Hardy spaces of solutions of generalized Riesz and Moisil-Teodorescu systems and associated analytic signals

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Abstract

Hardy spaces of solutions of generalized Riesz and generalized Moisil–Teodorescu systems in half space $\mathbb{R}^{m+1}_+$, and of their non–tangential L_2–boundary values in $\mathbb{R}^m$ are characterized.

1 Introduction

In the mid 1980's, the classical Hardy spaces on the real line and the upper half plane were generalized to the Euclidean space $\mathbb{R}^m$ and the upper half space $\mathbb{R}^{m+1}_+$, within the framework of Clifford analysis. Clifford analysis is a multidimensional function theory on functions defined in Euclidean space and taking values in a Clifford algebra or subspaces thereof, which is at the same time a generalization of the theory of holomorphic functions in the complex plane and a refinement of classical harmonic analysis. The H^p–spaces of several real variables as considered by Stein and Weiss in [40], could be fully incorporated in this Clifford analysis generalization, and, even more, this new approach enabled the study of Hardy spaces on Lipschitz domains. For an account of this theory of Hardy spaces in Euclidean space we refer to the monographs [28, 35, 18], to the lecture notes [34] and the papers [14, 16, 2, 29]. In the same context, the Hilbert transform, as well as more general singular integral operators, have been studied in higher dimensional Euclidean space (see [38, 30, 20, 21, 5]), in particular on Lipschitz hypersurfaces (see [36, 33, 32]) and also on smooth closed hypersurfaces, in particular the unit sphere (see [19, 6, 13, 12]).

The so–called analytic signals in the Hardy space $H^2(\mathbb{R}^m)$ appear as the non–tangential L_2–limit functions of the monogenic functions in the Hardy space $H^2(\mathbb{R}^{m+1}_+)$. Monogenic means that the considered function is a null solution of the Dirac operator ∂_x or, equivalently, the Cauchy-Riemann operator D_x, in $\mathbb{R}^{m+1}$. Depending on the value space this equation gives rise to specific systems of first order differential equations. Classical choices for the value space are the whole Clifford algebra $\mathbb{R}_{0,m+1}$ (or its complexification $\mathbb{C}_{m+1}$) and its complex spinor space. Recently there has arisen a lot of interest in the study of monogenic functions with values in a subspace of the Clifford algebra consisting of r-vectors ($0 < r < m+1$) or a direct sum of such Grassmann multivector subspaces, the corresponding system of equations being termed generalized Riesz systems or generalized Moisil–Teodorescu systems, respectively (see [22, 31, 1, 3, 4, 15, 23, 25, 37, 41]).

In this paper new classes of analytic signals on $\mathbb{R}^m$ are introduced and characterized, which are the non–tangential L_2–limit functions of the solutions in the half space $\mathbb{R}^{m+1}_+$ of those generalized Riesz systems (Section 4) and generalized Moisil–Teodorescu systems (Section 5). In order to make the paper self–contained, some basic notions from Clifford analysis (Section 2) and from the higher dimensional Hardy spaces in this framework (Section 3) are recalled.

2 Clifford analysis basics

Let $\mathbb{R}^{0,m+1}$ be the real vector space $\mathbb{R}^{m+1}$ ($m \geq 1$) provided with a non-degenerate quadratic form of signature $(0, m+1)$ and let $e = (e_0, \ldots, e_m)$ be an orthonormal basis of $\mathbb{R}^{0,m+1}$. Then e generates the universal Clifford algebra $\mathbb{R}_{0,m+1}$ over $\mathbb{R}^{0,m+1}$ and, embedded in $\mathbb{R}_{0,m+1}$, $\underline{e} = (e_1, \ldots, e_m)$ generates the universal Clifford algebra $\mathbb{R}_{0,m}$ over $\mathbb{R}^{0,m}$. The multiplication in the Clifford algebra $\mathbb{R}_{0,m+1}$ is non–commutative; it is governed by the rules

$$\begin{cases} e_i^2 = -1, & i = 0, \ldots, m; \\ e_i e_j + e_j e_i = 0, & i \neq j, i, j = 0, \ldots, m \end{cases}$$

A basis for $\mathbb{R}_{0,m+1}$ is given by $(e_A)_{A \subset \{0,\ldots,m\}}$ where for $A = (i_1, \ldots, i_r)$ with $0 \leq i_1 < i_2 < \ldots i_r \leq m$, we put $e_A = e_{i_1} e_{i_2} \ldots e_{i_r}$, while $e_\emptyset = 1$ is the identity element. Conjugation in $\mathbb{R}_{0,m+1}$ is defined as the anti–involution $a \to \overline{a}$ for which $\overline{e_i} = -e_i$, $i = 0, 1, \ldots, m$.

For each $r = 0, 1, \ldots, m+1$ we define the space of r–vectors $\mathbb{R}_{0,m+1}^{(r)}$ by

$$\mathbb{R}_{0,m+1}^{(r)} = \operatorname{span}_{\mathbb{R}} (e_A \,:\, |A| = r)$$

Clearly $\mathbb{R} \simeq \mathbb{R}_{0,m+1}^{(0)} = \mathbb{R}1$, $\mathbb{R}^{0,m+1} = \mathbb{R}_{0,m+1}^{(1)}$ and

$$\mathbb{R}_{0,m+1} = \bigoplus_{r=0}^{m+1} \mathbb{R}_{0,m+1}^{(r)}$$

An element $a \in \mathbb{R}_{0,m+1}$ may thus be written as $a = \sum_{r=0}^{m+1} [a]_r$, where $[a]_r$ is the projection of a on $\mathbb{R}_{0,m+1}^{(r)}$, also called the r–vector part of a. A norm $|\cdot|$ on $\mathbb{R}_{0,m+1}$ is defined by $|a|^2 = [a\overline{a}]_0$, $a \in \mathbb{R}_{0,m+1}$.

By singling out the basis vector e_0, the Clifford algebra $\mathbb{R}_{0,m+1}$ clearly admits the splitting

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e_0}\, \mathbb{R}_{0,m} \tag{1}$$

The Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ are identified with the subspaces of 1-vectors in the respective Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{R}_{0,m+1}$, by putting $x = \sum_{i=0}^m e_i x_i$ and $\underline{x} = \sum_{j=1}^m x_j e_j$. It follows that $x = x_0 e_0 + \underline{x}$ and also $\overline{e}_0 x = x_0 + \overline{e}_0 \underline{x}$, the latter expression being in accordance with the splitting (1).

Observe that if $x \in \mathbb{R}_{0,m+1}^{(1)}$ and $v \in \mathbb{R}_{0,m+1}^{(r)}$ ($0 < r < m+1$), then the Clifford product xv splits into a *dot* and a *wedge* product:

$$xv = x \bullet v + x \wedge v \tag{2}$$

where

$$x \bullet v = [xv]_{r-1}$$

and

$$x \wedge v = [xv]_{r+1}$$

Now let $\Omega \subset \mathbb{R}^{m+1}$ be open, let S be a subspace of $\mathbb{R}_{0,m+1}$, and let $F \in \mathcal{E}(\Omega; S)$ be an S–valued smooth function on Ω. The function F is said to be an S–valued (left) monogenic function in Ω if $\partial_x F = 0$ in Ω or, equivalently, $D_x F = 0$ in Ω, where $\partial_x = \sum_{i=0}^m e_i \partial_{x_i}$ and $D_x = \overline{e_0} \partial_x = \partial_{x_0} + \overline{e_0} \partial_{\underline{x}}$ are the Dirac and the Cauchy–Riemann operators in $\mathbb{R}^{m+1}$, respectively. From $\partial_x^2 = -\Delta_x$ or $\overline{D}_x D_x = D_x \overline{D}_x = \Delta_x$, it follows that S–valued (left) monogenic functions in Ω are harmonic. For a full account on this function theory we refer e.g. to [7, 26].

The Dirac operator $\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}$ is an example of a Clifford vector operator the components of which, in casu the partial derivatives, are mutually commuting. More generally, let $\mathcal{V} = \sum_{i=0}^{m} e_i \mathcal{V}_i$ be a Clifford vector operator, its components $\mathcal{V}_i$ being scalar operators. When acting, from the left, on a Clifford algebra valued function, the Clifford multiplication is tacitly understood. As the Clifford product splits into a *dot* and a *wedge* part (see (2)), it is always possible to define two associated operators

$$\mathcal{V}^+ = \mathcal{V} \wedge \quad \text{and} \quad \mathcal{V}^- = \mathcal{V} \bullet$$

such that

$$\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$$

As

$$\mathcal{V}^2 = -\sum_{i=o}^{m} \mathcal{V}_i^2 + \sum_{j<k} e_j e_k (\mathcal{V}_j \mathcal{V}_k - \mathcal{V}_k \mathcal{V}_j)$$

we observe that, on condition that the scalar operators $\mathcal{V}_i, i = 0, \ldots, m$, are mutually commuting, $\mathcal{V}^2$ becomes a scalar operator:

$$\mathcal{V}^2 = -\sum_{i=0}^{m} \mathcal{V}_i^2$$

In the case of the Dirac operator, this is the well–known relation

$$\partial_x^2 = -\sum_{i=0}^{m} \partial_{x_i}^2 = -\Delta_{m+1}$$

Still under the same condition of mutual commutativity of the scalar components of $\mathcal{V}$, we have, on purely algebraic grounds, that

$$(\mathcal{V}^+)^2 = 0 \quad \text{and} \quad (\mathcal{V}^-)^2 = 0$$

and hence

$$\mathcal{V}^2 = -\sum_{i=0}^{m} \mathcal{V}_i^2 = \mathcal{V}^+ \mathcal{V}^- + \mathcal{V}^- \mathcal{V}^+$$

For the Dirac operator $\partial_x = \partial_x \wedge + \partial_x \bullet = \partial_x^+ + \partial_x^-$ this means (see also [9])

$$(\partial_x^+)^2 = 0 \quad \text{and} \quad (\partial_x^-)^2 = 0$$

and

$$\partial_x^2 = -\Delta_{m+1} = \partial_x^+ \partial_x^- + \partial_x^- \partial_x^+$$

As has been observed in [9], through the natural isomorphism Θ between, on the one side, smooth multivector functions and, on the other, differential forms in open regions of Euclidean space:

$$\Theta : \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}) \longrightarrow \mathcal{E}(\Omega; \textstyle\bigwedge \mathbb{R}^{m+1})$$

the action of ∂_x on $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1})$ corresponds to the action of $d + d^*$ on $\mathcal{E}(\Omega; \bigwedge \mathbb{R}^{m+1})$, more precisely the operator ∂_x^+ corresponds to the exterior derivative d, while ∂_x^- corresponds to the coderivative d^*. In such a way, the theory of (left) monogenic functions in Ω is equivalent to the theory of so–called self–conjugate differential forms in Ω (see [9, 17]) and, putting $\omega = \Theta(F)$, there holds:

$$\partial_x F = 0 \quad \Longleftrightarrow \quad (d + d^*)\omega = 0$$

In particular, for $F_r = \sum_{|A|=r} F_A^r e_A$ $(0 \le r \le m+1)$ and $\omega^r = \Theta F_r = \sum_{|A|=r} \omega_A^r dx^A$, where for all $A = (i_1, \ldots, i_r) \subset \{0, 1, \ldots, m\}$, $\omega_A^r = F_A^r$ and $dx^A = dx^{i_1} \wedge \ldots \wedge dx^{i_r}$, we have that

$$\partial_x F_r = 0 \quad \Longleftrightarrow \quad (\mathrm{GR}) \left\{ \begin{array}{rcl} \partial_x^+ F_r & = & 0 \\ \partial_x^- F_r & = & 0 \end{array} \right. \tag{3}$$

The system (3) is called a generalized Riesz system of type (r); its solutions are called monogenic r–vector functions (see also [24]). Through the isomorphism Θ, we also have

$$(\mathrm{GR}) \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} d\omega^r & = & 0 \\ d^*\omega^r & = & 0 \end{array} \right. \tag{4}$$

As is well–known, the latter system (4) is called the Hodge–de Rham system for r–forms and its solutions are called harmonic r–forms.

3 The Hardy spaces $H^2(\mathbb{R}^m)$ and $H^2(\mathbb{R}_+^{m+1})$

We briefly recall some basic results from Hardy space theory on $\mathbb{R}^m$ and $\mathbb{R}_+^{m+1}$ in the Clifford analysis context. The fundamental solution $C(x)$ of the Cauchy–Riemann operator D_x in $\mathbb{R}^{m+1}$ being

$$C(x) = \frac{1}{a_{m+1}} \frac{x_0 - \overline{e_0}\underline{x}}{|x|^{m+1}}$$

with $a_{m+1} = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})}$ the area of the unit sphere in $\mathbb{R}^{m+1}$, for a function $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, its Cauchy transform $\mathcal{C}[f]$ is defined by

$$\mathcal{C}[f](x) = \int_{\mathbb{R}^m} C(x_0, \underline{x} - \underline{y}) f(\underline{y}) d\underline{y}, \quad \underline{x} \in \mathbb{R}^{m+1} \setminus \mathbb{R}^m$$

It shows a number of important properties.

Property 1. *For $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, its Cauchy transform $\mathcal{C}[f]$ is (left) monogenic in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$.*

Property 2. *For $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$*

$$\mathcal{C}^+[f](\underline{x}) \equiv \lim_{x_0 \to 0+} \mathcal{C}[f](x) = \frac{1}{2} f(\underline{x}) + \overline{e_0} \frac{1}{2} \mathcal{H}[f](\underline{x})$$

where the limit is taken nontangentially and where

$$\mathcal{H}[f](\underline{x}) = \frac{2}{a_{m+1}} Pv \int_{\mathbb{R}^m} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) d\underline{y} = \sum_{j=1}^{m} \overline{e_j} \mathcal{R}_j[f](\underline{x})$$

is the Hilbert transform on $\mathbb{R}^m$, $\mathcal{R}_j (j = 1, \ldots, m)$ denoting the j–th Riesz transform on $\mathbb{R}^m$ given by

$$\mathcal{R}_j[f](\underline{x}) = \frac{2}{a_{m+1}} Pv \int_{\mathbb{R}^m} \frac{x_j - y_j}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) d\underline{y}$$

This Hilbert transform enjoys the following properties.

Property 3. *The Hilbert transform $\mathcal{H}$ is a translation and dilation invariant, bounded and norm preserving linear operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, which is moreover invertible since $\mathcal{H}^2 = \mathbf{1}$.*

Property 4. *The scalar Riesz transforms $\mathcal{R}_j, j = 1, \ldots, m$, are mutually commuting, and defining*

$$\mathcal{H}^+[f](\underline{x}) = \mathcal{H} \wedge [f](\underline{x}) = \frac{2}{a_{m+1}} \, Pv \int_{\mathbb{R}^m} \frac{(\overline{\underline{x}} - \overline{\underline{y}}) \wedge f(\underline{y})}{|\underline{x} - \underline{y}|^{m+1}} \, d\underline{y} = \sum_{j=1}^m \overline{e_j} \wedge \mathcal{R}_j[f](\underline{x})$$

and

$$\mathcal{H}^-[f](\underline{x}) = \mathcal{H} \bullet [f](\underline{x}) = \frac{2}{a_{m+1}} \, Pv \int_{\mathbb{R}^m} \frac{(\overline{\underline{x}} - \overline{\underline{y}}) \bullet f(\underline{y})}{|\underline{x} - \underline{y}|^{m+1}} \, d\underline{y} = \sum_{j=1}^m \overline{e_j} \bullet \mathcal{R}_j[f](\underline{x})$$

one thus has

$$(\mathcal{H}^+)^2 = 0 \quad \text{and} \quad (\mathcal{H}^-)^2 = 0$$

while

$$\mathcal{H}^2 = \mathcal{H}^+\mathcal{H}^- + \mathcal{H}^-\mathcal{H}^+ = -\sum_{j=1}^m \mathcal{R}_j^2 = \mathbf{1}$$

Remark 1. *These associated Hilbert transforms $\mathcal{H}^+$ and $\mathcal{H}^-$ in fact originate by taking the non–tangential limit for $x_0 \to 0+$ in the Cauchy transform where the multiplication operator $\underline{x}$ is decomposed as $\underline{x} = \underline{x}\wedge + \underline{x}\bullet$, with $(\underline{x}\wedge)^2 = (\underline{x}\bullet)^2 = 0$ and $\underline{x}^2 = \underline{x}\wedge\, \underline{x}\bullet + \underline{x}\bullet\, \underline{x}\wedge = -|\underline{x}|^2$ (see also [9]).*

Remark 2. *It is known (see e.g. [10]) that the convolution kernel $\mathcal{H}\partial_{\underline{x}} = \partial_{\underline{x}}\mathcal{H}$ is nothing else but the scalar convolution kernel $-\frac{2}{a_{m+1}}\mathrm{Fp}\frac{1}{|\underline{x}|^{m+1}}$, leading to the scalar Hilbert–Dirac operator which equals the "square root of the negative Laplace operator":*

$$\mathcal{H}\partial_{\underline{x}}[f] = \partial_{\underline{x}}\mathcal{H}[f] = (-\Delta)^{1/2}[f]$$

In terms of the associated wedge and dot operators it is then obtained that

$$\mathcal{H}^+\partial_{\underline{x}}^+ = \partial_{\underline{x}}^+\mathcal{H}^+ = 0 \quad \text{and} \quad \mathcal{H}^-\partial_{\underline{x}}^- = \partial_{\underline{x}}^-\mathcal{H}^- = 0$$

while

$$\mathcal{H}^+\partial_{\underline{x}}^- + \mathcal{H}^-\partial_{\underline{x}}^+ = \partial_{\underline{x}}^+\mathcal{H}^- + \partial_{\underline{x}}^-\mathcal{H}^+ = \sum_{j=1}^m \mathcal{R}_j\partial_{x_j} = \sum_{j=1}^m \partial_{x_j}\mathcal{R}_j = -\frac{2}{a_{m+1}}\mathrm{Fp}\frac{1}{|\underline{x}|^{m+1}}$$

The Hardy space $H^2(\mathbb{R}^m)$ is the closed subspace of $L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1})$ characterized by either

$$H^2(\mathbb{R}^m) = \{g \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1}) \,:\, \overline{e}_0\mathcal{H}[g] = g\} \tag{5}$$

or

$$H^2(\mathbb{R}^m) = \left\{g \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1}) \,:\, \mathcal{C}^+[g] = g\right\} \tag{6}$$

As for each $f \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1})$,

$$\overline{e}_0\mathcal{H}[\mathcal{C}^+[f]] = \overline{e}_0\mathcal{H}[\frac{1}{2}f + \overline{e_0}\frac{1}{2}\mathcal{H}[f]] = \overline{e_0}\frac{1}{2}\mathcal{H}[f] + \frac{1}{2}f = \mathcal{C}^+[f]$$

the next property follows at once.

Property 5. *For $f \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1})$, $\mathcal{C}^+[f] \in H^2(\mathbb{R}^m)$.*

According to the decomposition (1) of $\mathbb{R}_{0,m+1}$, $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ admits the splitting

$$f = u + \overline{e_0} v \tag{7}$$

where $u, v \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$; we put $\mathbb{R}\mathrm{e} f = u$, $\mathbb{I}\mathrm{m} f = v$. We also introduce the operators

$$\mathcal{A} = \mathbf{1} + \overline{e_0}\, \mathcal{H} \quad \text{and} \quad \mathcal{B} = \mathcal{H} + \overline{e_0}\, \mathbf{1} = \mathcal{A}\, \mathcal{H}$$

Property 6. *Each function $g \in H^2(\mathbb{R}^m)$ may be written as*

$$g = \mathbb{R}\mathrm{e}\, g + \overline{e_0}\, \mathcal{H}[\mathbb{R}\mathrm{e}\, g] = \mathcal{A}[\mathbb{R}\mathrm{e}\, g]$$

and

$$g = \mathcal{H}[\mathbb{I}\mathrm{m}\, g] + \overline{e_0}\, \mathbb{I}\mathrm{m}\, g = \mathcal{B}[\mathbb{I}\mathrm{m}\, g]$$

Conversely, for any u and v in $L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$, the functions

$$\mathcal{A}[u] = u + \overline{e_0}\, \mathcal{H}[u] \quad \text{and} \quad \mathcal{B}[v] = \mathcal{H}[v] + \overline{e_0}\, v$$

belong to $H^2(\mathbb{R}^m)$.

Corollary 1. *The spaces $H^2(\mathbb{R}^m)$ and $L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$ are isomorphic.*

In engineering sciences the functions in the Hardy space $H^2(\mathbb{R})$ are termed *analytic signals*, since they are the non–tangential boundary values of holomorphic (or analytic) functions in the upper half complex plane. In the same order of ideas we speak of the functions in $H^2(\mathbb{R}^m)$ as analytic signals. The pairs $(u, \mathcal{H}[u])$ and $(\mathcal{H}[v], v)$ are called conjugate pairs or Hilbert pairs in $\mathbb{R}^m$.

On the half space $\mathbb{R}^{m+1}_+ = \{(x_0, \underline{x}) : x_0 > 0\}$, the Hardy space $H^2(\mathbb{R}^{m+1}_+)$ is defined as the space of (left) monogenic functions F in $\mathbb{R}^{m+1}_+$ satisfying the estimate

$$\sup_{x_0 > 0} \int_{\mathbb{R}^m} |F(x_0, \underline{x})|^2\, d\underline{x} \ < \ +\infty \tag{8}$$

Through the Cauchy transform $\mathcal{C}$, the Hardy spaces $H^2(\mathbb{R}^m)$ and $H^2(\mathbb{R}^{m+1}_+)$ are intimately related as is apparent from the following properties.

Property 7. *For $F \in H^2(\mathbb{R}^{m+1}_+)$, the non-tangential limit function*

$$\lim_{x_0 \to 0+} F(x_0, \underline{x}) = F^+(\underline{x}) \tag{9}$$

belongs to $H^2(\mathbb{R}^m)$, and, conversely, $F \in H^2(\mathbb{R}^{m+1}_+)$ is recovered from its boundary value F^+ by the Cauchy transform, i.e. $F = \mathcal{C}[F^+]$. In conclusion: the spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$ are isomorphic.

Property 8. *For a function $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ one has $\mathcal{C}[f] \in H^2(\mathbb{R}^{m+1}_+)$ and*

$$\mathcal{C}[f](x) = \mathcal{P}\left[\mathcal{C}^+[f]\right](x) = \frac{1}{2}\left(\mathcal{P}[f] + \overline{e_0}\mathcal{P}\left[\mathcal{H}[f]\right]\right)(x) \tag{10}$$

where $\mathcal{P}$ is the Poisson transform given by

$$\mathcal{P}[g](x) = \frac{1}{a_{m+1}} \int_{\mathbb{R}^m} \frac{2x_0}{|\underline{x} - \underline{y}|^{m+1}}\, g(\underline{y}) d\underline{y}, \quad g \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$$

Property 9. *For a function $g \in H^2(\mathbb{R}^m)$ one has $\mathcal{C}[g] = \mathcal{P}[g]$.*

4 The Hardy spaces $H^2_{(r)}(\mathbb{R}^m)$ and $H^2_{(r)}(\mathbb{R}^{m+1}_+)$

We recall that $\mathbb{R}^{(r)}_{0,m+1}$ is the space of r–vectors in $\mathbb{R}_{0,m+1}$ (see Section 2), and we define the following subspaces of the Hardy spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$.

Definition 1. *The space $H^2_{(r)}(\mathbb{R}^{m+1}_+), 0 < r < m+1$, consists of all functions in $H^2(\mathbb{R}^{m+1}_+)$ which are $\mathbb{R}^{(r)}_{0,m+1}$–valued. The space $H^2_{(r)}(\mathbb{R}^m), 0 < r < m+1$ consists of all functions in $H^2(\mathbb{R}^m)$ which are $\mathbb{R}^{(r)}_{0,m+1}$–valued.*

Take a function $F_r \in H^2_{(r)}(\mathbb{R}^{m+1}_+)$, in other words a function F_r satisfying

(i) $\partial_x F_r = 0$ in $\mathbb{R}^{m+1}_+$;

(ii) $\sup_{x_0>0} \displaystyle\int_{\mathbb{R}^m} |F_r(x_0, \underline{x})|^2 \, d\underline{x} < +\infty$.

We know from Section 3 that $F_r^+ = \lim_{x_0 \to 0+} F_r(x_0, \underline{x})$ belongs to $H^2(\mathbb{R}^m)$, implying that $\overline{e_0}\, \mathcal{H}\,[F_r^+] = F_r^+ = \mathcal{C}^+[F_r^+]$. Obviously F_r^+ is $\mathbb{R}^{(r)}_{0,m+1}$–valued and thus belongs to $H^2_{(r)}(\mathbb{R}^m)$. Conversely let $g_r \in H^2_{(r)}(\mathbb{R}^m)$. Then $\mathcal{C}[g_r] \in H^2(\mathbb{R}^{m+1}_+)$, and clearly $\mathcal{C}[g_r] = \mathcal{P}[g_r]$ is $\mathbb{R}^{(r)}_{0,m+1}$–valued, and so $\mathcal{C}[g_r] \in H^2_{(r)}(\mathbb{R}^{m+1}_+)$. These considerations lead to the following result.

Property 10. *The spaces $H^2_{(r)}(\mathbb{R}^{m+1}_+)$ and $H^2_{(r)}(\mathbb{R}^m)$ are isomorphic.*

Now, denoting $\mathbb{R}\mathrm{e}\, F_r^+ = u_r \in L_2(\mathbb{R}^m; \mathbb{R}^{(r)}_{0,m})$ and $\mathbb{I}\mathrm{m}\, F_r^+ = v_{r-1} \in L_2(\mathbb{R}^m; \mathbb{R}^{(r-1)}_{0,m})$, there holds, in view of Property 6, that

$$F_r^+ = u_r + \overline{e_0}\ \mathcal{H}[u_r] = \mathcal{H}[v_{r-1}] + \overline{e_0}\ v_{r-1}$$

from which it follows that

$$\mathcal{H}^+[u_r] = 0, \quad \mathcal{H}[u_r] = \mathcal{H}^-[u_r], \quad \mathcal{H}^+\mathcal{H}^-[u_r] = u_r$$

and

$$\mathcal{H}^-[v_{r-1}] = 0, \quad \mathcal{H}[v_{r-1}] = \mathcal{H}^+[v_{r-1}], \quad \mathcal{H}^-\mathcal{H}^+[v_{r-1}] = v_{r-1}$$

So we have proven the following result.

Proposition 1. *The non–tangential boundary value $F_r^+ \in H^2_{(r)}(\mathbb{R}^m)$ of the r–vector function $F_r \in H^2_{(r)}(\mathbb{R}^{m+1}_+)$ may be written as*

$$F_r^+ = \mathbb{R}\mathrm{e}\ F_r^+ + \overline{e_0}\ \mathcal{H}^-[\mathbb{R}\mathrm{e}\ F_r^+]$$

with

$$\mathbb{R}\mathrm{e}\ F_r^+ \in L_2(\mathbb{R}^m; \mathbb{R}^{(r)}_{0,m}) \quad \text{and} \quad \mathcal{H}^+[\mathbb{R}\mathrm{e}\ F_r^+] = 0$$

or, alternatively,

$$F_r^+ = \mathcal{H}^+[\mathbb{I}\mathrm{m}\ F_r^+] + \overline{e_0}\ \mathbb{I}\mathrm{m}\ F_r^+$$

with

$$\mathbb{I}\mathrm{m}\ F_r^+ \in L_2(\mathbb{R}^m; \mathbb{R}^{(r-1)}_{0,m}) \quad \text{and} \quad \mathcal{H}^-[\mathbb{I}\mathrm{m}\ F_r^+] = 0$$

Conversely, assume that $u_r \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$ is such that $\mathcal{H}^+[u_r] = 0$, and that $v_{r-1} \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$ is such that $\mathcal{H}^-[v_{r-1}] = 0$. Put

$$g_r = \mathcal{A}[u_r] = u_r + \overline{e_0}\ \mathcal{H}[u_r] = u_r + \overline{e_0}\ \mathcal{H}^-[u_r]$$

and

$$h_r = \mathcal{B}[v_{r-1}] = \mathcal{H}[v_{r-1}] + \overline{e_0}\ v_{r-1} = \mathcal{H}^+[v_{r-1}] + \overline{e_0}\ v_{r-1}$$

Then, in view of Property 6, the functions g_r and h_r both belong to $H^2(\mathbb{R}^m)$ and thus to $H^2_{(r)}(\mathbb{R}^m)$, since they are $\mathbb{R}_{0,m+1}^{(r)}$–valued. Putting

$$L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)}) = \{u_r \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)}) : \mathcal{H}^+[u_r] = 0\}$$

and

$$L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)}) = \{v_{r-1} \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)}) : \mathcal{H}^-[v_{r-1}] = 0\}$$

we thus have proven the following result.

Proposition 2. *Let $u_r \in L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$ and let $v_{r-1} \in L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$. Then the functions*

$$\widetilde{\mathcal{A}}[u_r] = u_r + \overline{e_0}\ \mathcal{H}^-[u_r] \quad \text{and} \quad \widetilde{\mathcal{B}}[v_{r-1}] = \mathcal{H}^+[v_{r-1}] + \overline{e_0} v_{r-1}$$

both belong to $H^2_{(r)}(\mathbb{R}^m)$.

Note that when $u_r \in L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$ then $\mathcal{H}^-[u_r] \in L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$, and that $v_{r-1} \in L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$ implies $\mathcal{H}^+[v_{r-1}] \in L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$. Moreover $\mathcal{H}^+\ \mathcal{H}^- = \mathbf{1}$ on $L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$, while $\mathcal{H}^-\ \mathcal{H}^+ = \mathbf{1}$ on $L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$.

Corollary 2. *The spaces $L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})$, $L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})$ and $H^2_{(r)}(\mathbb{R}^m)$ are isomorphic:*

$$H^2_r(\mathbb{R}^m) = \widetilde{\mathcal{A}}\left[L_2^+(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r)})\right] = \widetilde{\mathcal{B}}\left[L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1)})\right]$$

Remark 3. *In [27, Theorem 13.3.5], a characterization was obtained of all boundary values $\omega_+^r \in H^2(\mathbb{R}^m; \bigwedge^r \mathbb{R}^{m+1})$ of functions $\omega^r \in H^2(\mathbb{R}_+^{m+1}; \bigwedge^r \mathbb{R}^{m+1})$, i.e. harmonic r–forms ω^r in $\mathbb{R}_+^{m+1}$ satisfying*

$$\sup_{x_0>0} \int_{\mathbb{R}^m} |\omega^r(x_0, \underline{x})|^2\, d\underline{x} \quad < \quad +\infty$$

A straightforward analysis shows that this characterization corresponds to the one given in Proposition 1 for the boundary values F_r^+ of functions $F_r \in H^2(\mathbb{R}_+^{m+1}; \mathbb{R}_{0,m+1}^{(r)})$.

Remark 4. *Let $r = 1$ and consider the real–valued function $v_0 \in L_2^-(\mathbb{R}^m; \mathbb{R}_{0,m}^{(0)})$. According to Proposition 2, the function*

$$F_1^+ = \mathcal{H}^+[v_0] + \overline{e_0}\ v_0$$

belongs to $H^2_{(1)}(\mathbb{R}^m)$ and

$$\mathcal{C}[F_1^+] = \mathcal{P}[F_1^+] = \overline{e_0}\ \mathcal{P}[v_0] + \sum_{j=1}^m \overline{e_j}\ \mathcal{P}[\mathcal{R}_j[v_0]]$$

belongs to $H^2_{(1)}(\mathbb{R}_+^{m+1})$. The $(m+1)$–tuple $(\mathcal{P}[v_0], \mathcal{P}[\mathcal{R}_1[v_0]], \ldots, \mathcal{P}[\mathcal{R}_m[[v_0]])$ is then a system of conjugate harmonic functions in $\mathbb{R}_+^{m+1}$ in the sense of Stein and Weiss (see [40, §4.2]).

Now we turn our attention to the concept of conjugate harmonic r–vector functions.

Definition 2. *The space* $Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m}), 0 < r < m+1$, *consists of all functions in the Hardy space* $Harm^2(\mathbb{R}^{m+1}_+)$ *which are* $\mathbb{R}^{(r)}_{0,m}$*–valued.*

Take a function $U_r \in Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$, i.e. a harmonic $\mathbb{R}^{(r)}_{0,m}$–valued function satisfying the estimate

$$\sup_{x_0>0} \int_{\mathbb{R}^m} \mid U_r(x_0,\underline{x}) \mid^2 \; d\underline{x} < +\infty$$

It possesses a non–tangential L_2–boundary value $u_r \in L_2(\mathbb{R}^m;\mathbb{R}^{(r)}_{0,m})$, from which U_r may be recovered by means of the Poisson transform, i.e. $U_r = \mathcal{P}[u_r]$. It was shown in [1], Theorem 3.1, that U_r admits in $\mathbb{R}^{m+1}_+$ an $\mathbb{R}^{(r)}_{0,m}$–valued conjugate harmonic, in the sense of [8], if and only if $\partial^+_{\underline{x}} U_r = 0$, but this conjugate harmonic does not need to be in $Harm^2(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$. Now we can prove the following.

Proposition 3. *If the function* U_r *belongs to the Hardy space* $Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$ *and shows the non–tangential* L_2*–boundary value* $u_r \in L_2(\mathbb{R}^m;\mathbb{R}^{(r)}_{0,m})$*, then* U_r *admits a conjugate harmonic belonging to* $Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$ *if and only if* $\mathcal{H}^+[u_r] = 0$*.*

Proof.
It is clear that $\overline{e_0}\; V = \overline{e_0}\; \mathcal{P}[\mathcal{H}[u_r]]$ is conjugate harmonic to $\mathcal{P}[u_r] = U_r$, and moreover belongs to $Harm^2(\mathbb{R}^{m+1}_+)$. If $\mathcal{H}^+[u_r] = 0$, then V clearly takes its values in $\mathbb{R}^{(r-1)}_{0,m}$, implying that the conjugate harmonic $\overline{e_0}\; V$ belongs to $Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$. Conversely, if $\overline{e_0}\; V_{r-1}$ is a conjugate harmonic to U_r, which is in $Harm^2_{(r)}(\mathbb{R}^{m+1}_+;\mathbb{R}^{(r)}_{0,m})$, then the monogenic function $U_r + \overline{e_0}\; V_{r-1}$ belongs to the Hardy space $H^2(\mathbb{R}^{m+1}_+)$ and shows the boundary value $u_r + \overline{e_0}\; v_{r-1}$ which belongs to the Hardy space $H^2(\mathbb{R}^m)$. It follows that $\mathcal{H}[u_r] = v_{r-1}$, implying that $\mathcal{H}^+[u_r] = 0$.

Remark 5. *In the setting explained above, the condition* $\mathcal{H}^+[u_r] = 0$ *implies the condition* $\partial^+_{\underline{x}} U_r = 0$*, but not conversely.*

5 The Hardy spaces $H^2_{(r,p,q)}(\mathbb{R}^m)$ and $H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$

Take $0 < r < m+1$ fixed, take $p, q \in \mathbb{N}$ such that $p < q$ and $r + 2q \leq m+1$ and put

$$\mathbb{R}^{(r,p,q)}_{0,m+1} = \bigoplus_{j=p}^{q} \mathbb{R}^{(r+2j)}_{0,m+1}$$

Then we define the following subspaces of the Hardy spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$.

Definition 3. *The space* $H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$ *consists of all functions in* $H^2(\mathbb{R}^{m+1}_+)$ *which are* $\mathbb{R}^{(r,p,q)}_{0,m+1}$*–valued. The space* $H^2_{(r,p,q)}(\mathbb{R}^m)$ *consists of all functions in* $H^2(\mathbb{R}^m)$ *which are* $\mathbb{R}^{(r,p,q)}_{0,m+1}$*–valued.*

Take a function $W = \sum_{j=p}^{q} W_{r+2j} \in H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$, i.e. a function W satisfying

(i) $\partial_x W = 0$ in $\mathbb{R}^{m+1}_+$;

(ii) $\sup_{x_0>0} \int_{\mathbb{R}^m} |W(x_0,\underline{x})|^2 d\underline{x} < +\infty$.

Note that

$$\partial_x W = 0 \quad \Longleftrightarrow \quad (\text{GMT}) \left\{ \begin{array}{l} \partial_{\underline{x}}^- W_{r+2p} = 0 \\ \partial_{\underline{x}}^+ W_{r+2j} + \partial_{\underline{x}}^- W_{r+2(j+1)} = 0, \quad j = p, \ldots, q-1 \\ \partial_{\underline{x}}^+ W_{r+2q} = 0 \end{array} \right.$$

The system (GMT) is called a generalized Moisil–Teodorescu system of type (r,p,q) (see also [1]).

We know from Section 3 that $W^+ = \lim_{x_0 \to 0+} W(x_0,\underline{x})$ belongs to $H^2(\mathbb{R}^m)$, implying that $\overline{e_0}\ \mathcal{H}[W^+] = W^+ = \mathcal{C}^+[W^+]$. Obviously, $W^+ = \sum_{j=p}^q W^+_{r+2j}$ is $\mathbb{R}_{0,m+1}^{(r,p,q)}$–valued, and thus belongs to $H^2_{(r,p,q)}(\mathbb{R}^m)$. Conversely let $w \in H^2_{(r,p,q)}(\mathbb{R}^m)$. Then $\mathcal{C}[w] \in H^2(\mathbb{R}^{m+1}_+)$, and clearly $\mathcal{C}[w] = \mathcal{P}[w]$ is $\mathbb{R}_{0,m+1}^{(r,p,q)}$–valued, and so $\mathcal{C}[w] \in H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$. These considerations lead to the following result.

Property 11. *The spaces $H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$ and $H^2_{(r,p,q)}(\mathbb{R}^m)$ are isomorphic.*

Now, denoting $\mathbb{R}\mathrm{e}\, W^+ = u \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r,p,q)})$ and $\mathbb{I}\mathrm{m}\, W^+ = v \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m}^{(r-1,p,q)})$, there holds, in view of Property 6, that

$$W^+ = \mathcal{A}[u] = u + \overline{e_0}\ \mathcal{H}[u]$$

and also

$$W^+ = \mathcal{B}[v] = \mathcal{H}[v] + \overline{e_0}\ v$$

from which it follows that

$$\mathcal{H}^-[u_{r+2p}] = v_{r+2p-1} \tag{11}$$

$$\mathcal{H}^+[u_{r+2j}] + \mathcal{H}^-[u_{r+2j+2}] = v_{r+2j+1} \quad (j = p, \ldots, q-1) \tag{12}$$

$$\mathcal{H}^+[u_{r+2q}] = 0 \tag{13}$$

and

$$\mathcal{H}^-[v_{r+2p-1}] = 0 \tag{14}$$

$$\mathcal{H}^+[v_{r+2j-1}] + \mathcal{H}^-[v_{r+2j+1}] = u_{r+2j} \quad (j = p, \ldots, q-1) \tag{15}$$

$$\mathcal{H}^+[v_{r+2q-1}] = u_{r+2q} \tag{16}$$

So we have proven the following result.

Proposition 4. *For the non–tangential boundary $W^+ \in H^2_{(r,p,q)}(\mathbb{R}^m)$ of the function $W \in H^2_{(r,p,q)}(\mathbb{R}^{m+1}_+)$ there exist functions $u_{r+2j} \in L_2(\mathbb{R}^m; R_{0,m}^{(r+2j)}), j = p, \ldots, q-1$ and a function $u_{r+2q} \in L_2^+(\mathbb{R}^m; R_{0,m}^{(r+2q)})$ such that*

$$W^+ = (u_{r+2p} + \cdots + u_{r+2q-2} + u_{r+2q}) + \overline{e_0}\ \left(\mathcal{H}[u_{r+2p}] + \cdots + \mathcal{H}[u_{r+2q-2}] + \mathcal{H}^-[u_{r+2q}]\right)$$

At the same time there exist functions $v_{r+2j+1} \in L_2(\mathbb{R}^m; R_{0,m}^{(r+2j+1)}), j = p, \ldots, q-1$ and a function $v_{r+2p-1} \in L_2^-(\mathbb{R}^m; R_{0,m}^{(r+2p-1)})$ such that

$$W^+ = \left(\mathcal{H}^+[v_{r+2p-1}] + \mathcal{H}[v_{r+2p+1}] + \cdots + \mathcal{H}[v_{r+2q-1}]\right) + \overline{e_0}\ (v_{r+2p-1} + v_{r+2p+1} + \cdots + v_{r+2q-1})$$

Moreover the functions u_{r+2j} and v_{r+2j-1}, $j = p, \ldots, q$ are intertwinned by the relations (11),(12),(15),(16).

In a similar way as for Proposition 2 the converse result is obtained.

Proposition 5. *Let* $u_{r+2j} \in L_2(\mathbb{R}^m; R_{0,m}^{(r+2j)})$ *and* $v_{r+2j+1} \in L_2(\mathbb{R}^m; R_{0,m}^{(r+2j+1)})$, $j = p, \ldots, q-1$ *and let* $u_{r+2q} \in L_2^+(\mathbb{R}^m; R_{0,m}^{(r+2q)})$ *and* $v_{r+2p-1} \in L_2^-(\mathbb{R}^m; R_{0,m}^{(r+2p-1)})$. *Then the functions*

$$\sum_{j=p}^{q-1} \mathcal{A}[u_{r+2j}] + \widetilde{\mathcal{A}}[u_{r+2q}]$$

and

$$\widetilde{\mathcal{B}}[v_{r+2p-1}] + \sum_{j=p}^{q-1} \mathcal{B}[v_{r+2j+1}]$$

both belong to $H^2_{(r,p,q)}(\mathbb{R}^m)$.

Corollary 3. *The following decompositions of* $H^2_{(r,p,q)}(\mathbb{R}^m)$ *hold:*

$$\begin{aligned} H^2_{(r,p,q)}(\mathbb{R}^m) &= \bigoplus_{j=p}^{q-1} \mathcal{A}\left[L_2(\mathbb{R}^m; R_{0,m}^{(r+2j)})\right] \oplus \widetilde{\mathcal{A}}\left[L_2^+(\mathbb{R}^m; R_{0,m}^{(r+2q)})\right] \\ &= \bigoplus_{j=p}^{q-1} \mathcal{A}\left[L_2(\mathbb{R}^m; R_{0,m}^{(r+2j)})\right] \oplus H^2_{(r+2q)}(\mathbb{R}^m) \end{aligned}$$

and

$$\begin{aligned} H^2_{(r,p,q)}(\mathbb{R}^m) &= \widetilde{\mathcal{B}}\left[L_2^-(\mathbb{R}^m; R_{0,m}^{(r+2p-1)})\right] \oplus \bigoplus_{j=p}^{q-1} \mathcal{B}\left[L_2(\mathbb{R}^m; R_{0,m}^{(r+2j+1)})\right] \\ &= H^2_{(r+2p)}(\mathbb{R}^m) \oplus \bigoplus_{j=p}^{q-1} \mathcal{B}\left[L_2(\mathbb{R}^m; R_{0,m}^{(r+2j+1)})\right] \end{aligned}$$

6 The Hardy spaces $H^2_E(\mathbb{R}^m)$ and $H^2_E(\mathbb{R}^{m+1}_+)$

Considering functions taking their values in the even subalgebras

$$\mathbb{R}^+_{0,m+1} = \bigoplus_{k \text{ even}} \mathbb{R}^{(k)}_{0,m+1}$$

and

$$\mathbb{R}^+_{0,m} = \bigoplus_{k \text{ even}} \mathbb{R}^{(k)}_{0,m}$$

of the respective Clifford algebras $\mathbb{R}_{0,m+1}$ and $\mathbb{R}_{0,m}$, we may introduce the following subspaces of the Hardy spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$.

Definition 4. *The space* $H^2_E(\mathbb{R}^{m+1}_+)$ *consists of all functions in* $H^2(\mathbb{R}^{m+1}_+)$ *which are* $\mathbb{R}^+_{0,m+1}$*–valued. The space* $H^2_E(\mathbb{R}^m)$ *consists of all functions in* $H^2(\mathbb{R}^m)$ *which are* $\mathbb{R}^+_{0,m+1}$*–valued.*

It is clear that these spaces fit into the framework of Section 5 by making the appropriate choices $r = 0, p = 0$ and $q = \lfloor \frac{m+1}{2} \rfloor$. In view of Property 11 the following result is then immediate.

Property 12. *The spaces* $H^2_E(\mathbb{R}^m)$ *and* $H^2_E(\mathbb{R}^{m+1}_+)$ *are isomorphic.*

Now observe that when $m+1$ is even, then the space $L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{m+1})$ is the null space. When $m+1$ is odd, then the condition $\mathcal{H}^+[u_m]=0$ is trivially fulfilled for $u_m \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{(m)})$, in other words $L_2^+(\mathbb{R}^m;\mathbb{R}_{0,m}^{(m)}) = L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{(m)})$. The result of Corollary 3 now takes the following form.
When $m+1$ is even:

$$H_E^2(\mathbb{R}^m) = \bigoplus_{j=0}^{\frac{m-1}{2}} \mathcal{A}\left[L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{(2j)})\right] = \mathcal{A}\left[L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{+})\right]$$

When $m+1$ is odd:

$$H_E^2(\mathbb{R}^m) = \bigoplus_{j=0}^{\frac{m}{2}-1} \mathcal{A}\left[L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{(2j)})\right] + \widetilde{\mathcal{A}}\left[L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{(m)})\right] = \mathcal{A}\left[L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{+})\right]$$

This leads to the following conclusion, which is a refinement of [26, Theorem 5.33] for L_2–spaces.

Property 13. *The spaces $H_E^2(\mathbb{R}^m)$ and $L_2(\mathbb{R}^m;\mathbb{R}_{0,m}^{+})$ are isomorphic.*

In the particular case where $m+1=3$, the Hardy spaces $H_E^2(\mathbb{R}_+^3)$ and $H_E^2(\mathbb{R}^2)$ are obtained. It was shown in [3] that the space of solutions to the classical Moisil-Teodorescu system in Ω, Ω open in $\mathbb{R}^3$, is isomorphic to the space of (left) monogenic $\mathbb{R}_{0,3}^{+}$– valued functions in Ω. Taking $\Omega = \mathbb{R}_+^3$, the space $H_E^2(\mathbb{R}_+^3)$ thus consists of all solutions W to the classical Moisil-Teodorescu system in $\mathbb{R}_+^3$ satisfying the condition

$$\sup_{x_0>0} \int_{\mathbb{R}^2} |W(x_0,\underline{x})|^2 d\underline{x} \quad < \quad +\infty$$

Its corresponding space of non–tangential boundary values $H_E^2(\mathbb{R}^2)$ is thus characterized by

$$H_E^2(\mathbb{R}^2) = \mathcal{A}\left[L_2(\mathbb{R}^2;\mathbb{R}_{0,2}^{+})\right] = \left\{u + \overline{e_0}\mathcal{H}[u] \mid u \in L_2(\mathbb{R}^2;\mathbb{R}_{0,2}^{+})\right\}$$

The latter space was called in [3] the space of analytic signals on $\mathbb{R}^2$.

Finally notice that in the particular case where $m+1=2$, there holds $\mathbb{R}_{0,2}^{+} \simeq \mathbb{C}$, $\mathbb{R}_+^2 = \mathbb{C}_+$ and $\mathbb{R}_{0,1}^{+} \simeq \mathbb{R}$, so that the classical Hardy space in the upper half complex plane $H^2(\mathbb{C}_+)$ and the Hardy space $H^2(\mathbb{R})$ of analytic signals on the real line are reobtained.

References

[1] R. Abreu, J. Bory, R. Delanghe, F. Sommen, Generalized Moisil–Théodoresco Systems and Cauchy Integral Decompositions, *Int. J. Math. Math. Sci.*, 2008, article ID 746946, DOI:10.1155/2008/746946, 19 pp.

[2] S. Bernstein, L. Lanzani, Szegö projections for Hardy spaces of monogenic functions and applications, *Int. J. Math. Math. Sci.* **29(10)**, 2002, 613–624.

[3] J. Bory Reyes, R. Delanghe, On the solutions of the Moisil–Théodoresco systems, *Math. Meth. Appl. Sci* **31**, 2008, 1427–1439.

[4] J. Bory Reyes, R. Delanghe, On the structure of solutions of the Moisil-Théodoresco system in Euclidean space, *Adv. Appl. Clifford Algebr.* **19(1)**, 2009, 15–28.

[5] F. Brackx, B. De Knock, H. De Schepper, Hilbert Transforms in Clifford Analysis. In: *Geometric algebra computing in engineering and computer science*, E. Bayro–Corrochano, G. Scheuermann (eds.), Springer Verlag (London 2010), 163–187.

[6] F. Brackx, B. De Knock, H. De Schepper, D. Eelbode, On the interplay between the Hilbert transform and conjugate harmonic functions, *Math. Meth. Appl. Sci.* **29(12)**, 2006, 1435–1450.

[7] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Advanced Publishing Program (Boston–London–Melbourne, 1982).

[8] F. Brackx, R. Delanghe, F. Sommen, On Conjugate Harmonic Functions in Euclidean Space, *Math. Meth. Appl. Sci.* **25(16–18)**, 2002, 1553–1562.

[9] F. Brackx, R. Delanghe, F. Sommen, Differential forms and/or multi–vector functions, *Cubo* **7** (2005), 139–169.

[10] F. Brackx, H. De Schepper, Hilbert–Dirac Operators in Clifford Aanalysis, *Chin. Ann. Math.* **26B:1**, 2005, 1–14.

[11] F. Brackx, H. De Schepper, The Hilbert Transform on a Smooth Closed Hypersurface, *Cubo* **10(2)** (2008), 83–106.

[12] F. Brackx, H. De Schepper, The Hilbert Transform on the Unit Sphere in $\mathbb{R}^m$. In: Quaternionic and Clifford Analysis, Trends in Mathematics, Birkhäuser Verlag (Basel, 2008), 11–36.

[13] F. Brackx, H. De Schepper, D. Eelbode, A new Hilbert transform on the unit sphere in $\mathbb{R}^m$, *Comp. Var. Ell. Equ.* **51(5–6)**, 2006, 453–462.

[14] F. Brackx, N. Van Acker, H^p spaces of monogenic functions. In: A. Micali et al. (eds.), *Clifford Algebras and their Applications in Mathematical Physics*, Kluwer Academic Publishers (Dordrecht, 1992), 177–188.

[15] I. Cacão, Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in $\mathbb{R}^3$, *Numer. Algorithms* **55(2–3)**, 2010, 191–203.

[16] P. Calderbank, *Clifford analysis for Dirac operators on manifolds–with–boundary*, Max Planck–Institut für Mathematik (Bonn, 1996).

[17] A. Cialdea, On the theory of self–conjugate differential forms, *Alti Sem. Math. Fis. Univ. Modena* **46** (1998), 595–620.

[18] J. Cnops, *An introduction to Dirac operators on manifolds*, Birkhäuser Verlag (Basel, 2002).

[19] R. Delanghe, On the Hardy spaces of harmonic and monogenic functions in the unit ball of $\mathbb{R}^{m+1}$. In: *Acoustics, mechanics and the related topics of mathematical analysis*, World Scientific Publishing (River Edge, New Jersey, 2002), 137–142.

[20] R. Delanghe, Some remarks on the principal value kernel in $\mathbb{R}^m$, *Comp. Var. Theory Appl.* **47**, 2002, 653–662.

[21] R. Delanghe, On Some Properties of the Hilbert Transform in Euclidean Space, *Bull. Belg. Math. Soc. – Simon Stevin* **11**, 2004, 163–180.

[22] R. Delanghe, On homogeneous polynomial solutions of the Riesz system and their harmonic potentials, *Complex Var. Elliptic Equ.* **52(10–11)**, 2007, 1047–1061.

[23] R. Delanghe, On homogeneous polynomial solutions of the Moisil-Théodoresco system in $\mathbb{R}^3$, *Comput. Methods Funct. Theory* **9(1)**, 2009, 199–212.

[24] R. Delanghe, F. Sommen, On the structure of harmonic multi–vector functions, *Adv. Appl. Cliff. Alg.* **17** (2007), 395–410.

[25] R. Delanghe, On homogeneous polynomial solutions of generalized Moisil-Théodoresco systems in Euclidean space, *Cubo* **12(2)**, 2010, 145–167.

[26] R. Delanghe, F. Sommen, V. Souček, *Clifford algebra and spinor values functions*, Kluwer (Dordrecht, 1992).

[27] J.E. Gilbert, J.A. Hogan, J.D. Lakey, Frame decomposition of form–valued Hardy spaces. In: J. Ryan (ed.), *Clifford algebras in analysis and related topics*, CRC Press (Boca Raton – New York, 1996), 239–259.

[28] J.E. Gilbert, M.A.M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press (Cambridge, 1991).

[29] S. Hofmann, E. M. Olea, M. Mitrea, S. P. Esteva, M. Taylor, Hardy Spaces, Singular Integrals and the Geometry of Euclidean Domains of Locally Finite Perimeter, Geom. Funct. Anal. **19 (3)**, 2009, 842–882.

[30] V.V. Kravchenko, M.V. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics*, Pitman Research Notes in Mathematics Series 351, Longman Scientific and Technical (Harlow, 1996).

[31] Le Hung Son, Tran Minh Hoang, Hartogs' extension theorem for a modified Riesz system. In: *Function spaces in complex and Clifford analysis*, Natl. Univ. Publ. Hanoi (Hanoi, 2008), 90–99.

[32] C. Li, A. McIntosh, T. Qian, Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces, *Rev. Math. Iberoamer.* **10**, 1994, 665–721.

[33] C. Li, A. McIntosh, S. Semmes, Convolution singular integrals on Lipschitz surfaces, *J. Amer. Math. Soc.* **5**, 1992, 455–481.

[34] A. Mc Intosh, Clifford algebras, Fourier theory, singular integrals and harmonic functions on Lipschitz domains. In: J. Ryan (ed.), *Clifford algebras in analysis and related topics*, CRC Press (Boca Raton – New York, 1996), 33–87.

[35] M. Mitrea, *Clifford wavelets, singular integrals and Hardy spaces*, Lecture Notes in Mathematics **1575**, Springer Verlag (Berlin, 1991).

[36] M. Murray, The Cauchy integral, Calderon commutation, and conjugation of singular integrals in $\mathbb{R}^n$, *Trans. of the AMS* **298**, 1985, 497–518.

[37] E. N. Sattorov, Regularization of the solution of the Cauchy problem for the generalized Moisil-Teodorescu system, *Differ. Uravn.* **44(8)**, 2008, 1100–1110; translation in *Differ. Equ.* **44(8)**, 2008, 1136–1146.

[38] M.V. Shapiro, N.L. Vasilevski, Quaternionic ψ–holomorphic functions, singular integral operators and boundary value problems, Parts I and II, *Complex Variables: Theory and Application* **27**, 1995, 14–46 and 67–96.

[39] E.M. Stein, *Harmonic analysis*, Princeton University Press, (Princeton, 1993).

[40] E.M. Stein, G. Weiss, On the theory of harmonic functions of several variables, I. The theory of H^p–spaces, *Acta Math.* **103**, 1960, 25–62.

[41] Yang Pi Wen, Li Man Li, Riemann-Hilbert boundary value problems for nth-order Moisil-Théodoresco type equations, *Acta Math. Sinica* (Chin. Ser.) **53(3)**, 2010, 585–596.