Deformed 𝔰𝔲(1,1) Algebra
as a Model for Quantum Oscillators

Elchin I. JAFAROV †‡, *Neli I. STOILOVA* § *and Joris VAN DER JEUGT* †

† *Department of Applied Mathematics and Computer Science, Ghent University,*
  Krijgslaan 281-S9, B-9000 Gent, Belgium
  E-mail: *Joris.VanderJeugt@UGent.be*

‡ *Institute of Physics, Azerbaijan National Academy of Sciences,*
  Javid Av. 33, AZ-1143 Baku, Azerbaijan
  E-mail: *ejafarov@physics.ab.az*

§ *Institute for Nuclear Research and Nuclear Energy,*
  Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria
  E-mail: *stoilova@inrne.bas.bg*

Abstract. The Lie algebra $\mathfrak{su}(1,1)$ can be deformed by a reflection operator, in such a way that the positive discrete series representations of $\mathfrak{su}(1,1)$ can be extended to representations of this deformed algebra $\mathfrak{su}(1,1)_\gamma$. Just as the positive discrete series representations of $\mathfrak{su}(1,1)$ can be used to model a quantum oscillator with Meixner–Pollaczek polynomials as wave functions, the corresponding representations of $\mathfrak{su}(1,1)_\gamma$ can be utilized to construct models of a quantum oscillator. In this case, the wave functions are expressed in terms of continuous dual Hahn polynomials. We study some properties of these wave functions, and illustrate some features in plots. We also discuss some interesting limits and special cases of the obtained oscillator models.

Key words: oscillator model; deformed algebra $\mathfrak{su}(1,1)$; Meixner–Pollaczek polynomial; continuous dual Hahn polynomial

2010 Mathematics Subject Classification: 81R05; 81Q65; 33C45

1 Introduction

The $\mathfrak{su}(1,1)$-model of a quantum oscillator [19] is a model that obeys the dynamics of a harmonic oscillator, but with the position and momentum operators and the Hamiltonian being elements of the Lie algebra $\mathfrak{su}(1,1)$ instead of the Heisenberg algebra.

There are many algebraic constructions to model a quantum oscillator. The difficulty for such models is often to determine the spectra of observables and an explicit form of their eigenfunctions. Only for some models, one can develop such a complete theory. One of these models is the q-oscillator, a q-deformation of the standard quantum oscillator [11, 18, 24, 31]. The q-oscillator has many interesting properties, both from the mathematics and physics point of view. But it also has some drawbacks, in particular the Newton–Lie (or Hamilton–Lie) equations are not satisfied.

Following this, new oscillator models were developed such that the same dynamics as in the classical or quantum case is satisfied, and in such a way that the operators corresponding to position, momentum and Hamiltonian are elements of some algebra different from the traditional Heisenberg (or oscillator) Lie algebra. In the one-dimensional case, there are three (essentially self-adjoint) operators: a position operator $\hat{q}$, its corresponding momentum operator $\hat{p}$ and a (pseudo-)Hamiltonian $\hat{H}$ which is the generator of time evolution. These operators

should satisfy the Hamilton–Lie equations (or the compatibility of Hamilton's equations with the Heisenberg equations):

$$[\hat{H}, \hat{q}] = -i\hat{p}, \qquad [\hat{H}, \hat{p}] = i\hat{q}, \tag{1}$$

in units with mass and frequency both equal to 1, and $\hbar = 1$. Contrary to the canonical case, the commutator $[\hat{q}, \hat{p}] = i$ is not required. Apart from (1) and the self-adjointness, it is then common to require the following conditions [3]:

- all operators $\hat{q}$, $\hat{p}$, $\hat{H}$ belong to some (Lie) algebra (or superalgebra) $\mathcal{A}$;
- the spectrum of $\hat{H}$ in (unitary) representations of $\mathcal{A}$ is equidistant.

A very interesting model occurs for $\mathcal{A} = \mathfrak{su}(2)$ (or its enveloping algebra) [3, 4, 5]. In that case, the relevant representations are the well known $\mathfrak{su}(2)$ representations labeled by an integer or half-integer j. Since these representations are finite-dimensional, one is dealing with "finite oscillator models", of potential use in optical image processing [5]. Up to a constant, the Hamiltonian $\hat{H}$ is the diagonal $\mathfrak{su}(2)$ operator with a linear spectrum $n + \frac{1}{2}$ ($n = 0, 1, \ldots, 2j$). Also $\hat{q}$ and $\hat{p}$ have a finite spectrum, given by $\{-j, -j + 1, \ldots, +j\}$ [3]. The (discrete) position wave functions have been constructed, and are given by Krawtchouk functions (normalized symmetric Krawtchouk polynomials) [3], tending to the canonical wave functions in terms of Hermite polynomials when $j \to \infty$. In the terminology of quantum theory of angular momentum, these discrete position wave functions are just Wigner D-functions [6], and their relation to Krawtchouk polynomials was first given by Koornwinder [23].

In two previous papers [16, 17], the $\mathfrak{su}(2)$ model for the finite one-dimensional harmonic oscillator was extended. The underlying algebra is a deformation of $\mathfrak{su}(2)$ with an extra reflection (or parity) operator and an additional parameter α (>-1). In the even-dimensional representations [16] (j half-integer), the spectrum of the position operator is of the form

$$-\alpha - j - \frac{1}{2}, -\alpha - j + \frac{1}{2}, \ldots, -\alpha - 1; \alpha + 1, \alpha + 2, \ldots, \alpha + j + \frac{1}{2},$$

and the position wave functions could be constructed in terms of normalized Hahn (or dual Hahn) polynomials (with parameters $(\alpha, \alpha+1)$ or $(\alpha+1, \alpha)$). In the odd-dimensional representations [17] (j integer), the spectrum of the position operator is

$$0, \quad \pm\sqrt{k(2\alpha + k + 1)}, \qquad k = 1, \ldots, j.$$

The position (and momentum) wave functions are again Hahn polynomials (in this case with parameters (α, α) or $(\alpha + 1, \alpha + 1)$).

Models of quantum oscillators with continuous spectra of position and momentum operators were constructed based on the positive discrete series representations of $\mathfrak{su}(1, 1)$ [19]. In such a representation, labeled by a positive number $a > 0$, the spectrum of the position operator is $\mathbb{R}$. The position wave function, when the oscillator is in the nth eigenstate of the Hamiltonian, is given by

$$\phi_n^{(a)}(x) = \frac{2^a \sqrt{n!}}{\sqrt{2\pi \, \Gamma(n + 2a)}} |\Gamma(a + ix)| \, P_n^{(a)}(x; \pi/2),$$

where $P_n^{(\lambda)}(x; \phi)$ is the Meixner–Pollaczek polynomial [20]. Many interesting properties of these $\mathfrak{su}(1, 1)$ oscillators were described by Klimyk [19].

One type of deformation of this $\mathfrak{su}(1, 1)$ model was offered by its q-deformation. The $\mathfrak{su}_q(1, 1)$ model was investigated in [2]. The position and momentum operators have spectra covered by

a finite interval of the real line, which depends on the value of q, and the wavefunctions are given in terms of q-Meixner–Pollaczek polynomials.

In the current paper, we consider an extension of the Lie algebra $\mathfrak{su}(1,1)$ by a parity or reflection operator R. In this extension or deformation, the common $\mathfrak{su}(1,1)$ commutator $[J_+, J_-] = -2J_0$ is replaced by $[J_+, J_-] = -2J_0 - \gamma R$, where γ is a (real) deformation parameter. The positive discrete series representations of $\mathfrak{su}(1,1)$, labeled by a positive number a, can be extended to representations of the deformed algebra $\mathfrak{su}(1,1)_\gamma$, provided γ can be written in the form $\gamma = (2a-1)(2c-1)$ for some positive c-value (sometimes c rather than γ will be referred to as the deformation parameter). Section 2 recalls some known formulas for the $\mathfrak{su}(1,1)$ case, and in Section 3 the algebra $\mathfrak{su}(1,1)_\gamma$ and its representations are given. The core of the paper comes in Section 4, where models for a quantum oscillator are built using $\mathfrak{su}(1,1)_\gamma$ representations. Just as for the canonical oscillator, the Hamiltonian has a discrete but infinite equidistant spectrum in these models, and the position operator has spectrum $\mathbb{R}$. We have managed to obtain explicit expressions for the orthonormal wave functions $\psi_n^{(a,c)}(x)$, where a is the representation label and c the deformation label. These wave functions involve the class of so-called continuous dual Hahn polynomials [20, 14]. Their properties and shapes are studied in Section 4. In Section 5 we consider some limits and special cases. For $c = 1/2$ (or $\gamma = 0$), our functions reduce to those studied by Klimyk [19]. Another interesting case is when c tends to $+\infty$: then the model reduces to that of the paraboson oscillator. So for $a = 1/2$ and $c \to +\infty$ the model coincides with the canonical oscillator. In Section 6, we give a differential-reflection operator realization of the deformed algebra $\mathfrak{su}(1,1)_\gamma$, and in Section 7 we discuss the possibility of considering further parameters in the deformation. Our paper closes with some remarks in a concluding section.

2 The algebra $\mathfrak{su}(1,1)$, positive discrete series representations and oscillator models

The Lie algebra $\mathfrak{su}(1,1)$ [19] can be defined by its basis elements J_0, J_+, J_- with commutator relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0.$$

The positive discrete series representations of $\mathfrak{su}(1,1)$ are labeled [8, 19] by a positive real number $a > 0$ (the Bargmann index), and are infinite-dimensional. The action of the $\mathfrak{su}(1,1)$ generators on a set of basis vectors $|a, n\rangle$ (with $n = 0, 1, 2, \dots$) is given by

$$
\begin{aligned}
J_0|a, n\rangle &= (n + a)\,|a, n\rangle, \\
J_+|a, n\rangle &= \sqrt{(n+1)(n+2a)}\,|a, n+1\rangle, \\
J_-|a, n\rangle &= \sqrt{n(n+2a-1)}\,|a, n-1\rangle.
\end{aligned}
\tag{2}
$$

This action satisfies $J_0^\dagger = J_0$, $J_\pm^\dagger = J_\mp$. The representation space $\mathcal{H}_a$ is a Hilbert space with orthonormal basis $|a, n\rangle$ (with $n = 0, 1, 2, \dots$).

In the $\mathfrak{su}(1,1)$ oscillator model, the position, momentum and Hamiltonian are chosen as follows:

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0.\tag{3}$$

These operators satisfy (1). The spectrum of $\hat{H}$ is thus $(n+a)$ ($n = 0, 1, 2, \dots$), an equidistant and infinite spectrum. One can just as well choose the representation-dependent operator $J_0 - a + 1/2$

for $\hat{H}$ in order to get the spectrum of the standard quantum oscillator, but in fact this shift is not so relevant. A more interesting aspect is the determination of the spectrum of the position operator $\hat{q}$ and its eigenvectors. In fact, the matrix elements for $\hat{q}$ can be deduced from the explicit computations of Basu and Wolf [9], and were (for the current representations) identified with Meixner–Pollaczek polynomials by Koornwinder [22]. Let us briefly describe the method followed by Klimyk [19]. Denoting a formal eigenvector of $\hat{q}$, for the eigenvalue x, by

$$v(x) = \sum_{n=0}^{\infty} A_n(x) |a, n\rangle,$$

the equation $\hat{q}v(x) = xv(x)$ leads by means of (2) and (3) to

$$2xA_n(x) = \sqrt{(n+1)(n+2a)}A_{n+1}(x) + \sqrt{n(n+2a-1)}A_{n-1}(x).$$

It was shown in [19, 21] that this recurrence relation is satisfied by normalized Meixner–Pollaczek polynomials $P_n^{(a)}(x; \pi/2)$, i.e.

$$A_n(x) = \phi_n^{(a)}(x) = \frac{2^a \sqrt{n!}}{\sqrt{2\pi\Gamma(n+2a)}} |\Gamma(a+ix)| P_n^{(a)}(x; \pi/2), \tag{4}$$

where these Meixner–Pollaczek polynomials are given in terms of the hypergeometric function as

$$P_n^{(a)}(x; \pi/2) = \frac{(2a)_n}{n!} i^n {}_2F_1 \left(\begin{matrix} -n, a+ix \\ 2a \end{matrix} ; 2 \right). \tag{5}$$

We have used here the common notation for Pochhammer symbols [1, 7, 29] $(a)_k = a(a + 1)\cdots(a + k - 1)$ for $k = 1, 2, \ldots$ and $(a)_0 = 1$. These (real) functions are orthonormal,

$$\int_{-\infty}^{+\infty} \phi_n^{(a)}(x)\phi_m^{(a)}(x) = \delta_{nm},$$

with support $\mathbb{R}$. Hence the spectrum of $\hat{q}$ is $\mathbb{R}$ [10, 19, 28], and the functions (4) have an interpretation as position wave functions [28].

3   The deformed algebra $\mathfrak{su}(1,1)_\gamma$ and its representations

The extension and deformation of the Heisenberg algebra with a reflection operator was performed in [26]. In that case, the representations of the deformed algebra correspond to representation of the Lie superalgebra $\mathfrak{osp}(1|2)$. Recently, there has been further interest in extending algebras by reflection operators [32], or even more basically in extending differential operators by reflection operators [27, 33]. It is in this context that we present an extension and deformation of $\mathfrak{su}(1,1)$.

The universal enveloping algebra of $\mathfrak{su}(1,1)$ can be extended by a parity (or reflection) operator R, with action $R|a,n\rangle = (-1)^n |a,n\rangle$ in the representation $\mathcal{H}_a$. This means that R commutes with J_0, anticommutes with J_+ and J_-, and $R^2 = 1$. This extended algebra can be deformed by a parameter γ, leading to the definition of $\mathfrak{su}(1,1)_\gamma$.

Definition 1. Let γ be a parameter. The algebra $\mathfrak{su}(1,1)_\gamma$ is a unital algebra with basis elements J_0, J_+, J_- and R subject to the following relations:

- R is a parity operator satisfying $R^2 = 1$ and

$$[R, J_0] = RJ_0 - J_0R = 0, \qquad \{R, J_\pm\} = RJ_\pm + J_\pm R = 0. \tag{6}$$

- The $\mathfrak{su}(1,1)$ relations are deformed as follows:

$$[J_0, J_\pm] = \pm J_\pm, \tag{7}$$
$$[J_+, J_-] = -2J_0 - \gamma R. \tag{8}$$

Clearly, for $\gamma = 0$ this is just $\mathfrak{su}(1,1)$ extended by a parity operator, and the positive discrete series representations $\mathcal{H}_a$ are the same as (2) with extra action $R|a,n\rangle = (-1)^n |a,n\rangle$. The interesting property is that also for $\gamma \neq 0$, the representation $\mathcal{H}_a$ ($a > 0$) can be deformed in such a way that it becomes a representation for $\mathfrak{su}(1,1)_\gamma$, provided some conditions are satisfied for a. This is described in the following proposition. From now on we shall assume that γ is a given nonzero real number.

Proposition 1. *Let a be a positive real number with $a \neq 1/2$ and consider the space $\mathcal{H}_a$ with basis vectors $|a, n\rangle$ ($n = 0, 1, 2, \ldots$). Assume that γ can be written as*

$$\gamma = (2a-1)(2c-1) \qquad with \quad c > 0. \tag{9}$$

Then the following action turns $\mathcal{H}_a$ into an irreducible representation space of $\mathfrak{su}(1,1)_\gamma$

$$R|a,n\rangle = (-1)^n |a,n\rangle, \tag{10}$$

$$J_0|a,n\rangle = \left(n + a + c - \frac{1}{2}\right)|a,n\rangle, \tag{11}$$

$$J_+|a,n\rangle = \begin{cases} \sqrt{(n+2a)(n+2c)}\,|a,n+1\rangle, & \text{if } n \text{ is even,} \\ \sqrt{(n+1)(n+2a+2c-1)}\,|a,n+1\rangle, & \text{if } n \text{ is odd,} \end{cases} \tag{12}$$

$$J_-|a,n\rangle = \begin{cases} \sqrt{n(n+2a+2c-2)}\,|a,n-1\rangle, & \text{if } n \text{ is even,} \\ \sqrt{(n+2a-1)(n+2c-1)}\,|a,n-1\rangle, & \text{if } n \text{ is odd.} \end{cases} \tag{13}$$

So, for a given $\gamma \neq 0$ (i.e. for a fixed algebra $\mathfrak{su}(1,1)_\gamma$), not all positive a-values are allowed as representation parameter. From (9), it follows that:

- if $\gamma < 0$ then $a \in]0, \frac{1}{2}[\cup]\frac{1-\gamma}{2}, +\infty[$,
- if $0 < \gamma < 1$ then $a \in]0, \frac{1-\gamma}{2}[\cup]\frac{1}{2}, +\infty[$,
- if $\gamma \geq 1$ then $a \in]\frac{1}{2}, +\infty[$.

The proof of Proposition 1 is essentially by direct computation: it is a simple task to verify that the actions (10)–(13) do indeed satisfy the relations (6)–(8). The conditions $a > 0$ and $c > 0$ ensure that the factors under the square roots are positive. The irreducibility follows from the fact that $(J_+)^m |a,n\rangle$ is nonzero and proportional to $|a, n+m\rangle$ for $n, m = 0, 1, 2, \ldots$, and that $(J_-)^m |a, n+m\rangle$ is nonzero and proportional to $|a,n\rangle$.

Note also that the representation given in this proposition is unitary under the star conditions $R^\dagger = R$, $J_0^\dagger = J_0$, $J_\pm^\dagger = J_\mp$.

4 A one-dimensional oscillator model based on $\mathfrak{su}(1,1)_\gamma$

Following the choice in [19] (as in Section 2) let us choose the position, momentum and Hamiltonian operators as follows:

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 - \left(c - \frac{1}{2}\right). \tag{14}$$

with c determined by (9). The choice of subtracting a constant $c - \frac{1}{2}$ is just for convenience, and in fact this shift in the spectrum is not so relevant. The operators (14) satisfy (1). In the representation space $\mathcal{H}_a$, $\hat{H}|a, n\rangle = (n + a)|a, n\rangle$, therefore the spectrum of $\hat{H}$ is linear and given by

$$n + a, \qquad n = 0, 1, 2, \ldots.$$

Just as for the undeformed $\mathfrak{su}(1, 1)$ models, an interesting question is the determination of the spectrum of the position operator $\hat{q}$ (resp. momentum operator $\hat{p}$) and its eigenvectors. Writing the action of $\hat{q}$ on even and odd states separately, one has from (12)–(13):

$$\hat{q}|a, 2n\rangle = \sqrt{n(n + a + c - 1)}\,|a, 2n - 1\rangle + \sqrt{(n + a)(n + c)}\,|a, 2n + 1\rangle,$$
$$\hat{q}|a, 2n + 1\rangle = \sqrt{(n + a)(n + c)}\,|a, 2n\rangle + \sqrt{(n + 1)(n + a + c)}\,|a, 2n + 2\rangle. \tag{15}$$

If we denote the formal eigenvector of $\hat{q}$, for the eigenvalue x, again by

$$v(x) = \sum_{n=0}^{\infty} A_n(x)\,|a, n\rangle, \tag{16}$$

the equation $\hat{q}v(x) = xv(x)$ leads due to (15) to

$$xA_{2n}(x) = \sqrt{(n + a)(n + c)}A_{2n+1}(x) + \sqrt{n(n + a + c - 1)}A_{2n-1}(x), \tag{17}$$
$$xA_{2n+1}(x) = \sqrt{(n + 1)(n + a + c)}A_{2n+2}(x) + \sqrt{(n + a)(n + c)}A_{2n}(x). \tag{18}$$

The task is now to solve (17)–(18). The solution can again be found in the context of orthogonal polynomials. However, (17)–(18) is not simply the recurrence relation of a known (normalized) set of orthogonal polynomials. Instead, we will have to combine two sets of such orthogonal polynomials.

For this purpose, let us recall the continuous dual Hahn polynomials $S_n(x^2; a; b; c)$ in the variable x^2, with at least two positive parameters a, b and c [1, 14, 20]. These polynomials of degree n ($n = 0, 1, 2, \ldots$) in the variable x^2 are defined by [14, 20]:

$$\frac{S_n(x^2; a, b, c)}{(a + b)_n(a + c)_n} = {}_3F_2\left(\begin{matrix} -n, a + ix, a - ix \\ a + b, a + c \end{matrix}; 1\right), \tag{19}$$

in terms of the generalized hypergeometric series ${}_3F_2$ of unit argument [1, 7, 29]. For $a, c > 0$ and $b \geq 0$, the orthogonality relation of these polynomials reads [12, 14, 20]:

$$\frac{1}{2\pi}\int_0^{+\infty}\left|\frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)}{\Gamma(2ix)}\right|^2 S_m(x; a, b, c)S_n(x; a, b, c)\,dx$$
$$= \Gamma(n + a + b)\Gamma(n + a + c)\Gamma(n + b + c)n!\delta_{mn}. \tag{20}$$

Like all orthogonal polynomials, the continuous dual Hahn polynomials satisfy a 3-term recurrence relation. However, it is not this relation that can be used here. Instead, we shall need to make use of difference relations, given in the following proposition. Note that these difference relations appeared already in [12, equation (2.13)]; we give a more general proof here.

Proposition 2. *Continuous dual Hahn polynomials satisfy the following difference relations:*

$$\left(x^2 + b^2\right)S_n\left(x^2; a, b + 1, c\right) = (n + a + b)(n + b + c)S_n\left(x^2; a, b, c\right) - S_{n+1}\left(x^2; a, b, c\right), \tag{21}$$
$$S_n\left(x^2; a, b, c\right) = S_n\left(x^2; a, b + 1, c\right) - n(n + a + c - 1)S_{n-1}\left(x^2; a, b + 1, c\right). \tag{22}$$

Proof. Let us start with the following identity:

$$_3F_2\left(\begin{matrix}-n,A,B\\C,D\end{matrix};z\right) - {}_3F_2\left(\begin{matrix}-n-1,A,B\\C,D\end{matrix};z\right) = \frac{zAB}{CD}\,_3F_2\left(\begin{matrix}-n,A+1,B+1\\C+1,D+1\end{matrix};z\right).$$

To see that this identity holds, compare coefficients of z^k in left and right hand side. Putting $A = a + ix$, $B = a - ix$, $C = a + b$, $D = a + c$, $z = 1$ and following (19), this can be written as

$$\left(a^2 + x^2\right) S_n\left(x^2; a+1, b, c\right) = (n+a+b)(n+a+c)S_n\left(x^2; a, b, c\right) - S_{n+1}\left(x^2; a, b, c\right).$$

Since the continuous dual Hahn polynomials are symmetric in (a, b, c), (21) follows by permuting a and b. To prove (22), one can start from the following contiguous relation between $_3F_2$'s:

$$(n+C)\,_3F_2\left(\begin{matrix}-n,A,B\\C+1,D\end{matrix};z\right) - n\,_3F_2\left(\begin{matrix}-n+1,A,B\\C+1,D\end{matrix};z\right) = C\,_3F_2\left(\begin{matrix}-n,A,B\\C,D\end{matrix};z\right). \tag{23}$$

To verify that this relation is correct, one simply compares powers of z^k in the left and right hand side. Finally, putting $A = a + ix$, $B = a - ix$, $C = a + b$, $D = a + c$ and $z = 1$ in (23) yields (22). ∎

In particular, it follows from the previous proposition that

$$x^2 S_n\left(x^2; a, 1, c\right) = (n+a)(n+c)S_n\left(x^2; a, 0, c\right) - S_{n+1}\left(x^2; a, 0, c\right), \tag{24}$$

$$S_n(x^2; a, 0, c) = S_n\left(x^2; a, 1, c\right) - n(n+a+c-1)S_{n-1}\left(x^2; a, 1, c\right). \tag{25}$$

Comparing with the relations (17)–(18), we can propose:

$$A_{2n}(x) = \frac{(-1)^n S_n(x^2; a, 0, c)}{\sqrt{\Gamma(n+a)\Gamma(n+c)\Gamma(n+a+c)n!}},$$

$$A_{2n+1}(x) = \frac{(-1)^n x S_n(x^2; a, 1, c)}{\sqrt{\Gamma(n+a+1)\Gamma(n+c+1)\Gamma(n+a+c)n!}}.$$

It is now rather trivial to see that (17) follows from (25), and that (18) follows from (24). So we have found a solution for the recurrence relations (17)–(18). Note that the solution is in terms of *two* sets of orthogonal polynomials: continuous dual Hahn polynomials with parameters $(a, 0, c)$ for the even degree polynomials, and with parameters $(a, 1, c)$ for the odd degree polynomials.

The expression for $A_n(x)$ is a real polynomial of degree n in x. For the appropriate function

$$w(x) = \frac{1}{4\pi}\left|\frac{\Gamma(a+ix)\Gamma(ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2$$

we have

$$\int_{-\infty}^{+\infty} w(x)A_m(x)A_n(x)dx = \delta_{mn}. \tag{26}$$

Indeed, for m and n even, one writes $\int_{-\infty}^{+\infty} = 2\int_0^{+\infty}$, and then the result follows directly from (20). For m and n odd, one uses again $\int_{-\infty}^{+\infty} = 2\int_0^{+\infty}$. The extra appearance of x^2 as a factor is taken care of by $|\Gamma(ix)|^2 x^2 = |ix\Gamma(ix)|^2 = |\Gamma(1+ix)|^2$, and then (26) follows again from (20). Finally, for m even and n odd (or vice versa), (26) follows trivially from the fact

that $w(x)$ is an even function and $A_m(x)A_n(x)$ is an odd function of x. Note that, due to the identity $\Gamma(ix)/\Gamma(2ix) = 2^{1-2ix}\sqrt{\pi}/\Gamma(\frac{1}{2} + ix)$, the function w can be rewritten as

$$w(x) = \left| \frac{\Gamma(a + ix)\Gamma(c + ix)}{\Gamma(\frac{1}{2} + ix)} \right|^2. \tag{27}$$

Similarly as in (4), the support of the orthogonal polynomials appearing in (26) is $\mathbb{R}$, so the spectrum of $\hat{q}$ is $\mathbb{R}$. Rewriting $\sqrt{w(x)}A_n(x)$ by $\psi_n^{(a,c)}(x)$ (and rescaling the eigenvectors (16) by $\sqrt{w(x)}$), we have the following:

Theorem 1. *In the $su(1,1)_\gamma$ representation $\mathcal{H}_a$, with $\gamma = (2a-1)(2c-1)$, the position operator $\hat{q}$ has formal eigenvectors*

$$v(x) = \sum_{n=0}^{\infty} \psi_n^{(a,c)}(x)\,|a; n\rangle$$

with

$$\psi_{2n}^{(a,c)}(x) = \sqrt{w(x)} \frac{(-1)^n S_n(x^2; a, 0, c)}{\sqrt{\Gamma(n+a)\Gamma(n+c)\Gamma(n+a+c)n!}}, \tag{28}$$

$$\psi_{2n+1}^{(a,c)}(x) = \sqrt{w(x)} \frac{(-1)^n x S_n(x^2; a, 1, c)}{\sqrt{\Gamma(n+a+1)\Gamma(n+c+1)\Gamma(n+a+c)n!}}, \tag{29}$$

where $w(x)$ is given in (27). The spectrum of $\hat{q}$ is $\mathbb{R}$. The functions $\psi_n^{(a,c)}(x)$ are orthonormal:

$$\int_{-\infty}^{+\infty} \psi_m^{(a,c)}(x)\psi_n^{(a,c)}(x)dx = \delta_{mn}. \tag{30}$$

The formal eigenvectors satisfy the Dirac delta orthonormality:

$$\sum_{n=0}^{\infty} \psi_n^{(a,c)}(x)\psi_n^{(a,c)}(x') = \delta(x - x'), \qquad \int_{-\infty}^{+\infty} v(x)v(x')dx = \delta(x - x').$$

The last assertion of the theorem follows from the uniqueness of the weight function for continuous dual Hahn polynomials [15] (or, otherwise said, the Hamburger moment problem for continuous dual Hahn polynomials is determinate).

This result also implies that the functions $\psi_n^{(a,c)}(x)$ can be interpreted as position wave functions for the $\mathfrak{su}(1,1)_\gamma$ oscillator model, when the oscillator is in the stationary state $|a, n\rangle$ with energy $n+a$. This interpretation makes physically sense provided $|\psi_n^{(a,c)}(x)|^2 \leq 1$. Investigating plots of the functions $\psi_n^{(a,c)}(x)$ indicate that the maximum value for $|\psi_n^{(a,c)}(x)|^2$ is obtained for $|\psi_0^{(a,c)}(0)|^2$. It is easy to see from (27) and (28) that

$$\psi_0^{(a,c)}(0) = \sqrt{\frac{\Gamma(a)\Gamma(c)}{\pi\Gamma(a+c)}} = \sqrt{\frac{B(a,c)}{\pi}},$$

in terms of the Beta function. So the parameters should satisfy $B(a,c) \leq \pi$. As can be seen from the Beta integral formula [1, (1.1.21)], this is certainly satisfied when both $a \geq \frac{1}{2}$ and $c \geq \frac{1}{2}$.

It is interesting to study some plots of the wave functions $\psi_n^{(a,c)}(x)$. First of all, note that this function is symmetric in a and c, so one can keep one parameter a fixed and let the other one c vary. In Fig. 1 we plot the wave functions for $n = 0$ and in Fig. 2 for $n = 1$, both for
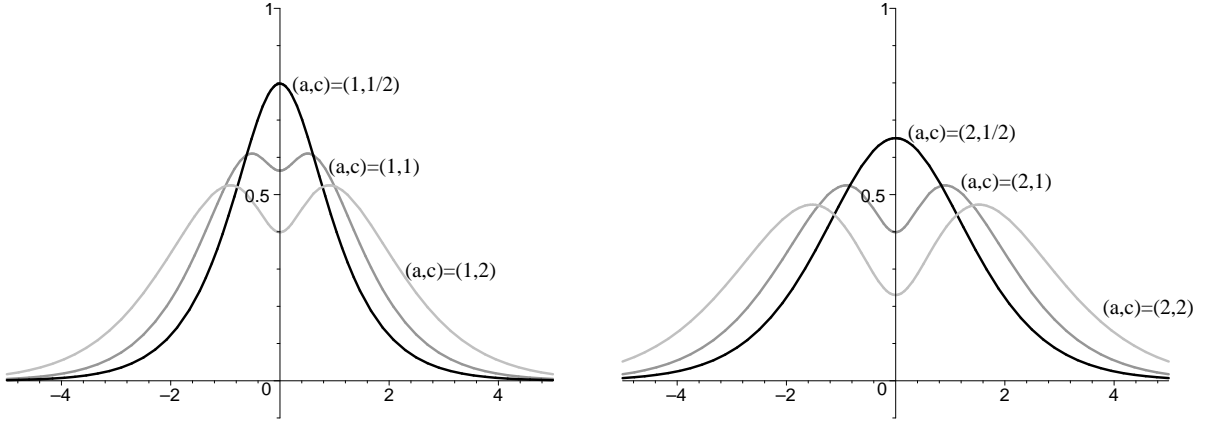
Figure 1. Plots of the wave functions $\psi_n^{(a,c)}(x)$ for $n = 0$ (ground level), for $a = 1$ (left figure) and for $a = 2$ (right figure). The values of c varies: $c = 1/2$ (black), $c = 1$ (dark gray) and $c = 2$ (light gray).
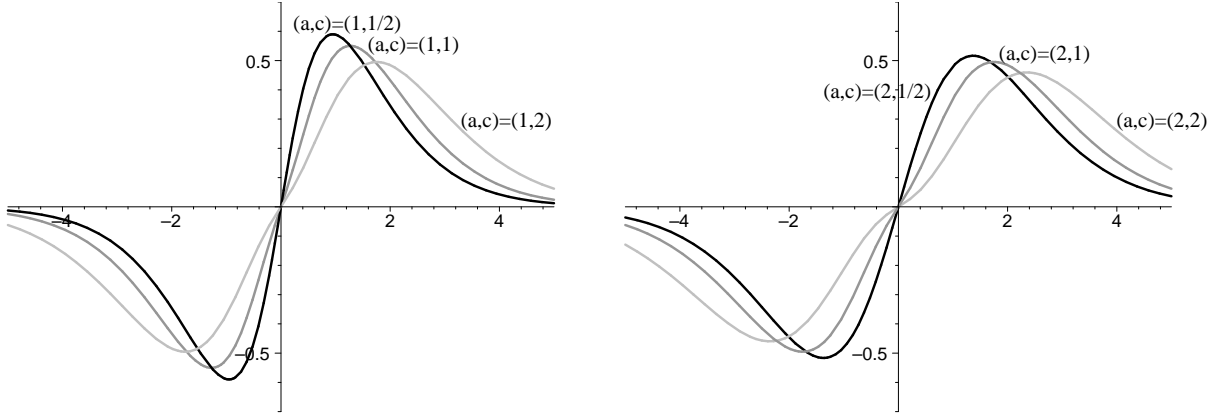
Figure 2. Plots of the wave functions $\psi_n^{(a,c)}(x)$ for $n = 1$, for $a = 1$ (left figure) and for $a = 2$ (right figure). The values of c are again $c = 1/2$ (black), $c = 1$ (dark gray) and $c = 2$ (light gray).

$a = 1$ and for $a = 2$, for some values of c. For $c = 1/2$ (undeformed algebra), one observes for the ground state wave function a shape similar to the Gaussian function, but with increasing variance as a increases. When c increases, the bell shapes are deformed in the middle, and the position probability decreases around the origin. The deformations for $n > 0$ follow a similar pattern.

In a similar way, the spectrum and eigenvalues of the momentum operator $\hat{p}$ can be determined. Denoting the formal eigenvector of $\hat{p}$, for the eigenvalue p, by

$$\bar{v}(p) = \sum_{n=0}^{\infty} \bar{A}_n(p) \, |a, n\rangle,$$

the equation $\hat{p}\bar{v}(p) = p\bar{v}(p)$ leads, using (14), to a set of recurrence relations similar to (17)–(18). The solution to these equations is the same as before, up to a multiple of i. So one finds that the formal eigenvectors of $\hat{p}$ are given by

$$\bar{v}(p) = \sum_{n=0}^{\infty} i^n \psi_n^{(a,c)}(p) \, |a, n\rangle,$$

where the functions $\psi_n^{(a,c)}$ are the same as in Theorem 1. The spectrum of $\hat{p}$ is $\mathbb{R}$.

5   Limits and special cases

For $c = 1/2$ we have that $\gamma = 0$ and hence the algebra $\mathfrak{su}(1,1)_\gamma$ is undeformed. So for $c = 1/2$, the position wave functions $\psi_n^{(a,c)}(x)$ should reduce to the functions $\phi_n^{(a)}(x)$ of Section 2. This can indeed be verified explicitly, although it follows already from the fact that putting $c = 1/2$ in the weight function (27) reduces to the corresponding weight function of Meixner–Pollaczek polynomials.

Explicitly, let us first consider the case with an even index, i.e. expression (28) for $\psi_{2n}^{(a,1/2)}(x)$. For the Gamma functions in the denominator of (28), of the form $\Gamma(n+a)\Gamma(n+a+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(n+1)$, one can use the famous multiplication formula

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

For the function $S_n(x^2; a, 0, 1/2)$ in the numerator of (28), one uses

$$
{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a, a+1/2 \end{matrix}; 1\right) = {}_2F_1\left(\begin{matrix} -2n, a+ix \\ 2a \end{matrix}; 2\right),
\tag{31}
$$

and this last expression ${}_2F_1$ yields the Meixner–Pollaczek polynomial in (5). So one finds indeed:

$$\psi_{2n}^{(a,1/2)}(x) = \phi_{2n}^{(a)}(x).$$

Note that the identity (31) is obtained by first applying a Thomae–Weber–Erdelyi transformation on the ${}_3F_2$ [30, Appendix] (sometimes called a Shephard transformation [1, Corollary 3.3.4]):

$$
{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a, a+1/2 \end{matrix}; 1\right) = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}\, {}_3F_2\left(\begin{matrix} -n, ix, -ix \\ a, 1/2 \end{matrix}; 1\right),
$$

and then one can turn this last ${}_3F_2$ into a ${}_2F_1$ by [17, equation (39)].

For an odd index, one uses expression (29) for $\psi_{2n+1}^{(a,1/2)}(x)$. The computation is similar, the main difference coming from the appearance of $S_n(x^2; a, 1, 1/2)$ in the numerator of (29). Here, one can use

$$
{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+1, a+1/2 \end{matrix}; 1\right) = \frac{ia}{x}\, {}_2F_1\left(\begin{matrix} -2n-1, a+ix \\ 2a \end{matrix}; 2\right).
\tag{32}
$$

The last identity (32) is again obtained by first applying a Thomae–Weber–Erdelyi transformation on the ${}_3F_2$ [30, Appendix]:

$$
{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+1, a+1/2 \end{matrix}; 1\right) = \frac{(\frac{3}{2})_n}{(a+\frac{1}{2})_n}\, {}_3F_2\left(\begin{matrix} -n, 1+ix, 1-ix \\ a+1, 3/2 \end{matrix}; 1\right),
$$

and then using [17, equation (40)].

A second interesting case is the limit $c \to +\infty$. Note from the action (12)–(13) that the operators

$$b^+ = \lim_{c\to\infty} \frac{J_+}{\sqrt{2c}}, \qquad b^- = \lim_{c\to\infty} \frac{J_-}{\sqrt{2c}},$$

have the same action on $\mathcal{H}_a$ as the paraboson oscillator creation and annihilation operators [16, equation (A7)]. Under this same limit, the position and momentum operators become

$$\hat{Q} = \lim_{c\to\infty} \frac{J_+ + J_-}{2\sqrt{c}}, \qquad \hat{P} = \lim_{c\to\infty} i\frac{J_+ - J_-}{2\sqrt{c}}
\tag{33}$$

and these operators satisfy, from (8) and (14),

$$[\hat{Q}, \hat{P}] = \lim_{c \to \infty} \left(\frac{i}{c} J_0 + \frac{i\gamma}{2c} R \right) = \lim_{c \to \infty} \left(\frac{i}{c} \left(\hat{H} + c - \frac{1}{2} \right) + \frac{i(2a-1)(2c-1)}{2c} R \right)$$
$$= i + (2a - 1)iR,$$

which is the common commutator in the paraboson case [25, 26, 32]. Hence one can also expect the wave functions $\psi_n^{(a,c)}(x)$ to tend to the paraboson wave functions $\Psi_n^{(a)}(x)$ [16, equation (A.11)] when c tends to ∞. In order to consider this limit, note that the position operator has been divided by $\sqrt{c}$ in (33), so for its eigenvalue we should introduce a new variable by putting $x = \sqrt{c}\xi$. Then according to (30), we need to compute the limit

$$\lim_{c \to +\infty} c^{1/4} \psi_n^{(a,c)}(\sqrt{c}\xi).$$

Let us consider the even case $\psi_{2n}^{(a,c)}$ (the odd case is similar). For the polynomial part, (28) and (19) lead to the following limit:

$$\lim_{c \to \infty} {}_3F_2 \left(\begin{array}{c} -n, a + i\sqrt{c}\xi, a - i\sqrt{c}\xi \\ a, a + c \end{array} ; 1 \right) = {}_1F_1 \left(\begin{array}{c} -n \\ a \end{array} ; \xi^2 \right) = \frac{n!}{(a)_n} L_n^{(a-1)}(\xi^2),$$

where $L_n^{(a-1)}$ is a Laguerre polynomial [20]. Taking care of the other factors, one finds

$$\lim_{c \to \infty} c^{1/4} \psi_{2n}^{(a,c)}(\sqrt{c}\xi) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+a)}} |\xi|^{a-1/2} e^{-\xi^2/2} L_n^{(a-1)}(\xi^2).$$

So one finds indeed

$$\lim_{c \to +\infty} c^{1/4} \psi_n^{(a,c)}(\sqrt{c}\xi) = \Psi_n^{(a)}(\xi),$$

in terms of the paraboson wave functions [16, equation (A.11)].

Note that for $a = 1/2$ the paraboson wave functions reduce to the canonical oscillator wave functions, as the corresponding Laguerre polynomials become Hermite polynomials [16, Appendix]. So therefore, one can say that under the limit $(a,c) \to (1/2, +\infty)$ the new oscillator models introduced in this paper reduce to the canonical quantum oscillator.

6 Differential-reflection operator realization of $\mathfrak{su}(1,1)_\gamma$

Just as $\mathfrak{su}(1,1)$ has a differential operator realization, and a realization of the Hilbert space $\mathcal{H}_a$ [19], the analogue can be constructed for $\mathfrak{su}(1,1)_\gamma$ as well. The basis vectors can be realized as monomials in a variable z:

$$|a, 2n\rangle = \sqrt{\frac{(a)_n (a+c)_n}{(c)_n n!}} \, z^{2n}, \qquad |a, 2n+1\rangle = \sqrt{\frac{(a)_{n+1} (a+c)_n}{(c)_{n+1} n!}} \, z^{2n+1}. \tag{34}$$

Clearly, the 'abstract' reflection operator R is realized by the concrete reflection operator $\check{R}$ with action

$$\check{R} f(z) = f(-z).$$

Then it is a simple exercise to verify that the following differential-reflection operators satisfy (11)–(13) when acting on the monomials (34)

$$J_0 = z\frac{d}{dz} + a + c - \frac{1}{2}, \qquad J_- = \frac{d}{dz} + \left(c - \frac{1}{2}\right)\frac{1 - \check{R}}{z},$$

$$J_+ = z^2\frac{d}{dz} + 2az + \left(c - \frac{1}{2}\right)z(1 - \check{R}).$$

In order to characterize the Hilbert space $\mathcal{H}_a$ as a space of analytic functions, a description of the scalar product should be found in such a way that the monomials (34) are orthonormal. So far, we have not been able to construct this. Let us nevertheless mention that the position (and momentum) wave functions have a more explicit expression in this realization. Indeed,

$$v(x) = \sum_n \psi_n^{(a,c)}(x)\,|a,n\rangle = \sum_n \psi_{2n}^{(a,c)}(x)\,|a,2n\rangle + \sum_n \psi_{2n+1}^{(a,c)}(x)\,|a,2n+1\rangle. \qquad (35)$$

For the first sum, one finds

$$\sum_n \psi_{2n}^{(a,c)}(x)\,|a,2n\rangle = \sqrt{\frac{w(x)}{\Gamma(a)\Gamma(c)\Gamma(a+c)}}\sum_n \frac{S_n(x^2;a,0,c)}{(c)_n n!}\left(-z^2\right)^n$$

$$= \sqrt{\frac{w(x)}{\Gamma(a)\Gamma(c)\Gamma(a+c)}}\left(1 + z^2\right)^{-a+ix}{}_2F_1\left(\begin{array}{c}ix, c+ix \\ c\end{array}; -z^2\right),$$

by using [20, (9.3.14)]. In a similar way, the second sum of (35) equals

$$\sqrt{\frac{w(x)}{\Gamma(a)\Gamma(c)\Gamma(a+c)}}\frac{xz}{c}\left(1 + z^2\right)^{-a+ix}{}_2F_1\left(\begin{array}{c}1+ix, c+ix \\ 1+c\end{array}; -z^2\right).$$

7   A further deformation of $\mathfrak{su}(1,1)_\gamma$

The model in Section 4 has two parameters a and c: one coming from the representation label a, and one coming from the deformation parameter $\gamma = (2a-1)(2c-1)$. The position operator eigenfunctions are in terms of continuous dual Hahn polynomials $S_n(x^2;a,0,c)$ or $S_n(x^2;a,1,c)$, due to the relations (24)–(25). However, continuous dual Hahn polynomials $S_n(x^2;a,b,c)$ have in general three parameters a, b and c, and furthermore the relations (21)–(22) preceding (24)–(25) are indeed in terms of three parameters. So one may wonder whether a third deformation parameter b could be introduced in the algebraic relation (8). This is in fact the case; however, we shall see that it leads to 'unphysical' wave functions.

Assume that we define a new deformed algebra as a unital algebra with elements J_0, J_+, J_- and R subject to the relations of Definition 1, but with (8) replaced by:

$$[J_+, J_-] = -2J_0 - \gamma R - 4bJ_0 R,$$

where as before $\gamma = (2a-1)(2c-1)$. The actions of Proposition 2 on a representation space $\mathcal{H}_a$ can then be generalized to

$$J_0|a,n\rangle = \left(n + a + b + c - \frac{1}{2}\right)|a,n\rangle,$$

$$J_+|a,n\rangle = \begin{cases} \sqrt{(n+2a+2b)(n+2b+2c)}\,|a,n+1\rangle, & \text{if } n \text{ is even,} \\ \sqrt{(n+1)(n+2a+2c-1)}\,|a,n+1\rangle, & \text{if } n \text{ is odd,} \end{cases}$$

$$J_-|a,n\rangle = \begin{cases} \sqrt{n(n+2a+2c-2)}\,|a,n-1\rangle, & \text{if } n \text{ is even,} \\ \sqrt{(n+2a+2b-1)(n+2b+2c-1)}\,|a,n-1\rangle, & \text{if } n \text{ is odd,} \end{cases}$$

and the action of R is unchanged. We shall also assume that $a, c > 0$ and $b \geq 0$.

In this case, the expressions (14) can still be used, and thus a formal eigenvector of $\hat{q}$ for the eigenvalue x, of the form (16), leads to:

$$xA_{2n}(x) = \sqrt{(n+a+b)(n+b+c)}A_{2n+1}(x) + \sqrt{n(n+a+c-1)}A_{2n-1}(x),$$
$$xA_{2n+1}(x) = \sqrt{(n+1)(n+a+c)}A_{2n+2}(x) + \sqrt{(n+a+b)(n+b+c)}A_{2n}(x),$$

reminiscent of the more general equations (21)–(22). These equations can indeed be solved by taking

$$A_{2n}(x) = \frac{(-1)^n S_n(x^2 - b^2; a, b, c)}{\sqrt{\Gamma(n+a+b)\Gamma(n+b+c)\Gamma(n+a+c)n!}},$$
$$A_{2n+1}(x) = \frac{(-1)^n\, x\, S_n(x^2 - b^2; a, b+1, c)}{\sqrt{\Gamma(n+a+b+1)\Gamma(n+b+c+1)\Gamma(n+a+c)n!}}.$$

Although this is a formal solution, the appearance of $x^2 - b^2$ as an argument of the continuous dual Hahn polynomials spoils the orthogonality relation (20), and we would not obtain $\mathbb{R}$ as spectrum of the position operator (but rather the values for which $|x| > |b|$, plus possibly some discrete points according to the orthogonality [20, (9.3.3)], depending on the values of a, b, c). Because of the unphysical nature of the corresponding eigenfunctions, we will not consider this further.

8   Conclusion

The one-dimensional quantum harmonic oscillator is a central problem and model in quantum mechanics. The simplest and standard model, the non-relativistic quantum harmonic oscillator in the canonical approach, has an attractive solution for its wave functions of stationary states in terms of Hermite polynomials. The dynamical algebra of this standard oscillator (or "Hermite oscillator") is the usual Heisenberg algebra.

This standard model can be extended both for continuous and discrete measures (for the wave functions), and in both cases some elegant models with analytic solutions for the wave functions exist.

In the continuous case, there are two well-known ways of extending the Hermite oscillator: these are the Meixner–Pollaczek oscillator [19] and the paraboson oscillator [25]. Both extensions have the same equidistant energy spectrum, where the ground state energy is some positive value a instead of $1/2$ in the case of the Hermite oscillator. The dynamical algebra is quite different though. For the Meixner–Pollaczek oscillator, the Hamiltonian together with the position and momentum operator form a basis of the Lie algebra $\mathfrak{su}(1,1)$. The ground state energy a corresponds to the representation label (lowest weight) of a positive discrete series representation. For the paraboson oscillator, the position and momentum operators are considered as odd generators of the Lie superalgebra $\mathfrak{osp}(1|2)$. The ground state energy a is again a representation label for a unitary $\mathfrak{osp}(1|2)$ representation. Note that in this case the wave functions are given in terms of (generalized) Laguerre polynomials.

By constructing models for the quantum harmonic oscillator based upon a deformation of $\mathfrak{su}(1,1)$, we have been able to unify the previous well-known extensions in the continuous case. The algebra $\mathfrak{su}(1,1)_\gamma$ is an extension of $\mathfrak{su}(1,1)$ by a parity operator R, and involves a deformation parameter γ. The proposed models for the oscillator involve two parameters a and c: a is again a representation label, and c is a deformation label related to γ by $\gamma = (2a-1)(2c-1)$. The energy spectrum is again equidistant, with ground state energy equal to a. Our main result is that the stationary wave functions of such models have elegant closed form expressions in

terms of continuous dual Hahn polynomials. Some properties of these wave functions have been described in Section 4.

Clearly, the dynamical algebra for these new models is $\mathfrak{su}(1,1)_\gamma$. For $c = 1/2$, $\mathfrak{su}(1,1)_\gamma$ becomes $\mathfrak{su}(1,1)$ and also the wave functions for the new models reduce to those of the Meixner–Pollaczek oscillator. For $c \to \infty$, the algebra reduces to the paraboson algebra, and the wave functions become the known paraboson oscillator wave functions.

As far as potential applications to physical models are concerned, we can offer the argument that our proposed model has two parameters a and c (one more that the Meixner–Pollaczek oscillator and the paraboson oscillator, and two more than the Hermite oscillator). These parameters should be greater than or equal to $1/2$ in order to deal with physically acceptable wave functions. Having two parameters available in a mathematical model for the quantum oscillator, opens the way to more flexibility in applications. For possible applications (of the deformed algebra and its representations) in quantum field theory, see the ideas presented in [13].

Acknowledgments

References

[1] Andrews G.E., Askey R., Roy R., Special functions, *Encyclopedia of Mathematics and its Applications*, Vol. 71, Cambridge University Press, Cambridge, 1999.

[2] Atakishiyev M.N., Atakishiyev N.M., Klimyk A.U., On $su_q(1,1)$-models of quantum oscillator, *J. Math. Phys.* **47** (2006), 093502, 21 pages.

[3] Atakishiyev N.M., Pogosyan G.S., Vicent L.E., Wolf K.B., Finite two-dimensional oscillator. I. The Cartesian model, *J. Phys. A: Math. Gen.* **34** (2001), 9381–9398.

[4] Atakishiyev N.M., Pogosyan G.S., Vicent L.E., Wolf K.B., Finite two-dimensional oscillator. II. The radial model, *J. Phys. A: Math. Gen.* **34** (2001), 9399–9415.

[5] Atakishiyev N.M., Pogosyan G.S., Wolf K.B., Finite models of the oscillator, *Phys. Part. Nuclei* **36** (2005), 247–265.

[6] Atakishiyev N.M., Suslov S.K., Difference analogues of the harmonic oscillator, *Theoret. and Math. Phys.* **85** (1990), 1055–1062.

[7] Bailey W.N., Generalized hypergeometric series, *Cambridge Tracts in Mathematics and Mathematical Physics*, no. 32, Stechert-Hafner, Inc., New York, 1964.

[8] Bargmann V., Irreducible unitary representations of the Lorentz group, *Ann. of Math. (2)* **48** (1947), 568–640.

[9] Basu D., Wolf K.B., The unitary irreducible representations of SL(2, **R**) in all subgroup reductions, *J. Math. Phys.* **23** (1982), 189–205.

[10] Berezans'kiĭ Ju.M., Expansions in eigenfunctions of selfadjoint operators, *Translations of Mathematical Monographs*, Vol. 17, American Mathematical Society, Providence, R.I., 1968.

[11] Biedenharn L.C., The quantum group $SU_q(2)$ and a q-analogue of the boson operators, *J. Phys. A: Math. Gen.* **22** (1989), L873–L878.

[12] Groenevelt W., Koelink E., Meixner functions and polynomials related to Lie algebra representations, *J. Phys. A: Math. Gen.* **35** (2002), 65–85, math.CA/0109201.

[13] Horváthy P.A., Plyushchay M.S., Valenzuela M., Bosons, fermions and anyons in the plane, and supersymmetry, *Ann. Physics* **325** (2010), 1931–1975, arXiv:1001.0274.

[14] Ismail M.E.H., Classical and quantum orthogonal polynomials in one variable, *Encyclopedia of Mathematics and its Applications*, Vol. 98, Cambridge University Press, Cambridge, 2005.

[15] Ismail M.E.H., Letessier J., Valent G., Quadratic birth and death processes and associated continuous dual Hahn polynomials, *SIAM J. Math. Anal.* **20** (1989), 727–737.

[16] Jafarov E.I., Stoilova N.I., Van der Jeugt J., Finite oscillator models: the Hahn oscillator, *J. Phys. A: Math. Theor.* **44** (2011), 265203, 15 pages, arXiv:1101.5310.

[17] Jafarov E.I., Stoilova N.I., Van der Jeugt J., The $\mathfrak{su}(2)_\alpha$ Hahn oscillator and a discrete Fourier–Hahn transform, *J. Phys. A: Math. Theor.* **44** (2011), 355205, 18 pages, arXiv:1106.1083.

[18] Klimyk A.U., On position and momentum operators in the q-oscillator, *J. Phys. A: Math. Gen.* **38** (2005), 4447–4458.

[19] Klimyk A.U., The $su(1,1)$-models of quantum oscillator, *Ukr. J. Phys.* **51** (2006), 1019–1027.

[20] Koekoek R., Lesky P.A., Swarttouw R.F., Hypergeometric orthogonal polynomials and their q-analogues, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2010.

[21] Koelink H.T., Van Der Jeugt J., Convolutions for orthogonal polynomials from Lie and quantum algebra representations, *SIAM J. Math. Anal.* **29** (1998), 794–822, q-alg/9607010.

[22] Koornwinder T.H., Group theoretic interpretations of Askey's scheme of hypergeometric orthogonal polynomials, in Orthogonal Polynomials and their Applications (Segovia, 1986), *Lecture Notes in Math.*, Vol. 1329, Springer, Berlin, 1988, 46–72.

[23] Koornwinder T.H., Krawtchouk polynomials, a unification of two different group theoretic interpretations, *SIAM J. Math. Anal.* **13** (1982), 1011–1023.

[24] Macfarlane A.J., On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, *J. Phys. A: Math. Gen.* **22** (1989), 4581–4588.

[25] Ohnuki Y., Kamefuchi S., Quantum field theory and parastatistics, University of Tokyo Press, Tokyo, 1982.

[26] Plyushchay M.S., Deformed Heisenberg algebra with reflection, *Nuclear Phys. B* **491** (1997), 619–634, hep-th/9701091.

[27] Post S., Vinet L., Zhedanov A., Supersymmetric quantum mechanics with reflections, *J. Phys. A: Math. Theor.* **44** (2011), 435301, 15 pages, arXiv:1107.5844.

[28] Regniers G., Van der Jeugt J., Wigner quantization of some one-dimensional Hamiltonians, *J. Math. Phys.* **51** (2010), 123515, 21 pages, arXiv:1011.2305.

[29] Slater L.J., Generalized hypergeometric functions, Cambridge University Press, Cambridge, 1966.

[30] Srinivasa Rao K., Van der Jeugt J., Raynal J., Jagannathan R., Rajeswari V., Group theoretical basis for the terminating ${}_3F_2(1)$ series, *J. Phys. A: Math. Gen.* **25** (1992), 861–876.

[31] Sun C.P., Fu H.C., The q-deformed boson realisation of the quantum group $SU(n)_q$ and its representations, *J. Phys. A: Math. Gen.*, **22** (1989), L983–L986.

[32] Tsujimoto S., Vinet L., Zhedanov A., From $sl_q(2)$ to a parabosonic Hopf algebra, *SIGMA* **7** (2011), 093, 13 pages, arXiv:1108.1603.

[33] Tsujimoto S., Vinet L., Zhedanov A., Jordan algebras and orthogonal polynomials, *J. Math. Phys.* **52** (2011), 103512, 8 pages, arXiv:1108.3531.