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Abstract

I present a complete list of hypersurface homogeneous space–times admitting a non-null valence two Killing spinor, including a new class admitting only exceptional Killing tensors. A connection is established with the classification of locally rotationally symmetric or boost symmetric space–times.

1. Introduction

This paper is a continuation of the results on homogeneous KS space times obtained in [14]. KS space–times were defined [8] as non-conformally flat space–times ($\mathcal{M}$, **g**), admitting a non-null valence two Killing spinor **X** or, equivalently, as the Petrov type D conformal Killing–Yano space–times. The square of their conformal Killing–Yano two-form is a conformal Killing tensor **K** of Segre type $[(11)(11)]$. In the conformal representant in which the Killing spinor is of modulus one, the so called *unitary representant*, **K** is necessarily an *exceptional* Killing tensor, i.e. a non-trivial ($\mathbf{K} \not\propto \mathbf{g}$) Killing tensor possessing two constant eigenvalues. The classification of the corresponding space–times depends crucially on the existence of conformal representants admitting *regular* or *semi-regular* Killing tensors (these being defined as Killing tensors admitting two non-constant eigenvalues or a single non-constant eigenvalue respectively), as then the eigenvalues can be used to set up a preferred coordinate system. The regular case, which includes a wide range of physically interesting metrics, was dealt with successfully in [6], where also a classification of the KS space–times was presented, based on the properties of the spin coefficients in a Weyl-aligned Geroch–Held–Penrose (GHP)-tetrad for the unitary representant: they were said to be of class 1, 1_N, 2, 3, 3_N or 4 according to whether $\rho\rho'\tau\tau' \neq 0$ (class 1), $\rho\tau\tau' \neq 0 = \rho'$ (class 1_N), $\tau\tau' \neq 0 = \rho = \rho'$(class 2), $\rho\rho' \neq 0 = \tau = \tau'$ (class 3), $\rho \neq 0 = \rho' = \tau = \tau'$ (class 3_N) or $\rho = \rho' = \tau = \tau' = 0$ (class 4). As a conformal

transformation affects the imaginary parts of ρ and ρ' only by a rescaling, the conformal representants of classes 1_N, 2, 3_N and 4 belong to the Robinson–Trautman or Kundt families of algebraically special space–times; for classes 1 and 3 the real principal null directions always have a non-vanishing twist. Semi-regular and exceptional Killing tensors can only occur [8] in classes 1_N or 1. The possibility of semi-regular Killing tensors in class 1_N was first noticed in [8], the resulting space–times being hypersurface homogeneous or homogeneous [14]. Class 1 space–times with semi-regular Killing tensors were presented in [13] and all admitted at most a one-dimensional isometry group. Introducing the functions ϕ, ϕ' defined by

$$\tau'\Phi_{01} = -3\rho\tau\tau' - 2\rho\phi, \quad \tau\Phi_{21} = -3\rho'\tau\tau' - 2\rho'\phi', \tag{1}$$

class 1 was further sub-divided in classes KS_1 and KS_2, characterized by[1]

$$KS_1: \phi + \overline{\phi} = \phi' + \overline{\phi}' = 0, \tag{2}$$

$$KS_2: \phi + \overline{\phi}' = 0 : \tag{3}$$

the regular class 1 metrics discussed in [6] made up the set $KS_1 \cap KS_2$, while the metrics of [13] exhausted the symmetric difference $KS_1 \Delta KS_2$. The existence of KS space–times belonging to class $1 \setminus (KS_1 \cup KS_2)$ (hence admitting only exceptional Killing tensors) was demonstrated in [2]. They were found by investigating the purely Weyl-electric metrics in the class $KS_3 \supset KS_2$, characterized by

$$KS_3: \Im(\phi - \phi') = 0. \tag{4}$$

Note that $KS_3 \cap KS_1 = KS_2 \cap KS_1$. In section 2 it will be shown that the extra elements of $KS_3 \setminus KS_2$ are precisely the hypersurface homogeneous[2] members of class 1 (including the homogeneous sub-family discussed in [2]), the metrics of which will be explicitly constructed in section 3. The present work completely exhausts class KS_3. Whether any KS space–times exist in which condition $\Im(\phi - \phi') = 0$ does not hold, is at present still an open question.

KS space–times of classes 2, 3, 3_N and 4 are [6] conformally related to Killing–Yano space–times. All these have been described in [4] and their hypersurface homogeneous members can in principle be found by suitably restricting the Killing–Yano metrics. However it is more convenient to construct the complete family of hypersurface homogeneous KS space–times in a coordinate invariant way and the resulting hypersurface homogeneous metrics are presented in sections 5–7.

It turns out that the hypersurface homogeneous members of classes 3 and 4 are precisely given by the locally rotationally (or boost) symmetric space–times (13.1), (13.2) and (13.14) of [9] (see also [12]).

Notations and conventions follow [2], which is based on the GHP [5] and NP formalisms [10], as presented in [9]: the GHP weighted operators þ, $\text{þ}'$, $\eth$, $\eth'$ generalize the NP operators D, Δ, δ, $\overline{\delta}$, corresponding to the basis one-forms $(k^a, \ell^a, m^a, \overline{m}^a)$, while the GHP variables κ', σ', ρ', τ' replace the NP variables $-\nu$, $-\lambda$, $-\mu$, $-\pi$. All calculations were done with the aid of the Maple symbolic algebra package[3] and the properties of the obtained metrics were checked with Maple's differential geometry package [1].

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[1] There was a print error in equation (40) of [13].
[2] Obviously hypersurface homogeneity is not a conformal property, but the name 'hypersurface homogeneous KS space–times' appearing in the title may be justified as follows: if a Petrov type D conformal representant is hypersurface homogeneous then (because of the alignment of the Killing spinor with the Weyl spinor) the modulus of the Killing spinor is a geometric invariant and is therefore constant on the surfaces of homogeneity, such that also the unitary representant is hypersurface homogeneous. Vice versa, when the unitary representant is hypersurface homogeneous then any conformal factor which is constant on the hypersurfaces of homogeneity will generate a conformal representant which is hypersurface homogeneous too.
[3] A set of special purpose routines for the GHP and NP formalisms can be obtained from the author.

2. Main equations

I first present the main equations describing a KS space–time; for the details the reader is referred to [6, 13].

Choosing the spinor basis such that

$$X_{AB} = Xo_{(A'B)}, \tag{5}$$

the Killing spinor equation

$$\nabla_{A'(A}X_{BC)} = 0, \tag{6}$$

reduces to

$$\kappa = \sigma = 0 \tag{7}$$

and

$$\text{þ}X = -\rho X, \tag{8}$$

$$\text{ð}X = -\tau X. \tag{9}$$

These equations are accompanied by their 'primed versions', namely (as $X' = X$) $\kappa' = \sigma' = 0$ and $\text{þ}'X = -\rho'X$, $\text{ð}'X = -\tau'X$. The Weyl tensor is then of Petrov type D (or O) and $\boldsymbol{k}$, $\boldsymbol{\ell}$, $\boldsymbol{m}$, $\overline{\boldsymbol{m}}$[4] are its principal null directions ($\varPsi_0 = \varPsi_1 = \varPsi_3 = \varPsi_4 = 0$).

In the unitary representant $|X|$ is constant and hence

$$\rho + \overline{\rho} = \tau + \overline{\tau}' = 0. \tag{10}$$

Herewith one obtains

(a) the integrability conditions expressing the existence of the Killing spinor:

$$\text{þ}'\rho - \text{þ}\rho' = 0, \quad \text{ð}\tau' - \text{ð}'\tau = 0, \quad \text{þ}\tau' - \text{ð}'\rho = 0, \tag{11}$$

(b) the GHP equations:

$$\text{þ}\rho = 0, \tag{12}$$

$$\text{ð}\rho = 2\rho\tau + \varPhi_{01}, \tag{13}$$

$$\text{þ}\tau = 2\rho\tau + \varPhi_{01}, \tag{14}$$

$$\text{ð}\tau = 0, \tag{15}$$

$$\text{þ}\rho' - \text{ð}\tau' = -\rho\rho' - \tau\overline{\tau} - \varPsi_2 - \frac{1}{12}R, \tag{16}$$

$$\varPhi_{00} = -\rho^2, \ \varPhi_{02} = -\tau^2, \tag{17}$$

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[4] $k^a = o^A\overline{o}^{\dot{B}}$, $\ell^a = \iota^A\overline{\iota}^{\dot{B}}$, $m^a = o^A\overline{\iota}^{\dot{B}}$.

$$E = -\frac{R}{12} - \rho\rho' + \tau\tau', \tag{18}$$

E being the real part of $\Psi_2 = E + iH$, and
 (c) the Bianchi equations:

$$\text{þ}\Phi_{01} = -\rho\left(4\Phi_{01} + 5\tau\rho\right), \tag{19}$$

$$\text{þ}'\Phi_{01} = \rho'\Phi_{01} - \rho\Phi_{12} + \tau\left(3\overline{\Psi_2} + \tau\overline{\tau} - 2\Phi_{11}\right) + \text{ð}\left(\overline{\Psi_2} - \frac{R}{24} + \Phi_{11}\right), \tag{20}$$

$$\text{ð}\Phi_{01} = -\tau\left(4\Phi_{01} + 5\tau\rho\right), \tag{21}$$

$$\text{ð}'\Phi_{01} = -\overline{\tau}\Phi_{01} - \tau\Phi_{10} - \rho\left(3\overline{\Psi_2} + \rho\rho' + 2\Phi_{11}\right) - \text{þ}\left(\overline{\Psi_2} - \frac{R}{24} - \Phi_{11}\right), \tag{22}$$

$$\text{þ}\left(\mu^2\right) = 3\rho'\left(\overline{\Psi_2} - \Psi_2\right) - 3\overline{\tau}\Phi_{12} - 3\tau\Phi_{21} + \text{þ}'\left(4\Psi_2 - 2\overline{\Psi_2} + \frac{R}{24} - \Phi_{11}\right), \tag{23}$$

$$\text{ð}\left(\overline{\tau}^2\right) = 3\overline{\tau}\left(\overline{\Psi_2} - \Psi_2\right) + 3\rho'\Phi_{10} - 3\rho\Phi_{21} - \text{ð}'\left(\Psi_2 + \overline{\Psi_2} + \frac{R}{24} + \Phi_{11}\right). \tag{24}$$

All these equations must be read as being accompanied by their 'primed' and complex conjugated analogues. They can be simplified by introducing the (real) 0-weighted quantities U and V, defined by

$$R = 8(U - V) - 16\rho\rho', \tag{25}$$

$$\Phi_{11} = U + V - 2\rho\rho', \tag{26}$$

$(U' = U, V' = V)$. One has then, by (18)

$$\Psi_2 = \frac{1}{3}(\rho\rho' - 2U + 2V) + \tau\tau' + iH. \tag{27}$$

3. Hypersurface homogeneous KS space–times of class 1

3.1. General properties

It is advantageous here to define, besides ϕ and ϕ', new 0-weighted variables w (real) and ζ (complex), with $w' = w$ and $\zeta' = \overline{\zeta}$, by

$$\text{þ}\rho' = \text{þ}'\rho = -iw, \tag{28}$$

$$\text{þ}'\phi = \rho'\left(2\frac{\phi\phi'}{|\tau|^2} + \zeta\right) + i\frac{w\phi}{\rho}. \tag{29}$$

Herewith one rewrites the Bianchi equations in the following form:

$$\text{þ}\phi = \frac{2\rho}{|\tau|^2}\left(|\tau|^4 - |\phi|^2\right), \tag{30}$$

$$\text{ð}\phi = -\frac{1}{\tau'}\left(2|\tau|^4 + i\phi(w - H) - 2\phi^2\right), \tag{31}$$

$$\eth'\phi = -2\frac{\left|\phi^2\right|}{\tau} - \tau'\left(2\phi + 2\overline{\phi} + \rho\rho' - 2V + \frac{\mathrm{i}}{2\rho}\flat H\right), \tag{32}$$

$$\flat U = \rho\left(\phi - \overline{\phi} - 3\mathrm{i}H\right), \tag{33}$$

$$\eth V = \frac{\rho\rho'}{\tau'}\left(\phi - \overline{\phi}'\right) + \mathrm{i}\tau\left(H + 2w\right), \tag{34}$$

$$\eth H = 2\mathrm{i}\tau\left(|\tau|^2 + 2U - 4\rho\rho'\right) - 2\mathrm{i}\frac{\rho\rho'}{\tau'}\left(2\phi + 2\overline{\phi}' + \zeta\right), \tag{35}$$

together with their primed and complex conjugated analogues (taking into account $\rho' = -\rho$ and $\tau' = -\overline{\tau}$). Applying the commutators involving $\flat'\rho$ and using (28) yields two further relations, namely

$$\flat w = 2\mathrm{i}\rho\left(\rho\rho' + 2\phi + 2\overline{\phi} - 2V - 2|\tau|^2\right), \tag{36}$$

$$\eth w = -2\mathrm{i}\frac{\rho\rho'}{\tau'}\left(2|\tau|^2 + \zeta\right). \tag{37}$$

Herewith all 'first level' integrability conditions on ρ, ρ', τ, τ' are identically satisfied. It is now easy to investigate the relation between hypersurface homogeneity, the electric and magnetic parts of the Weyl tensor and the functions ϕ, ϕ'.

First note that in a hypersurface homogeneous space–time all 0-weighted GHP quantities, such as $\rho\rho'$, $\tau\tau'$, U, V, w and H are functions of the 0-weighted scalar[5] X. This suggests to define real and 0-weighted scalars r and m (r, $m > 0$) by

$$r^2 = Q\rho\rho' \quad (Q = \pm 1), \tag{38}$$

$$m^2 = |\tau|^2. \tag{39}$$

Expressing that $\eta_{[,a}X_{,b]} = m_{[,a}X_{,b]} = 0$ implies

$$H - w + 2\mathrm{i}\left(\overline{\phi} - \phi\right) = 0, \tag{40}$$

$$Qm^2w - 2\mathrm{i}r^2\left(\phi - \overline{\phi}'\right) = 0, \tag{41}$$

$$Qm^2w - 2\mathrm{i}r^2\left(\phi - \overline{\phi}\right) = 0, \tag{42}$$

from which one immediately infers $\phi' = \phi$. Denoting with $\mathcal{H}_i$ the set of hypersurface homogeneous space–times of class i ($i = 1, 1_N, 2, 3, 3_N, 4$), it follows that $\mathcal{H}_1 \subset KS_3$.

We now show that $KS_3 \backslash KS_2 \subset \mathcal{H}_1$: evaluating the imaginary part of $\eth$ (4) with (20, 21), one finds

$$\left(H - w + 2\mathrm{i}\left(\overline{\phi} - \phi\right)\right)\left(\overline{\phi}' + \phi\right) = 0, \tag{43}$$

such that, provided solutions do not belong to KS_2, again (40) follows. Herewith (36) implies

$$\flat H = 4\mathrm{i}\rho\left(m^2 + \frac{Q}{2}r^2 + \phi + \overline{\phi} - V - \frac{2}{m^2}|\phi|^2\right), \tag{44}$$

together with a similar equation for $\flat'H$, with which the real part of $\eth$ (4) simplifies to $(\overline{\phi}' + \phi)(\overline{\phi}' - \overline{\phi}) = 0$ and hence

[5] By (8, 9) X is a constant in class 4 only.

$$\phi' = \phi. \tag{45}$$

Acting with the operators þ, þ′ on (45) gives then

$$H = -2i\frac{Qr^2 + m^2}{m^2}\left(\overline{\phi} - \phi\right), \tag{46}$$

$$\zeta = \frac{2}{m^2}\left(m^4 - 2|\phi|^2\right), \tag{47}$$

the ð, ð′ derivatives of which lead to

$$U = 3Qr^2 + \frac{3}{2}m^2 - \frac{Qr^2}{m^2}\left(\phi + \overline{\phi}\right) - 2\frac{|\phi|^2}{m^2}, \tag{48}$$

$$V = -\frac{3}{2}Qr^2 - m^2 + \phi + \overline{\phi} + 2\frac{Qr^2}{m^4}|\phi|^2. \tag{49}$$

All 0-weighted quantities (and hence all invariants) become then algebraic functions of m, r and ϕ, with

$$dr = \frac{r\left(\phi - \overline{\phi}\right)}{m^2}d\log X, \quad dm = \frac{\overline{\phi} - \phi}{m}d\log X, \quad d\phi = 2\frac{|\phi|^2 - m^4}{m^2}d\log X, \tag{50}$$

implying that the corresponding space–times are hypersurface homogeneous.

Herewith we have demonstrated that the hypersurface homogeneous members of class 1 are precisely the solutions for which $\Im(\phi - \phi') = 0 \neq \phi' + \overline{\phi}$, together with the hypersurface homogeneous members of the regular family $KS_1 \cap KS_2$. Denoting the latter as $\mathcal{H}_{1,r}$ we have:

Property 1

$$\mathcal{H}_1 = \left(KS_3 \backslash KS_2\right) \bigcup \mathcal{H}_{1,r}.$$

In [14] we showed already that the purely Weyl-electric members of KS_3 are (space–time) homogeneous. It is easy to see now that a stronger result holds:

Property 2

Purely Weyl electric KS space–times of class 1 belong to KS_3 and are homogeneous.

Proof. Substitute $H = 0$ in (35), to obtain an expression for ζ

$$\zeta = 4m^2 - Q\frac{m^2}{r^2}\left(2U + m^2\right) - 2\left(\phi + \overline{\phi}'\right). \tag{51}$$

This enables one to simplify the [þ, þ′]ϕ commutator relation to

$$m^2E\left(\phi + m^2\right) = 0, \tag{52}$$

implying, for non-conformally flat space–times, that ϕ is real. Similarly ϕ' is real and the KS_3 condition (4) holds trivially.

With a bit of extra work we can also demonstrate:

Property 3

Purely Weyl-magnetic solutions of class 1 do not exist.

Proof. Putting $E = 0$ in (18) yields

$$V - U = \frac{3}{2}m^2 - \frac{1}{2}Qr^2, \tag{53}$$

which allows one to obtain, together with (33, 34), all derivatives of U and V:

$$\eth U = \frac{1}{2}\mathrm{i}\tau(5H + w), \tag{54}$$

$$\þV = -\frac{1}{2}\rho\left(8\left(\overline{\phi} - \phi\right) + 6\mathrm{i}H - \mathrm{i}w\right). \tag{55}$$

Applying the $[\eth', \eth]$ and $[\eth, \þ]$ commutators to U leads then to expressions, the imaginary part of which reduces to the KS_1 condition (8) and the real part of which results in

$$\þH = 2\mathrm{i}\rho\left(4Qr^2 - m^2 - Q\frac{r^2}{m^2}\left(\zeta + 2\phi + 2\overline{\phi}'\right) - 2U + 2\mathrm{i}\frac{\phi H}{m^2}\right). \tag{56}$$

Taking the $\þ$ and $\eth$ derivatives of (8) shows that also the KS_2 condition (9) holds and that

$$\zeta = 2\,m^2 + \mathrm{i}Q\frac{\phi}{r^2}(3H - w). \tag{57}$$

Herewith the $[\eth, \þ']\phi$ commutator relation yields an expression for w

$$w = \mathrm{i}\frac{m^2}{\phi}\left(2Qr^2 - m^2 - 2U + \frac{\mathrm{i}H}{m^2\phi}\left(m^4 - 3\phi^2\right)\right), \tag{58}$$

substitution of which in (36), (37) leads to an inconsistency, namely $2\,m^4 + \phi^2 = m^4 - 2\phi^2 = 0$.

3.2. $KS_3 \backslash KS_2$

According to the results of the previous paragraph hypersurface homogeneous KS space–times of class 1 either admit only exceptional Killing tensors (and are then characterized by $\phi' = \phi$ and $\Re\phi \neq 0$), or they admit regular Killing tensors and have $\phi' = \phi$ and $\Re\phi = 0$. In both cases the explicit metrics can be found by first translating the previously obtained invariant information into Newman–Penrose language. Aligning the tetrad as before, we will fix a boost and a spatial rotation such that $\rho = \mathrm{i}Qr$ (hence $\mu = \mathrm{i}r$) and $\tau = m$. From the Newman–Penrose equations and (17), (18), (25), (26) one immediately obtains then

$$\varepsilon = \mathrm{i}Q\frac{r}{m}\alpha, \; \gamma = Q\varepsilon, \tag{59}$$

$$\beta = \alpha = \frac{1}{2m}\left(m^2 - 2\overline{\phi}\right), \tag{60}$$

together with

$$R = \frac{4}{m^4}\left(Qr^2 + m^2\right)\left(5\,m^4 - 4\,\phi\,\overline{\phi} - 2\left(\phi + \overline{\phi}\right)m^2\right), \tag{61}$$

$$\Psi_2 = \frac{4}{3\,m^4}\left(\phi + 2\,m^2\right)\left(\overline{\phi} - m^2\right)\left(Qr^2 + m^2\right). \tag{62}$$

Note that by (50) r/m and $\Re\phi$ are constants. We therefore put

$$m = kr, \tag{63}$$

$$\phi = k^2\left(-Qk^2 r_0^2 + \mathrm{i}\psi\right), \tag{64}$$

with k, r_0 constants ($k > 0$ and r_0 real or imaginary) and ψ a real function of r. Integrating (50c) gives then

$$(Q\psi - Lr)(Q\psi + Lr) = \left(r_0^2 - r^2\right)\left(r^2 - k^4 r_0^2\right), \tag{65}$$

with L a constant of integration, while (50a) gives

$$\mathrm{d}r = 2\psi\left(\mathrm{i}k\left(\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2\right) + \boldsymbol{\omega}^3 - Q\boldsymbol{\omega}^4\right). \tag{66}$$

As ψ is real, the positivity of the right hand side of (65) implies that only the following cases can occur:

(a) $0 \neq r_0$ and L are real and arbitrary, the coordinate domain in which ψ (and hence $\mathcal{A}$) is real, being the interval $]r_-, r_+[$ with

$$2r_\pm^2 = r_0^2\left(1 + k^4\right) + L^2 \pm \left[\left(r_0^2\left(1 + k^4\right) + L^2\right)^2 - 4k^4 r_0^4\right]^{1/2}, \tag{67}$$

(b) $0 \neq r_0$ is imaginary, $0 \neq L$ is real and

$$\left|L/r_0\right| > 1 + k^2, \quad r \in \left]r_-, r_+\right[, \tag{68}$$

(c) $0 \neq r_0$ is real, $0 \neq L$ is imaginary and

$$|L/r_0| < |1 - k^2|, \quad r \in \left]r_-, r_+\right[. \tag{69}$$

It is easy to check that $r_0 < r_-$ if $k > 1$ and $r_+ < r_0$ if $0 < k < 1$. The case $r_0 = 0$ belongs to $KS_1 \cap KS_2$: this is the regular situation, which will be dealt with in section 3.3.

Introducing new (real) basis one-forms by

$$\boldsymbol{\Omega}^1 = \mathrm{i}\left(\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2\right), \quad \boldsymbol{\Omega}^2 = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2, \quad \boldsymbol{\Omega}^3 = \boldsymbol{\omega}^3 + Q\boldsymbol{\omega}^4, \quad \boldsymbol{\Omega}^4 = \boldsymbol{\omega}^3 - Q\boldsymbol{\omega}^4, \tag{70}$$

it follows from (66) that $\boldsymbol{\Omega}^4 + k\boldsymbol{\Omega}^1$ is exact:

$$\boldsymbol{\Omega}^4 + k\boldsymbol{\Omega}^1 = \frac{1}{2\psi}\mathrm{d}r. \tag{71}$$

The line-element reads now

$$\mathrm{d}s^2 = 2\left(\boldsymbol{\Omega}^{1^2} + \boldsymbol{\Omega}^{2^2} - Q\boldsymbol{\Omega}^{3^2} + Q\boldsymbol{\Omega}^{4^2}\right), \tag{72}$$

and the Cartan equations become

$$\mathrm{d}\boldsymbol{\Omega}^1 = 2Qk^2\frac{r_0^2}{r}\boldsymbol{\Omega}^3 \wedge \boldsymbol{\Omega}^2, \tag{73}$$

$$\mathrm{d}\boldsymbol{\Omega}^2 = 2\frac{k}{r}\left(-\psi\boldsymbol{\Omega}^2 - Qkr_0^2\boldsymbol{\Omega}^3\right) \wedge \boldsymbol{\Omega}^1 - 2Qkr\boldsymbol{\Omega}^3 \wedge \boldsymbol{\Omega}^4, \tag{74}$$

$$\mathrm{d}\boldsymbol{\Omega}^3 = -2\frac{Q}{r}\left(r^2\boldsymbol{\Omega}^1 + k^3 r_0^2 \boldsymbol{\Omega}^4\right) \wedge \boldsymbol{\Omega}^2 - 2\frac{\psi}{r}\boldsymbol{\Omega}^3 \wedge \boldsymbol{\Omega}^4, \tag{75}$$

$$\mathrm{d}\boldsymbol{\Omega}^4 = 2Qk^3\frac{r_0^2}{r}\boldsymbol{\Omega}^2 \wedge \boldsymbol{\Omega}^3. \tag{76}$$

The fact [14] that in the homogeneous case a closed (constant) linear combination of $\boldsymbol{\Omega}^2$ and $\boldsymbol{\Omega}^3$ exists, suggests in the present case to look for a function $\mathcal{A}(r)$ such that $\mathcal{A}\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^3$ is closed. This leads to an over-determined set of two differential equations for $\mathcal{A}$

$$k\psi\left(r\mathcal{A}_{,r} + \mathcal{A}\right) - Q\left(r^2 + k^2 r_0^2 \mathcal{A}^2\right) = 0,$$

$$\psi\left(r\mathcal{A}_{,r} - \mathcal{A}\right) - Qk\left(r^2\mathcal{A}^2 + k^2 r_0^2\right) = 0, \tag{77}$$

which, remarkably, under (65) has a unique solution, determined by

$$k^2\mathcal{A}^2\left(r^2 - r_0^2\right) + 2Qk\psi\mathcal{A} + k^4 r_0^2 - r^2 = 0, \tag{78}$$

or, as L in (65) is only defined up to sign, by

$$\mathcal{A} = \frac{Q\psi + Lr}{k\left(r_0^2 - r^2\right)} \equiv \frac{r^2 - k^4 r_0^2}{k\left(Q\psi - Lr\right)} \tag{79}$$

and hence

$$k^2\mathcal{A}^2\left(r^2 - r_0^2\right) + 2kLr\mathcal{A} + r^2 - k^4 r_0^2 = 0. \tag{80}$$

Further integration depends on whether $\mathcal{A}$ is real or complex:

3.2.1. $\mathcal{A}$ real. $\mathcal{A}$ real can only occur in the cases (67), (68) above: L is real and r_0 is real or imaginary. Introducing real functions x, y such that $\mathcal{A}\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^3 = y\mathrm{d}x$, the second Cartan equation becomes

$$- Qkr_0^2\frac{y\mathcal{A}}{r\psi}\mathrm{d}r \wedge \mathrm{d}x - 2\frac{y}{r\left(k^2\mathcal{A}^2 - 1\right)}$$
$$\times \left(k^2\psi\mathcal{A}^2 + Qkr_0^2\left(k^4 - 1\right)\mathcal{A} + \psi\right)\mathrm{d}x \wedge \boldsymbol{\Omega}^4 - \mathrm{d}x \wedge \mathrm{d}y = 0. \tag{81}$$

When the coefficient of $\mathrm{d}x \wedge \boldsymbol{\Omega}^4$ is non-vanishing (which happens precisely when $L \neq 0$), $\boldsymbol{\Omega}^4$ can then be calculated easily. Proceeding in the same way with (76) to find $\boldsymbol{\Omega}^3$, we finally obtain

$$-2L\boldsymbol{\Omega}^4 = Qk^2 z\mathrm{d}x + 2Qk^2 r_0^2\frac{\mathcal{A}^2}{r\Sigma_-}\mathrm{d}r + \frac{Q}{y}\mathrm{d}y, \tag{82}$$

$$-2L\boldsymbol{\Omega}^3 = \left(\frac{y\left(r^2 - k^4 r_0^2\right)}{kr\mathcal{A}} - \frac{kr\mathcal{A}z^2}{4r_0^2 y} + \frac{Qr\mathcal{A}}{ky}W\right)\mathrm{d}x - \frac{r\mathcal{A}}{2kr_0^2 y}\mathrm{d}z, \tag{83}$$

with the auxiliary functions $\Sigma_\pm$ defined by

$$\Sigma_\pm = k^2\mathcal{A}^2\left(r^2 \pm r_0^2\right) \pm r^2 \mp k^4 r_0^2 \tag{84}$$

and with W a function, which by (75) is restricted to be independent of r, y and z: $W = W(x)$. However, as (1) the curvature components and the spin coefficients depend only on r, (2) the null tetrad is invariantly defined and (3) (71) holds, the function W cannot appear in the classification algorithm [7] and[6] one can put, without loss of generality, $W = 0$. The corresponding expressions for $\boldsymbol{\Omega}^1$ and $\boldsymbol{\Omega}^2$ read then

$$-2L\boldsymbol{\Omega}^1 = -Qkz\,\mathrm{d}x - \frac{Q\Sigma_+}{kr\Sigma_-}\mathrm{d}r - \frac{Q}{ky}\mathrm{d}y,$$

$$-2L\boldsymbol{\Omega}^2 = \left(\frac{k\left(r^2 - r_0^2\right)}{r}y + \frac{krz^2}{r_0^2 y} \right)\mathrm{d}x + \frac{r}{2kr_0^2 y}\mathrm{d}z. \tag{85}$$

When, on the other hand, $L = 0$ (which requires r_0 to be real), the construction of the tetrad basis vectors demands the explicit integration of the Cartan equations. The procedure is standard and after repeated use of a Cartan–Karlhede argument to eliminate free functions, one obtains

$$k\boldsymbol{\Omega}^1 = -C\mathrm{d}y - \mathrm{d}z - \frac{Qk^4 r_0^2}{2}\left|k^4 r_0^2 - r^2\right|^{-3/2}\left|r^2 - r_0^2\right|^{-1/2}\mathrm{d}r,$$

$$k\boldsymbol{\Omega}^2 = -QS\left|r^2 - r_0^2\right|^{1/2}\mathrm{d}y + k^2\left(2Qz\left|r^2 - r_0^2\right|^{1/2} + r\left|k^4 r_0^2 - r^2\right|^{-1/2}\right)\mathrm{d}r,$$

$$k\boldsymbol{\Omega}^3 = \left|k^4 r_0^2 - r^2\right|^{1/2}\left(-2Qkz\mathrm{d}x + \frac{QS}{k}\mathrm{d}y\right),$$

$$k\boldsymbol{\Omega}^4 = kC\mathrm{d}y + k\mathrm{d}z + \frac{kQ}{2}r^2\left|k^4 r_0^2 - r^2\right|^{-3/2}\left|r^2 - r_0^2\right|^{-1/2}\mathrm{d}r, \tag{86}$$

with $S = \sin(\mathrm{h})2k^2 r_0^2 x$ and $C = \cos(\mathrm{h})2k^2 r_0^2 x$ if $k < 1$ ($k > 1$).

Note that, because of (80) $k = 1$ is only allowed if $L \neq 0$; then also $Q = 1$, as otherwise the space–time is conformally flat ($\Psi_2 = 0$).

3.2.2. A not real. As remarked above, this is the case where r_0 is real and $0 \neq L$ is imaginary. In a way, this is easier than the real case, as the one-forms $\boldsymbol{\Omega}^2$ and $\boldsymbol{\Omega}^3$ are determined by taking the real and imaginary parts of $\mathcal{A}(\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^3) = P(\mathrm{d}x + \mathrm{i}\mathrm{d}y)$ (P complex and x, y real). The second and fourth Cartan equation allow then to find P and $\boldsymbol{\Omega}^4$. With ψ defined by (65), the solutions become

$$\boldsymbol{\Omega}^1 = -\frac{Q}{2k\lambda\mathcal{K}}\left(\mathcal{K}_{,y}\mathrm{d}x - \mathcal{K}_{,x}\mathrm{d}y\right) - \frac{1}{k}\mathrm{d}z - \frac{r^2}{2k\psi\left(r_0^2 - r^2\right)}\mathrm{d}r,$$

$$\boldsymbol{\Omega}^2 = \left(\frac{k\left|r_0^2 - r^2\right|}{\lambda}\right)^{1/2}\frac{\operatorname{sgn}(k-1)}{\mathcal{K}}(Q\sin 2\lambda z\,\mathrm{d}x - \cos 2\lambda z\,\mathrm{d}y),$$

$$\boldsymbol{\Omega}^3 = \left(\lambda k\left|r_0^2 - r^2\right|\right)^{-1/2}\frac{1}{\mathcal{K}}((\lambda r\cos 2\lambda z + \psi\sin 2\lambda z)\mathrm{d}x$$
$$+ Q(\lambda r\sin 2\lambda z - \psi\cos 2\lambda z)\mathrm{d}y),$$

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[6] This argumentation will be referred to henceforth as a '*Cartan–Karlhede argument*'.

$$\boldsymbol{\Omega}^4 = \frac{Q}{2\lambda\mathcal{K}}\left(\mathcal{K}_{,y}\mathrm{d}x - \mathcal{K}_{,x}\mathrm{d}y\right) + \mathrm{d}z + \frac{1}{2}\frac{r_0^2}{\psi\left(r_0^2 - r^2\right)}\mathrm{d}r, \tag{87}$$

where $\lambda = |L|$ and $-\log\mathcal{K}$ ($\mathcal{K} = \mathcal{K}(x, y)$) is an arbitrary solution of the Liouville equation

$$Z_{,xx} + Z_{,yy} + \mathrm{sgn}\,(k - 1)\lambda k^4 r_0^2 e^{2Z} = 0. \tag{88}$$

Although the general solution of (88) is defined up to an arbitrary analytic function, again the Cartan–Karlhede argument allows one to put

$$\mathcal{K} = 1 - \mathrm{sgn}\,(k - 1)\lambda k^3 r_0^2\left(x^2 + y^2\right). \tag{89}$$

3.3. $\mathcal{H}_{1,r}$

Having found all the exceptional hypersurface homogeneous KS space–times we now investigate, in view of property (1), the regular family $\mathcal{H}_{1,r}$. The relevant equations can be obtained by putting $r_0 = 0$ in equations (64)–(79). By (65) L (and hence also $\mathcal{A}$) is then necessarily real and $r \in]0, L[$, while, by (73), (76), $\boldsymbol{\Omega}^1$ and $\boldsymbol{\Omega}^4$ are exact. This guarantees that the first term of (82) can be absorbed in the last term ($z = z(x)$) and hence

$$-2L\boldsymbol{\Omega}^4 = \frac{Q}{y}\mathrm{d}y. \tag{90}$$

The fourth Cartan equation gives now no extra information and $\boldsymbol{\Omega}^3$ has to be calculated by explicit integration of (75). The resulting expression

$$2L\boldsymbol{\Omega}^3 = \frac{r^2\psi}{Ly\left(QL^2r - Qr^3 + L\psi\right)}\left(y^2\mathrm{d}x + \left(\frac{4(Lr + \psi)^2}{r^4}\right)^Q \mathrm{d}z\right), \tag{91}$$

suggests the coordinate transformation $\frac{2\psi}{r^2} = t - \frac{1}{t}$. By (65) this implies $\frac{2L}{r} = t + \frac{1}{t}$ and, after $y \to e^{Qy}$ and a rescaling by $2L$, the canonical null tetrad can then be written as follows:

(a) $Q = 1$:

$$\boldsymbol{\omega}^1 = \overline{\boldsymbol{\omega}^2} = \frac{\mathrm{i}}{4k}\left(\frac{\mathrm{d}t}{t} - \mathrm{d}y\right) - \frac{kt}{t^2 + 1}\left(\frac{e^y}{2}\mathrm{d}x - 2e^{-y}\mathrm{d}z\right),$$

$$\boldsymbol{\omega}^3 = \frac{\mathrm{d}y}{4} + \frac{1}{2\left(t^2 + 1\right)}\left(e^y\mathrm{d}x + 4t^2e^{-y}\mathrm{d}z\right),$$

$$\boldsymbol{\omega}^4 = -\frac{\mathrm{d}y}{4} + \frac{1}{2\left(t^2 + 1\right)}\left(e^y\mathrm{d}x + 4t^2e^{-y}\mathrm{d}z\right). \tag{92}$$

(b) $Q = -1$:

$$\boldsymbol{\omega}^1 = \overline{\boldsymbol{\omega}^2} = \frac{\mathrm{i}}{4k}\left(\frac{\mathrm{d}t}{t} - \mathrm{d}y\right) - \frac{kt}{t^2 + 1}\left(\frac{e^{-y}}{2}\mathrm{d}x - \frac{e^{-y}}{8}\mathrm{d}z\right),$$

$$\boldsymbol{\omega}^3 = -\frac{\mathrm{d}y}{4} + \frac{1}{2\left(t^2 + 1\right)}\left(e^{-y}t^2\mathrm{d}x + \frac{e^y}{4}\mathrm{d}z\right),$$

$$\omega^4 = -\frac{\mathrm{d}y}{4} - \frac{1}{2\left(t^2 + 1\right)}\left(e^{-y}t^2\mathrm{d}x + \frac{e^y}{4}\mathrm{d}z\right). \tag{93}$$

For all solutions discussed in the present paragraph, i.e. (86), (87), (92), (93), it is clear from (71), (72) that the hypersurfaces of homogeneity are space-like when $Q = -1$, $|k| < 1$ and time-like in all other cases (note that, by (62) the case $|k| = 1$, $Q = -1$ is conformally flat).

4. Hypersurface homogeneous KS space–times of class 1_N

When $\mu = 0$ (13') implies $\Phi_{12} = 0$. There are then always conformal representants [8] admitting a regular or semi-regular Killing tensor. The semi-regular case was treated in [8] and [14]: all solutions turned out to be T_3-homogeneous of Bianchi class VIII or (space–time) homogeneous. In the regular case there are no homogeneous solutions [14], but hypersurface homogeneous solutions do exist: they can be constructed explicitly, along the same lines of the previous paragraph, and turn out to be precisely the regular limits (i.e. in which the real part ϕ vanishes) of the space–times considered in [8]. Their canonical (Weyl-aligned) null-tetrad can be simplified to the following form:

$$\omega^1 = \overline{\omega^1} = \sin r\left(2e^{-2ky}k^2\mathrm{d}x - \frac{1}{2}e^{2ky}\mathrm{d}z\right) - \frac{\mathrm{i}}{2}\left(\mathrm{d}y + \frac{1}{2k}\frac{\mathrm{d}r}{\sin r}\right),$$

$$\omega^3 = 2k^2e^{-2ky}(1 + \cos r)\mathrm{d}x + \frac{1}{2}e^{2ky}(1 - \cos r)\mathrm{d}z,$$

$$\omega^4 = \mathrm{d}y. \tag{94}$$

The r, x or z = constant coordinate surfaces are time-like (while the y = constant surface is clearly null): the space–time is T_3-homogeneous, with Killing vectors given by ∂_x, ∂_z and $2z\partial_z - 2x\partial_x - \frac{1}{k}\partial_y$. The Bianchi type is VI$_0$.

In the next sections we briefly consider the remaining regular cases, classes 2, 3, 3_N and 4.

5. Hypersurface homogeneous KS space–times of class 2

When $\rho = \mu = 0$ the NP tetrad can be fixed, modulo a remaining boost, by requiring τ to be real. By (8), (9) hypersurface homogeneity implies then that the operators D, Δ and $\delta + \bar{\delta}$ are identically 0 (when acting on invariantly defined quantities). From the Bianchi and NP equations it follows then that ε and γ are real and $\beta = \bar{\alpha} - \mathrm{i}\tau/(2H)$, with

$$\delta\tau = -\mathrm{i}H/2, \quad \delta H = 2\mathrm{i}\tau\left(\tau^2 + 2U\right), \quad \delta V = \mathrm{i}\tau H. \tag{95}$$

Hence

$$V + \tau^2 = V_0 = \text{constant} \tag{96}$$

and

$$\omega^1 - \omega^2 = \frac{2\mathrm{i}}{H}\mathrm{d}\tau, \tag{97}$$

with $H = H(\tau)$ an arbitrary function. The Cartan equations show then that ω^3 and ω^4 are closed, allowing to partially fix a boost such that ω^3 is exact. We have then $\varepsilon = 0$ and $\alpha = \tau/2 - iH/(4\tau)$ and the remaining Cartan equations become

$$\mathrm{d}\left(\omega^1 + \omega^2\right) = -iH\big/\tau\omega^1 \wedge \omega^2 + 4\tau\omega^3 \wedge \omega^4, \tag{98}$$

$$\mathrm{d}\omega^4 = -2\gamma\omega^3 \wedge \omega^4, \tag{99}$$

while the NP equations reduce to $D\gamma = 2V_0$ and $\delta\gamma = \bar\delta\gamma = 0$. Like for the (space–time) homogeneous situation (occurring for $H = 0$, i.e. when τ is constant), $2\omega^3\omega^4$ is then the metric of a two-space of constant curvature. Distinguishing the flat ($V_0 = 0$) and non-flat ($V_0 \neq 0$) cases, one obtains the following solutions:

(a) $V_0 \neq 0$:

$$\omega^1 + \omega^2 = \frac{\tau}{2V_0}\left(\mathrm{d}x + \frac{\mathrm{d}y - \mathrm{d}z}{y}\right), \tag{100}$$

$$\omega^3 = \mathrm{d}z, \quad \omega^4 = \frac{1}{8V_0 y^2}(\mathrm{d}z - \mathrm{d}y) \tag{101}$$

(the τ, x or $y =$ constant surfaces are time-like and the Killing vectors are ∂_z, ∂_x, $y\partial_y + z\partial_z$ and $(z - y)\partial_x + (y - 2z)y\partial_y - z^2\partial_z$.

(b) $V_0 = 0$:

$$\omega^1 + \omega^2 = 4\tau(\mathrm{d}x + z\mathrm{d}y), \tag{102}$$

$$\omega^3 = \mathrm{d}z, \quad \omega^4 = \mathrm{d}y \tag{103}$$

(the τ or $x =$ constant surfaces are time-like and the Killing vectors are ∂_x, ∂_y, $y\partial_y - z\partial_z$ and $y\partial_x - \partial_z$).

In both cases we have a boost-isotropic space–time, with, by (97), a G_4 on T_3.

6. Hypersurface homogeneous KS space–times of classes 3 and 3_N

Solutions of class 3_N cannot be hypersurface-homogeneous, without being also space–time homogeneous. This can be easily seen by substituting the 3_N conditions $\rho \neq 0 = \rho' = \tau = \tau'$ (and hence, by the GHP equations, $\Phi_{01} = \Phi'_{01} = \Phi_{02} = \Phi'_{00}$) in the Bianchi equations, to derive that $\flat U = \flat' U = 0$ and $V = 0$. Imposing hypersurface homogeneity by putting $U = U(X)$ implies then by (9) that also $\eth U = \eth' U = 0$, such that U is constant. All 0-weighted quantities are then constants and we find ourselves again in the space–time homogeneous case, which was treated in [14].

The class 3 space–times can be obtained along the same lines as in class 2 (the two being related by a Sachs transform). We begin by fixing the NP tetrad, modulo a rotation, by requiring $\rho = iQt$ and $\mu = it$, with t real and $Q = \pm 1$. By (9) and hypersurface homogeneity we have $\delta I = (D + Q\Delta)I = 0$ for all invariantly defined quantities I.

From the Bianchi and NP equations one finds $\beta = -\bar\alpha$, together with the analogues of equations (96), namely $\Re\gamma = Q\Re\varepsilon = QH/t$ and

$$Dt = H/2, \quad DH = 2t\left(2QV - t^2\right), \quad DU = QtH, \tag{104}$$

implying

$$U - Qt^2 = U_0 = \text{constant} \tag{105}$$

and

$$\boldsymbol{\omega}^4 - Q\boldsymbol{\omega}^3 = \frac{2}{H}\mathrm{d}t, \tag{106}$$

with $H = H(t)$. The Cartan equations show that $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ are closed, enabling one to partially fix the rotation such that $\boldsymbol{\omega}^1 = P^{-1}\mathrm{d}\zeta$ with P real and ζ complex. From the NP equations it follows then that $P = P(\zeta, \bar{\zeta})$ and

$$\delta\alpha - \bar{\delta}\beta + 4\alpha\beta - 2U_0 = 0, \tag{107}$$

implying that $2\boldsymbol{\omega}^1\boldsymbol{\omega}^2$ is the metric of a two-space of constant curvature and suggesting to introduce coordinates x and y by

$$\boldsymbol{\omega}^1 = \overline{\boldsymbol{\omega}^2} = \frac{\mathrm{d}x + \mathrm{i}x\mathrm{d}y}{1 + kx^2/4} \left(k \equiv 8U_0\right). \tag{108}$$

With H an arbitrary function of t, the one remaining Cartan equation

$$\mathrm{d}\boldsymbol{\omega}^3 = 2\mathrm{i}Qr\boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^1 + \frac{H}{2t}\boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^3, \tag{109}$$

can be integrated to yield

$$\boldsymbol{\omega}^3 = \frac{4Qtxy}{\left(1 + \frac{k}{4}x^2\right)^2}\mathrm{d}x + t\mathrm{d}r. \tag{110}$$

Note that the $t = \text{constant}$ surface is time-like when $Q = 1$ and space-like when $Q = -1$. The x, y or $r = \text{constant}$ surfaces are always time-like. The Killing vectors are ∂_r, $\frac{k}{32}\partial_y + \frac{Q}{kx^2+4}\partial_r$ and

$$\frac{Qx}{kx^2 + 4}(y\cos y - \sin y)\partial_r - \frac{kx^2 + 4}{64}\cos y\partial_x - \frac{kx^2 - 4}{64}\sin y\partial_y,$$

$$\frac{Qx}{kx^2 + 4}(y\sin y + \cos y)\partial_r - \frac{kx^2 + 4}{64}\sin y\partial_x + \frac{kx^2 - 4}{64}\cos y\partial_y.$$

Clearly we have a locally rotationally symmetric (LRS) space–time, admitting a G_4 on T_3 or S_3.

7. Hypersurface homogeneous KS space–times of class 4

When $\rho = \rho' = \tau = \tau' = 0$ the only non-vanishing curvature components are R, Ψ_2 and Φ_{11}, while X is constant. The Bianchi identities immediately give $\flat U = \flat' U = 0$ and $\eth V = \eth' V = 0$, such that hypersurface-homogeneity implies that either U or V should be constants (or both, in which case we have space–time homogeneity). From the Cartan equations it follows that the four basis one-forms are closed, enabling us to partially fix a boost and rotation such that

$$\boldsymbol{\omega}^1 = P^{-1}\mathrm{d}\zeta, \quad \boldsymbol{\omega}^3 = Q^{-1}\mathrm{d}s, \quad \boldsymbol{\omega}^4 = Q^{-1}\mathrm{d}t, \tag{111}$$

with P and Q real. The Cartan equations immediately imply then $P = P(\zeta, \bar{\zeta})$ and $Q = Q(s, t)$, with

$$\alpha = -\bar{\beta} = \frac{1}{2}P_{,\bar{\zeta}}, \quad \varepsilon = -\frac{1}{2}Q_{,t}, \quad \gamma = \frac{1}{2}Q_{,s}. \tag{112}$$

The surviving Newman–Penrose equations are

$$D\gamma - \Delta\varepsilon + 4\varepsilon\gamma = 2V, \tag{113}$$

$$\delta\alpha + \bar{\delta}\bar{\alpha} - 4\alpha\bar{\alpha} = 2U, \tag{114}$$

showing that the space–time is a product of an arbitrary two-space $\mathcal{M}_1$ and a two-space $\mathcal{M}_0$ of constant curvature (the latter's metric being $2\omega^1\omega^2=$ if U is constant and $2\omega^3\omega^4$ if V is constant). Hypersurface-homogeneity implies

that also $\mathcal{M}_1$ should admit a Killing vector and hence the space–time is locally rotationally or boost symmetric (BS), with metric given by

$$ds^2 = e\left[-dt^2 + A^2(t)dx^2\right] + \frac{dy^2 + dz^2}{\left(1 + \frac{k}{4}\left(y^2 + z^2\right)\right)^2}, \quad (e = \pm1) \tag{115}$$

or

$$ds^2 = dx^2 + A^2(x)dy^2 + \frac{du\,dv}{\left(1 + \frac{k}{4}uv\right)^2}. \tag{116}$$

8. Discussion

A complete classification is obtained of the (non-conformally flat) hypersurface homogeneous space–times admitting a non-null Killing spinor of valence two, i.e. of the hypersurface-homogeneous Petrov type D conformal Killing–Yano space–times. In this classification two new families appear of space–times for which the associated Killing tensor is *exceptional*, namely those determined by the metrics (86) and (87). Both belong to class 1 and admit a G_3 on S_3 for $Q = -1$, $|k| < 1$; for all other values of Q and k they admit a G_3 on T_3. The regular members of class I (forming a subclass of the Carter [3] metrics discussed in [6]), are given by (92), (93). The causal character of the hypersurfaces of homogeneity is identical to that of the exceptional metrics. In all cases the isometry group has Bianchi type VI_0. Semi-regular Killing tensors only appear in class 1_N and were discussed in [8], with all solutions being T_3 homogeneous of Bianchi type VIII. The regular members of class 1_N are given by the metrics (94): all are T_3 homogeneous of Bianchi type VI_0. The hypersurface homogeneous members of class 2, 3, 3_N and 4 all admit isometry-groups of dimension >3. The 3_N solutions in particular are homogeneous and were discussed in [14]. The metrics of class 3, given by (108)–(110), turn out to be LRS, admitting a G_4 on T_3 (when $Q = 1$) or on S_3 (when $Q = -1$), while the class 2 metrics (100)–(103) are their BS analogues, admitting a G_4 on T_3. Finally the class 4 metrics are either LRS or BS and admit a G_4 on S_3 or T_3.

All this pertains to the metrics of the *unitary* representants, in which the Killing spinor has modulus 1 ($|K| = 1$), but, the existence of a Killing spinor being a conformally invariant property and hypersurface-homogeneity being preserved under suitable conformal transformations, stronger conclusions hold. First note that it can be easily verified that the LRS metrics (13.1) and (13.2) of [9], namely

$$ds^2 = e\left[-dt^2 + A^2(t)dx^2\right] + B^2(t)\left[dy^2 + \Sigma(y,k)^2 dz^2\right], \tag{117}$$

and

$$ds^2 = e\left[-dt^2 + A^2(t)(dx + \sigma(y,k)dz)^2\right] + B^2(t)\left[dy^2 + \Sigma(y,k)^2 dz^2\right], \tag{118}$$

($e = \pm 1$; $\Sigma(y,k) = \sin y$, $\sinh y$ or y and $\sigma(y,k) = \cos y$, $\cosh y$ or $y^2/2$ according to whether $k = 1, -1$ or 0) both admit a Killing spinor, with (117) belonging to class 4 and (118) to class 3. It follows that the LRS metrics (117) (and their BS counterparts, namely (13.14) in [9]) completely exhaust the hypersurface homogeneous class 4 space–times, while the LRS metrics (118) and their BS counterparts exhaust the hypersurface homogeneous classes 3 and 2 respectively. On the other hand, the third family of LRS space–times (of Bianchi type V or VII_h), given by the metrics (13.3) in [9]

$$ds^2 = e\left[-dt^2 + A^2(t)dx^2\right] + B^2(t)e^{2x}\left[dy^2 + dz^2\right], \quad (e = \pm 1) \tag{119}$$

does *not*[7] admit a Killing spinor: using a Weyl-adapted null tetrad the spin coefficients ρ and μ will be real in all conformal representants, contradicting the condition (10) which should hold in the unitary representant.

What remains to be done is to investigate the possible physical interpretation of any of the conformal representants of these solutions. While this has been partially successful [2, 14, 15] in the homogeneous case and for the regular and semi-regular families [6, 11], close to nothing is known so far about the exceptional metrics.

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[7] This corrects an earlier remark in [15] that all LRS perfect fluids should admit a Killing spinor!