Asymptotic ideals (ideals in the ring of Colombeau generalized constants with continuous parametrization)

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Abstract

We study the asymptotics at zero of continuous functions on $(0,1]$ by means of their asymptotic ideals, i.e., ideals in the ring of continuous functions on $(0,1]$ satisfying a polynomial growth condition at 0 modulo rapidly decreasing functions at 0. As our main result, we characterize maximal and prime ideals in terms of maximal and prime filters.

1 Introduction

In this paper, we study the asymptotic ideals of continuous functions $(0,1] \to \mathbb{K}$ (where $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$), i.e., ideals in the ring of continuous functions ϕ satisfying the following growth condition (usually called moderateness)

$$(\exists N \in \mathbb{N})(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)\, |\phi(\varepsilon)| \leq \varepsilon^{-N}$$

modulo the ideal of continuous functions ϕ satisfying

$$(\forall n \in \mathbb{N})(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)\, |\phi(\varepsilon)| \leq \varepsilon^{n}$$

(usually called negligibility). Apart from the obvious interest of such a study to asymptotic analysis, such equivalence classes of functions also naturally arise in generalized function theory as the ring of generalized constants $\widetilde{\mathbb{K}}_{\text{cnt}}$ of the algebra of Colombeau generalized functions (see §2).
The ring $\widetilde{\mathbb{K}}_{\text{cnt}}$ of generalized constants with continuous dependence on the parameter has been introduced and studied in [5], where it is also shown that this ring is isomorphic to the ring of generalized constants with smooth dependence. In fact, the study of the ring $\widetilde{\mathbb{K}}_{\text{cnt}}$ amounts to the study of the asymptotics at zero of moderate continuous functions on $(0,1]$.
In generalized function theory, the choice of continuous dependence comes from the observation that when one embeds distributions in an algebra of Colombeau generalized

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functions and when one solves nonlinear problems, one always encounters generalized functions represented by continuous (even smooth) nets of smooth functions.
The algebraic properties of the ring $\widetilde{\mathbb{K}}_{\rm cnt}$ are different from those of the ring $\widetilde{\mathbb{K}}$ of generalized constants without continuous dependence on the parameter, and many tools used in the study of $\widetilde{\mathbb{K}}$ cannot be used. Most strikingly, this is manifested by the fact that $\widetilde{\mathbb{K}}_{\rm cnt}$ does not have any nontrivial idempotent elements, in sharp contrast with the ring $\widetilde{\mathbb{K}}$ (which is a so-called exchange ring [13]). Thus the main tools used in [1] and [13] to study $\widetilde{\mathbb{K}}$ cannot be used.
In this paper, we study prime and maximal ideals by attaching a filter of closed subsets of $(0,1]$ to each ideal. The filter is analogous to the filter $\{S \subseteq (0,1] : e_{S^c} \in I\}$ attached to an ideal $I \lhd \widetilde{\mathbb{K}}$ ([13, §6]), and thus allows us to overcome the difficulty of the lack of idempotents. In this way, we obtain a classification of maximal and minimal prime ideals in terms of maximal and prime filters.
The methods used in this paper are inspired by the study of the ideals in $\widetilde{\mathbb{K}}$ [1, 13] and by the study of maximal ideals of rings of continuous functions by Gillman and Jerison [7]. Compared to [7], the main novelty is the adaptation to the asymptotic nature of the ring $\widetilde{\mathbb{K}}_{\rm cnt}$.

2 Preliminaries

The ring $\widetilde{\mathbb{K}}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (the field of real, resp. complex numbers), is defined as $\mathcal{M}_{\mathbb{K}}/\mathcal{N}_{\mathbb{K}}$, where

$$\mathcal{M}_{\mathbb{K}} = \{(x_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} : (\exists N \in \mathbb{N})(\exists \varepsilon_0 > 0)(\forall \varepsilon \le \varepsilon_0)\, |x_\varepsilon| \le \varepsilon^{-N}\}$$
$$\mathcal{N}_{\mathbb{K}} = \{(x_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} : (\forall n \in \mathbb{N})(\exists \varepsilon_0 > 0)(\forall \varepsilon \le \varepsilon_0)\, |x_\varepsilon| \le \varepsilon^{n}\}.$$

We denote by $[x_\varepsilon] \in \widetilde{\mathbb{K}}$ the element with representative $(x_\varepsilon)_\varepsilon$ and we denote $\rho := [\varepsilon]$.
$\widetilde{\mathbb{K}}$ is a complete topological ring with the so-called sharp topology, which can be defined as follows. Let $x = [x_\varepsilon] \in \widetilde{\mathbb{K}}$. Let

$$v(x) := \sup\{a \in \mathbb{R} : (\exists \varepsilon_0 > 0)(\forall \varepsilon \le \varepsilon_0)\, |x_\varepsilon| \le \varepsilon^{a}\}.$$

Then the ultrametric $d(x, y) := e^{-v(x-y)}$ induces a topology on $\widetilde{\mathbb{K}}$ which is called the sharp topology [12].
Denoting by $\mathcal{C}((0,1])$ (resp. $\mathcal{C}^\infty((0,1])$ the set of continuous (resp. smooth) maps in $\mathbb{K}^{(0,1]}$, the ring $\widetilde{\mathbb{K}}_{\rm cnt} := (\mathcal{M}_{\mathbb{K}} \cap \mathcal{C}((0,1]))/(\mathcal{N}_K \cap \mathcal{C}((0,1]))$ and $\widetilde{\mathbb{K}}_{\rm sm} := (\mathcal{M}_{\mathbb{K}} \cap \mathcal{C}^\infty((0,1]))/(\mathcal{N}_K \cap \mathcal{C}^\infty((0,1]))$. Clearly, $\widetilde{\mathbb{K}}_{\rm sm} \subseteq \widetilde{\mathbb{K}}_{\rm cnt} \subseteq \widetilde{\mathbb{K}}$. In [5], it is shown that $\widetilde{\mathbb{K}}_{\rm cnt} = \widetilde{\mathbb{K}}_{\rm sm}$.
We denote $I \lhd \widetilde{\mathbb{K}}_{\rm cnt}$ for a proper ideal I of $\widetilde{\mathbb{K}}_{\rm cnt}$ (i.e., $I \ne \widetilde{\mathbb{K}}_{\rm cnt}$).
$\widetilde{\mathbb{K}}$ is an exchange ring [13], i.e., for each $a \in \widetilde{\mathbb{K}}$, there exists an idempotent $e \in \widetilde{\mathbb{K}}$ such that $a + e$ is invertible. Unlike $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{K}}_{\rm cnt}$ is not an exchange ring [5, Lemma 4.3].
Like $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{K}}_{\rm cnt}$ is a Gelfand ring [5, Lemma 4.5], i.e., every prime ideal is contained in a unique maximal ideal.
Like $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{K}}_{\rm cnt}$ is a Bezout ring [5, Prop. 4.26], i.e., every finitely generated ideal is principal.
Like $\widetilde{\mathbb{R}}$, $\widetilde{\mathbb{R}}_{\rm cnt}$ is an l-ring (or lattice-ordered ring) [5, Prop. 4.13].

Let $I \trianglelefteq \widetilde{\mathbb{K}}_{\rm cnt}$ and $x \in I$. Then $|x| \in I$ [5, Lemma 4.24].
Let $I \trianglelefteq \widetilde{\mathbb{R}}_{\rm cnt}$. Then I is an l-ideal (or absolutely (order) convex), i.e., if $x \in I$, $x' \in \widetilde{\mathbb{R}}_{\rm cnt}$ and $|x'| \le |x|$, then $x' \in I$. [5, Prop. 4.25].
Let us point out explicitly the corollary that then also for $I \trianglelefteq \widetilde{\mathbb{C}}_{\rm cnt}$, $z \in I$, $z' \in \widetilde{\mathbb{C}}_{\rm cnt}$, $|z'| \le |z|$ implies that $z' \in I$. Indeed, $z \in I$ implies $|z| \in I \cap \widetilde{\mathbb{R}}_{\rm cnt}$ [5, Lemma 4.24]. As $I \cap \widetilde{\mathbb{R}}_{\rm cnt} \trianglelefteq \widetilde{\mathbb{R}}_{\rm cnt}$, $I \cap \widetilde{\mathbb{R}}_{\rm cnt}$ is an l-ideal in $\widetilde{\mathbb{R}}_{\rm cnt}$. Hence $|\Re z'| \le |z|$ implies that $\Re z' \in I \cap \widetilde{\mathbb{R}}_{\rm cnt}$. Similarly, $\Im z' \in I \cap \widetilde{\mathbb{R}}_{\rm cnt}$. Thus $z' = \Re z' + i \Im z' \in I$.
Hence the bijective correspondence of ideals in $\widetilde{\mathbb{K}}_{\rm cnt}$ takes the same form as for ideals in $\widetilde{\mathbb{K}}$ ([13]): the map $I \trianglelefteq \widetilde{\mathbb{C}}_{\rm cnt} \mapsto I \cap \widetilde{\mathbb{R}}_{\rm cnt} = \{\Re z : z \in I\} \trianglelefteq \widetilde{\mathbb{R}}_{\rm cnt}$ has as an inverse the map $J \trianglelefteq \widetilde{\mathbb{R}}_{\rm cnt} \mapsto \langle J \rangle = \{z \in \widetilde{\mathbb{C}}_{\rm cnt} : |z| \in J\} \trianglelefteq \widetilde{\mathbb{C}}_{\rm cnt}$ (where $\langle J \rangle$ is also the ideal generated by J in $\widetilde{\mathbb{C}}_{\rm cnt}$). It is an inclusion-preserving bijection between the lattice of ideals of $\widetilde{\mathbb{C}}_{\rm cnt}$ and the lattice of ideals of $\widetilde{\mathbb{R}}_{\rm cnt}$. In particular, arbitrary sums and intersections are preserved. One easily checks that the isomorphism also preserves products of ideals, principal, pseudoprime and irreducible ideals.
Let R be a commutative ring with 1. An ideal $I \trianglelefteq R$ is pure if [4, Prop. 7.2]

$$(\forall x \in I)(\exists y \in I)(x = xy).$$

We denote by $m(I)$ the pure part of $I \trianglelefteq R$, i.e., the largest pure ideal contained in I [4, Prop. 7.8]. By definition, I is pure iff $I = m(I)$. If R is a Gelfand ring, then [4, §8.2–3]

$$m(I) = \{x \in R : (\exists y \in I)(x = xy)\}.$$

An ideal $I \trianglelefteq R$ is idempotent if $I^2 = I$.
We denote the radical of $I \trianglelefteq R$ by $\sqrt{I} = \{x \in R : (\exists n \in \mathbb{N}) x^n \in I\} = \bigcap_{\substack{I \subseteq P \\ P \text{ prime}}} P$ (e.g., see [7, 0.18]).
$I \trianglelefteq R$ is radical (or semiprime) if $I = \sqrt{I}$, or equivalently, if $(\forall x \in R)(x^2 \in I \Rightarrow x \in I)$.
$I \trianglelefteq R$ is pseudoprime if for each $a, b \in R$, $ab = 0$ implies $a \in I$ or $b \in I$.
$I \trianglelefteq R$ is irreducible (or meet-irreducible) if for each $J, K \trianglelefteq R$, $I = J \cap K$ implies $I = J$ or $I = K$ [10, §6].

3 Characteristic sets

Definition 3.1. *A set $S \subseteq (0,1]$ such that $0 \in \overline{S}$ (closure in $\mathbb{R}$) is called a* **characteristic set** *[5]. We denote the set of all characteristic sets by $\mathcal{S}$.*
Let $S, T \in \mathcal{S}$. We say that T is an **extension** *of S if $\overline{S} \subseteq T^\circ$ (closure and interior in $(0,1]$) and denote this by $S \prec T$ (or equivalently, $T \succ S$). It is straightforward to check that $\prec$ is antireflexive and transitive on $\mathcal{S} \setminus \{(0,1]\}$, and hence defines a partial order on $\mathcal{S} \setminus \{(0,1]\}$. Notice that $(0,1] \prec (0,1]$, which will turn out to be convenient.*

Lemma 3.2. *Let $S, T \in \mathcal{S}$.*

1. *If $S \prec T$, there exists $U \in \mathcal{S}$ such that $S \prec U \prec T$.*
   In particular, $\prec$ is a dense order on $\mathcal{S} \setminus \{(0,1]\}$.

2. *$S \prec T$ iff $T^c \prec S^c$.*

Proof. 1. Let $S \prec T$. By Urysohn's lemma, there exists $\phi \in \mathcal{C}((0,1])$ such that $0 \le \phi \le 1$, $\phi_{|S} = 0$ and $\phi_{|T^c} = 1$. Let $U := \{\varepsilon \in (0,1] : \phi(\varepsilon) \le 1/2\}$. Then $S \prec U \prec T$.
2. $\overline{S} \subseteq T^\circ \iff \overline{(T^c)} = (T^\circ)^c \subseteq (\overline{S})^c = (S^c)^\circ$. ☐

Definition 3.3. *(cf. [5, 4.16]) Let $x \in \widetilde{\mathbb{K}}_{\rm cnt}$ and $S \in \mathcal{S}$. Then $x_{|S} = 0$ if*

$$(\forall n \in \mathbb{N})(\exists \delta > 0)(\forall \varepsilon \in S \cap (0,\delta))(|x_\varepsilon| \le \varepsilon^n).$$

where $(x_\varepsilon)_\varepsilon$ is any representative of x. We similarly write $x_{|S} = y_{|S}$ for $(x-y)_{|S} = 0$, $x_{|S} = 1$ for $(x-1)_{|S} = 0$, ...
We say that $x_{|S}$ is invertible if there exists $y \in \widetilde{\mathbb{K}}_{\rm cnt}$ such that $(xy)_{|S} = 1$.

Lemma 3.4. *Let $S \in \mathcal{S}$.*

1. *Let $x \in \widetilde{\mathbb{K}}_{\rm cnt}$. Then the following are equivalent:*
   (a) *$x_{|S}$ is invertible (in $\widetilde{\mathbb{K}}_{\rm cnt}$)*
   (b) *$x_{|S}$ is invertible in $\widetilde{\mathbb{K}}$*
   (c) *$x_{|S}$ is bounded away from zero, i.e., for some representative $(x_\varepsilon)_\varepsilon$ of x,*
   $$(\exists n \in \mathbb{N})(\exists \delta > 0)(\forall \varepsilon \in S \cap (0,\delta))(|x_\varepsilon| \ge \varepsilon^n).$$
   (the statement then automatically holds for any representative $(x_\varepsilon)_\varepsilon$ of x).
   (d) *for each characteristic set $T \subseteq S$, $x_{|T} \ne 0$.*
2. *$\{x \in \widetilde{\mathbb{K}}_{\rm cnt} : x_{|S}$ is invertible$\}$ is open.*
3. *$x_{|S} = 0$ iff for each characteristic set $T \subseteq S$, $x_{|T}$ is not invertible.*

Proof. 1. $(b) \Leftrightarrow (c) \Leftrightarrow (d)$: by [13, Lemma 4.1].
$(a) \Rightarrow (b)$: trivial.
$(c) \Rightarrow (a)$: let $T := \{\varepsilon \in (0,1] : |x_\varepsilon| > \varepsilon^n/2\}$. As $(x_\varepsilon)_\varepsilon$ is continuous, $S \cap (0,\delta) \prec T$. By Urysohn's lemma, there exists $\phi \in \mathcal{C}((0,1])$ such that $0 \le \phi \le 1$, $\phi_{|S\cap(0,\delta)} = 1$ and $\phi_{|T^c} = 0$. Let $y_\varepsilon := \phi(\varepsilon)/x_\varepsilon$, if $\varepsilon \in T$ and $y_\varepsilon := 0$, if $\varepsilon \in T^c$. Then $|y_\varepsilon| \le 2\varepsilon^{-n}$, $(y_\varepsilon)_\varepsilon \in$ is continuous and $x_\varepsilon y_\varepsilon = 1$ for each $\varepsilon \in S \cap (0,\delta)$. Hence $(y_\varepsilon)_\varepsilon$ is a representative of some $y \in \widetilde{\mathbb{K}}_{\rm cnt}$ with $(xy)_{|S} = 1$.
2. Let $x_{|S}$ be invertible. Let $n \in \mathbb{N}$ as in part 1(c). Then $y_{|S}$ is invertible for each $y \in \widetilde{\mathbb{K}}_{\rm cnt}$ with $|x-y| \le \rho^n/2$ (again by part 1(c)).
3. By [13, Lemma 4.1], since $\widetilde{\mathbb{K}}_{\rm cnt} \subseteq \widetilde{\mathbb{K}}$. ☐

Proposition 3.5. *Let $x \in \widetilde{\mathbb{K}}_{\rm cnt}$ and $S \in \mathcal{S}$.*

1. *If $x_{|S} = 0$, then $x_{|T} = 0$ for some $T \succ S$.*
2. *If $x_{|S}$ is invertible, then $x_{|T}$ is invertible for some $T \succ S$.*

Proof. 1. Let $(x_\varepsilon)_{\varepsilon\in(0,1]}$ be a (continuous) representative of x. Then for each $n \in \mathbb{N}$, there exist $\delta_n > 0$ (w.l.o.g. strictly decreasing and tending to 0) such that $|x_\varepsilon| \le \varepsilon^n$ for each $\varepsilon \in S$, $\varepsilon \le \delta_n$. Then let $T := \bigcup_{n\in\mathbb{N}}(\delta_{n+2}, \delta_n) \cap \{\varepsilon \in (0, 1] : |x_\varepsilon| \le 2\varepsilon^n\}$. Then also $x_{|T} = 0$. We show that $S \prec T$. Let $\varepsilon \in \overline{S}$. Then $\varepsilon \in (\delta_{n+2}, \delta_n)$ for some n. By continuity, also $|x_\varepsilon| \le \varepsilon^n$ for each $\varepsilon \in \overline{S}$, $\varepsilon < \delta_n$. Hence ε belongs to the open set $(\delta_{n+2}, \delta_n) \cap \{\varepsilon \in (0, 1] : |x_\varepsilon| < 2\varepsilon^n\} \subseteq T$. Thus $\varepsilon \in T^\circ$.
2. Let $n \in \mathbb{N}$ and $\delta > 0$ as in lemma 3.4.1(c). Let $T := \{\varepsilon \in (0, 1] : |x_\varepsilon| > \varepsilon^n/2\} \cup (\delta/2, 1)$. As $(x_\varepsilon)_\varepsilon$ is continuous, $S \prec T$. By lemma 3.4.1(c), $x_{|T}$ is invertible. □

Lemma 3.6. *Let $a, b \in \widetilde{\mathbb{K}}_{\rm cnt}$ and $S \in \mathcal{S}$. If $(ab)_{|S} = 0$, then there exist closed T, U with $S \subseteq T^\circ \cup U^\circ$ such that $a_{|T} = 0$ and $b_{|U} = 0$.*

Proof. As $a, b \in \widetilde{\mathbb{K}}$, there exists $V \subseteq S$ such that $a_{|V} = 0$ and $b_{|S\setminus V} = 0$ [13]. As $a, b \in \widetilde{\mathbb{K}}_{\rm cnt}$, there exist (w.l.o.g. closed) T, U with $V \prec T$, $S \setminus V \prec U$ such that $a_{|T} = 0$ and $b_{|U} = 0$ by Prop. 3.5. □

4 Asymptotic filters

In [7], to any ideal $I \lhd \mathcal{C}(X)$ (with X a topological space), a filter is associated consisting of the zero-sets of all $f \in I$ and conversely, to a filter $\mathcal{F}$ of zero-sets, an ideal I is associated. Taking into account that there is no largest zero-set for $x \in \widetilde{\mathbb{K}}_{\rm cnt}$, we proceed as follows:

Definition 4.1. *A filter of closed subsets of $(0, 1]$ is a family $\mathcal{F}$ of (relatively) closed subsets of $(0, 1]$ such that*

1. *$\emptyset \notin \mathcal{F}$*
2. *if $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$*
3. *if $S \in \mathcal{F}$, $T \subseteq (0, 1]$ is closed and $S \subseteq T$, then $T \in \mathcal{F}$.*

A closed characteristic subset of $(0, 1]$ is called an **asymptotic subset**. *We denote the set of all asymptotic subsets by $\mathcal{A}$.*
An **asymptotic filter** *or* **a-filter** *is a filter of closed subsets of $(0, 1]$ that contains $(0, \delta]$ for each $\delta > 0$. Notice that this implies that $\mathcal{F} \subseteq \mathcal{A}$.*
We define as follows a topology on $\mathcal{A}$. Denoting open intervals corresponding to $\prec$ by

$$(S, T)_\prec := \{U \in \mathcal{A} : S \prec U \prec T\},$$

the **extension topology** *is the topology on $\mathcal{A}$ with base $\{(S, T)_\prec : S, T \subseteq (0, 1]\}$. We will call $\prec$-open, $\prec$-closed, ... sets that are open, closed, ... for this topology. Notice that $\{(0, 1]\}$ is $\prec$-open, which will turn out to be convenient.*

Remark 4.2. A filter is called free (or non-principal) if $\bigcap_{S\in\mathcal{F}} S = \emptyset$. We can alternatively define an a-filter as a free filter of closed subsets of $(0, 1]$. For, if $\mathcal{F}$ is a filter of closed subsets of $(0, 1]$ and $(0, \delta] \notin \mathcal{F}$ for some $\delta > 0$, then $S \cap [\delta, 1] \ne \emptyset$ for each $S \in \mathcal{F}$. By compactness of $[\delta, 1]$, it would then follow that $\bigcap_{S\in\mathcal{F}} S \cap [\delta, 1] \ne \emptyset$.

Definition 4.3. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Then*

$$\mathcal{F}(I) := \{S \subseteq (0,1] \text{ closed}: (\exists x \in I)(x_{|S^c} \text{ is invertible})\}$$

(here it is understood that $x_{|S}$ is trivially invertible if $0 \notin \overline{S}$).
Let $\mathcal{F}$ be an a-filter on $(0,1]$. Then

$$I(\mathcal{F}) := \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (\exists S \in \mathcal{F})(x_{|S} = 0)\}.$$

Lemma 4.4. *For $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$,*

$$\mathcal{F}(I) = \{S \subseteq (0,1] \text{ closed}: (\exists x \in I)(x_{|S^c} = 1)\}.$$

Proof. If $x \in I$ and $x_{|S^c}$ is invertible, then there exists $y \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ such that $(xy)_{|S^c} = 1$, and $xy \in I$. ☐

Proposition 4.5. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ and $\mathcal{F}$ an a-filter on $(0,1]$.*

1. *$\mathcal{F}(I)$ is an a-filter on $(0,1]$.*
2. *$I(\mathcal{F}) \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.*
3. *$\mathcal{F}(I(\mathcal{F})) \subseteq \mathcal{F}$.*
4. *$I(\mathcal{F}(I)) \subseteq I$.*

Proof. 1. Since a proper ideal does not contain invertible elements, $\emptyset \notin \mathcal{F}(I)$.
If $S, T \in \mathcal{F}(I)$, then there exist $x, y \in I$ such that $x_{|S^c}$ and $y_{|T^c}$ are invertible. Hence also $|x|^2 + |y|^2 \in I$ and $(|x|^2 + |y|^2)_{|S^c \cup T^c}$ is invertible, so also $S \cap T \in \mathcal{F}(I)$.
If $S \in \mathcal{F}(I)$, $T \subseteq (0,1]$ is closed and $S \subseteq T$, then clearly $T \in \mathcal{F}(I)$.
If $\delta > 0$, then $0 \notin \overline{(0,\delta]^c}$, hence $x_{|(0,\delta]^c}$ is (trivially) invertible for each $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.
2. If $x, y \in I(\mathcal{F})$, then $x_{|S} = 0$ and $y_{|T} = 0$ for some $S, T \in \mathcal{F}$. Then also $x + y_{|S \cap T} = 0$ and $S \cap T \in \mathcal{F}$, so $x + y \in I(\mathcal{F})$. For $z \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, also $xz_{|S} = 0$, so $xz \in I(\mathcal{F})$. $1 \notin I(\mathcal{F})$, since $1_{|S} \neq 0$ for each $S \in \mathcal{S}$.
3. Let $S \in \mathcal{F}(I(\mathcal{F}))$. Then there exists $x \in I(\mathcal{F})$ such that $x_{|S^c} = 1$. So there exists $T \in \mathcal{F}$ such that $x_{|T} = 0$. Then $T \cap (0,\delta] \subseteq S$ for some $\delta > 0$. For otherwise, one constructs $V \subseteq T \cap S^c$ with $0 \in \overline{V}$ such that $x_{|V} = 0$, contradicting $x_{|V} = 1$. Thus $S \in \mathcal{F}$.
4. Let $x \in I(\mathcal{F}(I))$. Then there exists $S \in \mathcal{F}(I)$ such that $x_{|S} = 0$. So there exists $y \in I$ such that $y_{|S^c} = 1$. As $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, $|x| \le \rho^{-N}$ for some $N \in \mathbb{N}$. Then $|x| \le \rho^{-N}|y|$. As $\rho^{-N} y \in I$ and ideals in $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ are absolutely order convex [5, Prop. 4.25], $x \in I$. ☐

Proposition 4.6. *Let $\mathcal{F}$ be an a-filter on $(0,1]$. Then*

1. *$\mathcal{F}^\circ = \{S \in \mathcal{A} : (\exists T \prec S)(T \in \mathcal{F})\}$ ($\mathcal{F}^\circ$ denotes the $\prec$-interior).*
2. *$\mathcal{F}^\circ$ is an a-filter.*

Proof. 1. $\subseteq$: let $\mathcal{X} \subseteq \mathcal{F}$ be $\prec$-open. If $S \in \mathcal{X}$, then $S \in (T,U)_\prec \subseteq \mathcal{X}$, for some $T, U \subseteq (0,1]$. W.l.og., T is closed. We first show that there exists $V \prec S$ with $V \in \mathcal{A}$. Otherwise, $T \notin \mathcal{S}$, i.e., $T \cap (0,\delta] = \emptyset$ for some $\delta > 0$. As $S \in \mathcal{S}$, we can construct $W_1, W_2 \subseteq S$ with $W_1, W_2 \in \mathcal{A}$ and $W_1 \cap W_2 = \emptyset$. Then $W_j \cup [\delta/2, 1] \in (T,U)_\prec \subseteq \mathcal{F}$. Hence also $\emptyset = W_1 \cap W_2 \cap (0, \delta/3] \in \mathcal{F}$, a contradiction.
Since $\prec$ is a dense order, $T \prec W \prec S$ for some closed W. Hence also $T \prec V \cup W \prec S$, and $V \cup W \in \mathcal{A}$. Thus $V \cup W \in (T,U)_\prec \subseteq \mathcal{X} \subseteq \mathcal{F}$. Hence $\mathcal{X} \subseteq \{S \in \mathcal{A} : (\exists T \prec S)(T \in \mathcal{F})\}$.
$\supseteq$: $\{S \in \mathcal{A} : (\exists T \prec S)(T \in \mathcal{F})\} \subseteq \mathcal{F}$ and is $\prec$-open: if $T \prec S$ with $T \in \mathcal{F}$, then also $S \in (T, (0,1])_\prec \subseteq \{S \in \mathcal{A} : (\exists T \prec S)(T \in \mathcal{F})\}$.
2. As $\mathcal{F}^\circ \subseteq \mathcal{F}$, $\emptyset \notin \mathcal{F}^\circ$.
If $U \prec S$, $V \prec T$ with $U, V \in \mathcal{F}$, then also $U \cap V \prec S \cap T$ with $U \cap V \in \mathcal{F}$.
The other defining properties of an a-filter are immediately checked using part 1. □

Theorem 4.7.

1. *For each a-filter $\mathcal{F}$ on $(0,1]$, $\mathcal{F}(I(\mathcal{F})) = \mathcal{F}^\circ$.*
2. *$\{\mathcal{F}(I) : I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}\}$ is the set of $\prec$-open a-filters on $(0,1]$.*

Proof. First, let $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$. We show that $\mathcal{F}(I)$ is $\prec$-open:
Let $S \in \mathcal{F}(I)$. Then there exists $x \in I$ such that $x_{|S^c}$ is invertible. By proposition 3.5, there exists $T \succ S^c$ such that $x_{|T}$ is invertible. W.l.og. T is open. Then $T^c \in \mathcal{F}(I)$ and $T^c \prec S$.
In particular, $\mathcal{F}(I(\mathcal{F})) \subseteq \mathcal{F}$ is $\prec$-open, and hence $\mathcal{F}(I(\mathcal{F})) \subseteq \mathcal{F}^\circ$.
Conversely, we show that $\mathcal{F}^\circ \subseteq \mathcal{F}(I(\mathcal{F}))$:
Let $S \in \mathcal{F}^\circ$. Then there exists $T \prec S$ such that $T \in \mathcal{F}$. By Urysohn's lemma, there exists $x \in \widetilde{\mathbb{K}}_{\text{cnt}}$ such that $x_{|T} = 0$ and $x_{|S^c} = 1$. Hence $x \in I(\mathcal{F})$ and $S \in \mathcal{F}(I(\mathcal{F}))$.
Finally, if an a-filter $\mathcal{F}$ is $\prec$-open, then $\mathcal{F} = \mathcal{F}^\circ = \mathcal{F}(I(\mathcal{F}))$, hence $\mathcal{F} = \mathcal{F}(I)$ for some $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$. □

Theorem 4.8.

1. *For each $I \unlhd \widetilde{\mathbb{K}}_{\text{cnt}}$, $I(\mathcal{F}(I)) = m(I)$.*
2. *$\{I(\mathcal{F}) : \mathcal{F}$ is an a-filter on $(0,1]\}$ is the set of (proper) pure ideals in $\widetilde{\mathbb{K}}_{\text{cnt}}$.*

Proof. First, let $\mathcal{F}$ be an a-filter on $(0,1]$. We show that $I(\mathcal{F})$ is pure:
Let $x \in I(\mathcal{F})$. Then there exists $S \in \mathcal{F}$ such that $x_{|S} = 0$. By proposition 3.5, $x_{|T} = 0$ for some $T \succ S$. By Urysohn's lemma, there exists $y \in \widetilde{\mathbb{K}}_{\text{cnt}}$ such that $y_{|S} = 0$, $y_{|T^c} = 1$. Then $(xy)_{|T} = 0$ and $(xy)_{|T^c} = x_{|T^c}$. Hence $x = xy$ and $y \in I(\mathcal{F})$.
In particular, $I(\mathcal{F}(I)) \subseteq I$ is pure for each $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$, and hence $I(\mathcal{F}(I)) \subseteq m(I)$.
Conversely, we show that $m(I) \subseteq I(\mathcal{F}(I))$ for each $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$:
Let $x \in m(I)$, i.e., there exists $y \in I$ such that $x = xy$. As $x(1-y) = 0$, there exist (by lemma 3.6) closed $S, T \subseteq (0,1]$ with $S \cup T = (0,1]$ such that $x_{|S} = 0$ and $(1-y)_{|T} = 0$. Hence $y_{|S^c} = 1$, so $S \in \mathcal{F}(I)$, and $x \in I(\mathcal{F}(I))$.
Finally, if $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$ is pure, then $I = m(I) = I(\mathcal{F}(I))$, hence $I = I(\mathcal{F})$ for some a-filter $\mathcal{F}$ on $(0,1]$. □

5 Closed ideals and filters

We will denote $\overline{I}(\mathcal{F}) := \overline{I(\mathcal{F})}$ (closure in the sharp topology) and $\overline{\mathcal{F}}(I) := \overline{\mathcal{F}(I)}$ ($\prec$-closure).

Proposition 5.1. *Let $\mathcal{F}$ be an a-filter on $(0,1]$. Then*

1. $\overline{\mathcal{F}} = \{S \in \mathcal{A} : (\forall T \succ S,\, T \text{ closed})(T \in \mathcal{F})\}$.
2. $\overline{\mathcal{F}}$ *is an a-filter.*

Proof. 1. Call $\mathcal{F}^* := \{S \in \mathcal{A} : (\forall T \succ S,\, T \text{ closed})(T \in \mathcal{F})\}$.
$\subseteq$: $\mathcal{F} \subseteq \mathcal{F}^*$ and $\mathcal{F}^*$ is $\prec$-closed: if $S \in \mathcal{A} \setminus \mathcal{F}^*$, then there exists a closed $T \succ S$ with $T \notin \mathcal{F}$, hence also $(\emptyset, T)_\prec \subseteq \mathcal{A} \setminus \mathcal{F}^*$.
$\supseteq$: let $\mathcal{X} \supseteq \mathcal{F}$ be $\prec$-closed. Let $S \in \mathcal{A} \setminus \mathcal{X}$. Then $S \in (T,U)_\prec \subseteq \mathcal{A} \setminus \mathcal{X}$ for some $T, U \in \mathcal{A}$. As $\prec$ is a dense order, $S \prec V \prec U$ for some closed V, and $V \in (T,U)_\prec \subseteq \mathcal{A} \setminus \mathcal{X} \subseteq \mathcal{A} \setminus \mathcal{F}$. Thus $S \notin \mathcal{F}^*$. Hence $\mathcal{F}^* \subseteq \mathcal{X}$.
2. $\emptyset \notin \overline{\mathcal{F}}$, since $\emptyset \notin \mathcal{A}$.
Let S_1, $S_2 \in \overline{\mathcal{F}}$ and let $T \succ S_1 \cap S_2$. Let

$$\begin{aligned} U_1 &= \{\varepsilon \in (0,1] : d(\varepsilon, S_1) < d(\varepsilon, S_2)\} \\ U_2 &= \{\varepsilon \in (0,1] : d(\varepsilon, S_2) < d(\varepsilon, S_1)\}. \end{aligned}$$

Let $V_1 := U_1 \cup T^\circ$ and $V_2 := U_2 \cup T^\circ$. Then $S_1 = (S_1 \setminus S_2) \cup (S_1 \cap S_2) \subseteq U_1 \cup T^\circ = V_1$ since S_2 is closed. Since V_1 is open, $S_1 \prec V_1$. Hence $V_1 \in \mathcal{F}$. Similarly, $V_2 \in \mathcal{F}$. As $V_1 \cap V_2 \subseteq T$, $T \in \mathcal{F}$. We conclude that $S_1 \cap S_2 \in \overline{\mathcal{F}}$.
The other defining properties of an a-filter are immediately checked using part 1. ☐

Corollary 5.2. *If $\mathcal{F}$ is an a-filter on $(0,1]$, then $\overline{\mathcal{F}^\circ} = \overline{\mathcal{F}}$.*

Proof. $\supseteq$: since $\mathcal{F}^\circ \subseteq \mathcal{F}$.
$\subseteq$: it suffices to show that $\mathcal{F} \subseteq \overline{\mathcal{F}^\circ}$. Let $S \in \mathcal{F}$. Let $T \subseteq (0,1]$ be closed such that $T \succ S$. Then $T \in \mathcal{F}^\circ$. Hence $S \in \overline{\mathcal{F}^\circ}$. ☐

Theorem 5.3. *Let $\mathcal{F}$ be an a-filter. Then*

$$\overline{I}(\mathcal{F}) = \{x \in \widetilde{\mathbb{K}}_{\rm cnt} : (\forall S \in \mathcal{A})(x_{|S^c} \text{ invertible} \Rightarrow S \in \mathcal{F})\}.$$

Proof. Call $I^+(\mathcal{F}) := \{x \in \widetilde{\mathbb{K}}_{\rm cnt} : (\forall S \in \mathcal{A})(x_{|S^c} \text{ invertible} \Rightarrow S \in \mathcal{F})\}$.
We first show that $I^+(\mathcal{F})$ is closed:
If $a \in \widetilde{\mathbb{K}}_{\rm cnt} \setminus I^+(\mathcal{F})$, then there exists $S \in \mathcal{A} \setminus \mathcal{F}$ such that $a_{|S^c}$ is invertible. By lemma 3.4, $x_{|S^c}$ is invertible for each x in a certain neighborhood of a. Then such $x \notin I^+(\mathcal{F})$, too. Hence $\widetilde{\mathbb{K}}_{\rm cnt} \setminus I^+(\mathcal{F})$ is open.
We now show that $I(\mathcal{F}) \subseteq I^+(\mathcal{F})$:
Let $x \in I(\mathcal{F})$. Then $x_{|S} = 0$ for some $S \in \mathcal{F}$. Let $T \in \mathcal{A}$ such that $x_{|T^c}$ is invertible. Then $S \cap (0,\delta) \setminus T = \emptyset$ for some $\delta > 0$, for otherwise, $0 \in \overline{S \setminus T}$ and $x_{|S \setminus T} = 0$ and $x_{|S \setminus T}$ is invertible, a contradiction. Hence $S \cap (0,\delta) \subseteq T$, and $T \in \mathcal{F}$. Thus $x \in I^+(\mathcal{F})$.
Finally, we show that $I^+(\mathcal{F}) \subseteq \overline{I}(\mathcal{F})$:
Let $x = [x_\varepsilon] \in I^+(\mathcal{F})$. Consider the sets $L_n := \{\varepsilon : |x_\varepsilon| > \varepsilon^n\}$. As $x_{|L_n}$ is invertible, $L_n^c \in \mathcal{F}$, for each $n \in \mathbb{N}$. Further, $L_n \prec L_{n+1}$ for each $n \in \mathbb{N}$. By Urysohn's

lemma, there exist $y_n \in \widetilde{\mathbb{K}}_{\rm cnt}$ such that $y_{n|L_n} = 1$ and $y_{n|L^c_{n+1}} = 0$ and $0 \le y_n \le 1$. Then $|xy_n - x|_{|L_n} = 0$ and $|xy_n - x|_{|L^c_n} \le |x|_{|L^c_n} \le \rho^n$. Hence $|xy_n - x| \le \rho^n$, and $\lim_{n\to\infty} xy_n = x$. As $(xy_n)_{|L^c_{n+1}} = 0$, $xy_n \in I(\mathcal{F})$, for each n. □

Corollary 5.4. *Let $I \lhd \widetilde{\mathbb{K}}_{\rm cnt}$. Then $I(\mathcal{F}(I)) \subseteq I \subseteq \overline{I}(\mathcal{F}(I))$ and $\overline{I} = \overline{m(I)}$.*

Proof. $I \subseteq \overline{I}(\mathcal{F}(I))$: let $x \in I$. Let $S \in \mathcal{A}$ such that $x_{|S^c}$ is invertible. Then $S \in \mathcal{F}(I)$. Hence by theorem 5.3, $x \in \overline{I}(\mathcal{F}(I))$.
By proposition 4.5, $I(\mathcal{F}(I)) \subseteq I$. Hence $\overline{I} = \overline{I}(\mathcal{F}(I)) = \overline{m(I)}$ by theorem 4.8. □

Theorem 5.5. *Let $I \lhd \widetilde{\mathbb{K}}_{\rm cnt}$. Then*

$$\overline{\mathcal{F}}(I) = \{S \in \mathcal{A} : (\forall x \in \widetilde{\mathbb{K}}_{\rm cnt})(x_{|S} = 0 \Rightarrow x \in I)\}.$$

Proof. Call $\mathcal{F}^+(I) := \{S \in \mathcal{A} : (\forall x \in \widetilde{\mathbb{K}}_{\rm cnt})(x_{|S} = 0 \Rightarrow x \in I)\}$.
We show that $\mathcal{F}^+(I)$ is closed:
Let $S \in \overline{\mathcal{F}^+}(I)$, i.e. $S \in \mathcal{A}$ and $T \in \mathcal{F}^+(I)$, for each closed $T \succ S$. Let $x \in \widetilde{\mathbb{K}}_{\rm cnt}$ such that $x_{|S} = 0$. By lemma 3.5, there exists $T \succ S$ such that $x_{|T} = 0$. W.l.o.g, T is closed. Thus $x \in I$. Hence $S \in \mathcal{F}^+(I)$.
We now show that $\mathcal{F}(I) \subseteq \mathcal{F}^+(I)$:
Let $S \in \mathcal{F}(I)$. Then there exists $a \in I$ such that $a_{|S^c} = 1$. Now let $x \in \widetilde{\mathbb{K}}_{\rm cnt}$ such that $x_{|S} = 0$. Then $x_{|T} = 0$ for some $T \succ S$. By Urysohn's lemma, there exists $y \in \widetilde{\mathbb{K}}_{\rm cnt}$ with $y_{|S} = 0$ and $y_{|T^c} = 1$. Then $(xya)_{|T} = x_{|T} = 0$ and $(xya)_{|T^c} = x_{|T^c}$. Hence $x = xya \in I$.
Finally, we show that $\mathcal{F}^+(I) \subseteq \overline{\mathcal{F}}(I)$:
Let $S \in \mathcal{F}^+(I)$ and let $T \succ S$ be closed. By Urysohn's lemma, there exists $y \in \widetilde{\mathbb{K}}_{\rm cnt}$ such that $y_{|S} = 0$ and $y_{|T^c} = 1$. As $S \in \mathcal{F}^+(I)$, $y \in I$. Hence $T \in \mathcal{F}(I)$. □

Theorem 5.6. *If $I, J \lhd \widetilde{\mathbb{K}}_{\rm cnt}$, then $\mathcal{F}(I) = \mathcal{F}(J) \iff m(I) = m(J) \iff \overline{I} = \overline{J}$.*

Proof. 1. If $\mathcal{F}(I) = \mathcal{F}(J)$, then $m(I) = I(\mathcal{F}(I)) = I(\mathcal{F}(J)) = m(J)$ by theorem 4.8.
2. If $m(I) = m(J)$, then $\overline{I} = \overline{m(I)} = \overline{m(J)} = \overline{J}$ by corollary 5.4.
3. Let $S \in \mathcal{F}(\overline{I})$. Let $E := \{x \in \widetilde{\mathbb{K}}_{\rm cnt} : x_{|S^c}$ is invertible$\}$. Then $\overline{I} \cap E \ne \emptyset$. By lemma 3.4, E is open, hence also $I \cap E \ne \emptyset$, i.e., $S \in \mathcal{F}(I)$.
Hence, if $\overline{I} = \overline{J}$, then $\mathcal{F}(I) = \mathcal{F}(\overline{I}) = \mathcal{F}(\overline{J}) = \mathcal{F}(J)$. □

Corollary 5.7. *If $I \lhd \widetilde{\mathbb{K}}_{\rm cnt}$, then $m(\overline{I}) = m(I)$.*

Theorem 5.8. *Let $\mathcal{F}_1, \mathcal{F}_2$ be a-filters on $(0,1]$. Then $I(\mathcal{F}_1) = I(\mathcal{F}_2) \iff \mathcal{F}_1^\circ = \mathcal{F}_2^\circ \iff \overline{\mathcal{F}_1} = \overline{\mathcal{F}_2}$.*

Proof. 1. If $I(\mathcal{F}_1) = I(\mathcal{F}_2)$, then $\mathcal{F}_1^\circ = \mathcal{F}(I(\mathcal{F}_1)) = \mathcal{F}(I(\mathcal{F}_2)) = \mathcal{F}_2^\circ$ by theorem 4.7.
2. If $\mathcal{F}_1^\circ = \mathcal{F}_2^\circ$, then $\overline{\mathcal{F}_1} = \overline{\mathcal{F}_1^\circ} = \overline{\mathcal{F}_2^\circ} = \overline{\mathcal{F}_2}$ by corollary 5.2.
3. Let $x \in I(\overline{\mathcal{F}})$. Then $x_{|S} = 0$ for some $S \in \overline{\mathcal{F}}$. By proposition 3.5, there exists $T \succ S$ (w.l.o.g. T closed) such that $x_{|T} = 0$. So $T \in \mathcal{F}$, and $x \in I(\mathcal{F})$.
Hence, if $\overline{\mathcal{F}_1} = \overline{\mathcal{F}_2}$, then $I(\mathcal{F}_1) = I(\overline{\mathcal{F}_1}) = I(\overline{\mathcal{F}_2}) = I(\mathcal{F}_2)$. □

Corollary 5.9. *If $\mathcal{F}$ is an a-filter on $(0,1]$, then $(\overline{\mathcal{F}})^\circ = \mathcal{F}^\circ$.*

6 Maximal and prime ideals and filters

Definition 6.1. *An a-filter $\mathcal{F}$ on $(0,1]$ is called prime if for each $S,T \in \mathcal{A}$ with $S \cup T \in \mathcal{F}$, either $S \in \mathcal{F}$ or $T \in \mathcal{F}$.*
An a-filter $\mathcal{F}$ on $(0,1]$ is called pseudoprime if for each $S,T \in \mathcal{A}$ with $S^\circ \cup T^\circ = (0,1]$, either $S \in \mathcal{F}$ or $T \in \mathcal{F}$.

Remark 6.2. 1. In the definition of (pseudo)prime a-filter, we may also ask the condition for each closed $S,T \subseteq (0,1]$ (instead of for each $S,T \in \mathcal{A}$ only). For, if $S \notin \mathcal{A}$, then $S \notin \mathcal{S}$, i.e., $(0,\delta] \cap S = \emptyset$ for some $\delta > 0$. Hence $(S \cup T) \cap (0,\delta] \subseteq T$. So if $S \cap T \in \mathcal{F}$, then also $T \in \mathcal{F}$. The case $T \notin \mathcal{A}$ is symmetric.
2. An a-filter $\mathcal{F}$ on $(0,1]$ is prime if and only if for each $S,T \in \mathcal{A}$ with $S \cup T = (0,1]$, either $S \in \mathcal{F}$ or $T \in \mathcal{F}$. For, if $\mathcal{F}$ satisfies the latter condition and $S \cup T \in \mathcal{F}$, we consider

$$U := \{\varepsilon \in (0,1] : d(\varepsilon, S) \le d(\varepsilon, T)\} \quad \text{and} \quad V := \{\varepsilon \in (0,1] : d(\varepsilon, T) \le d(\varepsilon, S)\}.$$

Then U, V are closed with $U \cup V = (0,1]$. Hence $U \in \mathcal{F}$ or $V \in \mathcal{F}$. If $U \in \mathcal{F}$, then also $(S \cup T) \cap U \in \mathcal{F}$. As $(S \cup T) \cap U \subseteq S$, also $S \in \mathcal{F}$. The case $V \in \mathcal{F}$ is symmetric. This motivates our (less obvious) definition of pseudoprime a-filter.

Lemma 6.3. *Let $S, T, U \subseteq (0,1]$ be open and nonempty with $\overline{U} \subseteq S \cup T$. Then there exist $V \prec S$ and $W \prec T$ such that $U \subseteq V \cup W$.*

Proof. Let

$$V := \{\varepsilon \in (0,1] : \max(d(\varepsilon, U), d(\varepsilon, T^c)) \le d(\varepsilon, S^c)\}$$
$$W := \{\varepsilon \in (0,1] : \max(d(\varepsilon, U), d(\varepsilon, S^c)) \le d(\varepsilon, T^c)\}.$$

If $\varepsilon \in V \setminus S$, then $\varepsilon \in T^c \cap \overline{U} \subseteq T^c \cap (S \cup T) \subseteq S$. Hence $V \subseteq S$. As V is closed and S is open, also $V \prec S$. Similarly $W \prec T$.
Further, let $\varepsilon \in U$. Then either $d(\varepsilon, T^c) \le d(\varepsilon, S^c)$ (hence $\varepsilon \in V$) or $d(\varepsilon, S^c) \le d(\varepsilon, T^c)$ (hence $\varepsilon \in W$). So $U \subseteq V \cup W$. ◻

Lemma 6.4. *Let $\mathcal{F}$ be a pseudoprime a-filter on $(0,1]$. Then $I(\mathcal{F})$ is pseudoprime.*

Proof. Let $xy = 0$. By lemma 3.6, there exist closed T, U with $T^\circ \cup U^\circ = (0,1]$ such that $x_{|T} = 0$ and $y_{|U} = 0$. As $\mathcal{F}$ is pseudoprime, $T \in \mathcal{F}$ or $U \in \mathcal{F}$. Hence $x \in I(\mathcal{F})$ or $y \in I(\mathcal{F})$. ◻

Lemma 6.5. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ be pseudoprime. Then $\mathcal{F}(I)$ is pseudoprime.*

Proof. Let $S, T \in \mathcal{A}$ with $S^\circ \cup T^\circ = (0,1]$. Let $V \prec S^\circ$ and $W \prec T^\circ$ such that $V \cup W = (0,1]$ (lemma 6.3 with $U = (0,1]$). By Urysohn's lemma, there exist $x, y \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ such that $x_{|V} = 0$, $x_{|S^c} = 1$, $y_{|W} = 0$ and $y_{|T^c} = 1$. Then $xy = 0$. As I is pseudoprime, $x \in I$ or $y \in I$. Hence $S \in \mathcal{F}(I)$ or $T \in \mathcal{F}(I)$. ◻

Lemma 6.6. *Every closed ideal $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is radical.*

Proof. Let $S \in \mathcal{F}(\sqrt{I})$. Then there exists $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ and $n \in \mathbb{N}$ with $x^n \in I$ and $x_{|S^c} = 1$. Then also ${x^n}_{|S^c} = 1$, hence $S \in \mathcal{F}(I)$. Thus $\mathcal{F}(\sqrt{I}) = \mathcal{F}(I)$. By theorem 5.6, $I \subseteq \sqrt{I} \subseteq \overline{\sqrt{I}} = \overline{I} = I$. ◻

Proposition 6.7. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Then the following are equivalent:*

1. *I is pseudoprime*
2. *the set of ideals containing I is totally ordered (for $\subseteq$)*
3. *I is irreducible*
4. *$\sqrt{I}$ is prime*
5. *$\mathcal{F}(I)$ is pseudoprime.*

For $\widetilde{\mathbb{K}}_{\mathrm{cnt}} = \widetilde{\mathbb{R}}_{\mathrm{cnt}}$, this is still equivalent with

6. *$\widetilde{\mathbb{R}}_{\mathrm{cnt}}/I$ is totally ordered.*

Proof. $1 \Rightarrow 6$ (for $\widetilde{\mathbb{K}}_{\mathrm{cnt}} = \widetilde{\mathbb{R}}_{\mathrm{cnt}}$): let $a \in \widetilde{\mathbb{R}}_{\mathrm{cnt}}$. Since $a^2 = |a|^2$, we have $(a-|a|)(a+|a|) = 0$. As I is pseudoprime, $a - |a| \in I$ or $a + |a| \in I$. As $\widetilde{\mathbb{R}}_{\mathrm{cnt}}$ is an l-ring, it follows that $a + I \geq 0$ or $-a + I \geq 0$ in $\widetilde{\mathbb{R}}_{\mathrm{cnt}}/I$ (cf. [7, Thm. 5.3]).
$6 \Rightarrow 2$ (for $\widetilde{\mathbb{K}}_{\mathrm{cnt}} = \widetilde{\mathbb{R}}_{\mathrm{cnt}}$, cf. [8, 4.1]): the map $J \mapsto J/I$ is an order preserving bijection between the (l-)ideals of $\widetilde{\mathbb{R}}_{\mathrm{cnt}}$ containing I and the l-ideals of $\widetilde{\mathbb{R}}_{\mathrm{cnt}}/I$. As in any totally ordered ring, the l-ideals in $\widetilde{\mathbb{R}}_{\mathrm{cnt}}/I$ are totally ordered.
$1 \Rightarrow 2$ (for $\widetilde{\mathbb{K}}_{\mathrm{cnt}} = \widetilde{\mathbb{C}}_{\mathrm{cnt}}$): by the bijective correspondence of ideals in $\widetilde{\mathbb{R}}_{\mathrm{cnt}}$ and in $\widetilde{\mathbb{C}}_{\mathrm{cnt}}$ (section 2).
$2 \Rightarrow 3$: let $K = I \cap J$. Either $I \subseteq J$ or $J \subseteq I$, whence $K = I$ or $K = J$.
$3 \Rightarrow 1$: as in any commutative l-ring with 1 in which every ideal is an l-ideal, the irreducibility of $I \unlhd \widetilde{\mathbb{R}}_{\mathrm{cnt}}$ is equivalent with: for any $x, y \in \widetilde{\mathbb{R}}_{\mathrm{cnt}}$, $x\widetilde{\mathbb{R}}_{\mathrm{cnt}} \cap y\widetilde{\mathbb{R}}_{\mathrm{cnt}} \subseteq I$ implies $x \in I$ or $y \in I$ [2, Prop. 8.4.1]. So let $x, y \in \widetilde{\mathbb{R}}_{\mathrm{cnt}}$ with $xy = 0$. By lemma 3.6, there exist open T, U with $T \cup U = (0, 1]$ such that $x_{|T} = 0$ and $y_{|U} = 0$. Let $z \in x\widetilde{\mathbb{R}}_{\mathrm{cnt}} \cap y\widetilde{\mathbb{R}}_{\mathrm{cnt}}$. Then $z_{|T} = z_{|U} = 0$, hence $z = 0$. In particular, $x\widetilde{\mathbb{R}}_{\mathrm{cnt}} \cap y\widetilde{\mathbb{R}}_{\mathrm{cnt}} \subseteq I$, and hence $x \in I$ or $y \in I$. The bijective correspondence of ideals in $\widetilde{\mathbb{R}}_{\mathrm{cnt}}$ and $\widetilde{\mathbb{C}}_{\mathrm{cnt}}$ yields the result for $\widetilde{\mathbb{C}}_{\mathrm{cnt}}$.
$2 \Rightarrow 4$: the intersection of a chain of prime ideals is prime, hence $\sqrt{I} = \bigcap_{I \subseteq P, P \text{ prime}} P$ is prime.
$4 \Rightarrow 5$: by lemma 6.5, $\mathcal{F}(\sqrt{I})$ is pseudoprime. By the proof of lemma 6.6, $\mathcal{F}(I) = \mathcal{F}(\sqrt{I})$.
$5 \Rightarrow 1$: by lemma 6.4, $m(I) = I(\mathcal{F}(I))$ is pseudoprime. Hence $I \supseteq I(\mathcal{F}(I))$ is also pseudoprime. □

Theorem 6.8. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Then I is prime iff I is pseudoprime and radical.*

Proof. $\Rightarrow$: as I is prime, $\sqrt{I} = \bigcap_{I \subseteq P, P \text{ prime}} P = I$.
$\Leftarrow$: $I = \sqrt{I}$ is prime by proposition 6.7. □

Lemma 6.9. *Every pure ideal $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is radical.*

Proof. Let $x^n \in I$ for some $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ and $n \in \mathbb{N}$. As $I = m(I) = I(\mathcal{F}(I))$, there exists $S \in \mathcal{F}(I)$ such that ${x^n}_{|S} = 0$. Hence also $x_{|S} = 0$, and $x \in I(\mathcal{F}(I)) = I$. □

Proposition 6.10. *For $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, the following are equivalent:*

1. *I is pseudoprime*

2. *$m(I)$ is prime*

3. *I contains a prime ideal.*

Proof. $1 \Rightarrow 2$: by lemmas 6.4 and 6.5, $m(I) = I(\mathcal{F}(I))$ is pseudoprime. By lemma 6.9, $m(I)$ is radical. Hence $m(I)$ is prime.
$2 \Rightarrow 3$: $m(I) \subseteq I$.
$3 \Rightarrow 1$: if $P \subseteq I$ is prime and $xy = 0$, then $xy \in P$, so $x \in P \subseteq I$ or $y \in P \subseteq I$. □

Proposition 6.11. *Let $\mathcal{F}$ be an a-filter on $(0, 1]$. Then the following are equivalent:*

1. *$\mathcal{F}$ is pseudoprime*

2. *$I(\mathcal{F})$ is pseudoprime*

3. *$I(\mathcal{F})$ is prime.*

Proof. $1 \Rightarrow 2$: by lemma 6.4, $I(\mathcal{F})$ is pseudoprime.
$2 \Rightarrow 3$: as $I(\mathcal{F})$ is pure, $I(\mathcal{F})$ is radical (lemma 6.9). By theorem 6.8, $I(\mathcal{F})$ is prime.
$3 \Rightarrow 1$: by lemma 6.5, $\mathcal{F}(I(\mathcal{F}))$ is pseudoprime. As $\mathcal{F}(I(\mathcal{F})) \subseteq \mathcal{F}$, also $\mathcal{F}$ is pseudoprime. □

We now consider maximal ideals and a-filters:

Theorem 6.12. *Let $\mathcal{F}$ be an a-filter.*

1. *if $\mathcal{F}$ is pseudoprime, then $\overline{\mathcal{F}}$ is maximal.*

2. *$\mathcal{F}$ is maximal if and only if $\mathcal{F}$ is prime and $\prec$-closed.*

Proof. 1. Suppose $\overline{\mathcal{F}} \subsetneq \mathcal{F}'$ for some a-filter $\mathcal{F}'$. Let $S \in \mathcal{F}' \setminus \overline{\mathcal{F}}$. Then there exists a closed $T \succ S$ such that $T \notin \mathcal{F}$. As $\prec$ is a dense order, there exists an open V with $S \prec V \prec T$. Since $T^\circ \cup (V^c)^\circ = (0, 1]$ and $\mathcal{F}$ is pseudoprime, $V^c \in \mathcal{F}$. But then $\emptyset = S \cap V^c \in \mathcal{F}'$, a contradiction.
2. $\Rightarrow$: we show that $\mathcal{F}$ is closed: as $\mathcal{F} \subseteq \overline{\mathcal{F}}$, and $\overline{\mathcal{F}}$ is an a-filter, $\mathcal{F} = \overline{\mathcal{F}}$ by maximality. Further, we show that $\mathcal{F}$ is prime: let $S, T \in \mathcal{A}$ such that $S \cup T \in \mathcal{F}$. Suppose there exists $U \in \mathcal{F}$ such that $U \cap S = \emptyset$ and there exists $V \in \mathcal{F}$ such that $V \cap T = \emptyset$. Then $\emptyset = (U \cap V) \cap (S \cup T) \in \mathcal{F}$, a contradiction. We may thus assume that $U \cap S \neq \emptyset$, for each $U \in \mathcal{F}$. (The case $U \cap T \neq \emptyset$, for each $U \in \mathcal{F}$ is similar.) Then $\emptyset \notin \mathcal{F}' := \{U \subseteq (0, 1] \text{ closed: } (\exists V \in \mathcal{F})(S \cap V \subseteq U)\}$. As $\mathcal{F}'$ is an a-filter, $\mathcal{F} = \mathcal{F}'$ by maximality. Hence $S \in \mathcal{F}$.
2. $\Leftarrow$: by part 1, $\mathcal{F} = \overline{\mathcal{F}}$ is maximal. □

Theorem 6.13. *Let $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.*

1. *if I is pseudoprime, then $\overline{I}$ is maximal.*

2. *I is maximal if and only if I is prime and closed.*

Proof. 1. By proposition 6.7, $\mathcal{F}(I) = \mathcal{F}(\overline{I})$ is pseudoprime. Thus by theorem 6.12, $\overline{\mathcal{F}}(\overline{I})$ is maximal. Now let $\overline{I} \subseteq J \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$. Then $\overline{\mathcal{F}}(\overline{I}) \subseteq \overline{\mathcal{F}}(J)$, and hence $\overline{\mathcal{F}}(\overline{I}) = \overline{\mathcal{F}}(J)$ by maximality. Hence also $m(\overline{I}) = I(\mathcal{F}(\overline{I})) = I(\overline{\mathcal{F}}(\overline{I})) = I(\overline{\mathcal{F}}(J)) = I(\mathcal{F}(J)) = m(J)$, and hence $J \subseteq \overline{J} = \overline{I}$ by theorem 5.6.
2. $\Rightarrow$: let E denote the set of invertible elements in $\widetilde{\mathbb{K}}_{\text{cnt}}$. As I is a proper ideal, $I \cap E = \emptyset$. As E is open, also $\overline{I} \cap E = \emptyset$. Hence $\overline{I}$ is proper, and $I = \overline{I}$ by maximality. Maximal ideals are prime in any commutative ring with 1.
$\Leftarrow$: by part 1, $I = \overline{I}$ is maximal. □

Corollary 6.14.
1. The set of minimal prime ideals in $\widetilde{\mathbb{K}}_{\text{cnt}}$ equals

$$\{I(\mathcal{F}) : \mathcal{F} \textit{ is a max. a-filter on } (0,1]\} = \{I(\mathcal{F}) : \mathcal{F} \textit{ is a pseudoprime a-filter on } (0,1]\}.$$

2. The set of maximal ideals in $\widetilde{\mathbb{K}}_{\text{cnt}}$ equals

$$\{\overline{I}(\mathcal{F}) : \mathcal{F} \textit{ is a max. a-filter on } (0,1]\} = \{\overline{I}(\mathcal{F}) : \mathcal{F} \textit{ is a pseudoprime a-filter on } (0,1]\}.$$

Proof. 1.(a) Let $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$ be a minimal prime. Then $\mathcal{F}(I)$ is pseudoprime, and $I(\mathcal{F}(I)) \subseteq I$ is a prime ideal. By minimality, $I = I(\mathcal{F}(I)) = I(\overline{\mathcal{F}}(I))$ and $\overline{\mathcal{F}}(I)$ is maximal.
(b) Let $\mathcal{F}$ be a pseudoprime a-filter on $(0,1]$. Then $I(\mathcal{F})$ is prime by proposition 6.11. If $P \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$ is prime with $P \subseteq I(\mathcal{F})$, then $\mathcal{F}(P) \subseteq \mathcal{F}(I(\mathcal{F})) \subseteq \mathcal{F}$, and hence $\overline{\mathcal{F}}(P) \subseteq \overline{\mathcal{F}}$. As P is prime, $\mathcal{F}(P)$ is pseudoprime, and hence $\overline{\mathcal{F}}(P)$ is maximal by theorem 6.12. Hence $\overline{\mathcal{F}}(P) = \overline{\mathcal{F}}$. Consequently, $P \supseteq I(\mathcal{F}(P)) = I(\overline{\mathcal{F}}(P)) = I(\overline{\mathcal{F}}) = I(\mathcal{F})$.
2.(a) Let $I \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$ be maximal. Then I is pseudoprime, hence $\mathcal{F}(I)$ is pseudoprime, and thus $\overline{\mathcal{F}}(I)$ is maximal. Further, $I = \overline{I} = \overline{m(I)} = \overline{I}(\mathcal{F}(I)) = \overline{I}(\overline{\mathcal{F}}(I))$.
(b) Let $\mathcal{F}$ be a pseudoprime a-filter on $(0,1]$. Then $I(\mathcal{F})$ is pseudoprime, hence $\overline{I}(\mathcal{F})$ is maximal. □

Proposition 6.15. *Let $I \lhd \widetilde{\mathbb{K}}$. Then $\overline{I} = \bigcap_{\substack{I \subseteq M \\ M \text{ maximal}}} M$.*
In particular, an ideal $I \lhd \widetilde{\mathbb{K}}$ is closed iff it is an intersection of maximal ideals.

Proof. $\subseteq$: by theorem 6.13, maximal ideals are closed.
$\supseteq$: let $x \notin \overline{I} = \overline{m(I)} = \overline{I}(\mathcal{F}(I))$ (corollary 5.4). By theorem 5.3, there exists $S \in \mathcal{A} \setminus \mathcal{F}(I)$ such that $x_{|S^c}$ is invertible. Let $E := \{x \in \widetilde{\mathbb{K}}_{\text{cnt}} : x_{|S^c} \text{ is invertible}\}$. As E is closed under multiplication and $E \cap I = \emptyset$, there exists a prime $P \lhd \widetilde{\mathbb{K}}_{\text{cnt}}$ such that $I \subseteq P$ and $E \cap P = \emptyset$ (e.g., [7, 0.16]). As E is open (lemma 3.4), also $E \cap \overline{P} = \emptyset$. In particular, $\overline{P}$ is maximal and $x \notin \overline{P}$. □

Remark 6.16. In the previous, we showed that maximal ideals of $\widetilde{\mathbb{K}}_{\text{cnt}}$ are in bijective correpondence with maximal a-filters, which are in bijective correspondence with points of $\beta(0,1] \setminus (0,1]$, where $\beta(0,1]$ denotes the Stone-Čech compactification of $(0,1]$ (cf. [7, 6.5]).

7 Rapid a-filters

Definition 7.1. *An a-filter $\mathcal{F}$ is called* **rapid** *if for each sequence $(S_n)_n$ in $\mathcal{F}$ with $S_1 \succ S_2 \succ \dots$, there exists $T \in \mathcal{F}$ such that $T \setminus S_n \notin \mathcal{S}$.*

Theorem 7.2. *Let $\mathcal{F}$ be an a-filter. Then $I(\mathcal{F})$ is closed iff $\mathcal{F}$ is rapid.*

Proof. $\Leftarrow$: let $a \in \overline{I}(\mathcal{F})$ with continuous representative $(a_\varepsilon)_\varepsilon$. For each $n \in \mathbb{N}$, let $S_n := \{\varepsilon \in (0,1] : |a_\varepsilon| \le \varepsilon^n\}$. By theorem 5.3, $S_n \in \mathcal{F}$, and also $S_1 \succ S_2 \succ \dots$. As $\mathcal{F}$ is rapid, there exists $T \in \mathcal{F}$ such that $T \setminus S_n \notin \mathcal{S}$. Hence $|a|_{|T} \le \rho^n$, for each $n \in \mathbb{N}$, i.e., $a_{|T} = 0$. Hence $a \in I(\mathcal{F})$.
$\Rightarrow$: let $S_n \in \mathcal{F}$, and also $S_1 \succ S_2 \succ \dots$. By Urysohn's lemma, there exist $\phi_n \in \mathcal{C}((0,1])$ such that $0 \le \phi_n \le \varepsilon^n$, $\phi_{n|S_{n+1}} = 0$ and $\phi_{n|S_n^c} = \varepsilon^n$. Let $\phi := \sum_{n=1}^\infty \phi_n$ on $(0,1/2]$. By uniform convergence, ϕ is continuous and $\varepsilon^{n+1} \le \phi(\varepsilon) \le \varepsilon^n + \varepsilon^{n+1} + \dots \le 2\varepsilon^n$ on $(0,1/2] \cap S_n \setminus S_{n+1}$. Extend ϕ to a continuous map on $(0,1]$. Then $a := [\phi(\varepsilon)] \in \widetilde{\mathbb{K}}_{\text{cnt}}$. Let $T \in \mathcal{A}$ be such that $a_{|T^c}$ is invertible. Then there exists $n \in \mathbb{N}$ such that $|\phi(\varepsilon)| > 2\varepsilon^n$ for $\varepsilon \in T^c \cap (0,\delta]$ (some $0 < \delta \le 1/2$). Hence $S_n \cap (0,\delta] \subseteq T$, and $T \in \mathcal{F}$. By theorem 5.3, $a \in \overline{I}(\mathcal{F}) = I(\mathcal{F})$. Thus there exists $T \in \mathcal{F}$ such that $a_{|T} = 0$.
Let $n \in \mathbb{N}$. Then $|\phi(\varepsilon)| < \varepsilon^n$ for each $\varepsilon \in (0,\delta] \cap T$ (some $0 < \delta \le 1/2$). Hence $(0,\delta] \cap T \setminus S_n = \emptyset$. ☐

Remark 7.3. Recall that a filter $\mathcal{F}$ of subsets of $\mathbb{N}$ is called rapid if for any decreasing sequence $(S_n)_n$ in $\mathcal{F}$, there exists $S \in \mathcal{F}$ such that $S \setminus S_n$ is finite for every $n \in \mathbb{N}$. A free ultrafilter $\mathcal{U}$ of subsets of $\mathbb{N}$ is called weakly selective (or δ-stable or P-point of $\beta\mathbb{N} \setminus \mathbb{N}$) if for each sequence $(S_n)_n$ in $\mathcal{U}$, there exists $S \in \mathcal{U}$ such that $S \setminus S_n$ is finite for each $n \in \mathbb{N}$. There exist weakly selective free ultrafilters if we assume the continuum hypothesis [11, 6] (in fact, it satisfies to assume weaker axioms, e.g. ZFC+Martin's axiom [3, §4]). By definition, a weakly selective free ultrafilter is rapid.

Lemma 7.4. *There exists a rapid maximal a-filter, if we assume the continuum hypothesis.*

Proof. Let $\mathcal{U}$ be a rapid free ultrafilter on $\mathbb{N}$. Let

$$\mathcal{F} := \big\{S \in \mathcal{A} : \{n \in \mathbb{N} : 1/n \in S\} \in \mathcal{U}\big\}.$$

From the fact that $\mathcal{U}$ is a filter, it is straightforward to check that $\mathcal{F}$ is an a-filter. From the fact that $\mathcal{U}$ is rapid, resp. maximal, it is straightforward to check that $\mathcal{F}$ is a rapid, resp. prime a-filter. By theorem 6.12, it suffices to show that $\mathcal{F}$ closed. Let $S \in \overline{\mathcal{F}}$. As S is a closed set, there exists a closed $T \succ S$ such that $\{n \in \mathbb{N} : 1/n \in T\} = \{n \in \mathbb{N} : 1/n \in S\}$. Since $T \in \mathcal{F}$, $\{n \in \mathbb{N} : 1/n \in T\} \in \mathcal{U}$. Hence also $S \in \mathcal{F}$. ☐

Proposition 7.5. *There exists a prime ideal in $\widetilde{\mathbb{K}}_{\text{cnt}}$ which is both minimal and maximal, if we assume the continuum hypothesis.*

Proof. Let $\mathcal{F}$ be a rapid maximal a-filter. By theorem 7.2, $I(\mathcal{F})$ is closed, hence $I(\mathcal{F})$ is both a minimal and maximal prime ideal by corollary 6.14. ☐

8 z-ideals

As the notion of z-ideal in the ring $\mathcal{C}(X)$ of continuous functions on a topological space X can be expressed by a purely algebraic condition [7, 4A], G. Mason [9] used this condition to define a z-ideal of any commutative ring R with 1.

Definition 8.1. *Denoting by $\mathcal{M}(a) = \{M$ max. ideals of $R : a \in M\}$, $I \trianglelefteq R$ is a z-ideal if*

$$(\forall a \in R)(\forall b \in I)(\mathcal{M}(a) = \mathcal{M}(b) \Rightarrow a \in I).$$

We proceed to show a similar characterization as for z-ideals in $\widetilde{\mathbb{K}}$. As in [13], we denote $Z(a) := \{S \in \mathcal{S} : a_{|S} = 0\}$.

Theorem 8.2. *Let $a, b \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Then $\mathcal{M}(a) \subseteq \mathcal{M}(b) \iff Z(a) \subseteq Z(b)$.*

Proof. $\Rightarrow$: let $S \in Z(a) \setminus Z(b)$, i.e., $a_{|S} = 0$ and $b_{|S} \neq 0$. By lemma 3.4, there exists $T \in \mathcal{S}$ with $T \subseteq S$ such that $b_{|T}$ is invertible. Let M be a maximal ideal containing $I := \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : x_{|T} = 0\} \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Since $a_{|S} = 0$, also $a_{|T} = 0$, hence $a \in M$. Suppose that $b \in M$. Since $b_{|T}$ is invertible, $b_{|U}$ is invertible for some $U \succ T$. By Urysohn's lemma, there exists $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ such that $x_{|T} = 0$ and $x_{|U^c} = 1$. Hence $x \in I \subseteq M$, and $\bar{x}x + \bar{b}b = |x|^2 + |b|^2 \in M$ would be invertible, a contradiction. We conclude that $M \in \mathcal{M}(a) \setminus \mathcal{M}(b)$.
$\Leftarrow$: let $M \in \mathcal{M}(a) \setminus \mathcal{M}(b)$, so $a \in M$ and $b \notin M$. As M is maximal, $M + b\widetilde{\mathbb{K}}_{\mathrm{cnt}} = \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Let $m \in M$ and $c \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ such that $m + bc = 1$. As $bc, m \in \widetilde{\mathbb{K}}$, there exists $S \subseteq (0, 1]$ such that $(bc)_{|S}$ and $m_{|S^c}$ are invertible [13, Lemma 4.1]. Hence also $b_{|S}$ is invertible. Suppose that $a_{|S}$ is invertible. Then $\bar{a}a + \bar{m}m = |a|^2 + |m|^2 \in M$ would be invertible, a contradiction. By lemma 3.4, there exists $T \in \mathcal{S}$ with $T \subseteq S$ such that $a_{|T} = 0$. We conclude that $T \in Z(a) \setminus Z(b)$. ☐

Corollary 8.3. *$I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is a z-ideal iff*

$$(\forall a \in \widetilde{\mathbb{K}}_{\mathrm{cnt}})(\forall b \in I)(Z(a) = Z(b) \Rightarrow a \in I).$$

Proposition 8.4.

1. *For $I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$,*

$$\begin{aligned} I_z : &= \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (\exists a \in I)(Z(x) = Z(a))\} = \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (\exists a \in I)(Z(x) \supseteq Z(a))\} \\ &= \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (\exists a \in I)(\mathcal{M}(x) = \mathcal{M}(a))\} = \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (\exists a \in I)(\mathcal{M}(x) \supseteq \mathcal{M}(a))\} \end{aligned}$$

   is the smallest z-ideal containing I. We call it the z-closure of I. I is a z-ideal iff $I = I_z$.

2. *For $I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, $I \subseteq \sqrt{I} \subseteq I_z$. Hence $(\sqrt{I})_z = I_z$ and every z-ideal is radical. A (proper) z-ideal is prime iff it is pseudoprime.*

Proof. As in [13, Prop. 4.3]. ☐

Proposition 8.5. *Every closed ideal $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is a z-ideal.*

Proof. I is an intersection of maximal ideals (proposition 6.15), hence a z-ideal [9]. ☐

Proposition 8.6.

1. *For a family $(I_\lambda)_{\lambda\in\Lambda}$ of ideals $I_\lambda \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, $\left(\sum_{\lambda\in\Lambda} I_\lambda\right)_z = \sum_{\lambda\in\Lambda} (I_\lambda)_z$. In particular, the sum of a family of z-ideals is a z-ideal.*

2. *For* $I, J \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$*,* $I_z \cap J_z = (I \cap J)_z$*.*

3. *For* $I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$*,* $I^z := \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : (x\widetilde{\mathbb{K}}_{\mathrm{cnt}})_z \subseteq I\}$ *is the largest* z*-ideal contained in* I*. We call it the* z*-part of* I*.* I *is a* z*-ideal iff* $I = I^z$*.*

4. *For a family* $(I_\lambda)_{\lambda\in\Lambda}$ *of ideals* $I_\lambda \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$*,* $\bigcap_{\lambda\in\Lambda} I_\lambda^z = \left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)^z$*. In particular, the intersection of a family of* z*-ideals is a* z*-ideal.*

5. *For* $I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$*,* $m(I) \subseteq I^z \subseteq I^{\surd} \subseteq I$*. In particular, every pure ideal of* $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ *is a* z*-ideal. If* $I \triangleleft \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ *is pseudoprime, then* I^z *is prime.*

Proof. 1. First, we show that $(I+J)_z = I_z + J_z$.
Let $x \in (I+J)_z$. Hence there exist $a \in I$, $b \in J$ such that $Z(x) = Z(a+b)$. Let $(\alpha_\varepsilon)_\varepsilon$, resp. $(\beta_\varepsilon)_\varepsilon$, be representatives of $|a|$, resp. $|b|$, with $\alpha_\varepsilon \neq 0$ and $\beta_\varepsilon \neq 0$ for all ε. Let $S := \{\varepsilon \in (0,1] : \alpha_\varepsilon < 2\beta_\varepsilon\}$ and $T := \{\varepsilon \in (0,1] : \beta_\varepsilon < 2\alpha_\varepsilon\}$. As $\alpha_\varepsilon \neq 0$ and $\beta_\varepsilon \neq 0$, $S \cup T = (0,1]$. By lemma 6.3, there exist $V \prec S$, $U \prec T$ such that $U \cup V = (0,1]$. By Urysohn's lemma, there exists $y, z \in \widetilde{\mathbb{R}}_{\mathrm{cnt}}$ such that $y_{|V} = 1$, $y_{|S^c} = 0$, $z_{|U} = 1$ and $z_{|T^c} = 0$ and $0 \le y, z \le 1$. Then $y + z \ge 1$. Hence there exists $u \in \widetilde{\mathbb{R}}_{\mathrm{cnt}}$ such that $(y+z)u = 1$.
Now let $W \in Z(a)$, i.e., $a_{|W} = 0$. As $|b|_{|T} \le 2|a|_{|T}$, also $b_{|T\cap W} = 0$. Hence $T \cap W \in Z(a+b) = Z(x)$, i.e. $x_{|T\cap W} = 0$. Hence $xzu_{|W} = xzu_{|(W\cap T)\cup(W\setminus T)} = 0$. Thus $Z(a) \subseteq Z(xzu)$. As $a \in I$, $xzu \in I_z$. Similarly, $xyu \in J_z$. Hence $x = xyu + xzu \in I_z + J_z$.
For arbitrary sums, the result follows as in [13, Prop. 4.4].
2–4. As in [13, Prop. 4.4].
5. We show that $m(I) \subseteq I^z$. Let $x \in m(I) = I(\mathcal{F}(I))$. Then there exists $S \in \mathcal{F}(I)$ such that $x_{|S} = 0$. Let $y \in (x\widetilde{\mathbb{K}}_{\mathrm{cnt}})_z$. Then also $y_{|S} = 0$, so $y \in I(\mathcal{F}(I)) \subseteq I$. Thus $(x\widetilde{\mathbb{K}}_{\mathrm{cnt}})_z \subseteq I$. The other statements follow as in [13, Prop. 4.4] (using [5, Prop. 4.29]). □

Remark 8.7. There are z-ideals that are not closed (e.g., consider a minimal prime ideal that is not maximal).

It is well known that $\widetilde{\mathbb{K}}$ is complete for the sharp topology [12]. Similarly, we have:

Theorem 8.8. $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ *is complete for the sharp topology.*

Proof. Since $\widetilde{\mathbb{K}}_{\mathrm{cnt}} \subseteq \widetilde{\mathbb{K}}$ and $\widetilde{\mathbb{K}}$ is complete, we show that $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is closed in $\widetilde{\mathbb{K}}$. Let $x_n \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ with continuous representative $(x_{n,\varepsilon})_\varepsilon$ such that $x_n \to x \in \widetilde{\mathbb{K}}$. By taking a subsequence, we may assume that for each $n \in \mathbb{N}$,

$$|x_{n,\varepsilon} - x_\varepsilon| \le \varepsilon^n, \quad \forall \varepsilon \le \varepsilon_n.$$

W.l.o.g., $(\varepsilon_n)_n$ is strictly decreasing and tends to 0. Then let $u_{1,\varepsilon} := x_{1,\varepsilon}$ and

$$u_{n,\varepsilon} := \begin{cases} x_{n,\varepsilon} - x_{n-1,\varepsilon} & \varepsilon \le \varepsilon_{n+1} \\ 0, & \varepsilon > \varepsilon_n \end{cases}$$

in such a way that $u_{n,\varepsilon}$ is continuous in ε and $|u_{n,\varepsilon}| \le |x_{n,\varepsilon} - x_{n-1,\varepsilon}|$ for each $\varepsilon \in (0,1]$. Then $s_\varepsilon := \sum_{n=1}^{\infty} u_{n,\varepsilon}$ is a locally finite sum. Hence $(s_\varepsilon)_\varepsilon$ is continuous and for each

$\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$,

$$|s_\varepsilon - x_\varepsilon| = \Big|\sum_{k=1}^{n} u_{k,\varepsilon} - x_\varepsilon\Big| \le |u_{n,\varepsilon}| + |x_{n-1,\varepsilon} - x_\varepsilon| \le |x_{n,\varepsilon} - x_\varepsilon| + 2\,|x_{n-1,\varepsilon} - x_\varepsilon| \le 3\varepsilon^{n-1}.$$

Hence $x = [s_\varepsilon] \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. □

Theorem 8.9. *Let $I \trianglelefteq \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ be a finitely generated ideal.*

1. *If I is radical (in particular, if I is closed, pure or a z-ideal), then $I \in \{0, \widetilde{\mathbb{K}}_{\mathrm{cnt}}\}$.*
2. *$I_z = \overline{I}$*
3. *$m(I) = I^z$.*

Proof. By [5, Lemma 4.5], I is principal, i.e. $I = a\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ for some $a \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.
1. By [5, Prop. 4.28], I is idempotent. Hence $a = a^2 b$ for some $b \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Thus ab is idempotent. So either $ab = 0$, whence $a = a^2 b = 0$ and $I = 0$, or $ab = 1$, whence $I = \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.
2. Let $x \in \overline{I}$, i.e. $x = \lim_{n\to\infty} x_n$ for some $x_n \in I$. Let $S \in Z(a)$, i.e. $a_{|S} = 0$. Then also $x_{n|S} = 0$ for each $n \in \mathbb{N}$, hence also $x_{|S} = 0$, i.e. $S \in Z(x)$. Thus $x \in I_z$. The converse inclusion holds by proposition 8.5.
3. Let $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} \setminus m(I) = I(\mathcal{F}(I))$. Then for each $S \in \mathcal{F}(I)$, $x_{|S} \ne 0$. In particular, let $(a_\varepsilon)_\varepsilon$ be a (continuous) representative of a and $L_n := \{\varepsilon \in (0,1] : |a_\varepsilon| > \varepsilon^n\}$. Then $L_n^c \in \mathcal{F}(I)$, so $x_{|L_n^c} \ne 0$. By lemma 3.4, there exist $T_n \in \mathcal{S}$ with $T_n \subseteq L_{n+1}^c$ and $x_{|T_n}$ is invertible. By lemma 3.5, there exist $S_n \succ T_n$ such that $x_{|S_n}$ is invertible (as $T_n \prec L_n^c$, we may assume $S_n \prec L_n^c$). By Urysohn's lemma, there exist $y_n \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ with $y_{n|T_n} = (\sqrt{|a|})_{|T_n}$, $y_{n|S_n^c} = 0$ and $0 \le y_n \le \sqrt{|a|}$. As $|y_n|_{|S_n} \le \sqrt{|a|}_{|S_n} \le \rho^{n/2}$, $y := \sum_{n=1}^{\infty} y_n \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ exists ($\widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is a complete ultrametric space). We show that $y \in (x\widetilde{\mathbb{K}}_{\mathrm{cnt}})_z$.
Let $U \in Z(x)$, i.e., $x_{|U} = 0$. Then $0 \notin \overline{U \cap S_n}$, since $x_{|S_n}$ is invertible. Hence $y_{n|U} = 0$. Then also $y_{|U} = 0$, i.e., $U \in Z(y)$.
Also $y \notin I$: $|y|_{|T_n} \ge |y_n|_{|T_n} = \sqrt{|a|}_{|T_n} \ge (\rho^{-n/2}\,|a|)_{|T_n}$ for each $n \in \mathbb{N}$. Hence $|y| \not\le \rho^{-N}\,|a|$ for any $N \in \mathbb{N}$, and thus $y \notin I$. Hence $(x\widetilde{\mathbb{K}}_{\mathrm{cnt}})_z \not\subseteq I$, i.e., $x \notin I^z$. □

Let $I \triangleleft \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Let $I^\perp = \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : xy = 0, \forall y \in I\}$. As in $\widetilde{\mathbb{K}}$, we have:

Proposition 8.10. *Let $I \triangleleft \widetilde{\mathbb{K}}_{\mathrm{cnt}}$. Then*

1. *$I^\perp$ is closed.*
2. *$\overline{I} \subseteq I^{\perp\perp}$.*
3. *$\overline{I} \cap I^\perp = \{0\}$.*
4. *If I is pseudoprime, then $I^\perp = \{0\}$. In particular, $\overline{I} \subsetneq I^{\perp\perp} = \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.*

Proof. 1. Let $x = \lim_{n\to\infty} x_n$, with $x_n \in I^\perp$. Then $x_n y = 0$, $\forall n \in \mathbb{N}$, hence also $xy = 0$, $\forall y \in I$. Thus $x \in I^\perp$.
2. If $x \in I$, then $xy = 0$, $\forall y \in I^\perp$, so $I \subseteq I^{\perp\perp}$. By part 1, also $\overline{I} \subseteq I^{\perp\perp}$.
3. If $x \in I \cap I^\perp$, then $x^2 = 0$, hence $x = 0$. Hence also $I^\perp \cap \overline{I} \subseteq I^\perp \cap I^{\perp\perp} = \{0\}$.
4. Let $x \in I^\perp$. If $x \ne 0$, then there exists $T \in \mathcal{S}$ such that $x_{|T}$ is invertible. By lemma 3.5, there exists $S \succ T$ such that $x_{|S}$ is invertible. W.l.o.g. S is closed, T is open and $S^c \in \mathcal{S}$. As $(T^c)^\circ \cup S^\circ = (0, 1]$ and $\mathcal{F}(I)$ is pseudoprime, either $T^c \in \mathcal{F}(I)$ or $S \in \mathcal{F}(I)$. In the first case, there exists $y \in I$ such that $y_{|T} = 1$. As $x \in I^\perp$, $xy = 0$, contradicting the fact that $(xy)_{|T}$ is invertible. In the second case, there exists $y \in I$ such that $y_{|S^c} = 1$. Hence $xy = 0$, and thus $x_{|S^c} = 0$. As $(xz)_{|S} = 1$ for some $z \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, and $(xz)_{|S^c} = 0$, $xz \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is idempotent, and hence $xz = 0$ (contradicting $(xz)_{|S} = 1$) or $xz = 1$ (contradicting $(xz)_{|S^c} = 0$). Thus $x = 0$. □

Lemma 8.11. *There exists $J \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ such that $J \ne \{0\}$ and $J^\perp \ne \{0\}$.*

Proof. Let $S := \bigcup_{n\in\mathbb{N}} (a_n, b_n)$ with $1 > b_1 > a_1 > b_2 > a_2 > \dots$ and $a_n \to 0$. Then there exists $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} \setminus \{0\}$ such that $x_{|S} = 0$ and there exists $y \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} \setminus \{0\}$ such that $y_{|S^c} = 0$. Let $J = \{x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}} : x_{|S} = 0\}$. Then $x \in J$ and $y \in J^\perp$. □

Also as in $\widetilde{\mathbb{K}}$, the Hahn-Banach extension property does not hold in the following sense:

Theorem 8.12. *Let $J \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ with $J \ne \{0\}$ and $J^\perp \ne \{0\}$. Let $I := J + J^\perp$. Then there exists a continuous $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$-linear map ϕ: $I \to \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ that cannot be extended to a $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$-linear map ψ: $\widetilde{\mathbb{K}}_{\mathrm{cnt}} \to \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.*

Proof. Let $\phi(x + y) := x$, for each $x \in J$ and $y \in J^\perp$. As $J \cap J^\perp = \{0\}$, ϕ is defined unambiguously and is $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$-linear. Further, $|\phi(x + y)|^2 = |x|^2 \le |x|^2 + |y|^2 = (x + y)(\bar{x} + \bar{y}) = |x + y|^2$, for each $x \in J$ and $y \in J^\perp$. Hence ϕ is also continuous.
Now suppose that ψ: $\widetilde{\mathbb{K}}_{\mathrm{cnt}} \to \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is a $\widetilde{\mathbb{K}}_{\mathrm{cnt}}$-linear extension of ϕ. Then for any $x \in J$, $x\psi(1) = \psi(x) = \phi(x) = x$. Hence $x(\psi(1) - 1) = 0$. Thus $\psi(1) - 1 \in J^\perp$. Hence $\psi(1)\psi(1) - \psi(1) = \psi(\psi(1) - 1) = \phi(\psi(1) - 1) = 0$. It follows that $\psi(1) \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ is idempotent, hence $\psi(1) = 0$ or $\psi(1) = 1$. If $\psi(1) = 0$, then $\psi = 0$, and thus also $\phi = 0$, whence $J = \{0\}$. If $\psi(1) = 1$, then $\psi(x) = x$ for each $x \in \widetilde{\mathbb{K}}_{\mathrm{cnt}}$, and thus also $\phi(y) = y$ for each $y \in J^\perp$, whence $J^\perp = \{0\}$. □

Corollary 8.13. *If $I \lhd \widetilde{\mathbb{K}}_{\mathrm{cnt}}$ with $I \ne \{0\}$ and $I^\perp \ne \{0\}$, then $I + I^\perp \ne \widetilde{\mathbb{K}}_{\mathrm{cnt}}$.*

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