarrow ([B_2\vee]A_2 \blacklozenge\!\!\rightarrow (...([B_{n-1}\vee]A_n \blacklozenge\!\!\rightarrow \bigvee(\Delta))...)),$$

with $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{W}$ and $\Delta \subset \Omega^*$.

[11]It is also possible to use an alternative strategy.

As I have already characterized both the lower limit logic and the set of abnormalities of $\mathbf{AR_d^{\Diamond}}$, I can directly move on to the proof theory and the semantics.[12]

A. Proof Theory of $\mathbf{AR_d^{\Diamond}}$

Because of its rather unusual abnormalities, the $\mathbf{AR_d^{\Diamond}}$–proof theory is not completely standard. To characterize it more easily, I first introduce a new way of referring to *Dab*–formulas. Besides the usual $Dab(\Delta)$, I will also make use of $Dab(C_1, ..., C_n)$, which is taken to stand for $(C_1 \wedge C_1^{\blacklozenge}) \vee ... \vee (C_n \wedge C_n^{\blacklozenge})$.

Deduction Rules. It's now possible to present the deduction rules of $\mathbf{AR_d^{\Diamond}}$. Notice that the conditional rule **RC** is non–standard.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\mathbf{R_d^{\Diamond}}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

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[12] Remark that it is possible to construct inconsistency–adaptive (normal) modal logics in approximately the same way. Their **LLL** is a modal paralogic (see chapter 4, section 4.4). Their set of abnormalities is the set $\Omega = \Omega^* \cup \Omega^{\blacklozenge}$, with

(1) $\Omega^* = \{A \wedge {\sim}A \mid A \in \mathcal{S}\}$, and
(2) $\Omega^{\blacklozenge} = \{A \wedge A^{\blacklozenge} \mid A \in \mathcal{W}^{\Omega}$ and $A \in Cn_{\mathbf{LLL}}(\Gamma)\}$.

Their strategy is any of the known adaptive strategies.

Of course, the set $\mathcal{W}^{\Omega}$ is not defined as for relevant logics. It is now defined as the set of all formulas of the standard modal language $\mathcal{W}^{\mathcal{M}}$, having the following form:

$$[B_1\vee]\mathbf{M}(B_2 \vee \mathbf{M}(...\mathbf{M}(B_n \vee \bigvee(\Delta))...)),$$

with $B_1, ..., B_n \in \mathcal{W}^{\mathcal{M}}$, $1 \leqslant n$, $\Delta \subset \Omega^*$, and $\mathbf{M}$ denoting an arbitrary string of modal operators.

As a consequence, also the formula $A^{\blacklozenge}$ gets a different characterization. First, a new (defined) connective is introduced:

Definition 10.15 $\blacklozenge A =_{df} \Box A \vee \Diamond A$.

Next, a formula $A^{\blacklozenge}$ is now obtained from a formula $A \in \mathcal{W}^{\Omega}$ in the following way:

1. replace in A all modal operators outside $B_1, ..., B_n$ by the modal operator $\blacklozenge$, and
2. replace in A the subformula $B_n \vee \bigvee(\Delta)$ by the formula $\bigvee(\Delta)$.

Hence, a formula $A^{\blacklozenge}$ will be of the following form:

$$[B_1\vee]\blacklozenge...\blacklozenge(B_2 \vee \blacklozenge...\blacklozenge(...(B_{n-1} \vee \blacklozenge...\blacklozenge(\bigvee(\Delta))...)),$$

with $B_1, ..., B_{n-1} \in \mathcal{W}^{\mathcal{M}}$ and $\Delta \subset \Omega^*$.

RC1 If $A_1, ..., A_n \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} B \vee Dab(\Theta)$ $(\Delta \subset \Omega^*)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

RC2 If $A_1, ..., A_n \vdash_{\mathbf{R}^{\Diamond}_{\mathbf{d}}} B \vee Dab(\{C_1 \wedge C_1^{\blacklozenge}, ..., C_m \wedge C_m^{\blacklozenge}\})$ and each of $A_1, ..., A_n, C_1, ..., C_m$ occurs in the proof on lines $i_1, ..., i_n, j_1, ..., j_m$ that have conditions $\Delta_1, ..., \Delta_n, \emptyset, ..., \emptyset$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n, j_1, ..., j_m$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_m \wedge C_m^{\blacklozenge}\}$.

Marking Criterium. The marking criterium only consists of one marking rule, namely the standard marking rule for adaptive logics based on the normal selections strategy. However, the way to decide whether or not a *Dab*–formula is also a *Dab*–consequence of a premise set, is not standard.

Definition 10.16 *$Dab(C_1, ..., C_n)$ is a Dab–consequence of Γ at stage s of the proof iff $Dab(C_1, ..., C_n), C_1, ..., C_{n-1}$ and C_n have all been derived on the condition $\emptyset$ at stage s of the proof.*

Definition 10.17 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ is a Dab–consequence of Γ at stage s.*

Final Derivability. The definitions for final $\mathbf{AR}^{\Diamond}_{\mathbf{d}}$–derivability remain standard and as such, I do not consider it necessary to mention them again.

Example. Consider the example below, it is based on the premise set $\Gamma = \{p \to (r \vee s), p \to {\sim}r, q \vee (s \to (r \vee t)), s \to {\sim}r, q \vee (s \to r)\}$.

1	$p \to (r \vee s)$	—;PREM	$\emptyset$	
2	$p \to {\sim}r$	—;PREM	$\emptyset$	
3	$q \vee (s \to (r \vee t))$	—;PREM	$\emptyset$	
4	$s \to {\sim}r$	—;PREM	$\emptyset$	
5	$q \vee (s \to r)$	—;PREM	$\emptyset$	
6	$p \to (s \vee (r \wedge {\sim}r))$	1,2;RU	$\emptyset$	
7	$q \vee (s \to (t \vee (r \wedge {\sim}r)))$	3,4;RU	$\emptyset$	
8	$p \to s$	6;RC	$\{(p \to (s \vee (r \wedge {\sim}r))) \wedge (p \diamond\!\!\to (r \wedge {\sim}r))\}$	
9	$q \vee (s \to t)$	7;RC	$\{(q \vee s \to (t \vee (r \wedge {\sim}r))) \wedge (q \vee s \diamond\!\!\to (r \wedge {\sim}r))\}$	✓
10	$q \vee (s \to (t \vee (r \wedge {\sim}r))) \wedge (q \vee s \diamond\!\!\to (r \wedge {\sim}r))$	4,5;RU	$\emptyset$	

At stage 10 of the proof, the formula on line 9 has been marked, because of the *Dab*–consequence that was derived at line 10.

Remark that it is now also possible to show why it is necessary to demand of abnormal formulas that a part of them is $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–derivable. Suppose the proof above is extended in the following way:

11	$(q \vee s \to u) \vee ((s \to (u \vee (r \wedge \sim r))) \wedge (s \Diamond\!\!\to (r \wedge \sim r)))$	4,5;RU	$\emptyset$

If it were not demanded that $(s \to (u \vee (r \wedge \sim r))$ is $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–derivable in order to consider $(s \to (u \vee (r \wedge \sim r))) \wedge (s \Diamond\!\!\to (r \wedge \sim r))$ as an abnormal formula, the formula $q \vee s \to u$ would be $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$–derivable. Even more, it would be finally derivable! But, this is an irrelevant consequence of the premise set! Hence, it should definitely not be derivable.

B. Semantics of $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$

Despite the fact that the $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$–proof theory is not completely standard, its semantic characterization remains completely standard. As such, it first selects the minimally abnormal $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–models of a premise set. After that, those models are grouped together in normal sets. Finally, semantic consequence is defined by means of those normal sets.

Definition 10.18 *Where M is a $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–model: its abnormal part is the set $Ab(M) = \{A \in \Omega \mid M \models A\}$.*

Definition 10.19 *An $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–model M of Γ is a minimally abnormal model iff there is no $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–model M' of Γ for which $Ab(M') \subset Ab(M)$.*

Definition 10.20 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 10.21 *A set Σ of $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–models of Γ is a normal set iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.*

Definition 10.22 $\Gamma \models_{\mathbf{AR}_{\mathbf{d}}^{\Diamond}} A$ *iff A is verified by all members of at least one normal set of $\mathbf{R}_{\mathbf{d}}^{\Diamond}$–models of Γ.*

Soundness and Completeness. Also the soundness and completeness proof for $\mathbf{AR}_{\mathbf{d}}^{\Diamond}$ is completely equivalent to the one for standard adaptive logics based on the normal selections strategy. As a consequence:

Theorem 10.3 $\Gamma \vdash_{\mathbf{AR}_{\mathbf{d}}^{\Diamond}} A$ *iff* $\Gamma \models_{\mathbf{AR}_{\mathbf{d}}^{\Diamond}} A$.

10.4 Conclusion

In this chapter, I showed that it is possible to turn the relevant logic $\mathbf{R_d}$ into an inconsistency–adaptive logic. As such, the inference rule **DS** is reintroduced in $\mathbf{R_d}$, without also reintroducing any of the fallacies of relevance. Moreover, it is easily conceived that this can also be done for a lot of other relevant logics, if not for all of them.

However, although **DS** is reintroduced in **RL** as presented in this chapter, the obtained **IARL** do not capture deductive relevance yet. They will nevertheless turn out to be a decisive step in the right direction.

Chapter 11

Relevant Relevance Logic

11.1 Introduction

In this chapter, I will finally show how the theory of first degree relevance (**FDR**) that was presented in part III can be combined with the relevant implication from standard Relevance Logic. More specifically, this will be done by "combining" the logic $\exists\mathbf{CL\bar{o}Ns^s}$ from chapter 9, with the inconsistency–adaptive logics (**IARL**) from the foregoing chapter. As will be shown in section 11.5, this will result in relevant logics (**RL**) that truly solve the **DS**–problem in Relevance Logic, and hence capture relevant deduction in an adequate way.

Two Possibilities. Just as there are two possible ways to reintroduce **DS** into **RL** (see ch. 10), there are also two possible ways to combine **FDR** with a relevant implication: one that only introduces **FDR** outside the scope of an implication, and one that also introduces **FDR** inside the scope of an implication.

Moreover, in the foregoing chapter, I mentioned that I strongly prefer **RL** that allow for both **RDS** and **HDS**, because these treat hypothetical and non–hypothetical reasoning on a par. For the same reason, I also prefer **RL** that both express **FDR** outside and inside the scope of an implication.

Overview. In this chapter, I will characterize the relevant logics $\mathbf{R}^\gamma_\mathbf{d}$ and $\mathbf{R}^*_\mathbf{d}$. The former expresses **FDR** only outside the scope of an implication, while the latter also expresses **FDR** inside the scope of an implication. As such, I take the latter to explicate *relevant deduction*.

Both $\mathbf{R}^\gamma_\mathbf{d}$ and $\mathbf{R}^*_\mathbf{d}$ are defined by means of a translation to an ambiguity–adaptive logic. More specifically, if $\mathcal{W}$ is the set of wffs of the relevant logic **R** (see ch. 5), they are defined as follows:

Definition 11.1 $\Gamma \vdash_{\mathbf{R}^\gamma_\mathbf{d}} A$ *iff* $\Gamma^{\exists i} \vdash_{\exists\mathbf{R}^\gamma_\mathbf{d}} (\exists i)A^{(i)}$*, for* $\Gamma \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$*.*

Definition 11.2 $\Gamma \vdash_{\mathbf{R}^*_{\mathbf{d}}} A$ *iff* $\Gamma^{\exists i} \vdash_{\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}} (\exists i)A^{(i)}$, *for* $\Gamma \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$.

Consequently, in order to characterize the logics $\mathbf{R}^{\gamma}_{\mathbf{d}}$ and $\mathbf{R}^*_{\mathbf{d}}$, I need to present the ambiguity–adaptive logics $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$ and $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$. This will be done respectively in section 11.3 and section 11.4.

Preliminary Remarks. In order to be complete, three preliminary remarks have to be made. First, I want to repeat that I am only interested in relevant deduction, and that I am not interested in entailments (the implicational theorems of **RL**). Hence, the latter will not be discussed. Secondly, I will restrict myself to adaptive logics based on the relevant logic **R**. Adaptive logics based on other **RL** can be obtained along the same lines. Thirdly, that I restrict myself to adaptive logics based on the relevant logic **R**, doesn't mean that I (implicitly) take stand in the discussion on which (relevant) implication best expresses the natural–language implication (see e.g. Lance [63], Brady [41]).

11.2 The Lower Limit Logic

Both the adaptive logics $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$ and $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$ are based on the same lower limit logic, namely the logic $\exists\mathbf{R}_{\mathbf{d}}$, an ambiguity logic based on the logic $\mathbf{R}_{\mathbf{d}}$ (see chapter 5, section 5.4.4).

Language Schema. Let $\mathcal{L}^{\exists i}$ be the language of $\exists\mathbf{R}_{\mathbf{d}}$. It is defined from $\langle \mathcal{S}^{\mathcal{I}}, \mathcal{V}\rangle$, with $\mathcal{S}^{\mathcal{I}} = \{A^i \mid A \in \mathcal{S} \text{ and } i \in \mathbb{N}\}$ the set of indexed (sentential) letters, and $\mathcal{V}$ the set of variables. The set of well–formed formulas $\mathcal{W}^{\exists i}$ of the language $\mathcal{L}^{\exists i}$ is constructed as follows:

(i) $\mathcal{S}^{\mathcal{I}} \subset \mathcal{W}^{\exists i}$.
(ii) When $A \in \mathcal{W}^{\exists i}$ then $\neg_! A, {\sim}A \in \mathcal{W}^{\exists i}$.
(iii) When $A, B \in \mathcal{W}^{\exists i}$ then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \Diamond\!\!\rightarrow B) \in \mathcal{W}^{\exists i}$.
(iv) When $A \in \mathcal{W}^{\exists i}$ and $i \in \mathcal{V}$ then $(\exists i)A[i], (\forall i)A[i] \in \mathcal{W}^{\exists i}$.

As in chapter 10, the implication connective $\Diamond\!\!\rightarrow$ is a defined connective:

Definition 11.3 $A \Diamond\!\!\rightarrow B =_{df} \neg_!(A \rightarrow \neg_! B) \vee (\neg_! A \vee B)$.

Overview. Consider table 11.1. It clearly states the relations between the languages $\mathcal{L}$, $\mathcal{L}^{\mathcal{I}}$ and $\mathcal{L}^{\exists i}$. All of them will be used in the remaining of this chapter.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\sim, \wedge, \vee, \rightarrow$	$\mathcal{W}$
$\mathcal{L}^{\mathcal{I}}$	$\mathcal{S}^{\mathcal{I}}$	$\neg!, \sim, \wedge, \vee, \rightarrow$	$\mathcal{W}^{\mathcal{I}}$
$\mathcal{L}^{\exists i}$	$\mathcal{S}^{\mathcal{I}}$	$\neg!, \sim, \wedge, \vee, \rightarrow, \exists, \forall$	$\mathcal{W}^{\exists i}$

Table 11.1: Relations between $\mathcal{L}$, $\mathcal{L}^{\mathcal{I}}$ and $\mathcal{L}^{\exists i}$.

Classes of Well–formed Formulas. For the semantic characterization of $\exists\mathbf{R_d}$, I will also make use of $\mathfrak{a}$– and $\mathfrak{b}$–formulas. I've put them in table 11.2 below.

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\sim(A \wedge B)$	$\sim A$	$\sim B$
$\sim(A \vee B)$	$\sim A$	$\sim B$		$A \vee B$	A	B
$\sim\sim A$	A	A				
$\sim(\exists i)A$	$(\forall i)\sim A$	$(\forall i)\sim A$				
$\sim(\forall i)A$	$(\exists i)\sim A$	$(\exists i)\sim A$				

Table 11.2: $\mathfrak{a}$– and $\mathfrak{b}$–formulas for $\exists\mathbf{R_d}$.

Semantics. The semantics of $\exists\mathbf{R_d}$ is constructed along the lines of the semantics of $\mathbf{R}$ (see chapter 5). As such, the $\exists\mathbf{R_d}$–models of a premise set will be a subset of the $\exists\mathbf{R_d^{LLL}}$–models of that premise set.

Let $\mathbb{N}'$ be a denumerable set of pseudo–indices, e.g. $1', 2', 3', \ldots$ The pseudo–language $\mathcal{L}_+^{\exists i}$ is now defined from $\langle \mathcal{S}_+^{\mathcal{I}}, \mathcal{V}\rangle$, with $\mathcal{S}_+^{\mathcal{I}} = \{A^i \mid A \in \mathcal{S}$ and $i \in \mathbb{N} \cup \mathbb{N}'\}$ the extended set of indexed letters and $\mathcal{V}$ the set of variables. The set of well–formed formulas $\mathcal{W}_+^{\exists i}$ is defined for $\mathcal{L}_+^{\exists i}$ in the same way as $\mathcal{W}^{\exists i}$ is defined for $\mathcal{L}^{\exists i}$. Finally, also consider the following sets of formulas:

- $^{\sim}\mathcal{S}_+^{\mathcal{I}} = \{\sim A \mid A \in \mathcal{S}_+^{\mathcal{I}}\}$,
- $^{\sim}\mathcal{I} = \{\sim(A \rightarrow B) \mid A, B \in \mathcal{W}_+^{\exists i}\}$,
- $^{\sim}\mathcal{N} = \{\sim\neg! A \mid A \in \mathcal{W}_+^{\exists i}\}$, and

An $\exists\mathbf{R_d^{LLL}}$–model for the language $\mathcal{L}_+^{\exists i}$ is a 6–tuple $< g, d, W, R, \leqslant, v >$, where W is a set of worlds, with $g \in W$ the base world, $d \in W$ the deductive world, R a ternary relation on W, satisfying

FP0 For all $a, b \in W$: $Rgab$ iff $a = b$,
FP7 For all $a, b, c, d \in W$: if R^2abcd then $R^2b(ac)d$.
FP8 For all $a, b, c, d \in W$: if R^2abcd then $R^2a(bc)d$.
FP9 For all $a, b, c \in W$: if $Rabc$ then R^2abbc.
FP10 For all $a, b, c \in W$: if $Rabc$ then $\exists x \in W$: $a \leqslant x$, $x \leqslant a$ and $Rbxc$.

$\leqslant$ a reflexive and transitive binary (containment) relation on W, satisfying:

C1 For all $a, b \in W$ such that $a \leqslant b$: if $v(A, a) = 1$ then $v(A, b) = 1$.
C2 For all $a, b, c, d \in W$ $(a \neq g)$ such that $a \leqslant b$: if $Rbcd$ then $Racd$.
C3 For all $a, b, c \in W$ such that $g \leqslant a$: if $Rabc$ then $b \leqslant c$.

and v an assignment function such that:

AP1 v: $\mathcal{S} \times W \mapsto \{0, 1\}$.
AP2 v: ${}^{\sim}\mathcal{S} \times W \mapsto \{0, 1\}$.
AP3 v: ${}^{\sim}\mathcal{I} \times W \mapsto \{0, 1\}$.
AP4 v: ${}^{\sim}\mathcal{N} \times W \mapsto \{0, 1\}$.

The valuation function v_M based on the interpretation $\mathbf{M}$ is characterized as follows:

SP0 v_M: $\mathcal{W}_{+}^{\exists i} \times W \mapsto \{0, 1\}$.
SP1 For $A \in \mathcal{S}_{+}^{\mathcal{I}}$: $v_M(A, a) = 1$ iff $v(A, a) = 1$.
SP2 For $A \in \mathcal{S}_{+}^{\mathcal{I}}$: $v_M({\sim}A, a) = 1$ iff $v({\sim}A, a) = 1$.
SP3 $v_M(\mathfrak{a}, a) = 1$ iff $v_M(\mathfrak{a}_1, a) = 1$ and $v_M(\mathfrak{a}_2, a) = 1$.
SP4 $v_M(\mathfrak{b}, a) = 1$ iff $v_M(\mathfrak{b}_1, a) = 1$ or $v_M(\mathfrak{b}_2, a) = 1$.
SP5 $v_M(A \to B, a) = 1$ iff for all $b, c \in W$: if $Rabc$ then $v_M(A, b) = 0$ or $v_M(B, c) = 1$.
NP1 $v_M({\sim}(A \to B), a) = 1$ iff $v({\sim}(A \to B), a) = 1$.
BP1 $v_M(\neg_! A, a) = 1$ iff $v_M(A, a) = 0$.
BP2 $v_M({\sim}\neg_! A, a) = 1$ iff $v({\sim}\neg_! A, a) = 1$.
EP1 For $\xi \in \mathcal{S}$, $v_M((\exists i)A[i], a) = 1$ iff $v_M((\exists i)A[\xi^i], a) = 1$, for at least one $i \in \mathbb{N} \cup \mathbb{N}$.
EP2 For $\xi \in \mathcal{S}$, $v_M((\forall i)A[i], a) = 1$ iff $v_M((\exists i)A[\xi^i], a) = 1$ for all $i \in \mathbb{N} \cup \mathbb{N}$.

The definitions for semantical validity and semantical consequence are the same as for $\mathbf{R_d}$ (see chapter 5). First, the $\exists\mathbf{R_d^{LLL}}$–models of a premise set are characterized.

Definition 11.4 *A valuation function v_M d–verifies A iff $v_M(A, d) = 1$, and d–falsifies A iff $v_M(A, d) = 0$.*

Definition 11.5 *A valuation function v_M is an $\exists\mathbf{R_d^{LLL}}$–model of Γ iff it d–verifies all $A \in \Gamma$.*

Next, the set of abnormalities Ω is defined as the union of the following two sets:

a) $\Omega_1 = \{(A \to {\sim}B) \to (B \to {\sim}A) \mid A, B \in \mathcal{W}^{\exists i}\}$

b) $\Omega_2 = \{(A \rightarrow \sim A) \rightarrow \sim A \mid A \in \mathcal{W}^{\exists i}\}$

The $\exists\mathbf{R_d}$–models are now characterized as those $\exists\mathbf{R_d^{LLL}}$–models of the premise set that do not falsify any abnormalities.

Definition 11.6 *A valuation function v_M g–verifies A iff $v_M(A, g) = 1$, and g–falsifies A iff $v_M(A, g) = 0$.*

Definition 11.7 *For each $\exists\mathbf{R_d^{LLL}}$–model M, Ab(M) $= \{A \in \Omega \mid M$ g–falsifies $A\}$.*

Definition 11.8 *An $\exists\mathbf{R_d^{LLL}}$–model of Γ is an $\exists\mathbf{R_d}$–model of Γ iff Ab(M) $= \emptyset$.*

Finally, semantic consequence is defined by means of the $\exists\mathbf{R_d}$–models of a premise set.

Definition 11.9 $\Gamma \models_{\exists\mathbf{R_d}} A$ *iff A is d–verified by all $\exists\mathbf{R_d}$–models of Γ.*

Proof Theory. To be honest, I don't know exactly which axioms should be added to the axiom system of **R** in order to obtain the proof theory of $\exists\mathbf{R_d}$. Nevertheless, the axioms and rule stated in table 10.2 below, definitely have to be added. They govern the behavior of the classical negation and the second implication.

BA1	$A \rightarrow (B \rightarrow (C \vee \neg_! C))$
BA2	$\neg_!(A \rightarrow B) \vee (\neg_! A \vee B)$
BA3	$(A \wedge \neg_! A) \rightarrow B$
BR1	$(A \wedge B) \rightarrow C \blacktriangleright (A \wedge \neg_! C) \rightarrow \neg_! B$
P1	$(A \diamondminus\!\!\rightarrow B) \rightarrow (\neg_!(A \rightarrow \neg_! B) \vee (\neg_! A \vee B))$
P2	$(\neg_!(A \rightarrow \neg_! B) \vee (\neg_! A \vee B)) \rightarrow (A \diamondminus\!\!\rightarrow B)$

Table 11.3: Axioms for $\exists\mathbf{R_d}$ 1.

As a consequence, the problem I have with the $\exists\mathbf{R_d}$–proof theory concerns the behavior of the quantifiers. Anyway, I will leave this for further research and only state the axioms of which I am sure that they should be added. They are stated in the table below.

QA1	$\sim(\exists i)A \rightarrow (\forall i)\sim A$; $(\forall i)\sim A \rightarrow \sim(\exists i)A$
QA2	$\sim(\forall i)A \rightarrow (\exists i)\sim A$; $(\exists i)\sim A \rightarrow \sim(\forall i)A$
QA3	$(\forall i)A[i] \rightarrow A[i/j]$ $(j \in \mathbb{N})$
QA4	$A[i/j] \rightarrow (\exists i)A[i]$ $(j \in \mathbb{N})$

Table 11.4: Axioms for $\exists\mathbf{R_d}$ 2.

Soundness and Completeness. As a result of what I have stated above, also soundness and completeness have not been proven yet. This is also left for further research.

Final Remark. The fact that I do not have characterized the proof theory of $\exists\mathbf{R_d}$ completely will not affect the characterization of the adaptive logics that are based on $\exists\mathbf{R_d}$, as these make use of generic deduction rules.

However, the fact that soundness and completeness for $\exists\mathbf{R_d}$ have not been proven yet, does constitute a problem, but only for the metatheoretical characterization of the **AL**. Their soundness and completeness proofs presuppose soundness and completeness of the **LLL**. As such, in those sections, I will just presuppose the soundness and completeness of $\exists\mathbf{R_d}$, so that once it has been proven, it is also immediately proven that the **AL** that are based on it are also sound and complete.

11.3 The Adaptive Logic $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$

The adaptive logic $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$ is a simple combined adaptive logic (see chapter 3, section 3.3.1). It is characterized by the following three components:

(1) Its **LLL** is the logic $\exists\mathbf{R_d}$ from section 11.2.
(2) Its set of abnormalities Ω is the union of the following two sets:

   a) $\Omega_1 = \{(\exists i)A^{(i)} \wedge \neg_! A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}}\}$.
   b) $\Omega_2 = \{A^i \wedge {\sim}A^j \mid A \in \mathcal{S} \text{ and } i, j \in \mathbb{N}\}$.

(3) The adaptive strategy is the normal selections strategy.

11.3.1 Proof Theory of $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$

The proof theory of $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$ is the standard proof theory for **AL** based on the normal selections strategy. First, consider the deduction rules.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists\mathbf{R_d}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\exists\mathbf{R_d}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Next, consider the marking criterium, it consists of one marking rule.

Definition 11.10 *Marking for* Normal Selections*: Line i is marked at stage s iff, where Δ is its condition, $Dab(\Delta)$ has been derived at stage s on a line with condition $\emptyset$.*

Finally, consider also the definitions for final $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$–derivability. Also these are completely standard.

Definition 11.11 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 11.12 $\Gamma \vdash_{\exists\mathbf{R}^{\gamma}_{\mathbf{d}}} A$ *iff A is finally derived on a line of a proof from Γ.*

Example. Before I state the example, remember that $\bigvee(\Delta)$ refers to the disjunction of the elements of Δ and that Ω_i refers to the adaptive condition of line i.

Next, consider the example below. It is based on the ambiguous premise set $\Gamma^{\exists\mathbf{i}} = \{(\exists i)(p \to (r \vee s))^{(i)}, (\exists i)(p \to {\sim}r)^{(i)}, (\exists i)(q)^{(i)}, (\exists i)({\sim}q)^{(i)}, (\exists i)(t \vee q)^{(i)}\}$.

1	$(\exists i)(p \to (r \vee s))^{(i)}$	—;PREM	$\emptyset$	
2	$(\exists i)(p \to {\sim}r)^{(i)}$	—;PREM	$\emptyset$	
3	$(\exists i)(q)^{(i)}$	—;PREM	$\emptyset$	
4	$(\exists i)({\sim}q)^{(i)}$	—;PREM	$\emptyset$	
5	$(\exists i)(t \vee q)^{(i)}$	—;PREM	$\emptyset$	
6	$p^1 \to (r^1 \vee s^1)$	1;RC	$\{(\exists i)(p \to (r \vee s))^{(i)} \wedge \neg_!(p^1 \to (r^1 \vee s^1))\}$	
7	$p^1 \to {\sim}r^2$	2;RC	$\{(\exists i)(p \to {\sim}r)^{(i)} \wedge \neg_!(p^1 \to {\sim}r^2)\}$	
8	q^2	3;RC	$\{(\exists i)(q)^{(i)} \wedge \neg_!(q^2)\}$	
9	${\sim}q^1$	4;RC	$\{(\exists i)({\sim}q)^{(i)} \wedge \neg_!({\sim}q^1)\}$	
10	$t^1 \vee q^2$	5;RC	$\{(\exists i)(t \vee q)^{(i)} \wedge \neg_!(t^1 \vee q^2)\}$	
11	${\sim}p^1 \vee (r^1 \vee s^1)$	6;RU	Ω_6	
12	${\sim}p^1 \vee {\sim}r^1$	7;RU	Ω_7	
13	$(\exists i)(t)^{(i)}$	9,10;RC	$\Omega_9 \cup \Omega_{10} \cup \{q^2 \wedge {\sim}q^1\}$	
14	$(\exists i)(u)^{(i)}$	8,9;RC	$\Omega_8 \cup \Omega_9 \cup \{q^2 \wedge {\sim}q^1\}$	✓
15	$(\exists i)({\sim}p \vee s)^{(i)}$	11,12;RC	$\Omega_{11} \cup \Omega_{12} \cup \{r^1 \wedge {\sim}r^1\}$	
16	$\bigvee(\Omega_8 \cup \Omega_9 \cup \{q^2 \wedge {\sim}q^1\})$	3,4;RU	$\emptyset$	
17	$(\exists i)(p \to (s \vee (r \wedge {\sim}r)))$	6,7;RU	$\Omega_6 \cup \Omega_7$	

This proof clearly shows how **FDR** is combined with a relevant implication. Moreover, notice that the example also shows that the logic $\exists\mathbf{R}^{\gamma}_{\mathbf{d}}$ only captures **FDR** outside the scope of the implication. This is obvious, as it is not possible to derive $(\exists i)(p \to s)$, despite the fact that $(\exists i)(p \to (s \vee (r \wedge {\sim}r))$ is derived on line 17.

11.3.2 Semantics of $\exists\mathbf{R}_{\mathbf{d}}^{\gamma}$

Just as the proof theory, also the semantics of $\exists\mathbf{R}_{\mathbf{d}}^{\gamma}$ is completely standard. As such, I will only give the necessary definitions.

Definition 11.13 *Where M is a $\exists\mathbf{R}_{\mathbf{d}}$–model: its abnormal part is the set $Ab(M) = \{A \in \Omega \mid M \models A\}$.*

Definition 11.14 *A $\exists\mathbf{R}_{\mathbf{d}}$–model M of Γ is a minimally abnormal model iff there is no $\exists\mathbf{R}_{\mathbf{d}}$–model M' of Γ for which $Ab(M') \subset Ab(M)$.*

Definition 11.15 $\Phi(\Gamma) = \{Ab(M) \mid M$ *is a minimally abnormal model of* $\Gamma\}$.

Definition 11.16 *A set Σ of $\exists\mathbf{R}_{\mathbf{d}}$–models of Γ is a normal set iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma; Ab(M) = \phi\}$.*

Definition 11.17 $\Gamma \models_{\exists\mathbf{R}_{\mathbf{d}}^{\gamma}} A$ *iff A is verified by all members of at least one normal set of $\exists\mathbf{R}_{\mathbf{d}}$–models of Γ.*

Soundness and Completeness. Again, as $\exists\mathbf{R}_{\mathbf{d}}^{\gamma}$ is a standard adaptive logic, soundness and completeness follows immediately.

Theorem 11.1 $\Gamma \vdash_{\exists\mathbf{R}_{\mathbf{d}}^{\gamma}} A$ *iff* $\Gamma \models_{\exists\mathbf{R}_{\mathbf{d}}^{\gamma}} A$

11.4 The Adaptive Logic $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$

The adaptive logic $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$ is a non–standard combined adaptive logic. Nevertheless, to characterize $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$, I will keep as close as possible to the standard format. Before I will give a general characterization, I first need to characterize its set of abnormalities.

Abnormal Formulas. The set of abnormalities of the adaptive logic $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$ is the union of the sets $\Omega^{!}$, Ω^{*} and $\Omega^{\blacklozenge}$, of which the former two sets are quite easily characterized:

Definition 11.18 $\Omega^{!} = \{(\exists i)A^{(i)} \wedge \neg_{!}A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists i}\}$.

Definition 11.19 $\Omega^{*} = \{A^{i} \wedge {\sim}A^{j} \mid A \in \mathcal{S}$ *and* $i, j \in \mathbb{N}\}$.

The set $\Omega^{\blacklozenge}$ is more demanding. First, consider the set $\mathcal{W}^{\Omega} \subset \mathcal{W}^{\mathcal{I}}$, which is the set of all formulas having the following form:

$$[B_1\vee]A_1 \to ([B_2\vee]A_2 \to (...([B_{n-1}\vee]A_n \to (B_n \vee \bigvee(\Delta)))...)),^{1}$$

[1] Remember that $\bigvee(\Delta)$ refers to the disjunction of the members of Δ. Moreover, also remember that the brackets occurring in the formula mean that those parts are not necessary for a formula to belong to $\mathcal{W}^{\Omega}$.

with $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{W}^{\mathcal{I}}$, $1 \leqslant n$ and $\Delta \subset \Omega^*$.

Next, every element A of $\mathcal{W}^{\Omega}$ has a counterpart $A^{\blacklozenge}$, which is the formula obtained from A in the following way:

1. replace in A the implication symbols outside $A_1, ..., A_n, B_1, ..., B_n$ by an implication symbol "$\Diamond\!\!\rightarrow$", and
2. replace in A the subformula $B_n \vee \bigvee(\Delta)$ by the formula $\bigvee(\Delta)$.

Hence, a formula $A^{\blacklozenge}$ will be of the following form:

$$[B_1\vee]A_1 \Diamond\!\!\rightarrow ([B_2\vee]A_2 \Diamond\!\!\rightarrow (...([B_{n-1}\vee]A_n \Diamond\!\!\rightarrow (\bigvee(\Delta)))...)),$$

with $A_1, ..., A_n, B_1, ..., B_{n-1} \in \mathcal{W}$ and $\Delta \subset \Omega^*$.

Finally, it is now possible to define the third set of abnormalities of the adaptive logic $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$, the set $\Omega^{\blacklozenge}$:

Definition 11.20 $\Omega^{\blacklozenge} = \{A \wedge A^{\blacklozenge} \mid A \in \mathcal{W}^{\Omega}\}$.

General Characterization of $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$. Despite the fact that the logic $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$ is a non–standard adaptive logic, it can be characterized by means of the usual three elements:

(1) Its **LLL** is the logic $\exists\mathbf{R}_{\mathbf{d}}$ from section 11.2.
(2) Its set of abnormalities Ω is the union of the following three sets:
   a) $\Omega_1 = \Omega^! \cup \Omega^*$, with $\Omega^!$ and Ω^* defined as in section 11.4, and
   b) $\Omega_2 = \Omega^{\blacklozenge}$, with $\Omega^{\blacklozenge}$ defined as in section 11.4.
(3) The adaptive strategy is a variant of the normal selections strategy. As there is however no need to change the name, I will just keep it.

11.4.1 Proof Theory of $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$

The proof theory of $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$ is not a standard **AL**–proof theory, which will become clear by considering the deduction rules and the marking criterium.

Deduction Rules. Instead of the usual three deduction rules, there are now four of them. Remarkable is that there are two conditional rules, one related to Ω_1 and one related to Ω_2.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists\mathbf{R}_{\mathbf{d}}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC1 If $A_1, ..., A_n \vdash_{\exists\mathbf{R_d}} B \vee Dab(\Theta)$ $(\Theta \subset \Omega_1)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RC1, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

RC2 If $A_1, ..., A_n \vdash_{\exists\mathbf{R_d}} B \vee Dab(\{C_1 \wedge C_1^\blacklozenge, ..., C_m \wedge C_m^\blacklozenge\})$ and each of $A_1, ..., A_n, C_1, ..., C_m$ occurs in the proof on lines $i_1, ..., i_n, j_1, ..., j_m$ that have conditions $\Delta_1, ..., \Delta_n, \Theta_1 \subset \Omega_1, ..., \Theta_m \subset \Omega_1$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n, j_1, ..., j_m$;RC2, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta_1 \cup ... \cup \Theta_m \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_m \wedge C_m^\blacklozenge\}$.

Marking Criterium. The marking criterium consists of one marking rule, the standard marking rule related to the normal selections strategy. However, the way in which a *Dab*–consequence is defined is not standard and should be considered first.

Definition 11.21 *$Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$ $(\Delta \subseteq \Omega_1)$ is a Dab–consequence of Γ at stage s of the proof iff $Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$ has been derived on the condition $\emptyset$ at stage s of the proof, and $C_1, ..., C_n$ have been derived respectively on the conditions $\Theta_1 \subset \Delta, ..., \Theta_n \subset \Delta$ at stage s of the proof.*

It is now possible to state the marking rule for $\exists\mathbf{R_d^\lozenge}$.

Definition 11.22 *Marking for* Normal Selections*: Line i is marked at stage s iff, where $\Delta \subset \Omega$ is its condition, $Dab(\Delta)$ is a Dab–consequence of Γ at stage s.*

Final Derivability. Despite the non–standardness of the $\exists\mathbf{R_d^\lozenge}$–proof theory, the definitions for final $\exists\mathbf{R_d^\lozenge}$–derivability are completely standard.

Definition 11.23 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 11.24 *$\Gamma \vdash_{\exists\mathbf{R_d^\lozenge}} A$ iff A is finally derived on a line of a proof from Γ.*

Example. The example below is based on the following premise set:

$$\Gamma^{\exists\mathbf{i}} = \{(\exists i)(p \rightarrow (r \vee s))^{(i)}, (\exists i)(p \rightarrow {\sim} r)^{(i)}, (\exists i)(q \vee (s \rightarrow (r \vee t)))^{(i)}, (\exists i)(s \rightarrow {\sim} r)^{(i)}, (\exists i)(q \vee (s \rightarrow r))^{(i)}\}.$$

1	$(\exists i)(p \to (r \vee s))^{(i)}$	—;PREM	$\emptyset$	
2	$(\exists i)(p \to \sim r)^{(i)}$	—;PREM	$\emptyset$	
3	$(\exists i)(q \vee (s \to (r \vee t)))^{(i)}$	—;PREM	$\emptyset$	
4	$(\exists i)(s \to \sim r)^{(i)}$	—;PREM	$\emptyset$	
5	$(\exists i)(q \vee (s \to r))^{(i)}$	—;PREM	$\emptyset$	
6	$p^1 \to (r^1 \vee s^1)$	1;RC1	$\{(\exists i)(p \to (r \vee s))^{(i)} \wedge \neg_!(p^1 \to (r^1 \vee s^1))\}$	
7	$p^1 \to \sim r^2$	2;RC1	$\{(\exists i)(p \to \sim r)^{(i)} \wedge \neg_!(p^1 \to \sim r^2)\}$	
8	$q^1 \vee (s^1 \to (r^1 \vee t^1))$	3;RC1	$\{(\exists i)(q \vee s \to (r \vee t))^{(i)} \wedge \neg_!(q^1 \vee s^1 \to (r^1 \vee t^1))\}$	
9	$s^1 \to \sim r^2$	4;RC1	$\{(\exists i)(s \to \sim r)^{(i)} \wedge \neg_!(s^1 \to \sim r^2)\}$	
10	$q^1 \vee (s^1 \to r^1)$	5;RC1	$\{(\exists i)(q \vee s \to r)^{(i)} \wedge \neg_!(q^1 \vee s^1 \to r^1)\}$	
11	$p^1 \to (s^1 \vee (r^1 \wedge \sim r^2))$	6,7;RU	$\Omega_6 \cup \Omega_7$	
12	$q^1 \vee (s^1 \to (t^1 \vee (r^1 \wedge \sim r^2)))$	8,9;RU	$\Omega_8 \cup \Omega_9$	
13	$(\exists i)(p \to s)^{(i)}$	11;RC2	$\Omega_{11} \cup \{(p^1 \to (s^1 \vee (r^1 \wedge \sim r^2))) \wedge (p^1 \Diamond\!\!\to (r^1 \wedge \sim r^2))\}$	
14	$(\exists i)(q \vee (s \to t))^{(i)}$	12;RC2	$\Omega_{12} \cup \{(q^1 \vee (s^1 \to (t^1 \vee (r^1 \wedge \sim r^2))))\wedge$ $(q^1 \vee (s^1 \Diamond\!\!\to (r^1 \wedge \sim r^1)))\}$	
15	$(\exists i)(q \vee (s \to u))^{(i)}$	9,10;RC2	$\Omega_9 \cup \Omega_{10} \cup \{(q^1 \vee (s^1 \to (t^1 \vee (r^1 \wedge \sim r^2))))\wedge$ $(q^1 \vee (s^1 \Diamond\!\!\to (r^1 \wedge \sim r^1)))\}$	✓
16	$q^1 \vee (s^1 \to (t^1 \vee (r^1 \wedge \sim r^2)))$	9,10;RC1	$\Omega_9 \cup \Omega_{10}$	
17	$\bigvee(\Omega_9 \cup \Omega_{10} \cup$ $\{(q^1 \vee (s^1 \to (t^1 \vee (r^1 \wedge \sim r^2))))\wedge$ $(q^1 \vee (s^1 \Diamond\!\!\to (r^1 \wedge \sim r^1)))\})$	4,5,15;RU	$\emptyset$	

11.4.2 Semantics of $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$

Also the semantics of $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$ is not a standard **AL**–semantics. Nevertheless, it comes quite close to the semantics of a prioritized adaptive logic.

Step One. First, consider the definition of the Ω_1–abnormal part of an $\exists\mathbf{R}_{\mathbf{d}}$–model.

Definition 11.25 *Where M is a $\exists\mathbf{R}_{\mathbf{d}}$–model: its Ω_1–abnormal part is the set $Ab^1(M) = \{A \in \Omega_1 \mid M \models A\}$.*

Now, the first step of the $\exists\mathbf{R}_{\mathbf{d}}^{\Diamond}$–semantics consists in selecting the $\exists\mathbf{R}_{\mathbf{d}}$–models of a premise set that are minimally abnormal with respect to Ω_1, and to divide them into normal sets. From now on, those normal sets will be called the normal Ω_1–sets of a premise set.

Definition 11.26 *An $\exists\mathbf{R}_{\mathbf{d}}$–model M of Γ is a minimally abnormal Ω_1–model iff there is no $\exists\mathbf{R}_{\mathbf{d}}$–model M' of Γ for which $Ab^1(M') \subset Ab^1(M)$.*

Definition 11.27 $\Phi^1(\Gamma) = \{Ab^1(M) \mid M$ *is a minimally abnormal Ω_1–model of* $\Gamma\}$.

Definition 11.28 *A set Σ of $\exists\mathbf{R}_{\mathbf{d}}$–models of Γ is a normal Ω_1–set iff for some $\phi \in \Phi^1(\Gamma)$, $\Sigma = \{M \mid M \models \Gamma;\ Ab^1(M) = \phi\}$.*

Step Two. The normal Ω_1–sets of a premise set can now be used to construct what I will call normal Ω_2–sets. First, consider the Ω_2–abnormal part of an $\exists\mathbf{R_d}$–model M of Γ. It is related to the normal Ω_1–set Σ to which M belongs, because Σ determines a set of Ω_2–abnormalities Ω_Σ.

Definition 11.29 $\Omega_\Sigma = \{A \wedge A^{\blacklozenge} \in \Omega_2 \mid \Sigma \models A\}$.

Definition 11.30 *Where M is an $\exists\mathbf{R_d}$–model of Γ that belongs to the normal Ω_1–set Σ: its Ω_2–abnormal part is the set $Ab^2_\Sigma(M) = \{A \in \Omega_\Sigma \mid M \models A\}$.*

Next, within each normal Ω_1–set Σ those models are selected that are minimally abnormal with respect to Ω_Σ.

Definition 11.31 *An $\exists\mathbf{R_d}$–model M of Γ that belongs to the normal Ω_1–set Σ is a minimally abnormal Ω_2–model of Σ iff there is no $\exists\mathbf{R_d}$–model $M' \in \Sigma$ for which $Ab^2_\Sigma(M') \subset Ab^2_\Sigma(M)$.*

The minimally abnormal Ω_2–models of a normal Ω_1–set Σ are now grouped together in a normal Ω_2–set Ξ_Σ.

Definition 11.32 $\Phi^2_\Sigma(\Gamma) = \{Ab^2_\Sigma(M) \mid M$ *is a minimally abnormal Ω_2–model of Σ, with Σ a normal Ω_1–set of Γ*$\}$.

Definition 11.33 *A set Ξ_Σ of $\exists\mathbf{R_d}$–models of Γ is a normal Ω_2–set iff for some $\phi \in \Phi^2_\Sigma(\Gamma)$, $\Xi_\Sigma = \{M \in \Sigma \mid M \models \Gamma; Ab^2_\Sigma(M) = \phi\}$.*

Finally, semantic consequence for $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is defined with respect to the normal Ω_2–sets of a premise set Γ.

Definition 11.34 $\Gamma \models_{\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}} A$ *iff A is verified by all members of at least one normal Ω_2–set of $\exists\mathbf{R_d}$–models of Γ.*

11.4.3 Soundness and Completeness

As the adaptive logic $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is not a standard adaptive logic, soundness and completeness are not immediately given. Hence, I will prove them below. First, consider the lemmas below.

Lemma 11.1 $\Gamma \vdash_{\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}} A$ *iff there is a $\Delta \subset \Omega_1$ and there are $C_1, ..., C_n \in \mathcal{W}^\Omega$ such that $\Gamma \vdash_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$, $\Gamma \vdash_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$,..., $\Gamma \vdash_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$, and $\Gamma \nvdash_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$.*

Proof. $\Rightarrow$ Suppose $\Gamma \vdash_{\exists\mathbf{R}_{\mathbf{d}}^{\lozenge}} A$. Hence, A is finally derived on a line i of an $\exists\mathbf{R}_{\mathbf{d}}^{\lozenge}$–proof from Γ. Let $\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\}$ $(\Delta \subset \Omega_1)$ be the condition of line i. But then $\Gamma \vdash_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$, $\Gamma \vdash_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$ $(\dagger_1)$,..., $\Gamma \vdash_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$ $(\dagger_n)$ (proven in the same way as [26, lemma 1]).

Suppose $\Gamma \vdash_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$. As a consequence, it is possible to extend the proof in such a way that it contains a line at which $Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$ is derived on the condition $\emptyset$. Moreover, from $(\dagger_1)$,...,$(\dagger_n)$, it follows that the proof can be extended such that $C_1, ..., C_n$ are derived on lines with condition Δ. From this extension on, $Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$ is a *Dab*–consequence of Γ (by definition 11.21), which means that in all extensions of this extension, line i is marked in view of definition 11.22. But this contradicts that A is finally derived at stage s on a line i with condition $\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\}$.

$\Leftarrow$ Suppose there is a $\Delta \subset \Omega_1$ and there are $C_1, ..., C_n \in \mathcal{W}^{\Omega}$ such that $\Gamma \vdash_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$, $\Gamma \vdash_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$,..., $\Gamma \vdash_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$, and $\Gamma \nvdash_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$. Because of [26, lemma 1], there is a $\exists\mathbf{R}_{\mathbf{d}}^{\lozenge}$–proof from Γ such that $A \vee Dab(\{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\}), C_1, ..., C_{n-1}$ and C_n have been derived on lines with condition Δ. In view of the deduction rules RC1 and RC2, it is now possible to extent the proof in such a way that A has been derived on a line with condition $\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}$. Moreover, because $\Gamma \nvdash_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$, $Dab(\Delta \cup \{C_1 \wedge C_1^{\blacklozenge}, ..., C_n \wedge C_n^{\blacklozenge}\})$ cannot be derived on a line with the condition $\emptyset$. Hence, by definitions 11.21, 11.22, 11.23 and 11.24, $\Gamma \vdash_{\exists\mathbf{R}_{\mathbf{d}}^{\lozenge}} A$. ■

Lemma 11.2 *A model M verifies $Dab(\Delta)$ $(\Delta \subset \Omega)$ iff $\Delta \cap Ab(M) \neq \emptyset$.*

Proof. Obvious and left to the reader. ■

Lemma 11.3 *Where $\mathcal{M}$ is a set of models, $\mathcal{M} \models Dab(\Delta)$ $(\Delta \subset \Omega)$ iff $Dab(\Delta)$ is verified by all members of $\mathcal{M}$ that are minimally abnormal with respect to Ω.*

Proof. $\Rightarrow$ Suppose $\mathcal{M} \models Dab(\Delta)$ $(\Delta \subset \Omega)$. Hence, all elements of $\mathcal{M}$ verify $Dab(\Delta)$. As the set of minimally abnormal elements of $\mathcal{M}$ is a subset of all elements of $\mathcal{M}$, $Dab(\Delta)$ is verified by all members of $\mathcal{M}$ that are minimally abnormal with respect to Ω.

$\Leftarrow$ Suppose $Dab(\Delta)$ is verified by all members of $\mathcal{M}$ that are minimally abnormal with respect to Ω. All those models verify at least one element of Δ. As this set is the set of minimal abnormal models with respect to Ω, all elements of $\mathcal{M}$ will at least also verify one element of Δ (by the definition of a minimally abnormal model). As a consequence, $\mathcal{M} \models Dab(\Delta)$. ■

Let $Neg(\Delta) = \{\neg_! A \mid A \in \Delta\}$.

Lemma 11.4 *$\Gamma \models_{\exists\mathbf{R}_\mathbf{d}^\Diamond} A$ iff there is a $\Delta \subset \Omega_1$ and there are $C_1, ..., C_n \in \mathcal{W}^\Omega$ such that $\Gamma \models_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$, $\Gamma \models_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$,..., $\Gamma \models_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$, and $\Gamma \not\models_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$.*

Proof. $\Rightarrow$ Suppose $\Gamma \models_{\exists\mathbf{R}_\mathbf{d}^\Diamond} A$. Hence, by definitions 11.32, 11.33 and 11.34, there is a $\phi \in \Phi^2_\Sigma(\Gamma)$ such that all members of the Ω_2–set $\Xi_\Sigma = \{M \in \Sigma \mid M \models \Gamma;\ Ab^2_\Sigma(M) = \phi\}$ verify A. Now, because Σ is a normal Ω_1–set of Γ, there is some $\psi \in \Phi^1(\Gamma)$ such that $\Sigma = \{M \mid M \models \Gamma;\ Ab(M) = \psi\}$ (by definitions 11.27 and 11.28). From this, it follows that all $\exists\mathbf{R_d}$–models of $\Gamma \cup Neg(\Omega_1 - \psi) \cup Neg(\Omega_\Sigma - \phi)$ verify A. This means that $\Gamma \cup Neg(\Omega_1 - \psi) \cup Neg(\Omega_\Sigma - \phi) \vdash_{\exists\mathbf{R_d}} A$ (by the completeness of $\exists\mathbf{R_d}$ with respect to its semantics). By the compactness of $\exists\mathbf{R_d}$, there is a finite $\Gamma' \subseteq \Gamma$, a finite $\psi' \subset (\Omega_1 - \psi)$, and a finite $\{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\} \subset (\Omega_\Sigma - \phi)$ (‡) such that $\Gamma' \cup Neg(\psi' \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\}) \vdash_{\exists\mathbf{R_d}} A$ (†).

Consequence 1. From (†), it follows that $\Gamma' \vdash_{\exists\mathbf{R_d}} A \vee Dab(\psi' \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$ and hence that $\Gamma \vdash_{\exists\mathbf{R_d}} A \vee Dab(\psi' \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$.

Consequence 2. From (‡), it follows that $\Sigma \models C_1$,...,$\Sigma \models C_n$ (by definition 11.29). As a consequence, by the same reasoning as above, there are finite $\psi_1 \subset (\Omega_1 - \psi)$,...,$\psi_n \subset (\Omega_1 - \psi)$ such that $\Gamma \vdash_{\exists\mathbf{R_d}} C_1 \vee Dab(\psi_1)$,..., $\Gamma \vdash_{\exists\mathbf{R_d}} C_n \vee Dab(\psi_n)$.

Result. Let $\Delta = \psi' \cup \psi_1 \cup ... \cup \psi_n$. Now, as $(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\}) \cap (\psi \cup \phi) = \emptyset$, it follows that $\Gamma \not\models Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$. This, together with consequence 1 and 2, gives us that $\Gamma \models_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$, $\Gamma \not\models_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$, $\Gamma \models_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$,..., $\Gamma \models_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$ (because of the completeness of $\exists\mathbf{R_d}$ with respect to its semantics).

$\Leftarrow$ Suppose there is a $\Delta \subset \Omega_1$ and there are $C_1, ..., C_n \in \mathcal{W}^\Omega$ such that (1) $\Gamma \models_{\exists\mathbf{R_d}} A \vee Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$, (2) $\Gamma \models_{\exists\mathbf{R_d}} C_1 \vee Dab(\Delta)$,..., $\Gamma \models_{\exists\mathbf{R_d}} C_n \vee Dab(\Delta)$, and (3) $\Gamma \not\models_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$. From (3), it follows that $\Gamma \not\models_{\exists\mathbf{R_d}} Dab(\Delta)$ (otherwise also $\Gamma \not\models_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$). By lemmas 11.2 and 11.3, it follows that there are some $\psi \in \Phi^1(\Gamma)$ such that $\psi \cap \Delta = \emptyset$. Suppose that this is the case for $\psi_1, ..., \psi_n$. As a consequence, every member of $\Sigma_i = \{M \mid M \models \Gamma;\ Ab^1(M) = \psi_i\}$ $(1 \leqslant i \leqslant n)$ falsifies $Dab(\Delta)$, and hence also verifies $A \vee Dab(\{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\}), C_1, ..., C_{n-1}$ and C_n.

Moreover, for some Σ_i, it has to be the case that $\Sigma_i \not\models Dab(\{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$ (otherwise $\Gamma \models_{\exists\mathbf{R_d}} Dab(\Delta \cup \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\}$, which contradicts (3)). Suppose $\Sigma_1 \not\models Dab(\{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\})$. By lemmas 11.2 and 11.3, it now follows that there is a $\phi \in \Phi^2_{\Sigma_1}(\Gamma)$ such that $\phi \cap \{C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge\} = \emptyset$. As a consequence, every member of $\{M \in \Sigma \mid M \models \Gamma;\ Ab^2_{\Sigma_1}(M) = \phi\}$ falsifies $Dab(C_1 \wedge C_1^\blacklozenge, ..., C_n \wedge C_n^\blacklozenge)$, and hence also verifies A. So, $\Gamma \models_{\exists\mathbf{R}_\mathbf{d}^\Diamond} A$ (by definitions 11.32 and 11.34). ■

Now, because of the soundness and completeness of the logic $\exists\mathbf{R_d}$, lemma 11.1 and lemma 11.4, it immediately follows that the logic $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$ is sound and complete with respect to its semantics.

Theorem 11.2 $\Gamma \vdash_{\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}} A$ *iff* $\Gamma \vdash_{\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}} A$.

11.5 Relevant Deduction?

In chapter 2 (section 2.5), I presented the conditions a solution to the **DS**–problem should satisfy. As I explicitly claim that the logic $\mathbf{R}^*_{\mathbf{d}}$ provides a nice solution to the problem, it is necessary to look whether it satisfies all the conditions.

(1) As the disjunctions in the logic $\mathbf{R}^*_{\mathbf{d}}$ are extensionally characterized disjunctions,[2] it is immediately clear that no relevance between the disjuncts has to be presupposed in order to be able to apply **DS**.
(2) The logic $\mathbf{R}^*_{\mathbf{d}}$ is obtained by combining the relevant implication with the theory of first degree relevance that was presented in chapter 6. As the latter doesn't restrict **DS** to an unjustified extent, also the logic $\mathbf{R}^*_{\mathbf{d}}$ doesn't do so.
(3) The logic $\mathbf{R}^*_{\mathbf{d}}$ does reintroduce **DS** in relevant logics, but it does not reintroduce any of the fallacies of relevance. First of all, in $\mathbf{R}^*_{\mathbf{d}}$ the implication behaves exactly as in the relevant logic $\mathbf{R_d}$. Hence, irrelevant implications cannot be derived. Secondly, as none of the **EQV**– and **EFQ**–paradoxes are derivable at the first degree, also these can not be obtained in $\mathbf{R}^*_{\mathbf{d}}$.
(4) In my approach, the **DS**–problem is solved by logical means. Consequently, no reference is made to extra–logical features of reasoning.
(5) In $\mathbf{R}^*_{\mathbf{d}}$, the inference rule **DS** is taken to express some deductive connection. As such, it is considered as a deductive rule, and not a heuristic one.
(6) That the logic $\mathbf{R}^*_{\mathbf{d}}$ treats hypothetical and non–hypothetical reasoning on a par, is clearly stated in section 11.1.

11.6 Conclusion

In this chapter, I have shown how the theory of first degree deductive relevance that was presented in chapter 6, can be combined with the relevant implication of the logic **R**. Moreover, I have also shown that this adequately solves the **DS**–problem of **RL**. Consequently, it can be claimed that the logics presented in this chapter capture deductive relevance in a better way than the standard **RL**.

[2]Here, I obviously refer to the semantic characterization of the disjunction in the adaptive logic $\exists\mathbf{R}^{\Diamond}_{\mathbf{d}}$!

Part V

Variations and Applications

The Aim of Part V

This final part of my dissertation is more or less unrelated to the other parts, as it does not deal with deductive relevance. There are however two reasons why it should nevertheless be included in this dissertation. Firstly, it treats with a different kind of relevance, namely relevant insight in the premises. Secondly, most of the adaptive logics that will be presented in this part, are based on logics presented in earlier chapters.

Overview of Part V

In chapter 12, I will show how adaptive logics can be characterized whose strategy is not based on the minimal *Dab*–consequences, but on the *Dab*–consequences that are truly contained in the premises (they can be obtained from the premises by mere analyzing steps).

In chapter 13, I will present adaptive logics that are able to explicate abductive reasoning processes based on both consistent and inconsistent background theories.

Chapter 12

Relevant Insight in the Premises

12.1 Introduction

In chapter 3, I already mentioned that adaptive logics (**AL**) express a two–fold dynamics: an external and an internal one. The external dynamics is expressed by the non–monotonicity of the **AL**–consequence relation: if a premise set is extended, some **AL**–consequences of the original premise set may not be derivable anymore.[1] The internal dynamics is expressed by the **AL**–proof theory. At every stage of an **AL**–proof, some earlier drawn consequences may be withdrawn, and some earlier withdrawn consequences may be considered as derivable again. As a consequence, **AL** are able to capture interesting reasoning processes that cannot be captured by means of other logics:

> Several aspects of real–life reasoning (argumentation) are dynamic. We not only drop conclusions after obtaining more information, but also after we analyzed the premises better.[2]

Remark that these are normally reasoning processes for which there doesn't exist a positive test, such as for example some reasoning processes based on inconsistent theories, induction, abduction, compatibility,...

Internal Dynamics. It has often been claimed by adaptive logicians that the internal dynamics is guided by a growing insight in the premises:

> In the case of the *internal* dynamics, it is caused by the reasoning process itself: as it proceeds, our insight into the premises increases.[3]

[1]Formally: there are Γ, Δ and A such that $\Gamma \vdash_{\mathbf{AL}} A$ and $\Gamma \cup \Delta \nvdash_{\mathbf{AL}} A$.

[2]See Batens [11, p. 286].

[3]Sic, see Batens [20, p. 149]. This claim was defended most thoroughly (and satisfactorily) in Batens [11].

In this chapter, I will explain what adaptive logicians normally mean by "an increasing insight in the premises." This will make clear that **AL** are usually based on only a partial insight in the premises. However, I will show that it is also possible to construct **AL** that are based on a complete insight in the premises.

Preliminary Remark. I will restrict the discussion below to inconsistency–adaptive logics.[4] Nevertheless, I remain hopeful that the obtained results can be extended to other kinds of **AL** as well. Moreover, in the remaining of this chapter, all inconsistency–adaptive logics will be represented by means of the logic $\mathbf{ACL\bar{u}Ns^r}$.[5] However, all remarks made also hold for all (standard) inconsistency–adaptive logics.

12.2 Insight in the Premises

In order to show what is usually meant by "gaining a better insight in the premises," consider the $\mathbf{ACL\bar{u}Ns^r}$–proof below. It is based on the premise set $\Gamma = \{p, \neg p, \neg q, p \vee q, q \vee r, \neg s, s \vee t\}$.

1	$q \vee r$	PREM	$\emptyset$	
2	$\neg q$	PREM	$\emptyset$	
3	r	1,2;RC	$\{q \wedge \neg q\}$	✓
4	$\neg s$	PREM	$\emptyset$	
5	$s \vee t$	PREM	$\emptyset$	
6	t	4,5;RC	$\{s \wedge \neg s\}$	
7	$p \vee q$	PREM	$\emptyset$	
8	$\neg p$	PREM	$\emptyset$	
9	$(p \wedge \neg p) \vee (q \wedge \neg q)$	2,7,8;RU	$\emptyset$	

Remark that line 3 is marked at stage 9 of the proof, because its condition contains one of the disjuncts of a minimal *Dab*–formula derived at stage 9 of the proof (the *Dab*–formula on line 9). Formally, this comes down to $\{q \wedge \neg q\} \cap U_9(\Gamma) \neq \emptyset$,[6] which means that the formula on line 3 is not considered as derivable at stage 9. However, suppose the proof is extended in the following way:

[4]For an introduction into inconsistency–adaptive logics, see Batens [12]. Other interesting papers concerning inconsistency–adaptive logics are Batens [13] and Batens & Meheus [32].

[5]This inconsistency–adaptive logic is a standard flat adaptive logic based on the paralogic $\mathbf{CL\bar{u}Ns}$ (the **LLL**), the set $\Omega = \{A \wedge \neg A \mid A \in \mathcal{S}\}$ (the set of abnormalities), and the reliability strategy.

[6]See chapter 3, section 3.2.1

10	p	PREM	$\emptyset$
11	$p \wedge \neg p$	8,10;RU	$\emptyset$

The formula on line 3 has become unmarked, as the *Dab*–formula on line 9 is no longer a minimal *Dab*–consequence of the premise set at stage 11. As a consequence, $\{q \wedge \neg q\} \cap U_{11}(\Gamma) = \emptyset$ such that at stage 11, the formula on line 3 is considered as derivable from the premise set Γ.

This example clearly illustrates why only the derivation of minimal *Dab*–consequences can change the markings in an **AL**–proof: only their derivation can alter the set $U_i(\Gamma)$ (for i a stage of the proof), which is necessary to change the markings. As a consequence, the internal dynamics of **AL** is completely dependent upon the *Dab*–formulas that are derivable from a premise set. This also means that "gaining a better insight in the premises" actually comes down to "obtaining a clearer view on the minimal *Dab*–formulas that are derivable from the premise set."

Moreover, remark that in an **AL**–proof, the insight in the premises can never decrease. If no extra minimal *Dab*–consequence has been derived after moving on to a next stage of the proof, the insight in the premises remains the same as in the previous stage. But, on the contrary, if some extra minimal *Dab*–consequence has been derived, the insight in the premises definitely has increased. As such, extending an **AL**–proof might really improve the insight in the premises.

Partial Insight. Despite the fact that the insight in the premises can never decrease, it can now easily be shown that the obtained insight is only a partial insight. In order to see this, consider again the $\mathbf{ACL\bar{u}Ns^r}$–proof from above. At stage 11 of the proof, line 3 becomes unmarked because the *Dab*–consequence on line 9 is not a minimal *Dab*–consequence anymore. However, it is easily verified that this *Dab*–formula is really "contained in" the premises, which means that it can be derived by analyzing the premises (in other words, no inconsistencies were added by means of the inference rule addition). This makes it difficult to see why the abnormality $q \wedge \neg q$ is not considered as unreliable. True, the minimal *Dab*–consequences are more likely to lead us to the flaw(s) in an inconsistent theory,[7] but they do not guarantee that the overlooked inconsistencies have nothing to do with it. Why? Because all *Dab*–consequences that are contained in the premises are

[7] I here presuppose that people usually consider inconsistent theories as problematic, and that they will aim to change inconsistent theories into consistent ones. Moreover, I also presuppose that the inconsistencies themselves are not taken as the core problem of a problematic theory. If this were the case, then an inconsistent theory could simply be turned consistent by rejecting one half of the inconsistency, which is obviously not a good strategy. As a consequence, I think inconsistencies are interpreted more as symptoms of a sick theory which can help us to diagnose correctly.

partly or completely independent of each other, and as such, they are also partly or completely justified on different grounds (which is obviously not the case for *Dab*–consequences obtained from smaller *Dab*–consequences by means of addition). In the $\mathbf{ACL\bar{u}Ns^r}$–proof above for example, the *Dab*–consequences derived on lines 9 and 11 are derived partly from different premises and might be considered as independent of each other for that reason.

As a consequence, it is necessary to look for a way to construct **AL** that are based on a *relevant insight* in the premises, which means that their marking criterium is not based solely on the minimal *Dab*–consequences of a premise set, but on the *relevant Dab*–consequences of a premise set, which are those *Dab*–consequences that are contained in the premise set and can be obtained by analyzing the premises.

Irrelevant Abnormalities. In order to construct **AL** that are based on a complete insight in the premises, a serious technical problem has to be overcome. Consider again the $\mathbf{ACL\bar{u}Ns^r}$–proof above and suppose it is extended by the following line:

12	$(p \wedge \neg p) \vee (s \wedge \neg s)$	11;RU	$\emptyset$

The formula on this line is obtained from the formula on line 11 by means of the inference rule addition. Because of the simplicity of the premise set, it can now easily be verified that the added inconsistency $(s \wedge \neg s)$ is not contained in the premises, so that it should not be considered as unreliable. But, how can we in general sensibly distinguish between those inconsistencies that were added to existing ones by means of addition and those that are really contained in the premise set? In fact, we can't, which is probably why adaptive logicians originally opted to consider only minimal *Dab*–consequences of a premise set. Of these, at least one can be sure that they are contained in the premises.

Final Remark. Remark that partial insight in the premises has nothing to do with what I will call fallible insight in the premises. The latter is unavoidable, while the former is not.

Fallible insight is the result of the fact that in **AL**–proofs, it always remains possible that by extending the proof, some new (minimal or relevant) *Dab*–consequences are derived. As such, it is fundamentally impossible to reach a complete insight in the premises, or better: it is fundamentally impossible to *know* if or when a complete insight in the premises has been reached (unless logical omniscience is presupposed, which is not the case).

Remark that this also implies that for some conditionally derived formulas (formulas derived at a certain stage), it will be impossible to decide whether or not they are also finally derived.[8] This might seem quite damaging at first, but actually, this is exactly what one might expect from logics capturing reasoning processes that lack a positive test. As a consequence, anyone who would consider this to be a problem, should blame human reasoning and not the logics trying to capture them.[9] Moreover, as the insight in the premises can never decrease (as was explained above), derivability at a stage offers a sensible estimate of final derivability. As a consequence, at any stage of the proof, one has to choose whether to continue the proof or to rely on the obtained insights. This is not so bad either, as it is completely in accordance with contemporary theories of rationality (for example Batens [**?**]): as absolute knowledge cannot be obtained by humans, human decisions are never based on certain or complete knowledge, but "merely" on the best insights available at a certain moment. These insights are fallible, but that doesn't prevent them from being the best ones around at that particular time.

12.3 Relevant Insight in the Premises

In this section, I will show that it is possible to obtain **AL** based on relevant insight in the premises. I will do so by presenting the inconsistency–adaptive logic $\mathbf{ACL\bar{u}Ns^i}$, obtained by translation to the ambiguity–adaptive logic $\exists\mathbf{CL\bar{u}Ns^{sr}}$:[10]

Definition 12.1 *$\Gamma \vdash_{\mathbf{ACL\bar{u}Ns^i}} A$ iff $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns^{sr}}} (\exists i)A^{(i)}$, for $\Gamma \cup \{A\} \subset \mathcal{W}$.*

The logic $\mathbf{ACL\bar{u}Ns^i}$ is an adaptive logic based on the reliability strategy. Although I will not present a similar adaptive logic based on any other adaptive strategy, they can be obtained in a similar way.

12.3.1 The Adaptive Logic $\exists\mathbf{CL\bar{u}Ns^{sr}}$

The adaptive logic $\exists\mathbf{CL\bar{u}Ns^{sr}}$ is a somewhat unusual adaptive logic, as it is somewhere in between a simple combined and a prioritized adaptive logic. Nevertheless, it is characterized by the usual three elements:

(1) Its **LLL** is the logic $\exists\mathbf{CL\bar{u}Ns}$ (see chapter 8, section 8.2).
(2) Its set of abnormalities Ω is the union of the following two sets:

[8] This has caused a search for efficient criteria that can allow us to decide whether or not a conditionally derived formula is also finally derived (see especially Batens [11, 16, 18, 23]). However, these criteria always remain *metatheoretical*, which means that they do not solve the undecidability problem at the proof theoretical level.

[9] See also Batens et al. [30].

[10] $\mathcal{W}$ is defined as for $\exists\mathbf{CL\bar{u}Ns}$ (see chapter 8, section 8.2.2), $\Gamma^{\exists \mathbf{i}}$ and $(\exists i)A^{(i)}$ are defined as for $\exists\mathbf{CL\bar{u}Ns}$ (see chapter 7, section 7.2).

a) $\Omega_1 = \{(\exists i)A^{(i)} \wedge \neg_! A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}}\}$.
b) $\Omega_2 = \{A^i \wedge \neg A^j \mid A \in \mathcal{S} \text{ and } i, j \in \mathbb{N} \text{ such that } i \neq j\}$.

(3) Its adaptive strategy is the *reliable normal selections strategy*, which is in fact a combination of the reliability strategy and the normal selections strategy.

It is immediately clear that the logic $\exists$**CLūNs**$^{\mathbf{sr}}$ will only differ from the logic $\exists$**CLūNs**$^{\mathbf{s}}$ (see chapter 8, section 8.3) with respect to its adaptive strategy. As a consequence, the proof theory and the semantics of both logics have a lot in common.

12.3.2 Proof Theory of $\exists$CLūNs$^{\mathbf{sr}}$

The proof theory of $\exists$**CLūNs**$^{\mathbf{sr}}$ is not the standard proof theory of flat **AL**. It demands some slight modifications, all related to the marking criterium.

Deduction Rules. The deduction rules for $\exists$**CLūNs**$^{\mathbf{sr}}$ are equivalent to those for $\exists$**CLūNs**$^{\mathbf{s}}$, and as such, they are completely standard.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{u}Ns}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\exists \mathbf{CL\bar{u}Ns}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Marking Rules. The distinctive feature of this adaptive logic is its marking criterium. It consists of two marking rules, each related to a specific subset of the set of abnormalities Ω. The subsets referred to are of course Ω_1 and Ω_2.

Readers well–acquainted with adaptive logics, might now suspect that the minimal *Dab*–consequences will be defined with respect to a particular set of abnormalities, as is the case for prioritized adaptive logics (see chapter 3, section 3.3.2). This is however not the case here. Minimal *Dab*–consequences are instead defined as usual, with respect to the complete set of abnormalities Ω.

Definition 12.2 $Dab(\Delta)$ $(\Delta \subset \Omega)$ *is a minimal Dab–consequence at stage s of the proof iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also a Dab–consequence at stage s of the proof.*

The first marking rule is a *normal selections*–marking rule, based on those minimal *Dab*–consequences that only consist of elements of Ω_1.

Definition 12.3 **NS**–*marking for* $\exists$**CLūNs**$^{\mathbf{sr}}$*: Line i is marked at stage s of the proof iff where* Δ *is its condition,* $Dab(\Theta)$ *has been derived at stage s,* $\Theta \subset \Omega_1$ *and* $\Theta \subseteq \Delta$*.*

The second marking rule is a *reliability*–marking rule, based on the set $U_s(\Gamma)$, the set of elements of Ω_2 that are considered unreliable at stage s of the proof.

Definition 12.4 $U_s(\Gamma) = \{A \in \Omega_2 \mid A \in \Delta$ *and* $Dab(\Delta)$ *is a minimal Dab–consequence of* Γ *at stage s of the proof*$\}$*.*

The reliability–marking rule itself is now plainly straightforward:

Definition 12.5 **R**–*marking for* $\exists$**CLūNs**$^{\mathbf{sr}}$*: Line i is marked at stage s of the proof iff where* Δ *is its condition,* $\Delta \cap U_s(\Gamma) \neq \emptyset$*.*

Final Derivability. As for all adaptive logics, I should also mention the definitions for final derivability. They are completely standard, and do not need any further explanation.

Definition 12.6 *A is* finally derived *from* Γ *on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 12.7 $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns^{sr}}} A$ *(A is finally* $\exists$**CLūNs**$^{\mathbf{sr}}$*–derivable from* Γ*) iff A is finally derived on a line of a proof from* Γ*.*

Example. Consider again the premise set $\Gamma = \{p, \neg p, \neg q, p \vee q, q \vee r, \neg s, s \vee t\}$. In section 12.2, it was shown that the formula r is **ACLūNs**$^{\mathbf{r}}$–derivable from Γ, even though its derivation is based on the supposition that $q \wedge \neg q$ is a reliable abnormality. However, as the latter occurs in a relevant *Dab*–consequence of Γ, it should not be considered as reliable.

So, if **ACLūNs**$^{\mathbf{i}}$ really captures relevant insight in the premises, it should be able to identify $q \wedge \neg q$ as an unreliable abnormality. As such, it should also not allow the derivation of the formula r. This is indeed the case, as the $\exists$**CLūNs**$^{\mathbf{sr}}$–proof below shows us.

First, construct the premise set $\Gamma^{\exists\mathbf{i}}$ from the premise set Γ above:

$$\Gamma^{\exists\mathbf{i}} = \{(\exists i)(p)^{(i)}, (\exists i)(\neg p)^{(i)}, (\exists i)(\neg q)^{(i)}, (\exists i)(p \vee q)^{(i)}, (\exists i)(q \vee r)^{(i)}, (\exists i)(\neg s)^{(i)}, (\exists i)(s \vee t)^{(i)}\}.$$

1	$(\exists i)(q \vee r)^{(i)}$	PREM	$\emptyset$
2	$(\exists i)(\neg q)^{(i)}$	PREM	$\emptyset$
3	$(\exists i)(p \vee q)^{(i)}$	PREM	$\emptyset$
4	$(\exists i)(\neg p)^{(i)}$	PREM	$\emptyset$
5	$(\exists i)(p)^{(i)}$	PREM	$\emptyset$
6	$q^1 \vee r^1$	1;RC	$\{(\exists i)(q \vee r)^{(i)} \wedge \neg_!(q^1 \vee r^1)\}$
7	$\neg q^2$	2;RC	$\{(\exists i)(\neg q)^{(i)} \wedge \neg_!(\neg q^2)\}$
8	$(\exists i)(r)^{(i)}$	6,7;RC	$\Omega_3 \cup \Omega_4 \cup \{q^1 \wedge \neg q^2\}$

At this stage of the proof, the formula $(\exists i)(r)^{(i)}$ is $\exists$**CLūNs**$^{\mathbf{sr}}$–derived on line 8. But, $(\exists i)(r)^{(i)}$ is not finally $\exists$**CLūNs**$^{\mathbf{sr}}$–derivable, which can be shown by extending the proof in the following way:

8	$(\exists i)(r)^{(i)}$	6,7;RC	$\Omega_3 \cup \Omega_4 \cup \{q^1 \wedge \neg q^2\}$	✓
9	$((\exists i)(\neg q)^{(i)} \wedge \neg_!(\neg q^2))$			
	$\vee((\exists i)(p \vee q)^{(i)} \wedge \neg_!(p^3 \vee q^1))$			
	$\vee((\exists i)(\neg p)^{(i)} \wedge \neg_!(\neg p^4))$			
	$\vee((p^3 \wedge \neg p^4))$			
	$\vee((q^1 \wedge \neg q^2))$	2,6,7;RU	$\emptyset$	
10	$((\exists i)(p)^{(i)} \wedge \neg_!(p^3))$			
	$\vee((\exists i)(\neg p)^{(i)} \wedge \neg_!(\neg p^4))$			
	$\vee(p^3 \wedge \neg p^4)$	7,8;RU	$\emptyset$	

It can now easily be verified that the *Dab*–formula derived on line 9 is a minimal *Dab*–formula, so that line 8 will remain marked.

11	$(\exists i)(\neg s)^{(i)}$	PREM	$\emptyset$
12	$((\exists i)(s \vee t)^{(i)}$	PREM	$\emptyset$
13	$\neg s^1$	11;RC	$\{(\exists i)(\neg s)^{(i)} \wedge \neg_!(s^1)\}$
14	$s^2 \vee t^1$	12;RC	$\{(\exists i)(s \vee t)^{(i)} \wedge \neg_!(s^2 \vee t^1)\}$
15	$(\exists i)(t)^{(i)}$	13,14;RC	$\Omega_{13} \cup \Omega_{14} \cup \{s^2 \wedge \neg s^1\}$

At stage 15 of the proof, line 15 is unmarked, and it is easily verified that it will remain unmarked. Hence, the formula on line 15 is finally $\exists$**CLūNs**$^{\mathbf{sr}}$–derivable from $\Gamma^{\exists \mathbf{i}}$.

Relevant Insight. In order to show that **ACLūNs**$^{\mathbf{i}}$ is based on relevant insight in the premises, it has to be shown that the marking criterium of $\exists$**CLūNs**$^{\mathbf{sr}}$ is based on the relevant *Dab*–consequences of the premise set Γ. Remark that this actually means that the *minimal Dab*–consequences derivable from an ambiguous premise set $\Gamma^{\exists \mathbf{i}}$ by means of $\exists$**CLūNs** (the **LLL**

of the logic $\exists$**CLūNs**$^{\mathbf{sr}}$) should correspond to the *relevant Dab*–consequences derivable from the premise set Γ by means of the logic **CLūNs** (the **LLL** of the adaptive logic **ACLūNs**$^{\mathbf{i}}$). Try to keep this in mind, otherwise the explanation below will be quite incomprehensible.

Now, let's show that these two kind of *Dab*–consequences indeed correspond to one another. First of all, as an ambiguous premise set $\Gamma^{\exists\mathbf{i}}$ never contains any formulas without existential quantifiers, the minimal *Dab*–consequences derivable from $\Gamma^{\exists\mathbf{i}}$ can only be obtained by combining the elements of $\Gamma^{\exists\mathbf{i}}$ with *minimal* $\exists$**CLūNs**–theorems of the form

$$\neg_! A_1^{\mathcal{I}} \vee ... \vee \neg_! A_n^{\mathcal{I}} \vee (B_1^{i_1} \wedge B_1^{i_2}) \vee ... \vee (B_m^{i_{2m-1}} \wedge B_m^{i_{2m}}),$$

with $A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}} \in \mathcal{W}^{\mathcal{I}}$ and $B_1^{i_1}, ..., B_m^{i_{2m}} \in \mathcal{S}^{\mathcal{I}}$.[11] Moreover, notice that the $A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}$ in these theorems are instantiations of elements of $\Gamma^{\exists\mathbf{i}}$ (by the definition of the set of abnormalities Ω).

Secondly, in some of the above statements, the formulas $A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}$ will be maximally ambiguous, which means that each index only occurs once in them. But, it can now easily be verified that for those *minimal* $\exists$**CLūNs**–theorems for which $A_1^{\mathcal{I}}, ..., A_n^{\mathcal{I}}$ are maximally ambiguous, the other part of the theorem (the disjunction of Ω_2–abnormalities) will always correspond to a *relevant Dab*–consequence of the premise set Γ. In order to make this claim more easily comprehensible, consider the premise set $\Gamma = \{p \wedge \neg p, p \vee q, \neg q\}$. It contains two relevant *Dab*–consequences, namely $p \wedge \neg p$ and $(p \wedge \neg p) \vee (q \wedge \neg q)$. Now, consider the ambiguous premise set based on Γ: $\Gamma^{\exists\mathbf{i}} = \{(\exists i)(p \wedge \neg p)^{(i)}, (\exists i)(p \vee q)^{(i)}, (\exists i)(\neg q)^{(i)}\}$. It will lead to minimal *Dab*–consequences when combined with the following *minimal* $\exists$**CLūNs**–theorems:

- $\neg_!(p^{i_0} \wedge \neg p^{i_1}) \vee \neg_!(q^{i_2} \vee p^{i_3}) \vee \neg_!(\neg q^{i_4}) \vee (p^{i_2} \wedge \neg p^{i_1}) \vee (q^{i_3} \wedge \neg q^{i_4})$
- $\neg_!(p^{i_0} \wedge \neg p^{i_1}) \vee (p^{i_0} \wedge \neg p^{i_1})$

Indeed, the disjunctions of Ω_2–abnormalities in these theorems correspond to the relevant *Dab*–consequences of the premise set Γ. Finally, remark that also the theorem below is derivable. It will however not give rise to a minimal *Dab*–consequence of $\Gamma^{\exists\mathbf{i}}$, because it is itself not a minimal $\exists$**CLūNs**–theorem.

- $\neg_!(p^{i_0} \wedge \neg p^{i_1}) \vee \neg_!(q^{i_2} \vee p^{i_3}) \vee \neg_!(\neg q^{i_4}) \vee (p^{i_0} \wedge \neg p^{i_1})$

12.3.3 Semantics of $\exists$CLūNs$^{\mathbf{sr}}$

Also the $\exists$**CLūNs**$^{\mathbf{sr}}$–semantics is not the standard **AL**–semantics. Nevertheless, it does work according to the same basic principle: the selection of preferred sets of **LLL**–models of a premise set.

[11] $\mathcal{S}^{\mathcal{I}}$ and $\mathcal{W}^{\mathcal{I}}$ are defined as in chapter 8, section 8.2.2.

First, we define the abnormal parts of the $\exists$**CLūNs**–models of a premise set Γ. Each model has two abnormal parts, one related to Ω_1 and one related to Ω_2:

Definition 12.8 *Where M is a $\exists$**CLūNs**–model of Γ, its Ω_1–abnormal part is the set $Ab^1(M) = \{A \in \Omega_1 \mid M \models A\}$.*

Definition 12.9 *Where M is a $\exists$**CLūNs**–model of Γ, its Ω_2–abnormal part is the set $Ab^2(M) = \{A \in \Omega_2 \mid M \models A\}$.*

In order to reach the preferred sets of $\exists$**CLūNs**–models of Γ, first consider the set $\mathcal{M}_0$, the set of all $\exists$**CLūNs**–models of Γ.

Definition 12.10 $\mathcal{M}_0 =_{df} \{M \mid M \models \Gamma\}$.

By means of the set $\mathcal{M}_0$ and the set $U(\Gamma)$ of unreliable elements of Ω_2, we can now construct the set $\mathcal{M}^r$ which contains the $\exists$**CLūNs**–models of Γ that are reliable with respect to Ω_2.

Definition 12.11 *$Dab(\Delta)$ $(\Delta \subset \Omega)$ is a minimal Dab–consequence of Γ iff $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta')$.*

Definition 12.12 *$U(\Gamma) = \{A \in \Omega_2 \mid A \in \Delta$ and $Dab(\Delta)$ is a minimal Dab–consequence of $\Gamma\}$.*

Definition 12.13 $\mathcal{M}^r =_{df} \{M \in \mathcal{M}_0 \mid Ab^2(M) \subseteq U(\Gamma)\}$.

Next, from the set $\mathcal{M}^r$, only those $\exists$**CLūNs**–models are selected that are minimally abnormal with respect to their Ω_1–abnormal part.

Definition 12.14 *$\mathcal{M}^m =_{df} \{M \in \mathcal{M}^r \mid$ for no $M' \in \mathcal{M}^r$, $Ab^1(M') \subset Ab^1(M)\}$.*

Finally, all elements of $\mathcal{M}^m$ that verify the same elements of Ω_1 are bundled into a *reliable normal set.* Semantic consequence is now defined with respect to those reliable normal sets of $\exists$**CLūNs**–models of Γ.

Definition 12.15 $\Phi(\Gamma) = \{Ab^1(M) \mid M \in \mathcal{M}^m\}$.

Definition 12.16 *A set Σ of **LLL**–models of Γ is a* reliable normal set *iff $\Sigma = \{M \in \mathcal{M}^m \mid Ab^1(M) = \phi$, for some $\phi \in \Phi(\Gamma)\}$.*

Definition 12.17 *$\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}} A$ iff A is verified by all members of at least one reliable normal set of $\exists$**CLūNs**–models of Γ.*

12.3.4 Soundness and Completeness

In this section, soundness and completeness for $\exists$**CLūNs**$^{\mathbf{sr}}$ is proven. This is necessary, as the logic $\exists$**CLūNs**$^{\mathbf{sr}}$ is not a standard adaptive logic, so that soundness and completeness are not provided by the standard format.

Consider the lemmas below. From them, soundness and completeness for $\exists$**CLūNs**$^{\mathbf{sr}}$ follows immediately.

Lemma 12.1 *If A is finally derived at line i of an $\exists$**CLūNs**–proof from Γ, and $\Delta \cup \Theta$ ($\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$) is the condition of line i, then $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$, and $\Theta \cap U(\Gamma) = \emptyset$.*

Proof. Suppose (1) the antecedent is true, but (2a) $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ or (2b) $\Theta \cap U(\Gamma) \neq \emptyset$. If (2a) is the case, line i will be NS–marked, and as it is impossible to become NS–unmarked again (as NS–marking does not depend on minimal *Dab*–consequences), it is also impossible for A to be finally derived at line i (by definition 12.6). As a consequence, (2b) will be the case, which means that there is a minimal *Dab*–consequence of Γ, say $Dab(\Delta' \cup \Theta')$ ($\Delta' \subset \Omega_1$ and $\Theta' \subset \Omega_2$), for which $\Theta \cap \Theta' \neq \emptyset$. So the $\exists$**CLūNs**–proof has an extension in which $Dab(\Delta' \cup \Theta')$ has been derived on an empty condition. But then, where s is the last stage of the extension, $\Theta' \subseteq U_s(\Gamma)$ and $\Theta \cap U_s(\Gamma) \neq \emptyset$. Hence, line i is marked at stage s (by definition 12.5). As $Dab(\Theta')$ is a minimal *Dab*–consequence of Γ, $\Theta' \subseteq U_{s'}(\Gamma)$ for all stages following s. So the extension has no further extension in which line i is unmarked again. In view of definition 12.6, this contradicts (1), which states that A is finally derived at line i of the $\exists$**CLūNs**–proof from Γ. ∎

Theorem 12.1 $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns^{sr}}} A$ *iff there are finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$, such that $\Gamma^{\exists i} \vdash_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$, $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$, and $\Theta \cap U(\Gamma) = \emptyset$.*

Proof. $\Rightarrow$ Suppose that $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns^{sr}}} A$. So A is finally derived on a line i of an $\exists$**CLūNs**$^{\mathbf{sr}}$–proof from Γ. Let $\Delta \cup \Theta$ ($\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$) be the condition of line i. But then $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$ (by [26, lemma 1]),[12] $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ and $\Theta \cap U(\Gamma) = \emptyset$ (by lemma 12.1).

$\Leftarrow$ Suppose that for some finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$, (1) $\Gamma^{\exists\mathbf{i}} \vdash_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$, (2) $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$, and (3) $\Theta \cap U(\Gamma) = \emptyset$. From (1), it follows that there is an $\exists$**CLūNs**–proof from Γ (containing only applications of PREM and RU) in which $A \vee Dab(\Delta \cup \Theta)$ is derived on the condition $\emptyset$.

[12] The lemma referred to is the following:

Lemma 12.2 $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$ *iff there is an **AL**–proof from Γ that contains a line on which A is derived on the condition Δ.*

It has been proven for all standard **AL**.

By an application of RC a line i can be added that has A as its formula and $\Delta \cup \Theta$ as its condition. Moreover, this line is unmarked. In any extension of this proof in which line i is marked, it is the case that (a) $Dab(\Delta)$ is derived on an empty condition, or (b) a $Dab(\Delta' \cup \Theta')$ ($\Delta' \subset \Omega_1$ and $\Theta' \subset \Omega_2$) is derived on an empty condition such that $\Theta \cap \Theta' \neq \emptyset$. But, (a) would mean that $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$, which is impossible in view of (2). As a consequence, only (b) remains possible. Now, as $\Theta \cap U(\Gamma) = \emptyset$, there will be a $(\Delta'' \cup \Theta'') \subset (\Delta' \cup \Theta')$ such that $Dab(\Delta'' \cup \Theta'')$ is a minimal Dab–consequence of Γ and $\Theta \cap \Theta'' = \emptyset$. So the proof–extension can be further extended in such a way that $Dab(\Delta'' \cup \Theta'')$ occurs in the proof on an empty condition. But then A is finally derived at line i in view of definition 12.6. ■

Lemma 12.3 *An* $\exists$**CLūNs**–*model* M *of* Γ *verifies* $Dab(\Delta)$ *(*$\Delta \subset \Omega_1$*) iff* $\Delta \cap Ab^1(M) \neq \emptyset$.

Proof. Obvious and left to the reader. ■

Lemma 12.4 $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ *(*$\Delta \subset \Omega_1$*) iff* $\mathcal{M}^m \models Dab(\Delta)$.

Proof. $\Rightarrow$ Suppose $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ ($\Delta \subset \Omega_1$). Hence, all $\exists$**CLūNs**–models of Γ verify $Dab(\Delta)$. As $\mathcal{M}^m$ is a subset of all $\exists$**CLūNs**–models of Γ, $\mathcal{M}^m \models Dab(\Delta)$.

$\Leftarrow$ Suppose $\mathcal{M}^m \models Dab(\Delta)$. All elements of $\mathcal{M}^m$ verify at least one element of Δ. As this set is the set of minimal abnormal models with respect to Ω_1, all $\exists$**CLūNs**–models of Γ will at least also verify one element of Δ. As a consequence, $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$. ■

Let $Neg(\Delta) = \{\neg_! A \mid A \in \Delta\}$.

Theorem 12.2 $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}^{\mathbf{sr}}} A$ *iff there are finite* $\Delta \subset \Omega_1$ *and* $\Theta \subset \Omega_2$, *such that* $\Gamma^{\exists i} \models_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$, $\Gamma \not\models_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$, *and* $\Theta \cap U(\Gamma) = \emptyset$.

Proof. $\Rightarrow$ Suppose that $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}^{\mathbf{sr}}} A$. Hence, there is a $\phi \in \Phi(\Gamma)$ such that all members of $\{M \in \mathcal{M}^m \mid Ab^2 \subseteq U(\Gamma)$ and $Ab^1(M) = \phi\}$ verify A (by definitions 12.13, 12.14, 12.15 and 12.16). As a consequence, all $\exists$**CLūNs**–models of $\Gamma \cup Neg(\Omega - U(\Gamma) - \phi)$ verify A, and, by the completeness of $\exists$**CLūNs** with respect to its semantics, $\Gamma \cup Neg(\Omega - U(\Gamma) - \phi) \vdash_{\exists\mathbf{CL\bar{u}Ns}} A$. By the compactness of $\exists$**CLūNs**, there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \cup \Theta \subseteq (\Omega - U(\Gamma) - \phi)$ ($\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$) such that $\Gamma' \cup Neg(\Delta \cup \Theta) \vdash_{\exists\mathbf{CL\bar{u}Ns}} A$. It follows that $\Gamma' \vdash_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$ and hence that $\Gamma \vdash_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$ (†). Moreover, as $\Delta \cap \phi = \emptyset$, it follows that $\Gamma \nvdash_{\exists\mathbf{CL\bar{u}Ns}} Dab(\Delta)$ (††). Finally, remark that it is also the case that $\Theta \cap U(\Gamma) = \emptyset$ († † †). From (†),(††) and († † †), it now follows that $\Gamma \models_{\exists\mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$,

$\Gamma \not\models_{\exists \mathbf{CL\bar{u}Ns}} Dab(\Delta)$, and $\Theta \cap U(\Gamma) = \emptyset$ (by the soundness and completeness of $\exists$**CLūNs**).

$\Leftarrow$ Suppose that there are finite $\Delta \subset \Omega_1$ and $\Theta \subset \Omega_2$, such that (1) $\Gamma^{\exists \mathbf{i}} \models_{\exists \mathbf{CL\bar{u}Ns}} A \vee Dab(\Delta \cup \Theta)$, (2) $\Gamma \not\models_{\exists \mathbf{CL\bar{u}Ns}} Dab(\Delta)$, and (3) $\Theta \cap U(\Gamma) = \emptyset$. $\Gamma \models_{\exists \mathbf{CL\bar{u}Ns^{sr}}} A$ holds vacuously if $\mathcal{M}_0 = \emptyset$. So, suppose $\mathcal{M}_0 \neq \emptyset$. From (1), it now follows that all members of $\mathcal{M}_0$ verify $A \vee Dab(\Delta \cup \Theta)$. By [26, theorem 5],[13] it follows that $\mathcal{M}^r \neq \emptyset$. As $\Theta \cap U(\Gamma) = \emptyset$ (because of (3)), all elements of $\mathcal{M}^r$ falsify Θ. So, all elements of $\mathcal{M}^r$ verify $A \vee Dab(\Delta)$ (†). From (2), it follows that there is a $\phi \in \Phi(\Gamma)$ such that $\phi \cap \Delta = \emptyset$ (by lemmas 12.3 and 12.4). From this, it follows that every member of $\{M \in \mathcal{M}^m \mid Ab^1(M) = \phi\}$ falsifies $Dab(\Delta)$. But, from (†), it also follows that every member of $\{M \in \mathcal{M}^m \mid Ab^1(M) = \phi\}$ verifies $A \vee Dab(\Delta)$. Hence, every member of $\{M \in \mathcal{M}^m \mid Ab^1(M) = \phi\}$ verifies A. As a consequence, $\Gamma \models_{\exists \mathbf{CL\bar{u}Ns^{sr}}} A$ (by definitions 12.16 and 12.17).

■

As $\exists$**CLūNs** was proven sound and complete (see theorem 8.3), the soundness and completeness of the adaptive logic $\exists$**CLūNs$^{\mathbf{sr}}$** now follows immediately from theorem 12.1 and theorem 12.2 above.

Corollary 12.1 *$\Gamma \vdash_{\exists \mathbf{CL\bar{u}Ns^{sr}}} A$ iff $\Gamma \models_{\exists \mathbf{CL\bar{u}Ns^{sr}}} A$.*

12.4 Conclusion

In this chapter, I have shown that the internal dynamics of adaptive logics is only based on a partial insight in the premises, namely the insight provided by the minimal *Dab*–consequences of the premise set. This however overlooks some of the *Dab*–consequences that are contained in the premises, which is particularly damaging, as there are not always good reasons not to consider those *Dab*–consequences as unreliable.

However, by presenting the adaptive logic **ACLūNs$^{\mathbf{i}}$**, I have shown that it is also possible to construct adaptive logics that are based on a complete insight in the premises.

[13] The theorem referred to states that strong reassurance is a feature of all adaptive logics based on the reliability strategy:

Theorem 12.3 (Strong Reassurance for Reliability) *If $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^r$, then there is a $M' \in \mathcal{M}_\Gamma^r$ such that $Ab(M') \subset Ab(M)$.*

Chapter 13

Relevance and Abductive Reasoning

13.1 Introduction

When searching an explanation for a (puzzling) phenomenon, people often *reason backwards*, from the phenomenon to be explained to possible explanations. As such, they perform a reasoning process usually called *abduction*:

> by abduction I mean the reasoning process from the evidence base and background knowledge to the hypothesis or explanans that explains the relevant evidence.[1]

Examples are legion: a physician in search of the right diagnose for a patients symptoms, a technician trying to find out why a machine broke down, a scientist trying to find an explanation for an empirical phenomenon contradicting some predictions derived from an accepted theory,...

Although not all abduction processes are similar, they all share a common element: inferences based on the argumentation schema known as *Affirming the Consequent* (**AC**):

[**AC**] $\quad A \sqsupset B, B \vdash A$

The Irrelevance Problem. If the classical inference relation is considered, **AC** is clearly not deductively valid. Moreover, because *Irrelevance* (**Irr**) is a valid inference step in Classical Logic (**CL**), adding **AC** to **CL** as an extra axiom would result in the trivial logic.

[**Irr**] $\quad B \vdash_{\mathbf{CL}} A \sqsupset B$

[1]See Kiikeri [62, p. 1].

I will call this the *irrelevance problem* for formal approaches towards abduction. In order to provide a nice formal account of abduction processes, this problem has to be faced.

In this chapter, I will first discuss some of the proposed solutions to this problem (see section 13.2), and I will also present my own solution, based on the adaptive logics programme (see section 13.3 and 13.4).

13.2 Formal Approaches towards Abduction

Although there are a lot of formal approaches towards abduction (for an overview, see Paul [82]), I will only discuss two of them, namely the so–called logic–based approaches and the adaptive approach.

13.2.1 Logic–Based Approaches

In order to avoid irrelevant abductions (abductions based on implications obtained by means of **Irr**), most logic–based approaches characterize abduction as a kind of *backwards deduction with additional conditions*, which means that a number of conditions is specified that enable one to decide whether or not a particular abductive inference is sound. The most common conditions are the following:

(i) $\Gamma \cup \{A\} \vdash B$
(ii) $\Gamma \nvdash \neg A$
(iii) $\Gamma \nvdash B$; $A \nvdash B$
(iv) A is 'minimal'
(v) A is the best among the possible explanations

Condition (i) states that adding A to the background knowledge (represented by Γ) has to make B derivable. Obviously, if it would not, one could hardly claim A to be an explanation of B. Condition (ii) states that an explanation has to be compatible with the background knowledge. Condition (iii) does not allow the explanandum to be derivable solely from the background theory or from the explanans. This in order to avoid self–explanations. Finally, also conditions (iv) and (v) place further restrictions upon possible explanations. However, as there are different possibilities to specify what is meant by them, I will leave it like that.

Remark that not all of the above conditions have to be fulfilled in order to characterize abduction processes. Different combinations of the conditions will lead to different kinds of abduction. A non–exhaustive taxonomy of these kinds of abduction is provided by Atocha Aliseda–Llera in [2, 3, 4].

Features of Abduction Processes. Remark also that, given the specified conditions, abduction processes will be non–monotonic: as new information becomes available, some abductive conclusions might be withdrawn.

For example, this will the case when a possible explanation for a phenomenon that was derived by means of abduction, is proven incorrect by further tests.

Moreover, abduction processes are not only non–monotonic, they also do not have a positive test, which means that it can not always be decided in a finite number of steps that a formula is a sound abductive conclusion. This is a consequence of the fact that **CL** lacks a negative test (for some formulas it can not be decided that they are not a **CL**–consequence of a premise set), and that some of the conditions above are stated in a negative way.

Problematic Features. Although the above described logic–based approaches succeed in specifying which formulas may count as valid consequences of abductive inference steps, they do nevertheless have some serious disadvantages. As they are satisfactorily discussed in Meheus [71], I will only state them.

First of all, in order to cope with the lack of a positive test, logic–based approaches are usually restricted to decidable fragments of **CL** (or some other logic). But, as most interesting (real–life) theories are undecidable, this seems an unjustified restriction.

Secondly, logic–based approaches do not explicate the way in which people actually reason by means of abductive inferences. This is most clearly shown by the absence of a decent proof theory. Even the search procedures that are sometimes provided to obtain the right abductive conclusions, are based on systems that do not even by far resemble human reasoning (for example, the tableaux method developed by Aliseda–Llera in [2, 4]).

13.2.2 The Adaptive Approach

Although the adaptive approach is strictly speaking also a logic–based approach, I distinguish it from the usual logic–based approaches because it avoids the disadvantages of the latter.

A. The Adaptive Logic $\mathbf{LA^r}$

In Meheus & Batens [74], Meheus and Batens presented the adaptive logic $\mathbf{LA^r}$. Shortly summarized, $\mathbf{LA^r}$ is a standard (flat) adaptive logic that has the logic **CL** as its **LLL** and the set Ω (see def. 13.1) as its set of abnormalities, and is based on the reliability strategy.

Definition 13.1 $\Omega = \{(\forall\alpha)(A(\alpha) \sqsupset B(\alpha)) \wedge B(\beta) \wedge \neg A(\beta) \mid B(\beta) \in \mathcal{W}^e, A(\beta) \in \mathcal{W}^a$ *and* $\nvdash_{\mathbf{CL}} (\forall\alpha)(A(\alpha) \sqsupset B(\alpha))\}$.[2]

[2] $\mathcal{W}^e$ and $\mathcal{W}^a$ are respectively the set of explananda and the set of explanantia.

In [74], Meheus and Batens show that $\mathbf{LA^r}$ only allows for the unproblematic applications of **AC**. Intuitively, this is done by accepting the consequences reached by applying **AC** only for as long as it has not been proven that they should be rejected (for example, because their negation is deductively derivable from the background theory).

Moreover, they also showed that $\mathbf{LA^r}$ not only incorporates the most important conditions of the usual logic–based approaches towards abduction ((i)–(iv)), but also avoids their disadvantages: $\mathbf{LA^r}$ can be applied to undecidable theories, and it provides us with a proof theory that comes much closer to actual reasoning processes.[3]

Disadvantage of $\mathbf{LA^r}$. Despite the mentioned advantages, the logic $\mathbf{LA^r}$ also has one major disadvantage: the possible abductive inferences are restricted to those of the form

$$(\forall\alpha)(A(\alpha) \sqsupset B(\alpha)), B(\beta) \vdash A(\beta).$$

This restriction was introduced in order to avoid the irrelevance problem. I leave out the technical details, but what it comes down to is the following: if the restriction would not have been introduced, and instead there was opted to conditionally allow abductive inferences of the form

$$A \sqsupset B,\ B \mid\!\!- A,$$

then, because of **Irr**, no abductive consequences would follow from a premise set. In proof theoretical terms: all conditionally derived formulas would get marked. That this doesn't happen when the abductive inferences are restricted to the ones above, is a consequence of the specific features of universally quantified formulas.[4]

But, despite the fact that there are good technical reasons to restrict the applications of **AC** to the ones above, it nevertheless remains a fact that as a consequence, $\mathbf{LA^r}$ only allows for a specific subset of the possible abductive inferences. This means that it can not cover all possible kinds of abduction processes. For example, abduction at the purely propositional level is not possible.

B. The Adaptive Logic AbL

In the remaining of this chapter, I will present the adaptive logic **AbL**. It avoids the irrelevance problem not by relying on the specific features of

[3] This was clearly illustrated by Meheus in [71].

[4] More specifically, it is a consequence of the fact that formulas that are not universally quantified can not lead to universally quantified formulas, so that **Irr** cannot affect the abductive situation.

universally quantified formulas as does $\mathbf{LA^r}$, but by introducing a specific modal operator (see section 13.3.1).

It will be shown that **AbL** is able to capture abductive inference in a more general way than $\mathbf{LA^r}$. For example, in **AbL**, abduction is also possible at the purely propositional level. Moreover, as **AbL** also keeps all the advantages of the logic $\mathbf{LA^r}$, I feel quite confident to claim that it should be preferred over $\mathbf{LA^r}$.

Practical and Theoretical Abduction. In [74], Meheus & Batens distinguish between two kinds of abduction. I will call them *practical* and *theoretical* abduction. The difference between both comes down to the following: in case there are multiple possible explanations for a particular phenomenon, practical abduction only allows to derive the disjunction of those explanations. Theoretical abduction on the contrary, allows to derive all of them. The distinction between practical and theoretical abduction is justified by Meheus & Batens by means of the following example:

> Consider the case of a patient a displaying some symptom P who consults a physician to get cured. Suppose that the physician's theoretical knowledge contains $(\forall x)(Qx \sqsupset Px)$ and $(\forall x)(Rx \sqsupset Px)$, and no other (sensible) candidates for an abductive step. It would be rather stupid of the physician to conclude to Qa and to act accordingly. This would be stupid because, if Ra is the case, rather than Qa, the patient would not be cured. So, the appropriate behaviour for the physician would be to draw the conclusion $Qa \vee Ra$ and to test whether Qa, Ra or both are true, or to act in such a way that the patient gets cured in either case.
>
> Compare this situation to one in which a 'theoretician' has the same knowledge, but is merely interested in finding and testing explanatory hypotheses for Pa. In this case, there would be no harm if the theoretician derived, say, Qa and tested it. if it turns out true, an explanation is produced. If it turns out false, Ra might be the next hypothesis derived.[5]

Despite the fact that two kinds of abduction are distinguished, the logic $\mathbf{LA^r}$ is only able to capture practical abduction, and no other logic has been presented by Meheus & Batens that captures theoretical abduction.

On the contrary, the logic **AbL** can capture both kinds of abduction. More precisely, I should say that **AbL** has two variants, $\mathbf{AbL^p}$ and $\mathbf{AbL^t}$, such that the former captures practical abduction and the latter captures theoretical abduction.

[5] See Meheus & Batens [74, p. 224].

13.3 The Adaptive Logic AbL

As the adaptive logic **AbL** was already introduced in the foregoing section, I can limit myself here to its characterization. First of all, **AbL** makes use of two distinct premise sets: Γ and Γ^e. The former represents the background theory, while the latter contains phenomena that are in need of an explanation (the explananda).

Secondly, if $\Gamma^{\Diamond}$ is defined as in definition 13.2, then **AbL** is characterized as in definition 13.3 below.[6]

Definition 13.2 $\Gamma^{\Diamond} = \{\Diamond A \mid A \in \Gamma^e\}$.

Definition 13.3 $\langle \Gamma, \Gamma^e \rangle \vdash_{\mathbf{AbL}} A$ *iff* $\langle \Gamma, \Gamma^{\Diamond} \rangle \vdash_{\mathbf{CL^{abd}}} A$.

Finally, as there are two variants of the logic **AbL** (one for practical and one for theoretical abduction), there will also be two variants of the logic $\mathbf{CL^{abd}}$. I will however use $\mathbf{CL^{abd}}$ to refer to both of them. As such, I do not always have to distinguish them explicitly.

Preliminary Remark 1. To retain the overall coherence of this dissertation, I will restrict myself to the propositional case. Nevertheless, the extension to the predicative case is completely straightforward.

Preliminary Remark 2. For reasons of simplicity, I will not refer to the premises as to a couple. I will just treat them as a single set. It should however be kept in mind that this is only a matter of speech.

13.3.1 The Lower Limit Logic $\mathbf{CL}^{\Diamond}$

The lower limit logic of the logic $\mathbf{CL^{abd}}$ is the "modal" logic $\mathbf{CL}^{\Diamond}$. First, consider its language schema.

Language Schema. $\mathbf{CL}^{\Diamond}$ is based on the language $\mathcal{L}^{\mathcal{M}}$, which is the standard propositional language $\mathcal{L}$ (see chapter 4, section 4.2.1), extended with the modal operator $\Diamond$. The set $\mathcal{W}^{\mathcal{M}}$ of well–formed formulas of the language $\mathcal{L}^{\mathcal{M}}$ is constructed as usual. Table 13.1 below clearly states the relation between $\mathcal{L}$ and $\mathcal{L}^{\Diamond}$.

[6] The elements of Γ, Γ^e and the formula A are all elements of the standard propositional language $\mathcal{L}$ (see also section 13.3.1). Hence, they do not contain the modal operator $\Diamond$.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\neg, \wedge, \vee, \sqsupset$	$\mathcal{W}$
$\mathcal{L}^{\mathcal{M}}$	$\mathcal{S}$	$\neg, \wedge, \vee, \sqsupset, \Diamond$	$\mathcal{W}^{\mathcal{M}}$

Table 13.1: Relation between $\mathcal{L}$ and $\mathcal{L}^{\Diamond}$.

The set of *primitive formulas* of the language $\mathcal{L}$ is the set $\mathcal{S} \cup^{\neg} \mathcal{S}$, with $^{\neg}\mathcal{S} = \{\neg A \mid A \in \mathcal{S}\}$. Now, consider the following definitions, they will turn out to be useful in the remaining of this chapter:

Definition 13.4 *A formula is in disjunctive normal form (***DNF***) when it is a disjunction of conjunctions of primitive formulas.*

Definition 13.5 $DNF(\mathcal{L}) = \{A \in \mathcal{W} \mid A$ *is in* **DNF**$\}$.

Definition 13.6 $CON(\mathcal{L}) = \{A_1 \wedge ... \wedge A_n \in \mathcal{W} \mid A_1, ..., A_n \in \mathcal{S} \cup^{\neg} \mathcal{S}\}$.

Semantics. Semantically, the logic $\mathbf{CL}^{\Diamond}$ is characterized as follows: a $\mathbf{CL}^{\Diamond}$–model for the language $\mathcal{L}$, with $\mathcal{S}$ the set of sentential letters, $^{\neg}\mathcal{S} = \{\neg A \mid A \in \mathcal{S}\}$ the set of negated sentential letters, and $\mathcal{W}^{\Diamond} = \{\Diamond A \mid A \in \mathcal{W}^{\mathcal{M}}\}$, is an assignment function v:

AP1 $v: \mathcal{S} \mapsto \{0,1\}$.
AP2 $v: {}^{\neg}\mathcal{S} \mapsto \{0,1\}$.
AP3 $v: \mathcal{W}^{\Diamond} \mapsto \{0,1\}$.

The valuation function v_M determined by the model M is defined as follows:[7]

SP1 $v_M : \mathcal{W} \mapsto \{0,1\}$.
SP2 For $A \in \mathcal{S}$: $v_M(A) = 1$ iff $v(A) = 1$.
SP3 For $A \in \mathcal{S}$: $v_M(\neg A) = 1$ iff $v_M(\neg A) = 0$.
SP4 $v_M(\mathfrak{a}) = 1$ iff $v_M(\mathfrak{a}_1) = 1$ and $v_M(\mathfrak{a}_2) = 1$.
SP5 $v_M(\mathfrak{b}) = 1$ iff $v_M(\mathfrak{b}_1) = 1$ or $v_M(\mathfrak{b}_2) = 1$.
SP6 $v_M(\Diamond A) = 1$ iff $v(\Diamond A) = 1$.

Truth in a model, semantical consequence, and validity are defined as usual.

Proof Theory. The $\mathbf{CL}^{\Diamond}$–proof theory is the same as the one for **CL** (see chapter 4, section 4.2.3). Hence, there are no inference rules related to the modal operator, which means that even Replacement of Identicals is not possible within the reach of a modal operator. In other words: "modal formulas have no modal consequences."[8] What this comes down to, will become clear by considering some examples:

[7] $\mathfrak{a}$– and $\mathfrak{b}$–formulas are defined as in chapter 4, table 4.1.

[8] In fact, $\mathbf{CL}^{\Diamond}$ resembles the modal logic **IM** from Batens & Haesaert [31]. Semantically, **IM** is obtained from $\mathbf{CL}^{\Diamond}$ by replacing clause **SP6** by the following one:

(1) $\Diamond(A \wedge B) \nvdash_{\mathbf{CL}^\Diamond} \Diamond A$
(2) $\Diamond A \nvdash_{\mathbf{CL}^\Diamond} \Diamond(A \vee B)$
(3) $\Diamond A, \Diamond B \vdash_{\mathbf{CL}^\Diamond} \Diamond A \wedge \Diamond B$
(4) $\Diamond(A \wedge B) \nvdash_{\mathbf{CL}^\Diamond} \Diamond(B \wedge A)$
(5) $\Diamond A \nvdash_{\mathbf{CL}^\Diamond} \Diamond(B \sqsupset A)$
(6) $\Diamond A \vdash_{\mathbf{CL}^\Diamond} B \sqsupset \Diamond A$

The last two examples make clear how the irrelevance problem will be avoided in the adaptive logics based on $\mathbf{CL}^\Diamond$: if the abductive inferences are limited to those of the following form

$$A \sqsupset B, \Diamond B \vdash A,$$

then the implication $A \sqsupset B$ cannot have been derived from the formula $\Diamond B$, but has to follow from the background theory.

13.3.2 The Adaptive Logic $\mathbf{CL}^{\mathbf{abd}}$

As stated above, the logic $\mathbf{CL}^{\mathbf{abd}}$ has two variants, one for practical abduction and one for theoretical abduction. They will be called respectively $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$. Both have the logic $\mathbf{CL}^\Diamond$ as their **LLL**, but differ concerning the set of abnormalities and the adaptive strategy.

Practical Abduction. The logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ is a standard (flat) adaptive logic, with the set Ω_p as its set of abnormalities.

$$\Omega_p = \{(A \sqsupset B) \wedge (\Diamond B \wedge \neg A) \mid A \in DNF(\mathcal{L}) \text{ and } \nvdash_{\mathbf{CL}^\Diamond} A \sqsupset B\}.$$

Its adaptive strategy is the standard reliability strategy.

Theoretical Abduction. The logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ is also a standard adaptive logic, with the set Ω_t as its set of abnormalities.

$$\Omega_t = \{(A \sqsupset B) \wedge (\Diamond B \wedge \neg A) \mid A \in CON(\mathcal{L}) \text{ and } \nvdash_{\mathbf{CL}^\Diamond} A \sqsupset B\}.$$

The adaptive strategy of the logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ is the *relevant reliability* strategy, a particular variant of the standard reliability strategy.

SP6' $v_M(\Diamond A) = 1$ iff $v_M(A) = 1$ or $v(\Diamond A) = 1$.

Proof theoretically, the following inference rule should be added to the $\mathbf{CL}^\Diamond$–proof theory:

Refl$\Diamond$ $A \blacktriangleright \Diamond A$.

As a consequence, one could claim that **IM** is $\mathbf{CL}^\Diamond$ with “a reflexive accessibility relation”.

13.3.3 Proof Theory of $\mathbf{CL^{abd}}$

The proof theories of $\mathbf{CL_p^{abd}}$ and $\mathbf{CL_t^{abd}}$ are obviously quite resemblant. First, consider the deduction rules, which are the same for both logics.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\mathbf{CL}^{\Diamond}} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\mathbf{CL}^{\Diamond}} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Next, consider the marking criterium. For both logics, it consists of one marking rule. Nevertheless, there is an important difference between both logics.

Practical Abduction. The logic $\mathbf{CL_p^{abd}}$ is based on the standard reliability strategy, so that at every stage of the proof, the set of unreliable formulas is constructed by means of the minimal *Dab*–consequence derived at that stage.

Definition 13.7 *$Dab(\Delta)$ $(\Delta \subset \Omega)$ is a minimal Dab–consequence at stage s of the proof iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also a Dab–consequence at stage s of the proof.*

Definition 13.8 *$U_s(\Gamma) = \{A \in \Omega \mid A \in \Delta$ and $Dab(\Delta)$ is a minimal Dab–consequence of Γ at stage s of the proof$\}$.*

As usual, whether or not a line is marked at a stage of the proof, now depends on the intersection of its condition and the set of unreliable formulas at that stage.

Definition 13.9 *Line i is marked at stage s of the proof iff where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

Theoretical Abduction. Marking for $\mathbf{CL_t^{abd}}$ is slightly more complicated. First, I need to introduce the notion of *homogenous Dab*–formulas, which are *Dab*–formulas based on the same explanandum, and constructed by means of the same sentential letters.

Definition 13.10 *$s(A) = \{B \in \mathcal{S} \mid B$ occurs in the formula $A\}$.*

Definition 13.11 *Dab–formulas* $(A \sqsupset B) \wedge (\Diamond B \wedge \neg A)$ *and* $(A' \sqsupset B') \wedge (\Diamond B' \wedge \neg A')$ *are homogenous iff* $B = B'$ *and* $s(A) = s(A')$.

In order to make this more concrete, consider the examples below.

Example 13.1 *The following Dab–formulas are homogenous:*

- $((p \wedge \neg r \wedge s) \sqsupset q) \wedge \Diamond q \wedge \neg(p \wedge \neg r \wedge s)$ *and* $((\neg p \wedge r \wedge s) \sqsupset q) \wedge \Diamond q \wedge \neg(\neg p \wedge r \wedge s)$
- $((\neg p \wedge s) \sqsupset \neg(q \wedge r)) \wedge \Diamond\neg(q \wedge r) \wedge \neg(\neg p \wedge s)$ *and* $((p \wedge \neg s) \sqsupset \neg(q \wedge r)) \wedge \Diamond\neg(q \wedge r) \wedge \neg(p \wedge \neg s)$

Next, it is possible to separate the homogenous from the non–homogenous *Dab*–consequences of a premise set. The former obviously being those *Dab*–consequences which consist of homogenous abnormalities.

Definition 13.12 $Dab(\Delta)$ *is a homogenous Dab–consequence of* Γ *at stage* s *of the proof iff* $Dab(\Delta)$ *is a minimal Dab–consequence of* Γ *at stage* s *of the proof and all elements in* Δ *are homogenous.*

Finally, the set of unreliable abnormalities at a stage of the proof can be constructed by relying on the *relevant Dab*–consequences of a premise set at that stage.

Definition 13.13 $Dab(\Delta)$ $(\Delta \subset \Omega)$ *is a relevant Dab–consequence of* Γ *at stage* s *of the proof iff* $Dab(\Delta)$ *is a minimal Dab–consequence at stage* s *of the proof and there is no* Δ' *such that* $Dab(\Delta')$ *is a homogenous Dab–consequence of* Γ *at stage* s *of the proof and* $\Delta \cap \Delta' \neq \emptyset$.

Definition 13.14 $U^r_s(\Gamma) = \{A \in \Omega \mid A \in \Delta$ *and* $Dab(\Delta)$ *is a homogenous or a relevant Dab–consequence of* Γ *at stage* s *of the proof*$\}$.

The marking rule for the relevant reliability strategy is now plainly straightforward, as it is the same as for the standard reliability strategy:

Definition 13.15 *Line* i *is marked at stage* s *of the proof iff where* Δ *is its condition,* $\Delta \cap U^r_s(\Gamma) \neq \emptyset$.

Final Derivability. As for all adaptive logics, I should also mention the definitions for final $\mathbf{CL^{abd}}$–derivability. As they are completely standard, they do not need any further explanation.

Definition 13.16 A *is* finally derived *from* Γ *on line* i *of a proof at stage* s *iff (i)* A *is the second element of line* i*, (ii) line* i *is not marked at stage* s*, and (iii) every extension of the proof in which line* i *is marked may be further extended in such a way that line* i *is unmarked again.*

Definition 13.17 $\Gamma \vdash_{\mathbf{CL^{abd}}} A$ *(*A *is finally* $\mathbf{CL^{abd}}$*–derivable from* Γ*) iff* A *is finally derived on a line of a proof from* Γ.

Example. Because the deduction rules of $\mathbf{CL^{abd}_p}$ and $\mathbf{CL^{abd}_t}$ are the same, the example below will do for both of them. Consider the premise couple $\langle \Gamma, \Gamma^{\Diamond} \rangle$, with $\Gamma = \{p \sqsupset q, r \sqsupset q, s \sqsupset t, u \sqsupset z, s, \neg z\}$ and $\Gamma^{\Diamond} = \{\Diamond q, \Diamond t\}$.

Finally, some terminological remarks. Marking for $\mathbf{CL^{abd}_p}$ and $\mathbf{CL^{abd}_t}$ will be denoted by respectively $\checkmark^p$ and $\checkmark^t$, and *Dab*–formulas will be abbreviated as follows $\langle A, B \rangle =_{df} (A \sqsupset B) \wedge (\Diamond B \wedge \neg A)$.

1	$p \sqsupset q$	–;PREM	$\emptyset$
2	$r \sqsupset q$	–;PREM	$\emptyset$
3	$s \sqsupset t$	–;PREM	$\emptyset$
4	$u \sqsupset z$	–;PREM	$\emptyset$
5	s	–;PREM	$\emptyset$
6	$\neg z$	–;PREM	$\emptyset$
7	$\Diamond q$	–;PREM	$\emptyset$
8	$\Diamond t$	–;PREM	$\emptyset$
9	p	$1, 7$;RC	$\langle p, q \rangle$
10	r	$2, 7$;RC	$\langle r, q \rangle$
11	s	$3, 8$;RC	$\langle s, t \rangle$
12	u	$4, 8$;RC	$\langle u, z \rangle$

At stage 12 of the proof, line 9 to 12 have been derived conditionally. None of them is marked yet. This however changes when the proof is extended as follows.

11	s	$3, 8$;RC	$\langle s, t \rangle$	$\checkmark^p$ $\checkmark^t$
12	u	$4, 8$;RC	$\langle u, t \rangle$	$\checkmark^p$ $\checkmark^t$
13	$\langle u, z \rangle$	$4, 6, 8$;RU	$\emptyset$	
14	$\langle s, t \rangle \vee \langle \neg s, t \rangle$	$3, 5, 8$;RU	$\emptyset$	

At stage 14 of the proof, line 11 and 12 are marked for both reliability and relevant reliability. Moreover, as both *Dab*–consequences on line 13 and 14 are relevant *Dab*–consequences, line 11 and 12 will remain marked.

9	p	$1, 7$;RC	$\langle p, q \rangle$	$\checkmark^p$ $\checkmark^t$
10	r	$2, 7$;RC	$\langle r, q \rangle$	$\checkmark^p$ $\checkmark^t$
...	...	...	...	
15	$\langle p, q \rangle \vee \langle r \wedge \neg p, q \rangle$	$1, 2, 7$;RU	$\emptyset$	
16	$\langle r, q \rangle \vee \langle p \wedge \neg r, q \rangle$	$1, 2, 7$;RU	$\emptyset$	

At stage 16 of the proof, line 9 and 10 have also become marked for both reliability and relevant reliability. Moreover, the *Dab*–consequences on line

15 and 16 are minimal *Dab*–consequences, which means that line 9 and 10 will not become unmarked again for reliability. Hence, the formulas on those lines are not finally derivable for the logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$. But, the *Dab*–consequences on line 15 and 16 are not homogenous, and it is possible to extent the proof in such a way that they are not considered relevant *Dab*–consequences anymore.

9	p	$1,7$;RC	$\langle p,q\rangle$	$\checkmark^p$
10	r	$2,7$;RC	$\langle r,q\rangle$	$\checkmark^p$
...	...	...	...	
15	$\langle p,q\rangle \vee \langle r \wedge \neg p, q\rangle$	$1,2,7$;RU	$\emptyset$	
16	$\langle r,q\rangle \vee \langle p \wedge \neg r, q\rangle$	$1,2,7$;RU	$\emptyset$	
17	$\langle r \wedge p,q\rangle \vee \langle r \wedge \neg p, q\rangle$	$1,2,7$;RU	$\emptyset$	
18	$\langle p \wedge r,q\rangle \vee \langle p \wedge \neg r, q\rangle$	$1,2,7$;RU	$\emptyset$	

As a consequence, at stage 18 of the proof, line 9 and 10 have become unmarked again with respect to the relevant reliability strategy. Hence, the formulas on those lines are finally derivable for the logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$. Moreover, their disjunction is finally derivable for the logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$.

19	$r \vee p$	$1,2,7$;RC	$\langle r \vee p, q\rangle$

13.3.4 Semantics of $\mathbf{CL}^{\mathbf{abd}}$

As the semantics of $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ differ in some respect, they will be treated separately.

Practical Abduction. The $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$–semantics is the standard semantics for adaptive logics based on the reliability strategy. This means that semantical $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$–consequence is defined with respect to the reliable $\mathbf{CL}^{\Diamond}$–models of a premise set Γ.

Definition 13.18 *Where M is a $\mathbf{CL}^{\Diamond}$–model, $Ab(M) = \{A \in \Omega_p \mid M \models A\}$.*

Definition 13.19 *$Dab(\Delta)$ is a minimal Dab–consequence of Γ iff $\Gamma \models_{\mathbf{CL}^{\Diamond}} Dab(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \not\models_{\mathbf{CL}^{\Diamond}} Dab(\Delta')$.*

Definition 13.20 *$U(\Gamma) = \Delta_1 \cup \Delta_2 \cup ...$, with $Dab(\Delta_1), Dab(\Delta_2), ...$ the minimal Dab–consequences of Γ.*

Definition 13.21 *A $\mathbf{CL}^{\Diamond}$–model M of Γ is reliable iff $Ab(M) \subseteq U(\Gamma)$.*

Definition 13.22 *$\Gamma \models_{\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}} A$ iff A is verified by all reliable models of Γ.*

Theoretical Abduction. The $\mathbf{CL_t^{abd}}$–semantics is obviously a little more complicated, as it is not based on the reliable $\mathbf{CL}^\Diamond$–models of a premise set, but on the relevant reliable models of a premise set.

Definition 13.23 *Where M is a $\mathbf{CL}^\Diamond$–model, $Ab(M) = \{A \in \Omega_t \mid M \models A\}$.*

Definition 13.24 *$Dab(\Delta)$ is a minimal Dab–consequence of Γ iff $\Gamma \models_{\mathbf{CL}^\Diamond} Dab(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \not\models_{\mathbf{CL}^\Diamond} Dab(\Delta')$.*

Definition 13.25 *$Dab(\Delta)$ is a homogenous Dab–consequence of Γ iff $Dab(\Delta)$ is a minimal Dab–consequence of Γ and all elements in Δ are homogenous.*

Definition 13.26 *$Dab(\Delta)$ is a relevant Dab–consequence of Γ iff $Dab(\Delta)$ is a minimal Dab–consequence of Γ and there is no Δ' such that $Dab(\Delta')$ is a homogenous Dab–consequence of Γ and $\Delta \cap \Delta' \neq \emptyset$.*

Definition 13.27 *$U^r(\Gamma) = \Delta_1 \cup \Delta_2 \cup ...$, with $Dab(\Delta_1), Dab(\Delta_2), ...$ the homogenous and relevant Dab–consequences of Γ.*

Definition 13.28 *A $\mathbf{CL}^\Diamond$–model M of Γ is relevant reliable iff $Ab(M) \subseteq U^r(\Gamma)$.*

Definition 13.29 *$\Gamma \models_{\mathbf{CL_t^{abd}}} A$ iff A is verified by all relevant reliable models of Γ.*

13.3.5 Soundness and Completeness

As the logic $\mathbf{CL_p^{abd}}$ is a standard adaptive logic, soundness and completeness are implied by the standard format (see chapter 3).

Theorem 13.1 *$\langle \Gamma, \Gamma^\Diamond \rangle \vdash_{\mathbf{CL_p^{abd}}} A$ iff $\langle \Gamma, \Gamma^\Diamond \rangle \models_{\mathbf{CL_p^{abd}}} A$.*

Soundness and completeness for the logic $\mathbf{CL_t^{abd}}$ is a bit more demanding. In fact, the soundness and completeness proof given in Batens [26, pp. 232–233] for adaptive logics based on the reliability strategy will also work for $\mathbf{CL_t^{abd}}$, but only if *Reassurance* can be proven for relevant reliability. As a consequence, I will do this below.

Theorem 13.2 (Reassurance) *If Γ has $\mathbf{CL}^\Diamond$–models, Γ also has $\mathbf{CL_t^{abd}}$–models.*

Proof. Suppose Γ has $\mathbf{CL}^\Diamond$–models. Hence, where $\mathcal{M}_\Gamma^{\mathbf{CL}^\Diamond}$ and $\mathcal{M}_\Gamma^{rr}$ are respectively the set of $\mathbf{CL}^\Diamond$–models of Γ and the set of relevant reliable models of Γ, the theorem obviously holds if $\mathcal{M}_\Gamma^{rr} = \mathcal{M}_\Gamma^{\mathbf{CL}^\Diamond}$. Next, consider $\Gamma \cup \Delta$, with $\Delta = Neg(\Omega - U^r(\Gamma))$.[9] The proof now proceeds in two steps:

[9] Remember that $Neg(\Delta) = \{\neg A \mid A \in \Delta\}$ (see chapter 12).

1. $\Gamma \cup \Delta$ has $\mathbf{CL}^{\Diamond}$–models.
   Suppose $\Gamma \cup \Delta$ has no $\mathbf{CL}^{\Diamond}$–models. This can only be true when an inconsistency is derivable from $\Gamma \cup \Delta$. As Γ has $\mathbf{CL}^{\Diamond}$–models, no inconsistencies follow from Γ alone. This means that some of the elements of Δ should lead to an inconsistency. This implies that the negation of a minimal *Dab*–consequence is derivable from Δ. But, this is impossible, as for all minimal *Dab*–consequences of Γ, there is at least one disjunct in $U^r(\Gamma)$ (by definition 13.26), which means that its negation has not been allowed in Δ. This contradicts the supposition, which means that $\Gamma \cup \Delta$ has $\mathbf{CL}^{\Diamond}$–models.
2. If M is a $\mathbf{CL}^{\Diamond}$–model of $\Gamma \cup \Delta$, then $M \in \mathcal{M}^{rr}_{\Gamma}$.
   Suppose M is a $\mathbf{CL}^{\Diamond}$–model of $\Gamma \cup \Delta$ and $M \notin \mathcal{M}^{rr}_{\Gamma}$. From the latter, it follows that $Ab(M) \nsubseteq U^r(\Gamma)$, which means that $Ab(M) \cap (\Omega - U^r(\Gamma)) \neq \emptyset$. Now, suppose $D \in (Ab(M) \cap (\Omega - U^r(\Gamma)))$. Hence, (1) $D \in Ab(M)$ and (2) $D \in (\Omega - U^r(\Gamma))$. From (1), it follows that M verifies D (by definition 13.23), and from (2), it follows that $\neg D \in \Delta$, which means that M verifies $\neg D$ (as M is a model of $\Gamma \cup \Delta$). But, in view of the $\mathbf{CL}^{\Diamond}$–semantics, this is impossible, so that our supposition is rejected.

From the above, it now follows that Γ also has $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$–models. ■

Now, because of theorem 13.2, soundness and completeness also follows for the logic $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$:

Theorem 13.3 $\langle \Gamma, \Gamma^{\Diamond} \rangle \vdash_{\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}} A$ *iff* $\langle \Gamma, \Gamma^{\Diamond} \rangle \models_{\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}} A$.

13.4 Paraconsistent Abduction

The adaptive logic **AbL** presented in the foregoing section, can only capture abductive inferences based on a consistent background theory. When applied to an inconsistent background theory, the logic **AbL** will unavoidably lead to the trivial consequence set. This is a consequence of the fact that **AbL** is characterized by translation to the logic $\mathbf{CL}^{\mathbf{abd}}$, which has an explosive **LLL** (see section 13.3.1).

However, it is a well–known fact that most of the interesting real–life theories are inconsistent. Moreover, this will not prevent anyone from trying to use them to explain certain (puzzling) phenomena, especially if those theories appear to be the best ones around. As a consequence, if one wants to explicate abductive reasoning, one should also consider paraconsistent abduction, abduction based on an inconsistent background theory.

The Logic PAbL. In this section, I will present the adaptive logic **PAbL**, the paraconsistent version of the logic **AbL**. It captures abductive inference based on an inconsistent background theory. As for **AbL**, there are also two variants of **PAbL**, one that captures practical abduction (the logic $\mathbf{PAbL^p}$) and one that captures theoretical abduction (the logic $\mathbf{PAbL^t}$).

When $\Gamma^{\exists\mathbf{i}}$ and $\Gamma^{\exists\mathbf{i}\Diamond}$ are defined respectively as in definitions 13.30 and 13.31, the logics $\mathbf{PAbL^p}$ and $\mathbf{PAbL^t}$ are characterized respectively as in definitions 13.32 and 13.33 below.[10]

Definition 13.30 $\Gamma^{\exists i} = \{(\exists i)A^{(i)} \mid A \in \Gamma\}$.

Definition 13.31 $\Gamma^{\exists i\Diamond} = \{\Diamond(\exists i)A^{(i)} \mid A \in \Gamma^e\}$.

Definition 13.32 $\langle\Gamma, \Gamma^e\rangle \vdash_{\mathbf{PAbL^p}} A$ *iff* $\langle\Gamma^{\exists i}, \Gamma^{\exists i\Diamond}\rangle \vdash_{\exists\mathbf{CL^{abd}_p}} (\exists i)A^{(i)}$.

Definition 13.33 $\langle\Gamma, \Gamma^e\rangle \vdash_{\mathbf{PAbL^t}} A$ *iff* $\langle\Gamma^{\exists i}, \Gamma^{\exists i\Diamond}\rangle \vdash_{\exists\mathbf{CL^{abd}_t}} (\exists i)A^{(i)}$.

In the remaining of this section, I will present the adaptive logics $\exists\mathbf{CL^{abd}_p}$ and $\exists\mathbf{CL^{abd}_t}$, starting with their lower limit logic, the logic $\exists\mathbf{CL}^\Diamond$.

Preliminary Remark. As for the logic **AbL**, I will not refer to the premises as to a couple, but will treat them as a single set. Again, this is only a matter of speech.

13.4.1 The Lower Limit Logic $\exists\mathbf{CL}^\Diamond$

The **LLL** of the logics $\exists\mathbf{CL^{abd}_p}$ and $\exists\mathbf{CL^{abd}_t}$ is the logic $\exists\mathbf{CL}^\Diamond$, the ambiguity logic based on the logic $\mathbf{CL}^\Diamond$ from section 13.3.1.

Language Schema. The logic $\exists\mathbf{CL}^\Diamond$ is based on the language $\mathcal{L}^{\exists\mathbf{i}\Diamond}$. This is the language $\mathcal{L}^{\exists\mathbf{i}}$ (see chapter 7, section 7.3.1), extended with the peculiar modal operator $\Diamond$ from above. The set of well–formed formulas of $\mathcal{L}^{\exists\mathbf{i}}$ is constructed in the usual way.

language	letters	connectives	set of wffs
$\mathcal{L}$	$\mathcal{S}$	$\neg, \wedge, \vee, \supset$	$\mathcal{W}$
$\mathcal{L}^{\exists\mathbf{i}}$	$\mathcal{S}^{\mathcal{I}}$	$\neg, \wedge, \vee, \supset, \exists, \forall$	$\mathcal{W}^{\exists\mathbf{i}}$
$\mathcal{L}^{\mathcal{I}}$	$\mathcal{S}^{\mathcal{I}}$	$\neg, \wedge, \vee, \supset$	$\mathcal{W}^{\mathcal{I}}$
$\mathcal{L}^{\exists\mathbf{i}\Diamond}$	$\mathcal{S}^{\mathcal{I}}$	$\neg, \wedge, \vee, \supset, \exists, \forall, \Diamond$	$\mathcal{W}^{\exists\mathbf{i}\Diamond}$
$\mathcal{L}^{\mathcal{I}\Diamond}$	$\mathcal{S}^{\mathcal{I}}$	$\neg, \wedge, \vee, \supset, \Diamond$	$\mathcal{W}^{\mathcal{I}\Diamond}$

Table 13.2: Relations between $\mathcal{L}^{\exists\mathbf{i}}$, $\mathcal{L}^{\mathcal{I}}$, $\mathcal{L}^{\exists\mathbf{i}\Diamond}$ and $\mathcal{L}^{\mathcal{I}\Diamond}$.

Quite unsurprisingly, the set of primitive formulas of $\mathcal{L}^{\exists\mathbf{i}\Diamond}$ is the set $\mathcal{S}^{\mathcal{I}} \cup {}^{\neg}\mathcal{S}^{\mathcal{I}}$, with ${}^{\neg}\mathcal{S}^{\mathcal{I}} = \{\neg A \mid A \in \mathcal{S}^{\mathcal{I}}\}$. As such, also the following definitions can now be stated:

[10] Remark that in all those definitions, the elements of Γ,Γ^e and the formula A are wffs of the standard propositional language $\mathcal{L}$.

Definition 13.34 *A formula is in disjunctive normal form (***DNF***) when it is a disjunction of conjunctions of primitive formulas.*

Definition 13.35 $DNF(\mathcal{L}^{\mathcal{I}}) = \{A \in \mathcal{W}^{\mathcal{I}} \mid A \textit{ is in } \mathbf{DNF}\}$.

Definition 13.36 $CON(\mathcal{L}^{\mathcal{I}}) = \{A_1 \wedge ... \wedge A_n \in \mathcal{W}^{\mathcal{I}} \mid A_1, ..., A_n \in \mathcal{S}^{\mathcal{I}} \cup {}^{\neg}\mathcal{S}^{\mathcal{I}}\}$.

Semantics and Proof Theory. Both the semantics and the proof theory can easily be reconstructed by relying on the information from sections 7.3 (chapter 7) and 13.3.1. As such, I will not discuss them anymore.

13.4.2 The Adaptive Logic $\exists \mathbf{CL}^{\mathbf{abd}}$

Also for the paraconsistent case, there are two variants of $\exists \mathbf{CL}^{\mathbf{abd}}$, one for practical and one for theoretical abduction. In analogy with their consistent counterparts, they will be called $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$. Both are based on the logic $\exists \mathbf{CL}^{\Diamond}$, but differ concerning set of abnormalities and adaptive strategy.

Practical Abduction. The set of abnormalities of the logic $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ is the set Ω^p, the union of the sets Ω^p_1 and Ω^1_2.

a) $\Omega^p_1 = \{(\exists i)A^{(i)} \wedge \neg A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}} \text{ and } A^{\mathcal{I}} \in \mathcal{I}(A)\}$.
b) $\Omega^p_2 = \{(A^{\mathcal{I}} \sqsupset (\exists i)B^{(i)} \wedge (\Diamond(\exists i)B^{(i)} \wedge \neg A^{\mathcal{I}}) \mid A^{\mathcal{I}} \in DNF(\mathcal{L}^{\mathcal{I}}) \text{ and } \nvdash_{\exists \mathbf{CL}^{\Diamond}} (\exists i)(A \sqsupset B)^{(i)}\}$.

The adaptive strategy of the logic $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ is the reliable normal selections strategy (see chapter 12).

Theoretical Abduction. The set of abnormalities of the logic $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ is the set Ω^t, which is the union of the sets Ω^p_1 and Ω^1_2.

a) $\Omega^t_1 = \{(\exists i)A^{(i)} \wedge \neg A^{\mathcal{I}} \mid (\exists i)A^{(i)} \in \Gamma^{\exists \mathbf{i}} \text{ and } A^{\mathcal{I}} \in \mathcal{I}(A)\}$.
b) $\Omega^t_2 = \{(A^{\mathcal{I}} \sqsupset (\exists i)B^{(i)}) \wedge (\Diamond(\exists i)B^{(i)} \wedge \neg A^{\mathcal{I}}) \mid A^{\mathcal{I}} \in CON(\mathcal{L}^{\mathcal{I}}) \text{ and } \nvdash_{\exists \mathbf{CL}^{\Diamond}} (\exists i)(A \sqsupset B)^{(i)}\}$.

The adaptive strategy of $\exists \mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ is the *relevant reliable normal selections* strategy, in which the relevant reliability strategy (see section 13.3.2) is combined with the normal selections strategy in the same way as it is done for reliability and normal selections in the reliable normal selections strategy.

13.4.3 Proof Theory of $\exists\mathrm{CL}^{\mathbf{abd}}$

As for $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$, the proof theories of $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ are based on the same deduction rules, but have different marking criteria. First, consider the deduction rules.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM, (iv) $\emptyset$.

RU If $A_1, ..., A_n \vdash_{\exists\mathbf{CL}^\Diamond} B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n$.

RC If $A_1, ..., A_n \vdash_{\exists\mathbf{CL}^\Diamond} B \vee Dab(\Theta)$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$ that have conditions $\Delta_1, ..., \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU, (iv) $\Delta_1 \cup ... \cup \Delta_n \cup \Theta$.

Next, consider the marking criteria. As they differ, they will be treated separately.

Practical Abduction. As $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ is based on the reliable normal selections strategy, its marking criterium consists of two marking rules, a reliability marking rule and a normal selections marking rule. First, consider the *normal selections*–marking rule. It is based on those *Dab*–consequences of a premise set that only consist of elements of Ω^p_1.

Definition 13.37 NS–*marking for* $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$*: Line i is marked at stage s of the proof iff where Δ is its condition, $Dab(\Theta)$ has been derived at stage s, $\Theta \subset \Omega^p_1$ and $\Theta \subseteq \Delta$.*

Next, consider the *reliability*–marking rule of $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$. It is based on the set $U_s(\Gamma)$, the set of elements of Ω^p_2 that are considered unreliable at stage s of the proof.

Definition 13.38 *$Dab(\Delta)$ $(\Delta \subset \Omega)$ is a minimal Dab–consequence at stage s of the proof iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also a Dab–consequence at stage s of the proof.*

Definition 13.39 *$U_s(\Gamma) = \{A \in \Omega^p_2 \mid A \in \Delta$ and $Dab(\Delta)$ is a minimal Dab–consequence of Γ at stage s of the proof$\}$.*

Definition 13.40 R–*marking for* $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$*: Line i is marked at stage s of the proof iff where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

Theoretical Abduction. Also the marking criterium for $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ consists of two marking rules, a relevant reliability marking rule and a normal selections marking rule. First, consider the normal selections marking rule, it is the same one as for $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$:

Definition 13.41 **NS**–*marking for* $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$*: Line i is marked at stage s of the proof iff where Δ is its condition, $Dab(\Theta)$ has been derived at stage s, $\Theta \subset \Omega^p_1$ and $\Theta \subseteq \Delta$.*

Next, consider the relevant reliability marking rule. As for the relevant reliability marking rule for $\mathbf{CL}^{\mathbf{abd}}$(see section 13.3.3), it is based on the relevant *Dab*–consequences of Γ at a stage of the proof. But, to be able to select out the relevant *Dab*–consequences, first consider the minimal *Dab*–consequences of Γ:

Definition 13.42 *$Dab(\Delta)$ $(\Delta \subset (\Omega^t_1 \cup \Omega^t_2))$ is a minimal Dab–consequence at stage s of the proof iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is also a Dab–consequence at stage s of the proof.*

It is now also possible to separate the homogenous from the non–homogenous *Dab*–consequences of a premise set.

Definition 13.43 *For $A^{\mathcal{I}} \in \mathcal{W}^{\mathcal{I}}$, $s(A^{\mathcal{I}}) = \{B \in \mathcal{S}^{\mathcal{I}} \mid B$ occurs in the formula $A^{\mathcal{I}}\}$.*

Definition 13.44 *Two Dab–formulas $(A^{\mathcal{I}} \sqsupset (\exists i)B^{(i)}) \wedge (\Diamond(\exists i)B^{(i)} \wedge \neg A^{\mathcal{I}})$ and $(A'^{\mathcal{I}} \sqsupset (\exists i)B'^{(i)}) \wedge (\Diamond(\exists i)B'^{(i)} \wedge \neg A'^{\mathcal{I}})$ are homogenous iff $B = B'$ and $s(A^{\mathcal{I}}) = s(A'^{\mathcal{I}})$.*

Definition 13.45 *$Dab(\Delta \cup \Theta)$ is a homogenous Dab–consequence of Γ at stage s of the proof iff $Dab(\Delta \cup \Theta)$ is a minimal Dab–consequence of Γ at stage s of the proof and all elements in Θ are homogenous.*

Finally, it is possible to define the set $U^r_s(\Gamma)$, the set of unreliable abnormalities at a stage of the proof. It is constructed by relying on the *relevant* *Dab*–consequences of the premise set at that stage.

Definition 13.46 *$Dab(\Delta \cup \Theta)$ $(\Delta \subset \Omega^t_1$ and $\Theta \subset \Omega^t_2)$ is a relevant Dab–consequence of Γ at stage s of the proof iff $Dab(\Delta \cup \Theta)$ is a minimal Dab–consequence at stage s of the proof and there is no $\Delta' \cup \Theta'$ $(\Delta' \subset \Omega^t_1$ and $\Theta' \subset \Omega^t_2)$ such that $Dab(\Delta' \cup \Theta')$ is a homogenous Dab–consequence of Γ at stage s of the proof and $\Theta \cap \Theta' \neq \emptyset$.*

Definition 13.47 *$U^r_s(\Gamma) = \{A \in \Omega^t_2 \mid A \in \Delta$ and $Dab(\Delta)$ is a homogenous or a relevant Dab–consequence of Γ at stage s of the proof$\}$.*

The marking rule for the relevant reliability strategy (**RR**) is now plainly straightforward:

Definition 13.48 **RR**–*marking for* $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$*: line* i *is marked at stage* s *of the proof iff where* Δ *is its condition,* $\Delta \cap U^r_s(\Gamma) \neq \emptyset$.

Final Derivability. The definitions for final $\exists\mathbf{CL}^{\mathbf{abd}}$–derivability are completely standard, so that I only need to mention them.

Definition 13.49 *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked again.*

Definition 13.50 $\Gamma \vdash_{\exists\mathbf{CL}^{\mathbf{abd}}} A$ *(A is finally $\exists\mathbf{CL}^{\mathbf{abd}}$–derivable from Γ) iff A is finally derived on a line of a proof from Γ.*

Example. Before I start with the example, remember that $\bigvee(\Delta)$ refers to the disjunction of all members of Δ, and that Ω_i refers to the adaptive condition of line i of the proof.[11] Moreover, *Dab*–formulas will be abbreviated as follows: $\langle A^{\mathcal{I}}, B\rangle = (A^{\mathcal{I}} \sqsupset (\exists i)B^{(i)}) \wedge (\Diamond(\exists i)B^{(i)} \wedge \neg A^{\mathcal{I}})$.

Now, consider the example below, based on the premise couple $\langle\Gamma^{\exists\mathbf{i}}, \Gamma^{\exists\mathbf{i}\Diamond}\rangle = \langle\{(\exists i)(p \sqsupset q)^{(i)}, (\exists i)(r \sqsupset q)^{(i)}, (\exists i)(\neg q)^{(i)}, (\exists i)(s)^{(i)}, (\exists i)(\neg s)^{(i)}\}, \{(\exists i)(\Diamond q)^{(i)}\}\rangle$.

1	$(\exists i)(p \sqsupset q)^{(i)}$	—;PREM	$\emptyset$
2	$(\exists i)(r \sqsupset q)^{(i)}$	—;PREM	$\emptyset$
3	$(\exists i)(\neg q)^{(i)}$	—;PREM	$\emptyset$
4	$(\exists i)(s)^{(i)}$	—;PREM	$\emptyset$
5	$(\exists i)(\neg s)^{(i)}$	—;PREM	$\emptyset$
6	$(\exists i)(\Diamond q)^{(i)}$	—;PREM	$\emptyset$
7	$p^1 \sqsupset q^1$	1; *RC*	$\{(\exists i)(p \sqsupset q)^{(i)} \wedge \neg(p^1 \sqsupset q^1)\}$
8	$r^2 \sqsupset q^2$	2; *RC*	$\{(\exists i)(r \sqsupset q)^{(i)} \wedge \neg(r^2 \sqsupset q^2)\}$
9	$(\exists i)p^{(i)}$	6, 7;RC	$\Omega_7 \cup \{\langle p^1, q\rangle\}$
10	$(\exists i)r^{(i)}$	6, 8;RC	$\Omega_8 \cup \{\langle r^2, q\rangle\}$
11	s^1	4;RC	$\{(\exists i)(s)^{(i)} \wedge \neg(s^1)\}$
12	$\neg s^1$	5;RC	$\{(\exists i)(\neg s)^{(i)} \wedge \neg(\neg s^1)\}$

At stage 12 of the proof, lines 7–12 have been derived conditionally. No markings occur yet. This however changes when the proof is extended as follows:

[11]See chapter 7, section 7.4.1.

9	$(\exists i)p^{(i)}$	3, 7;RC	$\Omega_7 \cup \{\langle p^1, q\rangle\}$	$\checkmark^p$	$\checkmark^t$
10	$(\exists i)r^{(i)}$	3, 8;RC	$\Omega_8 \cup \{\langle r^2, q\rangle\}$	$\checkmark^p$	$\checkmark^t$
...	...	...	...		
13	$\bigvee(\Omega_7 \cup \Omega_8 \cup \{\langle p^1, q\rangle, \langle r^2 \wedge \neg p^1, q\rangle\})$	1, 2, 6;RU	$\emptyset$		
14	$\bigvee(\Omega_7 \cup \Omega_8 \cup \{\langle p^2, q\rangle, \langle p^1 \wedge \neg r^2, q\rangle\})$	1, 2, 6;RU	$\emptyset$		

At stage 14 of the proof, line 9 and 10 are marked for both reliability and relevant reliability. Moreover, as the *Dab*–consequences on line 13 and 14 are minimal *Dab*–consequences of the premise set, line 9 and 10 will remain marked for reliability. But, they do not remain marked for relevant reliability, for suppose the proof is extended in the following way:

9	$(\exists i)p^{(i)}$	3, 7;RC	$\Omega_7 \cup \{\langle p^1, q\rangle\}$	$\checkmark^p$
10	$(\exists i)r^{(i)}$	3, 8;RC	$\Omega_8 \cup \{\langle r^2, q\rangle\}$	$\checkmark^p$
...	...	...	...	
15	$\bigvee(\Omega_7 \cup \{\langle p^1 \wedge r^2, q\rangle, \langle p^1 \wedge \neg r^2, q\rangle\})$	1, 6;RU	$\emptyset$	
16	$\bigvee(\Omega_8 \cup \{\langle r^2 \wedge p^1, q\rangle, \langle r^2 \wedge \neg p^1, q\rangle\})$	2, 6;RU	$\emptyset$	

At stage 16 of the proof, the *Dab*–consequences on line 13 and 14 are no longer relevant *Dab*–consequences, so that line 9 and 10 become unmarked again for relevant reliability. As a consequence, the formulas on those lines are finally derivable for the logic $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$.

Also remark that the "disjunction" of the formulas on line 9 and 10 is finally derivable for $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$.

17	$(\exists i)(p \vee r)^{(i)}$	7, 8;RC	$\Omega_7 \cup \Omega_8 \cup \{\langle p^1 \vee r^2, q\rangle\}$

Finally, the lines below show us that the inconsistencies in the premise set do not influence the markings for reliability or relevant reliability. They can only lead to **NS**–markings.

18	$\bigvee(\Omega_{11} \cup \Omega_{12} \cup \{\langle p^1, q\rangle\})$	4, 5;RU	$\emptyset$
19	$\bigvee(\Omega_{11} \cup \Omega_{12})$	4, 5;RU	$\emptyset$

13.4.4 Semantics of $\exists\mathrm{CL}^{\mathbf{abd}}$

The semantics of both $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ resembles the semantics of the logic $\exists\mathbf{CL\bar{u}Ns}^{\mathbf{sr}}$ of chapter 12. As such, their semantics should be quite understandable.

As the semantics of $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ and $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ differ in some respect, I will treat them separately. However, I will first state some definitions that are equivalent for both logics.

First of all, consider the definitions for the Ω_1– and Ω_2–abnormal part of an $\exists\mathbf{CL}^{\Diamond}$–model of the premise set Γ.

Definition 13.51 *Where M is a $\exists\mathbf{CL}^{\Diamond}$–model of Γ, its Ω_1–abnormal part is the set $Ab^1(M) = \{A \in \Omega_1 \mid M \models A\}$.*

Definition 13.52 *Where M is a $\exists\mathbf{CL}^{\Diamond}$–model of Γ, its Ω_2–abnormal part is the set $Ab^2(M) = \{A \in \Omega_2 \mid M \models A\}$.*

Next, consider the definition of a minimal *Dab*–consequence of the premise set.

Definition 13.53 $Dab(\Delta)$ $(\Delta \subset (\Omega_1 \cup \Omega_2))$ *is a minimal Dab–consequence of Γ iff $\Gamma \vdash_{\exists\mathbf{CL}^{\Diamond}} Dab(\Delta)$ and for all $\Delta' \subset \Delta$, $\Gamma \nvdash_{\exists\mathbf{CL}^{\Diamond}} Dab(\Delta')$.*

Finally, consider the set of $\exists\mathbf{CL}^{\Diamond}$–models of the premise set.

Definition 13.54 $\mathcal{M}_0 =_{df} \{M \mid M \models \Gamma\}$.

Practical Abduction. As the semantics of the logic $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}$ is completely equivalent to the semantics of the logic $\exists\mathbf{CL\bar{u}Ns}^{\mathbf{sr}}$ (chapter 12), I will only state the necessary definitions.

Definition 13.55 $U(\Gamma) = \{A \in \Omega^p_2 \mid A \in \Delta$ *and $Dab(\Delta)$ is a minimal Dab–consequence of Γ*$\}$.

Definition 13.56 $\mathcal{M}^r =_{df} \{M \in \mathcal{M}_0 \mid Ab^2(M) \subseteq U(\Gamma)\}$.

Definition 13.57 $\mathcal{M}^m =_{df} \{M \in \mathcal{M}^r \mid$ *for no* $M' \in \mathcal{M}^r$, $Ab^1(M') \subset Ab^1(M)\}$.

Definition 13.58 $\Phi(\Gamma) = \{Ab^1(M) \mid M \in \mathcal{M}^m\}$.

Definition 13.59 *A set Σ of $\exists\mathbf{CL}^{\Diamond}$–models of Γ is a* reliable normal set *iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \in \mathcal{M}^m \mid Ab^1(M) = \phi\}$.*

Definition 13.60 $\Gamma \models_{\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{p}}} A$ *iff A is verified by all members of at least one reliable normal set of $\exists\mathbf{CL}^{\Diamond}$–models of Γ.*

Theoretical Abduction. The semantics of the logic $\exists\mathbf{CL}^{\mathbf{abd}}_{\mathbf{t}}$ is slightly more demanding. First, as for the proof theory, it is necessary to distinguish between homogenous and relevant *Dab*–consequences of a premise set.

Definition 13.61 $Dab(\Delta \cup \Theta)$ $(\Delta \subset \Omega^t_1$ *and* $\Theta \subset \Omega^t_2)$ *is a homogenous Dab–consequence of Γ iff $Dab(\Delta \cup \Theta)$ is a minimal Dab–consequence of Γ and all elements in Θ are homogenous.*

Definition 13.62 *$Dab(\Delta \cup \Theta)$ ($\Delta \subset \Omega_1^t$ and $\Theta \subset \Omega_2^t$) is a relevant Dab–consequence of Γ iff $Dab(\Delta \cup \Theta)$ is a minimal Dab–consequence of Γ and there is no $\Delta' \cup \Theta'$ ($\Delta' \subset \Omega_1^t$ and $\Theta' \subset \Omega_2^t$) such that $Dab(\Delta' \cup \Theta')$ is a homogenous Dab–consequence of Γ and $\Theta \cap \Theta' \neq \emptyset$.*

Next, the set of unreliable abnormalities $U^r(\Gamma)$ is based on the homogenous and the relevant *Dab*–consequences of a premise set.

Definition 13.63 *$U^r(\Gamma) = \{A \in \Omega_2^t \mid A \in \Delta$ and $Dab(\Delta)$ is a homogenous or a relevant Dab–consequence of $\Gamma\}$.*

From now on the characterization of the $\exists\mathbf{CL_t^{abd}}$–semantics is equivalent to the one for $\exists\mathbf{CL_p^{abd}}$, which means that from the set of $\exists\mathbf{CL}^{\Diamond}$–models of Γ those models are selected that are reliable with respect to Ω_2^t and minimally abnormal with respect to Ω_1^t.

Definition 13.64 *$\mathcal{M}^r =_{df} \{M \in \mathcal{M}_0 \mid Ab^2(M) \subseteq U^r(\Gamma)\}$.*

Definition 13.65 *$\mathcal{M}^m =_{df} \{M \in \mathcal{M}^r \mid$ for no $M' \in \mathcal{M}^r$, $Ab^1(M') \subset Ab^1(M)\}$.*

Next, the set $\mathcal{M}^r$ is subdivided into relevant normal sets.

Definition 13.66 *$\Phi(\Gamma) = \{Ab^1(M) \mid M \in \mathcal{M}^m\}$.*

Definition 13.67 *A set Σ of $\exists\mathbf{CL}^{\Diamond}$–models of Γ is a* relevant reliable normal set *iff for some $\phi \in \Phi(\Gamma)$, $\Sigma = \{M \in \mathcal{M}^m \mid Ab^1(M) = \phi\}$.*

Finally, semantic consequence is defined by means of those relevant normal sets of $\exists\mathbf{CL}^{\Diamond}$-models of the premise set.

Definition 13.68 *$\Gamma \models_{\exists\mathbf{CL_t^{abd}}} A$ iff A is verified by all members of at least one relevant reliable normal set of $\exists\mathbf{CL}^{\Diamond}$–models of Γ.*

13.4.5 Soundness and Completeness

As the logic $\exists\mathbf{CL_p^{abd}}$ is based on the relevant reliability theory, soundness and completeness is proven in the same way as for the logic $\exists\mathbf{CL\bar{u}Ns^{sr}}$ from chapter 12. As a consequence, I will not present the proofs here, but leave them to the reader.

Theorem 13.4 *$\Gamma \vdash_{\exists\mathbf{CL_p^{abd}}} A$ iff $\Gamma \models_{\exists\mathbf{CL_p^{abd}}} A$.*

Soundness and completeness for the logic $\exists\mathbf{CL_t^{abd}}$ has not been proven yet, but I suppose soundness and completeness for $\exists\mathbf{CL_t^{abd}}$ simply follows from the soundness and completeness of the logics $\exists\mathbf{CL\bar{u}Ns^{sr}}$ (chapter 12) and $\mathbf{CL_t^{abd}}$ (section 13.3.5).

Theorem 13.5 *$\Gamma \vdash_{\exists\mathbf{CL_t^{abd}}} A$ iff $\Gamma \models_{\exists\mathbf{CL_t^{abd}}} A$.*

13.5 Conclusion

In this final chapter, I have presented the adaptive logics **AbL** and **PAbL** that capture abductive reasoning respectively based on consistent and on inconsistent background theories. Moreover, of both adaptive logics, I have presented two variants, one that captures practical abduction and one that captures theoretical abduction.

Part VI

Conclusion

The Aim of this Dissertation Bis

The main objective of this dissertation was to solve (once and for all) the famous **DS**–problem in Relevance Logic. In fact, this objective was attained by presenting the adaptive logic $\mathbf{R}^*_\mathbf{d}$ in chapter 11. This logic succeeds in reintroducing **DS** in standard relevant logics without reintroducing any of the fallacies of relevance.

Other Results

Besides the main objective, some other results were reached in this dissertation. Consider a short overview stating the most important ones.

- Fitch–style proof theories were presented for a lot of paralogics (ch. 4).
- A non–truthfunctional semantic characterization was presented for standard relevant logics (ch. 5).
- A theory of classical and first degree relevance was presented and characterized proof theoretically (ch. 6).
- An adaptive logic for classical relevance was presented, and proven equivalent to Tennant's *Compassionate Relevantism* (ch. 7).
- A universal logic method was presented for capturing classical and first degree relevance (ch. 8).
- An adaptive logic for first degree relevance was presented (ch. 9).
- It was shown how the insights from inconsistency–adaptive logics can be put to work for standard relevant logics. This resulted in inconsistency-adaptive relevant logics (ch. 10).
- The logic for first degree relevance presented in ch. 9 was combined with the implication from standard relevant logics (ch. 11).
- An (inconsistency–)adaptive logic was presented whose adaptive strategy is not based on the minimal *Dab*–consequences of a premise set, but on its relevant *Dab*–consequences (ch. 12).
- Some adaptive logics were presented for explicating abductive reasoning processes (ch. 13).

Appendix A

After All, Disjunction is An Ambiguous Connective

Introduction. In chapter 2, I showed that the **DS**–problem in relevant logics (**RL**) cannot be coherently solved by interpreting the disjunction as an ambiguous connective. In this small appendix, I will however show that it is nevertheless possible to treat the disjunction in relevant logics as an ambiguous connective. But, in order to avoid the problems of the "ambiguous connective"–approach that was presented in chapter 2, the intensional disjunction cannot be defined by means of the relevant implication. I claim that it should be defined by means of the material implication:

Definition A.1 $A \oplus B =_{df} (\sim A \supset B) \wedge (\sim B \supset A) \wedge (A \vee B)$.

Moreover, as in Read's account, intensional disjunction merely expresses deductive dependency, and not some kind of relevance between the disjuncts.[1]

The Language Schema. In order to characterize relevant deduction by treating the disjunction as an ambiguous connective, first extend the language schema $\mathcal{L}$ of standard relevant logics with the material implication $\supset$ and with the intensional disjunction $\oplus$. This gives us the language $\mathcal{L}^{\oplus}$.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\sim, \wedge, \vee, \rightarrow$	$\mathcal{W}$
$\mathcal{L}^{\oplus}$	$\mathcal{S}$	$\sim, \wedge, \vee, \rightarrow, \supset, \oplus$	$\mathcal{W}^{\oplus}$

Table A.1: Relations between $\mathcal{L}$ and $\mathcal{L}^{\oplus}$.

[1] Do notice that Read would probably be horrified by definition A.1, which has to do with the fact that he completely turns away from classically oriented logic.

The Relevant Logics. The relevant logic $\mathbf{R}^{\oplus}$ based on the language $\mathcal{L}^{\oplus}$, is equivalent to the logic $\mathbf{R}$ extended with the material implication. As this logic was already presented in chapter 5 (section 5.4.5), I will not present the proof theory and the semantics anymore.

Moreover, as is done throughout this dissertation, the logic $\mathbf{R}^{\oplus}$ is taken to stand for classical $\mathbf{R}^{\oplus}$–derivability, while the logic $\mathbf{R}^{\oplus}_{\mathbf{d}}$ is taken to capture relevant $\mathbf{R}^{\oplus}$–derivability.

The Entailments. It is now easily verified that the logic $\mathbf{R}^{\oplus}$ validates some entailments that are not validated by the logic $\mathbf{R}$, among others the intensional version of **DS** (**IDS**).

Entailment	No Entailment
$((A \oplus B) \wedge {\sim}A) \rightarrow B$	$((A \vee B) \wedge {\sim}A) \rightarrow B$
$A \rightarrow (A \vee B)$	$A \rightarrow (A \oplus B)$
$({\sim}A \wedge {\sim}B) \rightarrow {\sim}(A \oplus B)$	${\sim}(A \oplus B) \rightarrow ({\sim}A \wedge {\sim}B)$

Relevant Deduction. Of course, characterizing the relevant entailments is not equal to characterizing relevant deduction. In order to characterize $\mathbf{R}^{\oplus}_{\mathbf{d}}$–derivability, first consider the definitions below. They determine whether a formula is a positive or a negative part of another formula.[2]

1. A is a positive part of A, $A \vee B$, $B \vee A$, $A \wedge B$, $B \wedge A$ and $B \rightarrow A$.
2. A is a negative part of $\neg A$ and $A \rightarrow B$.
3. If A is a positive part of B and B is a positive part of C, then A is a positive part of C.
4. If A is a positive part of B and B is a negative part of C, then A is a negative part of C.
5. If A is a negative part of B and B is a positive part of C, then A is a negative part of C.
6. If A is a negative part of B and B is a negative part of C, then A is a positive part of C.

For every formula A of the language $\mathcal{W}$, there can now be constructed an intensional counterpart $A^{\oplus}$.

Definition A.2 *The formula $A^{\oplus} \in \mathcal{L}^{\oplus}$ is obtained from the formula $A \in \mathcal{W}$ by replacing*

(i) every subformula $B \vee C$ (resp. $B \sqsupset C$) that is a positive part of A by the formula $B \oplus C$ (resp. $\neg B \oplus C$), and

[2] The notion of positive part (resp. negative part) which I present here, is the same as the one that was given by Batens & Provijn in [37].

(ii) every subformula $B \wedge C$ that is a negative part of A by the formula $\neg(\neg B \oplus \neg C)$.

Next, the premise set $\Gamma \subset \mathcal{W}$ should be translated to the premise set $\Gamma^{\oplus}$, which is defined in the following way:

Definition A.3 $\Gamma^{\oplus} = \{A^{\oplus} \mid A \in \Gamma\}$.

Remark that this translation is necessary in order to be able to actually use the inference rule **IDS** in actual proofs. In other words, if a premise set doesn't contain any intensional disjunctions, then the $\mathbf{R}^{\oplus}_{\mathbf{d}}$–consequence set of that premise set will be equal to its $\mathbf{R_d}$–consequence set.

Finally, relevant deduction can now be captured by presenting a proof theory for $\mathbf{R}^{\oplus}_{\mathbf{d}}$. Consider its deduction rules.

PREM	If $A \in \Gamma^{\oplus}$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) —;PREM.
RU	If $\vdash_{\mathbf{R}^{\oplus}} (A_1 \wedge ... \wedge A_n) \rightarrow B$ and each of $A_1, ..., A_n$ occurs in the proof on lines $i_1, ..., i_n$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, ..., i_n$;RU.

As an $\mathbf{R}^{\oplus}_{\mathbf{d}}$–proof is defined in the usual way, $\mathbf{R}^{\oplus}_{\mathbf{d}}$–derivability can now be defined as follows:

Definition A.4 $\Gamma^{\oplus} \vdash_{\mathbf{R}^{\oplus}_{\mathbf{d}}} A$ *iff there is a* $\mathbf{R}^{\oplus}_{\mathbf{d}}$*–proof of* A *from* $B_1, ..., B_n \in \Gamma^{\oplus}$.

Example. Consider the example below, based on the premise set $\Gamma^{\oplus} = \{p \rightarrow (s \oplus r), q \rightarrow {\sim}s, p, {\sim}r, {\sim}(q \oplus {\sim}p) \rightarrow s\}$.

1	$p \rightarrow (s \oplus r)$	—;**PREM**
2	$q \rightarrow {\sim}s$	—;**PREM**
3	p	—;**PREM**
4	${\sim}r$	—;**PREM**
5	${\sim}(q \oplus {\sim}p) \rightarrow s$	—;**PREM**
6	$(p \wedge q) \rightarrow ((s \oplus r) \wedge {\sim}s)$	1,2;**RU**
7	$(p \wedge q) \rightarrow r$	6;**RU**
8	$s \oplus r$	1,3;**RU**
9	s	4,8;**RU**
10	$({\sim}q \wedge p) \rightarrow s$	5;**RU**

Relation with $\mathbf{R}^*_{\mathbf{d}}$. Probably the most remarkable fact about $\mathbf{R}^{\oplus}_{\mathbf{d}}$–derivability, is that it appears to be equivalent to $\mathbf{R}^*_{\mathbf{d}}$–derivability.

Definition A.5 $\Gamma \vdash_{\mathbf{R}^*_{\mathbf{d}}} A$ *iff* $\Gamma^{\oplus} \vdash_{\mathbf{R}^{\oplus}_{\mathbf{d}}} A$*, for* $\Gamma \subset \mathcal{W}$ *and* $A \in \mathcal{W}$.

Although all evidence seems to support this theorem, I have not been able to prove it yet. As a consequence, the truth of this theorem remains uncertain.[3]

Conclusion. In this appendix, I have shown that in spite of appearances, the disjunction can be interpreted as an ambiguous connective. Nevertheless, this can only be done by defining the intensional disjunction by means of the material implication, and not as it is usually done (by means of the relevant implication).

[3]Suppose that relevant deduction is characterized in the same way as it is done in this appendix, but by taking the logic **CLoNs** as the underlying logic instead of the logic $\mathbf{R_d}$. The result will be the logic $\mathbf{CLoNs}^{\oplus}$, of which I claim that the following can be stated:

Definition A.6 $\Gamma \vdash_{\mathbf{CL\bar{a}Ns}^*} A$ *iff* $\Gamma^{\oplus} \vdash_{\mathbf{CLoNs}^{\oplus}} A$*, for* $\Gamma \subset \mathcal{W}$ *and* $A \in \mathcal{W}$.

Also for this theorem, no proof has been found yet, so it should remain provisional.

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