A study of (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$

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Abstract

We study (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$, i.e. minihypers with the same parameters as a weighted sum of x hyperplanes. We characterize these minihypers as a nonnegative rational sum of hyperplanes and we use this characterization to extend and improve the main results of several papers which have appeared on the special case $t = 2$. We establish a new link with coding theory and we use this link to construct several new infinite classes of (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$ that cannot be written as an integer sum of hyperplanes.

Keywords: minihypers, Griesmer codes, projective geometry, rational sums, linear codes

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1 Introduction and preliminaries

We start by introducing the notions and notations that will be used throughout the paper.

Notation 1.1. By $\mathbb{N}_0$, we denote the set of nonnegative integers. By $\mathcal{P}$, we denote the point set of the t-dimensional projective geometry $\mathrm{PG}(t,q)$ over the finite field $\mathbb{F}_q$ of order q. By $v_{u+1} = \frac{q^{u+1}-1}{q-1}$, we denote the number of points in any u-dimensional subspace of $\mathrm{PG}(t,q)$. The set of hyperplanes of $\mathrm{PG}(t,q)$ will be denoted by $\mathcal{H}$.

Definition 1.2. A *multiset* is a mapping $\mathfrak{K} : \mathcal{P} \to \mathbb{N}_0$. This mapping is extended additively to the power set of $\mathcal{P}$: for any $\mathcal{Q} \subseteq \mathcal{P}$, we put $\mathfrak{K}(\mathcal{Q}) = \sum_{x\in\mathcal{Q}} \mathfrak{K}(x)$. The image of a point or subset under this mapping is called the *multiplicity* of the point or subset. The *cardinality* of the multiset is $\mathfrak{K}(\mathcal{P})$. The *support* $\operatorname{supp}\mathfrak{K}$ of a multiset $\mathfrak{K}$ is defined as the set of all points of positive multiplicity:

$$\operatorname{supp}\mathfrak{K} = \{x \in \mathcal{P} \mid \mathfrak{K}(x) > 0\}.$$

Multisets with $\mathrm{Im}(\mathfrak{K}) = \{0, 1\}$ are called *non-weighted*, or *projective*, and can be viewed as sets by identifying them with their supports. A multiset $\mathfrak{K}$ is said to be *proper* if $\operatorname{supp}\mathfrak{K} \neq \mathcal{P}$.

Definition 1.3. An $(f,m;t,q)$-*minihyper* is a multiset of cardinality f in $\mathrm{PG}(t,q)$ such that each hyperplane has multiplicity at least m. If t and q are clear from the context, we will speak of an

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(f, m)-minihyper. Similarly, an $(n, w; t, q)$-*arc*, or (n, w)-arc for short, is a multiset of cardinality n in $\mathrm{PG}(t, q)$ such that each hyperplane has multiplicity at most w. A *proper minihyper* is a minihyper which is proper as a multiset. To avoid trivial cases, we will always assume $t \geq 2$ and $f > 0$.

The set of points of a u-dimensional subspace of $\mathrm{PG}(t, q)$ is an example of a (v_{u+1}, v_u)-minihyper. Note that (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t, q)$, with $x \leq q$, are always proper, since their total multiplicity is only $xv_t \leq qv_t < v_{t+1}$.

Definition 1.4. The *characteristic function* of a set $\mathcal{Q} \subseteq \mathcal{P}$ is denoted by

$$\chi_{\mathcal{Q}}(x) = \begin{cases} 1 & \text{for } x \in \mathcal{Q}, \\ 0 & \text{for } x \notin \mathcal{Q}. \end{cases}$$

Remark 1.5. Every multiset $\mathfrak{K}$ in $\mathrm{PG}(t, q)$ can be uniquely interpreted as a vector $w \in \mathbb{Q}^{\mathcal{P}}$ as $w = (\mathfrak{K}(u))_{u \in \mathcal{P}}$. There is a natural bijective correspondence between the set of all multisets in $\mathrm{PG}(t, q)$ and the subset $\mathbb{N}_0^{\mathcal{P}} \subset \mathbb{Q}^{\mathcal{P}}$.

Addition (often referred to as *sum* or *weighted sum*) and scalar multiplication of multisets can be defined by

$$(\mathfrak{K}_1 + \mathfrak{K}_2)(x) = \mathfrak{K}_1(x) + \mathfrak{K}_2(x), \quad (c\mathfrak{K})(x) = c\mathfrak{K}(x)$$

which is just the standard addition and multiplication for their corresponding vectors. Clearly, the sum of two minihypers with parameters (f_1, m_1) and (f_2, m_2) is an (f, m)-minihyper with $f = f_1 + f_2$ and $m \geq m_1 + m_2$.

The intersection of a multiset $\mathfrak{K}$ and a set S is defined as follows:

$$(\mathfrak{K} \cap S)(x) = \begin{cases} \mathfrak{K}(x) & \text{if } x \in S. \\ 0 & \text{if } x \notin S. \end{cases}$$

Definition 1.6. An (f, m)-minihyper $\mathfrak{F}$ is called *indecomposable* if it cannot be represented as the sum of two nonempty minihypers with parameters (f_1, m_1) and (f_2, m_2), respectively, for which $m = m_1 + m_2$ and $f = f_1 + f_2$.

Clearly, an (f, m)-minihyper which is not proper and which is not the point set of $\mathrm{PG}(t, q)$, is decomposable: it can be represented as the sum of a (v_{t+1}, v_t)-minihyper (namely the entire space $\mathrm{PG}(t, q)$) and an $(f - v_{t+1}, m - v_t)$-minihyper.

Minihypers represent a useful tool for describing the structure of linear codes meeting the Griesmer bound [6, 13]. Given a linear $[n, k, d]_q$-code, let m be the largest positive integer such that $d \geq q^m$, let g be the smallest nonnegative integer such that $d \leq (g+1)q^m$ (so that $0 \leq (g+1)q^m - d < q^m$), and denote by $[\mu_0, \mu_1, \ldots, \mu_{m-1}]$ the expansion of $(g+1)q^m - d$ in basis q (so that $(g+1)q^m - d = \sum_{i=0}^{m-1} \mu_i q^i$). It can be shown [7] that there exists a bijective correspondence between the set of all non-equivalent $[n, k, d]_q$-codes meeting the Griesmer bound, and the set of $\left(\sum_{i=0}^{m-1} \mu_i v_{i+1}, \sum_{i=0}^{m-1} \mu_i v_i\right)$-minihypers in $\mathrm{PG}(m, q)$ with each $\mu_i \leq q - 1$. The characterization of minihypers with the above parameters is equivalent to the characterization of the corresponding Griesmer codes (cf. [11] and the references there). These minihypers were investigated earlier in [2, 9, 10].

Definition 1.7. Let X be a finite set of size v (which we call the *points*) and let $\mathcal{B}$ be a family of k-element subsets of X (which we call the *blocks*) in which every unordered pair of elements of X is contained in exactly λ blocks of $\mathcal{B}$. Then $(X, \mathcal{B})$ is called a *balanced incomplete* $2-(v, k, \lambda)$

block design. It is easy to see that each point of X is contained in $r = \lambda(v-1)/(k-1)$ blocks of $\mathcal{B}$. Letting $b = |\mathcal{B}|$, an easy double-counting argument yields that $vr = bk$. If $b = v$, the design is called *symmetric*.

Definition 1.8. Let $\mathcal{D} = (X, \mathcal{B})$ be a $2-(v,k,\lambda)$-design and fix any ordering of the points and of the blocks. The *incidence matrix* of $\mathcal{D}$ is the $b \times v$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if the } j\text{th point is contained in the } i\text{th block,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, A can be interpreted as an isomorphism between $\mathbb{Q}^{\mathcal{P}}$ and $\mathbb{Q}^{\mathcal{H}}$.

Remark 1.9. It is easily checked that $A^T A = (r-\lambda)I + \lambda J$, where I and J are the unit matrix of order v and the all-one matrix of order v, respectively. Hence $\det A^T A = rk(r-\lambda)^{v-1}$ over $\mathbb{Q}$. Hence, when $r \neq \lambda$ (or, equivalently, when $v \neq k$), $A^T A$ is nonsingular and hence A is nonsingular. In $\mathrm{PG}(t,q)$, $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $\mathcal{B}$ the set of hyperplanes, is a symmetric $2-(v_{t+1}, v_t, v_{t-1})$-design. For proofs of these statements and an in-depth introduction to designs (and their links with finite geometry), we refer to [1, 3, 5].

The remainder of this paper is structured as follows. In Section 2, we present a new characterization of proper (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$ as rational sums of hyperplanes. We thereby generalize a result by Landjev and Storme [10, Theorem 5]. In Section 3, we extend and improve several key results that have appeared on the special case $n = 2$ [9, 10]. Most notably, we prove a strong modular result and a useful inequality between x, q and c (c is defined in Theorem 2.5). Finally, in Section 4, we establish a new connection between the code words of certain geometrically defined codes and indecomposable minihypers. We exploit this new connection to present a new non-trivial construction for (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$.

2 Rational sums

Lemma 2.1. Let $\mathfrak{K}$ be an arbitrary multiset in $\mathrm{PG}(t,q)$, $q = p^h$. Then its incidence vector w can uniquely be written as a linear combination over $\mathbb{Q}$ of incidence vectors of hyperplanes: $w = \sum_{H \in \mathcal{H}} r_H \chi_H$ with $r_H \in \mathbb{Q}$.

Proof. Let A be an incidence matrix of the points and hyperplanes of $\mathrm{PG}(t,q)$. By Remark 1.9, A is invertible. Hence, the rows of A form a $\mathbb{Q}$-basis for the vector space $\mathbb{Q}^{\mathcal{P}}$ and for any $w \in \mathbb{Q}^{\mathcal{P}}$, one can find a unique collection of rational coefficients $\{r_H\}_{H \in \mathcal{H}}$ such that $w = \sum_{H \in \mathcal{H}} r_H \chi_H$. Note that, with $r = (r_H)_{H \in \mathcal{H}}$, $w = rA$. □

Notation 2.2. From now on, if $\mathfrak{F}$ is an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t,q)$, we will denote by $r_H(\mathfrak{F})$ the coefficient r_H associated to the hyperplane H in the rational sum obtained in Theorem 2.3. If the minihyper $\mathfrak{F}$ is clear from the context, we will simply write r_H. Since the minihyper can be written as a rational sum in a unique way, this will often be the case.

Theorem 2.3. Let $\mathfrak{K}$ be a multiset in $\mathrm{PG}(t,q)$ and let $w = \sum_{H \in \mathcal{H}} r_H \chi_H$ be its incidence vector. Then $r_H \geq 0$ for each $H \in \mathcal{H}$ if and only if w is an (f,m)-minihyper with $m \geq \frac{v_{t-1}}{v_t} f$. If in addition, $\mathfrak{K}$ is proper, then $r_H \geq 0$ for each $H \in \mathcal{H}$ if and only if $\mathfrak{K}$ is an (xv_t, xv_{t-1})-minihyper, with $x = \sum_{H \in \mathcal{H}} r_H \in \mathbb{N}_0$.

Proof. Since A is invertible and $JA = rJ$, we may write $A^T A = (r-\lambda)I + \lambda J$ as $(A^T - \frac{\lambda}{r} J)A = (r-\lambda)I$, which yields $A^{-1} = \frac{1}{r-\lambda}(A^T - \frac{\lambda}{r} J)$.

Let now $w \in \mathbb{Q}^{\mathcal{P}}$ be the incidence vector of any multiset $\mathfrak{K}$ (as defined in Remark 1.5). Then $w = (wA^{-1})A$, which yields an explicit form for the rational coefficients: $w = \sum_{H\in\mathcal{H}}(wA^{-1})_H \chi_H$, and this form is unique by Lemma 2.1.

Hence, we want to determine when each of the elements of $wA^{-1} \in \mathbb{Q}^{\mathcal{H}}$ is non-negative. From the explicit form derived above, $wA^{-1} = \frac{1}{r-\lambda}\left(wA - \frac{\lambda}{r}Jw\right)$. However, Jw is a vector with each of its entries equal to the total size of the multiset, f. Hence, we need $(wA^T)_H \geq \frac{\lambda}{r}f$ for each $H \in \mathcal{H}$.

Now the element $(wA^T)_H$ represents the total multiplicity of the hyperplane H, $\mathfrak{K}(H)$, and hence this inequality is equivalent to saying that $\mathfrak{K}(H) \geq \frac{\lambda}{r}f$ for each $H \in \mathcal{H}$. In other words, this is true if and only if w is the incidence vector of an $(f, m; t, q)$-minihyper with $m \geq \frac{\lambda}{r}f$. This proves the first statement.

If the multiset $\mathfrak{K}$ is proper, then there is a point u with $\mathfrak{K}(u) = 0$. We define a new multiset $\mathfrak{K}'$ as follows: $\mathfrak{K}' = \sum_{H\ni u}\mathfrak{K}\cap H$. Then the total multiplicity of this new multiset is $f\lambda$, since for each point of $\mathfrak{K}$ there are λ hyperplanes through u and through this point. On the other hand, this number is at least m times the number r of such hyperplanes, since each hyperplane contains at least m points. Hence, we also have $m \leq \frac{\lambda}{r}f$ and thus $m = \frac{\lambda}{r}f$. However, $\gcd(\lambda, r) = 1$, so r divides f, and thus $f = xr$ for some positive integer x. Hence, $m = x\lambda$ and since $r = \frac{q^t-1}{q-1}$ and $\lambda = \frac{q^{t-1}-1}{q-1}$, we have $f = x\left(\frac{q^t-1}{q-1}\right) = xv_t$ and $m = x\left(\frac{q^{t-1}-1}{q-1}\right) = xv_{t-1}$. □

Remark 2.4. The last part of the proof of Theorem 2.3 shows that for every proper (f, m)-minihyper in $\mathrm{PG}(t, q)$, one has $\frac{f}{m} \geq \frac{v_t}{v_{t-1}}$. This provides an additional motivation for the study of (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t, q)$.

Theorem 2.5. For any proper (xv_t, xv_{t-1})-minihyper $\mathfrak{F} = \sum_{H\in\mathcal{H}} r_H\chi_H$ in $\mathrm{PG}(t, q)$, the smallest positive integer c for which $cr_H \in \mathbb{N}_0$ for all $H \in \mathcal{H}$, is a power of p and a divisor of q^{t-1}.

Proof. From the proof of Theorem 2.3, we know that the coefficients r_H are given by $(wA^{-1})_H = \frac{1}{r-\lambda}\left((wA^T)_H - \frac{\lambda f}{r}\right)$. Since Jw is a vector with all its entries equal to $f = rx$, $\frac{\lambda}{r}Jw$ is an integer vector which only consists of entries λx. Since the entries of wA^T are also integers, $wA^T - \frac{\lambda}{r}Jw$ is an integer vector, and $(r-\lambda)wA^{-1}$ only contains integer entries.

Since $r - \lambda = q^{t-1}$, the smallest positive integer c for which $cr_H \in \mathbb{N}_0$ for all $H \in \mathcal{H}$, is a divisor of q^{t-1}, and hence it is indeed a power of p. □

Note that $c = 1$ corresponds to the minihyper being a weighted sum of hyperplanes.

Notation 2.6. Similar to Remark 2.2, we will write $c(\mathfrak{F})$ for the integer c from Theorem 2.5. If the minihyper $\mathfrak{F}$ is clear from the context, we will simply write c.

Remark 2.7. A proper (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$ (with $x > 0$) cannot be decomposed into a hyperplane and an $((x-1)v_t, (x-1)v_{t-1})$-minihyper if and only if $r_\pi < 1$ for each hyperplane π. In this case, we call the minihyper *hyperplane-indecomposable.* For $x \leq q$, we will see in Section 3 that hyperplane-indecomposability is equivalent to indecomposability.

3 Generalizations of previous results

In this section, we will apply Theorem 2.3 to generalize and improve several key results from [9] and [10]. In what follows, we let $q = p^h$ with p prime; this defines p and h.

R. Hill and H.N. Ward [9] proved the following modular result via polynomial techniques for $t = 2$. This was extended to $t > 2$ in [8, Theorem 4.6], using similar techniques.

Theorem 3.1. Let $\mathfrak{F}$ be an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$, with $x \leq q - p^g$ for some nonnegative integer g. Then $\mathfrak{F}(\pi) \equiv xv_{t-1} \pmod{p^{g+1}q^{t-2}}$ for every hyperplane π in $\mathrm{PG}(t, q)$.

Using Theorem 2.3, we can present a sharper version of this modular result. We begin with an easy counting lemma. We recall that if $\mathfrak{F} = \sum_{H\in\mathcal{H}} r_H \chi_H$ is an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$, then $\sum_{H\in\mathcal{H}} r_H = x$. We also recall that whenever we write r_H or c, this has to be interpreted as in Remark 2.2.

Lemma 3.2. Let $\mathfrak{F}$ be an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$. Then a hyperplane π with rational coefficient r_π has multiplicity $\mathfrak{F}(\pi) = r_\pi q^{t-1} + xv_{t-1}$.

Proof. The hyperplane π contributes r_π to the multiplicity of each point in π, and hence contributes $r_\pi v_t$ to the total multiplicity of π. Every other hyperplane π' intersects π in $\lambda = v_{t-1}$ points, hence contributing $r_{\pi'} v_{t-1}$ to $\mathfrak{F}(\pi)$. Since the sum of all rational coefficients is x, this yields a total multiplicity in π of $r_\pi v_t + (x - r_\pi)v_{t-1}$. Since $v_t = q^{t-1} + v_{t-1}$, this proves the statement. □

From this it follows that for any s-dimensional subspace π, one has $\mathfrak{F}(\pi) = xv_s + q^s \sum_{H\supseteq\pi, H\in\mathcal{H}} r_H$. Moreover, if π contains a point u with multiplicity 0, then all hyperplanes through u (and hence all hyperplanes through π) have their rational coefficient equal to 0. Hence, in this case $\mathfrak{F}(\pi) = xv_s$.

Theorem 3.3. Let $\mathfrak{F}$ be a proper (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$. Then $\mathfrak{F}(\pi) \equiv xv_{t-1} \pmod{\frac{q^{t-1}}{c}}$ for every hyperplane π in $\mathrm{PG}(t, q)$. Moreover, if $x \leq q - p^g$, then p^{g+1} divides $\frac{q}{c}$, making this result stronger than Theorem 3.1.

Proof. Let π be an arbitrary hyperplane and let r_π be its rational coefficient. Then $\mathfrak{F}(\pi) = r_\pi q^{t-1} + xv_{t-1}$ by Lemma 3.2. Since the denominator of r_π is a divisor of c, the product $q^{t-1} r_\pi$ is an integer multiple of $\frac{q^{t-1}}{c}$, and hence the first part of the statement follows.

For the second part, it is sufficient to recall that c is the smallest integer such that for all r_H, $cr_H \in \mathbb{N}_0$. By Theorem 3.1, $r_\pi q^{t-1}$ is divisible by $p^{g+1}q^{t-2}$ and hence $\left(\frac{q}{p^{g+1}}\right) r_\pi$ is an integer. Since π was arbitrary, and since c is the smallest positive integer for which cr_π is an integer for all π, it follows that $c \leq \frac{q}{p^{g+1}}$. Since c is a power of p by Theorem 2.5, it follows that p^{g+1} divides $\frac{q}{c}$. □

In Theorem 3.3, we work modulo $\frac{q^{t-1}}{c} = \frac{q}{c} q^{t-2}$. In Theorem 3.1, the result is only valid modulo $p^{g+1}q^{t-2}$. Since we just have just proven that p^{g+1} divides $\frac{q}{c}$, Theorem 3.3 is a generalization of Theorem 3.1.

Corollary 3.4. Let $\mathfrak{F}$ be a nonempty (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$. Then $x > q - \frac{q}{c}$. In other words: if $x \leq q - \frac{q}{c_0}$ for some positive integer c_0, then $c < c_0$.

Proof. If $x \geq q$, then the statement is trivially fulfilled. Otherwise, let g be the largest nonnegative integer for which $x \leq q - p^g$. By this maximality assumption, $x > q - p^{g+1}$. Since p^{g+1} divides $\frac{q}{c}$, it indeed follows that $x > q - \frac{q}{c}$. □

As a special case of Corollary 3.4, we get the following corollary.

Corollary 3.5. For $x \leq q - \frac{q}{p}$ (and hence for $x < q$ when $q = p$), we have $c = 1$. Hence, if $x \leq q - q/p$ then any (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$ is a sum of x hyperplanes.

This special case was proven earlier for $t = 2$ in [9, Theorem 20] and for general t in [8, Corollary 4.8]. The sharpness of the bound in Corollary 3.5 had not yet been demonstrated. In Section 4, we will show the sharpness of this bound. This family of examples will show the sharpness of the bound in Corollary 3.4 in general when $c = p^e$ with $e|h$ (with $q = p^h$).

Corollary 3.6. If $x \leq 2q - 2\frac{q}{p} + 1$, then a proper (xv_t, xv_{t-1})-minihyper is decomposable if and only if it is hyperplane-decomposable.

Proof. Assume by contraposition that there exists a proper decomposable, but hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper $\mathfrak{F}$ with $x \leq 2q - 2\frac{q}{p} + 1$. Since it is proper and decomposable, it can be written as $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$, where $\mathfrak{F}_1$ is a nonempty (x_1v_t, x_1v_{t-1})-minihyper and $\mathfrak{F}_2$ is a nonempty (x_2v_t, x_2v_{t-1})-minihyper, and $x_1 + x_2 = x$. Since $x \leq 2q - 2\frac{q}{p} + 1$, it follows that $\min(x_1, x_2) \leq q - \frac{q}{p}$, and, by Corollary 3.5, this minihyper is a sum of hyperplanes. Hence, we can subtract any such hyperplane from $\mathfrak{F}$ and end up with an $((x-1)v_t, (x-1)v_{t-1})$-minihyper, contradicting the assumption that $\mathfrak{F}$ is hyperplane-indecomposable. □

Remark 3.7. Corollary 3.5 and its sharpness determine the smallest x for which there is a (hyperplane-)indecomposable (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$.

An upper bound on the largest x for which a proper, hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper exists, can easily be derived as follows. Fix a point u with multiplicity 0 in this minihyper. Since we assume that $\mathfrak{F}$ is hyperplane-indecomposable, $r_H < 1$ for all hyperplanes H. Since $cr_H \in \mathbb{N}_0$ and since c is a divisor of q^{t-1}, by Theorem 2.5, this yields $r_H \leq 1 - \frac{1}{c} \leq 1 - \frac{1}{q^{t-1}}$. Hence,

$$x = \sum_{H \ni u} r_H + \sum_{H \not\ni u} r_H = 0 + \sum_{H \not\ni u} r_H \leq \sum_{H \not\ni u} \left(1 - \frac{1}{q^{t-1}}\right) = q^t \left(1 - \frac{1}{q^{t-1}}\right) = q^t - q,$$

with equality if and only if all hyperplanes not through u have $r_H = 1 - \frac{1}{q^{t-1}}$. And indeed, this equality can occur; in that case $\mathfrak{F}$ is $q^{t-1} - 1$ times the setwise complement of u in $\mathrm{PG}(t, q)$, since each point different from u lies on q^{t-1} hyperplanes not containing u.

The largest x for which such a proper indecomposable minihyper exists is not known, not even for $t = 2$. A generalization of the result by Landjev and Storme [10] on the case $t = 2$ follows straightforwardly from the techniques in this paper; it is presented in Theorem 3.8. We however believe that this bound is not sharp at all.

Theorem 3.8. Let $\mathfrak{F}$ be a proper indecomposable (xv_t, xv_{t-1})-minihyper which is not the setwise complement of a point. Then $x \leq q^t - 2q + \frac{q}{p} - 1$ and the multiplicity of any point in $\mathfrak{F}$ is at most $q^{t-1} - 1$.

Proof. Assume that $\mathfrak{F}$ is a proper indecomposable (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$, and let u be a point of multiplicity 0. Hence, $r_H = 0$ for all hyperplanes H through u. Since we assume that $\mathfrak{F}$ is indecomposable, it is also hyperplane-indecomposable, which means that $r_H < 1$ for all hyperplanes. Since $cr_H \in \mathbb{N}_0$ and c is a divisor of q^{t-1} by Theorem 2.5, this yields $r_H \leq 1 - \frac{1}{c} \leq 1 - \frac{1}{q^{t-1}}$.

Let u' be an arbitrary point different from u. From the fact that $r_H \leq 1 - \frac{1}{q^{t-1}} = \frac{q^{t-1}-1}{q^{t-1}}$ and the fact that there are only q^{t-1} hyperplanes through u' and not through u, it follows that the multiplicity of this point u' is at most $q^{t-1}-1$. Since u' was arbitrary, this yields the second claim.

Now, we revisit the switching construction from [10] with respect to u. In our terminology, it reduces to the natural substitution

$$\psi : \begin{cases} r_H \mapsto r_H (= 0) & \text{if } H \ni u, \\ r_H \mapsto 1 - \frac{1}{q^{t-1}} - r_H & \text{if } H \not\ni u. \end{cases}$$

Clearly, since $0 \leq r_H(\mathfrak{F}) \leq 1 - \frac{1}{q^{t-1}}$, the same holds for $r_H(\psi(\mathfrak{F}))$, and since each point different from u lies on $v_t - v_{t-1} = q^{t-1}$ hyperplanes not through u, the fact that each point has an integer multiplicity is also preserved under ψ. Hence, $\psi(\mathfrak{F})$ is a (yv_t, yv_{t-1})-minihyper in $\mathrm{PG}(t,q)$ with $y = q^t\left(1 - \frac{1}{q^{t-1}}\right) - x$.

Since $\mathfrak{F}$ is not the setwise complement of u, $\psi(\mathfrak{F})$ is nonempty. Moreover, since $r_H(\psi(\mathfrak{F})) < 1$, the minihyper $\psi(\mathfrak{F})$ is hyperplane-indecomposable, which means $c > 1$ and hence $c \geq p$. By Corollary 3.4, $y \geq q - \frac{q}{p} + 1$, which means that

$$x = (q^t - q) - y \leq (q^t - q) - (q - \frac{q}{p} + 1) = q^t - 2q + \frac{q}{p} - 1. \qquad \square$$

Corollary 3.9. There does not exist a hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t,q)$ for $q^t - 2q + \frac{q}{p} - 1 < x < q^t - q$.

4 Another link with coding theory

We will now establish a new correspondence between hyperplane-indecomposable (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$ and the dual projective space code over the ring $\mathbb{Z}_c$, with c the number described in Theorem 2.5. Let $\mathbb{Z}_c$ be the ring of integers modulo c, i.e. $\mathbb{Z}_c = (\{0,1,2,\ldots,c-1\}, +_c, \cdot_c)$, where $a +_c b$ and $a \cdot_c b$ denote the remainder of respectively $a+b$ and $a \cdot b$ after division by c. Note that the set $\{0,1,2,\ldots,c-1\}$ is a set of integers, a subset of $\mathbb{Z}$. If $c = p$, then $\mathbb{Z}_c$ is a field, isomorphic to $\mathbb{F}_p$.

Let H be the hyperplane-by-point incidence matrix of $\mathrm{PG}(t,q)$. Let $C_c^\perp(t,q)$ be the linear $\mathbb{Z}_c$-code defined by H as a parity check matrix, where the positions of the code correspond to the hyperplanes:

$$C_c^\perp(t,q) = \{z = (z_H)_{H\in\mathcal{H}} \in \mathbb{Z}_{\mathcal{H}} : zH = \bar{0}\};$$

hereby, the matrix multiplication is done over $\mathbb{Z}_c$. For this code $C_c^\perp(t,q)$, we define a new weight function $\mathrm{wt}(z) = \sum_{H\in\mathcal{H}} z_H$, where z_H is interpreted as an integer in $\{0,1,\ldots,c-1\}$ and summation is done over $\mathbb{Z}$. In the special case that $c = p$, $C_c^\perp(t,q)$ is equivalent to the commonly studied projective space code of points and hyperplanes.

Geometrically, code words of C correspond to multisets of hyperplanes in $\mathrm{PG}(t,q)$, with hyperplane multiplicities in the set $\{0,1,\ldots,c-1\}$, such that for each point r we have $\sum_{H\ni r} z_H \equiv 0 \pmod{c}$. We will interpret z_H in the proof of Theorem 4.1 as $z_H = c \cdot r_H$, where r_H are (as always) the rational coefficients from Lemma 2.1 for the minihyper $\mathfrak{F}$.

Theorem 4.1. There is a natural bijective correspondence between the code words

$$z = (z_H)_{H\in\mathcal{H}} \in C_{c_0}^\perp(t,q)$$

and the hyperplane-indecomposable (xv_t, xv_{t-1})-minihypers $\sum_{H\in\mathcal{H}} r_H \chi_H$ with $c = c_0$; this correspondence is given by $z_H = c \cdot r_H$.

Proof. First assume that we have a code word $z = (z_H)_{H\in\mathcal{H}} \in C_{c_0}^{\perp}(t,q)$. By definition of the code $C_{c_0}^{\perp}(t,q)$, we have $\sum_{H\ni u} z_H \equiv 0 \pmod{c_0}$ for each point u. Hence, for each point u, the multiplicity $\frac{1}{c_0}\sum_{H\ni u} z_H$ of the point u is an integer. Since we also have that each weight is nonnegative (as $z_H \in \{0,1,\dots,c_0-1\}$), it follows from Theorem 2.3 that $\mathfrak{F} := \sum_{H\in\mathcal{H}} \frac{z_H}{c_0}\chi_H$ is an (xv_t, xv_{t-1})-minihyper for $x = \sum_{H\in\mathcal{H}} \frac{z_H}{c_0}$. Since for each $H \in \mathcal{H}$, $z_H \in \{0,1,\dots,c-1\}$, one has $\frac{z_H}{c_0} < 1$, and hence $\mathfrak{F}$ is a hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper.

For the other direction, assume that we have a hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper $\mathfrak{F}$ in $\mathrm{PG}(t,q)$. By Theorem 2.3, $\mathfrak{F} = \frac{1}{c}\sum_{H\in\mathcal{H}} r_H\chi_H$. By Remark 2.7, $r_H < 1$ for each $H \in \mathcal{H}$. Let $z_H = cr_H$, then the multiplicity at each point u is $\frac{1}{c}\sum_{H\ni u} z_H \in \mathbb{N}_0$. This implies that $\sum_{H\ni u} z_H \equiv 0 \pmod{c}$, which means that $z = (z_H)_{H\in\mathcal{H}}$ is a code word of $C_c^{\perp}(t,q)$. $\square$

Theorem 4.1 can be used in the construction of non-trivial (xv_t, xv_{t-1})-minihypers. Ball's construction, mentioned in [10], can be derived as a special case of this construction. The key is to dualize the setting: we start with an arbitrary multiset of points, dualize it to have an arbitrary multiset of hyperplanes, and take a rational sum of them to obtain a minihyper. This yields the following interesting constructions.

Lemma 4.2 (Ball's construction)**.** Let B be a set of points in $\mathrm{PG}(t,q)$ and let e be the largest nonnegative integer such that B meets each hyperplane in 0 modulo p^e points. Then there exists a $\left(\frac{|B|}{p^e}v_t, \frac{|B|}{p^e}v_{t-1}\right)$-minihyper in $\mathrm{PG}(t,q)$ with $c = p^e$.

Proof. Let B' be the dual set of hyperplanes of the points in B. By the self-duality of $\mathrm{PG}(t,q)$, each point is contained in 0 modulo p^e hyperplanes of B'. Associating a coefficient $r_H = \frac{1}{p^e}$ to each of these hyperplanes (and 0 to all other hyperplanes) yields a $\left(\frac{|B|}{p^e}v_t, \frac{|B|}{p^e}v_{t-1}\right)$-minihyper. By construction, $c|p^e$, and by the maximality of e, it follows that $c = p^e$. $\square$

More interestingly, we can also utilize 1 modulo p^e sets to construct new examples, as the following lemma demonstrates.

Lemma 4.3. Let A and B be sets of points in $\mathrm{PG}(t,q)$ and let e be the largest nonnegative integer such that A and B both meet each hyperplane in 1 modulo p^e points. Then for any $\lambda \in \{1,2,\dots,p^e-1\}$ there exists an (xv_t, xv_{t-1})-minihyper $\mathfrak{F}$ in $\mathrm{PG}(t,q)$ with $c = p^e$ and $x = |B\setminus A| + \lambda\frac{|A|-|B|}{p^e}$.

Proof. Since A and B represent point sets, we can consider their associated dual sets A' and B' of hyperplanes. Since A and B intersect each hyperplane in 1 modulo p^e points, their differences $A\setminus B$ and $B\setminus A$ intersect each hyperplane in 0 modulo p^e points. Therefore if we add λ times the incidence vector of each hyperplane in $A'\setminus B'$ and $p^e-\lambda$ times the incidence vector of each hyperplane in $B'\setminus A'$, the multiplicity of each point will be divisible by p^e. Hence, dividing this by p^e yields a minihyper with c a divisor of p^e. By the maximality of e, it follows that $c = p^e$.

The total weight in the multiset before dividing by p^e, is

$$\lambda|A\setminus B| + (p^e-\lambda)|B\setminus A| = p^e|B\setminus A| + \lambda(|A|-|B|).$$

Dividing out p^e yields $x = |B\setminus A| + \lambda\frac{|A|-|B|}{p^e}$ as claimed. $\square$

Several examples of 1 modulo p^e sets (with $e \geq 1$) are known: i-dimensional subspaces with $i \geq 1$, Baer subgeometries, unitals and Hermitian varieties, linear blocking sets and many, many other commonly studied structures in finite geometries. With Lemma 4.3, all of them can be used to obtain structurally new examples. In particular, we were able to construct a minimal nontrivial example, i.e. a minihyper with $x = q - \frac{q}{p} + 1$ which is not a sum of x hyperplanes. This shows the sharpness of Corollary 3.5 and can also be used to show the sharpness of Theorem 3.8. In some cases, the construction can also be used to show the sharpness of Corollary 3.4.

Theorem 4.4. For each divisor e of h (where $q = p^h$), there exists an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$ with $x = q - \frac{q}{p^e} + 1$.

Proof. Let $q = p^h$ and let e be a divisor of h. Let A be the line in $\mathrm{PG}(2, q)$ having $X_0 = 0$ as its equation, and let B be the set

$$B = \{(1, z, z^{p^e}) | z \in \mathbb{F}_q\} \cup \{(0, z, z^{p^e}) | z \in \mathbb{F}_q^*\}.$$

Then it is shown in [4] that $|B| = q + y$ and $|B \cap A| = y$, with $y = \frac{q-1}{p^e-1}$. Moreover, it is shown there that each line intersects B in 1 modulo p^e points. This set B is called a Rédei-type blocking set.

Applying Lemma 4.3 with this A and B and with $\lambda = p^e - 1$, one obtains an (xv_2, xv_1)-minihyper with $x = q - \frac{q}{p^e} + 1$ in $\mathrm{PG}(2, q)$. This proves the statement for $t = 2$.

For $t > 2$, the construction in the plane can easily be extended. Let π be a 2-dimensional subspace of $\mathrm{PG}(t, q)$ and let π' be a $(t-3)$-dimensional subspace skew to π. Let $\mathfrak{F}$ be the constructed example for $t = 2$ in the 2-dimensional space π. Now for each line L in π, let r_L be its rational coefficient in $\mathfrak{F}$ and let H_L be the hyperplane spanned by L and π'. Then $\mathfrak{F}' := \sum_{L \subset \pi} r_L \chi_{H_L}$ is a cone with π' as its vertex and $\mathfrak{F}$ as its base. Moreover, $\mathfrak{F}'$ is an (xv_t, xv_{t-1})-minihyper with $x = q - \frac{q}{p^e} + 1$ in $\mathrm{PG}(t, q)$. □

Remark 4.5. Let again $t = 2$ and let $q = p^2$ and $e = 1$. Repeating the construction in the proof of Theorem 4.4 with the same choices of A and B, but now varying $\lambda \in \{1, \ldots, p-1\}$, one obtains a spectrum result: a nontrivial (xv_2, xv_1)-minihyper for each $x \in \{q - \frac{q}{p} + 1, \ldots, q - 1\}$.

The construction in the proof of Theorem 4.4 was inspired by the construction of the smallest known code words (in terms of Hamming weight) in the dual code $C^{\perp}_{\mathrm{PG}(2,q)}$ associated to the projective plane $\mathrm{PG}(2, q)$ [12]. These code words are conjectured to be the smallest in Hamming weight. Corollary 3.5 shows that they are the smallest weight code words with respect to the modified weight function $w : C^{\perp}_{\mathrm{PG}(2,q)} \to \mathbb{N}_0 : (z_H)_{H \in \mathcal{H}} \mapsto \sum_{H \in \mathcal{H}} z_H$. It would be interesting to see if this can be used to prove that it is also the smallest weight code word with respect to the Hamming weight.

Corollary 4.6. The bound in Corollary 3.5 is sharp. When e divides h (with $c = p^e$ and $q = p^h$), the bound in Corollary 3.4 is also sharp.

Proof. Consider the $((q - \frac{q}{p^e} + 1)v_t, (q - \frac{q}{p^e} + 1)v_{t-1})$-minihyper in $\mathrm{PG}(t, q)$ obtained in Theorem 4.4. Its rational coefficients are 0, $\frac{1}{p^e}$ and $\frac{p^e-1}{p^e}$, and hence this minihyper has $c = p^e$. This shows the sharpness of Corollary 3.4 when e divides h.

For $e = 1$, this yields a $((q - \frac{q}{p} + 1)v_t, (q - \frac{q}{p} + 1)v_{t-1})$-minihyper in $\mathrm{PG}(t, q)$ which is a rational sum of hyperplanes with rational coefficients 0, $\frac{1}{p}$ and $\frac{p-1}{p}$. This minihyper is not a sum of hyperplanes (since $c = p > 1$) and has $x = q - \frac{q}{p} + 1$, showing the sharpness of Corollary 3.5. □

Open Problem 4.7. It is not known whether the bound in Corollary 3.4 is sharp for all c.

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