Characterizations of Veronese and Segre Varieties

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Dedicated to the memory of A. Barlotti

Abstract. We survey the most important characterizations of quadric Veroneseans and Segre varieties of the last thirty years, including some very recent results.

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1. Introduction

Quadric Veronese varieties and Segre varieties are classical varieties that become very important combinatorial objects when defined over a finite field. They are connected with many other geometric objects and in order to easily recognize these structures, it is important to have a good set of characterizations for them. In this paper, we survey the most important ones. Our motivation is a recent common characterization of Veronese and Segre varieties, which is surprising since these varieties have different behaviour: on a Veronese variety no three points are collinear, whereas a Segre variety can contain subspaces of large dimension.

1.1. Quadric Veroneseans

The *Veronese variety* $\mathcal{V}$ of all quadrics of $\mathsf{PG}(n,K)$, $n \geq 1$, and K any commutative field is the set of points

$$\begin{aligned}\mathcal{V} \;=\; &\{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2,\\ &\ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n) \,\|\\ &(x_0, x_1, \ldots, x_n) \text{ is a point of } \mathrm{PG}(n,K)\}\end{aligned}$$

in $\mathsf{PG}(N_n,K)$ with $N_n = n(n+3)/2$. This set is in fact an algebraic variety and has, as a variety, dimension n. It is sometimes called the *Veronesean of quadrics* of $\mathsf{PG}(n,K)$, or the *quadric Veronesean* of $\mathsf{PG}(n,K)$. The variety $\mathcal{V}$ is *absolutely irreducible* and *non-singular*. It has *order* 2^n.

Let $\zeta : \mathsf{PG}(n,K) \to \mathsf{PG}(N_n,K)$,

$$(x_0,x_1,\ldots,x_n) \mapsto (y_{00},y_{11},\ldots,y_{n-1,n}),$$

with $y_{ij} = x_ix_j$. Then ζ is a bijection of $\mathsf{PG}(n,K)$ onto $\mathcal{V}$. Hence $\mathcal{V}$ is *rational.* Quadrics of $\mathsf{PG}(n,K)$ are mapped by ζ onto all hyperplane sections of $\mathcal{V}$. Other notations for $\mathcal{V}$ are $\mathcal{V}_n$ or $\mathcal{V}_n^{2^n}$.

Some examples and special cases. For $n = 1$, $\mathcal{V}_1^2$ is a conic of $\mathsf{PG}(2,K)$. If $n = 2$, then $\mathcal{V}_2^4$ is a surface of order 4 in $\mathsf{PG}(5,K)$. If $n = 3$, then $\mathcal{V}_3^8$ is a variety of dimension 3 and order 8 in $\mathsf{PG}(9,K)$.
Let π_s be any s-dimensional subspace of $\mathsf{PG}(n,K)$. Then π_s^ζ is a quadric Veronesean $\mathcal{V}_s$. Conversely, if $(|K|,s) \neq (2,1)$ and $\mathcal{V}_s$ is a quadric Veronesean on $\mathcal{V}_n$, then

$$\mathcal{V}_s = \pi_s^\zeta \text{ for some subspace } \pi_s$$

Corollary: if $K \neq \mathsf{GF}(2)$, then any two points of $\mathcal{V}_n$ are on a unique conic of $\mathcal{V}_n$. The planes generated by these conics will be called *conic planes* (for any field K, a *conic plane* is the plane generated by the image of a line),

Below, we will assume K =GF(q). Some results, however, also hold in the infinite case, and we will mention this at the appropriate places.

The nucleus subspace. Since a Veronese variety contains a lot of conics, it is reasonable to wonder, in case q is even, what the structure is of the set of all nuclei of these conics.
To that end, consider $\mathcal{V}_n$, with q even. Let $\mathcal{G}_n$ be the subset of $\mathsf{PG}(N_n,q)$ consisting of all nuclei of the conics on $\mathcal{V}_n$ (for $q = 2$, consider all conics corresponding to lines of $\mathsf{PG}(n,2)$).

Theorem 1.1 ((e.g) Thas and Van Maldeghem [9]**).** *The set $\mathcal{G}_n$ is the Grassmannian of the lines of $\mathsf{PG}(n,q)$, hence generates a subspace of dimension $(n-1)(n+2)/2$ of $\mathsf{PG}(N_n,q)$. This space Φ is called the nucleus subspace of $\mathcal{V}_n$ and is the intersection of all hyperplanes of $\mathsf{PG}(N_n,q)$ which intersect $\mathcal{V}_n$ in a $\mathcal{V}_{n-1} = \pi_{n-1}^\zeta$, π_{n-1} a hyperplane of $\mathsf{PG}(n,q)$.*

1.2. Segre pairs and Segre varieties

Let Σ and $\widetilde{\Sigma}$ be two families of subspaces of $\mathsf{PG}(d,q)$ and assume that the elements of Σ have dimension m and the elements of $\widetilde{\Sigma}$ have dimension n. Suppose also that

$$|\Sigma| = q^n + q^{n-1} + \cdots q + 1$$
$$|\widetilde{\Sigma}| = q^m + q^{m-1} + \cdots q + 1$$

and that each member of Σ intersects each member of $\widetilde{\Sigma}$ in precisely one point. Moreover, we assume that distinct members of Σ and distinct members of $\widetilde{\Sigma}$ are disjoint. Finally, assume that both families generate $\mathsf{PG}(d,q)$.
Then $(\Sigma,\widetilde{\Sigma})$ is called a *Segre pair* with parameters $(q;m,n,d)$. In the infinite case one deletes the condition on the numbers and requires that both families of subspaces cover the same points. The parameters are then $(K;m,n,d)$, where K is the coordinatizing field of the underlying projective space.

The *Segre variety* of the spaces $\mathsf{PG}(n,K)$ and $\mathsf{PG}(m,K)$, $n \geq 1, m \geq 1, K$ any commutative field is the set of points

$$\begin{aligned}\mathcal{S}_{m;n} = \mathcal{S}_{n;m} = \{&(x_0y_0,\cdots,x_0y_m,x_1y_0,\ldots,x_1y_m,\\ &\ldots,x_ny_0,\ldots,x_ny_m)\|(x_0,x_1,\ldots,x_n) \text{ is a point}\\ &\text{of } \mathsf{PG}(n,K),(y_0,y_1,\ldots,y_m) \text{ is a point of } \mathsf{PG}(m,K)\}\end{aligned}$$

in $\mathsf{PG}(mn+m+n,K)$.
As an algebraic variety, $\mathcal{S}_{m;n}$ is *absolutely irreducible* and *non-singular*, with *order* $\frac{(m+n)!}{m!n!}$.
Fix a point $(x_0,x_1,\cdots,x_n) \in \mathsf{PG}(n,K)$ and fix a point $(y_0,y_1,\cdots,y_m) \in \mathsf{PG}(m,K)$. By varying the point of $\mathsf{PG}(m,K)$, we obtain an m-dimensional space on $\mathcal{S}_{m;n}$ through $(x_0y_0,,\cdots,x_0y_m,x_1y_0,\cdots,x_1y_m,\cdots,x_ny_0,\cdots,x_ny_m)$. We denote the family of subspaces thus obtained by Σ. Similarly we also obtain an n-dimensional space on $\mathcal{S}_{m;n}$ containing that point. All subspaces obtained like that are put in the set $\widetilde{\Sigma}$. Hence $(\Sigma,\widetilde{\Sigma})$ is a Segre pair.
The elements of $\Sigma \cup \widetilde{\Sigma}$ are called the *generators* of the Segre variety.

Some examples and special cases. If $n=m=1$, then $\mathcal{S}_{1;1}$ is a hyperbolic quadric in $\mathsf{PG}(3,K)$. If $n=1$, $m=2$, then $\mathcal{S}_{1;2}$ is a variety in $\mathsf{PG}(5,K)$ with order 3. If $n=m=2$, then $\mathcal{S}_{2;2}$ is a variety in $\mathsf{PG}(8,K)$ of order 6.

2. Characterizations using tangent spaces

2.1. Quadric Veroneseans

In the complex case, the Veronese surface $\mathcal{V}_2^4$ has a unique tangent plane at every of its points. The Veronesean $\mathcal{V}_2^4$ or one of its projections is characterized by simply requiring that it contains ∞^2 many irreducible conics, see e.g. [3] for references. In order to mimic this characterization for finite fields of odd characteristic, and generalize this to higher dimensional Veroneseans, Mazzocca and Melone introduced three axioms, which we will reproduce now in a slightly more general form.
Let $X \subseteq \Pi = \mathsf{PG}(M,q), M > 2$, with $\langle X\rangle = \Pi$. Let $\mathcal{P}$ be a collection of planes of Π such that for every $\pi \in \mathcal{P}$ the set $X \cap \pi$ is an oval (that is, a set of points no three of which are collinear and admitting at each point exactly one tangent line) in π. Consider the following axioms.

(Q1) Every pair of points $x,y \in X$, $x \neq y$, lies in a unique member of $\mathcal{P}$, denoted $[x,y]$;
(Q2) for every pair of planes $\pi_1,\pi_2 \in \mathcal{P}$, we have $\pi_1 \cap \pi_2 \subseteq X$;
(Q3) For every point $x \in X$ and every plane $\pi \in \mathcal{P}$ such that $x \notin \pi$, the tangent lines at x to the ovals $[x,y] \cap X$, $y \in X \cap \pi$, are contained in some plane of Π.

Then we call X a *Veronesean cap of index* n. This definition is easily extended to the infinite case, replacing $\mathsf{GF}(q)$ by any skewfield K.
Historically, the following results have been proven.

Theorem 2.1 (Mazzocca and Melone [4]**).** *If q is odd, then a Veronesean cap, together with its conics, forms the point-line structure of a projective space* $\mathsf{PG}(n,q)$*, and it is a quadric Veronesean.*

It should be noted that Hirschfeld and Thas [3] pointed out that Mazzocca and Melone forgot to explicitly assume $M \geq n(n+3)/2$. They used this assumption in their proof. Moreover, Hirschfeld and Thas generalized the result to the even case as follows.

Theorem 2.2 (Hirschfeld and Thas [3]**).** *Let X be a finite Veronesean cap in* $\mathsf{PG}(M,q)$ *such that all ovals $X \cap \pi$, with $\pi \in \mathcal{P}$, are conics. Then X, endowed with all its conics (for $q = 2$ only those conics that are in some member of $\mathcal{P}$) forms the point-line structure of a projective space* $\mathsf{PG}(n,q)$*, and if $M \geq n(n+3)/2$, then it is a quadric Veronesean.*

Then, Thas and Van Maldeghem [8] gave a complete classification, deleting the hypothesis that the ovals must be conics, and removing the bound on M. Moreover, their proof is independent and shorter than the previous ones.

Theorem 2.3 (Thas and Van Maldeghem [8]**).** *Let X be a Veronesean cap in* $\mathsf{PG}(M,q)$*, $M > 2$, with set of planes $\mathcal{P}$, as above. Then X together with the ovals $\pi \cap X$, $\pi \in \mathcal{P}$, is a* $\mathsf{PG}(n,q)$*. Also, there exists a* $\mathsf{PG}(n(n+3)/2,q)$ *containing* $\mathsf{PG}(M,q)$*, a subspace Γ of* $\mathsf{PG}(n(n+3)/2,q)$ *skew to* $\mathsf{PG}(M,q)$*, and a Veronesean $\mathcal{V}_n$ in* $\mathsf{PG}(n(n+3)/2,q)$ *with $\Gamma \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from Γ onto* $\mathsf{PG}(M,q)$*. If $M \geq n(n+3)/2$, this means that $M = n(n+3)/2$, $\Gamma = \emptyset$ and $X = \mathcal{V}_n$.*

Ferrara Dentice and Marino claim in [1] to have generalized the above theorem. However, they state as a fact that (Q3) implies that the tangent lines in that axiom fill a plane. Using this, they can follow very closely the proof of Thas and Van Maldeghem [8], essentially noting that everything remains valid in the infinite case. In particular, the fact that X endowed with the ovals $\pi \cap X$, is the point-line geometry of a projective space, is a direct consequence of the assumption that in (Q3) the tangent lines fill the whole plane.
Recently, Schillewaert and Van Maldeghem proved in a direct way that, in the infinite case, X endowed with the ovals $\pi \cap X$, is the point-line geometry of a projective space, thus extending the result verbatim to the infinite case.

Theorem 2.4 (Schillewaert and Van Maldeghem (unpublished)). *Let X be a Veronesean cap in* $\mathsf{PG}(M,K)$*, $M > 2$, K any skewfield, and with set of planes $\mathcal{P}$, as above. Then K is a field, and X together with the ovals $\pi \cap X$, $\pi \in \mathcal{P}$, is a* $\mathsf{PG}(n,K)$*. Also, there exists a* $\mathsf{PG}(n(n+3)/2,K)$ *containing* $\mathsf{PG}(M,K)$*, a subspace Γ of* $\mathsf{PG}(n(n+3)/2,K)$ *skew to* $\mathsf{PG}(M,K)$*, and a Veronesean $\mathcal{V}_n$ in* $\mathsf{PG}(n(n+3)/2,K)$ *with $\Gamma \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from Γ onto* $\mathsf{PG}(M,K)$*. If $M \geq n(n+3)/2$, this means that $M = n(n+3)/2$, $\Gamma = \emptyset$ and $X = \mathcal{V}_n$.*

Hence the problem is solved completely.

2.2. Segre varieties

A very similar characterization of Segre varieties is due to recent work of Thas and Van Maldeghem.
Let H be a hyperbolic quadric in a 3-dimensional projective space Σ. For every point $x \in H$, the unique plane π through x intersecting H in two intersecting lines contains all lines through x that meet H in only x. The plane π is called the *tangent plane* at x to H and denoted $T_x(H)$.
Let X be a spanning point set of $\mathrm{PG}(N,K)$, with K any field and $N \in \mathbb{N}$, and let Ξ be a nonempty collection of 3-dimensional projective subspaces of $\mathrm{PG}(N,K)$, called the *hyperbolic spaces* of X, such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is a hyperbolic quadric $X(\xi)$ in ξ; for $x \in X(\xi)$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$. We say that two points of X are *collinear in* X if all points of the joining line belong to X. A *maximal subspace* π of X is a subspace of $\mathsf{PG}(N,K)$ on X not contained in a subspace $\pi' \supseteq \pi$, $\pi' \neq \pi$, on X. We call X a *Segre geometry* if the following properties hold:

(S1) Any two points x and y lie in an element of Ξ, denoted by $[x,y]$ if it is unique.

(S2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subseteq X$.

From (S2) it follows that the element of Ξ in (S1) is unique as soon as x and y are not collinear in X.

(S3) If $x \in X$ and L is a line entirely contained in X such that no point on L is collinear in X with x, then each of the planes $T_x([x,y])$, $y \in L$, is contained in a common 3-dimensional subspace of $\mathrm{PG}(N,\mathbb{K})$, denoted by $T(x,L)$.

Now we have the following result.

Theorem 2.5 (Thas and Van Maldeghem [14]**).** *Let X be a Segre geometry in $\mathsf{PG}(N,K)$, $N > 2$, K any field. Then the set of maximal subspaces on X can be partitioned into two families defining a Segre pair, say with parameters $(K;m,n,N)$. Also, there exists a $\mathsf{PG}(nm+n+m,K)$ containing $\mathsf{PG}(N,K)$, a subspace Γ of $\mathsf{PG}(nm+n+m,K)$ skew to $\mathsf{PG}(N,K)$, and a Segre variety $\mathcal{S}_{m;n}$ in $\mathsf{PG}(nm+n+m,K)$ with $\Gamma \cap \mathcal{S}_{m;n} = \emptyset$, such that X is the (bijective) projection of $\mathcal{S}_{m;n}$ from Γ onto $\mathsf{PG}(N,K)$. If $N \geq nm+n+m$, this means that $M = nm+n+m$, $\Gamma = \emptyset$ and $X = \mathcal{S}_{m;n}$.*

3. Characterizations using subvarieties

Here, the central problem is the following. Recall that $\mathsf{PG}(n,q)$ gives rise to the quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathsf{PG}(N_n,q)$, $N_n = n(n+3)/2$. Every hyperplane of $\mathsf{PG}(n,q)$ gives rise to a subveronesean $\mathcal{V}_{n-1}^{2^{n-1}}$ in a subspace of dimension $N_{n-1} = (n-1)(n+2)/2$ of $\mathsf{PG}(N_n,q)$. Such a subspace is called a $\mathcal{V}_{n-1}$-*subspace* of $\mathcal{V}_n^{2^n}$ or of $\mathsf{PG}(N_n,q)$. Let $\mathfrak{S}$ be a set of subspaces of dimension N_{n-1}. Under which conditions is $\mathfrak{S}$ the set of all $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n$?

The following properties of the set $\mathfrak{S}_n$ of $\mathcal{V}_{n-1}$-subspaces of $\mathcal{V}_n$ are worth noting.

(VS1) Every two members of $\mathfrak{S}_n$ generate a hyperplane of $\mathsf{PG}(N_n, q)$.
(VS2) Every three members of $\mathfrak{S}_n$ generate $\mathsf{PG}(N_n, q)$.
(VS3) No point is contained in every member of $\mathfrak{S}_n$.
(VS4) The intersection of any nonempty collection of members of $\mathfrak{S}_n$ is a subspace of dimension $N_i = i(i+3)/2$, $i \in \{-1, 0, 1, \ldots, n-1\}$.
(VS5) If q is even, then there exist distinct members S, S', S'' of $\mathfrak{S}_n$ with $S \cap S' = S' \cap S'' = S'' \cap S$.

Theorem 3.1 (Tallini [7]**).** *With the above set-up, and in case $n = 2$, if q odd, then a set $\mathfrak{S}$ of $q^2 + q + 1$ planes satisfying* (VS1), (VS2) *and* (VS3) *is the set $\mathfrak{S}_1$ of conic planes of $\mathcal{V}_2^4$.*

The general problem was tackled by Thas and Van Maldeghem [9].

Theorem 3.2 (Thas and Van Maldeghem [9]**).** *Let $\mathfrak{S}$ be a collection of $q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $(n-1)(n+2)/2$ of $\mathsf{PG}(n(n+3)/2, q)$, $n \geq 2$, satisfying* (VS1) *up to* (VS5). *Then either $\mathfrak{S}$ is the set of $\mathcal{V}_{n-1}$-subspaces of a $\mathcal{V}_n$, or q is even and $\mathfrak{S} = (\mathfrak{S}_n \cup \{\Phi\}) \setminus \{S\}$, with Φ the nucleus subspace of $\mathcal{V}_n$, $\mathfrak{S}_n$ the set of $\mathcal{V}_{n-1}$-subspaces of $\mathcal{V}_n$ and $S \in \mathfrak{S}_n$.*
If $n = 2$, then the statement holds if $\mathfrak{S}$ satisfies (VS1),(VS2),(VS3) and (VS5).

Theorem 3.3 (Thas and Van Maldeghem [9]**).** *Let $\mathfrak{S}$ be a collection of $q^n + q^{n-1} + \cdots + q + 1$ subspaces of dimension $(n-1)(n+2)/2$ of $\mathsf{PG}(n(n+3)/2, q)$, $n \geq 2$, satisfying* (VS1),(VS2),(VS3). *If $q \geq n$, then $\mathfrak{S}$ satisfies* (VS4).

Remarks.

1. In [9], the authors also consider the case q even, $|\mathcal{S}| = q^n + q^{n-1} + \cdots + q + 1$ and $\mathfrak{S}$ satisfies (VS1) up to (VS4), or (VS1) up to (VS3) with $q \geq n$, leaving out (VS5).
2. In [9], also the case $|\mathcal{S}| = q^n + q^{n-1} + \cdots + q + 2$ is handled (then q is automatically even).
3. Finally, in [9], a slight generalization of Theorem 3.3 is proved, with assumptions $|\mathfrak{S}| \geq q^n + q^{n-1} + \cdots + q + 1$, the elements of $\mathfrak{S}$ generate $\mathsf{PG}(m, q)$, $m \geq n(n+3)/2$, and all elements of $\mathfrak{S}$ have dimension $m - n - 1$.

No similar characterizations of Segre varieties are known.

4. Characterizations by intersection numbers

Also here, only characterizations of (finite) Veroneseans are known.
The Veronesean $\mathcal{V}_2^4$ is a cap $\mathcal{K}$ (that is, a set of points no three of which are collinear) in $\mathsf{PG}(5, q)$ which satisfies :

(VC1) For every hyperplane π of $\mathsf{PG}(5,q)$, we have

$$|\pi \cap \mathcal{K}| = 1, q+1 \text{ or } 2q+1,$$

and there exists some hyperplane π such that $|\pi \cap \mathcal{K}| = 2q+1$.

(VC2) Any plane of $\mathsf{PG}(5,q)$ with four points in $\mathcal{K}$ has at least $q+1$ points in $\mathcal{K}$.

Theorem 4.1 (Ferri [2]**).** *If $\mathcal{K}$ is a cap in $\mathsf{PG}(5,q)$, q odd, $q \neq 3$, satisfying* (VC1), (VC2)*, then it is isomorphic to $\mathcal{V}_2^4$.*

Theorem 4.2 (Hirschfeld and Thas [3]**).** *If $\mathcal{K}$ is a cap in $\mathsf{PG}(5,3)$ satisfying* (VC1), (VC2)*, then it is isomorphic to $\mathcal{V}_2^4$.*

Theorem 4.3 (Thas and Van Maldeghem [10]**).** *If $\mathcal{K}$ is a cap in $\mathsf{PG}(5,q)$, q even, satisfying* (VC1), (VC2)*, then either it is isomorphic to $\mathcal{V}_2^4$, or $q=2$ and $\mathcal{K}$ is an elliptic quadric in some subspace $\mathsf{PG}(3,2)$.*

A generalization to higher order Veroneseans has been found by Schillewaert, Thas and Van Maldeghem [6].

Theorem 4.4 (Schillewaert, Thas and Van Maldeghem [6]**).** *A set $\mathcal{K}$ of $q^n + q^{n-1} + \cdots + q + 1$ points generating $\mathsf{PG}(N_n,q)$, $N_n = n(n+3)/2$, $q \geq 5$, $n \geq 2$, is a $\mathcal{V}_n^{2^n}$ if and only if the following conditions* (*i*), (*ii*) *and* (*iii*) *hold.*

- (*i*) *If a plane of $\mathsf{PG}(N_n,q)$ intersects $\mathcal{K}$ in more than three points, then it contains exactly $q+1$ points of $\mathcal{K}$. Any two distinct points are contained in plane containing $q+1$ points of $\mathcal{K}$.*
- (*ii*) *If a solid Π_3 of $\mathsf{PG}(N_n,q)$ intersects $\mathcal{K}$ in more than four points, then there are four points of $\Pi_3 \cap \mathcal{K}$ contained in a plane of Π_3. By* (*i*)*, this implies that if $|\Pi_3 \cap \mathcal{K}| > 4$, then $|\Pi_3 \cap \mathcal{K}| \geq q+1$.*
- (*iii*) *If a 5-dimensional subspace Π_5 of $\mathsf{PG}(N_n,q)$ intersects $\mathcal{K}$ in more than $2q+2$ points, then it intersects $\mathcal{K}$ in exactly q^2+q+1 points.*

Remark. There are counterexamples to the previous theorem for $q = 2, 3$.
A *k-arc*, $k \in \mathbb{N}$, $k \geq 4$, of a 3-dimensional projective space is a set of k points no four of which are coplanar.

Theorem 4.5 (Schillewaert, Thas and Van Maldeghem [6]**).** *A set $\mathcal{K}$ of $q^n + q^{n-1} + \cdots + q + 1$ points generating $\mathsf{PG}(N_n,q)$, $N_n = n(n+3)/2$, $q \geq 5$, $n > 2$, is a $\mathcal{V}_n^{2^n}$ if and only if the following conditions* (*i*), (*ii*) *and* (*iii*) *hold.*

- (*i*) *For any plane π of $\mathsf{PG}(N_n,q)$, the intersection $\pi \cap \mathcal{K}$ contains at most $q+1$ points of $\mathcal{K}$.*
- (*ii*) *If a solid Π_3 of $\mathsf{PG}(N_n,q)$ intersects $\mathcal{K}$ in more than four points, then $|\Pi_3 \cap \mathcal{K}| \geq q+1$ and $\Pi_3 \cap \mathcal{K}$ is not a $(q+1)$-arc.*
- (*iii*) *If a 5-dimensional subspace Π_5 of $\mathsf{PG}(N_n,q)$ intersects $\mathcal{K}$ in more than $2q+2$ points, then it intersects $\mathcal{K}$ in exactly q^2+q+1 points; furthermore, any two distinct points of $\mathcal{K}$ are contained in a 5-dimensional subspace of $\mathsf{PG}(N_n,q)$ containing q^2+q+1 points of $\mathcal{K}$.*

Remark. For $n = 2$, any q, there are counterexamples to the previous theorem.

5. Characterizations as incidence structures

Here, there are many recent results. We start with an older result, though, on Veroneseans.

5.1. Quadric Veroneseans

Theorem 5.1 (Thas and Van Maldeghem [9]**).** *Let X be a set of points in $\mathsf{PG}(m,q)$, $m \geq n(n+3)/2$, $n \geq 2$, $q > 2$, spanning $\mathsf{PG}(m,q)$, let $\mathcal{O}$ be a set of ovals on X, and assume that $(X,\mathcal{O})$ is the design of points and lines of a projective space of dimension n. Then $m = n(n+3)/2$, X is a $\mathcal{V}_n$, and $\mathcal{O}$ is the set of all conics on $\mathcal{V}_n$.*

Remark. The condition $q > 2$ is necessary since any spanning set of $2^{n+1} - 1$ points of $\mathsf{PG}(2^{n+1} - 2, 2)$ can be given the structure of $\mathsf{PG}(n, 2)$ by selecting appropriate triples of points, which automatically form plane ovals.

This theorem has been generalized to the infinite case by Schillewaert and Van Maldeghem (unpublished).

Theorem 5.2 (Schillewaert and Van Maldeghem (unpublished)). *Let X be a set of points in the projective space $\mathsf{PG}(d,K)$, with K any skew field of order at least 3. Suppose that*

(V1*) *for any pair of points $x, y \in X$, there is a unique plane denoted $[x,y]$ such that $[x,y] \cap X$ is an oval, denoted $X([x,y])$;*

(V2*) *the set X endowed with all subsets $X([x,y])$, has the structure of the point-line geometry of a projective space $\mathsf{PG}(n,F)$, for some skew field F, $n \geq 3$, or of any projective plane Π (and we put $n = 2$ in this case);*

(V3*) $d \geq \frac{1}{2}n(n+3)$.

Then $d = \frac{1}{2}n(n+3)$ and X is the point set of a quadric Veronesean of index n. In particular, $F \cong K$ if $n \geq 3$, and Π is isomorphic to $\mathsf{PG}(2,K)$ if $n = 2$.

Now, Thas and Van Maldeghem [12] generalize both theorems and include projections of $\mathcal{V}_n^{2^n}$ from subspaces spanned by subveroneseans of $\mathcal{V}_n^{2^n}$. These objects are natural generalizations of normal rational cubic scrolls. In [12] even more complex objects are characterized, namely, unions of such projections.

The main idea of this characterization is to replace the axiom that the lines of $\mathsf{PG}(n,q)$ form ovals in planes of $\mathsf{PG}(m,q)$ by the assumption that the points of these lines are just planar sets.

So let $\theta : \mathsf{PG}(n,q) \rightarrow \mathsf{PG}(d,q)$ $d \geq n(n+3)/2$, be an injective map such that the image of θ generates $\mathsf{PG}(d,q)$, and such that θ maps the set of points of each line of $\mathsf{PG}(n,q)$ onto a set of coplanar points of $\mathsf{PG}(d,q)$. We call the image of θ a *generalized Veronesean*, and θ is called a *generalized Veronesean embedding.*

Now we construct such an embedding θ. Let $\alpha : \mathsf{PG}(n,q) \rightarrow \mathsf{PG}(d,q)$ be the ordinary quadric Veronesean map, So $d = n(n+3)/2$. Let U be an i-dimensional subspace of $\mathsf{PG}(n,q)$, with $-1 \leq i \leq n-1$. Put $d' = i(i+3)/2$. Then the image of U under α spans a d'-dimensional subspace V of $\mathsf{PG}(d,q)$.

Let W be a $(d-d'-1)$-dimensional subspace of $\mathsf{PG}(d,q)$ skew to V and let $\theta' : U \to V$ be a generalized Veronesean embedding of U in V. Define $\theta : \mathsf{PG}(n,q) \to \mathsf{PG}(d,q)$ as $\theta(x) = \theta'(x)$ for $x \in U$, and $\theta(x) = \langle \alpha(x), V \rangle \cap W$ for $x \in \mathsf{PG}(n,q) \setminus U$.
Such a θ is called an $(i+1)$-*Veronesean embedding*, and U is called the *lid* of the embedding. Note that a 0-Veronesean embedding is just an ordinary quadric Veronesean embedding (with empty lid).
If θ is an i-Veronesean embedding, we say that its image is an i-*Veronesean*. We now have the following theorem.

Theorem 5.3 (Thas and Van Maldeghem [12]**).** *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be isomorphic to the point-line geometry of $\mathsf{PG}(n,q)$, $n \geq 2$, $q > 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d,q)$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d,q)$, $d \geq n(n+3)/2$, and such that the point set of every member of $\mathcal{L}$ is a subset of a plane of $\mathsf{PG}(d,q)$. Then $\mathcal{P}$ is an i-Veronesean, for some $i \in \{0,1,\cdots,n\}$.*

Remark. The case $q=2$ is a true exception.
This can be further generalized to finite dimensional infinite projective spaces.

Theorem 5.4 (Thas and Van Maldeghem [12]**).** *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be isomorphic to the point-line geometry of $\mathsf{PG}(n,K)$, K a skew field, $n \geq 2$, $|K| > 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d,K)$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d,K)$, $d \geq n(n+3)/2$, and such that the point set of every member of $\mathcal{L}$ is a subset of a plane of $\mathsf{PG}(d,K)$. Assume also that for each $L \in \mathcal{L}$ and each point $x \in L$, whenever the map $y \mapsto \langle x,y \rangle$, $y \in L \backslash \{x\}$, is injective, then there is a unique line T of $\mathsf{PG}(d,K)$ in $\langle L \rangle$ through x such that $T \cap L = \{x\}$. Then $\mathcal{P}$ is an i-Veronesean, for some $i \in \{0,1,\cdots,n\}$.*

Remark. If K is noncommutative, then there is no notion of Veronesean, and in this case only an n-Veronesean embedding of $\mathsf{PG}(n,K)$ exists, which is defined inductively as follows : choose an $\mathsf{AG}(n,K)$ in $\mathsf{PG}(n,K)$, embed it in a natural way in a new n-dimensional projective space over K, and take the "direct sum" of the latter affine space with an $(n-1)$-Veronesean embedding of the $(n-1)$-dimensional projective space $\mathsf{PG}(n,K) \setminus \mathsf{AG}(n,K)$.

A further generalization, getting rid of the technical condition involving the line T in the previous theorem, is proved by Akça, Bayar, Ekmekçi, Kaya, Thas and Van Maldeghem [13]. The precise formulation is rather technical, so we content ourselves with mentioning that these authors classified in an explicit way the point sets $\mathcal{P}$ having the property that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is isomorphic to the geometry of points and lines of $\mathsf{PG}(n,K)$, K a skew field, $n \geq 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d,F)$, F a skew field, $\langle \mathcal{P} \rangle = \mathsf{PG}(d,F)$, $d \geq n(n+3)/2$, and such that every member L of $\mathcal{L}$ is a subset of points of a plane of $\mathsf{PG}(d,F)$.
As a byproduct in the research leading to a proof of Theorem 5.3, Thas and Van Maldeghem [11] obtain the classification of all affine and projective planes $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order q^2, $q > 2$, where $\mathcal{P} \subseteq \mathsf{PG}(d,q)$, $d \geq 4$, the elements of $\mathcal{L}$ are coplanar sets of $\mathsf{PG}(d,q)$, and $\mathtt{I}$ is the incidence of $\mathsf{PG}(d,q)$. They prove that $d=4$, that in the affine case there are three possibilities

and that in the projective case there is just one possibility. This yields a characterization of the André-Bruck-Bose representation of affine planes.

5.2. Segre varieties

Here, the basic result is due to Zanella [15].

Theorem 5.5 (Zanella [15]**).** *Consider a Segre pair* $(\Sigma, \widetilde{\Sigma})$ *in* $\mathsf{PG}(d,q)$*, with* $\max(m,n) \geq 2$*. Consider any* $\pi_1, \pi_2 \in \Sigma$*,* $\pi_1 \neq \pi_2$*, and assume that* $\widetilde{\Sigma}$ *defines a collineation from* π_1 *onto* π_2*; consider any* $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \widetilde{\Sigma}$*,* $\widetilde{\pi}_1 \neq \widetilde{\pi}_2$*, and assume that* Σ *defines a collineation from* $\widetilde{\pi}_1$ *onto* $\widetilde{\pi}_2$*. Then the set of points covered by the Segre pair is the projection of some Segre variety* $\mathcal{S}_{m;n}$ *of* $PG(mn+m+n,q) \supseteq \mathsf{PG}(d,q)$ *onto* $\mathsf{PG}(d,q)$ *from some subspace* $\mathsf{PG}(mn+m+n-d-1,q)$ *of* $\mathsf{PG}(mn+m+n,q)$ *skew to* $\mathsf{PG}(d,q)$*. Also* Σ *and* $\widetilde{\Sigma}$ *are the projections of the systems of generators of* $\mathcal{S}_{m;n}$*.*

The proof of this theorem relies on results of Melone and Olanda [5]. The result also holds in the infinite case.
The following results are due to Thas and Van Maldeghem [14].

Theorem 5.6 (Thas and Van Maldeghem [14]**).** *Let* $q \geq \min(m,n)$ *and* $d \geq mn+m+n$*. Then the set of points covered by a Segre pair with parameters* $(q;m,n,d)$ *is a Segre variety (and hence* $d = mn+m+n$*).*

Theorem 5.7 (Thas and Van Maldeghem [14]**).** *Let* $(\Sigma, \widetilde{\Sigma})$ *be a Segre pair with parameters* $(q;m,n,d)$*,* $q > n$ *and* $m \geq n$*. Assume that the following conditions* (i)*,* (ii) *and* (iii) *hold.*

- (i) *If* π_i, π_j *are distinct elements of* Σ*, then the subspace* $\Delta_{ij} = \Delta_{ji} = \langle \pi_i, \pi_j \rangle$ *contains exactly* $q+1$ *elements of* Σ*.*
- (ii) *The set* Σ *together with the spaces* Δ_{ij} *is a* $\mathsf{PG}(n,q)$ *(denoted by* Δ*) for the natural incidence.*
- (iii) *If* τ, τ' *are distinct independent subsets of* Δ *with* $\tau \subseteq \tau'$*, then they generate distinct subspaces of* $\mathsf{PG}(d,q)$*.*

Then the set of points covered by the Segre pair is the projection of some Segre variety of $\mathsf{PG}(mn+m+n,q) \supseteq \mathsf{PG}(d,q)$ *onto* $\mathsf{PG}(d,q)$ *from some subspace* $\mathsf{PG}(mn+m+n-d-1,q)$ *of* $\mathsf{PG}(mn+m+n,q)$ *skew to* $\mathsf{PG}(d,q)$*.*

Remark. There are also interesting variations on Theorem 5.7 proved in [14].

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