The hyperplanes of finite symplectic dual polar spaces which arise from projective embeddings

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Abstract

We characterize the hyperplanes of the dual polar space $DW(2n-1,q)$ which arise from projective embeddings as those hyperplanes H of $DW(2n-1,q)$ which satisfy the following property: if Q is an ovoidal quad, then $Q \cap H$ is a classical ovoid of Q. A consequence of this is that all hyperplanes of the dual polar spaces $DW(2n-1,4)$, $DW(2n-1,16)$ and $DW(2n-1,p)$ (p prime) arise from projective embeddings.

Keywords: symplectic dual polar space, hyperplane, Grassmann-embedding, universal embedding
MSC2000: 51A45, 51A50

1 Introduction

Let Π be a polar space (Tits [32]) of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points, respectively lines, are the maximal, respectively next-to-maximal, singular subspaces of Π, with incidence given by reverse containment. Δ is called a *dual polar space* (Cameron [5]). Distances between points of Δ will be measured in the collinearity graph of Δ. This is the graph with vertices the points of Δ, two points being adjacent whenever they are collinear, i.e. whenever there is a line incident with them. There exists a bijective correspondence between the possibly empty singular subspaces of Π and the non-empty convex subspaces of Δ: if α is a singular subspace of Π of dimension $n-1-k$, then the set of all maximal singular subspaces containing α is a convex subspace of diameter k of Δ. These convex subspaces are called *quads* if $k=2$ and *maxes* if $k=n-1$. The points and lines contained in a quad define a so-called generalized quadrangle (Payne and Thas [24]).

A *hyperplane* of a point-line geometry $\mathcal{S}$ is a proper subspace meeting each line. A natural way to construct hyperplanes of a point-line geometry is to embed it (fully) in a projective space Σ and then intersect it with a hyperplane of Σ. (We give more formal definitions in Section 2.) An important question which arises in this context is the following:

> $(*)$ Given an embeddable point-line geometry $\mathcal{S}$ and a class $\mathcal{C}$ of hyperplanes of $\mathcal{S}$. Does any hyperplane of $\mathcal{C}$ arise from a hyperplane of a projective space in which $\mathcal{S}$ is embedded?

The answer to question $(*)$ is affirmative for many classes of hyperplanes of point-line geometries. E.g., the answer is affirmative for the class of all hyperplanes of any embeddable point-line geometry with three points per line (Ronan [27]). In the case of dual polar spaces not so much was known till very recently. In the case of dual polar spaces, the question whether all hyperplanes arise from embedding is only interesting in the finite case, due to constructions using transfinite recursion. These constructions easily yield hyperplanes which do not arise from embeddings, see Cameron [6] and Cardinali & De Bruyn [7, Section 4]. In [29], Shult and Thas proved that all hyperplanes of the orthogonal dual polar space $DQ(2n, q)$, q odd, arise from the so-called spin-embedding of $DQ(2n, q)$. The next result was obtained only recently by De Bruyn and Pralle [16] who classified all hyperplanes of the Hermitian dual polar space $DH(5, q^2)$, $q \neq 2$, and showed that they all arise from the so-called Grassmann-embedding of $DH(5, q^2)$. With the aid of techniques from diagram geometry (simple connectedness) and Ronan's paper [27], it was subsequently shown by Cardinali, De Bruyn and Pasini [8, Corollary 1.6] that also all hyperplanes of $DH(2n-1, q^2)$, $n \geq 4$ and $q \neq 2$, arise from its Grassmann-embedding. The case of the orthogonal dual polar space $DQ^-(2n+1, q)$ was treated in De Bruyn [11, Theorem 1.4] where necessary and sufficient conditions were given for a hyperplane of $DQ^-(2n-1, q)$ to arise from embedding.

The case which remains to be done is the one of the symplectic dual polar space $DW(2n-1, q)$, $n \geq 2$, associated with the polar space $W(2n-1, q)$. The singular subspaces of this polar space are the subspaces of the projective space $\mathrm{PG}(2n-1, q)$ which are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(2n-1, q)$. The quads of the dual polar space $DW(2n-1, q)$ are isomorphic to the generalized quadrangle $Q(4, q)$. The points and lines of this generalized quadrangle are the points and lines of $\mathrm{PG}(4, q)$ which lie on a given nonsingular parabolic quadric $Q(4, q)$ of $\mathrm{PG}(4, q)$ (natural incidence). An *ovoid* of $Q(4, q)$ (or more generally, of any generalized quadrangle) is a set of points meeting every line in a unique point. An ovoid of $Q(4,q)$ is called *classical* if it is obtained by intersecting $Q(4,q)$ with a hyperplane of $\mathrm{PG}(4,q)$, i.e. if it is a nonsingular elliptic quadric in a 3-space of $\mathrm{PG}(4,q)$. It is well-known that the dual polar space $DW(2n-1,q)$ has a full embedding into the projective space $\mathrm{PG}(\binom{2n}{n}-\binom{2n}{n-2}-1,q)$, see e.g. Bourbaki [4, 13.3] or De Bruyn [12]. We refer to this particular embedding as the *Grassmann-embedding* of $DW(2n-1,q)$. The following is the main result of this paper.

Main Theorem. *The hyperplanes of the dual polar space $DW(2n-1,q)$, $q \neq 2$, which arise from its Grassmann-embedding are precisely those hyperplanes H of $DW(2n-1,q)$ which satisfy the following property: if Q is a quad of $DW(2n-1,q)$ such that $Q \cap H$ is an ovoid of Q, then $Q \cap H$ is a classical ovoid of Q.*

For certain values of q it is known that all ovoids of $Q(4,q)$ are classical:

Proposition. (1) *([1]) All ovoids of $Q(4,q)$, q prime, are classical.*
(2) *([2], [23]) All ovoids of $Q(4,4)$ are classical.*
(3) *([21], [22]) All ovoids of $Q(4,16)$ are classical.*

Combining the previous proposition with the Main Theorem, we obtain

Corollary. *Let Δ be one of the following dual polar spaces of rank $n \geq 2$: $DW(2n-1,4)$, $DW(2n-1,16)$, $DW(2n-1,p)$ with $p \neq 2$ prime. Then every hyperplane of Δ arises from its Grassmann-embedding.*

Remarks. (1) If $n \geq 2$ and $q \neq 2$, then by results of Cooperstein [9] and Kasikova & Shult [19], the Grassmann-embedding of $DW(2n-1,q)$ is absolutely universal. [We refer to Section 2 for the definition of the notion "absolutely universal embedding".] This implies that the hyperplanes of $DW(2n-1,q)$, $n \geq 2$ and $q \neq 2$, which arise from embedding are precisely those hyperplanes of $DW(2n-1,q)$ which arise from its Grassmann-embedding.

(2) Since the dual polar space $\Delta = DW(2n-1,2)$, $n \geq 2$, is embeddable and has three points on each line, every hyperplane of $DW(2n-1,2)$ arises from its absolutely universal embedding, see Ronan [27]. Although all ovoids of $Q(4,2)$ are classical, not every hyperplane of Δ arises from its Grassmann-embedding. The Grassmann-embedding of Δ has vector dimension $\binom{2n}{n}-\binom{2n}{n-2}$, while the absolutely universal embedding of Δ has vector dimension $\frac{(2^n+1)(2^{n-1}+1)}{3} > \binom{2n}{n}-\binom{2n}{n-2}$, see Li [20] or Blokhuis and Brouwer [3].

(3) Let Δ be the dual polar space $DW(2n-1,q)$, where $n \geq 2$ and $q \neq 2$. If O is a non-classical ovoid in a quad Q of Δ, then the set H of points of

Δ at distance at most $n-2$ from O is a hyperplane of Δ. If Q' is a quad of Δ opposite to Q, i.e. at maximal distance $n-2$ from Q, then $Q' \cap H$ is a non-classical ovoid of Q' which is isomorphic to the non-classical ovoid O of Q. Combining this observation with the Main Theorem, we conclude that all hyperplanes of Δ arise from its Grassmann-embedding if and only if every ovoid of $Q(4,q)$ is classical. Non-classical ovoids of $Q(4,q)$ are known to exist for any $q = p^h$ where p is an odd prime and $h \geq 2$ ([18], [25], [30]) and any $q = 2^{2h+1}$ where $h \geq 2$ ([31]).

(4) If q is a prime power such that every ovoid of $Q(4,q)$ is classical, then by the Main Theorem, every hyperplane of $DW(5,q)$ arises from embedding. The hyperplanes of $DW(5,q)$ which arise from embedding have been classified in the papers [10], [13] and [26].

2 Further definitions

Let Δ be a dual polar space. If x and y are two points of Δ, then $\mathrm{d}(x,y)$ denotes the distance between x and y in the collinearity graph of Δ. For every point x of Δ and every $i \in \mathbb{N}$, $\Delta_i(x)$, respectively $\Delta_i^*(x)$, denotes the set of points of Δ at distance i, respectively distance at most i, from x. We denote $\Delta_1^*(x)$ also by $x^\perp$. If x is a point and F is a non-empty convex subspace of Δ, then F contains a unique point $\pi_F(x)$ nearest to x and $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every point y of F.

A *full (projective) embedding* of a point-line geometry $\mathcal{S}$ is an injective mapping e from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space Σ satisfying: (i) $\langle e(\mathcal{P})\rangle = \Sigma$ and (ii) $e(L) := \{e(x)\,|\,x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. The numbers $\dim(\Sigma)$ and $\dim(\Sigma)+1$ are respectively called the *projective dimension* and the *vector dimension* of e. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then for every hyperplane α of Σ, $e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \alpha)$ *arises from the embedding* e. Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if U is a subspace of Σ satisfying (C1) $\langle U, e(x)\rangle \neq U$ for every point x of $\mathcal{S}$ and (C2) $\langle U, e(x_1)\rangle \neq \langle U, e(x_2)\rangle$ for any two distinct points x_1 and x_2 of $\mathcal{S}$, then there exists a full embedding e/U of $\mathcal{S}$ in the quotient space Σ/U, mapping each point x of $\mathcal{S}$ to $\langle U, e(x)\rangle$. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two full embeddings, then we say that $e_1 \geq e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then by Ronan [27], there exists a unique (up to isomorphism) full embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ satisfying (i) $\widetilde{e} \geq e$ and (ii) if $e' \geq e$ for some embedding e' of

$\mathcal{S}$, then $\widetilde{e} \geq e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e'} \cong \widetilde{e}$ for any other embedding e' of $\mathcal{S}$ with the same underlying division ring, then $\widetilde{e}$ is called *absolutely universal.* By Tits [32, 8.6] and Kasikova & Shult [19, 4.6], every embeddable thick dual polar space has a unique (up to isomorphism) absolutely universal embedding.

Let Δ be a dual polar space of rank $n \geq 2$. The set H_x of points of Δ at non-maximal distance from a given point x of Δ is a hyperplane which is called the *singular hyperplane* of Δ with *deepest point* x. If F is a convex subspace of Δ of diameter $\delta \geq 1$ and if H_F is a hyperplane of F, then the set H of points of Δ at distance at most $n - \delta$ from H_F is a hyperplane of Δ, see e.g. [17, Proposition 1]. We call H the *extension* of H_F.

If H is a hyperplane of a thick dual polar space Δ, then H is a maximal subspace of Δ by Shult [28, Lemma 6.1]. Moreover, if Q is a quad of Δ, then one of the following cases occurs: (1) $Q \subseteq H$; (2) there exists a point x in Q such that $x^\perp \cap Q = H \cap Q$; (3) $Q \cap H$ is a subquadrangle of Q; (4) $Q \cap H$ is an ovoid of Q. If case (1), (2), (3), respectively (4), occurs, then we say that Q is *deep*, *singular*, *subquadrangular*, respectively *ovoidal, with respect to* H.

3 Proof of the Main Theorem in the case $n = 3$

The aim of this section is the proof of the following proposition which is precisely the Main Theorem in the case $n = 3$.

Proposition 3.1 *The hyperplanes of the symplectic dual polar space $DW(5, q)$, $q \geq 3$, which arise from its Grassmann-embedding are precisely those hyperplanes H of $DW(5, q)$ which satisfy the following property: if Q is a quad of $DW(5, q)$ which is ovoidal with respect to H, then $Q \cap H$ is a classical ovoid of Q.*

If $e : DW(5, q) \to \Sigma$ denotes the Grassmann-embedding of $DW(5, q)$ and if Q is a quad of $DW(5, q)$, then the embedding $e_Q : Q \to \langle e(Q) \rangle_\Sigma$ of Q induced by e is isomorphic to the Grassmann-embedding of Q. If H is a hyperplane of $DW(5, q)$ arising from a hyperplane α of Σ, then $H \cap Q = e_Q^{-1}(\langle e(Q) \rangle \cap \alpha \cap e(Q))$. Hence, $Q \cap H$ cannot be a non-classical ovoid of Q. This proves one direction of Proposition 3.1.

Definition. A hyperplane H of $DW(5, q)$ is said to be of *Type (∗)* if $Q \cap H$ is a classical ovoid of Q for every quad Q of $DW(5, q)$ which is ovoidal with respect to H.

In order to prove Proposition 3.1, we need to show that every hyperplane of Type $(*)$ of $DW(5,q)$, $q \geq 3$, arises from the Grassmann-embedding of $DW(5,q)$.

Definitions. (1) By Payne and Thas [24, 2.3.1], every hyperplane of the generalized quadrangle $Q(4,q)$ is either a singular hyperplane, a $(q+1)\times(q+1)$-subgrid or an ovoid. A hyperplane of the generalized quadrangle $Q(4,q)$ is called *classical* if it is a singular hyperplane, a $(q+1)\times(q+1)$-subgrid or a classical ovoid. The classical hyperplanes of $Q(4,q)$ are precisely those hyperplanes of $Q(4,q)$ which arise from the natural embedding of $Q(4,q)$ into $\mathrm{PG}(4,q)$.

(2) A set $\mathcal{H}$ of hyperplanes of a dual polar space Δ is called a *pencil of hyperplanes* if every point of Δ is contained in either 1 or all elements of $\mathcal{H}$. If $\mathcal{H}$ is a pencil of hyperplanes of Δ, then $\bigcup_{H\in\mathcal{H}} H$ coincides with the whole point-set of Δ and $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$ for any three distinct hyperplanes H_1, H_2 and H_3 of $\mathcal{H}$.

Lemma 3.2 *If G_1 and G_2 are two distinct classical hyperplanes of $Q(4,q)$, then through every point $x \in Q(4,q)\setminus(G_1\cup G_2)$, there exists a unique classical hyperplane G_x through x satisfying $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$.*

Proof. Let $Q(4,q)$ be embedded in the projective space $\mathrm{PG}(4,q)$. Let α_i, $i \in \{1,2\}$, be the unique hyperplane of $\mathrm{PG}(4,q)$ such that $G_i = \alpha_i \cap Q(4,q)$. Observe that $< \alpha_1 \cap \alpha_2, x > \cap\ Q(4,q)$ is a classical hyperplane of $Q(4,q)$ satisfying the required properties.

The plane $\alpha_1 \cap \alpha_2$ intersects $Q(4,q)$ in one of the following: (i) a point x; (ii) a line L; (iii) the union of two distinct lines; (iv) a non-degenerate conic. If case (i) occurs, then since $G_1 \cap G_2$ is a hyperplane of both G_1 and G_2 (regarded as point-line geometries), there exists an $i \in \{1,2\}$ such that G_i is a classical ovoid of $Q(4,q)$ containing x and G_{3-i} is either a classical ovoid of $Q(4,q)$ containing x or the singular hyperplane of $Q(4,q)$ with deepest point x. If case (ii) occurs, then since $G_1 \cap G_2$ is a hyperplane of both G_1 and G_2, G_1 and G_2 are necessarily singular hyperplanes of $Q(4,q)$ with deepest points on L. Suppose now that G is a classical hyperplane of $Q(4,q)$ through x satisfying $G_1 \cap G = G_1 \cap G_2 = G_2 \cap G$ and let α denote the unique hyperplane of $\mathrm{PG}(4,q)$ containing G.

If case (iii) or (iv) occurs, then α is necessarily equal to $\langle \alpha_1 \cap \alpha_2, x\rangle$. It follows that $G_x := \langle \alpha_1 \cap \alpha_2, x\rangle \cap Q(4,q)$ is the unique classical hyperplane of $Q(4,q)$ satisfying $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$.

If case (i) occurs, then without loss of generality, we may suppose that G_1 is a classical ovoid of $Q(4,q)$ containing x. Since $G_1 \cap G_2$ is a point, $\alpha_1 \cap \alpha_2$ is the tangent hyperplane at the point $G_1 \cap G_2$ of the elliptic quadric

$\alpha_1 \cap Q(4,q)$ of α_1. Similarly, since $G \cap G_1 = G_1 \cap G_2$ is a point, $\alpha \cap \alpha_1$ must be the tangent hyperplane at the point $G_1 \cap G_2$ of the elliptic quadric $\alpha_1 \cap Q(4,q)$ of α_1. Since $\alpha \cap \alpha_1 = \alpha_1 \cap \alpha_2$, we necessarily have $\alpha = \langle \alpha_1 \cap \alpha_2, x \rangle$. Hence, $G_x := \langle \alpha_1 \cap \alpha_2, x \rangle \cap Q(4,q)$ is the unique classical hyperplane of $Q(4,q)$ satisfying $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$.

If case (ii) occurs with $G_1 \cap G_2 = L$, then G_1 and G_2 must be singular hyperplanes with deepest point on L. Since $G \cap G_1 = G_1 \cap G_2 = L$, also G must be a singular hyperplane with deepest point on L. Since $x \in G$, G necessarily is the singular hyperplane of $Q(4,q)$ with deepest point $\pi_L(x)$. So, also in this case, there exists a unique classical hyperplane G_x in $Q(4,q)$ satisfying $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$. This hyperplane G_x coincides with $\langle \alpha_1 \cap \alpha_2, x \rangle \cap Q(4,q)$. $\square$

Corollary 3.3 *Any two distinct classical hyperplanes of $Q(4,q)$ are contained in a unique pencil of classical hyperplanes of $Q(4,q)$.*

Lemma 3.4 *Let G be a $(q+1) \times (q+1)$-subgrid of $Q(4,q)$ and let x_1, x_2, x_3 be three mutually non-collinear points of G. Then there exists a unique ovoid O in G such that if H is a classical hyperplane of $Q(4,q)$ containing x_1, x_2 and x_3, then $O \subseteq H$.*

Proof. Let $Q(4,q)$ be fully embedded into the projective space $\mathrm{PG}(4,q)$. If x_1, x_2, x_3 lie on a line L of $\mathrm{PG}(4,q)$, then since $|L \cap Q(4,q)| \geq 3$, we must have $L \subseteq Q(4,q)$, contradicting the fact that x_1, x_2, x_3 are three mutually non-collinear points of G. Hence, $\langle x_1, x_2, x_3 \rangle$ is a plane of $\mathrm{PG}(4,q)$ contained in the 3-space $\langle G \rangle$ of $\mathrm{PG}(4,q)$ generated by the points of G. Since $G \cong Q(3,q)$, every plane of $\langle G \rangle$ intersects G in either an ovoid of G or the union of two intersecting lines. Since x_1, x_2, x_3 are mutually non-collinear, $O := \langle x_1, x_2, x_3 \rangle \cap G$ is necessarily an ovoid of G containing x_1, x_2, x_3. Now, if H is a classical hyperplane of $Q(4,q)$ containing x_1, x_2, x_3, then the hyperplane $\langle H \rangle$ of $\mathrm{PG}(4,q)$ contains x_1, x_2, x_3 and hence also $\langle x_1, x_2, x_3 \rangle$. It follows that $O \subseteq H$. $\square$

Definition. Let $W(5,q)$ denote the polar space associated with $DW(5,q)$. The singular subspaces of $W(5,q)$ are the subspaces of $\mathrm{PG}(5,q)$ which are totally isotropic with respect to a given symplectic polarity ζ of $\mathrm{PG}(5,q)$. If L is a line of $\mathrm{PG}(5,q)$ such that $L \cap L^\zeta = \emptyset$, then the set $\mathcal{Q}_L$ of the $q+1$ (mutually disjoint) quads of $DW(5,q)$ which correspond with the points of L satisfy the following property: any line meeting two distinct quads of $\mathcal{Q}_L$ meets every quad of $\mathcal{Q}_L$ in a unique point. Any set of $q+1$ quads which can be obtained in this way will be called a *hyperbolic set of quads* of $DW(5,q)$. Every two disjoint quads Q_1 and Q_2 of $DW(5,q)$ are contained in a unique hyperbolic set of quads of $DW(5,q)$. We will denote this hyperbolic set of quads by $N(Q_1, Q_2)$.

Lemma 3.5 *Let $\{Q_1, Q_2, \dots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ and let H be a hyperplane of $DW(5,q)$ such that $H \cap Q_1$ and $\pi_{Q_1}(H \cap Q_2)$ are distinct hyperplanes of Q_1. Then $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\}$ is a pencil of hyperplanes of Q_1.*

Proof. Put $H_i := \pi_{Q_1}(H \cap Q_i)$, $i \in \{1, \dots, q+1\}$. It suffices to show that every point x of Q_1 is contained in either 1 or all the hyperplanes of the set $\{H_1, H_2, \dots, H_{q+1}\}$. Let L denote the unique line through x meeting $Q_1, Q_2, \dots, Q_{q+1}$. If $L \subseteq H$, then $x \in H_i$ for all $i \in \{1, \dots, q+1\}$. If $|L \cap H| = 1$, then there exists a unique $i^* \in \{1, \dots, q+1\}$ such that $L \cap H \subseteq Q_{i^*}$. Then $x \in H_{i^*}$ and $x \notin H_i$ for all $i \in \{1, \dots, q+1\} \setminus \{i^*\}$. $\square$

Lemma 3.6 *Let $\{Q_1, Q_2, \dots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ and let G_1 be a classical hyperplane of Q_1. Then there exists a subset $X \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = Q_2$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$.*

Proof. Put $X_1 := G_1$, $X_2 := Q_2$, $X_i := \pi_{Q_i}(G_1)$ for every $i \in \{3, \dots, q+1\}$ and $X := X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{q+1}$. Now, let H be a hyperplane of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = Q_2$. Let x be an arbitrary point of Q_i, $i \in \{3, \dots, q+1\}$, and let L denote the unique line through x meeting each Q_i, $i \in \{1, 2, \dots, q+1\}$, in a point. Since H is a subspace and $L \cap Q_2 \subseteq H$, $x \in H$ if and only if $L \cap Q_1 = \{\pi_{Q_1}(x)\} \subseteq H$, i.e. if and only if $x \in X_i$. This proves that $H \cap Q_i = X_i$ for every $i \in \{1, 2, \dots, q+1\}$. Hence, $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$. $\square$

Lemma 3.7 *Let $\{Q_1, Q_2, \dots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$, let G_1 be a classical hyperplane of Q_1 and put $G_2 := \pi_{Q_2}(G_1)$. Then there exist $q-1$ subsets $X_1, X_2, \dots, X_{q-1}$ of $Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = G_2$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) \in \{X_1, X_2, \dots, X_{q-1}\}$.*

Proof. By Lemma 3.6, there exists a subset X_{i-2}, $i \in \{3, \dots, q+1\}$, of $Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_i = Q_i$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X_{i-2}$.

Now, let H be a hyperplane of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = G_2$. Let L denote a line meeting each quad of $\{Q_1, Q_2, \dots, Q_{q+1}\}$ such that $L \cap Q_1$ is not contained in G_1. Then also $L \cap Q_2$ is not contained in G_2. Choose $i \in \{3, \dots, q+1\}$ such that the singleton $L \cap H$ is contained in Q_i. Since H is a subspace, every line meeting G_1 and G_2 is contained in H.

Hence, $\pi_{Q_i}(G_1) \subseteq H$. Since $\pi_{Q_i}(G_1)$ is a maximal subspace of Q_i and $L \cap H \subseteq (H \cap Q_i) \setminus \pi_{Q_i}(G_1)$, $Q_i \subseteq H$. It follows that $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X_{i-2}$. $\square$

Lemma 3.8 *Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$. For every $i \in \{1,2,3\}$, let G_i be a classical hyperplane of Q_i such that G_1, $\pi_{Q_1}(G_2)$ and $\pi_{Q_1}(G_3)$ are three distinct hyperplanes of Q_1 satisfying $\pi_{Q_1}(G_2) \cap G_1 = \pi_{Q_1}(G_3) \cap G_1 = \pi_{Q_1}(G_2) \cap \pi_{Q_1}(G_3)$. Then there exists a subset $X \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$, $H \cap Q_2 = G_2$ and $H \cap Q_3 = G_3$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$.*

Proof. We first prove the following claim.

Claim. *There exists a line $L_1 \subseteq Q_1$ such that (i) $L_1 \cap G_1$ is a singleton, (ii) $\pi_{Q_2}(L_1) \cap G_2$ is a singleton, (iii) the unique points in $L_1 \cap G_1$ and $\pi_{Q_2}(L_1) \cap G_2$ are not collinear.*

Proof. Suppose such a line does not exist.

The union of two hyperplanes of $Q(4,q)$ cannot cover $Q(4,q)$, see e.g. Cardinali, De Bruyn and Pasini [8, Lemma 3.1]. So, $Q_1 \setminus (\pi_{Q_1}(G_2) \cup G_1) \neq \emptyset$.

Let x and y be two distinct collinear points of $Q_1 \setminus G_1$ such that $x \in Q_1 \setminus \pi_{Q_1}(G_2)$. Consider the line $L_1 = xy$. Since L_1 cannot satisfy properties (i), (ii) and (iii) of the Claim, the points in $L_1 \cap G_1$ and $\pi_{Q_2}(L_1) \cap G_2$ are collinear, i.e. $L_1 \cap G_1 = L_1 \cap \pi_{Q_1}(G_2)$. It follows that $y \in Q_1 \setminus \pi_{Q_1}(G_2)$. Since $Q_1 \setminus (\pi_{Q_1}(G_2) \cup G_1) \neq \emptyset$ and $Q_1 \setminus G_1$ is connected (recall that G_1 is a maximal subspace of Q_1), we have $Q_1 \setminus G_1 \subseteq Q_1 \setminus \pi_{Q_1}(G_2)$, i.e. $\pi_{Q_1}(G_2) \subseteq G_1$. Since $\pi_{Q_1}(G_2)$ is a maximal subspace of Q_1, it would then follow that $\pi_{Q_1}(G_2) = G_1$, a contradiction.

Now, let L_1 be a line of Q_1 satisfying the properties (i), (ii) and (iii) of the previous Claim. Put $L_i := \pi_{Q_i}(L_1)$ for every $i \in \{2, \ldots, q+1\}$. Put $L_1 \cap G_1 = \{x_1\}$ and $\pi_{Q_2}(L_1) \cap G_2 = \{x_2\}$. Since $\pi_{Q_1}(G_2) \cap G_1 = \pi_{Q_1}(G_3) \cap G_1 = \pi_{Q_1}(G_2) \cap \pi_{Q_1}(G_3)$, $(\pi_{Q_1}(G_2) \cap L_1) \cap (L_1 \cap G_1) = (\pi_{Q_1}(G_3) \cap L_1) \cap (G_1 \cap L_1) = (\pi_{Q_1}(G_2) \cap L_1) \cap (\pi_{Q_1}(G_3) \cap L_1)$, i.e. $\{\pi_{Q_1}(x_2)\} \cap \{x_1\} = (\pi_{Q_1}(G_3) \cap L_1) \cap \{x_1\} = \{\pi_{Q_1}(x_2)\} \cap (\pi_{Q_1}(G_3) \cap L_1)$. Since x_1 and x_2 are not collinear, $\{\pi_{Q_1}(x_2)\} \cap \{x_1\} = \emptyset$. It follows that $\pi_{Q_1}(G_3) \cap L_1$ is a singleton distinct from $\{\pi_{Q_1}(x_2)\}$ and $\{x_1\}$. Put $L_3 \cap G_3 = \{x_3\}$. Then x_1, x_2 and x_3 are three mutually non-collinear points of the $(q+1) \times (q+1)$-subgrid $G := L_1 \cup L_2 \cup \cdots \cup L_{q+1}$. By Lemma 3.4, there exists a unique ovoid O of G such that if H' is a classical hyperplane of the $Q(4,q)$-quad $\langle G \rangle$ containing x_1, x_2 and x_3, then $O \subseteq H'$. Here, $\langle G \rangle$ denotes the unique $Q(4,q)$-quad of $DW(5,q)$ containing G. Put $G_1' := G_1$, $G_2' := \pi_{Q_1}(G_2)$, $G_3' := \pi_{Q_1}(G_3)$ and

$O \cap Q_i = \{x_i\}$ for every $i \in \{4, \ldots, q+1\}$. Then G'_1, G'_2 and G'_3 are classical hyperplanes of Q_1 satisfying $G'_1 \cap G'_2 = G'_1 \cap G'_3 = G'_2 \cap G'_3$. By Lemma 3.2 and Corollary 3.3, the hyperplanes G'_1, G'_2, G'_3 are contained in a unique pencil $\{G'_1, G'_2, \ldots, G'_{q+1}\}$ of classical hyperplanes of Q_1. Without loss of generality, we may suppose that $\pi_{Q_1}(x_i) \in G'_i$ for every $i \in \{1, 2, \ldots, q+1\}$. Put $X := G_1 \cup G_2 \cup \cdots \cup G_{q+1}$, where $G_i := \pi_{Q_i}(G'_i)$. Notice that $x_i \in G_i$ for every $i \in \{1, 2, \ldots, q+1\}$.

We claim that if H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$, $H \cap Q_2 = G_2$ and $H \cap Q_3 = G_3$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$. So, suppose H is such a hyperplane. Then $x_1, x_2, x_3 \in H$ and $H \cap \langle G \rangle$ is a classical hyperplane of $\langle G \rangle$. It follows that $O = \{x_1, x_2, \ldots, x_{q+1}\} \subseteq H$. Now, by Lemma 3.5, $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\}$ is a pencil of hyperplanes of Q_1 containing $\pi_{Q_1}(H \cap Q_1) = G'_1$, $\pi_{Q_1}(H \cap Q_2) = G'_2$ and $\pi_{Q_1}(H \cap Q_3) = G'_3$. It follows that $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\} = \{G'_1, G'_2, \ldots, G'_{q+1}\}$. Since $x_i \in H \cap Q_i$, we have $H \cap Q_i = G_i = \pi_{Q_i}(G'_i)$ for every $i \in \{1, 2, \ldots, q+1\}$. Hence, $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = (H \cap Q_1) \cup (H \cap Q_2) \cup \cdots \cup (H \cap Q_{q+1}) = G_1 \cup G_2 \cup \cdots \cup G_{q+1} = X$. $\square$

Lemma 3.9 *Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$. Let G_1 be a classical hyperplane of Q_1 and G_2 be a classical hyperplane of Q_2 such that $G_1 \neq \pi_{Q_1}(G_2)$. Then there exist $q-1$ subsets $X_1, X_2, \ldots, X_{q-1}$ of $Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = G_2$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) \in \{X_1, X_2, \ldots, X_{q-1}\}$.*

Proof. Put $G'_1 := G_1$ and $G'_2 := \pi_{Q_1}(G_2)$. Then $G'_1 \neq G'_2$. By Corollary 3.3, G'_1 and G'_2 are contained in a unique pencil $\{G'_1, G'_2, \ldots, G'_{q+1}\}$ of classical hyperplanes of Q_1. For every $i \in \{3, \ldots, q+1\}$, let X_{i-2} denote a subset of $Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that if H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$, $H \cap Q_2 = G_2$ and $H \cap Q_3 = \pi_{Q_3}(G'_i)$, then $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X_{i-2}$ (cf. Lemma 3.8).

Now, suppose H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap Q_1 = G_1$ and $H \cap Q_2 = G_2$. By Lemma 3.5 and the fact that $G_1 \neq \pi_{Q_1}(G_2)$, $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\}$ is a pencil of classical hyperplanes of Q_1. By Corollary 3.3, $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\} = \{G'_1, G'_2, \ldots, G'_{q+1}\}$. Hence, there exists an $i \in \{3, \ldots, q+1\}$ such that $\pi_{Q_1}(H \cap Q_3) = G'_i$, i.e. $H \cap Q_3 = \pi_{Q_3}(G'_i)$. For such an i, we have $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X_{i-2}$. $\square$

Definitions. (1) Let $W(3,q)$ be the symplectic generalized quadrangle whose points and lines are the points and lines of $\mathrm{PG}(3,q)$ which are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(3,q)$. A line of $\mathrm{PG}(3,q)$ which is not totally isotropic with respect to that symplectic polarity is called a *hyperbolic line* of $W(3,q)$. The point-line geometry whose points and lines are the points and hyperbolic lines of $W(3,q)$ (natural incidence) is called the *geometry of the hyperbolic lines of* $W(3,q)$.

(2) Let $\mathcal{N} = \{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$. Let $P_{\mathcal{N}}$ denote the set of all quads of $DW(5,q)$ which meet each quad of $\mathcal{N}$ (in a line). If R_1 and R_2 are two disjoint elements of $P_{\mathcal{N}}$, then $N(R_1, R_2) \subseteq P_{\mathcal{N}}$. Put $L_{\mathcal{N}} := \{N(R_1, R_2) \,|\, R_1, R_2 \in P_{\mathcal{N}} \text{ and } R_1 \cap R_2 = \emptyset\}$ and let $\mathcal{S}_{\mathcal{N}}$ be the point-line geometry with point-set $P_{\mathcal{N}}$, line-set $L_{\mathcal{N}}$ and natural incidence.

Lemma 3.10 *For every hyperbolic set $\mathcal{N}$ of quads of $DW(5,q)$, $\mathcal{S}_{\mathcal{N}}$ is isomorphic to the geometry of the hyperbolic lines of $W(3,q)$.*

Proof. Let Q_1 be an arbitrary element of $\mathcal{N}$ and let θ_1 be an isomorphism between the point-line dual of Q_1 (regarded as generalized quadrangle) and the generalized quadrangle $W(3,q)$. For every element $Q \in P_{\mathcal{N}}$, put $\theta_2(Q) = Q \cap Q_1$. Then for every $Q \in P_{\mathcal{N}}$, $\theta_1 \circ \theta_2(Q)$ is a point of $W(3,q)$. It is straightforward to verify that $\theta_1 \circ \theta_2$ defines an isomorphism between $\mathcal{S}_{\mathcal{N}}$ and the geometry of the hyperbolic lines of $W(3,q)$. $\square$

Lemma 3.11 *If $\mathcal{N}$ is a hyperbolic set of quads of $DW(5,q)$, then $\bigcup_{Q \in P_{\mathcal{N}}} Q$ coincides with the whole point-set of $DW(5,q)$.*

Proof. Let Q_1 be an arbitrary element of $\mathcal{N}$, let x be an arbitrary point of $DW(5,q)$ and let L denote the unique line through $\pi_{Q_1}(x)$ meeting each element of $\mathcal{N}$. Let Q be a quad through x and L (which is unique if $x \notin L$). Then Q intersects each element of $\mathcal{N}$ in a line. Hence, $x \in Q \in P_{\mathcal{N}}$. This proves the lemma. $\square$

Lemma 3.12 *Let $\mathcal{N}$ be a hyperbolic set of quads of $DW(5,q)$, $q \geq 3$. There exists a set X of 4 points of $\mathcal{S}_{\mathcal{N}}$ such that the subspace of $\mathcal{S}_{\mathcal{N}}$ generated by X (i.e. the smallest subspace of $\mathcal{S}_{\mathcal{N}}$ containing X) coincides with the whole point-set of $\mathcal{S}_{\mathcal{N}}$.*

Proof. By Cooperstein [9, Lemma 2.3], this property holds for the geometry of the hyperbolic lines of $W(3,q)$ and hence also for $\mathcal{S}_{\mathcal{N}}$ by Lemma 3.10. $\square$

Lemma 3.13 *Let $\mathcal{N} = \{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$, $q \geq 3$. Let X be a set of points of $Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}$ such that $X \cap Q_1$ is an ovoid of Q_1. Then there are at most q^4 hyperplanes H of Type $(*)$ satisfying $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$.*

Proof. We may suppose that there exists a hyperplane H^* of Type $(*)$ satisfying $H^* \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$.

Claim I. *Let Q be an arbitrary element of $P_{\mathcal{N}}$. Then there exist q subsets $Y_1, Y_2, \ldots, Y_q$ of Q such that if H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$, then $H \cap Q \in \{Y_1, Y_2, \ldots, Y_q\}$.*
Proof. Put $L = Q \cap Q_1$. Since $X \cap Q_1$ is an ovoid of Q_1, $X \cap L$ is a singleton. Clearly, $G := Q \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1})$ is a $(q+1) \times (q+1)$-subgrid of Q containing the line L. The set $X \cap G = H^* \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) \cap Q = H^* \cap G$ is either G or a hyperplane of G. The former case cannot occur since $L \cap H^* = L \cap X$ is a singleton. So, $X \cap G$ is either the union of two intersecting lines of G or an ovoid of G. Now, let e_Q denote the (up to isomorphism) unique embedding of $Q \cong Q(4,q)$ into $\mathrm{PG}(4,q)$. Then $\langle e_Q(G) \rangle$ is 3-dimensional and $\langle e_Q(X \cap G) \rangle = \langle e_Q(H^* \cap G) \rangle$ is 2-dimensional. Suppose now that H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$. Then $H \cap Q$ is either Q or a classical hyperplane of Q. The former case cannot occur since $H \cap G = X \cap G \neq G$. Hence, $\langle e_Q(H \cap Q) \rangle$ is one of the q hyperplanes of $\mathrm{PG}(4,q)$ through $\langle e_Q(X \cap G) \rangle$ distinct from $\langle e_Q(G) \rangle$. So, if $\alpha_1, \alpha_2, \ldots, \alpha_q$ denote the q hyperplanes of $\mathrm{PG}(4,q)$ through $\langle e_Q(X \cap G) \rangle$ distinct from $\langle e_Q(G) \rangle$ and $Y_i := e_Q^{-1}(\alpha_i \cap e_Q(Q))$ for every $i \in \{1, 2, \ldots, q\}$, then $H \cap Q \in \{Y_1, Y_2, \ldots, Y_q\}$.

Claim II. *Let R_1 and R_2 be two distinct elements of $P_{\mathcal{N}}$ and let $R_3 \in N(R_1, R_2) \setminus \{R_1, R_2\}$. If H is a hyperplane of Type $(*)$ of $DW(5,q)$ satisfying $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$, then $H \cap R_3$ is completely determined by the intersections $H \cap R_1$ and $H \cap R_2$.*
Proof. Since $H \cap Q_1 = X \cap Q_1$ is an ovoid of Q_1, $H \cap R_1 \cap Q_1$, $\pi_{R_1}(H \cap R_2 \cap Q_1)$ and $\pi_{R_1}(H \cap R_3 \cap Q_1)$ are mutually distinct points of $Q_1 \cap R_1$. This implies that $\pi_{R_1}(H \cap R_2) \neq H \cap R_1$. By Lemma 3.5, we have $\pi_{R_1}(H \cap R_3) \cap (H \cap R_1) = \pi_{R_1}(H \cap R_2) \cap (H \cap R_1) = \pi_{R_1}(H \cap R_3) \cap \pi_{R_1}(H \cap R_2)$. By Lemma 3.2, there exists a unique classical hyperplane G of R_1 satisfying $\pi_{R_1}(H \cap R_3 \cap Q_1) \subseteq G$ and $G \cap (H \cap R_1) = \pi_{R_1}(H \cap R_2) \cap (H \cap R_1) = G \cap \pi_{R_1}(H \cap R_2)$. Hence, $G = \pi_{R_1}(H \cap R_3)$, i.e. $H \cap R_3 = \pi_{R_3}(G)$. So, the intersection $H \cap R_3$ is completely determined by $H \cap R_1$ and $H \cap R_2$.

The following is an immediate consequence of Claim II and Lemma 3.11.

Corollary. *If $\{R_1, R_2, R_3, R_4\}$ is a generating set of the geometry $\mathcal{S}_{\mathcal{N}}$ (cf. Lemma 3.12), then any hyperplane H of Type $(*)$ of $DW(5,q)$ satisfying $H \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) = X$ is completely determined by $H \cap R_1$, $H \cap R_2$, $H \cap R_3$ and $H \cap R_4$.*

Lemma 3.13 immediately follows from Claim I and the previous corollary. □

The following lemma completes the proof of Proposition 3.1.

Lemma 3.14 *If H is a hyperplane of Type $(*)$ of $DW(5,q)$, $q \geq 3$, then H arises from the Grassmann-embedding of $DW(5,q)$.*

Proof. If H does not admit ovoidal quads, then by De Bruyn and Pralle [15, Proposition 4.2], H is either a singular hyperplane, the extension of a $(q+1) \times (q+1)$-grid in a quad or a so-called *hexagonal hyperplane* (which only exists if q is even). All these hyperplanes arise from the Grassmann-embedding of $DW(5,q)$, see De Bruyn [11], [14] and Shult and Thas [29]. In the sequel, we therefore suppose that there exists a quad Q which is ovoidal with respect to H. Put $O := Q \cap H$. Let $e : DW(5,q) \to \Sigma$ denote the Grassmann-embedding of $DW(5,q)$. Then $\dim(\langle e(O) \rangle) = 3$, $\dim(\langle e(Q) \rangle) = 4$ and $\dim(\Sigma) = 13$. The number of hyperplanes of Σ containing $\langle e(O) \rangle$ but not $\langle e(Q) \rangle$ is equal to q^9. Hence, there are q^9 hyperplanes of $DW(5,q)$ which arise from e and which intersect Q in O. All these hyperplanes are of Type $(*)$. We will now show that there are at most q^9 hyperplanes of Type $(*)$ which intersect Q in O. From this it immediately follows that the hyperplane H arises from the Grassmann-embedding e.

Let Q' be a quad disjoint from Q. By Lemmas 3.6 and 3.13, there are at most q^4 hyperplanes H' of Type $(*)$ of $DW(5,q)$ which satisfy $H' \cap Q = O$ and $H' \cap Q' = Q'$. Now, there are $\frac{q^5-1}{q-1}$ classical hyperplanes in Q'. If G' is one of these classical hyperplanes of Q', then by Lemmas 3.7, 3.9 and 3.13, there are at most $(q-1)q^4$ hyperplanes H' of Type $(*)$ of $DW(5,q)$ which satisfy $H' \cap Q' = G'$ and $H' \cap Q = O$. Since every hyperplane of Type $(*)$ of $DW(5,q)$ intersects Q' in either Q' or a classical hyperplane of Q', there are at most $q^4 + \frac{q^5-1}{q-1} \cdot (q-1)q^4 = q^9$ hyperplanes of Type $(*)$ of $DW(5,q)$ which intersect Q in O. This is precisely what we needed to show. □

4 Proof of the Main Theorem: the general case

The following proposition is the special case $n_0 = 3$ of Corollary 1.5 of Cardinali, De Bruyn and Pasini [8].

Proposition 4.1 *For every integer $n \geq 3$, let $\mathbf{D}_n$ be a class of thick dual polar spaces of rank n. For every $\Delta \in \mathbf{D} := \bigcup_{n=3}^{\infty} \mathbf{D}_n$, let $\mathcal{H}(\Delta)$ be a class of hyperplanes of Δ. We assume that every $\Delta \in \mathbf{D}$ is embeddable and we* *denote by e_Δ the absolutely universal embedding of Δ. Assume that for every $\Delta \in \mathbf{D}_3$, it holds that every $H \in \mathcal{H}(\Delta)$ arises from e_Δ. If, moreover, for $n > 3$ and $\Delta \in \mathbf{D}_n$ (i) any max of Δ belongs to $\mathbf{D}_{n-1}$, (ii) for any max A of Δ and every hyperplane H of $\mathcal{H}(\Delta)$, we either have $A \subseteq H$ or $H \cap A \in \mathcal{H}(A)$, then H arises from e_Δ, for every $\Delta \in \mathbf{D}$ and every $H \in \mathcal{H}(\Delta)$.*

We will now apply Proposition 4.1 to prove the Main Theorem. For every $n \geq 3$, let $\mathbf{D}_n$ denote the set of all dual polar spaces which are isomorphic to $DW(2n-1, q)$ for some prime power $q \geq 3$. For every $\Delta \in \mathbf{D} := \bigcup_{n=3}^{\infty} \mathbf{D}_n$, let $\mathcal{H}(\Delta)$ denote the class of all hyperplanes of Type $(*)$ of Δ. Recall that the absolutely universal embedding e_Δ of an element $\Delta \in \mathbf{D}$ is isomorphic to the Grassmann-embedding of Δ. By Proposition 3.1, H arises from e_Δ for every $\Delta \in \mathbf{D}_3$ and every $H \in \mathcal{H}(\Delta)$. Clearly, also conditions (i) and (ii) of Proposition 4.1 are satisfied. We conclude that every hyperplane H of $\mathcal{H}(\Delta)$, where Δ is an arbitrary element of $\mathbf{D}$, arises from the Grassmann-embedding of Δ.

Conversely, every hyperplane of the dual polar space $\Delta = DW(2n-1, q)$, $n \geq 2$ and $q \neq 2$, which arises from the Grassmann-embedding of Δ belongs to $\mathcal{H}(\Delta)$.

Acknowledgment

At the moment of the writing of this paper, the author was a Postdoctoral Fellow of the Research Foundation - Flanders (Belgium).

References

[1] S. Ball, P. Govaerts and L. Storme. On ovoids of parabolic quadrics. *Des. Codes Cryptogr.* 38 (2006), 131–145.

[2] A. Barlotti. Un'estensione del teorema di Segre-Kustaanheimo. *Boll. Un. Mat. Ital.* 10 (1955), 96–98.

[3] A. Blokhuis and A. E. Brouwer. The universal embedding dimension of the binary symplectic dual polar space. *Discrete Math.* 264 (2003), 3–11.

[4] N. Bourbaki. Lie groups and Lie algebras. Chapters 7–9. Springer-Verlag, Berlin, 2005.

[5] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[6] P. J. Cameron. Ovoids in infinite incidence structures. *Arch. Math.* 62 (1994), 189–192.

[7] I. Cardinali and B. De Bruyn. The structure of full polarized embeddings of symplectic and Hermitian dual polar spaces. *Adv. Geom.* 8 (2008), 111–137.

[8] I. Cardinali, B. De Bruyn and A. Pasini. On the simple connectedness of hyperplane complements in dual polar spaces. *Discrete Math.* 309 (2009), 294–303.

[9] B. N. Cooperstein. On the generation of dual polar spaces of symplectic type over finite fields. *J. Combin. Theory Ser. A* 83 (1998), 221–232.

[10] B. N. Cooperstein and B. De Bruyn. Points and hyperplanes of the universal embedding space of the dual polar space $DW(5,q)$, q odd. *Michigan Math. J.* 58 (2009), 195–212.

[11] B. De Bruyn. The hyperplanes of $DQ(2n,\mathbb{K})$ and $DQ^-(2n+1,q)$ which arise from their spin-embeddings. *J. Combin. Theory Ser. A* 114 (2007), 681–691.

[12] B. De Bruyn. A decomposition of the natural embedding spaces for the symplectic dual polar spaces. *Linear Algebra Appl.* 426 (2007), 462–477.

[13] B. De Bruyn. The hyperplanes of $DW(5,2^h)$ arising from embedding. *Discrete Math.* 309 (2009), 304–321.

[14] B. De Bruyn. On a class of hyperplanes of the symplectic and Hermitian dual polar spaces. *Electron. J. Combin.* 16 (2009), Research paper 1, 20pp.

[15] B. De Bruyn and H. Pralle. The hyperplanes of $DW(5,q)$ with no ovoidal quad. *Glasg. Math. J.* 48 (2006), 75–82.

[16] B. De Bruyn and H. Pralle. The hyperplanes of $DH(5,q^2)$. *Forum Math.* 20 (2008), 239–264.

[17] B. De Bruyn and P. Vandecasteele. Valuations and hyperplanes of dual polar spaces. *J. Combin. Theory Ser. A* 112 (2005), 194–211.

[18] W. M. Kantor. Ovoids and translation planes. *Canad. J. Math.* 34 (1982), 1195–1207.

[19] A. Kasikova and E. E. Shult. Absolute embeddings of point-line geometries. *J. Algebra* 238 (2001), 265-291.

[20] P. Li. On the universal embedding of the $Sp_{2n}(2)$ dual polar space. *J. Combin. Theory Ser. A* 94 (2001), 100–117.

[21] C. M. O'Keefe and T. Penttila. Ovoids of $\mathrm{PG}(3, 16)$ are elliptic quadrics. *J. Geom.* 38 (1990), 95–106.

[22] C. M. O'Keefe and T. Penttila. Ovoids of $\mathrm{PG}(3, 16)$ are elliptic quadrics, II. *J. Geom.* 44 (1992), 140–159.

[23] G. Panella. Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito. *Boll. Un. Mat. Ital.* 10 (1955), 507–513.

[24] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Research Notes in Mathematics 110. Pitman, Boston, 1984.

[25] T. Penttila and B. Williams. Ovoids of parabolic spaces. *Geom. Dedicata* 82 (2000), 1–19.

[26] H. Pralle. The hyperplanes of $DW(5, 2)$. *Experiment. Math.* 14 (2005), 373–384.

[27] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179-185.

[28] E. E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[29] E. E. Shult and J. A. Thas. Hyperplanes of dual polar spaces and the spin module. *Arch Math.* 59 (1992), 610–623.

[30] J. A. Thas and S. E. Payne. Spreads and ovoids in finite generalized quadrangles. *Geom. Dedicata* 52 (1994), 227–253.

[31] J. Tits. Ovoïdes et groupes de Suzuki. *Arch. Math.* 13 (1962), 187–198.

[32] J. Tits. *Buildings of Spherical Type and Finite BN-pairs.* Lecture Notes in Mathematics 386. Springer, Berlin, 1974.