The complete k-arcs of PG$(2, 27)$ and PG$(2, 29)$

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Abstract

A full classification (up to equivalence) of all complete k-arcs in the Desarguesian projective planes of order 27 and 29 was obtained by computer. The resulting numbers of complete arcs are tabulated according to size of the arc and type of the automorphism group, and also according to the type of algebraic curve into which they can be embedded. For the arcs with the larger automorphism groups, explicit descriptions are given.

The algorithm used for generating the arcs is an application of isomorph-free backtracking using canonical augmentation, an adaptation of an earlier algorithm by the authors.

Part of the computer results can be generalized to other values of q: two families of arcs are presented (of size 12 and size 18) for which the symmetric group S_4 is a group of automorphisms.

1 Introduction

Consider the Desarguesian projective plane PG$(2, K)$ over a field K. Let k be a positive integer. A *k-arc* of PG$(2, K)$ is defined to be a set of k points of the plane no three of which are collinear.

We shall be interested in the case where K is the finite field of q elements, and write PG$(2, q)$ = PG$(2, K)$ as customary. In that case it is easily seen that k cannot be larger than $q + 2$. For every even q examples of $(q + 2)$-arcs are known. When q is odd, it can be proved that $(q + 2)$-arcs do not exist. However, every conic is a $(q+1)$-arc, and a well-known theorem of Segre proves that also the converse is true when q is odd. For further information on the geometrical properties of k-arcs we refer to [9].

The subject of k-arcs is not only of interest in its purely geometrical setting. Arcs have applications in coding theory, where they can be interpreted as linear maximum distance separable (MDS) codes, and they are related to superregular matrices (i.e., matrices with entries in K where every minor is non-zero), to linearly independent sets of vectors in vector spaces over K and to optimal covering arrays.

A k-arc is called *complete* if and only if it is not contained in a $(k+1)$-arc. Two k-arcs are called PΓL-*equivalent* if there exists a collineation of $\mathrm{P\Gamma L}(3, q)$ mapping one of them to the other one. If there also exists a collineation of $\mathrm{PGL}(3, q)$ mapping one arc to the other one, then the arcs will be called PGL-*equivalent*. In this paper, we are interested in the finite fields of order $q = 27$ and $q = 29$. We have $[\mathrm{P\Gamma L}(3, 27) : \mathrm{PGL}(3, 27)] = 3$ and $\mathrm{P\Gamma L}(3, 29) = \mathrm{PGL}(3, 29)$.

Many attempts have been made to determine all complete k-arcs in $\mathrm{PG}(2, q)$ up to equivalence. For all but the smallest q this is infeasible without the use of a computer. Full classifications for $q \leq 25$ have been known for some time, a survey for $q \leq 19$ can be found in [6]. In [4], we presented a full classification of the complete arcs in $\mathrm{PG}(2, 23)$ and $\mathrm{PG}(2, 25)$.

Marcugini et al. [14] found the spectrum of all complete ars in $\mathrm{PG}(2, 27)$ (i.e., the list of values k for which a complete k-arc exists). They also found that in $\mathrm{PG}(2, 29)$ no arc of size $k < 13$ exists. G. Kéri [10] has obtained a classification of all arcs of size $k \geq q - 8$ for values of q up to 32, in the context of MDS codes. Our results agree with the partial results of [10]. This article finishes the full classification of all complete k-arcs in $\mathrm{PG}(2, 27)$ and $\mathrm{PG}(2, 29)$. As far as we know, we are the first to do this.

The algorithm we used is that of [4] with some small (but significant) adaptations (cf. Section 2). The same method can also be used to classify the full set of arcs, i.e., not necessarily only those that are complete, and in that case we also think we are the first to obtain a full classification for $q = 27$ and $q = 29$ (cf. Section 6).

One of the purposes of doing a computer classification of this type is to gain further insight into the general class of objects under investigation. In our case we hope to find patterns in the vast amount of data, which may for instance allow us, or other researchers, to derive new general constructions of arcs that also work for larger fields. Using our results, we already managed to discover two (general) types of arc with the symmetric group S_4 as a group of automorphisms. These two types are described in Section 3.

Most well-known constructions produce arcs that have an interesting (and often large) automorphism group. For this reason we have computed the automorphism groups of all complete arcs (cf. Tables 1 and 3). We have also studied the arcs with the larger automorphism groups in more detail, in order to describe them in a more elegant way than by just listing the coordinates of their points. (See Section 4 for $q = 27$ and Section 5 for $q = 29$.)

Sometimes general families of arcs can be described as special subsets of cubic curves or pairs of conics. For this reason we have also computed for each arc the type of algebraic curve of lowest degree into which it can be embedded (cf. Tables 2 and 4).

Our programs, which were written in Java, were run on a Debian Linux system with two quad core Intel Xeon X5355 2.66GHz processors (for $q = 27$) and on a cluster of Debian Linux systems with 56 quad core Intel Xeon X3220 2.40GHz processors (for $q = 29$). The generation of all complete arcs of $\mathrm{PG}(2, 27)$ up to equivalence takes approximately 33 days of CPU time. For $q = 29$ it took approximately 1870 days (five years) of CPU time. To store the results (in compressed form) we need about 130GByte of disk space.

We estimate that for the next case ($q = 31$) the same algorithm would need about 110 000 days (three centuries) and 9000 GByte of storage. This might be just feasible on a large cluster if the program were rewritten in C (which is still faster than Java for this type of application).

2 The algorithm

For the generation of the (complete) k-arcs up to equivalence in $\mathrm{PG}(2, 23)$ and $\mathrm{PG}(2, 25)$, we used an application of isomorph-free generation using canonical augmentation, as introduced by B. McKay [15], which we adapted to the special case of k-arcs in Desarguesian projective planes [4].

The basic idea behind this generation algorithm is the use of a function F wich singles out a special orbit in the set of all orbits of the stabilizer of the arc S on the points of S, and which is group invariant. Arcs of size $k + 1$ are then generated from arcs of size k by adding a single point s to an arc S but only in those cases where s belongs to the special orbit $F(S \cup \{s\})$ of the generated arc. Computations are speeded up by making careful use of a special *point invariant*, a function I_S that associates an integer $I_S(p)$ to every point p of the arc S, in such a way that $I_S(p) = I_S(p')$ whenever p and p' are in the same orbit of the stabilizer group of the arc.

In [4] we used two variants of the same algorithm. One variant makes use of the set stabilizer of the partial arc which we obtain, the other does not, but requires further checks to make sure that no two isomorphic arcs are ever generated. In the first algorithm, the set stabilizer of the arc has to be recomputed for every iteration, a non-trivial task.

However, in some cases too much work was done. Indeed, whenever Γ_S is trivial, and the point s has a unique value of $I_S(s)$, then also $\Gamma_{S'}$ must be trivial (with $S' = S \cup \{s\}$), and therefore there is no need to compute $\Gamma_{S'}$ explicitly. Also, when a sufficient number of points of S have a value of I_S that is unique (five points when $q = 27$, four when $q = 29$), this again implies that Γ_S is trivial.

Because we compute the values of I_S in the course of the algorithm anyway, these extra checks allow us to make some simple shortcuts. Note that a trivial stabilizer group implies trivial orbits, making it easy to ensure that we select not more than one point for each orbit, a crucial step in the algorithm. Moreover, it turns out that almost all arcs that are encountered have a trivial automorphism group.

This idea is essentially a toned down version of a similar technique Brinkmann and McKay used for generating posets up to isomorphism [1].

For the case $q = 27$ we made an additional change to the algorithm. In [4] generation was done using PGL-equivalence, as this was easier to program. Since then we have expanded the program to also use PΓL-equivalence. If the stabilizer group G_S of the arc S in $\mathrm{PGL}(3, q)$ is the same as the stabilizer group Γ_S of S in $\mathrm{P\Gamma L}(3, q)$, then an orbit of $\mathrm{P\Gamma L}(3, q)$ is the union of multiple orbits of $\mathrm{PGL}(3, q)$ (see also Section 4). This means that fewer arcs need to be considered during the course of the algorithm when using PΓL-equivalence, making the program significantly faster.

Note however that the program is still fast enough without the latter modification (it then takes approximately 100 days of CPU time instead of 33). We ran both versions of the program, and the fact that for each k they resulted in the same number of PGL-equivalent arcs is an additional indication that our programs work correctly.

We have also run a consistency check based on the principle of 'double counting', somewhat similar to the method used by Östergård and Pottonen in their generation of perfect binary one-error-correcting codes [16].

Let A_k denote the number of pairs (S, p) where S is an arc of size k and p is a point of S. We shall count A_k in two different ways. Clearly, A_k is k times the total number of arcs of size k. By the orbit-stabilizer theorem, we have

$$A_k = k \sum_{S \in \mathcal{S}_k} \frac{|\Gamma|}{|\Gamma_S|},$$

where $\mathcal{S}_k$ contains one representative for each equivalence class of arcs of size k.

We can also compute A_k in a different way, by counting all pairs (T, p) where $S = T \cup \{p\}$. This yields

$$A_k = \sum_{T \in \mathcal{S}_{k-1}} n(T) \frac{|\Gamma|}{|\Gamma_T|},$$

where $n(T)$ denotes the number of points of the plane that can be added to T to create a new arc. Both formulas should yield the same result.

In order to compute the values of these formulas, we need to know the size of the stablizer group Γ_S and the number $n(S)$ for each arc S generated by our program. These are not so difficult to compute.

Both formulas did indeed yield the same results. For $q = 27$, the values of A_k/k (i.e. the total number of flags of that size) are the following :

k	A_k/k	k	A_k/k	k	A_k/k
4	3917052594	13	1169034571740840255360	22	1808559140544
5	470046311280	14	691584387197093703612	23	470046311280
6	39718913303160	15	133307342955312766344	24	97926314850
7	2299990320671472	16	6989572944968438916	25	15668210376
8	87525520660097676	17	81563752062977184	26	1807870428
9	2071641098183444424	18	323122891215852	27	133916328
10	28492327671983422884	19	43797870737712	28	4782726
11	209860860998677307328	20	16416367421454		
12	751659456980684459844	21	5810667733728		

We ran the same test for all smaller values of q. For $q = 29$ this would have taken too long (two or three years of CPU-time).

3 Some arcs with automorphism group S_4

Before we proceed to the specific results for the cases $q = 27, 29$, we shall first describe two types of arc that we discovered as a result of the computer searches and that also exist for values of q other than 27 or 29. There are two types of arc, one of size 12 and one of size 18. Both accept the symmetric group S_4 as a group of automorphisms.

The arc of size 12 was already discovered by Marcugini et al. [14], for the special case $q = 27$ (see also Section 4.2). However, they did not provide a description for general q.

Theorem 1 *Let $a \in \mathrm{GF}(q)$, q odd. Let $S^*(a)$ denote the set of points of $\mathrm{PG}(2,q)$ with coordinates of the form $(a, \pm 1, \pm 1)$, $(\pm 1, a, \pm 1)$ or $(\pm 1, \pm 1, a)$, with independent choices of sign.*

Then $S^*(a)$ $(= S^*(-a))$ is a 12-arc of $\mathrm{PG}(2,q)$ if and only if

$$a \notin \{0, \pm 1, \pm 2, \pm\sqrt{-1}, \pm\sqrt{-3}, \frac{1}{2}(\pm 1 \pm \sqrt{-7})\}. \tag{1}$$

If these conditions hold, then

- *If $a^2 = -2$, the points of $S^*(a)$ lie on the conic C with equation*

$$C : x_0^2 + x_1^2 + x_2^2 = 0,$$

- *Otherwise, $S^*(a)$ is the disjoint union of three sets $C_0 \cap C_1$, $C_1 \cap C_2$, $C_2 \cap C_0$ of size 4 which are the pairwise intersections of the three conics C_0, C_1, C_2 with equations*

$$\begin{aligned}
C_0 : (a^2+1)x_0^2 &= x_1^2 + x_2^2, \\
C_1 : (a^2+1)x_1^2 &= x_2^2 + x_0^2, \\
C_2 : (a^2+1)x_2^2 &= x_0^2 + x_1^2.
\end{aligned}$$

Proof : We leave it to the reader to verify that $|S^*(a)| = 12$ if and only if $a \neq 0, 1$ or -1.

We first consider the case $a^2 = -1$. Note that in that case the four points with coordinates $(a, 1, \pm 1)$ and $(-1, a, \pm 1)$ lie on the line with equation $x_0 = ax_1$, and then $S^*(a)$ is not an arc.

If $a^2 \neq -1$, the conics C_0, C_1, C_2 are nondegenerate. It is easily seen that any point with coordinates of the form $(a, \pm 1, \pm 1)$ lies on the conic C_1 and C_2. Similarly $(\pm 1, a, \pm 1) \in C_2 \cap C_0$ and $(\pm 1, \pm 1, a) \in C_0 \cap C_1$. It also follows that $C_0 \cap C_1 \cap C_2$ will be nonempty if and only if $a^2 = 1$ or $a^2 = -2$. In the first case $|S^*(a)| < 12$, in the second case we have $C_0 = C_1 = C_2 = C$.

Because different (nondegenerate) conics can intersect in at most 4 points, this proves our claim that $S^*(a)$ is the disjoint union of these three intersections, when (1) holds and $a^2 \neq -2$.

The set $S^*(a)$ is not an arc if and only if there exist three different points of $S^*(a)$ that are collinear. Note that for any pair of points in $S^*(a)$ there is a conic $C_i, i = 0, 1, 2$ that contains this pair. Because conics are arcs, a third point of $S^*(a)$ collinear to this pair cannot lie on that same conic. It follows that any collinear triple must consist of one point with coordinates of the form $(a, \pm 1, \pm 1)$, one with coordinates of the form $(\pm 1, a, \pm 1)$ and one with coordinates of the form $(\pm 1, \pm 1, a)$.

In other words, $S^*(a)$ is not an arc if and only if

$$\begin{vmatrix} a & \pm 1 & \pm 1 \\ \pm 1 & a & \pm 1 \\ \pm 1 & \pm 1 & a \end{vmatrix} = 0,$$

for at least one of the 64 different sign combinations in this determinant.

By multiplying the second and third rows and columns of this determinant by -1 if necessary, we may reduce this condition to

$$\begin{vmatrix} a & 1 & 1 \\ 1 & \pm a & \pm 1 \\ 1 & \pm 1 & \pm a \end{vmatrix} = 0,$$

which, after multiplying the second and third row by a and subtracting the first row, reduces to

$$(1 \pm a^2)(1 \pm a^2) = (1 \pm a)(1 \pm a),$$

with 16 different combinations of signs. Note however that the left hand side of this equation can only take three different values and the same holds for right hand side. This leaves us 9 conditions in all. The following table lists the 9 differences between the possible values of the left hand sides (rows) and right hand sides (columns):

	$(1+a)^2$	$(1-a)^2$	$1-a^2$
$(1+a^2)^2$	$a(a-1)(a^2+a+2)$	$a(a+1)(a^2-a+2)$	$a^2(a^2+3)$
$(1-a^2)^2$	$a(a-2)(a+1)^2$	$a(a+2)(a-1)^2$	$a^2(a-1)(a+1)$
$1-a^4$	$-a(a+1)(a^2-a+2)$	$-a(a-1)(a^2+a+2)$	$-a^2(a-1)(a+1)$

(Note that the second column can be obtained from the first by substituting $-a$ for a. This is a consequence of the fact that $S^*(a) = S^*(-a)$.)

The set $S^*(a)$ is not an arc if and only if any of the 9 entries in this table becomes zero, or equivalently, if and only if at least one of the factors of one of these entries becomes zero. These factors are

$$a, a-1, a+1, a-2, a+2, a^2+3, a^2-a+2, a^2+a+2,$$

whence the values of a listed in (1). ∎

Clearly any permutation of the three coordinates fixes $S^*(a)$. Also, changing the sign of one or more of the coordinates fixes $S^*(a)$. The group generated by these transformations is therefore a group of automorphisms of $S^*(a)$. This group is isomorphic to the symmetric group S_4.

The arc $S^*(a)$ is the same as the arc $S(2/a)$ which was already described in [4, Proposition 3] (the three conics C_0, C_1, C_2 were however not mentioned there). In that paper a different representation is used: the arc is embedded in the hyperplane of $\mathrm{PG}(3,q)$ with equation $x_0 + x_1 + x_2 + x_3 = 0$ and consists of the points whose coordinates are the permutations of $(a, a, -a-2, -a+2)$. The group S_4 of automorphisms acts on this representation by permuting the four coordinates.

In general the arc $S^*(a)$ is not complete. The following theorem shows that for $q = 1 \bmod 4$ and for certain values of a, (at least) six additional points can be added.

Theorem 2 *Let $q = 1 \bmod 4$. Let $a, i \in \mathrm{GF}(q)$, such that $i^2 = -1$. Let $S^*(a)$ be defined as in Theorem 1. Let I denote the set of six points whose coordinates are permutations of $(1, i, 0)$. (I is a subset of the conic $C : x_0^2 + x_1^2 + x_2^2 = 0$.)*

Then $S^*(a) \cup I$ is an 18-arc of $\mathrm{PG}(2,q)$ if and only if

$$a \notin \{0, \pm 1, \pm 2, \pm i, \pm 2i, \pm i\sqrt{3}, \pm i \pm 1, \frac{1}{2}(\pm 1 \pm i\sqrt{7}), \frac{1}{2}(\pm i \pm \sqrt{-5 \pm 4i})\}. \qquad (2)$$

Proof : For $S^*(a) \cup I$ to be an arc $S^*(a)$ must be an arc, and therefore all conditions of Theorem 1 must be fullfilled. Also I must be an arc, but this is trivially true, as I is a subset of a conic.

Therefore, if $S^*(a)$ is an arc then $S^*(a) \cup I$ will not be an arc if and only it contains a collinear triple that intersects both $S^*(a)$ and I.

Because of the automorphisms of $S^*(a)$ we may without loss of generality assume that the collinear triple contains the point P with coordinates $(a, 1, 1)$. The following table lists all lines through P and through one of the points $Q_1, \dots, Q_6$ of I.

$Q_1 = (1, i, 0)$	$x_0 + ix_1 - (a+i)x_2 = 0$
$Q_2 = (1, -i, 0)$	$x_0 - ix_1 - (a-i)x_2 = 0$
$Q_3 = (1, 0, i)$	$x_0 - (a+i)x_1 + ix_2 = 0$
$Q_4 = (1, 0, -i)$	$x_0 - (a-i)x_1 - ix_2 = 0$
$Q_5 = (0, 1, i)$	$(i-1)x_0 - aix_1 + ax_2 = 0$
$Q_6 = (0, 1, -i)$	$(-i-1)x_0 + aix_1 + ax_2 = 0$

Note that the equation for Q_2 can be obtained from that for Q_1 by substituing $-i$ for i. The equations for Q_3 and Q_4 then result from interchanging x_1 and x_2.

We shall denote the equation for the line PQ_j as $f_j(x_0, x_1, x_2) = 0$, $j = 1, \dots, 6$. For ease of computation we introduce the products

$$\begin{aligned} f_1 f_2 &= & & (x_0 - ax_2)^2 &+& (x_1 - x_2)^2, \\ f_3 f_4 &= (x_0 - ax_1)^2 &+& & & (x_1 - x_2)^2, \\ f_5 f_6 &= (x_0 - ax_1)^2 &+& (x_0 - ax_2)^2. \end{aligned}$$

To check whether $S^*(a) \cup I$ is an arc, we need to check whether any of the points (x_0, x_1, x_2) of $S^*(a) \cup I$ yields a value zero for any of the linear functions f_j. This is equivalent to checking whether any of the products $f_1 f_2$, $f_3 f_4$, $f_5 f_6$ yields a zero value. Moreover, because of symmetry, we need not check (x_0, x_2, x_1) when we have already checked (x_0, x_1, x_2). Similarly, the conditions remain invariant under the simultaneous substitution of x_0 by $-x_0$ and a by $-a$. This will reduce the computations somewhat.

The results are listed in the following table

(x_0, x_1, x_2)	$f_1 f_2$	$f_3 f_4$	$f_5 f_6$
$(1, i, 0)$	0	$-a(a+2i)$	$-(a+i+1)(a+i-1)$
$(0, 1, i)$	$-(a+i-1) \cdot (a-i+1)$	$-(a-i-1)(a+i+1)$	0
$(a, -1, -1)$	$4a^2$	$4a^2$	$8a^2$
$(a, 1, -1)$	$4(a-i)(a+i)$	4	$4a^2$
$(1, a, 1)$	$2(a-1)^2$	$(a+1+i)(a+1-i) \cdot (a-1)^2$	$(a+1+i)(a+1-i) \cdot (a-1)^2$
$(-1, a, 1)$	$2(a-i)(a+i)$	$(a^2 - ai + 1 - i) \cdot (a^2 - ai + 1 + i)$	$(a^2 + ai + 1 - i) \cdot (a^2 + ai + 1 + i)$

The zero entries in this table were to be expected, as they correspond to the points we used to define the six lines. For these cases we need to check the linear functions separately: we obtain $f_2(1,i,0) = 2$ and $f_6(0,1,i) = 2ai$, yielding no new conditions on a.

From the above it now follows that apart from the conditions of Theorem 1, a also needs to satisfy $a \neq \pm 2i, \pm i \pm 1$ and $a^2 \pm ai + (1 \pm i) \neq 0$. ∎

(Note that $q = 1 \bmod 4$ if and only an element i exists in $\mathrm{GF}(q)$ that satisfies $i^2 = -1$.)

It is easily checked that the symmetric group S_4 leaves the arc $S^*(a) \cup I$ invariant. When $a^2 = -2$ all points of this arc lie on the conic C. When $a^2 \neq -2$ each of the conics C_0, C_1, C_2 (cf. Theorem 1) contains two points of I.

4 Results: the complete arcs of PG(2,27)

In Table 1 we present a full classification of the complete k-arcs in $\mathrm{PG}(2,27)$, up to $\mathrm{P\Gamma L}$-equivalence. For each of these arcs S we have determined both the stabilizer G_S for the group $G = \mathrm{PGL}(3,27)$ and the stabilizer Γ_S for the group $\Gamma = \mathrm{P\Gamma L}(3,27)$. Each column in this table corresponds to a different arc size k. N_k denotes the number of inequivalent complete arcs of size k (using $\mathrm{P\Gamma L}$-equivalence). There are no complete k-arcs when $k < 12$, $k = 20$, $k = 21$, $23 \leq k \leq 27$ or $k > 28$.

For each k we specify a list of possible automorphism groups Γ_S and G_S and the corresponding number of k-arcs that have automorphism groups of that type. (We use the 'Atlas'-notation for the groups [3].) The numbers listed refer to $\mathrm{P\Gamma L}$-inequivalent arcs and not to PGL-inequivalent arcs. There are essentially two cases:

1. If $G_S = \Gamma_S$, the orbit of S in Γ is the union of three disjoint orbits of G. The first one is S^G, a second one is of the form $S^{\sigma G}$ for some $\sigma \in \Gamma \setminus G$, which we may assume to be the Frobenius automorphism of the field $\mathrm{GF}(27)$, and a last one is $S^{\sigma^2 G}$.

   The arcs S, S^σ and S^{σ^2} are therefore $\mathrm{P\Gamma L}$-equivalent but PGL-inequivalent. Hence the number of PGL-inequivalent k-arcs with a group of that type is three times the number listed.

2. If $G_S \neq \Gamma_S$, then $[\Gamma_S : G_S] = 3$ and $S^G = S^\Gamma$ (and hence S, S^σ and S^{σ^2} are PGL-equivalent). In that case the number of inequivalent arcs of the given type is the same whether we regard $\mathrm{PGL}(3,27)$ or $\mathrm{P\Gamma L}(3,27)$ as the group defining equivalence.

Below we give geometric descriptions of the arcs whose automorphism groups are underlined in Table 1.

$k=12$			$k=13$			$k=14$			$k=15$		
$N_k=7$			$N_k=221429$			$N_k=106320273$			$N_k=198631499$		
Γ_S	G_S	#	Γ_S	G_S	#	Γ_S	G_S	#	Γ_S	G_S	#
S_3	S_3	6	1	1	221342	1	1	106238792	1	1	198614859
$\underline{S_4}$	$\underline{S_4}$	1	2	2	14	2	2	81129	2	2	15506
			3	1	31	3	1	15	3	1	192
			3	3	42	2^2	2^2	224	3	3	936
						4	4	101	6	2	2
						6	2	7	S_3	S_3	4
						12	4	3			
						$\underline{D_{14}}$	$\underline{D_{14}}$	2			

$k=16$			$k=17$			$k=18$		
$N_k=20335114$			$N_k=276112$			$N_k=950$		
Γ_S	G_S	#	Γ_S	G_S	#	Γ_S	G_S	#
1	1	20291521	1	1	274230	1	1	534
2	2	42834	2	2	1861	2	2	333
3	1	223	3	1	21	3	1	3
3	3	159				3	3	19
2^2	2^2	235				2^2	2^2	30
4	4	19				4	4	3
6	2	49				S_3	S_3	25
S_3	S_3	42				9	3	1
D_8	D_8	12				A_4	A_4	1
3^2	3	1				$\underline{3^2\colon 2}$	$\underline{3^2\colon 2}$	1
12	4	2						
6×2	2^2	12						
A_4	2^2	3						
$\underline{\mathrm{SL}(2,3)}$	$\underline{Q_8}$	1						
$\underline{13:6}$	$\underline{D_{26}}$	1						

$k=19$			$k=22$			$k=28$		
$N_k=5$			$N_k=1$			$N_k=1$		
Γ_S	G_S	#	Γ_S	G_S	#	Γ_S	G_S	#
2	2	2	$\underline{7\colon 6}$	$\underline{D_{14}}$	1	$\underline{\mathrm{P\Gamma L}(2,27)}$	$\underline{\mathrm{PGL}(2,27)}$	1
$\underline{6}$	$\underline{2}$	1						
S_3	S_3	2						

Table 1: Numbers of complete k-arcs in $\mathrm{PG}(2,27)$ listed according to size and automorphism group types

Many constructions of arcs have been described in the literature: some arcs are constructed by adding a small number of points to a subset of a conic [11, 13], some can be obtained as unions of subsets of two distinct conics [7] and others as subsets of points of cubic curves [8, 17, 19, 20]. For this reason we enumerate the complete arcs in Table 2 according to their size (rows) and to the type of algebraic curve into which they can be embedded (columns). Each arc is listed with its most specific type. For example, an arc all of whose points belong to an irreducible cubic can also be embedded on a quartic, but will only be listed in the row labelled 'irr. cubic'.

k	conic	irr. cubic	conic + 1	conic + 2	conic + 3	cubic + 1
12						
13		1		7	61	136
14		31		527	8792	4435
15		6	79	561	4689	2261
16		3	69	96	202	270
17					1	24
18		4			1	
19		1				
22						
28	1					
total	1	46	148	1191	13746	7126

k	conic + 4	cubic + 2	2 conics	irr. quartic	other
12	2	5			
13	1667	17104	48557	153896	
14	123432	387014	3713947	102082095	
15	43818	65361	1087165	7159569	190267990
16	1255	2272	21750	30705	20278492
17	3	48	104	61	275871
18		2	8	14	921
19				2	2
22					1
28					
total	170177	471806	4871531	109426342	210823277

Table 2: Algebraic classification of the complete k-arcs in $\mathrm{PG}(2,27)$

Note that any set of 5 (resp. 9, 14, 20) points always lies on a curve of degree 2 (resp. 3, 4, 5), and hence we have restricted ourselves to conics, cubics and quartics. Clearly the only complete arc which lies on a conic is the conic itself.

In what follows let α denote a primitive element of GF(27) which satisfies $\alpha^3-\alpha^2+1=0$. We have $\alpha^{13}=-1$. The Frobenius automorphism of the field corresponds to $k \mapsto k^3$.

4.1 Standard constructions

Two of the listed arcs have well-known constructions. First there is, of course, the conic, the unique (complete) arc of size 28, with $G_S \simeq \mathrm{PGL}(2,27)$ and $\Gamma_S \simeq \mathrm{P\Gamma L}(2,27)$.

Secondly, the unique complete arc of size 16 with $G_S \simeq D_{26}$ is also well-known, see [13]. (The full automorphism group Γ_S of the arc is isomorphic to the semidirect product 13: 6.)

This arc S can be constructed as follows: the elements of S are the points e_1, e_2 with coordinates $e_1(0,1,0)$, $e_2(0,0,1)$ and the 14 points with coordinates $(1,t,t^2)$ where t is a square in GF(27). All points except e_1 belong to the conic C with equation $x_0x_2 = x_1^2$, and e_1 is an external point of C. The automorphism group G_S of S is generated by the element $\phi_1 : t \mapsto \alpha^2 \cdot t$, together with the involution ϕ_2 that interchanges the first and last coordinates, i.e., maps t onto $1/t$ and interchanges $(1,0,0)$ and $(0,0,1)$. The stabilizer group Γ_S is generated by ϕ_1 and $\phi_3 : t \mapsto t^{-3}$. Note that $\phi_2 = \phi_3^3$ and $\phi_1^{\phi_3} = \phi_1^{-3}$. The stabilizer group G_S (resp. Γ_S) of S is a subgroup of index 2 of the subgroup of $\mathrm{PGL}(3,27)$ (resp. $\mathrm{P\Gamma L}(3,27)$) that fixes both C and e_1.

4.2 The unique complete arc of size 12 with $G_S = \Gamma_S \simeq S_4$

If we apply Theorem 1 to $q = 27$, there are 24 values of a which lead to an arc $S^*(a)$ of size 12 with an automorphism group isomorphic to the symmetric group on 4 elements. Only in the cases $a = \pm\alpha^7, \pm\alpha^8, \pm\alpha^{11}$ this arc turns out to be complete. (And these six cases yield equal or PΓL-equivalent arcs.) This example is of special significance because 12 is the smallest size for a complete arc in $\mathrm{PG}(2,27)$.

Coordinates for the points of this arc in $\mathrm{PG}(2,27)$ were already given by Marcugini et al. [14], where it is also mentioned that the arc consists of a single orbit of its group S_4 of automorphisms. They also report that there are three conics that each intersect the arc in 8 points.

4.3 The two complete arcs of size 14 with $G_S = \Gamma_S \simeq D_{14}$

The projective plane $\mathrm{PG}(2,27)$ has two inequivalent complete arcs of size 14 with the dihedral group of order 14 as group of automorphisms. Both arcs can be partitioned into two sets of size 7 and each of these sets is contained in a conic. If we take one of the conics of each arc to be the conic C with equation $x_0x_2 = x_1^2$, then we find the following representatives for the arcs: both arcs contain the points with coordinates

$(1, t, t^2)$ with t one of the elements in the following list:

$$\alpha, \alpha^2, -\alpha^5, \infty, \alpha^5, -\alpha^2, -\alpha,$$

where $t = \infty$ corresponds to the point $(0, 0, 1)$.

The remaining points of the first arc S_1 lie on the conic C_1 with equation $x_0^2 - \alpha^{11}x_1^2 - \alpha^{11}x_2^2 + \alpha^9 x_0 x_2 = 0$. These 7 arc points are

$$(1, -\alpha^{10}, \alpha^6), (1, \alpha^7, \alpha^3), (1, \alpha^3, 1), (1, 0, \alpha^{12}), (1, -\alpha^3, 1), (1, -\alpha^7, \alpha^3), (1, \alpha^{10}, \alpha^6).$$

The remaining points of the second arc S_2 lie on the conic C_2 with equation $x_0^2 - \alpha^8 x_1^2 - \alpha^{11}x_2^2 + \alpha^5 x_0 x_2 = 0$. These 7 arc points are

$$(1, \alpha^9, 0), (1, \alpha^3, \alpha^8), (1, -\alpha^4, -\alpha^{12}), (1, 0, -1), (1, \alpha^4, -\alpha^{12}), (1, -\alpha^3, \alpha^8), (1, -\alpha^9, 0).$$

The automorphism group of both arcs is the same, and can be generated by

$$\begin{aligned} \phi_1 : (x_0\ x_1\ x_2) &\mapsto (x_0\ -x_1\ x_2), \\ \phi_2 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2) \begin{pmatrix} 1 & -\alpha^9 & -\alpha^5 \\ \alpha^8 & -\alpha^{10} & \alpha^9 \\ -\alpha^3 & -\alpha^8 & 1 \end{pmatrix}. \end{aligned}$$

The transformation ϕ_1 fixes the points $(0, 0, 1)$, $(1, 0, \alpha^{12})$ and $(1, 0, -1)$, and reverses the order of the points of S_1 and S_2 as listed above. ϕ_2 has order 7 and permutes the 7 arc points of each conic.

4.4 The unique complete arc of size 22 with $G_S \simeq D_{14}$ and $\Gamma_S \simeq 7{:}6$

PG$(2, 27)$ has a unique complete arc of size 22 with D_{14} as automorphism group G_S and $7{:}6$ as Γ_S. This arc was already described by Chao and Kaneta [2]. It consists of 14 points of a conic, 7 external points to this conic and 1 internal point. This last point is a fixed point of the automorphism group.

4.5 The unique complete arc of size 16 with $G_S \simeq Q_8$ and $\Gamma_S \simeq \mathrm{SL}(2, 3)$

PG$(2, 27)$ also has a unique complete arc of size 16 with G_S isomorphic to the quaternion group of order 8. We list coordinates for the points of one representative of the arc below.

$$\begin{array}{r|r} (0, 1, \pm 1) & (\alpha^2, \alpha, \pm 1) \\ (1, 0, \pm 1) & (\alpha, -\alpha^2, \pm 1) \\ (\alpha^9, \alpha^{12}, \pm 1) & (\alpha^5, \alpha^7, \pm 1) \\ (\alpha^{12}, -\alpha^9, \pm 1) & (\alpha^7, -\alpha^5, \pm 1) \end{array} \tag{3}$$

All points of this arc lie on the quartic with equation $x_0^4+x_1^4-x_2^4-\alpha^7x_0^3x_1+\alpha^7x_0x_1^3=0$. The group G_S is generated by the following eight linear transformations:

$$\begin{aligned}
\pm 1 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \\
\pm i : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \\
\pm j : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2)\begin{pmatrix} \alpha^9 & \alpha^{12} & 0 \\ \alpha^{12} & -\alpha^9 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \\
\pm k : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2)\begin{pmatrix} -\alpha^{12} & \alpha^9 & 0 \\ \alpha^9 & \alpha^{12} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix},
\end{aligned}$$

such that $i^2=j^2=k^2=ijk=-1$. To obtain Γ_S we need to add the automorphism $\phi' : (x_0,x_1,x_2) \mapsto (\alpha^{12}x_0^3, x_1^3-\alpha^9x_0^3, x_2^3)$ which belongs to $\mathrm{P\Gamma L}(3,27) \setminus \mathrm{PGL}(3,27)$. The group Γ_S is isomorphic to $\mathrm{SL}(2,3)$.

4.6 A complete arc of size 18 with $G_S = \Gamma_S \simeq S_3$

There are 25 inequivalent complete arcs of size 18 with $G_S = \Gamma_S \simeq S_3$, but only one of them consists of 15 (=$(q+3)/2$) points of a conic together with 3 points external to this conic. This arc was already described by Davydov et al. [5].

4.7 The unique complete arc of size 18 with $G_S = \Gamma_S \simeq 3^2{:}\,2$

There is a unique complete arc of size 18 with an automorphism group of size 18. The arc can be partitioned into two sets of size 9 each of which is contained in a conic. We list coordinates of one representative of the arc below.

$$\begin{array}{ccc|ccc}
(1,0,0) & (0,1,0) & (0,0,1) & (\alpha^{14},1,1) & (1,\alpha^{14},1) & (1,1,\alpha^{14}) \\
(\alpha^9,\alpha^{16},1) & (1,\alpha^9,\alpha^{16}) & (\alpha^{16},1,\alpha^9) & (\alpha^{11},1,\alpha^8) & (\alpha^8,\alpha^{11},1) & (1,\alpha^8,\alpha^{11}) \\
(\alpha^9,1,\alpha^{16}) & (\alpha^{16},\alpha^9,1) & (1,\alpha^{16},\alpha^9) & (\alpha^{11},\alpha^8,1) & (1,\alpha^{11},\alpha^8) & (\alpha^8,1,\alpha^{11})
\end{array} \qquad (4)$$

The nine points in the left hand part of (4) lie on the conic with equation $xy+xz+yz=0$, those in the right hand part on the conic with equation $\alpha^2x^2+\alpha^2y^2+\alpha^2z^2+xy+xz+yz=0$.

The group $G_S=\Gamma_S$ is generated by the projective transformations ϕ_1,ϕ_2,ϕ_3, represented as follows:

$$\phi_1 : (x_0\ x_1\ x_2) \mapsto (x_2\ x_0\ x_1),$$

$$\begin{aligned} \phi_2 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_2\ x_1), \\ \phi_3 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2) \begin{pmatrix} \alpha^9 & \alpha^{16} & 1 \\ 1 & \alpha^9 & \alpha^{16} \\ \alpha^{16} & 1 & \alpha^9 \end{pmatrix}. \end{aligned}$$

We have $\phi_1^3 = \phi_2^2 = \phi_3^3 = 1$, $\phi_1\phi_3 = \phi_3\phi_1$, $\phi_1^{\phi_2} = \phi_1^{-1}$ and $\phi_3^{\phi_2} = \phi_3^{-1}$.

The transformation ϕ_1 permutes the coordinates cyclicly. This corresponds to a permutation of the columns in the left hand part and in the right hand part of (4), leaving the rows invariant. The transformation ϕ_3 has exactly the opposite effect: it permutes the rows and leaves invariant the columns in (4). The transformation ϕ_2 interchanges the second and last coordinate of a point.

4.8 The unique complete arc of size 19 lying on an irreducible cubic

There is a unique complete arc of size 19 in PG$(2, 27)$ that can be embedded onto a non-singular irreducible cubic curve with one rational inflexion point. This curve has equation $x_2^2x_1 + x_0^3 - \alpha^5x_0^2x_1 + \alpha^2x_1^3 = 0$, and is of type (ii)a, as classified in [9, Theorem 11.54]. The inflexion point has coordinates $(0, 0, 1)$. The abelian group of the 38 rational non-singular points of the cubic is isomorphic to the cyclic group of order 38 and can be generated by the element with coordinates $(1, \alpha^3, 1)$. The arc points are the 19 odd multiples of this generator, in other words they correspond to a coset of a subgroup of index two. It is well known that this construction always yields an arc [19].

The automorphism group G_S is a cyclic group of order 2, while Γ_S is a cyclic group of order 6.

5 Results: the complete arcs of PG(2,29)

The results for $q = 29$ are summarized in Table 3. Again, for each of these arcs S we have determined the stabilizer G_S for the group $G = \mathrm{PGL}(3, 29)$ (which is the same as $\Gamma = \mathrm{P\Gamma L}(3, 29)$ in this case).

As before, each column in the table corresponds to a different arc size k and N_k denotes the number of projectively distinct complete arcs of size k. For the arcs whose automorphism group is underlined, we give a geometric description below. There are no complete k-arcs when $k < 13$, $k = 22$, $k = 23$, $25 \le k \le 29$ or $k > 30$.

In Table 4 the arcs are enumerated according to their size (rows) and to the type of algebraic curve into which they can be embedded (columns). Again, each arc is

$k=13$, $N_k=708$		$k=14$, $N_k=171139332$		$k=15$, $N_k=7402140892$		$k=16$, $N_k=4776509549$	
G_S	#	G_S	#	G_S	#	G_S	#
1	688	1	170929611	1	7402054723	1	4775412456
3	19	2	208889	2	78862	2	1092537
$\underline{13{:}3}$	1	4	212	3	7266	3	2530
		$2{:}2$	612	5	11	4	104
		D_8	6	S_3	29	$2{:}2$	1643
		$\underline{D_{14}}$	2	D_{10}	1	5	7
						S_3	210
						7	1
						D_8	39
						Q_8	1
						D_{10}	11
						A_4	4
						D_{14}	5
						$\underline{D_{30}}$	1

$k=17$, $N_k=271929757$		$k=18$, $N_k=2457679$		$k=19$, $N_k=4190$		$k=20$, $N_k=57$	
G_S	#	G_S	#	G_S	#	G_S	#
1	271852322	1	2421150	1	3615	1	1
2	77365	2	35080	2	546	2	26
4	68	3	525	3	21	2^2	18
7	1	2^2	529	S_3	8	4	1
D_{14}	1	4	91			D_8	4
		S_3	263			D_{10}	6
		6	1			$\underline{D_{20}}$	1
		D_8	14				
		Q_8	4				
		A_4	13				
		D_{12}	5				
		$\underline{S_4}$	4				

$k=21$, $N_k=2$		$k=24$, $N_k=1$		$k=30$, $N_k=1$	
G_S	#	G_S	#	G_S	#
$\underline{S_3}$	2	$\underline{\mathrm{PSL}(2,7)}$	1	$\underline{\mathrm{PGL}(3,29)}$	1

Table 3: Numbers of complete k-arcs in PG$(2,29)$ listed according to size and automorphism group types

k	conic	irr. cubic	conic + 1	conic + 2	conic + 3	cubic + 1
13						
14		25		266	6518	4218
15		106		3761	64204	20893
16		18	508	2769	17739	8600
17		3	305	249	620	942
18		20		14	12	67
19		2				
20		3				
21						
24						
30	1					
total	1	177	813	7059	89093	34720

k	conic + 4	cubic + 2	2 conics	irr. quartic	other
13	2	36	140	530	
14	117712	486746	4295022	166228825	
15	849236	1292115	26135224	251767733	7122007620
16	139307	145957	2873149	5520429	4767801073
17	2755	4327	35581	12258	271872717
18	43	83	471	277	2456692
19		1	4	4	4183
20			4	1	46
21					2
24					
30					
total	1109055	1929265	33339595	423530057	12164142333

Table 4: Algebraic classification of the complete k-arcs in $\mathrm{PG}(2,29)$

listed with its most specific type.

5.1 Standard constructions

Two of the listed arcs have well-known constructions. First there is, of course, the conic, the unique (complete) arc of size 30, with $G_S \simeq \mathrm{PGL}(2,29)$.

There is a second unique complete arc whose construction is fairly well known [11]. It has size 16 and its automorphism group is isomorphic to the dihedral group of order 30. The arc consists of 15 points of a conic C together with an internal point p of that conic. An example of an arc S of this type is constructed as follows. We take C to have equation $x_0x_2 = x_1^2$ and p to have coordinates $(1,0,2)$. The remaining points of S then have coordinates $(1,t,t^2)$ with t one of the elements in the following list:

$$15, 11, 22, 27, 17, 6, 9, \infty, 20, 23, 12, 2, 7, 18, 14, \tag{5}$$

where, as customary, the case $t = \infty$ should be interpreted to correspond to the point with coordinates $(0, 0, 1)$. The automorpishm group G_S is generated by the elements $\phi_1 : t \to \frac{t+26}{16t+1}$ of order 15 and $\phi_2 : t \to -t$ of order 2, both fixing p. The order of the parameter values in (5) corresponds to consecutive applications of ϕ_1. This order is reversed by ϕ_2. We have $\phi_1^{\phi_2} = \phi_1^{-1}$.

5.2 The unique complete arc of size 13 with $G_S \simeq 13{:}\,3$

The smallest size for a complete arc in PG$(2, 29)$ is 13. There is a unique complete arc of that size with an automorphism group of size 39. It can be constructed as the orbit of the 67th power of a Singer cycle and is therefore a so-called *cyclic* arc [18].

If we take this Singer cycle to be

$$\phi : (x_0\ x_1\ x_2) \quad \mapsto \quad (x_0\ x_1\ x_2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 0 & -1 \end{pmatrix}.$$

then the arc is the orbit of the point with coordinates $(1, 0, 0)$ of the cyclic group generated by

$$\phi_1 = \phi^{67} : (x_0\ x_1\ x_2) \quad \mapsto \quad (x_0\ x_1\ x_2) \begin{pmatrix} 0 & 1 & 4 \\ -12 & 0 & -3 \\ 9 & -12 & 3 \end{pmatrix}.$$

(ϕ_1 has order 13.)

The automorphism group of the arc is isomorphic to the semi-direct product $13{:}\,3$ and is generated by ϕ_1 and

$$\phi_2 : (x_0\ x_1\ x_2) \quad \mapsto \quad (x_0\ x_1\ x_2) \begin{pmatrix} 1 & 0 & 0 \\ -9 & 14 & 11 \\ 13 & -6 & 14 \end{pmatrix},$$

of order 3. We have $\phi_1^{\phi_2} = \phi_1^3$.

5.3 The 2 complete arcs of size 14 with $G_S \simeq D_{14}$

Like PG$(2, 27)$ also PG$(2, 29)$ has two inequivalent complete arcs of size 14 with the dihedral group of order 14 as automorphism group. Again, both arcs can be partitioned into two sets of size 7 and each of these sets is contained in a conic. If we take one of the conics of each arc to be the conic C with equation $x_1^2 = x_0x_2$, then we find the following representatives for the arcs: both arcs contain the points with coordinates $(1, t, t^2)$ with t one of the elements of the following list:

$$1,\ 7,\ 7^2 = -9,\ 7^3 = -5,\ 7^4 = -6,\ 7^5 = -13,\ 7^6 = -4$$

The remaining points of the first arc S_1 lie on the conic C_1 with equation $x_1^2 = -4x_0x_2$. These are the points $(1, t, 7t^2)$ for the same values of t. The remaining points of the second arc S_2 lie on the conic C_2 with equation $x_1^2 = -9x_0x_2$. These are the points $(1, t, -13t^2)$, again for the same values of t. The automorphism group of both arcs is the same, and can be generated by

$$\begin{aligned} \phi_1 : (x_0\ x_1\ x_2) &\mapsto (x_0\ 7x_1\ 7^2x_2), \\ \phi_2 : (x_0\ x_1\ x_2) &\mapsto (x_2\ x_1\ x_0). \end{aligned}$$

We have $\phi_1^{\phi_2} = \phi_1^{-1}$.

ϕ_1 acts like $t \mapsto 7t$ on both arcs. ϕ_2 corresponds to $t \mapsto 1/t$ on the conic C, $t \mapsto -4/t$ on $S_1 \setminus C$ and $t \mapsto -9/t$ on $S_2 \setminus C$. It fixes the points $(1, 1, 1)$, $(1, -5, 1)$ of S_1 and $(1, 1, 1)$, $(1, 7, 1)$ of S_2.

5.4 The 4 complete arcs of size 18 with $G_S \simeq S_4$

Applying Theorem 2 to the case $q = 29$ yields twelve values of a for which $S^*(a) \cup I$ is an 18-arc. For eight of these the arc is complete, i.e. when $a = \pm4, \pm6, \pm9$ or ±10. This results in four inequivalent complete arcs of size 18 with automorphism group isomorphic to the symmetric group on 4 elements.

5.5 The unique complete arc of size 20 with $G_S \simeq D_{20}$

There is a unique complete arc of size 20 with the dihedral group of order 20 as group of automorphisms. The arc can be partitioned into two sets of size 10 and each of these sets is contained in a conic.

We may choose coordinates in such way that the first conic C_1 has equation $x^2 + y^2 + 10z^2 = 0$. The arc points on this conic are the following:

$$\begin{matrix} (1,4,6) & (1,10,4) & (1,-3,12) & (1,6,11) & (1,-7,-13) \\ (1,-4,6) & (1,-10,4) & (1,3,12) & (1,-6,11) & (1,7,-13) \end{matrix} \tag{6}$$

The second conic C_2 then has equation $-11xz + 5y^2 - z^2 = 0$, the arc points on C_2 are:

$$(1,0,0) \quad \begin{matrix} (1,-7,-14) & (1,-9,-5) & (0,1,11) & (1,10,8) \\ \\ (1,7,-14) & (1,9,-5) & (0,-1,11) & (1,-10,8) \end{matrix} \quad (1,0,-11) \tag{7}$$

The automorphism group of the arc can be generated by

$$\begin{aligned}
\phi_1 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2)\begin{pmatrix} 4 & 1 & 2 \\ -1 & -1 & -7 \\ -9 & 12 & 7 \end{pmatrix}, \\
\phi_2 : (x_0\ x_1\ x_2) &\mapsto (x_0\ -x_1\ x_2).
\end{aligned}$$

ϕ_1 has order 10 and permutes the 10 arc points of each conic in a clockwise order in (6) and (7). The involution ϕ_2 fixes the points $(1, 0, 0)$ and $(1, 0, -11)$ of C_2 and none of the points of C_1. We have $\phi_1^{\phi_2} = \phi_1^{-1}$.

5.6 The two complete arcs of size 21 with $G_S \simeq S_3$

The third largest size of a complete arc in PG(2, 29) is 21. There are two arcs of this size. The first arc consists of the points

$$\begin{array}{llll}
(1,0,0) & (1,5,10) & (1,4,9) & (1,-3,-2) \\
(0,1,0) & (1,10,5) & (1,9,4) & (1,-2,-3) \\
(0,0,1) & (5,1,10) & (4,1,9) & (-3,1,-2) \\
 & (10,1,5) & (9,1,4) & (-2,1,-3) \\
 & (5,10,1) & (4,9,1) & (-3,-2,1) \\
 & (10,5,1) & (9,4,1) & (-2,-3,1),
\end{array} \tag{8}$$

the second arc consists of the points

$$\begin{array}{llll}
(1,0,0) & (1,2,8) & (1,5,13) & (1,-3,-5) \\
(0,1,0) & (1,8,2) & (1,13,5) & (1,-5,-3) \\
(0,0,1) & (2,1,8) & (5,1,13) & (-3,1,-5) \\
 & (8,1,2) & (13,1,5) & (-5,1,-3) \\
 & (2,8,1) & (5,13,1) & (-3,-5,1) \\
 & (8,2,1) & (13,5,1) & (-5,-3,1).
\end{array} \tag{9}$$

The automorphism group of these arcs is the symmetric group of degree three, which is clearly visible in (8) and (9).

5.7 The unique complete arc of size 24

The unique complete arc of size 24 has an interesting structure which can be described in various ways. It consists of the points of the well-known Klein quartic [12] on GF(29). Its automorphism group is PSL(2, 7) ≡ PSL(3, 2), of order 168.

The Klein quartic can be represented by the simple equation

$$x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0.$$

The automorphism group of this curve is generated by the following elements:

$$\begin{aligned}
\phi_1 : (x_0, x_1, x_2) &\mapsto (x_2, x_0, x_1), \\
\phi_2 : (x_0, x_1, x_2) &\mapsto (7^4 x_0, 7^2 x_1, 7x_2), \\
\phi_3 : (x_0\ x_1\ x_2) &\mapsto (x_0\ x_1\ x_2) \begin{pmatrix} -7 & 8 & -2 \\ 8 & -2 & -7 \\ -2 & -7 & 8 \end{pmatrix}
\end{aligned}$$

(with $\phi_1^3 = \phi_2^7 = \phi_3^2 = 1$).

An alternative representation of this curve, in three dimensions, is given by

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 19x_0x_1x_2x_3, \quad x_0 + x_1 + x_2 + x_3 = 0,$$

which displays the action of the symmetric group S_4 (a subgroup of $\mathrm{PSL}(2,7)$) on the arc. In this representation, the points of the arc correspond to the 24 permutations of the coordinates $(1, 4, 9, 15)$.

Chao and Kaneta [2] had already discovered this arc (and the order of its automorphism group) by computer. However, they did not give an explicit description of its points or mention the connection with the Klein quartic.

6 Results: the arcs of PG(2,27) and PG(2,29)

Finally, in Table 5 we list the number of PΓL-inequivalent k-arcs in $\mathrm{PG}(2,27)$ and $\mathrm{PG}(2,29)$, not necessarily complete. This table supplements the results given in [10].

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	$q = 27$	$q = 29$
$k = 4$	1	1
$k = 5$	4	10
$k = 6$	174	682
$k = 7$	8261	41301
$k = 8$	311313	1933469
$k = 9$	7348659	58423579
$k = 10$	101047498	1072049736
$k = 11$	744145433	11123944005
$k = 12$	2665334400	60140705285
$k = 13$	4145194407	153994534160
$k = 14$	2452359922	167238862321
$k = 15$	472714330	67799467128
$k = 16$	24808360	8854773945
$k = 17$	290532	314349510
$k = 18$	1431	2540088
$k = 19$	183	7280
$k = 20$	82	1477
$k = 21$	32	646
$k = 22$	15	293
$k = 23$	4	98
$k = 24$	3	43
$k = 25$	1	10
$k = 26$	1	5
$k = 27$	1	1
$k = 28$	1	1
$k = 29$		1

Table 5: Numbers of PΓL-inequivalent k-arcs in $\mathrm{PG}(2, 27)$ and $\mathrm{PG}(2, 29)$

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