A study of $(x(q+1), x; 2, q)$-minihypers

I. Landjev and L. Storme

June 19, 2009

Abstract. In this paper, we study the weighted $(x(q+1), x; 2, q)$-minihypers. These are weighted sets of $x(q+1)$ points in $\mathrm{PG}(2, q)$ intersecting every line in at least x points. We investigate the decomposability of these minihypers, and define a switching construction which associates to an $(x(q+1), x; 2, q)$-minihyper, with $x \leq q^2 - q$, not decomposable in the sum of another minihyper and a line, a $(j(q+1), j; 2, q)$-minihyper, where $j = q^2 - q - x$, again not decomposable into the sum of another minihyper and a line. We also characterize particular $(x(q+1), x; 2, q)$-minihypers, and give new examples. Additionally, we show that $(x(q+1), x; 2, q)$-minihypers can be described as rational sums of lines. In this way, this work continues the research on $(x(q+1), x; 2, q)$-minihypers by Hill and Ward [9], giving further results on these minihypers.

1 Introduction

Let $\mathcal{P}$ be the set of points of the projective geometry $\mathrm{PG}(t, q)$. A *multiset* in $\mathrm{PG}(t, q)$ is a mapping $\mathfrak{K}\colon \mathcal{P} \to \mathbb{N}$. This mapping is extended in a natural way to the subsets of $\mathcal{P}$: for any subset $\mathcal{Q}$ of $\mathcal{P}$, we set $\mathfrak{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathfrak{K}(P)$. The integer $\mathfrak{K}(P)$ is called the *multiplicity* of the point P and $n = \sum_{P \in \mathcal{P}} \mathfrak{K}(P)$ is called the *cardinality* of $\mathfrak{K}$. The *support* $\operatorname{supp} \mathfrak{K}$ of a multiset $\mathfrak{K}$ is the set of all points of positive multiplicity. A multiset $\mathfrak{K}$ is said to be *projective* if $\mathfrak{K}(P) \in \{0, 1\}$ for all points P. Projective multisets can be considered as sets of points by identifying them with their supports.

Conversely, given a finite set $\mathcal{Q}$ of points in $\mathrm{PG}(t, q)$, we define the characteristic multiset $\chi_{\mathcal{Q}}$ by:

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{if } P \notin \mathcal{Q}. \end{cases}$$

A multiset in $\mathrm{PG}(t, q)$ is called an $(n, w; t, q)$-*multiarc* if

(a) $\mathfrak{K}(\mathcal{P}) = n$;

(b) $\mathfrak{K}(H) \leq w$ for any hyperplane H;

(c) there exists a hyperplane H_0 with $\mathfrak{K}(H_0) = w$.

A multiset $\mathfrak{F}$ in $\mathrm{PG}(t,q)$ is called an $(f,m;t,q)$-*blocking multiset* or $(f,m;t,q)$-*minihyper*, if

(a) $\mathfrak{F}(\mathcal{P}) = f$;

(b) $\mathfrak{F}(H) \geq m > 0$ for any hyperplane H;

(c) there exists a hyperplane H_0 with $\mathfrak{F}(H_0) = m$.

To avoid trivialities, we impose $m > 0$ in the definition of an $(f,m;2,q)$-minihyper.

We can speak of (n,w)-*multiarcs* or (f,m)-*minihypers* if the geometry we consider is clear from the context. The characteristic multiset of a subspace of dimension u in $\mathrm{PG}(t,q)$ is a minihyper with parameters (v_{u+1}, v_u), where $v_u = \frac{q^u-1}{q-1}$.

An (f,m)-minihyper $\mathfrak{F}$ is called *minimal* if there exists no $(f-1,m)$-minihyper $\mathfrak{F}'$ with $\mathfrak{F}'(P) \leq \mathfrak{F}(P)$ for all points P.

The *sum* $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$ of two minihypers $\mathfrak{F}_1$ and $\mathfrak{F}_2$, where $\mathfrak{F}_i$ has parameters $(f_i, m_i; t, q)$, $i = 1,2$, is the $(f,m;t,q)$-minihyper, with $f = f_1 + f_2$, $m = m_1 + m_2$, and with the multiplicity of a point P in $\mathfrak{F}$ equal to the sum of its multiplicities in $\mathfrak{F}_1$ and $\mathfrak{F}_2$. As a particular example of a minihyper which is the sum of minihypers, we note that the sum of any x (not necessarily distinct) lines is a $(x(q+1), x; 2, q)$-minihyper.

An $(f,m;t,q)$-minihyper $\mathfrak{F}$ is called *indecomposable* or *irreducible* [7, Definition 2.5] if it cannot be represented as a sum $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$ of two other minihypers $\mathfrak{F}_i$, $i = 1,2$, where $\mathfrak{F}_i$ has parameters $(f_i, m_i; t, q)$, $i = 1,2$. Note again that this implies that $m_1, m_2 > 0$.

Within this introduction, we also define two important substructures of a projective plane $\mathrm{PG}(2,q)$.

A *hyperoval* K in $\mathrm{PG}(2,q)$, q even, is a set of $q+2$ points, no three collinear. The classical example of a hyperoval in $\mathrm{PG}(2,q)$, q even, is the union of a conic and its nucleus; such a hyperoval is called a *regular* hyperoval. For q even, $q \geq 16$, in $\mathrm{PG}(2,q)$, there exist *irregular* hyperovals, i.e., hyperovals which are not the union of a conic and its nucleus. We refer to [11] for the list of the known infinite classes of hyperovals in $\mathrm{PG}(2,q)$, q even.

A *maximal arc* K of $\mathrm{PG}(2,q)$ is a set of points intersecting every line in zero or n points. For $1 < n < q$, this necessarily implies that n is a divisor of q. Ball, Blokhuis, and Mazzocca proved that such maximal arcs cannot exist for q odd [1, 2]. For q even, Denniston proved the existence of a maximal arc in $\mathrm{PG}(2,2^h)$ intersecting every line in 2^i points, for every i satisfying $1 \leq i \leq h-1$ [4].

We also refer to the standard reference of Hirschfeld [10] for more information on hyperovals and maximal arcs in $\mathrm{PG}(2,q)$, q even.

Consider a multiset $\mathfrak{K}$ of $\mathrm{PG}(2,q)$; then to $\mathfrak{K}$ corresponds the *spectrum* $(a_i)_{i\geq 0}$ of $\mathfrak{K}$. The spectrum $(a_i)_{i\geq 0}$ of $\mathfrak{K}$ is the sequence of numbers a_i, $i \geq 0$, with a_i the number of lines of $\mathrm{PG}(2,q)$ intersecting $\mathfrak{K}$ in i points.

2 Minihypers with parameters $(x(q+1), x)$

In [9], Hill and Ward consider minihypers with parameters $(x(q+1), x)$, $x < q$, in $\mathrm{PG}(2,q)$. They try to characterize all indecomposable minihypers with the above parameters. Hill and Ward restrict the values of x to $x < q$ so that the associated codes are Griesmer codes. In [3], this problem is studied for $x = q$ in the terms of multiarcs. The following result describes divisibility properties of $(x(q+1), x)$-minihypers and is a straightforward generalization of Theorem 5.1 in [3].

Theorem 1. ([9, Theorem 18]) *Let $\mathfrak{F}$ be an $(x(q+1), x)$-minihyper in $\Pi = \mathrm{PG}(2,q)$, $q = p^m$, p prime, $m \geq 1$, with $x < q$, where p^f divides x. Then for each line L in Π, $\mathfrak{F}(L) \equiv x \pmod{p^{f+1}}$.*

This theorem implies two useful corollaries.

Corollary 2. ([9, Theorem 20]) *Every $(x(q+1), x)$-minihyper in $\mathrm{PG}(2,q)$, $q = p^m$, p prime, $m \geq 1$, with $x \leq q - \frac{q}{p}$, is a sum of x lines. In particular, if $\mathfrak{F}$ is an indecomposable $(x(q+1), x)$-minihyper, then $x > q - \frac{q}{p}$ or $\mathfrak{F}$ is a line.*

We wish to stress that the preceding corollary implies that the only indecomposable $(x(q+1), x; 2, q)$-minihypers in $\mathrm{PG}(2,q)$, q prime, with $x < q$, are the $(q+1, 1; 2, q)$-minihypers, so are the lines of $\mathrm{PG}(2,q)$, q prime.

Corollary 3. ([9, Theorem 23]) *Let $\mathfrak{K}$ be an indecomposable $(x(q+1), x)$-minihyper in $\mathrm{PG}(2,q)$, $q = p^m$, p prime, $m \geq 1$, for which $x \leq y < q$ and p^f divides y.*

- *(i) For each line L, $\mathfrak{K}(L) \leq x + q - p^{f+1}$.*
- *(ii) For each point P, $\mathfrak{K}(P) \leq x - p^{f+1}$.*
- *(iii) If $q - p + 1 \leq x \leq q - 1$ and $\mathfrak{K}(P) > x - 2p$, then $\mathfrak{K}(P)$ is divisible by $\frac{q}{p} - 1$.*

A challenging problem is that of the classification of all indecomposable $(x(q+1), x)$-minihypers, $x < q$, in projective planes over arbitrary finite fields. This problem has been solved for $\mathrm{PG}(2,8)$ and $\mathrm{PG}(2,9)$ by Hill and Ward in [9]. In the same paper, they describe five families of indecomposable $(x(q+1), x)$-minihypers in the planes of square order.

3 Ball's construction

A nice class of indecomposable $(x(q+1), x)$-minihypers was found by Ball in an unpublished note. The class of Ball's minihypers is explained below.

Suppose that $q = 2^m$ and consider two hyperovals $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in $\mathrm{PG}(2,q)$ that meet in $q+2-x$ points. Let $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 \pmod 2$, i.e. $\operatorname{supp}\mathfrak{F}$ is the symmetric difference of the supports of the two hyperovals $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Clearly, $|\mathfrak{F}| = 2x$ and $\mathfrak{F}(L) = 0, 2$, or 4 for every line L. If we dualize and regard 4-lines as 2-points, 2-lines as 1-points, and 0-lines as 0-points, we get an $(x(q+1), x)$-minihyper with spectrum

$$a_x = q^2+q+1-2x,\ a_{x+\frac{q}{2}} = 2x, a_i = 0,\ \text{for } i \neq x, x+\frac{q}{2}.$$

This construction works in a more general setting. Assume that we are given a multiset $\mathfrak{K}$ with $|\mathfrak{K}| = sx$ such that $\mathfrak{K}(L) = is$, $i \in \mathbb{N}$, for every line L, i.e. the multiplicity of every line is a multiple of s. We define a multiset $\mathfrak{F}$ in the dual plane in which lines of multiplicity is become points of multiplicity i. Then $\mathfrak{F}$ is an $(x(q+1), x)$-minihyper. We prove this as follows.

Let $(a_i)_{i\geq 0}$ be the spectrum of $\mathfrak{K}$. By counting the flags (P, L), $P \in L$, we obtain

$$sa_s + 2sa_{2s} + \cdots = sx(q+1).$$

This implies that $\sum ia_{is} = x(q+1)$, which means that $\mathfrak{F}$ has the desired cardinality. Let P be an arbitrary point. It has to be checked that $\sum_i ib_{is}(P) \geq x$, where $b_j = b_j(P)$ denotes the number of lines through P that have multiplicity j. Assume that $\mathfrak{K}(P) = \varepsilon$. Then

$$\begin{aligned} b_s + b_{2s} + \cdots &= q+1, \\ (s-\varepsilon)b_s + (2s-\varepsilon)b_{2s} + \cdots &= |\mathfrak{K}| - \varepsilon = sx - \varepsilon, \end{aligned}$$

which implies that

$$b_s + 2b_{2s} + \cdots = x + \varepsilon\frac{q}{s}.$$

This means that each line in the dual plane has multiplicity at least x since $\varepsilon q/s \geq 0$.

Example 1. In $\mathrm{PG}(2,q)$, q even, let $\mathfrak{M}$ be a maximal arc of degree 2^i and let L be an external line to $\mathfrak{M}$. Define $\mathfrak{K} = \chi_{\mathcal{P}\setminus L} - \mathfrak{M}$. Clearly, $\mathfrak{K}(M) = 0$, $q - 2^i$, or q for every line M of $\mathrm{PG}(2,q)$ and

$$|\mathfrak{K}| = q^2 - q(2^i - 1) - 2^i = 2^i \left(\frac{q^2}{2^i} - q + \frac{q}{2^i} - 1 \right).$$

If we set $s = 2^i$, $x = \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1$, using Ball's construction, we get a minihyper with parameters

$$\left((\frac{q^2}{2^i} - q + \frac{q}{2^i} - 1)(q+1), \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1; 2, q \right).$$

Remark 4. In the general case, we do not know whether the minihypers constructed in Example 1 are indecomposable. This can however be proven in some special cases. For instance, if we start with a maximal arc of degree $q/2$, $q \geq 4$, we arrive at a minihyper with

$$x = \frac{q^2}{q/2} - q + \frac{q}{q/2} - 1 = q + 1.$$

Note that in this case, the minihyper has $q+1$ 2-points, one 0-point, and $q^2 - 1$ 1-points. The 0-point corresponds to the fixed external line L to the maximal arc $\mathfrak{M}$, and the $q+1$ 2-points correspond to the other $q+1$ external lines to the maximal arc $\mathfrak{M}$. These $q+2$ external lines to the original maximal arc $\mathfrak{M}$ form a dual hyperoval, and so consequently, the 0-point and the $q+1$ 2-points form a hyperoval in the plane where the $((q+1)(q+1), q+1; 2, q)$-minihyper $\mathfrak{F}$ is embedded.

Assume that $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$, with $\mathfrak{F}_1$ an $(f_1, x_1; 2, q)$-minihyper and $\mathfrak{F}_2$ an $(f_2, x_2; 2, q)$-minihyper. Since there exists a 0-point to $\mathfrak{F}$, then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ also have a 0-point. Considering all lines of $\mathrm{PG}(2,q)$ through this 0-point, implies that $f_1 \geq x_1(q+1)$ and $f_2 \geq x_2(q+1)$, with $q+1 = x_1 + x_2$. Since $(q+1)(q+1) = f_1 + f_2 \geq (q+1)(q+1)$, necessarily $f_1 = x_1(q+1)$ and $f_2 = x_2(q+1)$. So $\mathfrak{F}_1$ is an $(x_1(q+1), x_1; 2, q)$-minihyper and $\mathfrak{F}_2$ an $(x_2(q+1), x_2; 2, q)$-minihyper.

Moreover, since $x_1 + x_2 = q + 1$, necessarily $x_1 \leq q/2$ or $x_2 \leq q/2$. Assume that $x_1 \leq q/2$. Then Corollary 2 implies that $\mathfrak{F}_1$ is the sum of x_1 lines. Let L_1 be one of these lines. Then reducing the weight of every point of $\mathfrak{F}_1$ on L_1 by one, a new $((x_1-1)(q+1), x_1 - 1; 2, q)$-minihyper $\mathfrak{F}_1'$ is obtained. But then $\mathfrak{F}_1' + \mathfrak{F}_2$ is a $(q(q+1), q; 2, q)$-minihyper. This is only possible if originally for L_1, $\mathfrak{F}(L_1) \geq q + 1 + q = 2q + 1$. But L_1 can contain at most two points of weight two, so $\mathfrak{F}(L_1) \leq q + 1 + 2$. This contradicts $q \geq 4$. So $\mathfrak{F}$ is indecomposable.

Example 2. Take a $(q+t,t)$-arc $\mathfrak{K}'$ of type $(0,2,t=2^i)$. These $(q+t,t)$-arcs of type $(0,2,t=2^i)$ have been studied in detail by Korchmáros and Mazzocca [12], and by Gács and Weiner [6]. For instance, a particular property of a $(q+t,t)$-arc $\mathfrak{K}'$ of type $(0,2,t=2^i)$ is that all the t-secants pass through a common point.

Let L be an external line to $\mathfrak{K}'$. Set $\mathfrak{K} = \chi_{\mathcal{P}\setminus L} - \mathfrak{K}'$. Then

$$|\mathfrak{K}| = q^2 - q - 2^i = 2\cdot(\frac{q^2}{2} - \frac{q}{2} - 2^{i-1}).$$

We have one 0-line and secants of multiplicity $q, q-2$, and $q-2^i$. By the Ball construction, we get a minihyper with $x = \frac{q^2}{2} - \frac{q}{2} - 2^{i-1}$.

Example 3. The complement of a unital. Consider $\mathrm{PG}(2,q)$, where q is a square. Let $\mathfrak{U}$ be a unital and set $\mathfrak{K} = \chi_{\mathcal{P}} - \mathfrak{U}$. Here

$$|\mathfrak{K}| = \sqrt{q}\,(q\sqrt{q} - q + \sqrt{q})$$

with line multiplicities $q-\sqrt{q}$ and q. By Ball's construction, we get a minihyper with parameters

$$((q\sqrt{q} - q + \sqrt{q})(q+1), q\sqrt{q} - q + \sqrt{q}; 2, q)\,.$$

In order to describe the fourth example, we need to define a linear blocking set in $\mathrm{PG}(2,q)$. We first of all introduce the notion of a Desarguesian spread.

By what is sometimes called *field reduction*, the points of $\mathrm{PG}(2,q)$, $q = p^h$, p prime, $h \geq 1$, correspond to $(h-1)$-dimensional subspaces of $\mathrm{PG}(3h-1,p)$, since a point of $\mathrm{PG}(2,q)$ is a 1-dimensional vector space over $\mathbb{F}_q$, and so an h-dimensional vector space over $\mathbb{F}_p$. In this way, we obtain a partition $\mathcal{D}$ of the point set of $\mathrm{PG}(3h-1,p)$ by $(h-1)$-dimensional subspaces. In general, a partition of the point set of a projective space by subspaces of a given dimension k is called a *spread*, or a *k-spread* if we want to specify the dimension. The spread we have obtained here is called a *Desarguesian spread*. Note that the Desarguesian spread satisfies the property that each subspace spanned by two spread elements is again partitioned by spread elements.

Definition 1. Let $\mathcal{D}$ be the Desarguesian $(h-1)$-spread of $\mathrm{PG}(3h-1,p)$, corresponding to the points of $\mathrm{PG}(2,p^h)$ and let U be a subset of $\mathrm{PG}(3h-1,p)$, then $\mathcal{B}(U) = \{R \in \mathcal{D} || U \cap R \neq \emptyset\}$. We identify the spread elements of $\mathcal{B}(U)$ with the corresponding points of $\mathrm{PG}(2,p^h)$.

Definition 2. We denote the $(h-1)$-dimensional spread element of $\mathrm{PG}(3h-1,p)$ corresponding to a point P of $\mathrm{PG}(2,p^h)$ by $\mathcal{S}(P)$. If U is a subspace of $\mathrm{PG}(2,p^h)$, then $\mathcal{S}(U) := \{\mathcal{S}(P) || P \in U\}$.

Analogously to the correspondence between the points of $\mathrm{PG}(2,q)$, $q = p^h$, and the elements of a Desarguesian spread $\mathcal{D}$ in $\mathrm{PG}(3h-1,p)$, we obtain the correspondence between the lines of $\mathrm{PG}(2,q)$ and the $(2h-1)$-dimensional subspaces of $\mathrm{PG}(3h-1,p)$ spanned by two elements of $\mathcal{D}$. With this in mind, it is clear that any subspace U of dimension at least h of $\mathrm{PG}(3h-1,p)$ defines a blocking set $\mathcal{B}(U)$ in $\mathrm{PG}(2,q)$. A blocking set constructed in this way is called a *linear blocking set*. Linear blocking sets were first introduced by Lunardon [14], although there a different approach is used. For more on the approach explained here, we refer to [13].

Example 4. The complement of a linear blocking set in $\mathrm{PG}(2,q)$, $q = p^h$, p prime, $h \geq 1$.

Such a linear blocking set $\mathcal{B}(U)$ in $\mathrm{PG}(2,q)$ intersects every line in 1 (mod p) points. Suppose that e is the maximal integer such that every line of $\mathrm{PG}(2,q)$ intersects $\mathcal{B}(U)$ in 1 (mod p^e) points. Then the complement of $\mathcal{B}(U)$ intersects every line in 0 (mod p^e) points. If $|\mathcal{B}(U)| = q+k+1$, then the complement has size $q^2 - k = p^e(q^2/p^e - k/p^e)$, so defines a $\{(q^2/p^e - k/p^e)(q+1), q^2/p^e - k/p^e; 2, q)$-minihyper.

4 Rational sums of lines

It has been noted that an $(x(q+1), x)$-minihyper is not necessarily a sum of lines. However this is always the case if we assume rational multiplicities for the points.

Theorem 5. *Let $\mathfrak{F}$ be an $(x(q+1), x)$-minihyper in* $\mathrm{PG}(2,q)$. *Then there exist lines $L_1, \ldots, L_s$ and positive rational numbers $c_1, \ldots, c_s$, such that*

$$\mathfrak{F} = c_1\chi_{L_1} + \cdots + c_s\chi_{L_s},$$

with $\sum_{i=1}^{s} c_i = x$.

Proof. Assume that there exists a line L with $\mathfrak{F}(L) \geq x+q$. Then $\mathfrak{F}' = \mathfrak{F} - \chi_L$ is an $((x-1)(q+1), x-1)$-minihyper and we get the result by induction on x. Hence, without loss of generality, we can assume that all lines have multiplicity at most $x+q-1$.

Let P be a point of multiplicity ε. Denote by a_i the number of lines through P that have multiplicity i. We have

$$\begin{aligned} \sum_{i=0}^{q-1} a_{x+i} &= q+1, \\ \sum_{i=0}^{q-1}(x+i-\varepsilon)a_{x+i} &= x(q+1)-\varepsilon, \end{aligned}$$

which implies

$$\sum_{i=0}^{q-1} \frac{i}{q} a_{x+i} = \varepsilon.$$

Therefore,

$$\mathfrak{F} = \sum_{i=0}^{q-1} \sum_{L:\mathfrak{F}(L)=x+i} \frac{i}{q}\chi_L,$$

which had to be proven. □

Example 5. Consider PG$(2,q)$, q even. Then this projective plane contains hyperovals.

Consider a dual hyperoval $\{L_1,\ldots,L_{q+2}\}$ in PG$(2,q)$, q even. Then the rational sum

$$\frac{1}{2}L_1 + \cdots + \frac{1}{2}L_{q+2}$$

is a $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$-minihyper in PG$(2,q)$, q even.

We know from Theorem 1 that if $\mathfrak{F}$ is an $(x(q+1),x)$-minihyper in $\Pi = \mathrm{PG}(2,q)$, $q=p^m$, p prime, $m \geq 1$, with $x < q$ where p^f divides x, then for each line L in Π, $\mathfrak{F}(L) \equiv x \pmod{p^{f+1}}$.

This allows us to describe more in detail the rational coefficients of the rational sum.

Corollary 6. *Let $\mathfrak{F}$ be an indecomposable $(x(q+1),x)$-minihyper in $\Pi = \mathrm{PG}(2,q)$, $q=p^m$, p prime, $m \geq 1$, with $x < q$ where p^f is the maximal power of p that divides x, then*

$$\begin{aligned} \mathfrak{F} &= \sum_{L^{(1)}} \frac{1}{p^{m-f-1}}\chi_{L^{(1)}} + \sum_{L^{(2)}} \frac{2}{p^{m-f-1}}\chi_{L^{(2)}} + \cdots + \\ & \sum_{L^{(p^{m-f-1}-1)}} \frac{p^{m-f-1}-1}{p^{m-f-1}}\chi_{L^{(p^{m-f-1}-1)}}, \end{aligned}$$

with $L^{(1)}$ *the lines intersecting* $\mathfrak{F}$ *in* $x+p^{f+1}$ *points, with* $L^{(2)}$ *the lines intersecting* $\mathfrak{F}$ *in* $x+2p^{f+1}$ *points, ..., and with* $L^{(p^{m-f-1}-1)}$ *the lines intersecting* $\mathfrak{F}$ *in* $x+(p^{m-f-1}-1)p^{f+1}$ *points.*

5 A construction for $x = \frac{3q}{4}$

If $\mathfrak{K}$ is a $(q^2+q+2, q+2)$-arc in $\mathrm{PG}(2,q)$, then $\mathfrak{K}(P) \le 2$ for all points P [3]. Thus the correspondence $\mathfrak{K} \leftrightarrow 2\chi_{\mathcal{P}} - \mathfrak{K}$ establishes a one-to-one correspondence between such arcs and $(q(q+1), q; 2, q)$-minihypers $\mathfrak{F}$ for which $\mathfrak{F}(P) \le 2$ for all points P. Thus the examples of [2, Section 2] provide $(q(q+1), q; 2, q)$-minihypers having this added restriction. In particular, the 3-line construction from [3, Theorem 2.3] leads to the following results.

Theorem 7. *Let* G *be a subgroup of the additive group* $(\mathbb{F}_q, +)$*. Let* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$ *be cosets of* G *in* $(\mathbb{F}_q, +)$ *with* $\mathcal{A}+\mathcal{B}+\mathcal{C} \neq G$*. Set*

$$\mathcal{A}' = -\mathcal{B}-\mathcal{C},\ \mathcal{B}' = -\mathcal{A}-\mathcal{C},\ \text{and } \mathcal{C}' = -\mathcal{A}-\mathcal{B}.$$

Define in the following way a multiset $\mathfrak{K}$ *in* $\mathrm{PG}(2,q)$*:*

2-points	$(a,0,-1)$	*for* $a \in \mathcal{A}'$,
	$(b,-1,0)$	*for* $b \in \mathcal{B}'$,
	$(c,1,1)$	*for* $c \in \mathcal{C}'$,
0-points	$(a,0,-1)$	*for* $a \in \mathcal{A}$,
	$(b,-1,0)$	*for* $b \in \mathcal{B}$,
	$(c,1,1)$	*for* $c \in \mathcal{C}$,
	$(1,0,0)$	
1-points	*the remaining points.*	

Then the multiset $\mathfrak{K}$ *is a* (q^2+q, q)*-minihyper. Conversely, every* (q^2+q, q)*-minihyper for which the 2-points lie on three lines meeting in a 0-point is isomorphic to* $\mathfrak{K}$*.*

Now consider the special case $q = 2^r$, $r \ge 2$. Let G be a subgroup of $(\mathbb{F}_q, +)$ of order 2^{r-1}. If $\mathfrak{K}$ is the minihyper defined in Theorem 7, the lines in $\mathrm{PG}(2,q)$ have the types described in Table 1 below.

	multiplicity	# of such lines	# of 2-pts	# of 1-pts	# of 0-pts
(A)	q	3	$q/2$	0	$q/2+1$
(B)	q	$3q^2/4$	1	$q-2$	2
(C)	q	$q-2$	0	q	1
(D)	$q+4$	$q^2/4$	3	$q-2$	0

Table 1

The points of PG$(2,q)$ can be divided into four classes with respect to the types of the lines they are incident with. The possible types of the points are described in Table 2.

	multiplicity	# of such points	# lines of type (A)	(B)	(C)	(D)
(α)	2	$3q/2$	1	$q/2$	0	$q/2$
(β)	1	q^2-2q	0	$3q/4$	1	$q/4$
(γ)	0	$3q/2$	1	q	0	0
(δ)	0	1	3	0	$q-2$	0

Table 2

Now we define a new multiset $\mathfrak{F}$ in the dual plane of PG$(2,q)$ by

$$\mathfrak{F}(l) = \begin{cases} q/4 & \text{if } l \text{ is of type (A);} \\ 1 & \text{if } l \text{ is of type (B);} \\ 0 & \text{if } l \text{ is of type (C) or (D).} \end{cases} \qquad (1)$$

Theorem 8. *The multiset $\mathfrak{F}$ is a 2-weight minihyper with parameters $(\frac{3q}{4}(q+1), \frac{3q}{4})$ and spectrum*

$$a_{3q/4} = q^2 - \frac{q}{2} + 1,\ a_{5q/4} = \frac{3q}{2},\ a_i = 0 \text{ for } i \neq \frac{3q}{4}, \frac{5q}{4}.$$

Proof. The proof is immediate from Table 2. Points of type $(\alpha), (\beta)$, and (δ) become lines of multiplicity $3q/4$, while lines of type (γ) become lines of multiplicity $5q/4$. Table 2 implies also the values of $a_{3q/4}$ and $a_{5q/4}$. $\square$

For $r = 2$, this construction gives a projective $(15,3)$-minihyper with 3- and 5-lines which is the complement of the hyperoval in PG$(2,4)$. For $r = 3$, the theorem gives a $(54,6)$-minihyper with 6- and 10-lines which is an "orphan" minihyper obtained in [9, Theorem 31]. For $r = 4$, the construction gives a $(204,12)$-minihyper with lines of multiplicity 12 and 20. For $r \geq 4$, these minihypers do not come from Ball's symmetric difference construction since they have points of multiplicity $2^{r-2} > 2$, while in Ball's construction all points have multiplicity $0, 1$, and 2.

This construction can be reversed: from a minihyper with three collinear points of multiplicity $q/4$ (the remaining points on the line being 0-points), $3q^2/4$ points of multiplicity 1 and $q^2/4 + q - 2$ points of multiplicity 0, one can obtain back the $(q(q+1), q)$-minihyper and the $(q^2+q+2, q+2)$-arc from the 3-line construction.

6 Indecomposable minihypers

In [9], Hill and Ward consider plane $(x(q+1), x)$-minihypers with $x < q$ only. The reason is that such minihypers give rise to Griesmer codes via the well-known construction of Hamada [5, 8]. If $\mathfrak{F}$ is an $(x(q+1), x; 2, q)$-minihyper and s is the maximal multiplicity of a point, then $s\chi_{\mathcal{P}} - \mathfrak{F}$ is an $(s(q^2+q+1) - x(q+1), s(q+1) - x; 2, q)$-arc. The code associated to this arc has parameters $[s(q^2+q+1) - x(q+1), 3, sq^2 - xq]$ and is easily checked to meet the Griesmer bound for $x < q$.

In what follows, we consider minimal, indecomposable $(x(q+1), x; 2, q)$-minihypers without imposing an explicit restriction on x. It turns out that the indecomposability requirement implies an upper bound on x.

Theorem 9. *Every $(x(q+1), x; 2, q)$-minihyper, with $x \geq q^2 - q + 1$, is decomposable.*

Proof. Assume that $\mathfrak{F}$ is an indecomposable $(x(q+1), x)$-minihyper. There exists a point P in PG$(2, q)$ that is of multiplicity 0. Otherwise $\mathfrak{F}$ can be represented as the sum of $\chi_{\mathcal{P}}$ and an $((x-q-1)(q+1)+q, x-q-1)$-minihyper, a contradiction to the indecomposability condition.

Note that all lines through the 0-point P are x-lines. Moreover the multiplicity of any line L cannot be larger than $x+q-1$ (otherwise, the minihyper is represented as the sum of χ_L and an $((x-1)(q+1), x-1)$-minihyper). Counting the flags (P', L'), with $P' \in L'$, we get

$$x(q+1)^2 \leq x(q+1) + q^2(x+q-1).$$

This implies $x \leq q^2 - q$. □

Theorem 10. *For an indecomposable $(x(q+1), x; 2, q)$-minihyper with $x \leq q^2 - q$, all points of PG$(2, q)$ have weight at most $q - 1$.*

Proof. Consider a point P of weight e, and now consider all the lines through P. Since they all have weight at most $x + q - 1$, we obtain the inequality

$$qe + x(q+1) \leq q(x+q-1) + x,$$

where we used the fact that every line has weight at most $x + q - 1$, and there is at least one x-secant through P. Namely, there is at least one point P' having weight zero, and the line PP' is an x-secant to the minihyper.

This leads to $e \leq q - 1$. □

Corollary 11. *There is no indecomposable* $(x(q+1), x; 2, q)$*-minihyper with* $x = q^2 - q$.

Proof. Assume otherwise and let $\mathfrak{F}$ be an indecomposable $((q^2-q)(q+1), q^2-q; 2, q)$-minihyper. Then by the counting argument from Theorem 9, we get that all points other than the 0-point P have multiplicity $q-1$ and $\mathfrak{F} = (q-1)\chi_{\mathcal{P}\setminus\{P\}}$. But $\chi_{\mathcal{P}\setminus\{P\}}$ itself is a $(q(q+1), q; 2, q)$-minihyper, a contradiction to our initial assumption. □

Remark 12. Using the same arguments, we can improve on the bound for x if we assume the existence of a certain number of 0-points. For example, if we assume that there are two 0-points, then the number of x-lines becomes at least $2q+1$ and the double counting argument gives

$$x(q+1)^2 \leq x(2q+1) + (q^2-q)(x+q-1),$$

whence $x \leq q^2 - 2q + 1$.

7 A switching construction

Consider an indecomposable $(x(q+1), x; 2, q)$-minihyper with $x \leq q^2-q-1$. Then all points have weight smaller than or equal to $q-1$ (Theorem 10), and for every line L, $\mathfrak{F}(L) \leq x+q-1$. Then this minihyper has at least one 0-point; see the proof of Theorem 9. Let us fix such a 0-point, P say. All the lines through P are of multiplicity x. Set $x = q^2-q-y$, $0 < y$, and define a new minihyper $\mathfrak{F}'$ in the following way:

$$\mathfrak{F}'(Q) = \begin{cases} q-1-\mathfrak{F}(Q) & \text{if } Q \neq P; \\ 0 & \text{if } Q = P. \end{cases} \tag{2}$$

We say that $\mathfrak{F}'$ is obtained from $\mathfrak{F}$ by using *switching with respect to* P. We have

$$\begin{aligned} |\mathfrak{F}'| &= \sum_{Q:Q\neq P}(q-1-\mathfrak{F}(Q)) = (q^2+q)(q-1) - \sum_{Q:Q\neq P}\mathfrak{F}(Q) & (3)\\ &= (q^2+q)(q-1) - x(q+1) = y(q+1). & (4) \end{aligned}$$

Furthermore, all lines through P have multiplicity $y = q(q-1)-x$. For the remaining lines L, one has

$$\mathfrak{F}'(L) \geq (q+1)(q-1) - (x+q-1) = q(q-1) - x = y.$$

Hence, $\mathfrak{F}'$ is a $(y(q+1), y)$-minihyper.

It is clear that switching $\mathfrak{F}'$ with respect to P, we again obtain $\mathfrak{F}$.

Lemma 13. *Let $\mathfrak{F}$ be an $(x(q+1), x; 2, q)$-minihyper, with $x \leq q^2 - q$, having a 0-point P and such that $\mathfrak{F}(L) \leq x + q - 1$ for every line L. Let $\mathfrak{F}'$ be the $(y(q+1), y; 2, q)$-minihyper, $y = q^2 - q - x$, obtained from $\mathfrak{F}$ by switching with respect to P. Then $\mathfrak{F}'(L) \leq y + q - 1$ for every line L. In particular, $\mathfrak{F}'$ is not a sum of lines.*

Proof. For every point Q, $\mathfrak{F}'(Q) \leq q - 1 - \mathfrak{F}(Q)$, so

$$\mathfrak{F}'(L) \leq (q+1)(q-1) - \mathfrak{F}(L) \leq q^2 - 1 - x = y + q - 1.$$

□

Theorem 14. *Every $(x(q+1), x; 2, q)$-minihyper, with $x \geq q^2 - 2q + \frac{q}{p}$, is decomposable.*

Proof. Assume otherwise and let $\mathfrak{F}$ be an indecomposable minihyper with parameters $(x(q+1), x)$, $x \geq q^2 - 2q + \frac{q}{p}$. Then $x \leq q^2 - q$ by Theorem 9. By the switching construction, we get a $(y(q+1), y)$-minihyper $\mathfrak{F}'$ with $y \leq (q^2 - q) - (q^2 - 2q + \frac{q}{p}) = q - \frac{q}{p}$. Since $\mathfrak{F}$ is indecomposable, $\mathfrak{F}(L) \leq x + q - 1$ and, by Lemma 13, $\mathfrak{F}'(L) \leq y + q - 1$. This contradicts Corollary 2. □

8 Two characterization results

8.1 A first characterization result

Consider $\mathrm{PG}(2, q)$, q even; then there are two known ways to construct $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$-minihypers. First of all, there is the sum $L_1 + \cdots + L_{q/2+1}$ of $q/2+1$ lines $L_1, \ldots, L_{q/2+1}$, and secondly there is the rational sum $\frac{1}{2}(L_1 + \cdots + L_{q+2})$, where $\{L_1, \ldots, L_{q+2}\}$ is a dual hyperoval of $\mathrm{PG}(2, q)$.

We now show that all $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$-minihypers arise from these two constructions.

Theorem 15. *Every $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$-minihyper $\mathfrak{K}$ in PG$(2, q)$, q even, is either:*

(1) a sum $L_1 + \cdots + L_{q/2+1}$ of $q/2+1$ lines $L_1, \ldots, L_{q/2+1}$, or

(2) a rational sum $\frac{1}{2}(L_1 + \cdots + L_{q+2})$, where $\{L_1, \ldots, L_{q+2}\}$ is a dual hyperoval.

Proof. Let $x = q/2 + 1$ and $y = 3q/4$. Assume first of all that $\mathfrak{K}$ is indecomposable. This implies in particular that the weight $\mathfrak{K}(L)$ of every line L is at most $q/2 + 1 + q - 1$. Then the following properties are valid.

Add a sum of $q/4-1$ lines to $\mathfrak{K}$ to obtain a $((3q/4)(q+1), 3q/4; 2, q)$-minihyper $\mathfrak{K}'$. Then Theorem 1 implies that for every line L, $\mathfrak{K}'(L) \equiv 3q/4 \pmod{q/2}$. Subtracting the contribution of the sum of the $q/4-1$ lines in $\mathfrak{K}' - \mathfrak{K}$, this implies that for every line L, $\mathfrak{K}(L) \equiv x \equiv q/2+1 \pmod{q/2}$. Since for every line L, $\mathfrak{K}(L) \equiv x \pmod{q/2}$, and since $\mathfrak{K}(L) \leq x+q-1$, this implies that $\mathfrak{K}(L) \in \{q/2+1, q+1\}$.

Again, since $x = q/2+1 \leq y = 3q/4 = q/2+q/4$ and since $q/4$ divides y, for every point P, $\mathfrak{K}(P) \leq q/2+1-q/2 = 1$ (Corollary 3). So $\mathfrak{K}$ only has points of weight one.

Let P be a point of $\mathfrak{K}$, let P belong to α $(q/2+1)$-secants and to β $(q+1)$-secants to $\mathfrak{K}$, then

$$\begin{cases} \alpha+\beta &= q+1, \\ \alpha \cdot \frac{q}{2} + \beta \cdot q &= (\frac{q}{2}+1)(q+1) - 1 = \frac{q^2}{2} + \frac{3q}{2}. \end{cases}$$

This implies that $\beta = 2$. So every point of $\mathfrak{K}$ belongs to two $(q+1)$-secants to $\mathfrak{K}$. This implies that there are in total $2(q/2+1)(q+1)/(q+1) = q+2$ different $(q+1)$-secants. Denote them by $L_1, \ldots, L_{q+2}$. Then since every point of $\mathfrak{K}$ belongs to two of the lines $L_1, \ldots, L_{q+2}$, the lines $L_1, \ldots, L_{q+2}$ necessarily define a dual hyperoval of $\mathrm{PG}(2,q)$, q even.

Assume now that $\mathfrak{K}$ is decomposable. Then the same arguments as in Remark 4 show that $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$, with $\mathfrak{K}_1$ an $(x_1(q+1), x_1; 2, q)$-minihyper and with $\mathfrak{K}_2$ an $(x_2(q+1), x_2; 2, q)$-minihyper with $x_1 + x_2 = x = q/2+1$. But since $x_1 \leq q/2$ and $x_2 \leq q/2$, $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are respectively a sum of x_1 and x_2 lines (Corollary 2), so $\mathfrak{K}$ is a sum of $x = q/2+1$ lines. □

8.2 A second characterization result

Consider again $\mathrm{PG}(2,q)$, q even. Then we have already three constructions for $((q/2+2)(q+1), q/2+2; 2, q)$-minihypers. The first construction is via a sum of $q/2+2$ lines, the second construction via a $(q+4, 4)$-arc of type $(0, 2, 4)$ (Example 2), and the third construction is via the sum of a line and a $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$-minihyper arising from a dual hyperoval in $\mathrm{PG}(2,q)$, q even (Example 1).

We now prove that these are the only three constructions for $((q/2+2)(q+1), q/2+2; 2, q)$-minihypers.

Theorem 16. *Every $((\frac{q}{2}+2)(q+1), \frac{q}{2}+2; 2, q)$-minihyper $\mathfrak{K}$ in $PG(2,q)$, q even, $q \geq 8$, is either:*

(1) a sum $L_1 + \cdots + L_{q/2+2}$ of $q/2+2$ lines $L_1, \ldots, L_{q/2+2}$, or

(2) a $((\frac{q}{2}+2)(q+1), \frac{q}{2}+2; 2, q)$*-minihyper constructed via a* $(q+4,4)$*-arc of type* $(0,2,4)$*, or*

(3) the sum of a line and a $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$*-minihyper arising from a dual hyperoval in* $PG(2,q)$*,* q *even.*

Proof. Assume first of all that $\mathfrak{K}$ is indecomposable, then again in particular, the weight of every line is at most $q/2+2+q-1$. Then the following properties are valid.

We again use that $x = q/2+2 \le y = 3q/4 = q/2+q/4$. Since $q/4$ divides y, for every point P, $\mathfrak{K}(P) \le q/2+2-q/2 = 2$ (Corollary 3). So $\mathfrak{K}$ only has points of weight one and two.

Using the same technique as in the proof of the preceding theorem, Theorem 1 implies that for every line L, $\mathfrak{K}(L) \equiv q/2+2 \pmod{q/2}$, and since $\mathfrak{K}(L) \le q/2+2+q-1$, this implies that $\mathfrak{K}(L) \in \{q/2+2, q+2\}$. We first determine the numbers $a_{q/2+2}$ and a_{q+2} of $(q/2+2)$-secants and $(q+2)$-secants. The standard equations are:

$$\begin{aligned} a_{q/2+2} + a_{q+2} &= q^2+q+1, \\ (q/2+2)a_{q/2+2} + (q+2)a_{q+2} &= (q/2+2)(q+1)^2, \end{aligned}$$

leading to $a_{q/2+2} = q^2-3$ and $a_{q+2} = q+4$.

The third standard equation is [9]:

$$(q/2+2)^2 a_{q/2+2} + (q+2)^2 a_{q+2} = (q+1)^2(q/2+2)^2 + q(p_1+4p_2),$$

where p_1 is the number of points in $\mathfrak{K}$ of weight one and p_2 the number of points in $\mathfrak{K}$ of weight two. This leads to $p_1+4p_2 = q^2/2+3q+4$.

But we also have the equations

$$\begin{aligned} p_0+p_1+p_2 &= q^2+q+1, \\ p_1+2p_2 &= (q/2+2)(q+1), \end{aligned}$$

leading to $p_0 = q^2/2-5q/4$, $p_1 = q^2/2+2q$, and $p_2 = q/4+1$.

We now check how the secants pass through a point of weight zero, one, or two. A 0-point only lies on $(q/2+2)$-secants. Suppose that a 1-point lies on $x_{q/2+2}$ different $(q/2+2)$-secants and on x_{q+2} different $(q+2)$-secants. Then

$$\begin{aligned} x_{q/2+2} + x_{q+2} &= q+1, \\ (q/2+2)x_{q/2+2} + (q+2)x_{q+2} &= (q/2+2)(q+1)+q, \end{aligned}$$

leading to $x_{q+2} = 2$ and $x_{q/2+2} = q - 1$. So a point of weight one lies on exactly two of the $(q+2)$-secants.

Suppose that a 2-point lies on $x'_{q/2+2}$ different $(q/2+2)$-secants and on x'_{q+2} different $(q+2)$-secants. Then

$$\begin{aligned} x'_{q/2+2} + x'_{q+2} &= q+1, \\ (q/2+2)x'_{q/2+2} + (q+2)x'_{q+2} &= (q/2+2)(q+1) + 2q, \end{aligned}$$

leading to $x'_{q+2} = 4$ and $x'_{q/2+2} = q - 3$.

A $(q+2)$-line is completely contained in $\mathfrak{K}$, and hence contains one point of weight two and q points of weight one.

This all leads to the conclusion that a point of weight zero lies on zero of the $(q+2)$-lines, a point of weight one lies on exactly two of the $(q+2)$-secants, and a point of weight two lies on exactly four of the $(q+2)$-secants. This implies that the $(q+2)$-secants form a dual $(q+4, 4)$-arc of type $(0, 2, 4)$. This shows that the minihyper arises from the construction of Example 2.

Assume now that $\mathfrak{K}$ is decomposable, then $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$, with $\mathfrak{K}_1$ an $(x_1(q+1), x_1; 2, q)$-minihyper and with $\mathfrak{K}_2$ an $(x_2(q+1), x_2; 2, q)$-minihyper with $x_1 + x_2 = x = q/2 + 2$. Assume that $x_1 \geq x_2$. If $x_2 \geq 2$, then $x_1 \leq q/2$ and $x_2 \leq q/2$, so $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are a sum of respectively x_1 and x_2 lines, implying that $\mathfrak{K}$ is a sum of $x = x_1 + x_2 = q/2 + 2$ lines. If $x_1 = q/2 + 1$ and $x_2 = 1$, then $\mathfrak{K}_1$ is as described in the preceding theorem and $\mathfrak{K}_2$ is a line. □

Acknowledgements. This research was supported by the Project *Combined algorithmic and theoretical study of combinatorial structures* between the Fund for Scientific Research Flanders-Belgium and the Bulgarian Academy of Sciences. This research is also part of the FWO-Flanders project nr. G.0317.06 *Linear codes and cryptography.*

The authors also wish to thank the referees for their suggestions for a clearer exposition of the results discussed in this article.

References

[1] S. Ball and A. Blokhuis, An easier proof of the maximal arcs conjecture. *Proc. Amer. Math. Soc.* **126** (1998), 3377–3380.

[2] S. Ball, A. Blokhuis, and F. Mazzocca, Maximal arcs in Desarguesian planes of odd order do not exist. *Combinatorica* **17** (1997), 31–41.

[3] S. Ball, R. Hill, I. Landjev, and H. Ward, On $(q^2+q+2, q+2)$-arcs in the projective plane $\mathrm{PG}(2,q)$. *Des. Codes Cryptogr.* **24** (2001), 205–224.

[4] R.H.F. Denniston, Some maximal arcs in finite projective planes. *J. Comb. Theory* **6** (1969), 317–319.

[5] S. Ferret and L. Storme, A classification result on weighted $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers. *Discrete Appl. Math.* **154** (2006), 277–293.

[6] A. Gács and Zs. Weiner, On $(q+t,t)$-arcs of type $(0,2,t)$. *Des. Codes Cryptogr.* **29** (2003), 131–139.

[7] N. Hamada, A characterization of some $[n,k,d;q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry. *Discrete Math.* **116** (1993), 229-268.

[8] N. Hamada and F. Tamari, On a geometrical method of construction of maximal t-linearly independent sets. *J. Combin. Theory Ser. A* **25** (1978), 14–28.

[9] R. Hill and H. Ward, A geometric approach to classifying Griesmer codes. *Des. Codes Cryptogr.* **44** (2007), 169–196.

[10] J.W.P. Hirschfeld, Projective geometries over finite fields. Oxford Mathematical Monographs. Oxford: Clarendon Press 1998.

[11] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: update 2001. *Developments in Mathematics* Vol. **3**, Kluwer Academic Publishers. *Finite Geometries*, Proceedings of the *Fourth Isle of Thorns Conference* (Chelwood Gate, July 16-21, 2000) (Eds. A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel and J.A. Thas), pp. 201-246.

[12] G. Korchmáros and F. Mazzocca, On $(q+t,t)$-arcs of type $(0,2,t)$ in a desarguesian plane of order q. *Math. Proc. Camb. Phil. Soc.* **108** (1990), 445–459.

[13] M. Lavrauw, Scattered spaces with respect to spreads, and eggs in finite projective spaces. PhD Dissertation, Eindhoven University of Technology, Eindhoven, 2001. viii+115 pp.

[14] G. Lunardon, Normal spreads. *Geom. Dedicata* **75** (1999), 245–261.

Ivan Landjev ivan@moi.math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
8, Acad. G. Bonchev, 1113 Sofia, BULGARIA
New Bulgarian University i.landjev@nbu.bg
21 Montevideo str., 1618 Sofia, BULGARIA

Leo Storme ls@cage.ugent.be
Department of Pure Mathematics and Computer Algebra,
Ghent University, Krijgslaan 281-S22, Ghent 9000, BELGIUM
http://cage.ugent.be/∼ls