The smallest split Cayley hexagon has two symplectic embeddings

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Abstract

It is well known that the smallest split Cayley generalized hexagon H(2) can be embedded into the symplectic space $W(5,2)$, or equivalently, into the parabolic quadric Q$(6,2)$. We establish a second way to embed H(2) into the same space and describe a computer proof of the fact that these are essentially the only two embeddings of this type.

1 Introduction

The split Cayley generalized hexagon H(K) over a field K is usually described as a point–line geometry embedded into the parabolic quadric $\mathcal{Q}$ of type Q$(6,K)$ with quadratic form given by

$$Q(x) = x_1x_4 + x_2x_5 + x_3x_6 - x_7^2, \tag{1}$$

and associated bilinear form

$$B(x,y) \stackrel{\text{def}}{=} Q(x+y) - Q(x) - Q(y) = x_1y_4 + x_2y_5 + x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3 - 2x_7y_7. \tag{2}$$

The points of H(K) are exactly the points of $\mathcal{Q}$, i.e., the points of the projective space PG$(6,K)$ whose coordinates satisfy $Q(x) = 0$. The lines of H(K) are the lines of $\mathcal{Q}$ whose

Grassmann coordinates satisfy the following simple linear equations :

$$\begin{gathered} p_{62} = p_{17},\ p_{13} = p_{72},\ p_{24} = p_{37},\ p_{35} = p_{74},\ p_{46} = p_{57},\ p_{51} = p_{76}, \\ p_{14} + p_{25} + p_{36} = 0, \end{gathered} \tag{3}$$

(see for instance [3, Section 3.5]), where, as customary, $p_{ij} \stackrel{\text{def}}{=} x_i y_j - x_j y_i$ for the line joining the points with coordinates x and y.

Points that are at maximal distance in the collinearity graph of $\mathrm{H}(K)$ are called *opposite*. The points represented by x and y in $\mathcal{Q}$ are opposite if and only if $B(x, y) \neq 0$. In $\mathcal{Q}$, two points x and y lie on the same line if and only if $B(x, y) = 0$. It follows that points of $\mathrm{H}(K)$ are not opposite if and only if they are collinear in $\mathcal{Q}$. The set of points collinear to x in $\mathcal{Q}$ is denoted by $x^\perp$.

An *embedding* of a point-line geometry $\mathcal{A}$ into a point-line geometry $\mathcal{B}$ is an injective map φ preserving incidence which maps the points of $\mathcal{A}$ onto a subset of the points of $\mathcal{B}$ and the lines of $\mathcal{A}$ into a subset of the lines of $\mathcal{B}$. If φ is an embedding of $\mathcal{A}$ into $\mathcal{B}$, then so is $\varphi' = \sigma_A \varphi \sigma_B$, where σ_A (resp. σ_B) is an automorphism of the geometry $\mathcal{A}$ (resp. $\mathcal{B}$). Any φ' of this form is considered *equivalent* to φ.

The representation of the hexagon $\mathrm{H}(K)$ given above satisfies this definition of an embedding into $\mathcal{Q}$. We shall call this the *classical* embedding of the split Cayley hexagon, and denote it by φ_C.

It is natural to think that φ_C is essentially the only way to embed $\mathrm{H}(K)$ into $\mathcal{Q}$, in other words, that all other embeddings of $\mathrm{H}(K)$ into $\mathcal{Q}$ are necessarily equivalent to φ_C. In this paper we will show that, at least when K is the field $\mathrm{GF}(2)$ of two elements, this is not true. In Section 2 we will establish an embedding φ_S of $\mathrm{H}(2)$ into $\mathcal{Q}$ (called the *skew* embedding) which is not equivalent to φ_C.

When $\mathrm{char} K = 2$, we may project the points of $\mathcal{Q}$ onto $\mathrm{PG}(5, K)$ by dropping the coordinate x_7. When K is perfect, i.e., when every element of K is a square (e.g., when K is finite), this projection is a bijection, with inverse operation

$$(x_1, \ldots, x_6) \mapsto (x_1, \ldots, x_6, \sqrt{x_1 x_2 + x_3 x_4 + x_5 x_6}).$$

Lines of $\mathcal{Q}$ map to lines of $\mathrm{PG}(2, 5)$ that satisfy

$$0 = p_{14} + p_{25} + p_{26} (= x_1 y_4 + x_2 y_5 + x_3 y_6 + x_4 y_1 + x_5 y_2 + x_6 y_3),$$

i.e., lines of the symplectic space $W(5, K)$. The symplectic form on this space is the essentially the same as the bilinear form on $\mathcal{Q}$ (and we will use the same notation) :

$$B(x, y) = x_1 y_4 + x_2 y_5 + x_3 y_6 + x_4 y_1 + x_5 y_2 + x_6 y_3.$$

As a consequence, when $\mathrm{char}K = 2$, there is a one-to-one correspondence between the embeddings of $\mathrm{H}(K)$ into $\mathcal{Q}$ and the embeddings of $\mathrm{H}(K)$ into $W(5,K)$. In this paper it will sometimes be more convenient to consider the latter type of embedding, which we will call a *symplectic* embedding.

In Section 3 we describe a computer algorithm which can be used to find all embeddings of $\mathrm{H}(2)$ into $W(5,2)$. It turns out that, apart from the classical embedding and the skew embedding, no other embeddings of this type exist.

It is well known that $\mathrm{H}(2)$ has a *universal embedding* φ_U into $\mathrm{PG}(13,2)$. This embedding has the property that every other embedding of $\mathrm{H}(2)$ into any projective space over the same field (hence also φ_C and φ_S) can be obtained by first applying φ_U and then projecting the result onto a smaller projective subspace. Thas and Van Maldeghem [2] give an elegant description of φ_U and also classify the homogeneous embeddings of $\mathrm{H}(2)$, i.e., those for which all automorphisms of the geometry are represented by projectivities of the space into which it is embedded. (φ_C is homogeneous, but φ_S is not.)

In principle, the symplectic embeddings of $\mathrm{H}(2)$ could be enumerated by considering all possible projections of φ_U into a projective space of dimension 6 and then checking whether the resulting points and lines lie on a parabolic quadric of type $\mathrm{Q}(2,6)$. However, even for modern computers, the number of projections to consider is much too large to make this a feasible project.

2 The skew embedding of H(2)

In what follows we shall restrict ourselves to the field $K = \mathrm{GF}(2)$. Note that in this case $k^2 = k$ for all $k \in K$. Also the projective space $\mathrm{PG}(5,2)$ is essentially the same as the 6-dimensional space over $\mathrm{GF}(2)$ without the zero vector. Henceforth we shall abbreviate $\mathrm{H}(2)$ to $\mathcal{H}$.

Theorem 1 *The coordinate map*

$$\epsilon : (x_1, \ldots, x_7) \mapsto (x_1 + x_6 + f_5(x), x_2 + x_3 + f_4(x), x_3, x_4, x_5, x_6, x_7) \tag{4}$$

with

$$f_4(x) \stackrel{\mathrm{def}}{=} x_3x_5 + x_7x_4, \quad f_5(x) \stackrel{\mathrm{def}}{=} x_4x_6 + x_7x_5, \tag{5}$$

establishes an embedding $\varphi_S \stackrel{\mathrm{def}}{=} \varphi_C\epsilon$ of $\mathcal{H}$ into $\mathcal{Q}$.

Proof : Note that ϵ only changes the first two coordinates of x and that these coordinates do not occur in the expression for $f_4(x)$ and $f_5(x)$. As a consequence ϵ^2 is the identity map, and hence ϵ is a bijection.

Let x denote a point of $\mathcal{Q}$. We have

$$\begin{aligned}
&Q(x^\epsilon)\\
&= (x_1 + x_6 + f_5(x))x_4 + (x_2 + x_3 + f_4(x))x_5 + x_3x_6 + x_7^2\\
&= x_1x_4 + x_4x_6 + x_4(x_4x_6 + x_7x_5) + x_2x_5 + x_3x_5 + x_5(x_3x_5 + x_7x_4) + x_3x_6 + x_7^2\\
&= x_1x_4 + (x_4 + x_4^2)x_6 + x_4x_5x_7 + x_2x_5 + (x_5 + x_5^2)x_3 + x_4x_5x_7 + x_3x_6 + x_7^2\\
&= x_1x_4 + x_2x_5 + x_3x_6 + x_7^2 = Q(x).
\end{aligned} \tag{6}$$

This proves that ϵ maps points of $\mathcal{Q}$ onto points of $\mathcal{Q}$, and hence that φ_S maps points of $\mathcal{H}$ onto points of $\mathcal{Q}$.

Because ϵ is not linear, it does not necessarily map all lines of $\mathcal{Q}$ onto lines. However, at least the lines of $\mathcal{H}$ are mapped onto lines of $\mathcal{Q}$. Indeed, let $\{x, y, x+y\}$ denote the image by φ_C of a line of $\mathcal{H}$. In terms of coordinates, we find

$$\begin{aligned}
&(x+y)^\epsilon - x^\epsilon - y^\epsilon\\
&= (f_5(x+y) - f_5(x) - f_5(y), f_4(x+y) - f_4(x) - f_4(y), 0, 0, 0, 0, 0)\\
&= (x_3y_5 + x_7y_4 + y_3x_5 + y_7x_4, x_4y_6 + x_7y_5 + y_4x_6 + y_7x_5, 0, 0, 0, 0, 0)\\
&= (p_{35} + p_{74}, p_{46} + p_{75}, 0, 0, 0, 0, 0)
\end{aligned}$$

and this is zero, by (3). It follows that $\{x^\epsilon, y^\epsilon, (x+y)^\epsilon\}$ is a line of $\mathcal{Q}$ and hence that also φ_S maps lines of $\mathcal{H}$ onto lines of $\mathcal{Q}$. ■

As was already indicated in the introduction, the embedding φ_S will be called the *skew* embedding of $\mathcal{H}$ into $\mathcal{Q}$.

We denote the set of points of $\mathcal{H}$ that are not opposite to a given point a of $\mathcal{H}$ by $S(a)$. If φ is an embedding of $\mathcal{H}$ into $\mathcal{Q}$, then we call a point a of $\mathcal{H}$ *classical* with respect to φ if and only if $\varphi(S(a)) = \varphi(a)^\perp$. Note that for φ_C every point is classical (whence the terminology).

Theorem 2 *There are exactly* 3 *points of* $\mathcal{H}$ *which are classical with respect to* φ_S*. These points form a line of* $\mathcal{H}$*.*

Proof : Let a be a point of $\mathcal{H}$. Then $S(a)$ consists of those points b of $\mathcal{H}$ for which $B(\varphi_C(a), \varphi_C(b)) = 0$. Writing $x = \varphi_S(a)$, $y = \varphi_S(b)$, and using the fact that ϵ^2 is the identity (from the proof of Theorem 1), this is equivalent to $B(x^\epsilon, y^\epsilon) = 0$. On the other hand, the set of points for which $\varphi_S(b) \in \varphi_S(a)^\perp$ are exactly those that satisfy $B(x, y) = 0$. Hence a is classical with respect to φ_S if and only if $B(x^\epsilon, y^\epsilon)$ and $B(x, y)$ are zero for exactly the same values of y, or equivalently (as $B(x, y)$ must either be equal to 0 or 1), for which $B(x^\epsilon, y^\epsilon) = B(x, y)$. We shall investigate what conditions need to be satisfied by x for this to be true for all values of y that correspond to points of $\mathcal{H}$.

We have

$$\begin{aligned} &B(x^\epsilon, y^\epsilon) - B(x, y) \\ &\quad = (x_6 + f_5(x))y_4 + (x_3 + f_4(x))y_5 + (y_6 + f_5(y))x_4 + (y_3 + f_4(y))x_5 \end{aligned} \tag{7}$$

and this is zero for all y when $x_3 = x_4 = x_5 = x_6 = 0$.

To prove the converse, we first restrict ourselves to the case $y_4 = y_5 = 0$ (and hence $f_4(y) = f_5(y) = 0$). In that case (7) reduces to

$$B(x^\epsilon, y^\epsilon) - B(x, y) = y_6 x_4 + y_3 x_5,$$

which is identically zero for all possible values of y_6, y_3 only if $x_4 = x_5 = 0$. Hence a can be classical for φ_S only if $x_4 = x_5 = 0$. (Note that to every quadruple y_3, y_4, y_5, y_6 there correspond several points of $\mathcal{H}$.)

Taking this into account (7) now reduces to

$$B(x^\epsilon, y^\epsilon) - B(x, y) = y_4 x_6 + y_5 x_3,$$

for general y. It follows that also $x_3 = x_6 = 0$ if a is to be classical.

The points of $\mathcal{Q}$ satisfying $x_3 = x_4 = x_5 = x_6$ have coordinates

$$(1, 0, 0, 0, 0, 0, 0),\ (0, 1, 0, 0, 0, 0, 0),\ (1, 1, 0, 0, 0, 0, 0)$$

and it is easily verified that they form the image of a line of the hexagon through φ_S. ∎

The line in the statement of the theorem above will be called the *axis* of φ_S.

3 There are no other embeddings

To find out whether apart from φ_C and φ_S there exist any other embeddings of $\mathcal{H}$ into $\mathcal{Q}$, we had to resort to the computer. Below we shall describe the algorithm which we have used. It turned out to be more convenient to enumerate the (symplectic) embeddings of $\mathcal{H}$ into $W(5, 2)$ instead of $\mathcal{Q}$.

In what follows, let φ denote any embedding of $\mathcal{H}$ into $W(5,2)$. A set S of points of $\mathcal{H}$ is called a *geometric hyperplane* of $\mathcal{H}$ if and only if every line of $\mathcal{H}$ either has all of its points in S, or intersects S in exactly one point.

Lemma 3 *Let $a, b \in \mathcal{H}$. Let $H(a)$ denote the set of points b of $\mathcal{H}$ such that $B(\varphi(a), \varphi(b)) = 0$ in $W(5,2)$. Then*

1. *$H(a)$ is a geometric hyperplane of $\mathcal{H}$,*
2. *$|H(a)| = 31$,*
3. *$|H(a) \cap H(b)| = 15$, when $a \neq b$,*
4. *$a \in H(a)$,*
5. *if a, b are collinear in $\mathcal{H}$, then $b \in H(a)$,*
6. *$a \in H(b)$ if and only if $b \in H(a)$.*

Proof : By definition $H(a)$ is the image through φ^{-1} of $\varphi(a)^{\perp}$. Note that $x^{\perp}$ is a hyperplane of $W(5,2)$ for all $x \neq 0$.

1. Every hyperplane of $W(5,2)$ is a geometric hyperplane. As φ maps lines of $\mathcal{H}$ to lines of $W(5,2)$ and preserves incidence, it follows that also $H(a)$ is a geometric hyperplane.

2. A hyperplane of $W(5,2)$ contains exactly 31 points.

3. Two different hyperplanes in $W(5,2)$ intersect in 15 points. It is therefore sufficient to prove that $H(a) \neq H(b)$ when $a \neq b$. In $W(5,2)$ the point x is the only point that is collinear with all points of $x^{\perp}$, hence $x^{\perp} = y^{\perp}$ if and only if $x = y$, and the statement follows.

4. We have $B(x,x) = 0$ and hence $x \in x^{\perp}$ for every point x of $W(5,2)$.

5. Because φ is an embedding, two collinear points of $\mathcal{H}$ are mapped to collinear points of $W(5,2)$. In $W(5,2)$ two points x, y are collinear precisely if they are conjugate, and hence the statement follows.

6. We have $B(x,y) = B(y,x)$ and hence $x \in y^{\perp}$ if and only if $y \in x^{\perp}$. ∎

Note that $H(a)$ and $S(a)$ coincide if and only if a is classical with respect to φ.

The algorithm we used essentially generates all functions H that satisfy Lemma 3.

First we generate all geometric hyperplanes of $\mathcal{H}$. For this we have used the universal embedding of $\mathcal{H}$ in $\mathrm{PG}(13,2)$, because in that embedding the hyperplanes and the geometric hyperplanes coincide [1]. There are $2^{14}-1 = 16383$ hyperplanes in $\mathrm{PG}(13,2)$, a number that can easily be managed by computer.

Of these geometric hyperplanes we are only interested in those that contain exactly 31 points of the hexagon. There turn out to be 3591 geometric hyperplanes of this type. (This number is also listed in [1].) For each point a of the hexagon there are exactly 175 geometric hyperplanes that contain a and all points collinear to a.

Using a simple backtracking algorithm we assign to each point a of the hexagon a candidate geometric hyperplane $H(a)$ from this list of 175, making sure that all constraints of Lemma 3 are satisfied. It takes the computer about one minute to come up with the final answer, which turns out to be a list of 64 possible functions H.

Because we did not make any provisions in the algorithm for isomorph-free generation, several of these functions H may correspond to the same embedding. In fact, by applying a suitable automorphism σ of $\mathcal{H}$, we may obtain a skew embedding $\sigma\varphi_S$ with any of the lines of $\mathcal{H}$ as an axis. These account for 63 of the 64 functions H found (the automorphism group of $\mathcal{H}$ acts transitively on its 63 lines). The remaining function H then corresponds to the classical embedding φ_C (for which $H = S$).

Hence, every embedding of $\mathcal{H}$ into $W(5,2)$ is equivalent to either φ_C or φ_S.

References

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