Cluster formation in a time-varying multi-agent system

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Abstract

We introduce time-varying parameters in a multi-agent clustering model and we derive necessary and sufficient conditions for the occurrence of clustering behavior with respect to a given cluster structure. For periodically varying parameters the clustering conditions may be formulated in a similar way as for the time-invariant model. The results require the individual weights assigned to the agents to be constant. For time-varying weights we illustrate with an example that the obtained results can no longer be applied.

Key words: time-varying systems; self-organizing systems; pattern generation; synchronization

1 Introduction

Synchronization in its most general form may be considered as a process where several subsystems achieve similar long-term behavior, usually as a result of mutual interactions. Examples include systems of coupled oscillators [14], animal swarms [3,13,11], opinion formation [8].

When the subsystems are not identical and the dispersion of the parameters is large compared to the interaction strength, several clusters may arise. Each cluster is characterized by its own long-term behavior, which may correspond to a common phase [7,6] or average frequency [12] in the case of coupled oscillators, a common direction of motion in e.g. swarming [9,4], or a common opinion [5,4].

The corresponding models are usually investigated using simulations, or by combining simulations and local stability results. In [1,2,4] we have introduced a model with a behavior similar to the clustering behavior of models of coupled oscillators such as the Kuramoto model [10], but with an increased potential for analytical results. Furthermore, the model is also relevant for applications not related to coupled oscillators, as we have argued in [4] for swarming and opinion formation, and in [2] for compartmental systems.

The (time-invariant) model is decribed as follows. Each agent tries to follow its natural velocity, while it interacts with other agents by saturating interactions. As a result of these dynamics a cluster structure emerges, with the long-term behavior of each cluster depending on the coupling strength. When the coupling strength equals zero, all agents move at their natural velocities and can be considered as separate clusters (assuming no two natural velocities are equal). For small values of the coupling strength several clusters arise, each characterized by the same asymptotic velocity for its members. For larger values (and provided the interactions are attractive, and the interaction network is connected), distances between agents remain bounded, and all agents are contained in a single cluster, characterized by an asymptotic velocity that equals the average natural velocity of the agents.

Although the assumption of time-invariance may be a good starting point for investigating cluster formation in multi-agent systems, it is clear that for complex systems such as animal swarms and opinion formation processes this assumption is not realistic. The behavior of the individual entities of these systems (i.e. the agents) is influenced by a large number of variables, most of which will be time-varying. Their behavior in time may be either highly predictable, stochastic, or a mixture of both. E.g. the evolution of the outside temperature in time (which may be relevant for the behavior of both animals and humans) may be written as a sum of two periodic func-

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tions (with periods one day and one year) and a residual stochastic component. For systems where such variables play an important role, it is more appropriate to abandon the assumption of time-invariance.

Notice however that, as will be shown in this paper, for periodically varying parameters, conditions characterizing the clustering behavior can be formulated in the same way as for the time-invariant model. Consequently, other results from the time-invariant model may be extended to the periodic case, such as Theorem 5.4 from [4], which shows that the time-invariant model exhibits clustering behavior for all choices of the parameters, and the proof of which indicates how the cluster structure can be obtained.

In this paper we let all parameters of the aforementioned clustering model vary in time — except for the weights γ_i (see the model in the next section) — and we derive necessary and sufficient conditions for the clustering behavior. A numerical example shows that this result is no longer valid if the weights γ_i are also time-varying. Our analytical approach cannot easily be extended to include time-varying weights, and it is not clear to us how the model with time-varying weights could be dealt with and whether or not similar criteria may be derived for clustering behavior.

For the applications mentioned before (we refer to [2,4] for details) the assumption of constant weights does not seem to pose problems. Although for swarming and opinion formation this may not be easily assessed in an objective way, it seems acceptable to assume that the influence of one animal or person on the others changes on a slower time scale than the other parameters. For the application on compartmental systems (such as a system of interconnected water basins) the weights relate to the sizes of the compartments (or water basins), which may be assumed constant in most cases, while the time-variance of the flow into or out of the compartments is captured by the time-varying parameters of model.

In the next section we review the time-invariant model and the results from [4] that are relevant to this paper. Section 3 introduces the time-varying model and presents necessary and sufficient conditions for clustering behavior of solutions of this model. The proof of this analytical result is given in section 5. In section 4 we reformulate the results for periodically varying parameters, and section 6 deals with an example with (periodically) varying weights γ_i for which our main result can no longer be applied. In section 7 we discuss further extensions of the model and whether or not these may be treated by simply extending our analysis.

For an application of the time-varying clustering model on swarming we refer to [4], where the influence of different network structures on the emerging cluster structure is discussed.

2 Preliminary results

2.1 The time-invariant model

The model from [4] is described by the following differential equations.

$$\dot{x}_i(t) = b_i + K \sum_{j=1}^{N} \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall\, i \in \{1, \dots, N\}, \tag{1}$$

with $\gamma_j > 0$, $K \geq 0$, $N > 1$. The functions f_{ij} are non-decreasing, Lipschitz continuous and satisfy

$$\begin{aligned} &f_{ji}(x) = -f_{ij}(-x), \qquad \forall\, x \in \mathbb{R}, \\ &f_{ij}(x) = F_{ij}, \quad \text{and thus} \quad f_{ij}(-x) = -F_{ji}, \\ &\qquad \forall\, x \in [d, +\infty), \end{aligned}$$

for all i, j in $\{1, \dots, N\}$, for some $d > 0$. It follows that (for $i, j \in \{1, \dots, N\}$) $F_{ij} \geq -F_{ji}$, and therefore $F_{ij} + F_{ji} \geq 0$. (Notice that either F_{ij} or F_{ji} is allowed to be negative.)

The intervals $[-F_{ji}, F_{ij}]$ and $[-F_{ij}, F_{ji}]$ cover the range of the interaction between agents i and j. The extent to which each individual agent j tends to influence the behavior of other agents is reflected by the weight γ_j.

The dispersion of the natural velocities b_i and the interaction lead to the formation of different clusters. The clusters consist of agents whose natural velocities are sufficiently close to each other compared to the interaction strength, while the difference in natural velocity between agents from different clusters is too large to form a single group.

Remark 1 *The second condition imposed on f_{ij} is more strict than the condition from [4] (which only required that $\lim_{x\to+\infty} f_{ij}(x) = F_{ij}$). As a consequence we will formulate a stronger result in Theorem 2 (later on) than the corresponding result from [4]. Theorem 2 is easily proven by adapting the proof in [4] and may be considered as a special case of the result that we will derive for the time-varying model in Theorem 3. This adaptation makes it more convenient to compare the results for the time-varying model with those for the time-invariant model.*

2.2 Some notation

For $n \in \mathbb{N}_0$ denote by I_n the set $\{1, \dots, n\}$ and let $G = (G_1, \dots, G_M)$ be an ordered set partition of I_N. Let $G_k^<$ be a shorthand notation for $\bigcup_{k'<k} G_{k'}$, and similarly set $G_k^> \triangleq \bigcup_{k'>k} G_{k'}$.

We consider the following definition of *clustering behavior* of a solution x of (1) with respect to a cluster structure G:

- The distances between agents in the same cluster remain bounded (i.e. $|x_i(t) - x_j(t)|$ is bounded for all $i, j \in G_k$, for any $k \in I_M$, for $t \geq 0$).
- For any $D > 0$ there exists a time after which the distances between agents in different clusters are and remain at least D.
- The agents are ordered by their membership to a cluster: $k < l \Rightarrow x_i(t) < x_j(t)$, for all i in G_k and j in G_l, for all $t \geq T$, for some $T > 0$.

For a non-empty set $G_0 \subset I_N$ and a vector $w \in \mathbb{R}^N$ denote by $\langle w \rangle_{G_0}$ the weighted average of w over all elements in G_0, with weighting factors γ_i:

$$\langle w \rangle_{G_0} = \frac{\sum_{i \in G_0} \gamma_i w_i}{\sum_{i \in G_0} \gamma_i},$$

and define the function $\widetilde{v}$ as

$$\widetilde{v}(G_-, G_0, G_+) \triangleq \langle b \rangle_{G_0} + \frac{K}{\sum_{i \in G_0} \gamma_i} \sum_{i \in G_0} \gamma_i \times \left(\sum_{j \in G_+} \gamma_j F_{ij} - \sum_{j \in G_-} \gamma_j F_{ij} \right),$$

where $G_-, G_0, G_+ \subset I_N$ with G_0 non-empty.

If $\{G_-, G_0, G_+\}$ partitions I_N, then the function value $\widetilde{v}(G_-, G_0, G_+)$ represents the average velocity $\langle \dot{x}(t) \rangle_{G_0}$ (with x a solution of (1)) of the agents in G_0 when the agents in G_-, resp. G_+, have x_i-values smaller than, resp. larger than, the x_i-values of the agents in G_0, with the differences being larger than or equal to d.

For any set S, let $\mathcal{P}(S)$ denote the set of all ordered partitions of S in two subsets, i.e.

$$\mathcal{P}(S) = \{(S_1, S_2) : S_1, S_2 \subsetneq S \text{ with } S_2 = S \setminus S_1\}.$$

2.3 Necessary and sufficient conditions

The notation introduced in the previous section allows a concise formulation of the following theorem which is an adapted version of Theorem 5.1 from [4].

Notice that the conditions (2a) require the velocities associated with different clusters to be ordered according to the order of the clusters. The conditions (2b) require that the velocities associated with two subsets constituting a partition of a cluster should be such that the separation between these two subsets cannot increase once they are sufficiently (i.e. a distance d) far apart.

Theorem 2 *Consider the following inequalities*

$$\widetilde{v}(G_k^<, G_k, G_k^>) < \widetilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>), \quad \forall\, k \in I_{M-1}, \tag{2a}$$

$$\widetilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) \leq \widetilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}), \quad \forall\, (G_{k,1}, G_{k,2}) \in \mathcal{P}(G_k), \quad \forall\, k \in I_M. \tag{2b}$$

The conditions (2) *are* necessary and sufficient *for clustering behavior with respect to G of* all *solutions of the system* (1)*.*

It can be shown (by adapting Theorem 5.4 from [4] to the formulation of (2)) that there is always a unique cluster structure satisfying the conditions (2), and therefore the solutions of the model (1) always exhibit clustering behavior. Notice that for $K = 0$ the corresponding cluster structure is easily found, and then increase K to the proposed value; whenever, in the process of increasing K, one of the conditions (2a) (resp. (2b)) is no longer satisfied, a new cluster structure satisfying (2) (for this particular value of K) can be obtained by merging two clusters (resp. splitting a cluster). This procedure can be repeated until K attains the proposed value.

3 Time-varying parameters

From now on we assume that b_i, K and f_{ij} (and therefore also F_{ij}) are time-varying:

$$\dot{x}_i(t) = b_i(t) + K(t) \sum_{j=1}^{N} \gamma_i f_{ij} \left(x_j(t) - x_i(t), t \right), \forall\, i \in I_N, \tag{3}$$

with $\gamma_j > 0$, $K(t) \geq 0$ for all t in $\mathbb{R}$, K and b_i continuous, and f_{ij} is Lipschitz continuous with $f_{ij}(x, t)$ increasing in x and saturating for all t in $\mathbb{R}$. The functions f_{ij} attain their saturation values for $d > 0$, i.e.

$$f_{ij}(x, t) = F_{ij}(t), \quad \forall\, x \geq d, \quad \forall\, t \in \mathbb{R}, \quad \forall\, i, j \in I_N,$$

and satisfy

$$f_{ji}(x, t) = -f_{ij}(-x, t), \quad \forall\, x, t \in \mathbb{R}, \quad \forall\, i, j \in I_N.$$

Redefine the function $\widetilde{v}$ as

$$\widetilde{v}(G_-, G_0, G_+, t) \triangleq \langle b(t) \rangle_{G_0} + \frac{K(t)}{\sum_{i \in G_0} \gamma_i} \sum_{i \in G_0} \gamma_i \times \left(\sum_{j \in G_+} \gamma_j F_{ij}(t) - \sum_{j \in G_-} \gamma_j F_{ij}(t) \right),$$

for any $t \in \mathbb{R}$, where again $G_-, G_0, G_+ \subset I_N$ with G_0 non-empty.

With this notation we can extend Theorem 2 as follows.

Theorem 3 (Main Theorem) *The conditions*

$$\lim_{t\to+\infty}\int_0^t \big(\widetilde{v}(G_{k+1}^{<}, G_{k+1}, G_{k+1}^{>}, t') - \widetilde{v}(G_k^{<}, G_k, G_k^{>}, t')\big)\mathrm{d}t' = +\infty, \quad \forall\, k \in I_{M-1}, \tag{4a}$$

$$\exists\, c \in \mathbb{R} : \int_{t_0}^t \big(\widetilde{v}(G_k^{<} \cup G_{k,1}, G_{k,2}, G_k^{>}, t') - \widetilde{v}(G_k^{<}, G_{k,1}, G_k^{>} \cup G_{k,2}, t')\big)\mathrm{d}t' < c, \quad \forall\, t, t_0 \in \mathbb{R}^+ \text{ with } t > t_0, \quad \forall\,(G_{k,1}, G_{k,2}) \in \mathcal{P}(G_k), \qquad \forall\, k \in I_M. \tag{4b}$$

are necessary and sufficient *for clustering behavior with respect to* G *of* all *solutions of the system* (3).

For the proof we refer to section 5.

Remark 4 *There may not always be a cluster structure satisfying* (4). *The corresponding solutions of* (3) *will then also not exhibit clustering behavior.*

4 Periodically varying parameters

In this section we assume that the parameters b_i, K, and F_{ij} are periodic in time, with a common period for K and all F_{ij} (unless K is constant). We define $\overline{v}$ as the time-average of the function $\widetilde{v}$:

$$\overline{v}(G_-, G_0, G_+) \triangleq \lim_{t\to+\infty} \frac{1}{t}\int_0^t \widetilde{v}(G_-, G_0, G_+, t')\mathrm{d}t',$$

for all $G_-, G_0, G_+ \subset I_N$ with G_0 non-empty.

With these assumptions each of the integrals in (4a) and (4b) can be written as the sum of a linear function of time, with a coefficient equal to

$$\overline{v}(G_{k+1}^{<}, G_{k+1}, G_{k+1}^{>}) - \overline{v}(G_k^{<}, G_k, G_k^{>})$$

or

$$\overline{v}(G_k^{<} \cup G_{k,1}, G_{k,2}, G_k^{>}) - \overline{v}(G_k^{<}, G_{k,1}, G_k^{>} \cup G_{k,2}),$$

and a periodic function of time. As a result Theorem 3 can be rewritten in terms of $\overline{v}$ as follows.

Corollary 5 *The conditions*

$$\overline{v}(G_k^{<}, G_k, G_k^{>}) < \overline{v}(G_{k+1}^{<}, G_{k+1}, G_{k+1}^{>}), \quad \forall\, k \in I_{M-1}, \tag{5a}$$

$$\overline{v}(G_k^{<} \cup G_{k,1}, G_{k,2}, G_k^{>}) \leq \overline{v}(G_k^{<}, G_{k,1}, G_k^{>} \cup G_{k,2}), \quad \forall\,(G_{k,1}, G_{k,2}) \in \mathcal{P}(G_k), \qquad \forall\, k \in I_M. \tag{5b}$$

are necessary and sufficient *for clustering behavior with respect to* G *of* all *solutions of the system* (3) *with periodically varying parameters* b_i*,* K*, and* f_{ij}*.*

Notice that the formulation of this result is similar to the formulation of Theorem 2 regarding the time-invariant system (1). It can be shown that an adapted version of Theorem 5.4 from [4] may be applied to the conditions (5), implying that a unique cluster structure can be associated with (3), for any choice of the (periodic) parameters.

Remark 6 *When* K *and* F_{ij} *have no common period,* $\overline{v}$ *may still be well-defined, but the proof of the convergence of the limit is somewhat technical, and for convenience we consider a more restrictive assumption implying periodicity of the products* KF_{ij}*.*

Remark 7 *Notice that the reformulation of* (4) *as* (5) *may also hold for non-periodically varying parameters. (However, the convergence of the limits in the definition of* $\overline{v}$ *does not guarantee the reformulation to be valid: if the integral in* (4a) *behaves like* $\ln t$ *for large* t*, then* (4a) *will be satisfied, but in* (5a) *the left hand side and the right hand side will be equal.) Therefore the present results show that for a large class of time-varying systems one may simply consider the results of the time-invariant model by appriopriately defining the constant parameters in this model. E.g. the parameters* b_i *of the time-invariant model will correspond to the time-averages of the functions* b_i *of the time-varying model.*

5 Proof of the Main Theorem

5.1 Necessity of the conditions (4)

Assume there is a solution x of (3), exhibiting clustering behavior with respect to G. Choose $T > 0$ such that the distances between agents in different clusters are and remain at least d. It follows that for any $k \in I_{M-1}$ and $t > T$

$$\langle \dot{x}(t)\rangle_{G_{k+1}} - \langle \dot{x}(t)\rangle_{G_k} = \widetilde{v}(G_{k+1}^{<}, G_{k+1}, G_{k+1}^{>}, t) - \widetilde{v}(G_k^{<}, G_k, G_k^{>}, t).$$

From the clustering behavior of x immediately follows (4a).

Similarly one derives, for any $k \in I_M$ and $(G_{k,1}, G_{k,2}) \in \mathcal{P}(G_k)$, and taking into account that the functions f_{ij} are non-decreasing in their first argument, that, for $t \geq T$,

$$\langle \dot{x}(t)\rangle_{G_{k,2}} - \langle \dot{x}(t)\rangle_{G_{k,1}} \geq \widetilde{v}(G_k^{<} \cup G_{k,1}, G_{k,2}, G_k^{>}, t) - \widetilde{v}(G_k^{<}, G_{k,1}, G_k^{>} \cup G_{k,2}, t),$$

and since $\langle x(t)\rangle_{G_{k,2}} - \langle x(t)\rangle_{G_{k,1}}$ remains bounded, (4b) follows.

5.2 Sufficiency of the conditions (4) *for the existence of* a *solution exhibiting clustering behavior with respect to G*

Assume (4) holds. Set $\gamma_{\min} \triangleq \min_{i\in I_N} \gamma_i$. For any $D_1 \geq D_2 > 0$, consider the following region $R_{D_1,D_2} \subset \mathbb{R}^N$:

$$y \in R_{D_1,D_2} \Leftrightarrow \begin{cases} \langle y\rangle_{G_{k+1}} - \langle y\rangle_{G_k} \geq \dfrac{D_1 \sum_{i\in G_k\cup G_{k+1}} \gamma_i}{2\gamma_{\min}}, \\ \qquad \forall\, k \in I_{M-1}, \\ \langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}} \leq \dfrac{D_2 \sum_{i\in G_k} \gamma_i}{2\gamma_{\min}}, \\ \qquad \forall\, (G_{k,1}, G_{k,2}) \in \mathcal{P}(G_k), \qquad \forall\, k \in I_M. \end{cases}$$

We will show that a solution x with $x(t_0) \in R_{D_1,d}$, satisfies $x(t) \in R_{D_1',D_2'}$, $\forall t \geq t_0$, for some $D_1, D_1', D_2' \in \mathbb{R}$, with $D_1 \geq D_1' \geq D_2' \geq d$.

The second set of inequalities characterizing R_{D_1,D_2} can be rewritten as

$$\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_k} \leq \frac{D_2 \sum_{i\in G_{k,1}} \gamma_i}{2\gamma_{\min}}$$

or

$$\langle y\rangle_{G_k} - \langle y\rangle_{G_{k,1}} \leq \frac{D_2 \sum_{i\in G_{k,2}} \gamma_i}{2\gamma_{\min}},$$

for all $(G_{k,1}, G_{k,2})$ in $\mathcal{P}(G_k)$, for all k in I_M. Considering the inequalities for which $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$ together with the first set of inequalities characterizing R_{D_1,D_2}, it follows that if $y \in R_{D_1,D_2}$ then for any $k \in I_{M-1}$, all $i \in G_k$ and $j \in G_{k+1}$ satisfy $y_j - y_i \geq \frac{D_1(\gamma_i+\gamma_j)}{2\gamma_{\min}} \geq D_1$. As a consequence we can derive that, if $x(t) \in R_{D_1,D_2}$ for some $t \in \mathbb{R}$, with $D_1 \geq D_2 \geq d$, then

$$\begin{aligned} &\langle \dot{x}(t)\rangle_{G_{k+1}} - \langle \dot{x}(t)\rangle_{G_k} = \\ &\quad \widetilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t) - \widetilde{v}(G_k^<, G_k, G_k^>, t), \end{aligned} \tag{6}$$

and therefore (4a) implies that, given $t_0 \in \mathbb{R}$,

$$\langle x(t)\rangle_{G_{k+1}} - \langle x(t)\rangle_{G_k} - \left(\langle x(t_0)\rangle_{G_{k+1}} - \langle x(t_0)\rangle_{G_k}\right)$$

is bounded from below. If follows that for the choice

$$\begin{aligned} D_1 \geq D_2 - &\min_{k\in I_{M-1}} \frac{2\gamma_{\min}}{\sum_{i\in G_k\cup G_{k+1}} \gamma_i} \inf_{t\geq t_0} \int_{t_0}^{t} \mathrm{d}t' \\ \times &\left(\widetilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t') - \widetilde{v}(G_k^<, G_k, G_k^>, t')\right), \end{aligned} \tag{7}$$

and $D_2 \geq d$, a solution with $x(t_0) \in R_{D_1,d}$ cannot leave $R_{D_2,D_2} \supset R_{D_1,d}$ by making

$$\langle x(t)\rangle_{G_{k+1}} - \langle x(t)\rangle_{G_k} - \frac{D_2 \sum_{i\in G_k\cup G_{k+1}} \gamma_i}{2\gamma_{\min}}$$

negative for some $k \in I_{M-1}$.

In the appendix we show that, if $x(t_0) \in R_{D_1,d}$ and D_1 and D_2 are sufficiently large,

$$\langle x(t)\rangle_{G_{k,2}} - \langle x(t)\rangle_{G_{k,1}} - \frac{D_2 \sum_{i\in G_k} \gamma_i}{2\gamma_{\min}} \leq 0, \tag{8}$$

for all $t \geq t_0$, $(G_{k,1}, G_{k,2})$ in $\mathcal{P}(G_k)$, for all k in I_M. This implies that $x(t) \in R_{D_2,D_2}$, $\forall\, t \geq t_0$, and together with (6) and (4a) we can conclude that x exhibits clustering behavior with respect to G.

The proof is quite technical. It is based on the fact that, on the one hand, when $x(t) \in R_{D_2,D_2}$ and $x_i(t) \leq x_j(t) - d$ for all i in $G_{k,1}$, j in $G_{k,2}$, for some $(G_{k,1}, G_{k,2})$ in $\mathcal{P}(G_k)$, for some k in I_M, and for some interval for t, then $\langle x(t)\rangle_{G_{k,2}} - \langle x(t)\rangle_{G_{k,1}}$ cannot increase by a value larger than c in this interval for t (by (4b)), while on the other hand, when switching to another partition $(G_{k,1}', G_{k,2}') \in \mathcal{P}(G_k)$ for considering this property, the corresponding value is multiplied by a factor Γ smaller than 1. As a result, the total increase will be bounded by $\frac{c}{1-\Gamma}$, implying that $x_i(t) - x_j(t)$ is bounded.

5.3 Sufficiency of (4) *for clustering behavior (with respect to G) of* all *solutions of* (1)

Let x^* be a solution exhibiting clustering behavior with respect to G and let x be any other solution of (1). Consider the function

$$V : \mathbb{R} \to \mathbb{R} : t \mapsto V(t) = \sum_{i\in I_N} \gamma_i (x_i^*(t) - x_i(t))^2.$$

Then

$$\begin{aligned} \dot{V}(t) &= 2 \sum_{i\in I_N} \gamma_i (x_i^*(t) - x_i(t)) K(t) \sum_{j\in I_N} \gamma_j \\ &\qquad \times \left(f_{ij}(x_j^*(t) - x_i^*(t), t) - f_{ij}(x_j(t) - x_i(t), t)\right) \\ &= K(t) \sum_{i,j\in I_N} \gamma_i\gamma_j (x_i^*(t) - x_j^*(t) + x_j(t) - x_i(t)) \\ &\qquad \times \left(f_{ij}(x_j^*(t) - x_i^*(t), t) - f_{ij}(x_j(t) - x_i(t), t)\right) \\ &\leq 0, \end{aligned}$$

since the functions f_{ij} are non-decreasing in their first argument. It follows that V is non-increasing, and therefore $|x_i^*(t) - x_i(t)|$ is bounded, for any $i \in I_N$. This implies that x exhibits the same clustering behavior as x^*.

6 An example with time-varying weights γ_i

The proof of Theorem 3 is based on the fact that

$$\frac{\mathrm{d}\langle x\rangle_{G_0}}{\mathrm{d}t} = \langle \dot{x}\rangle_{G_0},$$

i.e. the time-derivative of the average over a set G_0 equals the average of the time-derivatives. If the weights γ_i would also be time-varying, this equality would no longer hold and the result of Theorem 3 would not be valid.

In this section we provide an example of a system with 3 agents with time-varying weights γ_i, and we show (numerically) that a cluster structure appears which is different from the cluster structure predicted by Theorem 3.

The functions f_{ij} are set equal to the function f, defined by $F_{ij} = F = 1$, $d = 1$, and $f(x) = x$ for $x \in [-d, d]$. Furthermore, $K(t) = 1$, $b_1(t) = -6$ and $\gamma_1(t) = 1$ for all $t \in \mathbb{R}$. The parameters $b_2(t)$ and $b_3(t)$ are switching repeatedly between 6 and -2, and $\gamma_2(t)$ and $\gamma_3(t)$ are switching between 1 and 4, in time intervals with length 5:

$$\begin{aligned} b_2(t) &= 6, & \gamma_2(t) &= 1, \\ b_3(t) &= -2, & \gamma_3(t) &= 4, \end{aligned}$$

for all t in $[10k, 10k+5)$, for some k in $\mathbb{Z}$,

$$\begin{aligned} b_2(t) &= -2, & \gamma_2(t) &= 4, \\ b_3(t) &= 6, & \gamma_3(t) &= 1, \end{aligned}$$

for all t in $[10k+5, 10k+10)$, for some k in $\mathbb{Z}$.

Remark 8 *Although the functions b_2, b_3, γ_2, and γ_3 are not continuous, they can be approximated by continuous functions, without affecting our conclusion. For convenience, we choose them to be piecewise continuous.*

We verify that for the choice $G = (\{1, 2, 3\})$ (i.e. there is only one cluster, containing the three agents) the conditions (5b) from Corollary 5 are satisfied. (There are no conditions corresponding to (5a) for a partition with only one cluster.)

We find that

$$\begin{aligned} \overline{v}(\{1\}, \{2,3\}, \emptyset) &= -\frac{7}{5} < \overline{v}(\emptyset, \{1\}, \{2,3\}) = -1, \\ \overline{v}(\{2\}, \{1,3\}, \emptyset) &= -\frac{39}{10} < \overline{v}(\emptyset, \{2\}, \{1,3\}) = \frac{11}{2}, \\ \overline{v}(\{3\}, \{1,2\}, \emptyset) &= -\frac{39}{10} < \overline{v}(\emptyset, \{3\}, \{1,2\}) = \frac{11}{2}, \\ \overline{v}(\{1,2\}, \{3\}, \emptyset) &= -\frac{3}{2} < \overline{v}(\emptyset, \{1,2\}, \{3\}) = \frac{11}{10}, \\ \overline{v}(\{1,3\}, \{2\}, \emptyset) &= -\frac{3}{2} < \overline{v}(\emptyset, \{1,3\}, \{2\}) = \frac{11}{10}, \\ \overline{v}(\{2,3\}, \{1\}, \emptyset) &= -11 < \overline{v}(\emptyset, \{2,3\}, \{1\}) = \frac{3}{5}, \end{aligned}$$

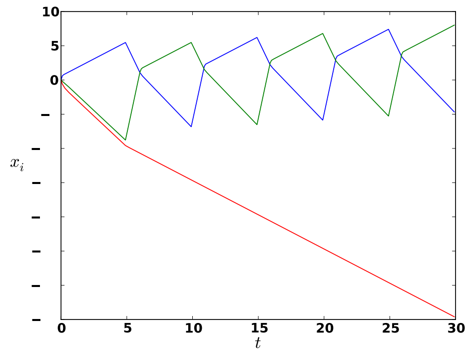

Figure 1. Simulation of the system described in section 6.

suggesting that there will indeed only be one cluster, equal to $\{1, 2, 3\}$. On the other hand, the simulation of the differential equations (3) shown in Fig. 1 reveals the emergence of two clusters: $G = (\{1\}, \{2, 3\})$. Although $\widetilde{v}(G_-, G_0, G_+, t)$ may still be considered as an average velocity over a set G_0 of agents, it is no longer equal to the time-derivative of the average x_i-value over G_0, since the latter may also vary as a result of changes in the weights γ_i.

7 Further extensions and conclusion

The previous example has illustrated that Theorem 3 cannot simply be extended for the case of time-varying weights. In fact, the expression for the time-derivative $\frac{\mathrm{d}\langle x\rangle_{G_0}}{\mathrm{d}t}(t)$ would also contain products of $\dot{\gamma}_i(t)$ and the positions $x_i(t)$, indicating that the distances between agents from the same cluster would also have an impact on the average cluster velocity, and therefore on the clustering behavior. Since these distances also depend on the shape of the interaction functions f_{ij} — and not only on the saturation values F_{ij} — a characterization of a potential emerging cluster structure would probably involve the functions f_{ij} and would therefore be much more complex than the necessary and sufficient conditions stated in this paper. It is not clear to us what such an exact characterization might look like if one wants to maintain the generality of the model, and even less how it may be derived.

Another extension of the model that would be relevant for applications is to consider a more general class of interaction functions. Removing the saturation assumption and allowing a function f_{ij} to grow unbounded could be easily incorporated into the analysis by setting the corresponding saturation value F_{ij} equal to $+\infty$, and introducing some new notation to avoid evaluating expressions equivalent to $\infty - \infty$. To investigate the case where the interactions become zero for large distances, which would be more realistic when modeling e.g. swarming behavior, one may consider removing the restriction that the functions f_{ij} are non-decreasing in their first argument. However, this would require a different analytical treatment; the independence of initial conditions will be lost and the possible cluster structures may depend on the shape of the interaction functions f_{ij} and not only on the saturation values, implying again that an analytical treatment with results as general as in this paper will be much harder.

In this paper we have studied a time-varying version of a clustering model and we have provided conditions characterizing the clustering behavior of its solutions. The results show that for a large class of time-varying systems these conditions may be formulated similarly as the conditions for the time-invariant system. This may for instance justify replacing some of the time-varying parameters by their time-averaged values.

Some properties of the system that are crucial in our analysis are the assumption of constant weights and non-decreasing interaction functions. For some of the applications that we have in mind these assumptions seem acceptable (e.g. for compartmental systems), for other applications this imposes some restrictions on the systems that may be investigated analytically. We have illustrated that an emerging cluster structure in a system with time-varying weights does not necessarily satisfy the conditions presented in this paper, and a different analytical approach may be required to handle this problem.

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A Proof of the inequalities (8)

We first introduce some notation. Fix a $k \in I_M$ and define $\tilde{D}_{2,k} : \mathbb{R}^N \to \mathbb{R}$ by

$$\tilde{D}_{2,k}(y) \triangleq \frac{2\gamma_{\min}}{\sum_{i\in G_k}\gamma_i} \max_{\substack{(G_{k,1},G_{k,2})\\ \in\mathcal{P}(G_k)}} \left(\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}}\right),$$

for all y in $\mathbb{R}^N$. Notice that $\tilde{D}_{2,k}$ is a linear function.

For $(G_{k,1}, G_{k,2})$ in $\mathcal{P}(G_k)$ and $\delta > 0$, set

$$\begin{aligned}
B_{G_{k,1},G_{k,2}} &\triangleq \Big\{y \in \tilde{D}_{2,k}^{-1}(2d) : \\
&\qquad \tfrac{2\gamma_{\min}}{\sum_{i\in G_k}\gamma_i}\left(\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}}\right) = 2d\Big\},\\
\widetilde{B}_{G_{k,1},G_{k,2}} &\triangleq \{y \in \tilde{D}_{2,k}^{-1}(2d) : \\
&\qquad \exists\, y' \in B_{G_{k,1},G_{k,2}} \text{ with } \|y-y'\|_\infty \le \tfrac{d}{2}\},\\
W_{G_{k,1},G_{k,2}} &\triangleq \{\lambda y \in \mathbb{R}^N \colon y \in B_{G_{k,1},G_{k,2}}, \lambda \ge 1\},\\
\widetilde{W}_{G_{k,1},G_{k,2}} &\triangleq \{\lambda y \in \mathbb{R}^N \colon y \in \widetilde{B}_{G_{k,1},G_{k,2}}, \lambda \ge 1\}.
\end{aligned}$$

The set $B_{G_{k,1},G_{k,2}}$ corresponds to the face of the convex polytope defined by $\tilde{D}_{2,k}(y) \leq 2d$ where the maximum in the definition of $\tilde{D}_{2,k}$ is attained for the partition $(G_{k,1}, G_{k,2})$. The set $\widetilde{B}_{G_{k,1},G_{k,2}}$ may be considered as a $\frac{d}{2}$-neighborhood of $B_{G_{k,1},G_{k,2}}$ on the boundary of this polytope. The sets $W_{G_{k,1},G_{k,2}}$ constitute a partition (disregarding the fact that they may have parts of their boundary in common) of the set $\{y \in \mathbb{R}^N : \tilde{D}_{2,k}(y) \geq 2d\}$.

In the sections B and C we prove the following lemmas.

Lemma 9 *For any $k \in I_M$, any $(G_{k,1}, G_{k,2})$ in $\mathcal{P}(G_k)$, and any i in $G_{k,1}$ and j in $G_{k,2}$,*

$$y_i \leq y_j - \tilde{D}_{2,k}(y), \qquad \forall\, y \in W_{G_{k,1},G_{k,2}},$$

and

$$y_i \leq y_j - \frac{\tilde{D}_{2,k}(y)}{2}, \qquad \forall\, y \in \widetilde{W}_{G_{k,1},G_{k,2}}.$$

Lemma 10 *There exists a $\Gamma \in (0,1)$ such that*

$$\frac{\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}}}{\langle y\rangle_{G'_{k,2}} - \langle y\rangle_{G'_{k,1}}} \leq \Gamma,$$

for all y in $W_{G'_{k,1},G'_{k,2}} \setminus \widetilde{W}_{G_{k,1},G_{k,2}}$, for all $(G_{k,1}, G_{k,2})$, $(G'_{k,1}, G'_{k,2}) \in \mathcal{P}(G_k)$.

Consider a solution x of (3), with $x(t_0) \in R_{D_1,d}$, where D_1 satisfies (7), with $D_2 > 2d$ to be determined later. Assume that x leaves the region R_{D_2,D_2} by making

$$\langle x(t)\rangle_{G_{k,2}} - \langle x(t)\rangle_{G_{k,1}} - \frac{D_2 \sum_{i\in G_k} \gamma_i}{2\gamma_{\min}}$$

positive for some t in $\mathbb{R}$, $(G'_{k,1}, G'_{k,2})$ in $\mathcal{P}(G_k)$, for some k in I_M.

Then there exist $t_1, t_2 \in \mathbb{R}$ with $t_0 < t_1 < t_2$, $\tilde{D}_{2,k}(x(t_1)) = 2d$, $\tilde{D}_{2,k}(x(t_2)) = D_2$ and $x(t) \in R_{D_2,D_2}$ and $2d \leq \tilde{D}_{2,k}(x(t)) \leq D_2$, $\forall\, t \in [t_1, t_2]$.

Since $\tilde{D}_{2,k}(x(t)) \geq 2d$, $\forall\, t \in [t_1, t_2]$, we can find a sequence $(G^1_{k,1}, G^1_{k,2}), \ldots, (G^n_{k,1}, G^n_{k,2})$ and times $\tau_0 = t_1, \tau_1, \ldots, \tau_n = t_2$, with the property that

$$\begin{aligned}
&x(t) \in \widetilde{W}_{G^l_{k,1},G^l_{k,2}}, && \forall\, t \in (\tau_{l-1}, \tau_l), \forall\, l \in I_n,\\
&x(\tau_l) \in W_{G^l_{k,1},G^l_{k,2}} \cap \partial\widetilde{W}_{G^{l+1}_{k,1},G^{l+1}_{k,2}}, && \forall\, l \in I_{n-1},\\
&x(\tau_n) \in W_{G^l_{k,1},G^l_{k,2}}.
\end{aligned}$$

(This sequence can be constructed easily by starting at time t_2 and reversing to t_1.)

For each $l \in I_n$, it follows from Lemma 9 that $x_i(t) \leq x_j(t) - d$, $\forall\, (i,j) \in G^l_{k,1} \times G^l_{k,2}$, $\forall\, t \in (\tau_{l-1}, \tau_l)$, and since also $x(t) \in R_{D_2,D_2}$ with $D_2 \geq 2d$, we can derive that

$$\begin{aligned}
\langle \dot{x}(t)\rangle_{G^l_{k,2}} - \langle \dot{x}(t)\rangle_{G^l_{k,1}} &= \widetilde{v}(G^<_k \cup G^l_{k,1}, G^l_{k,2}, G^>_k, t)\\
&\quad - \widetilde{v}(G^<_k, G^l_{k,1}, G^>_k \cup G^l_{k,2}, t),
\end{aligned}$$

$\forall\, t \in (\tau_{l-1}, \tau_l)$, and therefore, from (4b), that

$$\begin{aligned}
&\langle x(\tau_l)\rangle_{G^l_{k,2}} - \langle x(\tau_l)\rangle_{G^l_{k,1}}\\
&\qquad \leq \langle x(\tau_{l-1})\rangle_{G^l_{k,2}} - \langle x(\tau_{l-1})\rangle_{G^l_{k,1}} + c.
\end{aligned}$$

From Lemma 10 (and the continuity of x) it follows that for $l \in I_n$

$$\frac{\langle x(\tau_{l-1})\rangle_{G^l_{k,2}} - \langle x(\tau_{l-1})\rangle_{G^l_{k,1}}}{\langle x(\tau_{l-1})\rangle_{G^{l-1}_{k,2}} - \langle x(\tau_{l-1})\rangle_{G^{l-1}_{k,1}}} \leq \Gamma,$$

for some $\Gamma \in (0,1)$.

Consequently, $\forall\, l \in I_n$,

$$\begin{aligned}
\tilde{D}_{2,k}(x(\tau_l)) &= \frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} \left(\langle x(\tau_l)\rangle_{G^l_{k,2}} - \langle x(\tau_l)\rangle_{G^l_{k,1}}\right)\\
&\leq \frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} \left(\langle x(\tau_{l-1})\rangle_{G^l_{k,2}} - \langle x(\tau_{l-1})\rangle_{G^l_{k,1}} + c\right)\\
&\leq \frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} \left(\Gamma\left(\langle x(\tau_{l-1})\rangle_{G^{l-1}_{k,2}} - \langle x(\tau_{l-1})\rangle_{G^{l-1}_{k,1}}\right) + c\right)\\
&= \Gamma \tilde{D}_{2,k}(x(\tau_{l-1})) + \frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} c,
\end{aligned}$$

and thus

$$\begin{aligned}
\tilde{D}_{2,k}(x(t_2)) &\leq \Gamma^n \tilde{D}_{2,k}(x(t_1))\\
&\qquad + \frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} c \left(1 + \Gamma + \cdots + \Gamma^{n-1}\right)\\
&\leq 2d + \frac{\frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} c}{1 - \Gamma}.
\end{aligned}$$

Choosing D_2 larger than the right hand side will result in a contradiction, and therefore we have shown that a solution x of (3), with $x(t_0) \in R_{D_1,d}$, for some $t_0 \in \mathbb{R}$, D_1 satisfying (7), and

$$D_2 > 2d + \frac{\frac{2\gamma_{\min}}{\sum_{i\in G_k} \gamma_i} c}{1 - \Gamma},$$

will satisfy the inequalities (8) and $x(t) \in R_{D_2,D_2}$, $\forall\, t \geq t_0$.

Consequently, x exhibits clustering behavior with respect to G.

B Proof of lemma 9

For any $y \in W_{G_{k,1},G_{k,2}}$,

$$\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}} = \frac{\tilde{D}_{2,k}(y)\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}}. \qquad \text{(B.1)}$$

Assume $|G_{k,1}| \neq 1 \neq |G_{k,2}|$ and pick an $i_1 \in G_{k,1}$ and a $i_2 \in G_{k,2}$. Then

$$\langle y\rangle_{G_{k,2}\cup\{i_1\}} - \langle y\rangle_{G_{k,1}\setminus\{i_1\}} \leq \frac{\tilde{D}_{2,k}(y)\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}},$$
$$\langle y\rangle_{G_{k,2}\setminus\{i_2\}} - \langle y\rangle_{G_{k,1}\cup\{i_2\}} \leq \frac{\tilde{D}_{2,k}(y)\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}}.$$

Multiplying with $\left(\sum_{i\in G_{k,1}\setminus\{i_1\}}\gamma_i\right)\left(\sum_{i\in G_{k,2}\cup\{i_1\}}\gamma_i\right)$, resp. $\left(\sum_{i\in G_{k,1}\cup\{i_2\}}\gamma_i\right)\left(\sum_{i\in G_{k,2}\setminus\{i_2\}}\gamma_i\right)$, and subtracting $\left(\sum_{i\in G_{k,1}}\gamma_i\right)\left(\sum_{i\in G_{k,2}}\gamma_i\right)$ times (B.1) results in

$$\begin{aligned}
&y_{i_1}\gamma_{i_1}\sum_{i\in G_k}\gamma_i - \gamma_{i_1}\sum_{i\in G_k} y_i\gamma_i \leq \\
&\quad \gamma_{i_1}\left(\sum_{i\in G_{k,1}}\gamma_i - \sum_{i\in G_{k,2}}\gamma_i - \gamma_{i_1}\right)\frac{\tilde{D}_{2,k}(y)\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}},\\
&- y_{i_2}\gamma_{i_2}\sum_{i\in G_k}\gamma_i + \gamma_{i_2}\sum_{i\in G_k} y_i\gamma_i \leq \\
&\quad \gamma_{i_2}\left(\sum_{i\in G_{k,2}}\gamma_i - \sum_{i\in G_{k,1}}\gamma_i - \gamma_{i_2}\right)\frac{\tilde{D}_{2,k}(y)\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}}.
\end{aligned}$$

Adding both expressions, after dividing by $\gamma_{i_1}\sum_{i\in G_k}\gamma_i$, resp. $\gamma_{i_2}\sum_{i\in G_k}\gamma_i$, we obtain

$$y_{i_1} - y_{i_2} \leq -\frac{\tilde{D}_{2,k}(y)(\gamma_{i_1}+\gamma_{i_2})}{2\gamma_{\min}} \leq -\tilde{D}_{2,k}(y).$$

For the cases $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$, this reasoning is easily adapted.

Since for any $y \in \widetilde{B}_{G_{k,1},G_{k,2}}$, there exists a y' in $B_{G_{k,1},G_{k,2}}$ such that $y_{i_1} - y_{i_2} \leq y'_{i_1} - y'_{i_2} + d$, it follows by Lemma 9 (since $B_{G_{k,1},G_{k,2}} \subset W_{G_{k,1},G_{k,2}}$) that

$$y_{i_1} \leq y_{i_2} - d, \qquad \forall\, i_1 \in G_{k,1}, \forall\, i_2 \in G_{k,2},$$

which we may write as

$$y_{i_1} \leq y_{i_2} - \frac{\tilde{D}_{2,k}(y)}{2}, \qquad \forall\, i_1 \in G_{k,1}, \forall\, i_2 \in G_{k,2}.$$

Because of the linearity of the function $\tilde{D}_{2,k}$, this holds for any $y \in \widetilde{W}_{G_{k,1},G_{k,2}}$:

$$y_{i_1} \leq y_{i_2} - \frac{\tilde{D}_{2,k}(y)}{2}, \qquad \forall\, i_1 \in G_{k,1}, \forall\, i_2 \in G_{k,2}.$$

C Proof of lemma 10

Assume that there does not exist such a Γ. Then there exist $(G_{k,1}, G_{k,2}), (G'_{k,1}, G'_{k,2}) \in \mathcal{P}(G_k)$, and $(y^n)_{n\in\mathbb{N}}$ with y^n in $W_{G'_{k,1},G'_{k,2}} \setminus \widetilde{W}_{G_{k,1},G_{k,2}}$ for all n in $\mathbb{N}$, satisfying

$$\lim_{n\to\infty}\frac{\langle y^n\rangle_{G_{k,2}} - \langle y^n\rangle_{G_{k,1}}}{\langle y^n\rangle_{G'_{k,2}} - \langle y^n\rangle_{G'_{k,1}}} \geq 1.$$

Notice that we may assume that $\tilde{D}_{2,k}(y^n) = 2d$ (since we can divide both the numerator and the denominator by $\tilde{D}_{2,k}(y^n)/(2d)$), we may assume that $\langle y^n\rangle_{G_k} = 0$ (since we can subtract $\langle y^n\rangle_{G_k}$ from y_i^n with $i \in G_k$), and we can set $y_i^n = 0$ for all $i \notin G_k$ without loss of generality. Since this implies that the y^n are bounded, it follows that there exists a limit value y in the closure of $B_{G'_{k,1},G'_{k,2}} \setminus \widetilde{B}_{G_{k,1},G_{k,2}}$ with

$$\frac{\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}}}{\langle y\rangle_{G'_{k,2}} - \langle y\rangle_{G'_{k,1}}} \geq 1.$$

Since $y \in B_{G'_{k,1},G'_{k,2}}$ this inequality can also be written as

$$\frac{2\gamma_{\min}}{\sum_{i\in G_k}\gamma_i}\left(\langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}}\right) \geq \tilde{D}_{2,k}(y),$$

implying that this inequality is an equality (given the definition of $\tilde{D}_{2,k}$) and that $y \in B_{G_{k,1},G_{k,2}}$, contradicting the fact that y is in the closure of $B_{G'_{k,1},G'_{k,2}} \setminus \widetilde{B}_{G_{k,1},G_{k,2}}$.