CHROMATIC POLYNOMIALS OF SOME SUNFLOWER MIXED HYPERGRAPHS

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ABSTRACT. The theory of mixed hypergraphs coloring has been first introduced by Voloshin in 1993 and it has been growing ever since. The proper coloring of a *mixed hypergraph* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is the coloring of the vertex set X so that no $\mathcal{D}-$hyperedge is monochromatic and no $\mathcal{C}-$hyperedge is polychromatic. A mixed hypergraph with hyperedges of type $\mathcal{D}$, $\mathcal{C}$ or $\mathcal{B}$ is commonly known as a $\mathcal{D}-$, $\mathcal{C}-$, or $\mathcal{B}-$hypergraph respectively where $\mathcal{B} = \mathcal{C} = \mathcal{D}$. $\mathcal{D}-$hypergraph colorings are the classic hypergraph colorings which have been widely studied. The *chromatic polynomial* $P(\mathcal{H}, \lambda)$ of a mixed hypergraph $\mathcal{H}$ is the function that counts the number of proper $\lambda-$colorings, which are mappings $f : X \to \{1, 2, \dots, \lambda\}$. A *sunflower (hypergraph)* with l petals and a core S is a collection of sets $e_1, \dots, e_l$ such that $e_i \cap e_j = S$ for all $i \neq j$. Recently, Walter published [14] some results concerning the chromatic polynomial of some non-uniform $\mathcal{D}-$sunflower. In this paper, we present an alternative proof of his result and extend his formula to those of non-uniform $\mathcal{C}-$sunflowers and $\mathcal{B}-$sunflowers. Some results of a new but related member of sunflowers are also presented.

1. DEFINITIONS AND NOTATIONS

For basic definitions of graphs and hypergraphs we refer the reader to [1, 4, 13, 15]. A hypergraph $\mathcal{H}$ of order n is an ordered pair $\mathcal{H}$=$(X, \mathcal{E})$, where $|X| = n$ is a finite nonempty set of vertices and $\mathcal{E}$ is a collection of not necessarily distinct non empty subsets of X called (hyper)edges. $\mathcal{H}$ is said to be *$k-$uniform* if the size of each of its hyperedges is exactly k. A hypergraph is said to be *linear* if each pair of hyperedges has at most one vertex in common.

If the alternating sequence of vertices and distinct (hyper)edges $v_0, e_1, v_1, \dots, e_l, v_l$ is a (hyper)path of length $l \geq 2$, then the hypergraph induced by the sequence of hyperedges $e_1, \dots, e_l$ when $v_0 = v_l$ is called a *hypercycle* of length l. A hypergraph in which no set of hyperedges induce a hypercycle is said to be *acyclic*. In this paper all hypergraphs are assumed to be connected and acyclic.

The concept of mixed hypergraph colorings has been studied extensively by Voloshin [13]. A *mixed hypergraph* $\mathcal{H}$ with vertex set X is a triple $(X, \mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ and $\mathcal{D}$ are elements of $\mathcal{E}$, called $\mathcal{C}-$hyperedges and

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$\mathcal{D}-$hyperedges, respectively. Elements of $\mathcal{C} \cap \mathcal{D}$ are called $\mathcal{B}-$hyperedges. A proper coloring of $\mathcal{H}$ is a coloring of X such that each $\mathcal{C}-$hyperedge has at least two vertices with a $\mathcal{C}$ommon color and each $\mathcal{D}-$hyperedge has at least two vertices with $\mathcal{D}$istinct colors. When the vertices of a hyperedge are all colored with the same color, the hyperedge is said to be *monochromatic*. In the event the vertices are all colored differently, the hyperedge is said to be *polychromatic*. Given the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, when $\mathcal{C} = \emptyset$, we write $\mathcal{H} = (X, \mathcal{D})$ and $\mathcal{H}$ is often called a $\mathcal{D}-$hypergraph. In the case when $\mathcal{D} = \emptyset$, we write $\mathcal{H} = (X, \mathcal{C})$ and call the resulting hypergraph a $\mathcal{C}-$hypergraph. In the case when $\mathcal{C} = \mathcal{D}$, we write $\mathcal{H} = (X, \mathcal{B})$ and call $\mathcal{H}$ a $\mathcal{B}-$hypergraph.

The *chromatic polynomial* $P(\mathcal{H}, \lambda)$ of a mixed hypergraph $\mathcal{H}$ is the function that counts the number of proper $\lambda-$colorings, which are mappings, $f : X \to \{1, 2, \ldots, \lambda\}$ with the condition that no $\mathcal{C}-$hyperedge is polychromatic and no $\mathcal{D}-$hyperedge is monochromatic.

We note that while $\mathcal{D}-$(hyper)edges may be of size 2, hyperedges of types $\mathcal{C}$ or $\mathcal{B}$ are of size at least 3 since every $\mathcal{C}-$(hyper)edge of size 2 may be contracted to a single vertex and every $\mathcal{B}-$(hyper)edge of size 2 is uncolorable. We encourage the reader to refer to [5, 6, 10, 11, 13] for detailed information about chromatic polynomials, research, and applications of mixed hypergraph colorings.

For simplicity, throughout this paper, we denote the falling factorial $\lambda^{\underline{t}} = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-t+1)$.

2. Chromatic polynomial of some linear mixed hypergraphs

We begin this section with some known results ([2]) concerning the chromatic polynomials of some linear acyclic mixed hypergraphs also known as (linear) mixed hypertrees. We denote by $\mathcal{H}^l = (X, \mathcal{E})$, a hypergraph of length l, where $|\mathcal{E}|=l$.

Theorem 2.1. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be any $\mathcal{D}-$hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda \prod_{e \in \mathcal{D}} (\lambda^{|e|-1} - 1)$.

The result follows from the fact that there are $\lambda^{|e|} - \lambda = \lambda(\lambda^{|e|-1} - 1)$ ways to properly color each hyperedge $e \in \mathcal{D}$. Similar arguments as in the previous theorem are used in [2] to obtain the next two results.

Theorem 2.2. *Let $\mathcal{H}^l = (X, \mathcal{C})$ be any $\mathcal{C}-$hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda \prod_{e \in \mathcal{C}} (\lambda^{|e|-1} - (\lambda-1)^{\underline{|e|-1}})$.

Theorem 2.3. *Let $\mathcal{H}^l = (X, \mathcal{B})$ be any $\mathcal{B}-$hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda \prod_{e \in \mathcal{B}} (\lambda^{|e|-1} - (\lambda-1)^{\underline{|e|-1}} - 1)$.

The next corollaries concern uniform mixed hypertrees and they follow respectively from Theorems 2.1, 2.2 and 2.3 when $|e| = k$ for each $e \in \mathcal{E}$.

Corollary 2.3.1. *Let* $\mathcal{H}^l = (X, \mathcal{D})$ *be a* $k-$*uniform* $\mathcal{D}-$*hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - 1)^l$.

Corollary 2.3.2. *Let* $\mathcal{H}^l = (X, \mathcal{C})$ *be a* $k-$*uniform* $\mathcal{C}-$*hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - (\lambda - 1)^{\underline{k-1}})^l$.

Corollary 2.3.3. *Let* $\mathcal{H}^l = (X, \mathcal{B})$ *be a* $k-$*uniform* $\mathcal{B}-$*hypertree. Then* $P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - (\lambda - 1)^{\underline{k-1}} - 1)^l$.

3. Chromatic polynomial of some (strong) sunflowers

A *sunflower (hypergraph)* $\mathcal{H}^l = (X, \mathcal{E})$ (also known as a $\Delta-$ system in [7]) with l petals and a core S is a collection of sets $e_1, \ldots, e_l$ such that $e_i \cap e_j = S$ for all $i \neq j$. The elements of the core are called *seeds*. A Venn diagram of these sets would look like a sunflower. Observe that any family of pairwise disjoint sets is a sunflower (with an empty core) and a *hyperstar* is a sunflower with a core of size 1. Figure 1 is an example of a $5-$uniform sunflower with a core S of size 3 having e_1, e_2 and e_3 as petals.

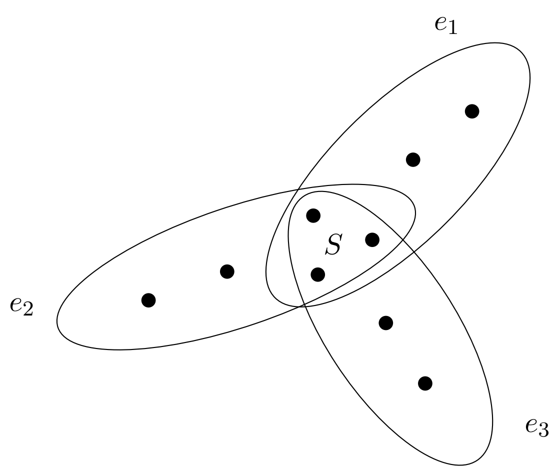

Figure 1: A $5-$uniform sunflower

Erdos-Rado sunflower Lemma [7] gives a necessary condition for the existence of a Sunflower given any collection of uniform sets. This condition is a lower bound for the size (or cardinality) of the collection although it is not known if the bound is best possible. We restate the lemma without proof, which is by induction on k.

Sunflower Lemma: *Given any collection of* n *distinct sets of size* k *(from a universal set) with* $n > k!(l-1)^k$*, there is a subcollection of* $l-$*sets that forms a sunflower.*

Our results assume sunflower mixed hypergraphs with a core S and petals $\{e_1, \ldots, e_l\}$ such that $|S| \leq |e_i| - 1$ for $1 \leq i \leq l$. We begin with a recent result of Walter [14] about sunflowers with petals of type $\mathcal{D}$. We note that when $|S| = 1$, it is easy to verify that we obtain the formula of a $\mathcal{D}-$hypertree, particularly the chromatic polynomial of a $\mathcal{D}-$hyperstar.

Theorem 3.1. [14] *Let $\mathcal{H}^l = (X, \mathcal{D})$ be any $\mathcal{D}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n - \lambda^{n-|S|+1} + \lambda \prod_{e \in \mathcal{D}} (\lambda^{|e|-|S|} - 1).$$

Proof. Suppose the core is monochromatic. Then there are $\lambda^{|e|-|S|} - 1$ ways to color the remaining vertices of each petal so that no petal is monochromatic, giving $\lambda \prod_{e \in \mathcal{D}} (\lambda^{|e|-|S|} - 1)$ proper colorings. Otherwise, there are $\lambda^{|S|} - \lambda$ ways to color the core. For each such coloring, there are $\lambda^{|e|-|S|}$ ways to color the remaining vertices of each petal $e \in \mathcal{D}$, giving $\prod_{e \in \mathcal{D}} \lambda^{|e|-|S|}(\lambda^{|S|} - \lambda)$ proper colorings. Further, since $n = |S| + \sum_{e \in \mathcal{D}}(|e| - |S|)$, the result gives all proper colorings. □

As mentioned earlier, the previous result was first established by Walter as the chromatic polynomial of non-uniform $\mathcal{D}-$sunflowers (albeit using a substantially different argument) to extend Tomescu's result for uniform $\mathcal{D}-$sunflowers [9]. We present Tomescu's result in the following corollary with a slightly different notation by assuming that $|S| = k - p$ for $2 \leq k < p$. This result follows from Theorem 3.1 when $|e| = k$ for each $e \in \mathcal{D}$.

Corollary 3.1.1. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be a $k-$uniform $\mathcal{D}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n - \lambda^{n-k+p+1} + \lambda(\lambda^p - 1)^l.$$

Similar results as in Theorem 3.1 and Corollary 3.1.1 have been recently established by White in the form of multivariate polynomials using a deletion-contraction-extraction recurrence. For detailed results of his technique and proofs, see [16].

Our next results address sunflower mixed hypergraphs with petals of the same type $\mathcal{C}$ or $\mathcal{B}$. A similar counting argument as in Theorem 3.1 is used to prove the next theorem. Further, when $|S| = 1$, it is easy to verify from the result that we obtain the formula of a $\mathcal{C}-$hypertree, particularly the chromatic polynomial of a $\mathcal{C}-$hyperstar.

Theorem 3.2. *Let $\mathcal{H}^l = (X, \mathcal{C})$ be any $\mathcal{C}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n - \lambda^{\underline{|S|}} \prod_{e \in \mathcal{C}} \lambda^{|e|-|S|} + \lambda^{\underline{|S|}} \prod_{e \in \mathcal{C}} (\lambda^{|e|-|S|} - (\lambda - |S|)^{\underline{|e|-|S|}}).$$

Proof. Suppose the core is polychromatic. There are $\lambda^{\underline{|S|}}$ ways to color its seeds. For each such coloring, there are $\lambda^{|e|-|S|} - (\lambda - |S|)^{\underline{|e|-|S|}}$ ways to color the remaining vertices of each petal so that no petal is polychromatic, giving the last term. Otherwise, there are $\lambda^{|S|} - \lambda^{\underline{|S|}}$ ways to color the core, giving the first two terms after an expansion. □

In the next theorem, we assume $|S| \geq 2$. Here, when $|S| = 1$, we refer the reader to the result of Theorem 2.3, which is equivalent to that of a $\mathcal{B}-$hyperstar.

Theorem 3.3. *Let $\mathcal{H}^l = (X, \mathcal{B})$ be any $\mathcal{B}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n - (\lambda^{\underline{|S|}} + \lambda) \prod_{e \in \mathcal{B}} \lambda^{|e|-|S|} + \lambda \prod_{e \in \mathcal{B}} (\lambda^{|e|-|S|} - 1) +$$

$$\lambda^{\underline{|S|}} \prod_{e \in \mathcal{B}} (\lambda^{|e|-|S|} - (\lambda - |S|)^{\underline{|e|-|S|}}).$$

Proof. Given any proper coloring of $\mathcal{H}^l$, one of the following is true:

(i) The core is polychromatic giving $\lambda^{\underline{|S|}}$ colorings. For each such coloring, there are $\lambda^{|e|-|S|} - (\lambda - |S|)^{\underline{|e|-|S|}}$ ways to color the remaining vertices of each petal so that no petal is polychromatic, giving the last term. The condition that $|S| \geq 2$ ensures that no hyperedge of $\mathcal{H}^l$ is monochromatic in this case.

(ii) The core is monochromatic giving λ colorings. For each such coloring, there are $\lambda^{|e|-|S|} - 1$ ways to color the remaining vertices of each petal so that no petal is monochromatic, giving the third term.

(iii) The core is neither polychromatic nor monochromatic giving $\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda$ colorings. For each such coloring, there are $\lambda^{|e|-|S|}$ ways to color the remaining vertices of each petal, giving the first two terms after an expansion. □

The following corollaries follow respectively from Theorems 3.2 and 3.3 when for each $e \in \mathcal{E}$, $|e| = k$ and $|S| = k - p$ with $k \geq 3$. We note that $n = k - p + pl$.

Corollary 3.3.1. *Let $\mathcal{H}^l = (X, \mathcal{C})$ be a $k-$uniform $\mathcal{C}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n + \lambda^{\underline{k-p}}((\lambda^p - (\lambda - k + p)^{\underline{p}})^l - \lambda^{pl}).$$

Corollary 3.3.2. *Let $\mathcal{H}^l = (X, \mathcal{B})$ be a $k-$uniform $\mathcal{B}-$sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda^n + \lambda^{\underline{k-p}}((\lambda^p - (\lambda - k + p)^{\underline{p}})^l - \lambda^{pl}) - \lambda^{pl+1} + \lambda(\lambda^p - 1)^l.$$

We define a $\mu-$*linear* set to be a collection of nonempty sets such that the intersection of any pair of its member is either trivial or μ. Thus, a sunflower with core size μ is a $\mu-$linear set (or hypergraph). The case when $\mu = 1$, a $1-$*linear* hypergraph is simply a *linear* hypergraph. On the other hand we define a set to be $\mu-$*nonlinear* when the cardinality of the intersection of any pair of its members is at most $\mu > 1$. Simply put, any collection of sets is either $\mu-$*linear* or $\mu-$*nonlinear* for some μ.

The problem of finding the chromatic polynomial of a $\mu-$linear or a $\mu-$nonlinear hypergraph of arbitrary length when $\mu > 1$ is NP hard [3].

Currently we are unaware of any known formulae of either family besides the ones we presented or quoted in this paper when $\mu > 1$. To encourage further work in this direction we introduce in the next section an acyclic hypergraph which chromatic polynomial we explicitly obtained when $\mu = 1$ and 2.

4. Chromatic polynomial of some weak sunflowers

A *transversal* (or *blocking set*) of $\mathcal{F} = \{e_1, \ldots, e_l\}$ is a set which intersects every member of $\mathcal{F}$. A transversal with the least member is often referred to as a *covering set* and its size is called a *covering number* (or *blocking number*). The core S of a sunflower is a transversal and S is a covering set if $|S| = 1$. Let $\mathcal{F} = \{e_1, \ldots, e_l\}$ be a collection of pairwise disjoint sets and we denote by S a transversal of the collection. We call S the core of the collection and its elements are the seeds. The members of the collection will be referred to as petals. Thus, it is natural to refer to the hypergraph $\mathcal{H}^l = (X, \mathcal{E})$ with $\mathcal{E} = S \cup \mathcal{F}$ as a *weak sunflower* with $l-$petals. Figure 2 is a representation of a $4-$uniform weak sunflower with a core S of size 6 having petals e_1, e_2 and e_3.

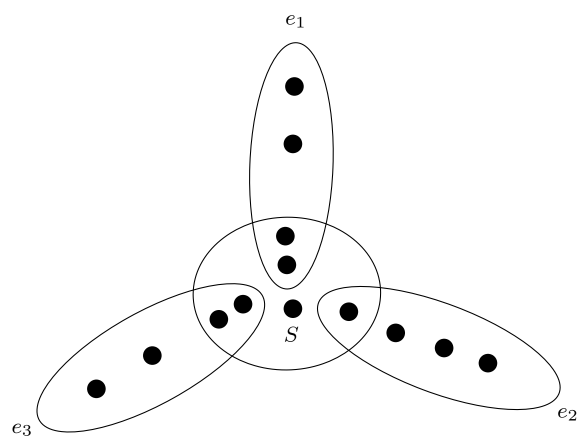

Figure 2: A $4-$uniform weak sunflower

Just as Erdos-Rado sunflower Lemma gives a necessary condition for the existence of a (strong) sunflower it will be interesting to find a condition on the existence of a weak sunflower given a collection of n distinct $k-$uniform sets.

We assume $|e_i \cap S| = \mu$ and discuss the chromatic polynomials of some weak sunflowers when $\mu = 1$ and 2. When $\mu = 1$, we obtain the formulae for linear acyclic mixed hypergraphs of length $l + 1$ in the next theorems. Also we assume $|S| = 2$ and each petal $e_1, \ldots, e_l$ is of the same type $\mathcal{D}$, $\mathcal{C}$

or $\mathcal{B}$. In the case when the petals are of type $\mathcal{D}$ we assume $|e_i| \geq 2$; $|e_i| \geq 3$ when each petal is of type $\mathcal{C}$ or $\mathcal{B}$.

Theorem 4.1. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be a linear weak $\mathcal{D}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = (\lambda^{|S|} - \lambda) \prod_{e \in \mathcal{D}} (\lambda^{|e|-1} - 1).$$

Proof. There are $\lambda^{|S|} - \lambda$ ways to color the seeds of the core such that they are not monochromatic. For each such coloring, there are $\lambda^{|e|-1} - 1$ ways to color the remaining $|e| - 1$ vertices of each disjoint petal so that no petal is monochromatic as well. □

The next two results follow a similar counting argument as in Theorem 4.1 when considering petals of types $\mathcal{C}$ and $\mathcal{B}$ respectively.

Theorem 4.2. *Let $\mathcal{H}^l = (X, \mathcal{C})$ be a linear weak $\mathcal{C}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = (\lambda^{|S|} - \lambda^{\underline{|S|}}) \prod_{e \in \mathcal{C}} (\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}}).$$

Proof. There are $\lambda^{|S|} - \lambda^{\underline{|S|}}$ ways to color the seeds so that they are not polychromatic. For each such coloring, there are $\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}}$ ways to color the remaining $|e| - 1$ vertices of each disjoint petal so that no petal is polychromatic. The result follows. □

Theorem 4.3. *Let $\mathcal{H}^l = (X, \mathcal{B})$ be a linear weak $\mathcal{B}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = (\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda) \prod_{e \in \mathcal{B}} (\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}} - 1).$$

Proof. There are $\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda$ ways to color the seeds so that they are neither monochromatic nor polychromatic. For each such coloring, there are $\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}} - 1$ ways to color the remaining $|e| - 1$ vertices of each disjoint petal. Hence the result. □

Suppose $\mathcal{H}^l = (X, \mathcal{E})$ is a weak sunflower mixed hypergraph with the set of petals $\{e_1, \ldots, e_l\}$ such that $|e| = k$ for every $e \in \mathcal{E}$. If $|S| = k$ then $\mathcal{H}^l$ is said to be *$k-$uniform*. The case when S is a covering set (i.e., when $|S| = l$), we simply let $l = k$. The next Corollaries follow respectively from Theorems 4.1, 4.2 and 4.3 when $|S| = |e| = k$ for each $e \in \mathcal{E}$.

Corollary 4.3.1. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be a linear $k-$uniform weak $\mathcal{D}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - 1)^{l+1}.$$

Corollary 4.3.2. *Let $\mathcal{H}^l = (X, \mathcal{C})$ be a linear $k-$uniform weak $\mathcal{C}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - (\lambda - 1)^{\underline{k-1}})^{l+1}.$$

Corollary 4.3.3. *Let $\mathcal{H}^l = (X, \mathcal{B})$ be a linear $k-$uniform weak $\mathcal{B}-$sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-1} - (\lambda - 1)^{\underline{k-1}} - 1)^{l+1}.$$

The arguments previously used in finding the chromatic polynomials of linear weak sunflowers cannot be extended to find the chromatic polynomial of any $2-$linear weak sunflower. We use an inductive argument to establish the recursion in the next theorem. The result assumes that each hyperedge/and core is of type $\mathcal{D}$, although the argument can be extended to hyperedge/and core of type $\mathcal{C}$ or $\mathcal{B}$ as well.

Theorem 4.4. *Suppose $\mathcal{H}^l = (X, \mathcal{D})$ is a $2-$linear weak sunflower. Then for all $l \geq 1$,*

$$P(\mathcal{H}^l, \lambda) = (\lambda^{|e_l|-2} - 1)P(\mathcal{H}^{l-1}, \lambda) + \lambda^{\underline{2}}\lambda^{|e_l|+|S|-4}\prod_{i=1}^{l-1}(\lambda^{|e_i|} - \lambda).$$

Proof. When $l = 1$ we let $S \cap e_1 = \{u, v\}$. We count the cases when $f(u) = f(v)$ and when $f(u) \neq f(v)$ respectively as follow: (i) we identify u and v as a new vertex giving $\lambda(\lambda^{|e_1|-2} - 1)(\lambda^{|S|-2} - 1)$ colorings. (ii) we connect u and v with an edge. In this case, having satisfied the necessary condition for a proper coloring of both S and e_1, we proceed to color their remaining vertices with no restriction to obtain $\lambda^{\underline{2}}\lambda^{|S|+|e_1|-4}$ colorings. Thus, the number of colorings when $l = 1$ is given by

$$P(\mathcal{H}^1, \lambda) = \lambda(\lambda^{|e_1|-2} - 1)(\lambda^{|S|-2} - 1) + \lambda^{\underline{2}}\lambda^{|S|+|e_1|-4}. \tag{1}$$

This formula is supported by the expression in the theorem when $l = 1$ since $P(\mathcal{H}^0, \lambda) = \lambda(\lambda^{|S|-2} - 1)$.

For $l \geq 2$, we let $S \cap e_l = \{u, v\}$ and assume that $P(\mathcal{H}^{l-1}, \lambda)$ counts the number of proper colorings of a weak sunflower with $l - 1$ petals.

Similarly as in the base case, in any proper coloring of H^l, one of the following must be true:

Case (i) : $f(u) = f(v)$. We identify u and v by a new vertex w as the intersection of both e_l and S. For each coloring f of the $P(\mathcal{H}^{l-1}, \lambda)$ proper colorings of $\mathcal{H}^{l-1}$, there exists $\lambda^{|e_l|-2} - 1$ colorings of $V(e_l) \setminus w$, giving the first term.

Case (ii) : $f(u) \neq f(v)$. For any proper coloring of the remaining $|e_l| + |S| - 4$ vertices of both S and e_l, we have $\lambda^{|e_i|} - \lambda$ ways to color the vertices of each disjoint petal $e_1, \ldots, e_{l-1}$, giving the second term. The result follows. □

The next corollary follows from Theorem 4.4 when equation (1) is used as the standard basis for the recursion.

Corollary 4.4.1. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be a 2−linear weak sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{|S|-2} - 1) \prod_{1 \le i \le l} (\lambda^{|e_i|-2} - 1) +$$

$$\sum_{1 \le i \le l} \Phi(\lambda, i) \prod_{\substack{i+1 \le j \le l \\ 1 \le k \le i-1}} (\lambda^{|e_j|-2} - 1)(\lambda^{|e_k|} - \lambda)$$

for all $l \ge 1$ where

$$\Phi(\lambda, i) = \lambda^{|e_i|+|S|-2i} - \lambda^{|e_i|+|S|-2i-1}.$$

When $|e_i| = |S| = k$ for all $i = 1, \ldots, l$, we obtain from Corollary 4.4.1 the next result.

Corollary 4.4.2. *Let $\mathcal{H}^l = (X, \mathcal{D})$ be a k−uniform 2−linear weak sunflower. Then*

$$P(\mathcal{H}^l, \lambda) = \lambda(\lambda^{k-2} - 1)^{l+1} + \sum_{1 \le i \le l} (\lambda^{2k-2i} - \lambda^{2k-2i-1})(\lambda^k - \lambda)^{i-1}(\lambda^{k-2} - 1)^{l-i}$$

for all $l \ge 1$.

Further chromatic polynomial formulae can be derived using similar arguments as in the previous theorems by considering the core of a sunflower to be of a different type ($\mathcal{D}$, $\mathcal{C}$ or $\mathcal{B}$) when compared to its petals.

Two mixed hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are *chromatically equivalent* or simply χ−*equivalent* if $P(\mathcal{H}_1, \lambda) = P(\mathcal{H}_2, \lambda)$ and we write $\mathcal{H}_1 \sim \mathcal{H}_2$. A mixed hypergraph $\mathcal{H}$ is χ−*unique* if $\mathcal{H}' \sim \mathcal{H}$ implies that $\mathcal{H}'$ is isomorphic to $\mathcal{H}$. The chromaticity of a mixed hypergraph studies whether a mixed hypergraph $\mathcal{H}$ is χ−unique. In other words, does $P(\mathcal{H}, \lambda)$ provide enough information so that $\mathcal{H}$ is uniquely determined? Attempting to answer this question for μ−(non)linear mixed hypergraphs is not easy, although Tomescu [8, 9] had shown that $\mathcal{D}$−sunflowers are chromatically unique in general. Further work still needs to be done with regard to the chromaticity of mixed sunflowers with hyperedges of type $\mathcal{C}$ or $\mathcal{B}$. Though it is not too difficult to see that acyclic mixed hypergraphs are not χ−*unique* in general, it remains however unclear to us what can be said about the class of mixed hypergraphs that are chromatically equivalent to μ−(non)linear acyclic mixed hypergraphs. Indeed, a weak sunflower with l petals is isomorphic to a μ−linear hyperpath of length $l \le 3$ and a strong sunflower with l petals is isomorphic to a μ−linear hyperpath of length $l \le 2$. We also hope that further work can be done to compare the chromatic polynomial of some 2−linear acyclic hypergraph of length $l > 3$ to that of a 2−linear weak sunflower with the goal of finding the formulae for other members of both μ−linear and μ−nonlinear mixed hypergraphs when $\mu > 1$.

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References

1. J.A. Allagan, *The chromatic polynomials of some linear uniform hypergraphs*, Congr. Numerantium **187** (2007), 156-160.
2. J.A. Allagan, D. Slutzky, *Chromatic polynomials of mixed hypercycles* (to appear).
3. J.A. Allagan, D. Slutzky, *Chromatic polynomials of mixed hypergraphs*, Australasian Journal of Combinatorics **58(1)** (2014), 197-213.
4. C. Berge, *Graphs and Hypergraphs*, North Holland, 1989.
5. Cs. Bujts, Zs. Tuza, *Mixed colorings of hypergraphs*, Electronic Notes in Discrete Mathematics **24** (2006), 273-275.
6. F.M. Dong, K.M. Koh, and K.L. Teo, *Chromatic polynomials and chromaticity of graphs*, Singapore: World Scientific Publishing. xxvii, 2005.
7. P. Erdös, R. Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. **35** (1960), 85-90.
8. I. Tomescu, *Sunflower hypergraphs are chromatically unique*, Discrete Math. **285** (2004), 355-357.
9. I. Tomescu, *On the chromaticity of sunflower hypergraphs SH(n,p,h)*, Discrete Math. **307 (6)** (2007), 781-786.
10. V.I. Voloshin, H. Zhou, *Pseudo-chordal mixed hypergraphs*, Discrete Math. **202** (1999), 239-248.
11. V.I. Voloshin, *The mixed hypergraphs*, Computer Science Journal of Moldova **1(1)** (1993), 45-52.
12. V. Voloshin, *On the upper chromatic number of a hypergraph*, Australasian Journal of Combinatorics **11** (1995), 25-45.
13. V. Voloshin, *Coloring mixed hypergraphs: Theory, Algorithms and Applications*, American Mathematical Society, 2002.
14. M. Walter, *Some results on chromatic polynomials of hypergraphs*, Electronic Journal of Combinatorics **16** (2009), R94.
15. D. West, *Introduction to Graph Theory*, Prentice Hall, 2001.
16. J.A. White, *On multivariate chromatic polynomials of hypergraphs and hyperedge elimination*, Electronic Journal of Combinatorics **18(1)** (2011), P160.

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