Tutte Polynomials of Some Graphs

by

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ABSTRACT

Given any graph G, there is a bivariate polynomial called Tutte polynomial which can be derived from G. We denote such polynomial by $T(G; x, y)$. This thesis introduces the two techniques commonly used to compute $T(G; x, y)$ along with several examples. Further, we determine $T(G; x, y)$ for various classes of graphs such as cycles, trees, cacti, $\theta(2, 2, 1)$, which is a multi-bridge graph, and the well-known Peterson graph. We plot these surfaces, their contours and, for each such graph G, we evaluate their $T(G; x, y)$ for some values (x, y) along a curve. We obtain important information about these graphs namely the number of spanning trees and number of spanning subgraphs. We also introduced some related polynomials such as the chromatic polynomial, the flow polynomial and the reliability polynomial.

DEDICATION This Thesis is dedicated to Anthony Meadows, Shander Meadows, Tiara Payton, Enid Kimble, and Melvin Kimble. Thank you all for the individual roles you have played in my journey to completing this research. Lastly this thesis is dedicated to the original and greatest mathematician, God Almighty.

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Firstly, I would like to express my sincere gratitude to my advisor Dr. Allagan for the continuous support of my study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor.Besides my advisor, I would like to thank the rest of my thesis committee for their insightful comments and encouragement.Lastly My sincere thanks goes to Dr. Jones who has provided me the greatest opportunity grow as a mathematician. Without his support it would not be possible to conduct this research.

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Chapter 1 Introduction

1.1 Background and Overview

The fundamental idea of graphs were first innovated in the 1700s by Swiss mathematician Leonhard Euler. His efforts and solution to the notable Königsberg bridge problem are ordinarily referred to as the root of graph theory. The German city of Königsberg, known today as Kaliningrad, Russia, is located on the Pregolya river. The geographical design is composed of four primary bodies of land joined together by seven bridges. The dilemma presented to Euler was a simple problem. Was it possible to travel across town in such a way that one would cross over every bridge once, and only once? This later became known as the Euler walk. Euler, understanding that the primary problems were the four bodies of land and the seven bridges, preceded to draw out the first known visual representation of a modern graph. A modern graph, is represented by a set of points, known as vertices or nodes, that are connected by a set of connecting lines known as edges. By first attempting to create paths in the graph, then later experimenting with multiple theoretical graphs with alternating number of lines and dots, or vertices and edges, he eventually concluded a general rule. In order to walk without repeating an edge, or in an Euler path , a graph can have none or two odd number of nodes. From there, the area of mathematics known as graph theory would lack much key progress for decades.

When William Thomas Tutte started his doctoral research in 1945, he had ideas in graph theory that originated in his study of squaring the square. During the beginning stages of his PhD research, Tutte told his supervisor Wylie that he had found a non-hamiltonian planar cubic graph, assuring Wylie that it was 3-connected and that it was a counterexample to Tait's conjecture. Tait's conjecture states that "Every 3-

connected planar cubic graph has a Hamiltonian cycle along the edges through all its vertices". His supervisor, however, was not impressed. In those times, graph theory still had a low reputation in the mathematical world. Because of this, Tuttes Ph.D. supervisor, advised him to drop graph theory and take up something that was actually respectable, such as differential equations. Nevertheless, Tutte's pursued doctoral research was in the mostly uncharted area of graph theory. He started with four papers, each of lasting significance in mathematics today. The first was the counterexample to Tait's conjecture. The second was a study of symmetry in graphs. The third contained what was to become the most famous of Tutte's discoveries, his 1-factor theorem. The last in this group, 'a ring in graph theory', identifies a function that satisfies a natural product rule, which he spoke of as a V-function. In a later paper, he specializes V-function to 'dichromate': other research workers preferred 'Tutte Polynomial', the name by which this function is now known. These research papers were predecessors to Tutte's PhD thesis. Tutte explained: 'My thesis attempted to reduce Graph Theory to Linear Algebra. It showed that many graph-theoretical results could be generalized as algebraic theorems about structures I called 'chain-groups'. Essentially, I was discussing a theory of matrices in which elementary operations could be applied to rows but not columns.' This is what we know today as matroid theory.

Tutte was able to express edges and vertices as algebraic equations using the Tutte polynomial. The Tutte polynomial of a graph is a 2-variable polynomial of significant importance in mathematics, as well as statistical physics, and biology. In a strong sense it “contains” every graphical invariant that can be computed by deletion and contraction. The Tutte polynomial can be evaluated at particular points (x,y) to give numerical graphical invariants, including the number of spanning trees, the number of forests, the number of connected spanning subgraphs, the dimension of the bicycle space and many more. The Tutte polynomial also specialises to a variety of single-variable graphical polynomials of independent combinatorial interest, including the chromatic polynomial, the flow polynomial and the reliability polynomial.

William Tutte was a famous mathematician and code breaker for Britain during

War World II. His contributions to mathematics as a whole are extensive, but his expertise with discrete mathematics, graph theory, and matroid theory was unparalleled. After his undergraduate time at Cambridge studying chemistry, he decided to join Bletchley Park to practice code breaking. During his time at Bletchley Park, he worked alongside prominent mathematicians such as Alan Turing. After this, he decided to return to Cambridge to study mathematics at the doctoral level where he pioneered the study of algebraic representations of graphs (later known as matroid theory), see [1].

His development of the Tutte Polynomial began when he was in undergraduate studies. He and three friends decided to look at the idea of a perfect rectangle. Specifically, "dissecting" a rectangle into unequal squares. This eventually would follow him as he found his way back to mathematics in his graduate degree. His curiosity began with a string of polynomials and their relationship to graphs. The first was the Kirchhoff equations for networks. He later found himself looking at Whitney's work with chromatic polynomials, which he later attributed much of his work in Tutte Polynomials to. Infatuated with the work, generalizing flow polynomials led him to the study of V-Functions and W-Functions. While working more with the W-Functions, Tutte began to simplify and develop the Tutte Polynomial that we know today, see [2]. Tutte polynomials have now found their way into other fields such as knot theory and statistical physics, see [3] and [4].

In the next section of this chapter (Chapter 1), we provide some essential definitions in the area of graph theory. Then, in Chapter 2, we introduce the tools to compute Tutte polynomials, namely a rank-definition technique and a recursive technique. Several examples are given for each technique. In Chapter 3, after introducing some fundamental definitions, we prove some results for several classes of graphs, namely some cyclic graphs including cacti and some 2-trees such as fan. In Chapter 4, we show three related univariate polynomials which are well-known for their applications: chromatic polynomial, flow polynomial and reliability polynomial. In Chapter 5, we introduce the reader to some applications of Tutte polynomials. Some of the evaluations of this function give important invariants of the graphs; the spanning subgraphs, the spanning trees, etc. We also added some 3-D and contour plots for several examples. We close this thesis with Chapter 6 where we introduce the computation process of the Tutte polynomial of multigraphs.

1.2 Graph Theory Preliminaries

A **simple graph** $G = (V, E)$ consists of $V = V(G)$, a nonempty set of objects called **vertices** (or nodes) and $E = E(G)$, a set of an unordered pair of distinct vertices called **edges**.

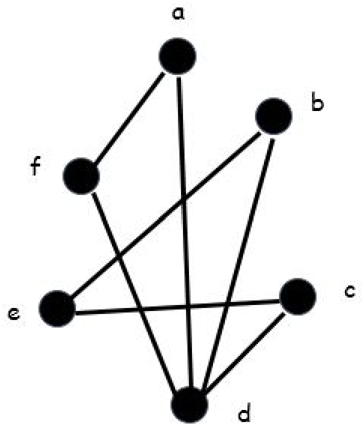


Figure 1.1: Example of a simple graph on 6 vertices

See Figure 1.1, for example. Vertices, say u and v that share an endpoint are said to be **adjacent**; u is also said to be a neighbor of v and vice-versa the edge denoted by uv is said to be incident to the vertices u and v. The **order** of the graph G is the size of its vertex set which we denote by $|V|$ and the size of the edge set, denoted by $|E|$, is called **size** of the graph G. The degree of vertex, v denoted by $deg(v)$, is the number of edges incident to v ; that is the size of its neighbor.

A vertex of degree 0 is said to be **isolated** while a vertex of degree 1 is called a **leaf**. The **minimum degree** of G, denoted by $\delta(G)$, is its smallest vertex degree, and the **maximum degree** of G denoted by $\Delta(G)$ is the largest degree among its vertices. A vertex u is said to be connected to a vertex v, in a graph G, if there exists a sequence of edges (or path) from u to v in G. A graph G is **connected** if there is a path that connects every two of its vertices. Otherwise, it is said to be **disconnected**. In which case, it has two or more *components*.

An edge $e \in E(G)$ with ends $u, v \in V(G)$ is denoted by $\{u, v\}$ or uv; e is said to be **incident** with u and v. An edge $\{u, u\}$ is called a *loop*. An edge $\{u, v\}$ that occurs more than once in E is called a **multiple** (or **parallel**) edge. A graph G is said to be **isomorphic** to a graph H if G can be obtained by relabelling the vertices of H; and we write $G \cong H$.

There are other types of graphs such as **multigraphs** (when multiple edges are allowed between vertices), **pseudographs** (when a vertex is allowed to be connected to itself, as in a loop) and **directed** graphs (when each edge is given an orientation, using an arrow).

1.2.1 Subgraphs

Given a graph G with vertex set $V(G)$ and edge set $E(G)$, we call a graph H a **subgraph** of G if the vertex set $V(H) \subseteq V(G)$ and the edge set $E(H) \subseteq E(G)$; H is obtained from G by deleting edges (including incident vertices) and/or vertices from G.

1.2.2 Spanning subgraphs

Suppose H is a subgraph of G. If $V(H) = V(G)$ and $E(H) \subseteq E(G)$, then H is said to **span** G. See Figure 1.2 shows some cyclic spanning subgraphs while 2.1 shows some examples of spanning trees and forests (not connected trees). See Figure

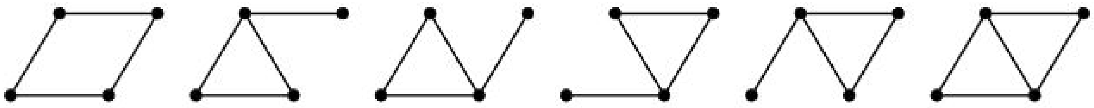


Figure 1.2: Some spanning subgraphs of $\theta(1, 2, 2)$

Chapter 2 Techniques for Computing Tutte Polynomials

Every graph G has an associated polynomial in two variables called the Tutte polynomial which we denote by $T(G;x,y)$. In this section we define the polynomial in two equivalent ways: a decomposition on all the subgraphs of a graph with the same set of vertices and a subset of the edges also known as *rank or nullity* and a recursive definition (or an algorithm) on the edges of the graph. We also give several examples of how to calculate the Tutte polynomial for graphs by either approach.

2.1 Rank Definition of Tutte Polynomials

The *Tutte polynomial* of a graph $G=(V,E)$ denoted by $T(G;x,y)=T_G(x,y)=T(G)$ is a bivariate (two variables) polynomial. Many problems in graph theory can be reduced to problems of finding and evaluating the Tutte polynomial at certain values. Here, we define the Tutte polynomial for graphs. Subsequent Chapters will rely on these graphs whether to plot Tutte polynomials, to evaluate Tutte polynomials, or to produce other related polynomials.

Definition 2.1.1 (Definition: Rank-Nullity)**.** *If $A \subseteq E$, we define the* ***rank*** *of A to be $r(A)=|V|-c(A)$, where $c(A)$ represents the number of connected components of the graph induced subgraph (V,A) and the* ***nullity*** *of A, is $|A|-r(A)$. Thus, Tutte polynomial is defined (rank-nullity) as*

$$T(G;x,y)=\sum_{S\subseteq E}(x-1)^{c(S)-c(E)}(y-1)^{c(S)+|S|-|V|},$$

where $c(S)$ is the number of components in the spanning subgraph (V,S), $|V|$ is the number of vertices in V, and $|S|$ is the number of edges in $S\subseteq E$.

A. Basic Notation Meanings

G	A graph.
V	The vertex set of G.
E	The edge set of G.
$G = (V, E)$	A graph with vertex set V and edge set E.
$S \subseteq E$	An edge set S that is a subset of E.
$C(S)$	The number of components in the spanning subgraph(s) (V, S).
$c(E)$	The number of components in the graph of $G = (V, E)$.
$\|S\|$	The number of edges in the spanning subgraph(s) (V, S).
$\|V\|$	The number of vertices in $G = (V, E)$.

B. Basic Steps

Step 1) Looking at the original graph, G, find the following: $c(E)$ and $|V|$.

Step 2) Allow $|S| = 0, 1, 2, ..., |E|$ and find all spanning subgraphs with the corresponding edge set.

Step 3) Calculate $c(S)$ and $|S|$ for each (set of) spanning subgraph(s).

Step 4) Calculate the partial Tutte Polynomial for each (set of) spanning subgraph(s).

Step 5) Multiply the corresponding partial Tutte Polynomials by the number of subgraphs in the set and sum all the partial Tutte Polynomials together to achieve the Tutte Polynomial for that specific graph.

2.1.1 Examples

Example 1: Tutte Polynomial of a cycle on 4 vertices

Here, we consider a cycle C_4, and apply the definition technique of computing the Tutte polynomial. We follow the steps outlined earlier. Figure 2.1 shows details of the following steps.

Step 1) Looking at the original graph, you can see that there are four vertices, $|\mathbf{V}| = \mathbf{4}$, and there is only one component, $\mathbf{c(E)} = \mathbf{1}$.

Step 2) Listed above.

Step 3) Look at the lists of spanning subgraphs and find how many components and edges represent that set of subgraphs.

$$\mathbf{c(S_0) = 4;\ |S_0| = 0}$$

$$\mathbf{c(S_1) = 3;\ |S_1| = 1}$$

$$\mathbf{c(S_2) = 2;\ |S_2| = 2}$$

$$\mathbf{c(S_3) = 1;\ |S_3| = 3}$$

$$\mathbf{c(S_4) = 1;\ |S_4| = 4}$$

Step 4) Calculate the partial Tutte Polynomial for each set of spanning subgraph(s).

$$\begin{aligned} T(S_0; x, y) &= (x-1)^{4-1}(y-1)^{4+0-4} \\ &= (x-1)^3 \end{aligned} \tag{2.1}$$

$$\begin{aligned} T(S_1; x, y) &= (x-1)^{3-1}(y-1)^{3+1-4} \\ &= (x-1)^2 \end{aligned} \tag{2.2}$$

$$\begin{aligned} T(S_2; x, y) &= (x-1)^{2-1}(y-1)^{2+2-4} \\ &= (x-1)^1 \end{aligned} \tag{2.3}$$

$$\begin{aligned} T(S_3; x, y) &= (x-1)^{1-1}(y-1)^{1+3-4} \\ &= 1 \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
T(S_4;x,y) &= (x-1)^{1-1}(y-1)^{1+4-4} \\
&= (y-1)
\end{aligned}
\tag{2.5}
$$

Step 5) Multiply the partial Tutte Polynomials by the number of subgraphs in each set and sum the results.

$$
\begin{aligned}
T(G;x,y) &= T(S_0;x,y)+4T(S_1;x,y)+6T(S_2;x,y)+4T(S_3;x,y)+T(S_4;x,y) \\
&= (x-1)^3+4(x-1)^2+6(x-1)+4\times 1+(y-1) \\
&= x^3+x^2+x+y.
\end{aligned}
\tag{2.6}
$$

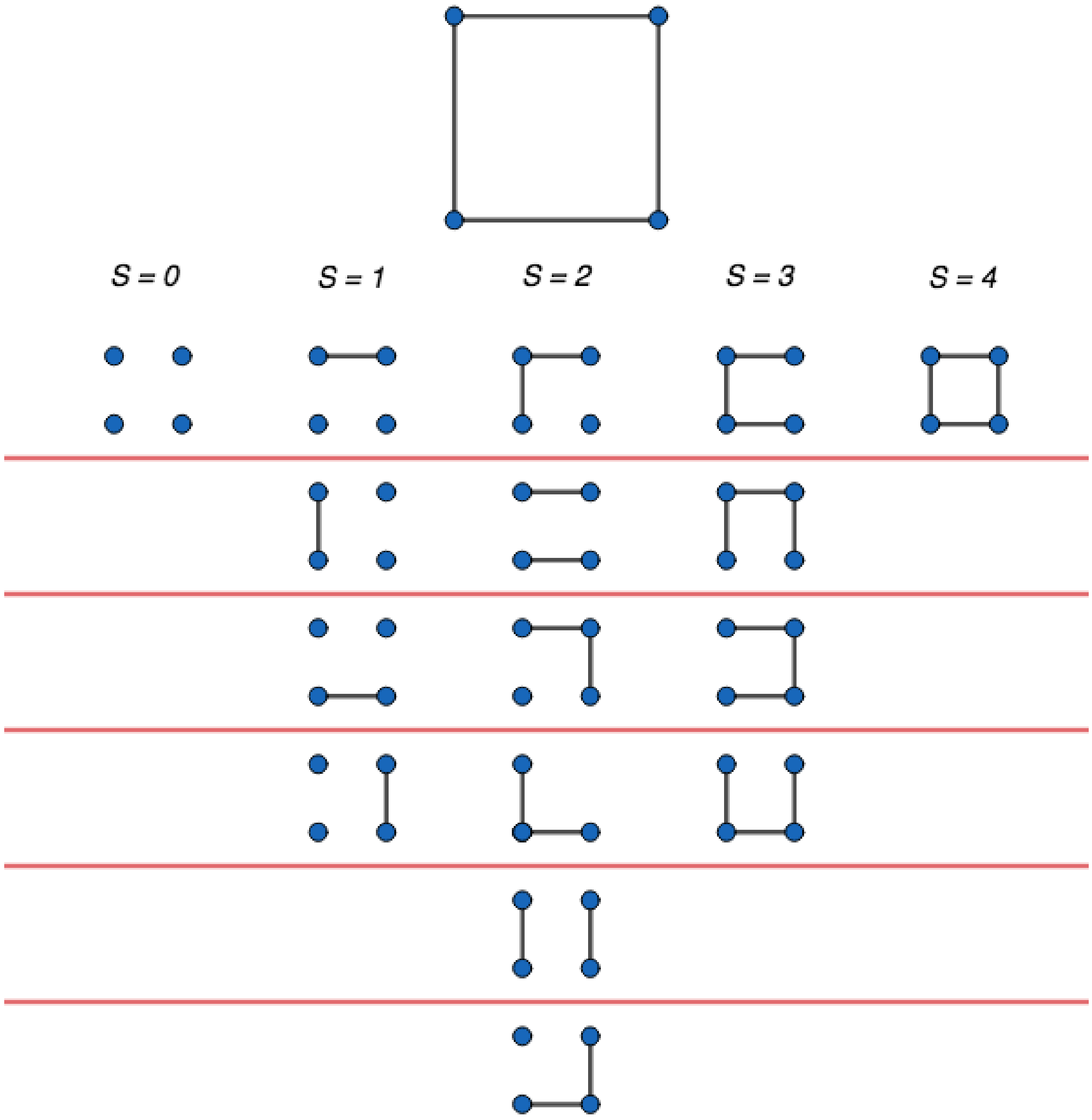


Figure 2.1: A cyclic graph G and it's spanning subgraphs

Example 2: Tutte Polynomial of a Peterson graph

Following Steps 1-4, we obtain the following:

$$\begin{aligned}
T_G(x,y) &= \sum_{A\subseteq E}(x-1)^{K(A)-K(E)}(y-1)^{K(A)-|A|-|V|} \\
&= 36x+120x^2+180x^3+170x^4+114x^5+21x^7+6x^8+x^9 \\
&+ 36y+84y^2+75y^3+35y^4+9y^5+y^6 \\
&+ 168xy+240x^2y+170x^3y+70x^4y+12x^5y \\
&+ 171xy^2+105x^2y^2+30x^3y^2 \\
&+ 65xy^3+15x^2y^3 \\
&+ 10xy^4.
\end{aligned} \tag{2.7}$$

Table of Peterson Graph Tutte Polynomial

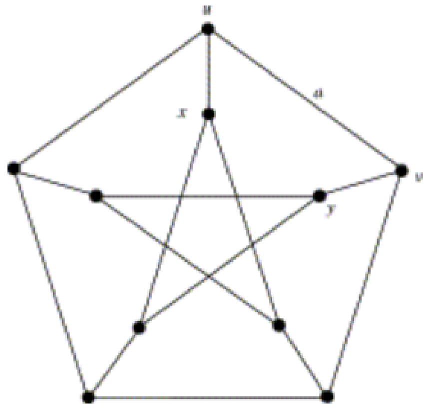


Figure 2.2: Peterson Graph

Table 3.2.1 illustrates the coefficients of the polynomial where matrix element (i,j) corresponds to the coefficient of x^iy^j for $0\leq i\leq 9$, $0\leq j\leq 6$.

0	**36**	**84**	**75**	**35**	**9**	**1**
36	**168**	**171**	**65**	**10**		
120	**240**	**105**	**15**			
180	**170**	**30**				
170	**70**					
114	**12**					
56						
21						
6						
1						

2.2 Recursion of Tutte polynomials

Definition 2.2.1. *The Tutte polynomial of a graph $G = (V, E)$ is defined alternatively by:*

$$T(G; x, y) = \begin{cases} 1 & E(G) = \emptyset \\ yT(G - e; x, y) & e \in E(G) \textit{ and } e \textit{ is a loop} \\ xT(G/e; x, y) & e \in E(G) \textit{ and } e \textit{ is a bridge} \\ T(G - e; x, y) + T(G/e; x, y) & \textit{otherwise.} \end{cases}$$

This definition provides a recursive algorithm also known as **deletion and contraction** method for computing $T(G; x; y)$. We illustrate this process in the next figure.

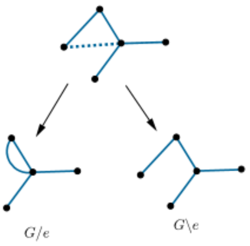


Figure 2.3: Deletion vs Contraction

2.2.1 Examples

Example 1: Cycle on 3 vertices or complete graph on 3 vertices See Figure 2.4 for details. Consider an edge $e \in C_3$. We apply the deletion-contraction algorithm to obtain:

$$\begin{aligned} T(C_3;x,y) &= T(C_3-e)+T(C_3/e) \\ &= T(P_2;x,y)+T(C_2;x,y) \end{aligned}$$

Further, from C_2, we apply the algorithm once again, to obtain:

$$\begin{aligned} T(C_2;x,y) &= T(C_2-e)+T(C_2/e) \\ &= T(P_1;x,y)+T(L;x,y) \end{aligned}$$

where L is a single loop. In which case,

$$T(C_3;x,y) = T(P_2;x,y)+T(P_1;x,y)+T(L;x,y)$$

Now, we know that $T(P_2;x,y)=x^2$, $T(P_1;x,y)=x$, and $T(L;x,y)=y$. Hence,

$$T(C_3;x,y) = x^2+x+y. \tag{2.8}$$

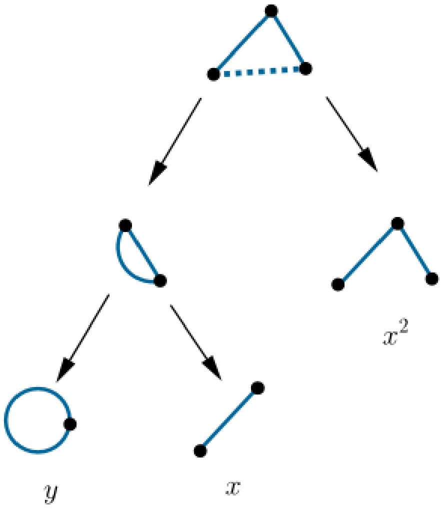


Figure 2.4: Recursion Technique on K_3

Example 2: Cycle on 4 vertices Consider the following graph, G, using the deletion-contraction algorithm find the corresponding Tutte Polynomial:

$$\begin{aligned} T(C_4;x,y) &= T(C_4-e)+T(C_4/e) \\ &= T(P_3;x,y)+T(C_3;x,y) \end{aligned} \tag{2.9}$$

Now, from C_3, we apply the algorithm once again, to obtain:

$$\begin{aligned} T(C_3;x,y) &= T(C_3-e)+T(C_3/e) \\ &= T(P_2;x,y)+T(C_2;x,y) \end{aligned}$$

Further, from C_2, we apply the algorithm once again, to obtain:

$$\begin{aligned} T(C_2;x,y) &= T(C_2-e)+T(C_2/e) \\ &= T(P_1;x,y)+T(L;x,y) \end{aligned}$$

where L is a single loop. In which case,

$$T(C_4;x,y) = T(P_3;x,y)+T(P_2;x,y)+T(P_1;x,y)+T(L;x,y)$$

Now, we know that

$T(P_3;x,y)=x^3$ $T(P_2;x,y)=x^2$, $T(P_1;x,y)=x$, and $T(L;x,y)=y$. Hence,

$$\begin{aligned} T(C_4;x,y) &= T(P_3;x,y)+T(P_2;x,y)+T(P_1;x,y)+T(L;x,y) \qquad (2.10) \\ &= x^3+x^2+x+y \end{aligned}$$

Later, we show using an inductive argument the general formula for the Tutte polynomial of C_n, for $n \geq 2$.

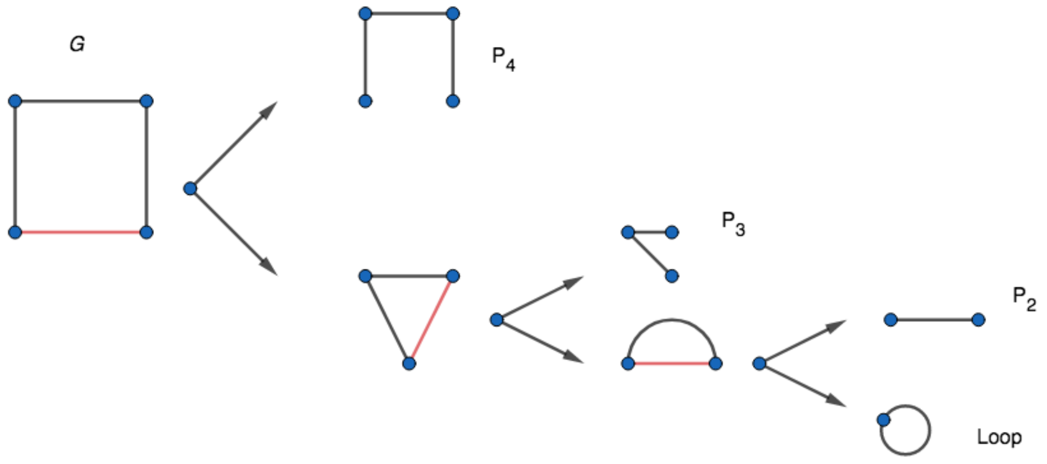


Figure 2.5: Deletion-Contraction Algorithm for Tutte Polynomials on C_4

As a generalization of a tree, a k-*tree* is a graph which arises from a k-clique by 0 or more iterations of adding n new vertices, each joined to a k-clique in the old graph; This process generates several non-isomorphic k-trees. Figure 1 shows two non-isomorphic 2-trees on 6 vertices. K-trees, when $k \geq 2$, are shown to be useful in constructing reliable network in [12]. Here, we denote by T_k^n, a k-tree on $n+k$ vertices which is obtained from a k-clique S, by repeatedly adding n new vertices and making them adjacent to all the vertices of S. When $k = 2$, this particular 2-tree is also known as an $(n+1)$-bridge $\theta(1, 2, \ldots, 2)$.

In the next section, we obtain some results of two members of this family. See Figure 3.1 as an example. Here, we present an example of a 2-tree graph in the case when the Fan and the bridge graphs are actually isomorphic. In which case, $\theta(1, 2, 2) \cong F^2$.

Example 3: A 2-tree graph

The dashed edge indicates the edge that is picked when applying the recurrence and Figure 2.6 shows that

$$T(G; x, y) \quad = \quad x^3 + 2x^2 + 2xy + x + y + y^2$$

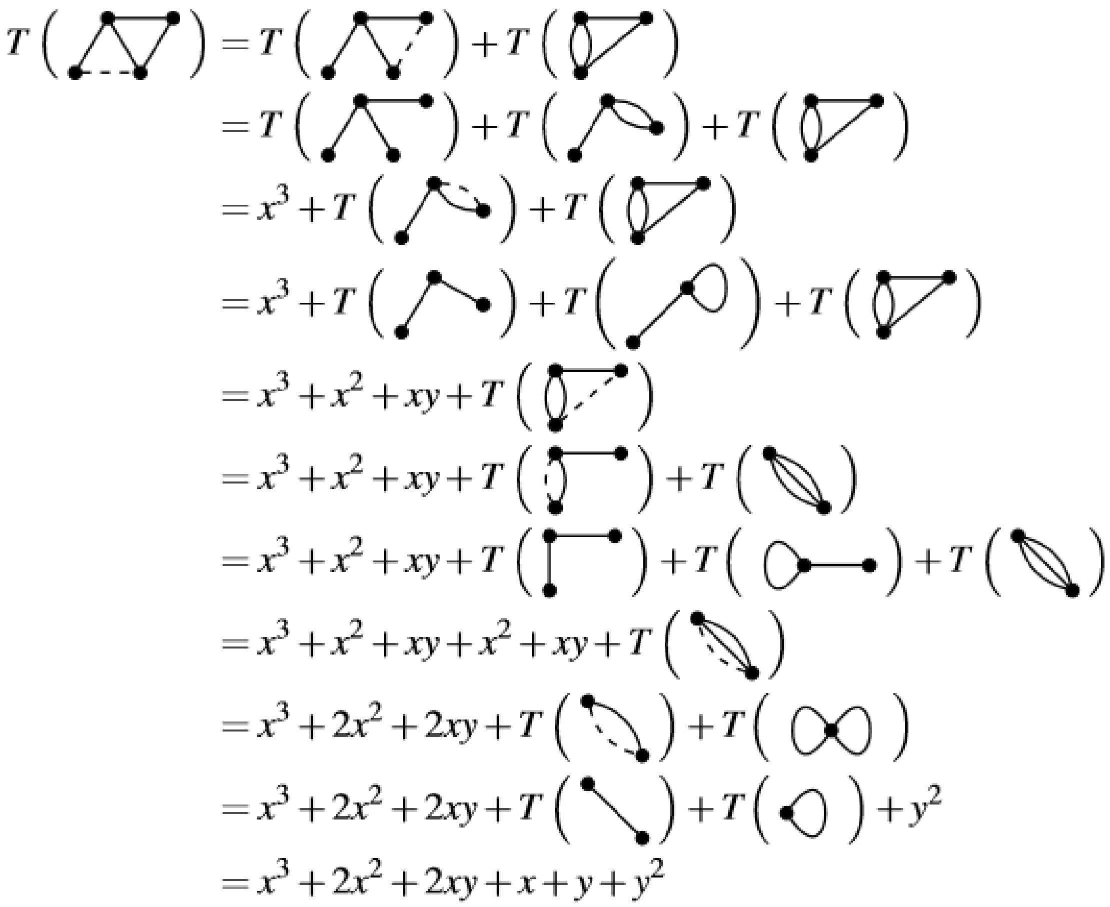


Figure 2.6: Deletion-Contraction Algorithm for a 2-tree graph

Given a path $P^l := v_1, e_1, v_2, \ldots, e_l, v_{l+1}$, when $v_1 = v_{l+1}$, then $P^l \cong C_l$ and we define a **wheel** graph by $W^l = C_l \vee \{w\}$ for all $l \geq 2$. Obviously, the case when $l = 1$, $W^2 \cong C_3$. C^l is often referred to as the *rim* of the wheel and the edges not in the rim are called *spokes*. We will call a wheel on l rim edges, an $l-$wheel, for short.

Example 4: A Wheel graph

We leave it to the reader to refer to Figure 2.7 to help establish that the Tutte polynomial of a Wheel W_4, is

$$T(W_4; x, y) \quad = \quad x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$$

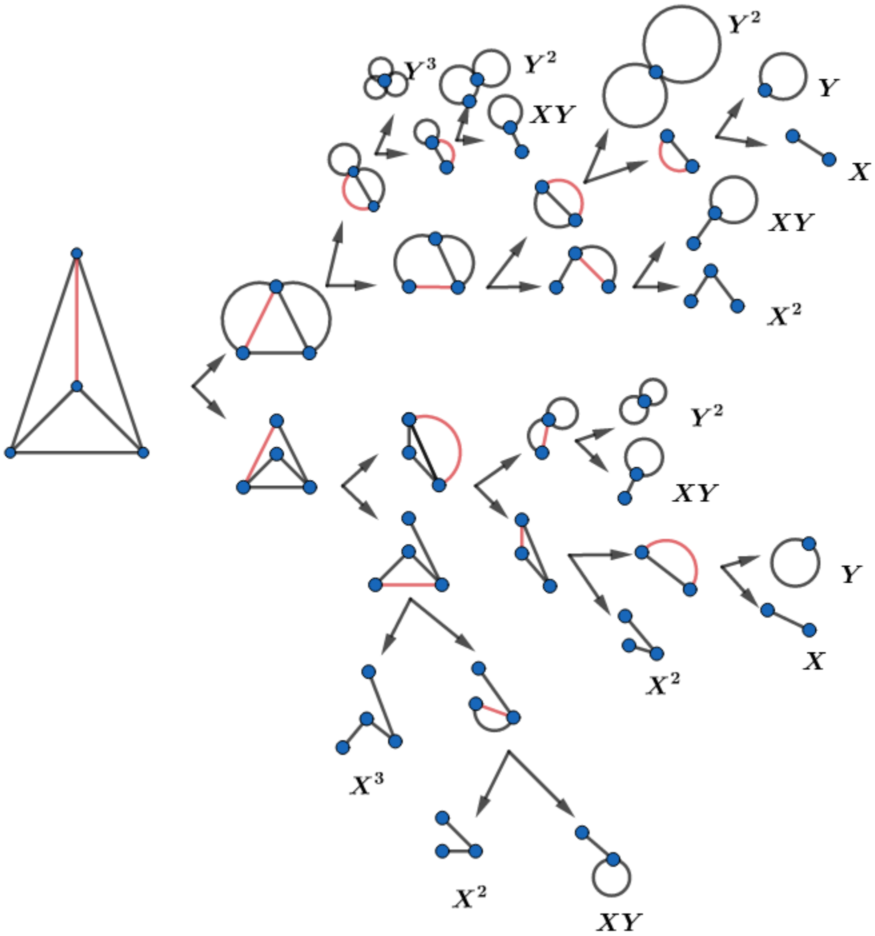


Figure 2.7: Deletion-Contraction Algorithm for Tutte Polynomials on W_4

Example 5: A Cactus

As an exercise, we leave it to the reader verify that Tutte polynomial of a cactus, shown in Figure 2.8 is

$$T(G;x,y) \quad = \quad x^7 + 2x^6 + x^5y + 2x^5 + 2x^4y + x^4 + 2x^3y + x^2y^2,$$

following a recursive argument. In Corollary 3.2.2, we present a general formula for any cactus.

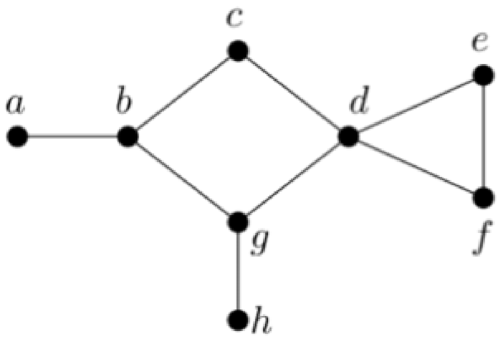


Figure 2.8: A cactus

Chapter 3 Tutte Polynomial of Some Graphs

We begin this chapter with some useful definitions.

3.1 Basic Definitions

Let G_1 and G_2 be two graphs. The **join** of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph H whose vertex set is $V(H) = V(G_1) \cup V(G_2)$, a disjoint union, and whose edge set is $E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2 \mid v_1 \in V(G_1),\ v_2 \in V(G_2)\}$. For example, $\overline{K}_{n_1} \vee \overline{K}_{n_2} \vee \ldots \vee \overline{K}_{n_k} = K(n_1, n_2, \ldots, n_k)$ is a complete $k-$partite graph with part sizes $n_1, \ldots, n_k$. For convenience, an $l-$*cycle*, written $C_l := (v_1, v_2, \ldots, v_l)$, consists of l distinct vertices $v_1, v_2, \ldots, v_l$, and l edges $e_j := \{v_j, v_{j+1}\}$, with $1 \leq j \leq l-1$, and $e_l := \{v_l, v_1\}$. When $e_l = \emptyset$, then we have an $(l-1)-$*path* which we denote by P^{l-1}.

Given $P^l := v_1, e_1, v_2, \ldots, e_l, v_{l+1}$, when $v_1 = v_{l+1}$, then $P^l \cong C_l$ and we define a **wheel** graph by $W^l = C_l \vee \{w\}$ for all $l \geq 2$. C^l is often referred to as the *rim* of the wheel and the edges not in the rim are called *spokes*. We will call a wheel on l rim edges, an $l-$wheel, for short.

A **multi-bridge** (or $m-$**bridge**) graph $G = \theta(a_1, \ldots, a_m)$ is the graph obtained by connecting two distinct vertices with $m \geq 2$ internally disjoint paths of lengths $a_1, \ldots, a_m$ respectively, with $a_i \geq 1$. See Figure 3.1(b) for an example of when $m = 4$. For instance, when $m = 2$, $\theta(a_1, a_2) \cong C_{a_1+a_2}$. For our result, we assume $m \geq 2$ and $a_i \geq 1$, though it is customary to define $\theta(a_1, \ldots, a_m)$ for $m \geq 3$ and $a_i \geq 2$. As such, a multi-bridge graph is a generalization of the well-known $\theta-$graph [23].

A **cactus** is a simple connected graph in which every pair of cycles share at most one vertex. A cactus with one cycle is called a **unicyclic** graph. Figure 2.8 shows a picture of a cactus with two cycles and two edges or bridges.

3.2 Some Results

Theorem 3.2.1. *If T_n is a tree with n vertices then $T(T_n; x, y) = x^{n-1}$.*

Proof. The result follows directly from the definition since each tree on n vertices has exactly $n-1$, edges, each form a bridge; they they account for x, in the Tutte polynomial formula. $\square$

Theorem 3.2.2. *The Tutte polynomial of a simple $n-$cycle is $T(C_n; x, y) = \sum_{i=1}^{n-1} x^i + y$, for all $n \geq 3$.*

Proof. Basis case: When $n = 3$. The result follows from 2.8. Let's assume the statement is true when $n = k$, i.e., that $T(C_k; x, y) = \sum_{i=1}^{k-1} x^i + y$. We now proceed to prove the statement when $n = k+1$. Consider a cycle on $k+1$ vertices, C_{k+1}. Take any edge $e \in C_{k+1}$. We apply the deletion-contraction algorithm on C_{k+1}, and obtain that,

$$\begin{aligned} T(C_{k+1}; x, y) &= T(C_{k+1} - e) + T(C_{k+1}/e) \\ &= T(P_k; x, y) + T(C_k; x, y) \\ &= x^k + \sum_{i=1}^{k-1} x^i + y, \qquad \textit{by the inductive hypothesis and Theorem } 1. \\ &= \sum_{i=1}^{k} x^i + y, \end{aligned} \tag{3.1}$$

giving the result for all $n \geq 3$. $\square$

Corollary 3.2.1. *The Tutte polynomial of a unicyclic graph G with an $n-$cycle and r bridges is $T(G; x, y) = \sum_{i=1}^{n+r-1} x^i + yx^r$, for all $n \geq 3$.*

Proof. Consider G. In which case, G is a cycle, with r bridges. Each bridge contributes to x^r to the Tutte polynomial. Hence

$$T(G; x, y) = x^r(\sum_{i=1}^{n-1} x^i + y), \tag{3.2}$$

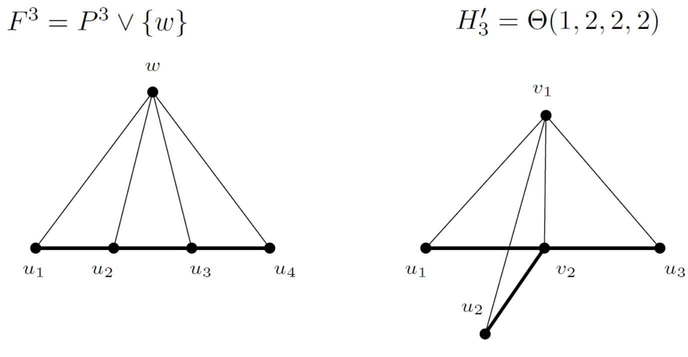
giving the result after an expansion, for all $n \geq 3$. $\square$

Corollary 3.2.2. *The Tutte polynomial of any cactus graph* G *with* l *distinct* m_j-*cycles and* r *bridges is* $T(G;x,y) = \prod_{j=1}^{l}(\sum_{i=1}^{m_j-1} x^i + y)x^r$, *for each* $m_j \geq 2$, $l \geq 1$.

Proof. Consider G. In which case, G is has l cycles, each of length $m_j \geq 2$. It follows from the definition of Tutte polynomial and Corollary 3.2.1 that

$$\begin{aligned} T(G;x,y) &= x^r(\prod_{j=1}^{l}(T(C_{m_j};x,y) \\ &= x^r\prod_{j=1}^{l}(\sum_{i=1}^{m_j-1} x^i + y), \end{aligned} \tag{3.3}$$

giving the result. $\square$



(a) Fan graph (b) 4−bridge graph

Figure 3.1: Two non-isomorphic 2-tree graphs

Corollary 3.2.3. *The Tutte polynomial of any graph* $G = \theta(1, a_1, a_2)$ *is* $T(G;x,y) = \sum_{i=1}^{a_1+a_2-2} x^i + y + (\sum_{j=1}^{a_1-1} x^j + y)(\sum_{k=1}^{a_2-1} x^k + y)$ *for each* $a_1, a_2 \geq 2$.

Proof. We can assume that $G := C_{a_1} \cup C_{a_2}$, together, sharing an edge e. Apply the deletion-contraction algorithm on e. In which case, $G - e$ produces a cycle of length $a_1 + a_2 - 2$ while G/e results into a cactus that we denote by $H^{a_1+a_2-2}$. Note that $H^{a_1+a_2-2}$ has exactly two cycles, each of length H^{a_i-1}, $i = 1, 2$. So, we have

$$\begin{aligned} T(G;x,y) &= T(G-e) + T(G/e) \\ &= T(C_{a_1+a_2-2};x,y) + T(H^{a_1+a_2-2};x,y) \\ &= \sum_{i=1}^{a_1+a_2-2} x^i + y + (\sum_{j=1}^{a_1-1} x^j + y)(\sum_{k=1}^{a_2-1} x^k + y). \end{aligned} \tag{3.4}$$

The result follows. Observe that the last equation follows from Corollary 3.2.2 when $r = 0$, we giving the result. $\square$

Let $P^l := (v_1, e_1, v_2, \ldots, e_l, v_{l+1})$ denote an alternating sequence of distinct vertices v_i and distinct edges e_i. We define an $l-$**fan** by $F^l = P^l \vee \{w\}$, with $w \neq v_i$ for $1 \leq i \leq l+1$. See Figure 3.1(a) for an example of a fan when $l = 3$. We note that F^0 is an edge of multiplicity 2 (or a $2-$edge) and $F^1 \cong C_3$ which Tutte polynomials are $x + y$ and $x^2 + x + y$ respectively. Thus, it is customary to define a fan graph on $l \geq 2$.

Theorem 3.2.3. *Suppose F^l is an $l-$fan. Then, $T(F^l; x, y) = xT(F^{l-1}) + \sum_{i=0}^{l-1} y^i T(F^{l-i-1})$ with $T(F^0) = x + y$ and $l \geq 2$.*

Proof. When $l = 2$, let's suppose $F^2 := (v_1, e_1, v_2, e_2, v_3) \vee \{w\}$. We apply the deletion/contraction method on e_2, giving that

$$\begin{aligned} T(F^2) &= T(F^2 - e_2) + T(F^2/e_2) && (3.5) \\ &= xT(F^1) + T(F^1_*), && (3.6) \end{aligned}$$

where $F^1_* := (v_1, e_1, v_2) \vee \{w\} \cup \{w, v_2\}$. Further, we apply again the deletion/contraction method on $\{w, v_2\}$ to obtain that

$$T(F^1_*) = T(F^1) + yT(F^0). \qquad (3.7)$$

Thus, from (2) and (3) together, we have

$$T(F^2) = xT(F^1) + T(F^1) + yT(F^0). \qquad (3.8)$$

Hence,

$$\begin{aligned} T(F^2) &= x(x^2 + x + y) + x^2 + x + y + y(x + y) \\ &= x^3 + 2x^2 + 2xy + x + y^2 + y. && (3.9) \end{aligned}$$

Moreover, for all $l \geq 2$, we have

$$\begin{aligned} T(F^l; x, y) &= T(F^l - e_l) + T(F^l/e_l) \\ &= xT(F^{l-1}) + T(F_*^{l-1}), \end{aligned} \tag{3.10}$$

where $F_*^{l-1} = F^{l-1} \cup \{w, v_l\}$.

Claim 3.2.3.1. *$T(F_*^r; x, y) = \sum_{i=0}^{r} y^i T(F^{r-i})$ for each $r \geq 1$.*

Proof. By induction on r. For $r = 1$ the result follows from Theorem 3.2.2.

Suppose $F_*^r = F^r \cup \{w, v_{r+1}\}$. Observe that $\{w, v_{r+1}\}$ becomes a $2-$edge. So, as one edge is deleted (in deletion), the other becomes a loop (in contraction). Thus, we apply the deletion/contraction method on $\{w, v_{r+1}\}$ to obtain $T(F_*^r; x, y) = T(F^r) + yT(F_*^{r-1})$. By the inductive hypothesis,

$$\begin{aligned} T(F_*^r; x, y) &= T(F^r) + y\Big(\sum_{i=0}^{r-1} y^i T(F^{r-i-1})\Big) \\ &= T(F^r) + \sum_{i=1}^{r} y^i T(F^{r-i}) \\ &= \sum_{i=0}^{r} y^i T(F^{r-i}). \end{aligned} \tag{3.11}$$

□

The result follows from (6) and Claim 3.2.3.1. □

Chapter 4 Other Related Graph Polynomials

Here, we look at three graph polynomials which are considered to be some specializations of the Tutte polynomial. Because each Tutte polynomial represents a surface, these restricted univariate polynomials are some intersection with the cartesian plane. For instance, the intersection of each surface with the plane $y = 0$ gives a curve along which we can obtain a scaled version of a well-known polynomial called *chromatic polynomial.* Likewise, when we restrict the surface to the values in a curve where $x = 0$, we obtain a polynomial known as the *flow polynomial.* We present these polynomials after they are defined and we give examples of such functions for different graphs that were introduced in Chapter 1.

We note that, throughout this chapter, we assume that G is a simple graph on n vertices with m edges and c components. In which case, when G is connected, $c = 1$ and when G is a null graph $m = 0$.

4.1 Chromatic Polynomial

A vertex coloring, also called a **proper coloring** of a graph assigns a color to each vertex so that no vertices connected by an edge share the same colour. The problem of finding such a graph colouring using λ colours (known as a λ-coloring) has been well-studied.

The **chromatic polynomial** $P(G, \lambda)$ gives the number of ways a graph G can be colored with λ-colours. For example, a graph of n isolated vertices has $P(G, \lambda) = \lambda^n$ since each vertex can be colored with any of the λ-colours. Likewise, a tree G with n vertices has chromatic polynomial

$$P(G, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

We can start at any vertex and color it any of the λ-colours, then each adjacent vertex can be colored any of the other $\lambda - 1$ colors, and we can repeat this process until the tree is completely colored. Any graph with a loop has chromatic polynomial 0, since there is no way to color the vertex at both ends of the loop with different colors.

The chromatic polynomial can be found by evaluating the Tutte polynomial $T(G; 1-\lambda, 0)$ and multiplying by a positive or negative monomial in λ that depends on the number of vertices and components of the graph G. Thus,

$$P(G,\lambda) \;=\; (-1)^{n-c}\lambda T(G; 1-\lambda, 0)$$

For the C_3 graph the chromatic polynomial is:

$$\begin{aligned} P(G,\lambda) &= (-1)^1\lambda T(G; 1-\lambda, 0) \\ &= (-1)(\lambda)(\lambda^2 - \lambda + 2) \\ &= -\lambda^3 + \lambda^2 - 2\lambda \end{aligned} \tag{4.1}$$

For the C_4 graph the chromatic polynomial is:

$$\begin{aligned} P(G,\lambda) &= (-1)^1\lambda T(G; 1-\lambda, 0) \\ &= (-1)(\lambda)(-\lambda^3 + 4\lambda^2 - 6\lambda + 3) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \end{aligned} \tag{4.2}$$

For the cactus shown in Figure 2.8 the chromatic polynomial is

$$\begin{aligned} P(G,\lambda) &= (-1)^8\lambda T(G; 1-\lambda, 0) \\ &= (\lambda)(-\lambda^7 + 9\lambda^6 - 35\lambda^5 + 76\lambda^4 - 99\lambda^3 + 77\lambda^2 - 33\lambda + 6) \\ &= -\lambda^8 + 9\lambda^7 - 35\lambda^6 + 76\lambda^5 - 99\lambda^4 + 77\lambda^3 - 33\lambda^2 + 6\lambda \end{aligned} \tag{4.3}$$

For the 2-tree graph shown in Figure 2.6, the chromatic polynomial is:

$$\begin{aligned} P(G,\lambda) &= (-1)^3\lambda T(G;1-\lambda,0) \\ &= -\lambda(1-\lambda)^3+2(1-\lambda)^2+(1-\lambda) \\ &= \lambda^4-5\lambda^3+8\lambda^2-4\lambda \end{aligned} \tag{4.4}$$

Note that this polynomial returns zero when λ is one or two, but $P(G,3)=6$. Thus, the graph G is 3-colorable, and can be colored in 6 ways using 3 colors.

For the Peterson graph shown in Figure 2.2, the chromatic polynomial is:

$$\begin{aligned} P(G,\lambda) &= (-1)^9\lambda T(G;1-\lambda,0) \\ &= (-1)\Big(\lambda(6\lambda^8-69\lambda^7+315\lambda^6-891\lambda^5+1895\lambda^4-3071\lambda^3+3429\lambda^2 \\ &- 2261\lambda+642+(-\lambda+1)^9\Big) \\ &= (-6\lambda^9+69\lambda^8-315\lambda^7+891\lambda^6-1895\lambda^5+3071\lambda^4-3429\lambda^3 \\ &+ 2261\lambda^2-642\lambda-\lambda)(-\lambda+1)^9 \end{aligned} \tag{4.5}$$

4.2 Flow Polynomial

Another essential area of graph theory concerns finding flows for graphs [20]. A flow is an assignment of a value to each edge of a directed graph so that, for each vertex, the sum of the values of all incident edges where the vertex is the tail (that is, “outgoing” edges) is equal to the sum of the values of all incident edges where the vertex is the head (“incoming” edges). A nowhere-zero flow also requires that each edge value be non-zero. If a graph has a flow assigning values of an abelian group H, it is called an H-flow. A k-flow is a $\mathbb{Z}$-flow where edges are assigned values between 0 (or 1, if nowhere-zero) and $k-1$. Thus, flow polynomial $F(G,\lambda)$ gives the number of nowhere-zero k-flows for a graph G. and abelian group H of order λ.

In general, we can obtain the flow polynomial by computing

$$F(G,k) = (-1)^{m-n+c}T(G;0,1-k).$$

For the C_3 graph the flow polynomial is:

$$\begin{aligned} F(G,k) &= (-1)^1 kT(G;0,1-k) \qquad (4.6)\\ &= (-1)k(1-k)\\ &= k^2-k \end{aligned}$$

For the C_4 graph the flow polynomial is:

$$\begin{aligned} F(G,k) &= (-1)^1 kT(G;0,1-k) \qquad (4.7)\\ &= (-1)k(1-k)\\ &= k^2-k \end{aligned}$$

For the cactus shown in Figure 2.8 the flow polynomial is

$$\begin{aligned} F(G,k) &= (-1)^1 kT(G;0,1-k) \qquad (4.8)\\ &= (-1)k(-k+1)\\ &= 0 \end{aligned}$$

We can calculate the flow polynomial of the 2-tree graph shown in Figure2.6 from the Tutte polynomial:

$$\begin{aligned} F(G,k) &= (-1)^2 kT(G;0,1-k) \qquad (4.9)\\ &= (1-k)+(1-k)^2\\ &= k^2-3k^3+2k \end{aligned}$$

For the Peterson graph shown in Figure 2.2, the Flow polynomial is:

$$\begin{aligned} F(G,k) &= (-1)^6 kT(G;0,1-k) \qquad (4.10)\\ &= k(k^6-15k^5+95k^4-324k^3+624k^2-620k+240)\\ &= k^7-15k^6+95k^5-324k^4+624k^3-620k^2+240k \end{aligned}$$

4.3 Reliability Polynomial

The **reliability polynomial** of G, denoted by $R(G;p)$, is the probability that G remains connected when each edge in G fails with probability p. We obtain the reliability polynomial of G by computing

$$R(G,p) \quad = \quad p^{n-c}(1-p)^{m-n+c}T(G;1,p^{-1}).$$

For the C_3 graph the Reliability polynomial is:

$$\begin{aligned} R(G,p) &= p^2(1-p)^1T(G;1,p^{-1}) \\ &= p^2(1-p)(2+p^{-1}) \\ &= p-p^2 \end{aligned} \tag{4.11}$$

For the C_4 graph the Reliability polynomial is:

$$\begin{aligned} R(G,p) &= p^3(1-p)^1T(G;1,p^{-1}) \\ &= p^3(1-p)(3-p^{-1}) \\ &= 3p^2-4p^3+p^4 \end{aligned} \tag{4.12}$$

For the cactus shown in Figure 2.8 the Reliability

$$\begin{aligned} R(G,p) &= p^8(1-p)^0T(G;1,p^{-1}) \\ &= p^8(p+11) \\ &= p^9+11p^8 \end{aligned} \tag{4.13}$$

We can calculate the reliability polynomial of the 2-tree graph shown in Figure2.6

from the Tutte polynomial:

$$\begin{aligned} R(G,p) &= p^3(1-p)^2T(G;1,p^{-1}). \quad (4.14)\\ &= p^3(1-p)^2(4+3p^{-1}+p)\\ &= p^6+2p^5-4p^4-2p^3+3p^2 \end{aligned}$$

For the Peterson graph shown in Figure 2.2, the reliability polynomial is:

$$\begin{aligned} R(G,p) &= p^{14}(1-p)^6T(G;1,p^{-1}). \quad (4.15)\\ &= p^{14}(1-p)^6(p^5+9p^4+45p^3+155p^2+390p+1344)\\ &= (1-p)^6(p^{19}+9p^{18}+45p^{17}+155p^{16}+390p^{15}+1344p^{14}) \end{aligned}$$

Chapter 5 Some Applications

The information encoded in the Tutte polynomial has a number of applications, and is useful in a wide variety of domains. One such piece of information is the number of spanning trees of a graph, which is important in the theory of electrical networks. Another is the number of colorings of a graph. A well-know application of this information is finding whether a map can be colored using four colors with each adjacent region (or country) a different color. However, many other applications exist. A graph might represent a scheduling problem where the edges correspond to items that cannot be scheduled at the same time. For example, consider a graph where the vertices correspond to exams, and there is an edge between vertices if there is at least one student taking both exams. Then, a vertex coloring, where each color corresponds to a different day for an exam, gives a schedule where no student has to sit two exams in the same day. Alternatively, a graph might correspond to a network of nodes, where something travels between the nodes. An obvious example is a computer network. Beyond these applications, here, we present two fundamental applications: one is the evaluation of Tutte polynomial which gives some characterizations of the graphs and the other is a representation of the function. Here, we present both applications.

5.1 Data Encoded in the Tutte polynomial

Here, we evaluate the Tutte Polynomials of the graphs that were introduced in Chapter 1.

5.1.1 Counting spanning trees

Given a connected graph G and it's corresponding Tutte Polynomial $T(G;x,y)$, evaluating $T(G;1,1)$ will produce the number of spanning trees in a connected graph.

Examples:

1. For a cycle graph $G = C_3$ where $T(G;x,y) = x^2 + x + y$, we will find that $T(G;1,1) = 3$. In which case, we can conclude that C_3 admits 3 spanning trees.

2. For a cycle graph $G = C_4$ we have $T(G;x,y) = x^3 + x^2 + x + y$. In which case $T(G;1,1) = 4$. So, we can conclude that C_4 admits 4 spanning trees.

3. For $G = \theta(1,2,2)$, a 3-bridge graph as shown in Figure 2.6, $T(G;x,y) = x^3 + 2x^2 + 2xy + x + y + y^2$ and $T(G;1,1) = 8$. In which case, we can conclude that $G = \theta(1,2,2)$ admits 8 spanning trees.

4. For the cactus G shown in Figure 2.8 with $T(G;x,y) = x^7 + 2x^6 + x^5y + 2x^5 + 2x^4y + x^4 + 2x^3y + x^2y^2$, we have $T(G;1,1) = 12$. In which case, we can conclude that G admits 12 spanning trees.

5. For the Peterson G shown in Figure 2.2 with $T(G;x,y) = 36x + 120x^2 + 180x^3 + 170x^4 + 114x^5 + 21x^7 + 6x^8 + x^9 + 36y + 84y^2 + 75y^3 + 35y^4 + 9y^5 + y^6 + 168xy + 240x^2y + 170x^3y + 70x^4y + 12x^5y + 171xy^2 + 105x^2y^2 + 30x^3y^2 + 65xy^3 + 15x^2y^3 + 10xy^4$, we have $T(G;1,1) = 1,791$. In which case, we can conclude that G admits $1,791$ spanning trees.

5.1.2 Counting acyclic subgraphs

Given a graph G and it's corresponding Tutte Polynomial $T(G;x,y)$, evaluating $T(G;2,1)$ will produce the number of forest (acyclic subgraphs) of graph G.

Examples:

1. For a cycle graph C_3 as shown in Figure 2.4 where $T(G;x,y) = x^2 + x + y$, we will find that $T(G;2,1) = 7$. In which case, we can conclude that C_3 will produce 7 forest (acyclic subgraphs) of graph G.

2. For a cycle graph C_4 as shown in Figure 2.5 where $T(G;x,y) = x^3+x^2+x+y$, we will find that $T(G;2,1) = 15$. In which case, we can conclude that C_4 will produce 15 forest (acyclic subgraphs) of graph G.

3. For $G = \theta(1,2,2)$ as shown in Figure 2.6, $T(G;x,y) = x^3+2x^2+2xy+x+y+y^2$ and $T(G;2,1) = 24$. In which case, we can conclude that $G = \theta(1,2,2)$ will produce 24 forest (acyclic subgraphs) of graph G.

4. For the cactus G shown in Figure 2.8 with $T(G;x,y) = x^7+2x^6+x^5y+2x^5+2x^4y+x^4+2x^3y+x^2y^2$, we have $T(G;2,1) = 420$. In which case, we can conclude that G admits 420 spanning trees.

5. For the Peterson G shown in Figure 2.2 with $T(G;x,y) = 36x+120x^2+180x^3+170x^4+114x^5+21x^7+6x^8+36x^9y+84y^2+75y^3+35y^4+9y^5+y^6+168xy+240x^2y+170x^3y+70x^4y+12x^5y+171xy^2+105x^2y^2+30x^3y^2+65xy^3+15x^2y^3+10xy^4$, we have $T(G;2,1) = 18,708$. In which case, we can conclude that G admits $18,708$ spanning trees.

5.1.3 Counting connected spanning subgraphs

Given a connected graph G and it's corresponding Tutte Polynomial $T(G;x,y)$, evaluating $T(G;1,2)$ will produce the number of connected subgraphs of G.

Examples:

1. For a cycle graph C_3 as shown in Figure 2.4 where $T(G;x,y) = x^2+x+y$, we will find that $T(G;1,2) = 4$. In which case, we can conclude that C_3 will produce 4 connected subgraphs of G.

2. For a cycle graph C_4 as shown in Figure 2.5 where $T(G;x,y) = x^3+x^2+x+y$, we will find that $T(G;1,2) = 5$. In which case, we can conclude that C_4 will produce 5 connected subgraphs of G. Some of these subgraphs are shown in Figure 1.2.

3. For $G = \theta(1,2,2)$ as shown in Figure 2.6, $T(G;x,y) = x^3+2x^2+2xy+x+y+y^2$ and $T(G;1,2) = 14$. In which case, we can conclude that $G = \theta(1,2,2)$ will

produce 14 connected subgraphs of G.

4. For the cactus G shown in Figure 2.8 with $T(G;x,y) = x^7 + 2x^6 + x^5y + 2x^5 + 2x^4y + x^4 + 2x^3y + x^2y^2$, we have $T(G;1,2) = 20$. In which case, we can conclude that G admits 20 spanning trees.

5. For the Peterson G shown in Figure 2.2 with $T(G;x,y) = 36x + 120x^2 + 180x^3 + 170x^4 + 114x^5 + 21x^7 + 6x^8 + 36x^9y + 84y^2 + 75y^3 + 35y^4 + 9y^5 + y^6 + 168xy + 240x^2y + 170x^3y + 70x^4y + 12x^5y + 171xy^2 + 105x^2y^2 + 30x^3y^2 + 65xy^3 + 15x^2y^3 + 10xy^4$, we have $T(G;1,2) = 5,912$. In which case, we can conclude that G admits $5,912$ spanning trees.

5.1.4 Counting acyclic orientations

Given a graph G and it's corresponding Tutte Polynomial $T(G;x,y)$, evaluating $T(G;2,0)$ will produce the number of acyclic orientations.

Examples:

1. For a cycle graph C_3 as shown in Figure 2.4 where $T(G;x,y) = x^2 + x + y$, we will find that $T(G;2,0) = 6$. In which case, we can conclude that C_3 will produce 6 acyclic orientations of G.

2. For a cycle graph C_4 as shown in Figure 2.5 where $T(G;x,y) = x^3 + x^2 + x + y$, we will find that $T(G;2,0) = 14$. In which case, we can conclude that C_4 will produce 14 acyclic orientations of G.

3. For $G = \theta(1,2,2)$ as shown in Figure 2.6, $T(G;x,y) = x^3 + 2x^2 + 2xy + x + y + y^2$ and $T(G;2,0) = 18$. In which case, we can conclude that $G = \theta(1,2,2)$ will produce 18 acyclic orientations of G.

4. For the cactus G shown in Figure 2.8 with $T(G;x,y) = x^7 + 2x^6 + x^5y + 2x^5 + 2x^4y + x^4 + 2x^3y + x^2y^2$, we have $T(G;2,0) = 336$. In which case, we can conclude that G admits 336 spanning trees.

5. For the Peterson G shown in Figure 2.2 with $T(G;x,y) = 36x + 120x^2 + 180x^3 + 170x^4 + 114x^5 + 21x^7 + 6x^8 + 36x^9y + 84y^2 + 75y^3 + 35y^4 + 9y^5 + y^6 + 168xy + 240x^2y +$

$170x^3y+70x^4y+12x^5y+171xy^2+105x^2y^2+30x^3y^2+65xy^3+15x^2y^3+10xy^4$, we have $T(G;2,0) = 13,096$. In which case, we can conclude that G admits $13,096$ spanning trees.

5.1.5 Counting all spanning subgraphs

Given a graph, G, and it's corresponding Tutte Polynomial $T(G;x,y)$, evaluating $T(G;2,2)$ will produce the number of spanning subgraphs. This number that can be written as $2^{|E|}$, where $|E|$ is the number of edges of G.

Examples:

1. For a cycle graph C_3 as shown in Figure 2.4 where $T(G;x,y) = x^2+x+y$, we will find that $T(G;2,2) = 8$. In which case, we can conclude that C_3 will produce 6 spanning subgraphs of G.

2. For a cycle graph C_4 as shown in Figure 2.5 where $T(G;x,y) = x^3+x^2+x+y$, we will find that $T(G;2,2) = 16$. In which case, we can conclude that C_4 will produce 16 spanning subgraphs of G.

3. For $G = \theta(1,2,2)$ as shown in Figure 2.6, $T(G;x,y) = x^3+2x^2+2xy+x+y+y^2$ and $T(G;2,2) = 32$. In which case, we can conclude that $G = \theta(1,2,2)$ will produce 32 spanning subgraphs of G.

4. For the cactus G shown in Figure 2.8 with $T(G;x,y) = x^7+2x^6+x^5y+2x^5+2x^4y+x^4+2x^3y+x^2y^2$, we have $T(G;2,2) = 512$. In which case, we can conclude that G admits 512 spanning trees.

5. For the Peterson G shown in Figure 2.2 with $T(G;x,y) = 36x+120x^2+180x^3+170x^4+114x^5+21x^7+6x^8+x^9+36y+84y^2+75y^3+35y^4+9y^5+y^6+168xy+240x^2y+170x^3y+70x^4y+12x^5y+171xy^2+105x^2y^2+30x^3y^2+65xy^3+15x^2y^3+10xy^4$, we have $T(G;2,2) = 29,184$. In which case, we can conclude that G admits $29,184$ spanning trees.

5.2 Plots of Tutte polynomials

The Tutte polynomial gives us a 3-dimensional surface that we can plot in the Cartesian coordinate system. Here, we plot the surfaces along with the contour plots of all the graphs discussed in Chapter 1. We leave it, as an exercise for the readers to compute other useful vectors such as the gradient, the normal vector and parameters such as the curvature for these surfaces.

5.2.1 $C_3 \cong K_3$

$T(C_3; x, y) = x^2 + x + y$.

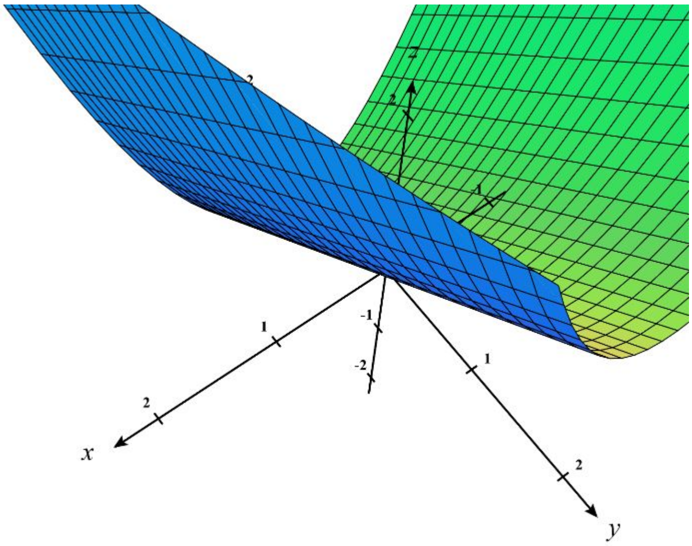


Figure 5.1: Surface of the Tutte polynomial of K_3

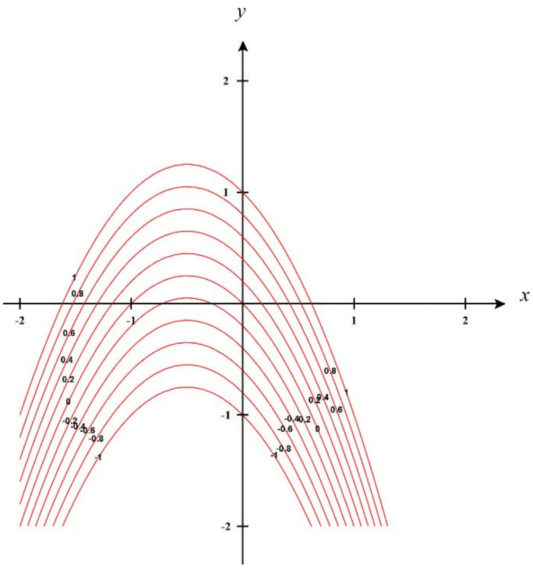


Figure 5.2: Contour plot of the Tutte polynomial of K_3

5.2.2 C_4

$T(C_4; x, y) = x^3 + x^2 + x + y.$

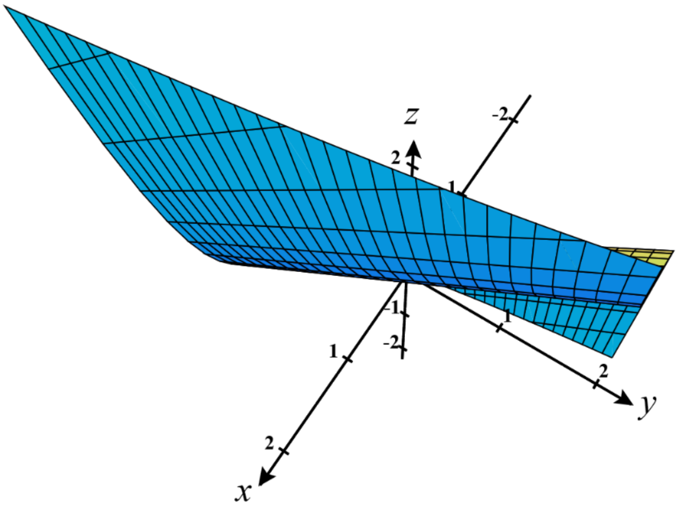


Figure 5.3: Surface of the Tutte polynomial of K_4

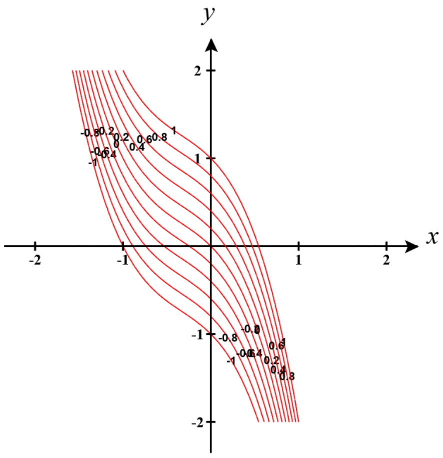


Figure 5.4: Contour plot of the Tutte polynomial of K_4

5.2.3 $\theta(1, 2, 2)$

Given a Theta graph $G = \theta(1, 2, 2)$, we have $T(G; x, y) = x^3 + 2x^2 + 2xy + x + y + y^2$.

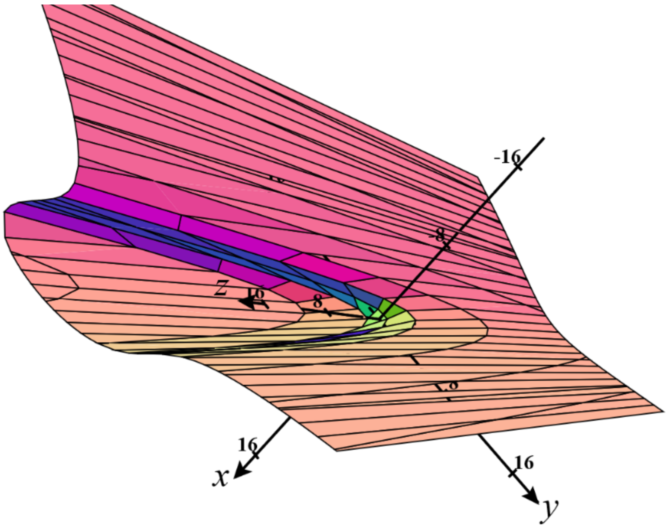


Figure 5.5: Surface of the Tutte polynomial of $\theta(1, 2, 2)$

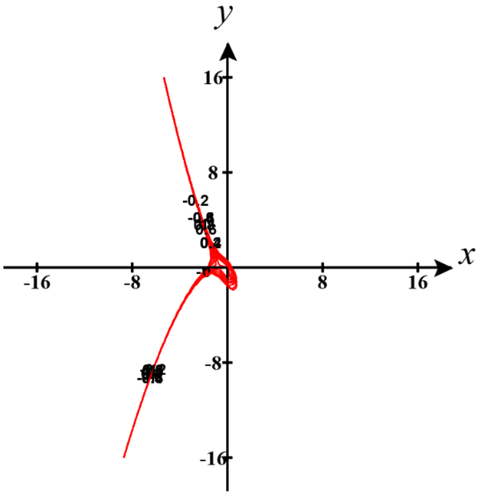


Figure 5.6: Contour plot of $\theta(1, 2, 2)$

5.2.4 Cactus

Given the Cactus as shown in in Figure 2.8, we have $T(G; x, y) = x^3 + 2x^2 + 2xy + x + y + y^2$.

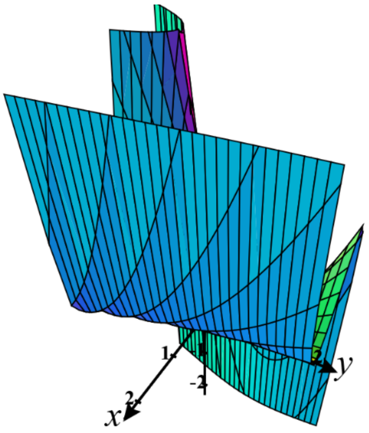


Figure 5.7: Surface of the Tutte polynomial of a *Cactus*

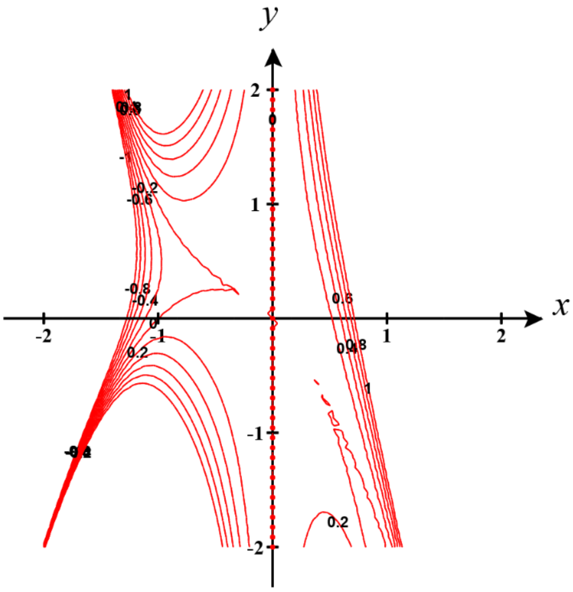


Figure 5.8: Contour plot of the Tutte polynomial of a *Cactus*

5.2.5 Peterson

Given the Peterson graph as shown in in Figure 2.2, we have $T(G;x,y) = 36x+120x^2+180x^3+170x^4+114x^5+21x^7+6x^8+x^9+36y+84y^2+75y^3+35y^4+9y^5+y^6+168xy+240x^2y+170x^3y+70x^4y+12x^5y+171xy^2+105x^2y^2+30x^3y^2+65xy^3+15x^2y^3+10xy^4$.

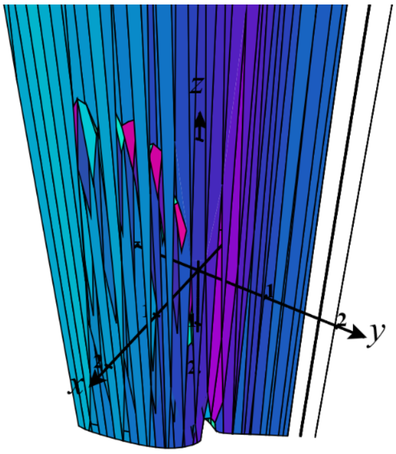


Figure 5.9: Surface of the Tutte polynomial of Peterson Graph

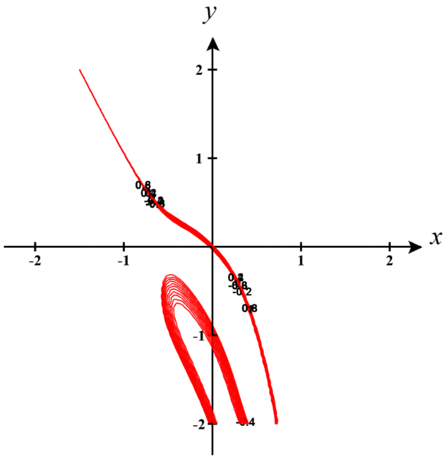


Figure 5.10: Contour plot of the Tutte polynomial of Peterson graph

Chapter 6 Conclusion and Future Research

The Tutte polynomial, originally known as dichromatic polynomial has been computed for several simple graphs including the famous Peterson graphs. Many examples were given and several of these results were proved using induction. Moreover, Tutte polynomial has a particular relation with a number of well-known univariate polynomials. For instance, the *reliability polynomial* of G, denoted by $R(G,p)$, is the probability that G remains connected when each edge in G fails with probability p. *The chromatic polynomial* of G, denoted by $P(G,\lambda)$, counts the number of ways the vertices of G can be colored using at most λ colors. The *flow polynomial* of G, denoted by $F(G,k)$, counts the number of nowhere-zero $k-$flows. From the Tutte polynomial of a loopless graph, we can recover the chromatic polynomial along $y=0$ and the flow polynomial along $x=0$. A survey of several related and unrelated polynomials can be found in [2, 15, 21]. Thus, for a graph G on n vertices with m edges and c components, the chromatic polynomial, the flow polynomial and the reliability polynomial of G are respectively obtained from the Tutte polynomial by:

$$\begin{aligned} P(G,\lambda) &= (-1)^{n-c}\lambda T(G;1-\lambda,0) \\ F(G,k) &= (-1)^{m-n+c}T(G;0,1-k) \\ R(G,p) &= p^{n-c}(1-p)^{m-n+c}T(G;1,\frac{1}{1-p}). \end{aligned}$$

Our research did not focus on these polynomials, but we showed through some results how these polynomials can be derived. We also showed how other important evaluations of $T(G;x,y)$ can be found at some specific points of the plane and also along several algebraic curves. We refer to [10, 13, 17] for details about the combinatorial interpretations of these evaluations.

Future research can focus on Multigraphs such as the one shown in Figure 6.1. Tutte polynomials of multigraphs can be computed using either of the techniques we outlined in this thesis although the process is rather lengthy as the number of vertices grow. Here, we present the Tutte polynomial of the multigraph shown in Figure 6.1.

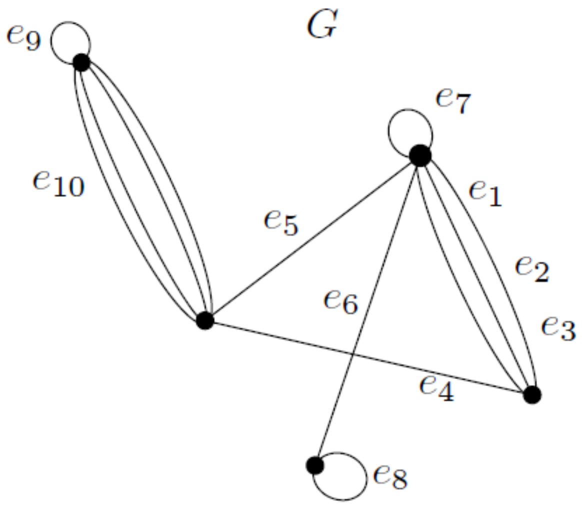


Figure 6.1: Multigraph

Consider the graph G with edges labelled $e_1 \dots e_{10}$, as shown. Note that e_7, e_8, e_9 are loops.

(i) G is reduced to $G^\dagger$ by removing, not necessarily in this order, e_9, e_{10}, e_8, e_7, and e_6, where e_{10} and e_6 represent a $4-$edge and a $1-$edge respectively and e_9, e_8, and e_7 are loops. From Proposition 3.1 (and the remark preceding it) follows that $T(G) = T(e_{10})T(e_9)T(e_8)T(e_7)T(e_6)T(G^\dagger) = xy^3T(e_{10})T(G^\dagger)$ since $T(e_9) = T(e_8) = T(e_7) = y$ and $T(e_6) = x$. We now apply the algorithm on the edges e_5, e_4, e_3 and e_2 of $G^\dagger$, giving respectively the following:

(ii) $T(G^\dagger) = T(G^\dagger - e_5) + T(G^\dagger/e_5) = T(G_1) + T(G_2)$. Note that $G_2 \cong e_{10}$.

(iii) $T(G_2) = T(G_2 - e_4) + T(G_2/e_4) = T(G_3) + y^3$.

(iv) $T(G_3) = T(G_3 - e_3) + T(G_3/e_3) = T(G_4) + y^2$.

(v) $T(G_4) = T(G_4 - e_2) + T(G_4/e_2) = x + y$.

Using (v), (iv) yields $T(G_3) = x+y+y^2$. From (iv), (iii) yields $T(G_2) = T(e_{10}) = x + y + y^2 + y^3$. Now we note that $T(G_1) = T(e_4)T(G_3) = x(x + y + y^2)$. So, (ii) yields (from (iii)) that $T(G^\dagger) = x(x + y + y^2) + x + y + y^2 + y^3$. Finally, from (ii) we get (i), namely that

$$T(G) = xy^3(x+y+y^2+y^3)\Big(x(x+y+y^2)+x+y+y^2+y^3\Big).$$

We point out that, in [1], the authors introduced two parameters, ζ and γ, to simplified such expressions. They are, for all $m \geq 1$:

$$\zeta_m := \Big(\sum_{k=0}^{m-1} y^k\Big), \tag{6.1}$$

and

$$\gamma_m := (x - 1 + \zeta_m). \tag{6.2}$$

Hence, $T(G) = y^3\gamma_1\gamma_4(\gamma_1\gamma_3 + \gamma_4)$.

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