12

Q_p-Spaces in Clifford Analysis

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1 The Unit Disk of $\mathbb{C}$

$\mathbf{Q_p}$-spaces were first introduced as interpolation spaces between two important function spaces on the unit disk of $\mathbb{C}$: the Bloch space and the Dirichlet space.

Let $\Delta = \{z : |z| < 1\}$ be the complex unit disk. The Bloch space is then defined by

$$\mathbf{B} = \{f : f \text{ analytic in } \Delta \text{ and } B(f) = \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty\}$$

while the Dirichlet space is given by

$$\mathbf{D} = \{f : f \text{ analytic in } \Delta \text{ and } \int_\Delta |f'(z)|^2 dxdy < \infty\}.$$

The group of direct conformal mappings of the unit disk is generated by rotations and by Möbius transformations of the form $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. With

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a conformal map of the disk φ we can associate a mapping of functions on the disk defined by

$$\Phi f = f \circ \varphi.$$

This is not the only possible transformation of holomorphic functions (see [7]), and the fact that in the higher dimensional analogue we construct here such a choice is impossible will influence the theory.

It is quite natural to introduce, for $a \in \Delta$, the weight function $g(z,a) = \ln |\frac{1-\overline{a}z}{a-z}| = -\ln |\varphi_a(z)|$ with logarithmic singularity in a, and the spaces

$$\mathbf{Q_p} = \{f : f \text{ holomorphic in } \Delta \text{ and } \sup_{a\in\Delta} \int_\Delta |f'(z)|^2 g^p(z,a)dxdy < \infty\}.$$

This way $\mathbf{D}$ equals $\mathbf{Q}_0$, while for $p \to \infty$ the mass of the weight function $g(a,z)$ is concentrated in a, and is proportional to $\sqrt{1-|a|^2}$, which indicates that $\mathbf{B}$ is the limiting space for $p \to +\infty$ of $\mathbf{Q}_p$. But it turns out that one must not take the limit $p \to +\infty$ to reach $\mathbf{B}$ [2]. More precisely the essential inclusions are:

$$\mathbf{D} \subset \mathbf{Q_p} \subset \mathbf{Q_q} \subset BMOA \quad 0 < p < q < 1 \quad [5]$$

$$\mathbf{Q_1} = BMOA \quad [5]$$

$$\mathbf{Q_p} = \mathbf{B} \quad \forall p > 1 \quad [2].$$

This means the spaces $\mathbf{Q_p}$ form a scale as desired and for special values of the scale parameter p these spaces are connected with other known and important spaces of analytic functions.

Another special property of these spaces is the conformal invariance under Möbius transformations.

Obviously both spaces $\mathbf{B}$ and $\mathbf{D}$ defined above are invariant under rotations for this kind of transformation, but it turns out that they are invariant for fractional transformations as well.

Indeed we can rewrite $B(f)$ as

$$B(f) = \sup_{a\in\Delta} (f \circ \varphi_a)'(0),$$

showing conformal invariance, and a simple change of variable in the defining integral of $\mathbf{D}$ also shows invariance.

There are several attempts to generalize these ideas and the corresponding approach to higher dimensions; independent of their method these approaches treat the case of the unit ball in $\mathbb{C}^n$ and not the case of the unit ball in $\mathbb{R}^n$. Basic ideas are to replace the derivative f' by the complex gradient of f and the measure $dxdy$ by a weighted measure $d\lambda(z) = \frac{dv}{(1-|z|^2)^{n+1}}$, where dv stands for the usual Lebesgue measure. Using an invariant Green's function some results similar to the complex one-dimensional case were proved. The most important results are that

$$\mathbf{Q_p} = \mathbf{B} \quad \text{for } 1 < p < \frac{n}{n-1} \quad \text{and } \mathbf{Q_1} = BMOA(\partial B),$$

where ∂B is the surface of the unit ball in $\mathbb{C}^n$. But, for $p \notin (\frac{n-1}{n}, \frac{n}{n-1})$ all $\mathbf{Q_p}$-spaces are trivial, i.e., only constant functions belong to $\mathbf{Q_p}$.
This is one of the reasons to look for other possibilities to generalize the complex one-dimensional ideas. Furthermore, using the $\mathbb{C}^n$-approach it is impossible in principle to consider $\mathbf{Q_p}$-spaces in odd real dimensions of the Euclidean space.
In this paper we study hypercomplex generalizations of $\mathbf{Q_p}$-spaces. We will follow two main lines for the generalization of the complex (one-dimensional) case.
In section 3 we consider the $\mathbf{Q_p}$-spaces as weighted spaces of Besov-type where the weight is defined by the fractional transformation and the function is measured by means of its "derivative". Instead of holomorphic functions in the unit disk we study monogenic functions $f : \mathbb{R}^n \mapsto \mathcal{C}\ell_{0,n-1}$ (i.e., solutions of generalized Cauchy-Riemann systems), which are a higher-dimensional generalization of holomorphic functions also applicable to the case of odd real dimensions of the Euclidean space. Important function classes like the solutions of the $div - rot$ system are included in the theory of monogenic functions.

With the generalized Cauchy-Riemann operator D, its adjoint $\bar{D}$, the hypercomplex Möbius transformation $\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$, and a modified fundamental solution g of the real Laplacian we consider generalized $\mathbf{Q_p}$-spaces defined by

$$\mathbf{Q_p} = \{f \in \ker D : \sup_{a \in B_1(0)} \int_B |\bar{D}f(x)|^2 (g(\varphi_a(x)))^p dx < \infty\}.$$

where $B_1(0)$ stands for the unit ball in $\mathbb{R}^n$. This definition seems to be natural because

- It has a deep structural analogy with the complex (one-dimensional) definition.
- All items used generalize definitions (analyticity, derivative, Möbius transformations and Green's functions) from the complex one-dimensional case.
- Generalized $\mathbf{Q_p}$-spaces have properties analogous to those of complex $\mathbf{Q_P}$-spaces.

To prove these analogous properties is the aim of this paper. We remark that for the case of functions $f : \mathbb{R}^4 \mapsto \mathbb{H}$ it is already known from [15] that $\bar{D}$ may be interpreted as a derivative. In [13] it is proved that $\bar{D}$ is a derivative for any real dimension. In section 3 we restrict ourselves to the case $n = 3$, the lowest non-commutative case, as a model case of general Clifford analysis. Moreover, we will identify the Clifford-Algebra $\mathcal{C}\ell_{0,2}$ with the skew field of quaternions. Thus we consider functions $f : \mathbb{R}^3 \mapsto \mathbb{H}$.

Beginning with Section 4 we generalize the complex $\mathbf{Q_p}$-spaces using a conformally invariant way. This approach can be extended to spaces of harmonic functions. The part on harmonic functions is also related to results of Leutwiler who considered in [14] spaces of harmonic functions with bounded mean oscillation. One of the main ideas is to use weighted Sobolev norms and that means we measure all the single partial derivatives of our functions and not the "derivative" of the (in section 3 monogenic) functions. It should be mentioned explicitly that both approaches generalize the complex one-dimensional case in the sense that the restriction to this case will describe the same class of functions. This is caused by the fact that for holomorphic functions the norm defined by the help of the complex derivative is equivalent to the usual Sobolev norm.

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2 Preliminaries

Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Consider the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ generated from $\mathbb{R}^n$ equipped with a negative inner product. Then we have the anti-commutation relationship $e_i e_j + e_j e_i = -2\delta_{ij} e_0, i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker delta symbol and $e_0 = 1$ is the identity of $\mathcal{C}\ell_{0,n}$. It may be observed that each element of the algebra can be represented in the form

$$a = \sum_A a_A e_A,$$

where a_A are real numbers and $e_A, A \subseteq \{1, \ldots, n\}$, with $e_A = e_{i_1} \ldots e_{i_k}$, $e_{\{i\}} = e_i, i = 1, \ldots, n$, and $e_\emptyset = e_0$, are the basis elements of $\mathcal{C}\ell_{0,n}$.

In what follows identify each element $\mathbf{x} = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ with the element

$$x = \sum_{k=1}^{n} x_k e_k$$

of the Clifford algebra. In this way the vector space $\mathbb{R}^n$ is embedded in $\mathcal{C}\ell_{0,n}$ and we will call these elements x of $\mathcal{C}\ell_{0,n}$ vectors.

By

$$\bar{a} = \sum_A a_A \bar{e}_A,$$

where $\bar{e}_A = \bar{e}_{i_k} \ldots \bar{e}_{i_1}, \bar{e}_j = -e_j, j = 1, \ldots, n$, we define a conjugate element. We also define reversion by

$$a^* = \sum_A a_A e_A^*,$$

where $e_A^* = e_{i_k} \ldots e_{i_1}$.

For C^1-functions defined on a domain $\Omega \subset \mathbb{R}^n$ we introduce a generalized Dirac operator by

$$D = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}.$$

Note that $-D^2 = \Delta$, where Δ is the Laplacian in $\mathbb{R}^n$.

A function $f : \Omega \mapsto \mathcal{C}\ell_{0,n}$ is said to be *left-monogenic* if it satisfies the equation $(Df)(x) = 0$ for each $x \in \Omega$.

In section 3 we will work in $\mathbb{H}$, the skew field of quaternions. As usual we identify $\mathbb{H}$ with $\mathcal{C}\ell_{0,2}$ and write $\{1, i, j, k\}$ instead of $\{e_0, e_1, e_2, e_1 e_2\}$. Points of $\mathbb{R}^3$ have coordinates (x_0, x_1, x_2), and instead of the Dirac operator we use the Cauchy-Riemann operator

$$D = \partial_0 + i\partial_1 + j\partial_2.$$

This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In this case $D\overline{D} = \overline{D}D = \Delta$, where $\overline{D} = \partial_0 - i\partial_1 - j\partial_2$ is the conjugate Cauchy-Riemann operator. Again an $\mathbb{H}$-valued function satisfying $Df = 0$ in a domain will be called monogenic, or left monogenic.

Using the fundamental solution $e(x) = \frac{1}{\omega} \frac{\bar{x}}{|x|^3}$ of D in $\mathbb{R}^3$ we introduce the Cauchy-type operator

$$(F_\Gamma u)(x) := \int_\Gamma e(x - y)\alpha(y)u(y)\, d\Gamma_y, x \notin \Gamma,$$

where $\alpha(y) = \sum_{k=0}^{2} e_k \alpha_k(y)$ is the outward pointing normal unit vector to Γ at the point y and ω stands for the surface area of the unit ball in $\mathbb{R}^3$.

Also, in what follows we will work in $B_1(0) \subset \mathbb{R}^3$, the unit ball in real three-dimensional space. Moreover, we will consider functions f defined on $B_1(0)$ with values in $\mathbb{H}$. The contents of this section follow the line and basic ideas of [12].

3 Definition of $\mathbf{Q_p}$-Spaces in $\mathbb{R}^3$

For $|a| < 1$ we will denote by

$$\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x, a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. Especially, we denote for all $p > 0$

$$g^p(x, a) = \frac{1}{4^p\pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p .$$

Let $f : B_1(0) \mapsto \mathbb{H}$ be a monogenic function. We will use, as in [12], the seminorms

- $B(f) = \sup_{x\in B_1(0)}(1 - |x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 g^p(x, a) dB_x$,

which lead to the following definitions:

Definition 3.1 *The spatial (or three-dimensional) Bloch space* $\mathbf{B}$ *is the right* $\mathbb{H}$*-module of all monogenic functions* $f : B_1(0) \mapsto \mathbb{H}$ *with* $B(f) < \infty$.

Definition 3.2 *The right* $\mathbb{H}$*-module of all quaternion-valued functions* f *defined on the unit ball, which are monogenic and satisfy* $Q_p(f) < \infty$*, is called* $\mathbf{Q_p}$*-space.*

Remark 3.1 *Because of the special structure of* $g(x, a)$ *the seminorms* $Q_p(f)$ *make sense for* $p < 3$ *only. Consequently, we will consider in this section* $\mathbf{Q_p}$*-spaces for* $p < 3$ *only. In subsection 3.2 we will describe another characterization of* $\mathbf{Q_p}$*-spaces which is equivalent with the definition under consideration for* $p < 3$ *and which makes sense for* $p \geq 3$ *also.*

Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighbourhood of the origin U_ϵ, with an arbitrary but fixed $\epsilon > 0$, then we can add the L_1-norm of f over U_ϵ to our seminorms and $\mathbf{B}$ as well as $\mathbf{Q_p}$ will become Banach spaces. Because this additional term is independent of p we will consider in the following only the spaces with the corresponding seminorm, but we have to keep in mind that all our results are also true in the case of the norm.

Definition 3.3 *The right* $\mathbb{H}$*-module of monogenic functions* $f : B_1(0) \mapsto \mathbb{H}$ *with*

$$\int_{B_1(0)} |\overline{D}f(x)|^2 dB_x < \infty,$$

is called spatial (or three-dimensional) Dirichlet space $\mathbf{D}$.

Remark 3.2 *Since $g(x,a)$ is non-negative in $B_1(0)$ we have, obviously,*

$$\mathbf{D} \subset \mathbf{Q_p}, \qquad 0 \leq p < 3.$$

3.1 Properties of $\mathbf{Q_p}$-spaces

First we show that the $\mathbf{Q_p}$-spaces form a range of Banach $\mathbb{H}$-modules (with our additional term added to the seminorm), connecting the spatial Dirichlet space with the spatial Bloch space. For doing this several lemmas are needed. Although these lemmas are only of technical nature we will at least state these results to show that the approach to $\mathbf{Q_p}$-spaces in higher dimensions which is sketched in this section is strongly based on properties of monogenic functions. From the properties of the Cauchy integral we obtain the following estimate.

Lemma 3.1 *Let f be monogenic in the unit ball. Then we have for all $r < 1$*

$$\int_{S_r(0)} |\overline{D}f(x)| dS_x \geq 4\pi r^2 |\overline{D}f(0)|,$$

where $S_r(0)$ is the surface of the ball $B_r(0)$ with centre at 0 and radius r.

Lemma 3.2 *Under the same conditions as in lemma 3.1 we have that for any fixed $R < 1$*

$$\int\limits_{B_R(0)} |\bar{D}f(x)|^2 dx \geq \frac{4\pi R^3}{3} |\overline{D}f(0)|^2$$

holds.

For the proof we have to use lemma 3.1 and the Cauchy-Schwarz inequality. For details see [12].

Proposition 3.1 *Let f be monogenic and $0 < p < 3$, then we have*

$$(1-|a|^2)^3 |\overline{D}f(a)|^2 \leq C_1 \int_{B_1(0)} |\overline{D}f(x)|^2 \left(\frac{1}{|\varphi_a(x)|} - 1 \right)^p dB_x, \qquad (1)$$

where the constant C_1 does not depend on a and f.

The inequality has the same structure as in the complex one-dimensional case. Only the exponent 3 on the left hand side shows how the real dimension of the space influences the estimate. To prove this proposition we need the previous lemmas, some geometrical properties of the Möbius transformation and the equality

$$\frac{1-|\varphi_a(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{|1-\bar{a}x|^2} \qquad (2)$$

which links properties of the (special) Möbius transformation φ_a with the more general weight function $1-|x|^2$. This equality generalizes in a direct way the corresponding property from the complex one-dimensional case. Considering on both sides of (1) the supremum we obtain the following corollary.

Corollary 3.3 *For $0 < p < 3$ we have $\mathbf{Q_p} \subset \mathbf{B}$.*

This corollary means that all $\mathbf{Q_p}$-spaces are subspaces of the Bloch space. We recall that in the complex one-dimensional case all $\mathbf{Q_p}$-spaces with $p > 1$ are equal and coincide with the Bloch space. This leads to a corresponding question in the three-dimensional case considered here. Basis of the necessary consideration is again a more technical result which proves at the end that we have an analogous property, but according to the expectations it is dependent on the dimension.

Proposition 3.2 *If f is monogenic in $B_1(0)$ and $2 < p < 3$, then for all $|a| < 1$*

$$\int_{B_1(0)} |\overline{D}f(x)|^2 g^p(x,a)\,dB_x \leq J(p)B(f)^2,$$

where $J(p) = 4\pi \int_0^1 \frac{r^{2-p}}{(1-r)^{3-p}(1+r)^3}dr$ is finite.

Theorem 3.4 *Let f monogenic in the unit ball. Then the following conditions are equivalent:*

1. *$f \in \mathbf{B}$.*
2. *$Q_p(f) < \infty$ for all $2 < p < 3$.*
3. *$Q_p(f) < \infty$ for some $p > 2$.*

Proof. The implication (1. $\Rightarrow$ 2.) follows from proposition 3.2. It is obvious that (2. $\Rightarrow$ 3.). From corollary 3.3 we have that 3. implies 1.
Theorem 3.4 means that all $\mathbf{Q_p}$-spaces for $p > 2$ coincide and are identical with the Bloch space.

3.2 Another characterization of $\mathbf{Q_p}$-spaces

The one-dimensional analogue of definition 3.2 was the first definition of $\mathbf{Q_p}$-spaces. This was motivated by the idea to have a range of spaces "around" the space BMOA. Comparing the original definition and one of the equivalent characterizations of BMOA in [6] it is obvious that $\mathbf{Q_1} = BMOA$. Another motivation is given by invariance properties of the Green function used in the definition. Recent papers of Aulaskari and co-authors (see e.g. [1]) show that the ideas of these weighted spaces can be generalized in a very direct way to the case of Riemannian manifolds. Caused by the singularity of the Green function difficulties arise in proving some properties of the scale. One of these properties is the inclusion property with respect to the index p.

In this subsection we discuss another possibility to characterise $\mathbf{Q_p}$-spaces, which is often easier to handle. Among others, this new characterization implies the proof of the fact that the $\mathbf{Q_p}$-spaces are a scale of function spaces with the Dirichlet space at one extreme point and the Bloch space at the other.

Lemma 3.5

$$\int_{B_1(0)} |\overline{D}f(x)|^2(1-|x|^2)^p dB_x \simeq \int_{B_1(0)} |\overline{D}f(x)|^2 g^p(x,0) dB_x$$

with $1 < p < 2.99$.

The idea to relate the Green function with more general weight functions of the type $(1-|x|^2)^p$ is not new. For the complex case it was already mentioned in [5], [3]. Another idea is to prove also a relation of $g^p(x,a)$ with $(1-|\varphi_a|^2)^p$. This way saves on the one hand the advantages of the simple term $(1-|x|^2)^p$ and preserves on the other hand a special behaviour of the weight function under Möbius transforms. As a hint we refer to equation (2).

Theorem 3.6 *Let f be monogenic in $B_1(0)$. Then, for $1 \le p < 2.99$,*

$$f \in \mathbf{Q_p} \Leftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2(1-|\varphi_a(x)|^2)^p dB_x < \infty.$$

At first glance, the condition $p < 2.99$ looks strange. But we have to keep in mind that theorem 3.4 means that all $\mathbf{Q_p}$-spaces for $p > 2$ are the same, so in fact this condition is only of technical nature caused by the singularity of $g^p(x,a)$ for $p = 3$.

Especially for the proof of this theorem we need the properties of monogenic functions and of the Möbius transformation. The main idea is a change of variables $w = \varphi_a(x)$ (the Jacobian determinant $\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3$ has no singularities) to come back to the situation of the previous lemma. The problem here is that, while $\overline{D}_x f(x)$ is monogenic, after the change of variables $\overline{D}_x f(\varphi_a(w))$ is not monogenic. But we know from [16] that $\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}_x f(\varphi_a(w))$ is again monogenic. We also refer to Sudbery [17] who studied this problem for the four-dimensional case already in 1979.

The above theorem makes it possible to state the same characterization also in the case of $p < 1$.

Proposition 3.3 *Let f be monogenic in $B_1(0)$. Then, for $0 < p \le 1$,*

$$f \in \mathbf{Q_p} \Leftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2(1-|\varphi_a(x)|^2)^p dB_x < \infty.$$

Using the alternative definition of $\mathbf{Q_p}$-spaces it will be shown that the $\mathbf{Q_p}$-spaces form a scale of Banach spaces.

Proposition 3.4 *For* $0 < p < q$ *we have:* $\mathbf{Q_p} \subset \mathbf{Q_q}$.

Proof. Let $f \in \mathbf{Q_p}$. Then

$$\sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^p dB_x < \infty.$$

Because of $(1 - |\varphi_a(x)|^2)^p \geq (1 - |\varphi_a(x)|^2)^q$ if $|x| \leq 1$ we have that

$$\int_{B_1(0)} |\overline{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^q dB_x \leq \int_{B_1(0)} |\overline{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^p dB_x.$$

Therefore,

$$\sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^q dB_x$$

$$\leq \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 (1 - |\varphi_a(x)|^2)^p dB_x.$$

This means $f \in \mathbf{Q_q}$.

4 Conformally Invariant Generalizations

In generalizing the results of the preceding section we have kept in mind two important paradigms of the theory in the unit disk of $\mathbb{C}$:

1. the theory is conformally invariant: all spaces are invariant for conformal maps of the unit disk. Moreover all conformal maps generate an isometry on each $\mathbf{Q_p}$ space;
2. the theory can be extended to include harmonic functions.

As already remarked, in complex analysis there is a certain freedom regarding the representation of the conformal group on spaces of holomorphic functions. In higher-dimensional theory this is much more restricted, and one cannot hope to obtain isometries.

4.1 Conformal mappings of the unit ball

Clifford algebras are extremely well suited to describe conformal mappings in more than two dimensions in a way quite similar to the one used in the complex plane. The counterpart of the transformations φ_a of the complex case are of the form $\phi_{\vec{a}}(\vec{x}) = (\vec{x} - \vec{a})(\vec{a}\vec{x} + 1)^{-1}$ which we can write in the more classical form

$$\phi_{\vec{a}}(\vec{x}) = \frac{\vec{x} - \vec{a}}{\vec{a}\vec{x} + 1}, \tag{3}$$

where $\vec{a}$ is an arbitrary element of the unit ball in $\mathbb{R}^n$. Each conformal mapping of the unit ball can be written as the product of an orthogonal transformation and a mapping of the form (3). Again we have that $\phi_{\vec{a}}(\vec{a}) = 0$ and $(\phi_{\vec{a}})^{-1} = \phi_{-\vec{a}}$.

More generally, an arbitrary Möbius transformation of $\mathbb{R}^n$ can be written in the form

$$g(\vec{x}) = \frac{a\vec{x} + b}{c\vec{x} + d},$$

where a, b, c and d are Clifford numbers satisfying specific properties (the so-called Vahlen conditions, see e.g. [8]). Specifically, these numbers can be chosen in such a way that $ad^* - bc^* = \pm 1$. For the mapping $\phi_{\vec{a}}$ we have the coefficients

$$\begin{array}{ll} a = 1/\sqrt{1 + \vec{a}^2} & b = -\vec{a}/\sqrt{1 + \vec{a}^2} \\ c = \vec{a}/\sqrt{1 + \vec{a}^2} & b = 1/\sqrt{1 + \vec{a}^2} \end{array}$$

The mappings share many properties with complex fractional mappings which can be written in a similar way. It is well known (see again [8]) that the differential satisfies

$$|dg(\vec{x})| = \frac{1}{|c\vec{x} + d|^2}|d\vec{x}|,$$

and we define the local contraction factor $\mu_g(\vec{x})$ by

$$\mu_g(\vec{x}) = \frac{1}{|c\vec{x} + d|^2},$$

which leads to the formula

$$\int_{B_1(0)} f(\vec{y})\, d\vec{y} = \int_{B_1(0)} f(\phi_{\vec{a}}(\vec{x}))\mu_{\phi_{\vec{a}}}(\vec{x})^n \, d\vec{x}$$

for a change of the integration variable.

4.2 Conformal mappings of harmonic and monogenic functions

Transformation of harmonic and monogenic functions is more rigid here than in the complex case, since the product of monogenic functions is not necessarily monogenic. We do have:

1. if f is monogenic, then so is

$$\gamma_g f(\vec{x}) = \frac{(c\vec{x} + d)^*}{|c\vec{x} + d|^n} f(g(\vec{x}));$$

2. if h is harmonic, then so is

$$\kappa_g h(\vec{x}) = (\mu_g(\vec{x}))^{n/2-1} h(g(\vec{x})).$$

For the transformations $\phi_{\vec{a}}$ these can be expressed as

$$\begin{aligned}\gamma_{\vec{a}} f(\vec{x}) &= (1+\vec{a}^2)^{\frac{n-1}{2}} \frac{\vec{x}\vec{a}+1}{|\vec{x}\vec{a}+1|^n} f(\phi_{\vec{a}}(\vec{x})) \\ \kappa_{\vec{a}} h(\vec{x}) &= (\mu_{\vec{a}}(\vec{x}))^{n/2-1} h(\phi_{\vec{a}}(\vec{x})).\end{aligned}$$

For a rotation A we have $b = c = 0$ and $d = a^{*-1}$, with the normalization $|d| = |a| = 1$, and the formulae become

$$\begin{aligned}\gamma_A f(\vec{x}) &= a f(A\vec{x}), \\ \kappa_A h(\vec{x}) &= h(A\vec{x}).\end{aligned}$$

If we denote by $\mathcal{H}$ the space of harmonic functions in the unit ball, and by $\mathcal{M}$ the space of monogenic functions we can write this as follows:

For arbitrary g in the Möbius group of the unit ball $\mathcal{H}$ is invariant under κ_g and $\mathcal{M}$ is invariant under γ_g.

5 Some Useful Estimates on Harmonic Functions in the Unit Ball

In what follows we will be interested in several norms on monogenic or harmonic functions, and on their derivatives. Apart from the Bloch and Dirichlet norms, defined by

$$\begin{aligned}B(g) &= \sup_{\vec{x}\in B_1(0)} (1-|\vec{x}|^2)^n |g(\vec{x})|^2 \\ D(g) &= \int_{B_1(0)} |g(\vec{x})|^2 \, d\vec{x},\end{aligned}$$

these norms will all be defined by a formula of the form

$$q(g,\mu_p) = \sup_{a\in B_1(0)} \int_{B_1(0)} |g(\vec{x})|^2 \mu_p(\phi_{\vec{a}}(\vec{x}))\, d\vec{x},$$

where μ_p is a suitable weight function on the unit ball. It should be noted that $B(g)$, $D(g)$ and $q_p(g,\mu_p)$ are all *squares* of norms rather than norms.

We now define for a parameter $0 \le q < +\infty$ the class of weight functions W_p as being the class of functions μ_p for which there exist strictly positive κ and k such that

$$\textstyle\int_{B_1(0)} \mu_p(\vec{x})\, dx < \infty, \tag{4}$$

$$\mu_p(\vec{x}) \ge k \text{ if } |\vec{x}| \le \tfrac{1}{2}, \tag{5}$$

$$k(1+\vec{x}^2)^p < \mu_p(\vec{x}) < \kappa(1+\vec{x}^2)^p \text{ if } |\vec{x}| \ge \tfrac{1}{2}. \tag{6}$$

The pivoting constant 1/2 is of course completely arbitrary, and can be replaced by any R strictly between 0 and 1.

We have the following theorem:

Theorem 5.1 *Let μ_p and ν_p be two weight functions in the class W_p. Then the norms defined by $q(\cdot, \mu_p)$ and $q(\cdot, \nu_p)$ are equivalent on $\mathcal{H}$. More exactly, there are strictly positive finite constants C_1 and C_2 such that*

$$C_1 q(g, \mu_p) \leq q(g, \nu_p) \leq C_2 q(g, \mu_p),$$

for all $g \in \mathcal{H}$, implying that $q(g, \mu_p) = +\infty$ if and only if $q(g, \nu_p) = +\infty$.

Of course we could immediately have taken $\mu_p(\vec{x}) = (1 + \vec{x}^2)^p$, but in the past other functions have been used, as in the complex case where $\mu_p(z) = (-\ln |z|)^p$, and we want to allow for an integrable singularity in 0. Anyway the example $\mu_p(\vec{x}) = (1 + \vec{x}^2)^p$ shows that the norms grow weaker as the index p increases.

Obviously $D(\cdot)$ equals $q(\cdot, 1)$, and a constant function is element of W_0. The Bloch norm is defined using a pointwise estimate rather than an integral. However, it is possible to prove the following estimate for harmonic functions:

Lemma 5.2 *Let μ_p be a weight function in any of the classes W_p. Then there exists a constant C such that for any arbitrary point $\vec{a}$ in $B_1(0)$, and any g in $\mathcal{H}$*

$$(1 + \vec{a}^2)^n |g(\vec{a})|^2 \leq C \int_{B_1(0)} |g(\vec{x})|^2 \mu_p(\phi_{\vec{a}}(\vec{x}))\, d\vec{x}$$

This already shows the Bloch norm is weaker than any $q(\cdot, \mu_p)$ norm. However, looking at the Bloch norm we have the pointwise estimate

$$|g(\vec{x})|^2 \leq (1 + \vec{x}^2)^{-n} B(g).$$

It can be proved that this is sufficient to give a bound on the defining integral for $q(g, \mu_p)$ if $p > n - 1$, proving that

Lemma 5.3 *For $p > n - 1$, the Bloch norm is equivalent with any norm defined by a weight function in W_p.*

6 Conformally Invariant Spaces

In Section 2 we have used the operators D and $\overline{D}$ to define $\mathbf{Q_p}$-spaces. Obviously, this definition is not rotationally invariant, as it emphasizes the x_0-axis. In this framework, a monogenic function independent of x_0 satisfies $\overline{D} f = 0$, while it is not necessary that the function be constant. Before dealing with derivatives however, we take a look at the interplay between the q_p norms and the actions κ and γ.

6.1 q_p norms applied to harmonic functions.

It is obvious from the definition of the q_p that they are invariant for rotations. Therefore it is only necessary to look at what happens with transformations of the form $\Phi_{\vec{a}}$ with $|\vec{a}| < 1$. We obtain the following estimates, using the function

$$K(g) = \frac{(1 + |g^{-1}(0)|)^2}{1 - |g^{-1}(0)|^2} = \frac{1 + |g^{-1}(0)|}{1 - |g^{-1}(0)|}.$$

Theorem 6.1 *Let f be a measurable function in the unit ball. Then, for arbitrary $p \geq 0$ and g in the Möbius group of the unit ball we have*

$$\begin{aligned} q_p(\kappa_g f) &\leq K(g)^2 q_p(f), \\ q_p(\gamma_g f) &\leq K(g) q_p(f). \end{aligned}$$

We can now conveniently define a few subspaces of $\mathcal{H}$:

1. The Bloch-like space
$$\mathbf{P} = \{f \in \mathcal{H} : B(f) < +\infty\},$$
2. The harmonic Bergman space
$$\mathbf{T} = \{f \in \mathcal{H} : D(f) < +\infty\},$$
3. The $\mathbf{K}_p$ spaces, for $0 \leq p < \infty$,
$$\mathbf{K}_p = \{f \in \mathcal{H} : q_p(f) < +\infty\}.$$

The Bergman space can of course be identified with $\mathbf{K}_0$. The corresponding spaces of monogenic functions (i.e. the intersections with $\mathcal{M}$) will be written as $\mathcal{M}\mathbf{P}$, $\mathcal{M}\mathbf{T}$ and $\mathcal{M}\mathbf{K}_p$ respectively. Obviously all these spaces are Banach spaces for the corresponding norm. From the definition of q_p it follows that, with increasing p the norms grow weaker, while Lemma 5.3 assures us that all norms with $p > n-1$ are equivalent. On the other hand, the transforms κ and γ are continuous with respect to each of the q_p norms, and we can summarize our results into one theorem:

Theorem 6.2 *For $0 \leq p \leq q$*

$$\mathbf{T} \subset \mathbf{K}_p \subset K_q \subset \mathbf{P}.$$

For $p > n-1$, $\mathbf{P} \subset \mathbf{K}_p$. Hence

$$\mathbf{P} = \mathbf{K}_p \qquad p > n-1.$$

Similar relations hold between the monogenic subspaces.

Each g in the Möbius group of the unit ball leads to two associated transforms:

1. *the transform κ_g leaving the spaces $\mathbf{K}_p$ invariant, and having transformation norm at most $K(g)^2$ there;*

2. *the transform γ_g leaving the spaces $\mathcal{M}\mathbf{K}_p$ invariant, and having transformation norm at most $K(g)$.*

It should be remarked that, even if $\mathcal{M}\mathbf{K}_p$ is a subspace of $\mathbf{K}_p$, it is not invariant for transformations of the form κ_g, and that it therefore makes no sense to apply κ_g to this space. On the other hand the space $\mathbf{K}_p$ is not invariant –in general– for γ_g: both spaces therefore carry their own transformations. This is very much unlike the complex case, where the κ_g can act both on harmonic and on holomorphic functions.

6.2 The q_p-norms applied to derivatives of functions.

Until now we have only considered L_2-like norms on harmonic and monogenic functions. In this section we shall take into account derivatives, which results in a Sobolev-like structure. Unlike the complex case, where only derivatives of functions appear in the norm (and constant functions are eliminated to obtain a norm instead of a seminorm), here it will be necessary to include both the functions' values and the derivatives into the norm. Contrary to holomorphic ones, monogenic functions do not have a directional derivative of equal size in all directions, a fact we describe more formally.

Take $\mathbb{R}^n$, the unit ball there, $B_1(0)$, the space $\mathcal{M}$ of monogenic functions in $B_1(0)$, and define the directional derivative as before: if ξ is a unit vector then

$$\partial_\xi f(\vec{x}) = \partial_t \; f(\vec{x} + t\xi)|_{t=0}\,.$$

The mapping $\xi \to \partial_\xi f(\vec{x})$ is a linear function, and we want a norm on this function. The space of linear functions in n variables is of course finite dimensional, so all norms are equivalent. All considerations which follow hence will be valid for an arbitrary norm, and it is only for our convenience that we take the norm given by

$$\begin{aligned}(Mf(\vec{x}))^2 &:= \sum_{i=1}^{n} |\partial_i f(\vec{x})|^2 \\ &= \frac{1}{B_n}\int_{S^{n-1}} |\partial_\xi f(\vec{x})|^2\, dS_\xi,\end{aligned}$$

where B_n is a normalizing constant. This constant can be obtained by substituting $\partial_\xi f$ by a suitable linear function in ξ, e.g. the first coordinate function. Hence

$$B_n = \int_{S^{n-1}} x_1^2\, dS_\xi.$$

Notice that $\partial_\xi f$ is monogenic whenever f itself is. For harmonic functions Mf has a nice property. If we define, for two Clifford numbers a and b, $[a,b]$ to be the scalar part of $\bar{a}b$, we have that

$$\begin{aligned} \Delta|f|^2 &= \Delta[f,f] \\ &= [\Delta f, f] + [f, \Delta f] + 2\sum_{i=1}^{n}[\partial_i f, \partial_i f] \\ &= 2(Mf)^2. \end{aligned}$$

In the complex case the directional derivative of a holomorphic function has the same norm in all directions:

$$|\partial_\xi f(z)| = |\partial_z f(z)|,$$

and so $(Mf(z))^2 = 2|\partial_z f(z)|^2$.

The counterparts of the complex Bloch, Dirichlet and $\mathbf{Q_p}$ spaces are

1. The Bloch space

$$\mathbf{B} = \left\{ f \in \mathcal{H} : B(Mf) + B(f) < +\infty \right\},$$

2. The Dirichlet space

$$\mathbf{D} = \left\{ f \in \mathcal{H} : D(Mf) + D(f) < +\infty \right\},$$

3. The $\mathbf{Q_p}$ spaces

$$\mathbf{Q_p} = \left\{ f \in \mathcal{H} : q_p(Mf) + q_p(f) < +\infty \right\},$$

and the monogenic subspaces $\mathcal{M}\mathbf{B}$, $\mathcal{M}\mathbf{D}$ and $\mathcal{M}\mathbf{Q_p}$, where again $\mathbf{D} = \mathbf{Q_0}$. Of course, Mf is not harmonic whenever f is. However, since $(Mf)^2 = \sum_i |\partial_i f|^2$, where $\partial_i f$ is harmonic, and so $q_p(\partial_i f) \le q_p(Mf) \le \sum_j q_p(\partial_j f)$ for any i it is easy to adapt to Mf the estimates obtained before and the counterpart of Theorem 6.2 is:

Theorem 6.3 *For* $0 \le p \le q$

$$\mathbf{D} \subset \mathbf{Q_p} \subset \mathbf{Q_q} \subset \mathbf{B}.$$

For $p > n-1$, $\mathbf{B} \subset \mathbf{Q_p}$. *Hence*

$$\mathbf{B} = \mathbf{Q_p} \qquad p > n-1.$$

It is when looking at the invariance under Möbius transformations of these spaces that we see a profound difference between the two- and the

n-dimensional case. In $\mathbb{C}$ we only have the transformations $\kappa_g f = f \circ g$, and so we have the pointwise estimate

$$|M\kappa_g f(z)| = |Mf(gz)||g'(z)|.$$

In the higher dimensional case however the pointwise estimates on $|M\kappa_g f(\vec{x})|$ and $|M\gamma_g f(\vec{x})|$ will have to include the function value of f itself in $g(\vec{x})$. Therefore the Möbius group cannot be made to act in a unitary way on the spaces defined above, although we still have continuity. We obtain the following estimates for harmonic, resp. monogenic functions:

$$\begin{aligned} q_p(M\kappa_{\vec{a}}h) &\leq n^2\left((n-2)K(\vec{a})+1\right)^2\left(q_p(h)+q_p(Mh)\right) \\ q_p(M\gamma_{\vec{a}}f) &\leq n^2\left[\frac{(1+n)|\vec{a}|}{\sqrt{1+\vec{a}^2}}+\sqrt{K(\vec{a})}\right]^2\left(q_p(f)+q_p(Mf)\right). \end{aligned}$$

This finally leads to

Theorem 6.4 *The following invariance relations hold:*

1. *The spaces* $\mathbf{B}$, $\mathbf{D}$ *and* $\mathbf{Q_p}$ *are invariant under Möbius transformations of the unit ball for the representation* κ*. Moreover each* $\kappa_{\vec{a}}$ *is a bounded operator on each of these spaces.*
2. *The spaces* $\mathcal{M}\mathbf{B}$, $\mathcal{M}\mathbf{D}$ *and* $\mathcal{M}\mathbf{Q_p}$ *are invariant under Möbius transformations of the unit ball for the representation* γ*, and each* $\gamma_{\vec{a}}$ *is a bounded operator on each of these spaces.*

Proof.
Consider e.g. the harmonic case, and let f be in $\mathbf{Q_p}$ ($0 \leq p \leq \infty$, where $\mathbf{Q}_\infty$ is identified with $\mathbf{B}$). Take an arbitrary $\vec{a} \in B_1(0)$. Since $\mathbf{Q_p}$ is a subspace of $\mathbf{K}_p$, we know that $\kappa_{\vec{a}} f$ is in $\mathbf{K}_p$, and $q_p(f) < +\infty$. From the preceding calculations it follows that $M\kappa_{\vec{a}} f$ is majorated by two functions both having finite q_p norm, and so $q_p(M\kappa_{\vec{a}} f) < +\infty$. Therefore $\kappa_{\vec{a}} f$ is in Q_p. Since the estimates on $M\kappa_{\vec{a}} f$ show constants independent of f, $\kappa_{\vec{a}}$ is a continuous operator on Q_p. ■

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