Duality for Hermitean Systems in $\mathbb{R}^{2n}$

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Abstract

In this paper, using the algebraic structure of the space of circulant (2×2) matrix, we characterize the dual of the (Frechet) space of germs of left Hermitean monogenic matrix functions in a compact set $\mathbf{E} \subset \mathbb{R}^{2n}$.

As an application we describe the dual space of the so-called h-monogenic functions satisfying simultaneously two Dirac type equations.

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1 Introduction

Hermitean Clifford analysis is a new and successful branch of standard (or orthogonal) Clifford analysis, centred, in its original setting, around the concept of h-monogenic functions, i.e. null solutions of a system of two complex mutually adjoint Dirac operators (see [7, 3, 5, 8, 9, 10, 11]). Notice that this system leads in a particular case to the system defining holomorphic functions in several complex variables and that it is also related to the Dolbeault complex (see [21]). For an approach of h-monogenic functions within the theory of $\mathbf{H}$-monogenic functions, i.e. mull solutions of a matrix Dirac operator, we refer to [1, 2, 11, 12, 13].

For convenience of the reader, let us now recall some definitions (see also Section 2).

Let $\mathbb{C}_{2n}$ be the complex Clifford algebra constructed over $\mathbb{C}^{2n}$ and let $\mathbf{CM}^{2\times 2}$ be the ring of circulant (2×2)-matrices over $\mathbb{C}_{2n}$.

Furthermore let $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ be the Hermitean Dirac operators in $\mathbb{R}^{2n}$ and let $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$ be the circulant (2×2)-matrix Dirac operator given by

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}.$$

If $\Omega \subset \mathbb{R}^{2n}$ is open, then a $\mathbb{C}_{2n}$-valued function g in Ω is said to be (left) h-monogenic in Ω if $\partial_{\underline{Z}} g = 0$, $\partial_{\underline{Z}^\dagger} g = 0$ in Ω. A $\mathbf{CM}^{2\times 2}$-valued function G_2^1 in Ω is called (left) $\mathbf{H}$-monogenic in Ω if $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} G_2^1 = 0$ in Ω.

For $\mathbf{E} \subset \mathbb{R}^{2n}$ compact, we may thus consider the spaces $h_l(\mathbf{E})$ and $\mathbf{H}_l(\mathbf{E})$ of, respectively, germs of h-monogenic and $\mathbf{H}$-monogenic functions on $\mathbf{E}$.

The aim of this paper is to characterize the duals $h_l^*(\mathbf{E})$ and $\mathbf{H}_l^*(\mathbf{E})$ of $h_l(\mathbf{E})$ and $\mathbf{H}_l(\mathbf{E})$. It tourns out that a characterization of $h_l^*(\mathbf{E})$ may be obtained by making use of the one established for $\mathbf{H}_l^*(\mathbf{E})$.

Notice that the characterization obtained for $\mathbf{H}_l^*(\mathbf{E})$ (see Theorem 4) relies essentially upon a Cauchy-integral type formula for $\mathbf{H}$-monogenic functions and upon determining the set of annihilators of $\mathbf{H}$-monogenic functions (see Theorem 2).

Notice also that $h_l^*(\mathbf{E})$ is fully characterized by the space $\widetilde{\mathcal{H}}_r(\mathbf{E})$ of pairs $(\omega, \omega^\dagger)$ of functions satisfying the system

$$\begin{cases} \omega\partial_{\underline{Z}} \ + \omega^\dagger \partial_{\underline{Z}^\dagger} \ = \ 0 \text{ in } \mathbb{R}^{2n} \setminus \mathbf{E} \\ \omega\partial_{\underline{Z}^\dagger} = 0, \, \omega^\dagger \partial_{\underline{Z}} \ = 0 \text{ in } \mathbb{R}^{2n}. \end{cases}$$

(see Theorem 5).

The kernel $\operatorname{Ker} J$ of the surjective map $J : \widetilde{\mathcal{H}}_r(\mathbf{E}) \mapsto h_l^*(\mathbf{E})$ is described in Theorem 6. It determines at the same time the set of annihilators of h-monogenic functions in bounded domains of $\mathbb{R}^{2n}$. In such a way, a new method is developed in characterizing annihilators of harmonic fields. It is different from the one presented in [17] in terms of differential forms.

A further step is thus made in studying dual spaces of spaces of null solutions to first order systems in Euclidean space which generalize generalize the classical Cauchy-Riemann system in the plane.

As is well known, the classical result of Köthe in [19] states that the dual of the space of germs of holomorphic functions on a compact set $\mathbf{E}$ in $\mathbb{R}^2$, endowed with the inductive limit topology, is isomorphic to the space of holomorphic functions on $\mathbb{R}^2 \setminus \mathbf{E}$. In [15] a generalization to $\mathbb{R}^3$ is proved concerning harmonic vector fields thus introducing at the same time the concept of pairs of harmonic fields (see also [18]). A multidimensional extension was established for harmonic differential forms in [16] and [4]. let us recall that in [4] were obtained within the framework of standard (real) Clifford analysis. For a complex Clifford analysis context, we refer to [22] where the author characterizes the dual and the bidual of the space of left regular functions in convenient domains. In [6] the dual of the space of k-monogenic functions was studied. Some related results in the case of Clifford hyperfunctions may be found in [20].

2 Preliminaries

Let $(e_1, \dots, e_m)$ be an orthonormal basis of the Euclidean space $\mathbb{R}^m$. Consider the complex Clifford algebra $\mathbb{C}_m$ constructed over $\mathbb{R}^m$. The non-commutative multiplication in $\mathbb{C}_m$ is governed by the rules:

$$\begin{aligned} e_j^2 &= -1, \qquad & j &= 1, \dots, m, \\ e_j e_k + e_k e_j &= 0, \qquad & j &\neq k. \end{aligned}$$

The Clifford algebra $\mathbb{C}_m$ is generated additively by elements of the form

$$e_A = e_{j_1} \dots e_{j_k},$$

where $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$ is such that $j_1 < \dots < j_k$, and so the dimension of $\mathbb{C}_m$ is 2^m. For $A = \emptyset$, one puts $e_\emptyset = 1$, the identity element.
Any Clifford number $a \in \mathbb{C}_m$ may thus be written as

$$a = \sum_A a_A e_A, \; a_A \in \mathbb{C}.$$

Any $a \in \mathbb{C}_m$ may also be written as

$$a = \sum_{k=0}^{m} [a]_k,$$

where $[a]_k = \sum_{|A|=k} a_A e_A$ is the so-called k-vector part of a.

The Hermitian conjugate $a^\dagger$ of a is defined by

$$a^\dagger = \sum_A a_A^c \overline{e}_A, \quad |A| = k,$$

where the bar denotes the usual real Clifford algebra conjugation, i.e., the main anti-involution for which $\overline{e}_j = -e_j$ and a_A^c is the standard complex conjugation.

The Hermitean conjugation also leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$, namely

$$(a, b) = [a^\dagger \, b]_0 \;\text{ and }\; |a| = \sqrt{[a^\dagger \, a]_0}.$$

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{C}_m$ by identifying $(x_1, \dots, x_m) \in \mathbb{R}^m$ with the real Clifford vector $\underline{X}$ given by

$$\underline{X} = \sum_{j=1}^{m} e_j x_j.$$

The functions under consideration are defined on an open subset of $\mathbb{R}^m$ and take values in the Clifford algebra $\mathbb{C}_m$. They are of the form

$$g = \sum_A g_A e_A,$$

where the functions g_A are $\mathbb{C}$-valued.

Whenever a property such as continuity, differentiability, etc. is ascribed to g it is understood that all the components g_A possess the cited property.

In the even dimensional case $m = 2n$ the real Clifford vector $\underline{X}$ and its twisted counterpart $\underline{X}|$ are given by

$$\underline{X} = \sum_{j=1}^{n} (e_{2j-1}x_{2j-1} + e_{2j}x_{2j})$$

$$\underline{X}| = \sum_{j=1}^{n} (e_{2j-1}x_{2j} - e_{2j}x_{2j-1}),$$

The Fischer dual of the vectors $\underline{X}$ and $\underline{X}|$, called the Dirac operators $\partial_{\underline{X}}$, $\partial_{\underline{X}|}$, are given by

$$\partial_{\underline{X}} = \sum_{j=1}^{n} (e_{2j-1}\partial_{x_{2j-1}} + e_{2j}\partial_{x_{2j}}),$$

$$\partial_{\underline{X}|} = \sum_{j=1}^{n} (e_{2j-1}\partial_{x_{2j}} - e_{2j}\partial_{x_{2j-1}}).$$

We notice that the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ anticommute and factorize the Laplacian, that is $-\partial_{\underline{X}}^2 = \Delta_{2n} = -\partial_{\underline{X}|}^2$. A continuously differentiable null solution g, of the Dirac operator $\partial_{\underline{X}}$ i.e., $\partial_{\underline{X}} g = 0$ (resp. $g\partial_{\underline{X}} = 0$), is called a left (resp. right) monogenic function. Thus, monogenicity with respect to that operator can be regarded as a refinement of harmonicity.

The Hermitean Clifford variables $\underline{Z}$ and $\underline{Z}^\dagger$ introduced in [13], are given by

$$\begin{aligned} \underline{Z} &= \tfrac{1}{2}\,(\underline{X} + i\,\underline{X}|) \\ \underline{Z}^\dagger &= -\tfrac{1}{2}\,(\underline{X} - i\,\underline{X}|) \end{aligned}$$

Finally, the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are derived from the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$:

$$\begin{aligned} \partial_{\underline{Z}^\dagger} &= \tfrac{1}{4}(\partial_{\underline{X}} + i\,\partial_{\underline{X}|}) \\ \partial_{\underline{Z}} &= -\tfrac{1}{4}(\partial_{\underline{X}} - i\,\partial_{\underline{X}|}) \end{aligned}$$

Observe further that the Hermitean vector variables are isotropic, i.e. $(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0$, which implies

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\underline{Z} \;=\; |\underline{Z}|^2 \;=\; |\underline{Z}^\dagger|^2 \;=\; |\underline{X}|^2 \;=\; |\underline{X}||^2.$$

Consequently

$$\begin{aligned} (\partial_{\underline{Z}})^2 &= (\partial_{\underline{Z}^\dagger})^2 = 0 \\ \Delta_{2n} &= 4\,(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}}). \end{aligned} \tag{1}$$

These facts motivate the following definition (see e.g. [11, 13, 14, 23]).

Definition 1 *A $\mathbb{C}_{2n}$-valued continuously differentiable function g in $\Omega \subset \mathbb{R}^{2n}$ open is called left Hermitean monogenic (or h-monogenic) in Ω, if and only if it satisfies in Ω the system*

$$\partial_{\underline{Z}} g = 0 = \partial_{\underline{Z}^\dagger} g.$$

or, equivalently,

$$\partial_{\underline{X}} g = 0 = \partial_{\underline{X}|} g.$$

In a similar way right h-monogenicity is defined. Functions that are both left and right h-monogenic are called two-sided h-monogenic

In other words, the h-monogenic functions (either left or right) are monogenic functions w.r.t. both Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$. Thus the h-monogenic functions constitute a subclass of the class of the monogenic functions.

The fundamental solutions of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ are respectively given by

$$E(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X}}{|\underline{X}|^{2n}}, \qquad E|(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X}|}{|\underline{X}|^{2n}}, \ \ \underline{X} \in \mathbb{R}^{2n} \setminus \{0\},$$

were σ_{2n} is the surface area of the unit sphere in $\mathbb{R}^{2n}$.

Starting from the pair of fundamental solutions $(E, E|)$ of the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$, we introduce the functions $\mathcal{E} = -(E + i\,E|)$ and $\mathcal{E}^\dagger = (E - i\,E|)$, or more explicitly:

$$\mathcal{E}(\underline{Z}) \;=\; \frac{2}{\sigma_{2n}} \frac{\underline{Z}}{|\underline{Z}|^{2n}} \qquad \text{and} \qquad \mathcal{E}^\dagger(\underline{Z}) \;=\; \frac{2}{\sigma_{2n}} \frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}}$$

Notice that

$$\lim_{|\underline{Z}|\to\infty} \mathcal{E}(\underline{Z}) \;=\; 0 \qquad \text{and} \qquad \lim_{|\underline{Z}|\to\infty} \mathcal{E}^\dagger(\underline{Z}) \;=\; 0.$$

Note that $\mathcal{E}$ and $\mathcal{E}^\dagger$ are not the fundamental solutions to the respective Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$.However introducing the particular circulant (2×2) matrices

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \;=\; \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} \;=\; \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\delta} \;=\; \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},$$

where δ is the Dirac delta distribution, one obtains that $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{\mathcal{E}}(\underline{Z}) = \boldsymbol{\delta}(\underline{Z})$, so that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$ in a matricial context.

Observe that $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$, $\boldsymbol{\mathcal{E}}$ and δ are (2×2)-circulant matrices with entries in $\mathbb{C}_{2n}$. We denote the set of (2×2)-circulant matrices over $\mathbb{C}_{2n}$ by $\mathbf{CM}^{2\times 2}$.

It is easily checked that for $\mathbf{A}, \mathbf{B} \in \mathbf{CM}^{2\times 2}$, $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}\,\mathbf{B}$ both belong to $\mathbf{CM}^{2\times 2}$. Moreover, defining for

$$\mathbf{A} \;=\; \left(\begin{array}{cc} a & b \\ b & a \end{array}\right) \in \mathbf{CM}^{2\times 2}$$

its conjugate $\mathbf{A}^\dagger$ in the classical way by

$$\mathbf{A}^\dagger \;=\; \left(\begin{array}{cc} a^\dagger & b^\dagger \\ b^\dagger & a^\dagger \end{array}\right),$$

then clearly $(\mathbf{A}^\dagger)^\dagger = \mathbf{A}$; $(\mathbf{A}\,a)^\dagger = a^\dagger \mathbf{A}^\dagger$ for $a \in \mathbb{C}_{2n}$; $(\mathbf{A}+\mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$ and $(\mathbf{A}\mathbf{B})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$. In other words, $\mathbf{CM}^{2\times 2}$ is a right linear associative algebra with involution over $\mathbb{C}_{2n}$.

Notice that $\mathbb{C}_{2n}$ may be embedded into $\mathbf{CM}^{2\times 2}$ by identifying $a \in \mathbb{C}_{2n}$ with

$$\left(\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right) \in \mathbf{CM}^{2\times 2}.$$

A norm on $\mathbf{CM}^{2\times 2}$ is defined by

$$\|\mathbf{A}\| = \max\{|a|, |b|\},\; \mathbf{A} \;=\; \left(\begin{array}{cc} a & b \\ b & a \end{array}\right).$$

The following observations lead to the introduction of so-called $\mathbf{H}$-monogenic funtions, which appear to be much suitable for establishing Cauchy-integral type formulae in the Hermitean setting (see also [13, 14]).

Definition 2 *A $\mathbf{CM}^{2\times 2}$-valued continuously differentiable function $\boldsymbol{G}_2^1$ in $\Omega \subset \mathbb{R}^{2n}$ open is said to be left (resp. right) $\mathbf{H}$-monogenic in Ω if*

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{G}_2^1 \;=\; \boldsymbol{O} \;\; (\textit{resp. } \boldsymbol{G}_2^1\,\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \;=\; \boldsymbol{O}) \textit{ in } \Omega.$$

Denote by $\mathbf{H}_l(\Omega)$ (resp. $\mathbf{H}_r(\Omega)$) the set of left (resp. right) $\mathbf{H}$-monogenic functions in Ω. Then clearly $\mathbf{H}_l(\Omega)$ (resp. $\mathbf{H}_r(\Omega)$) is a right (resp. a left) module over $\mathbf{CM}^{2\times 2}$.

Remarks

(1) Saying that

$$\boldsymbol{G}_2^1 = \left(\begin{array}{cc} g_1 & g_2 \\ g_2 & g_1 \end{array}\right)$$

is left $\mathbf{H}$-monogenic is equivalent with saying that the pair (g_1, g_2) satisfies in Ω the equations

$$\left\{\begin{array}{rcl} \partial_{\underline{Z}}\,[g_1] + \partial_{\underline{Z}^\dagger}[g_2] &=& 0 \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}\,[g_2] &=& 0. \end{array}\right.$$

Taking into account the relations (1) the equations above imply that (g_1, g_2) is a pair of $\mathbb{C}_{2n}$-valued harmonic functions in Ω.

(2) Let again

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \in \mathbf{H}_l(\Omega).$$

Then, the harmonicity of the pair (g_1, g_2) also follows from the following remarkable factorization of the Hermitean Laplacian

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix},$$

$$4\,\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right)^\dagger = 4\left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right)^\dagger \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \boldsymbol{\Delta}.$$

(3) Associate with the $\mathbb{C}_{2n}$-valued function g in Ω the diagonal matrix

$$\boldsymbol{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Then the (left) $\mathbf{H}$-monogenicity of G_0 is equivalent with the (left) h-monogenicity of g.

3 Hermitean integral representation formulas

In the following sections, it is understood that ω is bounded with a sufficiently smooth boundary Γ. Moreover we shall use the temporary notation $\Omega^+ = \Omega$, and $\Omega^- = \mathbb{R}^{2n} \setminus \overline{\Omega}$.

From now on we reserve the notations $\underline{Y}$ and $\underline{Y}|$ for Clifford vectors associated to points in $\Omega^\pm$. Their Hermitean counterparts are denoted by

$$\begin{aligned} \underline{V} &= \tfrac{1}{2}(\underline{Y} + i\,\underline{Y}|) \\ \underline{V}^\dagger &= -\tfrac{1}{2}(\underline{Y} - i\,\underline{Y}|). \end{aligned}$$

In [14] the following Hermitean Borel-Pompeiu formula was established for $\boldsymbol{G}_2^1 \in C^1(\overline{\Omega})$:

$$\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{G}_2^1(\underline{Y}) + \boldsymbol{\mathcal{T}}_\Omega \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}}(2i)^n \boldsymbol{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+, \\ 0, & \underline{Y} \in \Omega^-, \end{cases}$$

where $\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{G}_2^1$ is the Hermitean Cauchy transform given by

$$\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{G}_2^1(\underline{Y}) = \int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X})d\mathcal{H}^{2n-1},\ \underline{Y} \in \Omega^\pm.$$

Here $\mathcal{H}^{2n-1}$ stands for the $(2n-1)$-dimensional Hausdorff measure and $\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}$ is the (2×2)-circulant matrix

$$\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \underline{N} & -\underline{N}^\dagger \\ -\underline{N}^\dagger & \underline{N} \end{pmatrix},$$

where $\underline{N}$ and $\underline{N}^\dagger$ are (up to a constant factor) the Hermitean projections of the unit normal vector $\underline{n}(\underline{X})$ at the point $\underline{X} \in \Gamma$:

$$\underline{N} = -\frac{1}{4}(-1)^{n(n+1)/2}(\underline{n}(\underline{X}) - i\underline{n}|(\underline{X})),$$
$$\underline{N}^\dagger = -\frac{1}{4}(-1)^{n(n+1)/2}(\underline{n}(\underline{X}) + i\underline{n}|(\underline{X})).$$

Moreover, $\boldsymbol{\mathcal{T}}_\Omega$ is the Hermitean Thoedoresco transform, acting on $\boldsymbol{F}_2^1 \in C^1(\Omega)$ as follows:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{\Omega}}\boldsymbol{F}_2^1(\underline{Y}) = -\int_\Omega \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\boldsymbol{F}_2^1(\underline{X})dW(\underline{Z},\underline{Z}^\dagger),$$

where $dW(\underline{Z},\underline{Z}^\dagger)$ is the associated volume element given by

$$dV(\underline{X}) = (-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n dW(\underline{Z},\underline{Z}^\dagger).$$

For (left) $\mathbf{H}$-monogenic functions, the Hermitean Borel-Pompeiu formula reduces to the Cauchy integral formula

$$\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}}(2i)^n \boldsymbol{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+, \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

When Γ is sufficiently regular (e.g. an Ahlfors-David regular surface, see [1]) and $\boldsymbol{G}_2^1 \in \boldsymbol{\mathcal{C}}^{0,\nu}(\Gamma)$, $(0<\nu<1)$, then $\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{G}_2^1$ has boundary values in $\boldsymbol{\mathcal{C}}^{0,\nu}(\Gamma)$ and the following Hermitean Plemelj-Sokhotzki formula holds

$$\boldsymbol{\mathcal{C}}_\Gamma^\pm \boldsymbol{G}_2^1(\underline{Z}) = (-1)^{\frac{n(n+1)}{2}}(2i)^n\big(\pm\frac{1}{2}\boldsymbol{G}_2^1(\underline{Z}) + \frac{1}{2}\boldsymbol{\mathcal{H}}_\Gamma \boldsymbol{G}_2^1(\underline{Z})\big), \tag{2}$$

where the operator $\boldsymbol{\mathcal{H}}_\Gamma$ appearing in (2) is the Hermitean Hilbert transform

$$\boldsymbol{\mathcal{H}}_\Gamma \boldsymbol{G}_2^1(\underline{Y}) = \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n}\int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}(\boldsymbol{G}_2^1(\underline{X}) - \boldsymbol{G}_2^1(\underline{Y}))d\mathcal{H}^{2n-1} + \boldsymbol{G}_2^1(\underline{Y}).$$

for $\underline{Y} \in \Gamma$ (see [1, 13]).

The following theorem is a consequence of the above Plemelj-Sokhotzki formula. A detailed proof can be found in [1].

Theorem 1 *Let* $\boldsymbol{g}_2^1 \in \boldsymbol{\mathcal{C}}^{0,\nu}(\Gamma)$, $0<\nu<1$. *Then* *(a)* $\boldsymbol{g}_2^1$ *is the boundary value of a circulant matrix function* $\boldsymbol{G}_2^1 \in \mathbf{H}_l(\Omega^+)$ *if and only if*

$$\boldsymbol{\mathcal{H}}_\Gamma \boldsymbol{g}_2^1 = \boldsymbol{g}_2^1$$

(b) $\boldsymbol{g}_2^1$ *is the boundary value of a circulant matrix function* $\boldsymbol{G}_2^1 \in \mathbf{H}_l(\Omega^-)$ *vanishing at* ∞ *if and only if*

$$\boldsymbol{\mathcal{H}}_\Gamma \boldsymbol{g}_2^1 = -\boldsymbol{g}_2^1$$

One can state a quite analogous results for the case of right $\mathbf{H}$-monogenic functions.

The following theorem determines the annihilators of (right) $\mathbf{H}$-monogenic functions.

Theorem 2 *Let* $\boldsymbol{g}_2^1 \in \boldsymbol{\mathcal{C}}^{0,\nu}(\Gamma)$, $0 < \nu < 1$. *Then*

$$\int_\Gamma \boldsymbol{f}_2^1(\underline{X}) \mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{g}_2^1(\underline{X}) d\mathcal{H}^{2n-1} = \boldsymbol{O} \tag{3}$$

for all $\boldsymbol{f}_2^1 \in \mathbf{H}_r(\Omega^+)$ *which are continuous up to* Γ *if and only if* $\boldsymbol{g}_2^1$ *is the boundary value on* Γ *of a function* $\boldsymbol{G}_2^1 \in \mathbf{H}_l(\Omega^+)$

Proof: The if part follows directly from the Hermitean Cauchy Theorem (see [14]).

Now let $\underline{Y} \in \Omega^+$. Then the Hermitean Cauchy kernel $\boldsymbol{\mathcal{E}}$ is right $\mathbf{H}$-monogenic in Ω^+ and continuous in $\overline{\Omega^+}$. Consequently, by (3)

$$\int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{g}_2^1(\underline{X}) d\mathcal{H}^{2n-1} = \boldsymbol{O}$$

or equivalently

$$\boldsymbol{\mathcal{C}}_\Gamma \boldsymbol{g}_2^1(\underline{Y}) = \boldsymbol{O}, \ \underline{Y} \in \Omega^-.$$

In accordance with the Plemelj-Sokhotzki formula (2) we thus obtain

$$\boldsymbol{\mathcal{H}}_\Gamma \boldsymbol{g}_2^1 = \boldsymbol{g}_2^1.$$

The desired result then follows from Theorem 1. $\square$

A quite analogous result is the following

Theorem 3 *Let* $\boldsymbol{f}_2^1 \in \boldsymbol{\mathcal{C}}^{0,\nu}(\Gamma)$, $0 < \nu < 1$. *Then*

$$\int_\Gamma \boldsymbol{f}_2^1(\underline{X}) \mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{g}_2^1(\underline{X}) d\mathcal{H}^{2n-1} = \boldsymbol{O} \tag{4}$$

for all $\boldsymbol{g}_2^1 \in \mathbf{H}_l(\Omega^+)$ *which are continuous up to* Γ *if and only if* $\boldsymbol{f}_2^1$ *is the boundary value on* Γ *of a matrix function* $\boldsymbol{F}_2^1 \in \mathbf{H}_r(\Omega^+)$

4 Duality for H-monogenic functions

As we mentioned before the set $\mathbf{H}_l(\Omega)$ ($\mathbf{H}_l(\Omega)$) is a right (left) $\mathbf{CM}^{2\times 2}$- module and, equipped with the topology of uniform convergence on compacta, it becomes a right (left) Fréchet $\mathbf{CM}^{2\times 2}$- module.

Let $\mathbf{E} \subset \mathbb{R}^{2n}$ be a compact set such that $\mathbb{R}^{2n} \setminus \mathbf{E}$ is connected associate with it the space

$$\widetilde{\mathbf{H}_l}(\mathbf{E}) = \bigcup_{\Omega \supset \mathbf{E}} \mathbf{H}_l(\Omega), \ \ \Omega \text{ open set}$$

On $\widetilde{\mathbf{H}_l}(\mathbf{E})$ it is natural to define the equivalence relation $\sim$ where$\boldsymbol{g}_2^1 \sim \mathbf{f}_2^1$ if and only if $\boldsymbol{g}_2^1 = \mathbf{f}_2^1$ in some open set Ω' containing $\mathbf{E}$.

It makes thus sense to introduce the quotient right $\mathbf{CM}^{2\times 2}$- module

$$\mathbf{H}_l(\mathbf{E}) = \widetilde{\mathbf{H}_l}(\mathbf{E})/\sim .$$

The elements $\dot{\boldsymbol{g}}_2^1$ of this quotient space will be called germs.

We provide $\mathbf{H}_l(\mathbf{E})$ with the finest topology that makes all the natural mappings ρ_Ω, $\Omega \supset \mathbf{E}$ continuous, where ρ_Ω is defined by

$$\begin{aligned} \rho_\Omega : \mathbf{H}_l(\Omega) &\mapsto \mathbf{H}_l(\mathbf{E}) \\ \boldsymbol{g}_2^1 &\mapsto \dot{\boldsymbol{g}}_2^1 \end{aligned}$$

The quotient left $\mathbf{CM}^{2\times 2}$-module $\mathbf{H}_r(\mathbf{E})$ can be defined analogously.

In view of the foregoing considerations we have

Lemma 1 *Let Λ be a right $\mathbf{CM}^{2\times 2}$-linear functional on $\mathbf{H}_l(\mathbf{E})$. Then it is continuous if and only if the right $\mathbf{CM}^{2\times 2}$-linear functionals $\Lambda\, o\, \rho_\Omega$ are continuous on $\mathbf{H}_l(\Omega)$ for all $\Omega \supset \mathbf{E}$.*

Note that a right $\mathbf{CM}^{2\times 2}$-linear functional $\mathcal{L}$ is bounded on $\mathbf{H}_l(\Omega)$ if there exist $C > 0$ and a compact set $\mathbf{K} \subset \Omega$ such that for all $\boldsymbol{g}_2^1 \in \mathbf{H}_l(\Omega)$

$$\|\mathcal{L}(\boldsymbol{g}_2^1)\| \leq C \max_{\underline{X} \in \mathbf{K}} \|\boldsymbol{g}_2^1(\underline{X})\|.$$

The following theorem gives a unique integral representation for each right $\mathbf{CM}^{2\times 2}$-linear bounded functional on $\mathbf{H}_l(\mathbf{E})$.

Theorem 4 *Let Λ be a continuous right $\mathbf{CM}^{2\times 2}$-linear functional on $\mathbf{H}_l(\mathbf{E})$. Then, there exists a unique $\mathbf{CM}^{2\times 2}$-valued function $\boldsymbol{W}_2^1 \in \mathbf{H}_r(\mathbb{R}^{2n} \setminus \mathbf{E})$ with $\boldsymbol{W}_2^1(\infty) = \boldsymbol{O}$ such that for all $\dot{\boldsymbol{g}}_2^1 \in \mathbf{H}_l(\mathbf{E})$*

$$\Lambda \dot{\boldsymbol{g}}_2^1 = \int_{\Gamma_{\boldsymbol{g}}} \boldsymbol{W}_2^1(\underline{X}) \mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{g}_2^1(\underline{X}) d\mathcal{H}^{2n-1} \tag{5}$$

where $\boldsymbol{g}_2^1$ is any representative of the germ $\dot{\boldsymbol{g}}_2^1$ and $\Gamma_{\boldsymbol{g}}$ is a smooth surface surrounding $\mathbf{E}$.

Proof: Firstly it should be noted that in virtue of the Hermitean Stokes Theorem (see [14]) the right hand side in (5) does depend neither on the choice of the representative $\boldsymbol{g}_2^1$ nor on the choice of $\Gamma_{\boldsymbol{g}}$. Obviously, all the functionals given by (5) are continuous right $\mathbf{CM}^{2\times 2}$-linear functionals on $\mathbf{H}_l(\mathbf{E})$.

Next let Λ be a right $\mathbf{CM}^{2\times 2}$-linear functional on $\mathbf{H}_l(\mathbf{E})$ and let $\dot{\boldsymbol{g}}_2^1 \in \mathbf{H}_l(\mathbf{E})$.

Then $\boldsymbol{g}_2^1 \in \mathbf{H}_l(\Omega)$ for some $\Omega \supset \mathbf{E}$. Let now $\Gamma_{\boldsymbol{g}}$ be a closed smooth surface contained in Ω surrounding $\mathbf{E}$ and let Ω' be the open domain bounded by $\Gamma_{\boldsymbol{g}}$. Then, for each $\underline{Y} \in \Omega'$

$$\boldsymbol{g}_2^1(\underline{Y}) = \int\limits_{\Gamma_{\boldsymbol{g}}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{g}_2^1(\underline{X})d\mathcal{H}^{2n-1}$$

and

$$\Lambda\dot{\boldsymbol{g}}_2^1 = \Lambda\, o\, \rho_{\Omega'}\Big(\int\limits_{\Gamma_{\boldsymbol{g}}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{g}_2^1(\underline{X})d\mathcal{H}^{2n-1}\Big).$$

Since $\Lambda\, o\, \rho_{\Omega'}$ is a continuous right $\mathbf{CM}^{2\times 2}$-linear functional on $\mathbf{H}_l(\Omega')$, we have

$$\Lambda\dot{\boldsymbol{g}}_2^1 = \int\limits_{\Gamma_{\boldsymbol{g}}} \Lambda\, o\, \rho_{\Omega'}(\boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}))\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{g}_2^1(\underline{X})d\mathcal{H}^{2n-1}.$$

In order to verify that $\boldsymbol{W}_2^1$ given by

$$\boldsymbol{W}_2^1 = \Lambda\, o\, \rho_{\Omega'}(\boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}))$$

is right $\mathbf{H}$-monogenic in $\mathbb{R}^{2n} \setminus \mathbf{E}$ and vanishes at ∞, it is sufficient to use the continuity and the right $\mathbf{CM}^{2\times 2}$-linearity of $\Lambda\, o\, \rho_{\Omega'}$.

The uniqueness of the above representation follows easily from Theorem 2 and the Liouville Theorem. $\square$

5 Duality for h-monogenic functions

Let again Ω be a bounded open subset of $\mathbb{R}^{2n}$ and denote by $h_l(\Omega)$ ($h_r(\Omega)$) the right (left) $\mathbb{C}_{2n}$-module of all left (right) h-monogenic functions in Ω.

If $\mathbf{E} \subset \mathbb{R}^{2n}$ is compact, then in a quite similar way as in the foregoing section we introduce the right (left) $\mathbb{C}_{2n}$-module $h_l(\mathbf{E})$ ($h_r(\mathbf{E})$).

The aim of this section is to characterize the space $h_l^*(\mathbf{E})$ of bounded right $\mathbb{C}_{2n}$-linear functionals on $h_l(\mathbf{E})$ and thus by using the results of the previous section.

We start by the simple observation that $h_l(\Omega)$ is a subspace of the space $\mathrm{Harm}(\Omega)$ of all $\mathbb{C}_{2n}$-valued componentwise harmonic functions in Ω and this relation is of course also valid for the corresponding spaces of germs $h_l(\mathbf{E})$ and $\mathrm{Harm}(\mathbf{E})$.

By virtue of the Hahn-Banach Theorem each bounded (right) $\mathbb{C}_{2n}$-linear functional λ on $h_l(\mathbf{E})$ can be extended in a unique way to a bounded $\mathbb{C}_{2n}$-linear functional on $\mathrm{Harm}(\mathbf{E})$, which we still denoted by λ.

The next step is to construct the following right $\mathbf{CM}^{2\times 2}$-linear functional acting on $\mathbf{H}_l(\mathbf{E})$

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

where for $\boldsymbol{G}_2^1 \in \mathbf{H}_l(\mathbf{E})$, $\Lambda \boldsymbol{G}_2^1$ is defined by the matrix product $\Lambda \boldsymbol{G}_2^1$.

In particular, for $\dot{g} \in h_l(\mathbf{E})$ we have

$$\lambda\,\dot{g} = \frac{1}{2}\mathrm{tr}\big(\Lambda\,\dot{\boldsymbol{G}}_0\big), \tag{6}$$

where

$$\dot{\boldsymbol{G}}_0 = \begin{pmatrix} \dot{g} & 0 \\ 0 & \dot{g} \end{pmatrix}.$$

In view of Theorem 4

$$\Lambda\,\dot{\boldsymbol{G}}_0 = \int\limits_{\Gamma_{\boldsymbol{G}_0}} \boldsymbol{W}_2^1(\underline{X})\mathbf{N}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_0(\underline{X})d\mathcal{H}^{2n-1},$$

where $\boldsymbol{G}_0$ is, as before, any representative of the germ $\dot{\boldsymbol{G}}_0$, $\Gamma_{\boldsymbol{G}_0}$ is a sufficiently smooth surface surrounding $\mathbf{E}$ and

$$\boldsymbol{W}_2^1 = \Lambda\,(\dot{\boldsymbol{\mathcal{E}}}) = \begin{pmatrix} \lambda(\dot{\mathcal{E}}\ ) & \lambda(\dot{\mathcal{E}}^\dagger) \\ \lambda(\dot{\mathcal{E}}^\dagger) & \lambda(\dot{\mathcal{E}}\ ) \end{pmatrix}$$

By virtue of (6) and some simple computations we get:

$$\lambda\,\dot{g} = \int\limits_{\Gamma_g} (\omega\underline{N} - \omega^\dagger\underline{N}^\dagger)g\,d\mathcal{H}^{2n-1},$$

where $\omega = \frac{1}{2}\lambda(\dot{\mathcal{E}}\ )$ and $\omega^\dagger = \frac{1}{2}\lambda(\dot{\mathcal{E}}^\dagger)$ satisfy the system

$$\left\{\begin{array}{l} [\omega]\partial_{\underline{Z}} \ + [\omega^\dagger]\partial_{\underline{Z}^\dagger} \ = \ 0 \text{ in } \mathbb{R}^{2n}\setminus\mathbf{E} \\ [\omega]\partial_{\underline{Z}^\dagger} = 0,\ [\omega^\dagger]\partial_{\underline{Z}} \ = 0 \text{ in } \mathbb{R}^{2n}. \end{array}\right. \tag{7}$$

In view of the foregoing considerations we have

Theorem 5 *Let λ be a bounded right $\mathbb{C}_{2n}$-linear functional on $h_l(\mathbf{E})$. Then there exists a pair $(\omega, \omega^\dagger)$, vanishing at ∞ and satisfying the system (7), such that for all $\dot{g} \in h_l(\mathbf{E})$*

$$\lambda\,\dot{g} = \int\limits_{\Gamma_g} (\omega\underline{N} - \omega^\dagger\underline{N}^\dagger)g\,d\mathcal{H}^{2n-1}, \tag{8}$$

where g is any representative of the germ $\dot{g}$ and Γ_g is a sufficiently smooth surface surrounding $\mathbf{E}$.

From the above theorem, we may conclude that there exists a surjective $\mathbb{C}_{2n}$-linear map J between the space $h_l^*(\mathbf{E})$ and the space $\widetilde{\mathcal{H}}_r(\mathbf{E})$ of pairs $(\omega, \omega^\dagger)$ satisfying (7) with $\omega(\infty) = \omega^\dagger(\infty) = 0$.

The following theorem describes $\operatorname{Ker} J$.

Theorem 6 *If $(\omega_1, \omega_2) \in \widetilde{\mathcal{H}}_r(\mathbf{E})$ belongs to $\operatorname{Ker} J$, then ω_1 and ω_2 are both right h-monogenic functions in $\mathbb{R}^{2n} \setminus \mathbf{E}$.*

Proof: Suppose $(\omega_1, \omega_2) \in \operatorname{Ker} J$ and let $\mathbf{E}_\delta$ be an arbitrary δ-neighborhood of $\mathbf{E}$.

By abuse of notation, we continue to write (ω_1, ω_2) for a C^∞ extension to $\mathbb{R}^{2n}$ of the pair $(\omega_1, \omega_2)|_{\mathbb{R}^{2n}\setminus \mathbf{E}_\delta}$.

Next choose $\epsilon > \delta$ such that $\mathbf{E} \subset \mathbf{E}_\delta \subset \mathbf{E}_\epsilon$.

By assumption we have

$$\int\limits_{\partial \mathbf{E}_\epsilon} (\omega_1 \underline{N} - \omega_2 \underline{N}^\dagger) g \, d\mathcal{H}^{2n-1} = 0$$

for any (left) h-monogenic function g in $\mathbf{E}_\epsilon$.

By virtue of the Hermitean Stokes Theorem (see [14])

$$\int\limits_{\mathbf{E}_\epsilon} ([\omega_1]\partial_{\underline{Z}} + [\omega_2]\partial_{\underline{Z}^\dagger}) g \, dW = 0$$

for any (left) h-monogenic g in $\mathbf{E}_\epsilon$.

Since $g = \partial_{\underline{Z}^\dagger} f$ is (left) h-monogenic for any (left) monogenic f in $\mathbf{E}_\epsilon$, i.e. for any f satisfying $\partial_{\underline{X}} f = 0$ in $\mathbf{E}_\epsilon$, then

$$\int\limits_{\mathbf{E}_\epsilon} ([\omega_1]\partial_{\underline{Z}} + [\omega_2]\partial_{\underline{Z}^\dagger})(\partial_{\underline{Z}^\dagger} f) \, dW = 0$$

for any (left) monogenic f.

The fact that $[\omega_1]\partial_{\underline{Z}} + [\omega_2]\partial_{\underline{Z}^\dagger} = 0$ in $\mathbb{R}^{2n} \setminus \mathbf{E}_\epsilon$, yields

$$\int\limits_{\mathbf{E}_\epsilon} (([\omega_1]\partial_{\underline{Z}} + [\omega_2]\partial_{\underline{Z}^\dagger})\partial_{\underline{Z}^\dagger}) f \, dW = 0,$$

or equivalently

$$\int\limits_{\mathbf{E}_\epsilon} ([\omega_1]\partial_{\underline{Z}} \, \partial_{\underline{Z}^\dagger}) \, f \, dW = 0$$

for any (left) monogenic f in $\mathbf{E}_\epsilon$.

Consequently, there exists a test function $\psi \in \mathcal{D}(\mathbf{E}_\epsilon)$ such that $[\omega_1]\partial_{\underline{Z}} \, \partial_{\underline{Z}^\dagger} = \psi \partial_{\underline{X}}$

The relation $2\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} = \partial_{\underline{Z}}\partial_{\underline{X}}$ implies that

$$(2\psi - \omega_1\partial_{\underline{Z}})\partial_{\underline{X}} = 0.$$

By the Clifford Liouville Theorem we have that $2\psi = \omega_1\partial_{\underline{Z}}$, and hence that $\omega_1\partial_{\underline{Z}} = 0$ outside $\mathbf{E}_\epsilon$, which together with (7), implies the right h-monogenicity of ω_1 and ω_2 in $\mathbb{R}^{2n} \setminus \mathbf{E}_\epsilon$ and finally outside $\mathbf{E}$, by analytic continuation. $\square$

Remarks

1. The h-monogenicity in $\mathbb{R}^{2n} \setminus \mathbf{E}$ of ω_1 and ω_2 is a necessary condition for $(\omega_1, \omega_2) \in \operatorname{Ker} J$. The converse is in general not true. The study of the converse requires the consideration of h-monogenics of the form fI, I being the primitive idempotent and f an l-vector valued function; and it is related to Dolbeault complex. Only in case the Dolbeault cohomology vanishes, the obtained condition is also sufficient.

   In the matrix language, the above condition means that the entries of the right $\mathbf{H}$-monogenic matrix function in $\mathbb{R}^{2n} \setminus \mathbf{E}$

   $$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$$

   are right h-monogenic there, which is not true in general.

2. A careful look at (8) reveals that λ has two equivalent Euclidean representations, i.e.,

   $$\lambda(\dot{g}) = \int_{\Gamma_g} u(\underline{X})\underline{n}(\underline{X})\, g(\underline{X}) d\mathcal{H}^{2n-1} = \int_{\Gamma_g} u|(\underline{X})\underline{n}|(\underline{X})\, g(\underline{X}) d\mathcal{H}^{2n-1},$$

   where u and $u|$ are (up to a complex factor) equal to $\omega^\dagger - \omega$ and $\omega^\dagger + \omega$, respectively and satisfy $u\partial_{\underline{X}} = 0$ and $u|\partial_{\underline{X}|} = 0$ in $\mathbb{R}^{2n} \setminus \mathbf{E}$.

   It thus means that to any functional λ on $h_l(\mathbf{E})$ we can alternatively associate either a $\partial_{\underline{X}}$-monogenic function u in $\mathbb{R}^{2n} \setminus \mathbf{E}$ or a $\partial_{\underline{X}|}$-monogenic function $u|$ in $\mathbb{R}^{2n} \setminus \mathbf{E}$. The kernel of both these mappings is characterized by the space of right h-monogenic functions in $\mathbb{R}^{2n} \setminus \mathbf{E}$.

3. Note that in connection with Remark 2 it is possible to follow a real approach to determine the dual space of $h_l(\mathbf{E})$.

   Indeed, any right linear bounded functional on $h_l(\mathbf{E})$ can be extended to a right linear bounded functional on the space of germs of (left) monogenic functions in $\mathbf{E}$, the latter being completely characterized (see e.g. [4]) by the space of right monogenic functions in $\mathbb{R}^{2n} \setminus \mathbf{E}$ vanishing at infinity. Similar arguments to those used in the proof of Theorem 6 show that if a right monogenic function in $\mathbb{R}^{2n} \setminus \mathbf{E}$ vanishing at infinity annihilates each $\dot{g} \in h_l(\mathbf{E})$, then it is also right h-monogenic there.

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