18
Clifford Analysis on Super-Space

F. Sommen[*]
University of Gent,
Department of Mathematical Analysis
Galglaan 2, B-9000 Gent,
Belgium

ABSTRACT In this paper we further elaborate an extension of Clifford analysis towards super-symmetry, started in our paper [So1]. We discuss the generalized spingroup, the Fischer decomposition and give several examples of canonically defined super-manifolds.

Introduction

Many of the fundamental special functions in Clifford analysis are functions of zonal type i.e. functions depending of several Clifford vector variables like $\underline{x} = \sum e_j x_j,\ \underline{u} = \sum e_j uj, \ldots$, and their inner products $\underline{x}^2, \underline{x}\underline{u} + \underline{u}\underline{x}, \underline{u}^2, \ldots$ whereby $e_1, \ldots, e_m$ satisfy the Clifford algebra defining relations $e_i\ e_j + e_j\ e_i = -2\delta_{ij}$. Moreover these functions are in principle the same in all dimensions m, whereby the dimension m is given by $\partial_{\underline{x}}[\underline{x}] = -m,\ \partial_{\underline{x}} = \sum e_j \partial_{x_j}$ being the Dirac operator. This lead to the idea to define an algebra $R(S)$ of abstract vector variables which is the free associative algebra generated by a set S of "abstract vector variables" $x, y, z, \ldots$ together with the axiom: $\{x, y\}z = z\{x, y\}$, and to redefine Dirac operators as endomorphisms on $R(S)$, i.e. as vector derivatives denoted by $\partial_x,\ x \in S$. This theory

[*]Senior Research Associate, FWO, Belgium

Advances in Applied Clifford Algebras **11 (S1)** 291-304, 2001

was inspired by "geometric calculus" as presented in [HS] and developed to some extent in our papers [So2], [So3].
Important are the facts that

(i) $R(S)$ does not depend on any particular dimension m

(ii) using the assignment $x \to \underline{x} = \sum e_j x_j,\ x \in S$, the algebra $R(S)$ is represented by an algebra $R(\underline{S})$ of Clifford polynomials and this map is injective provided S is finite and $M \geq \text{Card } S$,

(iii) the vector derivative ∂_x leads to the introduction of the abstract scalar parameter $\partial_x[x] = M$, called abstract dimension; using the representation $x \to \underline{x}$ of abstract vectors by m-dimensional Clifford vectors, after identification $M = m$, ∂_x is mapped on the operator $-\partial_{\underline{x}}$

(iv) the algebra $R(S)$ is in fact independent of the choice of a quadratic form.

Moreover, in our paper [So1] we have already shown that the algebra $R(S)$ leads to an extension of Clifford analysis to super-symmetry. Hereby one uses an assignment of the form $x \to \underline{\grave{x}} + \underline{x}$, whereby $\underline{x} = \sum x_j\ e_j$ is a usual Clifford vector as before and $\underline{\grave{x}} = \sum e_{\grave{j}}\ x_{\grave{j}}$ is a so called "fermionic Clifford vector", i.e. the coordinates $x_{\grave{j}}$ are anti-commuting and the elements $e_{\grave{j}}$ are such that the abstract axioms for $R(S)$ remain satisfied in the representation. It turns out that then $e_{\grave{1}}, \ldots, e_{2\grave{n}}$ are generators of a Weyl algebra or symplectic Clifford algebra (see [Cr][Ha]). One hence obtains a canonical extension of Clifford analysis to the super-space as introduced in e.g. [Be], [VV] and our approach is also related to abstract approaches to super-symmetry as developed in [CRS]. In our paper [So1] we also presented a treatment of abstract super-forms which may be of importance in connection with Stokes theorem in super-symmetry (see also [Pa]).
In section one we study more in detail the extension of the spingroup to super-space, in which the super-sphere plays an essential role. Hereby the super-sphere is the solution set in super-space of the equation $x^2 = -1$ which exists on the abstract level of radial algebra $R(S)$. One also obtains an extension of the symplectic spingroups introduced in [Cr].
In section two we study in detail the Fischer decomposition for polynomials on super space, leading to a theory of spherical monogenics on super-space. Also this can be done to some extent on the abstract vector variable level.
Finally we introduce spaces of super-multivectors, super-Grassmannians etc. based upon the notion of k-vectors which already exists once again on the level of abstract vector variables.
For further information on Clifford analysis we refer to [DSS]

1 Super-Vector Variables

The notion of super-space and super-analysis is well established (see e.g. [Be], [VV]). To define it we start from a number of commuting variables

$x_1, \ldots, x_m$ and a number of anti-commuting variables $x_{\grave{1}}, \ldots, x_{\grave{p}}$ and these coordinates vary over a super-algebra an example of which is provided by any Grassmann algebra $L_N = \mathrm{Alg}\{f_1, \ldots, f_N\}$ whereby $f_i\ f_j + f_j\ f_i = 0$. For this algebra one has the splitting $L_N = L_{N+} + L_{N-}$ and it is understood that commuting variables like $x_1, \ldots, x_m$ take their values in L_{N+} while $x_{\grave{1}}, \ldots, x_{\grave{p}}$ take their values in L_{N-}. Hereby the elements $f_1, \ldots, f_N$ are interpreted as fixed anti-commutative numbers or, what is the same, anti-commutative variables for which there is only one value. In particular one could indeed take $N = p$ and $x_{\grave{1}} = f_1, \ldots, x_{\grave{p}} = f_p$ which would mean that the whole space with coordinates $(x_{\grave{1}}, \ldots, x_{\grave{p}})$ contains a canonical point namely $(f_1, \ldots, f_p)$. In other words, there are many possibilities to assign values to anti-commuting variables and also infinite dimensional "super algebras" are used in the literature. But to develop the basic ideas it suffices to consider L_N and to consider completions later on.

To set up the language of Clifford analysis on super-space we need to go over from a vector like $(x_1, \ldots, x_m)$ to a Clifford vector $\underline{x} = \sum x_j\ e_j$ whereby for example $e_j\ e_k + e_k\ e_j = -2\delta_{jk}$.
As we already pointed out in [So2], Clifford vector variables may be seen as representations of the radial algebra $R(S)$generated by a set S of abstract vector variables x, y, using the assignment

$$x \to \underline{x} = \sum x_j\ e_j$$

and the fact that $xy + yx$ is scalar readily leads to the statement that $e_j\ e_k + e_k\ e_j = -2g_{jk}$ is scalar, i.e. the definition of Clifford algebras.
Moreover in [So] we also saw how this procedure may be generalized to produce super-vector variables like $\underline{x}\grave{} + \underline{x} = \sum x_{\grave{j}}\ e_{\grave{j}} + \sum x_j\ e_j$ whereby we used the "canonical defining relations" valid for $p = 2n$

$$e_i\ e_j + e_j\ e_i = -2\delta_{ij},$$

$$e_i\ e_{\grave{j}} = -e_{\grave{j}}\ ei$$

$$e_{\grave{i}}\ e_{\grave{j}} - e_{\grave{j}}\ e_{\grave{i}} = h_{ij}, \quad h_{2i-1,2j} = \delta_{ij}, \quad h_{2i-1,2j-1} = h_{2i,2j} = 0,$$

i.e. $\mathrm{Alg}\{e_{\grave{1}}, \ldots, e_{\grave{2n}}\}$ is the Crumeyrolle Clifford algebra (see [Cr]) or Weyl algebra. An explicit realization may be obtained starting from the Clifford algebra R_{m+1} generated by putting

$$e_{\grave{2j-1}} = e_{m+1}\ \partial_{a_j}, \quad e_{\grave{2j}} = -e_{m+1}\ a_j.$$

In general the radial algebra assumption $xy + yx$ =scalar together with the relations $x_i\ x_j = x_j\ x_i,\ x_i\ x_{\grave{j}} = x_{\grave{j}}\ x_i,\ x_{\grave{i}}\ x_{\grave{j}} = -x_{\grave{j}}\ x_{\grave{i}}$ lead to Clifford algebra defining relations of the form

$$\begin{aligned}
&e_i\ e_j + e_j\ e_i = -2g_{ij} = -2g_{ji} = \text{ fixed commutative (scalar)}\\
&e_{\grave{i}}\ e_{\grave{j}} - e_{\grave{j}}\ e_{\grave{i}} = h_{ij} = -h_{ji} = \text{ scalar}\\
&e_i\ e_{\grave{j}} + e_{\grave{j}}\ e_i = a_{i\grave{j}} = \text{ fixed anti-commutative object.}
\end{aligned}$$

Formally one can quite well work with but to fix the ideas we'll stick to the canonical case $g_{ij} = \delta_{ij},\ h_{ij} =$ simplectic form , $a_{i\grave{j}} = 0$. Moreover we stick to the notation $e_i,\ e_{\grave{j}}$ rather than represent $e_{\grave{j}}$ by Weyl algebra because we'll use Clifford algebra nomenclature during the process of building up the super-extension of Clifford analysis.
The next thing we need is a proper replacement for the action of the spingroup Spin(m) on R^m. In our previous paper [So1] we pointed out that the infinitesimal elements of this "super-spin-group" are of the form

$$s = \exp\ \varepsilon B, \quad B = \sum B_{ij}\ e_{ij}, \quad B_{ij} = -B_{ij} = \text{ commutative}$$
$$s' = \exp\ \varepsilon B', \quad B' = \sum B'_{ij}\ e_{\grave{i}}\, e_{\grave{j}}, \quad B'_{ij} = B'_{ji} = \text{ commutative}$$
$$\grave{s} = \exp\ \varepsilon \grave{B}, \quad \grave{B} = \sum B_{i\grave{j}}\, e_i\ e_{\grave{j}}, \quad B_{i\grave{j}} = \text{ anti-commutative}$$

whereby ε is infinitesimal, and one also has to consider compositions leading to a definition of a super-spingroup which also depends on the algebra $L_N = L_{N+} + L_{N-}$ in which commuting and anti-commuting objects take their values. Note that in case where B_{ij} and B'_{ij} are real valued, the group of elements s leads to the spingroup Spin(m) while the group of elements s' leads to the Crumeyrolle spingroup Spin'(2n) which is a double covering of the symplectic group (see also [Cr]). In all these cases, the action of a super-spingroup element S on a super-vector variable $\underline{\grave{x}} + \underline{x}$ is given by the mapping

$$\underline{\grave{x}} + \underline{x} \to S(\underline{\grave{x}} + \underline{x})S^{-1}$$

whereby in the infinitesimal cases,

$$s^{-1} = 1 - \varepsilon B, \quad s^{'-1} = 1 - \varepsilon B', \quad \grave{s}^{-1} = 1 - \varepsilon \grave{B}.$$

This generalized group action preserves the anti commutator $xy + yx$.
In our treatment we only considered the infinitesimal group elements because for many applications in Clifford analysis this is sufficient. One can also consider the super groups themselves but, as pointed out in [Cr] this cannot be done in the infinite dimensional algebra $\mathrm{Alg}\{e_{\grave{1}}, \ldots, e_{\grave{2n}},\ e_1, \ldots, e_m\}$ which is only the freely generated associative algebra with these generators. One also has to consider formal series, completions and more general functions but we think it is good politics to postpone this till later on.
Next in Clifford algebra the spingroup is also defined as the set of even products of the form

$$s = w_1 \ldots w_{2h} \qquad \text{whereby} \qquad w_j \in S^{m-1}, \quad \text{i.e.} \quad w_j^2 = -1.$$

To prove this it is in fact sufficient to consider the infinitesimal case of products of the form $s = w_1\ w_2$ whereby $w_2 \in S^{m-1}$ is infinitesimally close to $-w_1 \in S^{m-1}$ and to prove that they generate the exponentials of infinitesimal bivectors. We'll investigate the same here for the super-spingroup but it turns out that one obtains only a proper subgroup. To that end we first define the unit super-sphere to be the super-surface with equation

$$(\underline{\grave{x}} + \underline{x})^2 = -1.$$

Examples of points on this object are:

(i) the basis elements $e_1, \ldots, e_m$,

(ii) if f is an anti-commutative fixed object, then also $e_i + f e_{\grave{j}}$ satisfies $(e_i + e_{\grave{j}} f)^2 = e_i^2 + f^2 e_{\grave{j}}^2 = e_i^2 = -1$,

(iii) suppose that $f_1,\ f_2$ are two anti-commuting fixeds and recall that $e_{2\grave{j}-1}\, e_{2\grave{j}} - e_{2\grave{j}}\, e_{2\grave{j}-1} = 1$ are the only nonzero products and the square $(f_1\ e_{2\grave{j}-1} + f_2\ e_{2\grave{j}})^2 = f_1\ f_2$ while also the square $((1 + \frac{1}{2} f_1\ f_2) e_1)^2 = -1 - f_1\ f_2$ so that $f_1\ e_{2\grave{j}-1} + f_2\ e_{2\grave{j}} + (1 + \frac{1}{2} f_1 f_2) e_1$ lies on the super-sphere,

(iv) similarly, if $f_1,\ f_2, \ldots, f_{2n}$ are anti-commuting fixeds, then the element

$$f_1 e_{\grave{1}} + f_2\ e_{\grave{2}} + \ldots + f_{2n} e_{2\grave{n}} +$$

$$+ n^{-1/2}(1 + \frac{n}{2} f_1 f_2) e_1 + \ldots + n^{-1/2}(1 + \frac{n}{2} f_{2n-1} f_{2n}) e_n,$$

is a canonical element of the super-sphere, canonical because $(f_1, \ldots f_{2n})$ are thought of as anti-commuting variables with only one value. In other words, the above element is in fact a super-surface inside the super-sphere which consists of just one point.

(v) the above example raises the question whether purely fermionic unit vectors like the canonical vector $f_1 e_{\grave{1}} + \ldots + f_{2n} e_{2\grave{n}}$ exist; the answer is negative for that would lead to an identity of the form

$$f_1 f_2 + \ldots + f_{2n-1} f_{2n} = -1,$$

while $f_1 f_2 + \ldots + f_{2n-1} f_{2n}$ is nilpotent. This also has to do with the fact that no finite dimensional fermionic representation of R(S) is isomorphic.

Now putting $w_1 = e_i,\ w_2 = -e_i + \varepsilon\ e_j,\ j \neq i, \varepsilon$ infinitesimal, then $s = w_1\ w_2 = 1 + \varepsilon\ e_{ij}$ which shows that e_{ij} is in the Lie algebra of the group generated by the super-sphere.
One can also take $w_2 = -e_i + \varepsilon f\ e_{\grave{j}}$ leading to the product $s = 1 + \varepsilon(f\ e_i\ e_{\grave{j}})$ showing that also $f\ e_i\ e_{\grave{j}}$ is in the Lie algebra of the super-sphere.
But there seems to be no way to arrive at the products $e_{\grave{i}}\ e_{\grave{j}} + e_{\grave{j}}\ e_{\grave{i}}$ as elements of this Lie algebra. This seems disappointing at first and raises the question for a more complete super-space. But the anti-commutator $f_1\ e_1\ e_{\grave{j}}\ f_2\ e_1\ e_{\grave{k}} - f_2\ e_1\ e_{\grave{k}}\ f_1\ e_1\ e_{\grave{j}} = f_1\ f_2(e_{\grave{j}}\ e_{\grave{k}} + e_{\grave{k}}\ e_{\grave{j}})$ does belong to the Lie algebra and hence so does the element (for N even)

$$(f_1\ f_2 + \ldots + f_{N-1}\ f_N)\ (e_{\grave{j}}\ e_{\grave{k}} + e_{\grave{k}}\ e_{\grave{j}})$$

which in some sense a "surrogate" for $-(\grave{e}_j \grave{e}_k + \grave{e}_k \grave{e}_j)$ and again brings us to make the forbidden identification

$$f_1\ f_2 + \ldots + f_{N-1}\ f_N + 1 = 0.$$

Now in the ideal case where N could be infinite, the canonical element $f_1\ f_2 + \ldots + f_{N-1}\ f_N$ whould no longer be a zero-devisor and the above identification would be no longer forbidden. We think of $f_1\ f_2 + \ldots + f_{N-1}\ f_N$ as an approximation for -1. In this sense the group generated by the super-sphere remains different from the super-spingroup, but in any case the super-spingroup itself is also dependent upon the number N and in some sense all these groups are approximations of an ideal infinite dimensional group.
Note also that the symplectic group, generated by $\grave{e}_j\ \grave{e}_k + \grave{e}_k\ \grave{e}_j$ is the "most far away" from the group of the super-sphere. The representation of elements of the spingroup as products of an even number of unit vectors has to do with the Hamilton principle for rotations: any rotation is the composition of an even number of reflections.

2 The Dirac Operator on Super-Space

Also the definition of a Dirac operator on super space was already provided in [So]. What we need is a good representation for the endomorphism ∂_x on $R(S)$, called vector derivative and in [So] we came up with a solution assuming the canonical identities $e_i\ e_j + e_j\ e_i = -2\delta_{ij},\ e_i\ \grave{e}_j = -\grave{e}_j\ e_i$ and $\grave{e}_{2j-1} = \partial_{a_j}\ e_{m+1},\ \grave{e}_{2j} = -a_j\ e_{m+1}$.
First denote by $\partial_{\grave{x}_j}$ the derivative with respect to the anti-commuting variable $\grave{x}_j$ determined by

$$\partial_{\grave{x}_j}[F] = 0, \quad \partial_{\grave{x}_j}[\grave{x}_j F] = F$$

in case $\grave{x}_j$ doesn't occur in F.
Next we define the fermionic Dirac operator

$$\partial_{\grave{\underline{x}}} = 2\sum \partial_{\grave{x}_{2j-1}}\grave{e}_{2j} - 2\sum \partial_{\grave{x}_{2j}}\grave{e}_{2j-1}$$

and if we define for $F \in R(S)$, the left and right action of ∂_x by the assignment

$$x \to \grave{\underline{x}}, \quad \partial_x[F] \to \partial_{\grave{\underline{x}}}[F], \quad [F]\partial_x \to -[F]\partial_{\grave{\underline{x}}},$$

whereby F is the element corresponding to F under $x \to \grave{\underline{x}}$, then the operator ∂_x satisfies the correct axioms for an abstract vector derivative given in [So2] (see also [HS]).
The same is true for the standard Dirac operator $\partial_{\underline{x}} = \sum e_j \partial_{x_j}$ if we make the assignments

$$x \to \underline{x}, \quad \partial_x[F] \to -\partial_{\underline{x}}[F], \quad [F]\partial_x \to -[F]\partial_{\underline{x}}.$$

Moreover, in the standard case we have the dimension formula

$$\partial_x[x] = -\partial_{\underline{x}}[\underline{x}] = m$$

whereas in the anti-commutative (fermionic) case we obtain negative dimension

$$\partial_x[x] = \partial_{\underline{\grave{x}}}[\underline{\grave{x}}] = -2n.$$

Hence on the super-space we make the assignments

$$\begin{aligned} &x \to \underline{\grave{x}} + \underline{x}, \qquad F \to F \\ &\partial_x[F] \to (\partial_{\underline{\grave{x}}} - \partial_x)[F], \\ &[F]\partial_x \to [F](-\partial_{\underline{\grave{x}}} - \partial_x) \end{aligned}$$

leading to the correct axioms and dimension formula

$$\partial_x[x] = (\partial_{\underline{\grave{x}}} - \partial_x)[\underline{\grave{x}} + \underline{x}] = -2n + m.$$

One may now start to produce generalizations to super-space of classical results in Clifford analysis. We only discuss here the Fischer decomposition for elements in the algebra generated by the set

$$\grave{x}_1, \ldots, \grave{x}_{2n}, x_1, \ldots, x_m; \grave{e}_1, \ldots, \grave{e}_{2n}, e_1, \ldots, e_m,$$

called Clifford polynomials.

This is a small class of functions defined on super-space; normally one may consider general functions in the coordinates $\grave{x}_1, \ldots, \grave{x}_{2n}, x_1, \ldots, x_m$ and also the infinite dimensional algebra $\mathrm{Alg}\{\grave{e}_1, \ldots, \grave{e}_{2n}, e_1, \ldots, e_m\}$ in which functions take their values may be completed in many ways (see also [Cr]). Moreover, functions may also take their values in spaces on which the elements $\grave{e}_1, \ldots, \grave{e}_{2n},\ e_1, \ldots, e_m,\ \grave{e}_{2j-1} = e_{m+1}\partial_{a_j}, \grave{e}_{2j} = -e_{m+1}\ a_j$ act as endomorphisms. Note that $e_1, \ldots, e_{m+1}$ act as endomorphisms on spinor spaces while the elements a_j, ∂_{a_j} of the Weyl algebra act as endomorphisms on e.g. $S'(R^n)$ or $L_2(R^n)$ etc. Hence there are several analysis problems associated with monogenic function theory on super-space and in our paper [So1] we gave the formulation a fermionic Cauchy-Kowalewski extension for tempered distributions $f(a; \grave{x}_1, \ldots, \grave{x}_{2n}) \in S'(R^n; \mathrm{Alg}\{\grave{x}_1, \ldots, \grave{x}_{2n}\})$ with "values in" the Grassmann algebra $\mathrm{Alg}\{\grave{x}_1, \ldots, \grave{x}_{2n}\}$.

2.1 The bosonic and fermionic Fischer decompositions

Let $R(\underline{x})$ and $S(\underline{x})$ be homogeneous polynomials of degree k in R^m with values in the Clifford algebra R_m; then the Fischer inner product is defined by

$$(R(\underline{x}), S(\underline{x})) = [\bar{R}(\partial_{\underline{x}})S(\underline{x})]o$$

whereby $R(\partial_{\underline{x}})$ means replacing x_j by $\partial_{x_j},\ a \to \bar{a}$ is the main anti-involution and $[a]_o$ is the scalar part of $a \in R_m$. This inner product is positive definite on the space P_k of all homogeneous Clifford polynomials of degree

k and the orthogonal complement of the subspace $\underline{x}\ P_k$ inside P_k is the space M_k of spherical monogenics of degree k, i.e. homogeneous solutions of $\partial_{\underline{x}} P_k(\underline{x}) = 0$. One thus arrives at a unique orthogonal Fischer decomposition

$$R_k(\underline{x}) = M(R_k)(\underline{x}) + \underline{x} R_{k-1}(\underline{x})$$

with $\partial_{\underline{x}} M(R_k)(\underline{x}) = 0$ and recursive application of this result leads to the complete Fischer decomposition.

Hence in the Bosonic setting there is a Fischer decomposition like this and it is the question to what extent does this result extend to the super space. Already now we can say that the result is negative because in the case where $m = 2n$, we have the identity $\partial_x[x] = 0$ which means that x is itself monogenic and the Fischer decomposition fails to exist or to be unique for linear functions. Indeed, if $R(x)$ would be a linear function admitting a Fischer decomposition, then it has the form $R(x) = P(x) + x\ a$ whereby both $P(x)$ and $x\ a$ would be monogenic. Therefore also $R(x)$ would have to be monogenic, which is not true in general.

Hence things are not so straightforward and first question is: how about the validity of Fischer decomposition in the purely fermionic case. In that case we have to define a proper Fischer inner product which is positive definite and for which the adjoint of the multiplyer

$$\grave{\underline{x}} = \sum \grave{e}_{2j-1}\, \grave{x}_{2j-1} + \sum \grave{e}_{2j}\, \grave{x}_{2j}$$

is the operator

$$-\partial_{\grave{\underline{x}}} = 2 \sum \grave{e}_{2j-1}\, \partial_{\grave{x}_{2j}} - 2 \sum \grave{e}_{2j}\, \partial_{\grave{x}_{2j-1}}$$

so that already $(\grave{\underline{x}}, \grave{\underline{x}}) = 2n > 0$.

Hereby one may use the Weyl algebra representation $\grave{e}_{2j-1} = \partial_{a_j}, \grave{e}_{2j} = a_j$. To arrive at this inner product we introduce certain operations on the Weyl algebra inspired by similar operations for the Clifford algebra.

First of all we need a fermionic analogue of the anti involution $a \to \bar{a}$ on R_m. This can be done by defining on this algebra the "adjoint mapping"

$$a_j^+ = \partial_{a_j}, \quad \partial_{a_j}^+ = -a_j, \quad (ab)^+ = b^+\ a^+$$

and to prove that these axioms are consistent (see section 3).

Next we need the analogue of the scalar part projection $a \to [a]_o,\ a \in R_m$. To define this we first define the analogue of k-vectors for this algebra. This can be done by using the definition of the wedge product

$$\grave{\underline{x}}_1 \wedge \ldots \wedge \grave{\underline{x}}_k = \frac{1}{k!} \sum\ \mathrm{sgn}\ \pi\ \grave{\underline{x}}_{\pi(1)} \ldots \grave{\underline{x}}_{\pi(k)},$$

which comes from the wedge product on the radial algebra (see [So]). By deriving this relation with respect to the coordinates $\grave{x}_{ij}$ of $\grave{\underline{x}}_i$ one automatically arrives at the correct definition of the wedge product for the

generators $\partial_{a_j},\ a_j$ of the Weyl algebra and to the definition of k-vectors. Moreover, it is possible to write any element in the Weyl algebra in a unique way as $a = [a]_o + [a]_1 + [a]_2 + \ldots$ whereby $[a]_o$ is a real number and $[a]_k$ is a k-vector (see section 3.). In this way also the scalar part $[a]_o$ is well defined and one may consider the inner product on the Weyl algebra $(a, b) = [a^+\, b]_o$. We haven't obtained the full proof that this inner product is positive definite but checked this in several special cases.
Next one extends the definition of the adjoint to elements belonging to $\mathrm{Alg}\{\grave{x_1}, \ldots, \grave{x_{2n}}, \partial_{a_j}, a_j\}$ (which is the fermionic analogue of the algebra of Clifford polynomials) by putting

$$(\grave{x_j})^+ = \partial_{\grave{x_j}}, \quad (ab)^+ = b^+\, a^+$$

and one may introduce a positive definite Fischer inner product by putting

$$(R_k(\grave{\underline{x}}), S_k(\grave{\underline{x}})) = [R_k(\grave{\underline{x}})^+ S_k(\grave{\underline{x}})]_o$$

whereby both R_k and S_k are assumed to be homogeneous of degree k in the anti-commuting variables $\grave{x_j}$. Then it follows that every fermionic homogeneous polynomial of degree k, $R_k(\grave{\underline{x}})$ admits a unique orthogonal decomposition of the form

$$R_k(\grave{\underline{x}}) = M(R_k)(\grave{\underline{x}}) + \grave{\underline{x}} R_{k-1}(\grave{\underline{x}})$$

called Fischer decomposition, which after iteration leads to a complete Fischer decomposition.

2.2 Fischer decomposition on the level of radial algebra

Let S be a finite set of vector variables $S = \{u_1, \ldots, u_l\}$ then for $m \geq l$ the vector variable representation $x \to \underline{x} = \sum e_j\, x_j$ leads to an isomorphic embedding of the radial algebra $R(S)$ into the Clifford polynomial algebra (of several vector variables) in m dimensions. Hence the Fischer inner product on $R(S)$ may be inherited from this embedding and it is positive definite. Hence if we put $x = u_1$, any element $F \in R(S)$ may be written in a unique way as $F = M(F) + x\, G$ for some $G \in R(S)$ whereby $M(F) \in R(S)$ is monogenic in the sense that $\partial_x[M(F)] = 0$ with ∂_x the abstract vector derivative. Indeed, under the application $x \to \underline{x}$, the abstract vector derivative ∂_x with $\partial_x[x] = m$ corresponds to the Dirac operator $-\sum e_j \partial_{x_j}$ which is the adjoint with respect to the Fischer inner product of the vector variable $\underline{x}$. Now for a given fixed $F \in R(S)$ not depending on the extra scalar parameter m, both $M(F)$ and F will be available for any $m \geq l$ and they will be in fact functions of the dimension m, defined for integer values of m not less than l. Using the standard way to compute Fischer decomposition (using the action of powers of ∂_x on the identity $F = M(F) + x\, F'$) it is not hard to see that, as function of the parameter m, $M(F)$ is extendable to a meromorphic function. This means that the Fischer decomposition on the level of radial algebra exists for almost all complex values of m but

as we already know, there may be isolated poles which may belong to the set Z of integers. Hence the Fischer decomposition on the level of radial algebra may provide an answer for the Fischer decomposition on super-space, where in general the dimension $m \in Z$. Moreover in case a given dimension $m_o \in Z$ is a pole of $M(F)$ one may always take the average value of $M(F)$ and F' in the point $m = m_o$ to arrive at a decomposition $F = M(F)(m = m_o) + x\ F'(m = m_o)$ but the problem is that this average is no longer monogenic in x.
Finally, by adding sufficiently many parameters it is always possible to represent any Clifford polynomial on super-space by an element in some sufficiently large radial algebra (exercise). Note that one may also define the adjoint mapping on $R(S)$ directly by $(a\ b)^+ = b^+\ a^+,\ x^+ = \partial_x,\ x \in S$ and consider the Fischer inner product $(F, G)J = J\ F^+ G\ J$, with J the endomorphism which projects $F \in R(S)$ on its constant part (see also [So]). But this inner product is only positive definite for $m \geq l,\ m \in N$.

3 Super-Multivector Space, More Canonical Super-Manifolds.

In this section we'll treat more systematically the generalization of the conjugation $a \to \bar{a}$ and the notion of k-vector in R_m to the free associative algebra $\mathrm{Alg}\{e_{\grave{1}}, \ldots, e_{2\grave{n}}, e_1, \ldots, e_m\}$ generated by the basis elements used in our Clifford algebra administration. We also use the same notation on the extended Clifford algebra as for R_m, because for elements of the Weyl algebra $\bar{a} = -a^+$ rather that $\bar{a} = a^+$.
We first define the conjugate on the space $V = \mathrm{Span}\{e_{\grave{1}}, \ldots, e_{2\grave{n}}; e_1, \ldots, e_m\}$ simply by putting:

$$\bar{e}_{2\grave{j}-\grave{1}} = e_{2\grave{j}}, \quad \bar{e}_{2\grave{j}} = -e_{2\grave{j}-\grave{1}}, \quad \bar{e}_j = -e_j.$$

Next this mapping may be extended in a unique way to an anti-morphism on the tensor algebra TV which is the free associative algebra generated by V together with 1 with no additional relations. Now the generalized Clifford algebra $\mathrm{Alg}\{e_{\grave{1}}, \ldots, e_{2\grave{n}}, e_1, \ldots, e_m\}$ is simply the quotient of the tensor algebra with respect to the two sided ideal generated by the elements

$$\begin{array}{rcl} A_j & = & e_{2\grave{j}-\grave{1}}\, e_{2\grave{j}} - e_{2\grave{j}}\, e_{2\grave{j}-\grave{1}} - 1 \\ B_{jk} & = & e_{\grave{k}}\, e_{\grave{l}} - e_{\grave{l}}\, e_{\grave{k}} \quad \text{for} \quad \{k, l\} \neq \{2j-1, 2j\} \quad \text{some } j, \\ C_{jk} & = & e_{\grave{k}}\, e_l + e_l\, e_{\grave{k}} \\ D_{jk} & = & e_k\, e_l + e_l\, e_k + 2\delta_{kl}. \end{array}$$

Hence the mapping $a \to \bar{a}$ will be well defined as an anti-morphism $\mathrm{Alg}\{e_{\grave{1}}, \ldots, e_{2\grave{n}}, e_1, \ldots, e_m\}$, i.e. $\overline{a\, b} = \bar{b}\, \bar{a}$, provided that the generators of the two sided ideal are mapped on generators in a bijective way.
As $\bar{1} = 1$ the elements a_j and D_{jk} are invariant under conjugation while

the elements B_{jk} are permuted and C_{jk} are (up to the sign) permuted. Hence we have a well defined conjugation.
To define the notion of a k-vector we use the canonical notion of a k-vector defined for the radial algebra $R(S)$ by (see [So2]):

$$x_1 \wedge \ldots \wedge x_k = \frac{1}{k!} \sum \text{ sgn } \pi \; x_{\pi(1)} \ldots x_{\pi(k)}$$

and simply replace abstract vector variables by vector variables on super-space using the assignment

$$x_j \rightarrow \underline{x}_j\grave{} + \underline{x}_j = \sum e_k\grave{} \, x_{jk}\grave{} + \sum e_k \; x_{jk}.$$

Next one may derive the obtained formula for the wedge product

$$(\underline{x}_1\grave{} + \underline{x}_1) \wedge \ldots \wedge (\underline{x}_k\grave{} + \underline{x}_k) = \frac{1}{k!} \sum \ldots$$

left and with respect to all the present super coordinates $x_{jk}\grave{}, x_{jk}$ to end up with the definition of an associative wedge product for all the basis elements $e_j\grave{}, j = 1, \ldots, 2n, \; e_j, \; j = 1, \ldots, m$.
Note that in particular $e_j \wedge e_k = -e_k \wedge e_j$ as usual, $e_j \wedge e_k\grave{} = -e_k\grave{} \wedge e_j$ and $e_j\grave{} \wedge e_k\grave{} = e_k\grave{} \wedge e_j\grave{}$ and that

$$\begin{aligned} e_{a_1} \wedge \ldots \wedge e_{a_k} &= \frac{1}{k!} \sum \text{ sgn } \pi e_{a_{\pi(1)}} \ldots e_{a_{\pi(k)}} \\ e_{b_1}\grave{} \wedge \ldots \wedge e_{b_l}\grave{} &= \frac{1}{k!} \sum e_{b_{\pi(1)}}\grave{} \ldots e_{b_{\pi(l)}}\grave{}. \end{aligned}$$

Moreover any product of vector variables may be decomposed as

$$x_1 \ldots x_k = x_1 \wedge \ldots \wedge x_k + \sum \text{ scalars l.o.t},$$

whereby l.o.t. means lower order terms and similar decompositions remain valid for vector variables on super-space and also after derivation with respect to super-coordinates. It follows that every product of basis elements

$$e_{b_1}\grave{} \ldots e_{b_l}\grave{} \, e_{a_1} \ldots e_{a_k} = e_{b_1}\grave{} \wedge \ldots \wedge e_{b_l}\grave{} \wedge e_{a_1} \wedge \ldots \wedge e_{a_k} + \text{ l.o.m}$$

whereby l.o.m. means lower order multivectors (k'-vectors of order $k' < k + l$). It is also clear that the algebra $\text{Alg}\{e_1\grave{}, \ldots, e_{2n}\grave{}, \; e_1, \ldots, e_m\}$ decomposes as infinite direct sum of spaces $\text{Alg}_{l,k}$ of generalized l-vectors in the basis elements $e_1\grave{}, \ldots, e_{2n}\grave{}$ and k-vectors in the basis elements $e_1, \ldots, e_m$, and by $[a]_{l,k}$ we denote the projection of a general element of the real algebra $\text{Alg}\{e_1\grave{}, \ldots, e_{2n}\grave{}, \; e_1, \ldots, e_m\}$ onto $\text{Alg}_{l,k}$ and the projection onto the scalar part is denoted by $[a]_o = [a]_{o,o}$.
Next a non-degenerate bilinear form on the whole algebra is given by $(a, \; b) = [\bar{a} \; b]_o$ as for the Clifford algebra R_m, but it is not positive definite.

Note however that the vector variable is denoted by $\grave{\underline{x}} + \underline{x}$ while the Dirac operator acting on super-space $R_{2n,m}$ is given by

$$\partial_{\grave{\underline{x}}} - \partial_{\underline{x}} = \sum \grave{\bar{e}}_j \partial_{\grave{x}_j} + \sum \bar{e}_j \partial_{x_j},$$

i.e. one replaces the basis elements by their conjugate and the variables by the corresponding derivatives. One could extend this to a Fischer adjoint by considering the map $R(\grave{x}_j, x_j) \to \bar{R}(\partial_{\grave{x}_j}, \partial_{x_j})$ and consider the corresponding Fischer inner product. But it is no positive definite inner product so that one has to be careful with orthogonality arguments.
We haven't defined super-multivector space yet; to define it we first introduce the super-space of (l,k)-vectors $R_{2n,m;l,k}$ by stating that a variable of that space has the form

$$x = x_{\langle l,k \rangle} = \sum x_{b_1} \ldots {}_{b_l; a_1} \ldots {}_{a_k} \quad \grave{e}_{b_1} \wedge \ldots \wedge \grave{e}_{b_l} \wedge e_{a_1} \wedge \ldots \wedge e_{a_k}$$

whereby for l even, $x_{b_1} \ldots b_l; a_1 \ldots a_k$ is a commuting variable while for l odd it is an anti-commuting variable. The super-space $R_{2n,m;K}$ of super K-vectors is then formally given by $R_{2n,m;K} = \sum\limits_{k+l=K} R_{2n,m;l,k}$ which in fact means that the general variable of this space is given by

$$x = x_{\langle R \rangle} = \sum_{l+k=K} x_{\langle l,k \rangle}$$

whereby the multivector variable $x_{\langle l,k \rangle}$ is as stated above.
This notion of super K-vector corresponds to the definition of an abstract K-vector given in [So3], i.e. it is in accordance with the radial algebra. In particular for any set of abstract vector variables, $\{x_1, \ldots, x_K\} \subset S$, the wedge product $x_1 \wedge \ldots \wedge x_K$ is called a K-vector in radial algebra, but it is a K-vector of a very special type as opposed to the K-vector variables $x = x_{\langle R \rangle}$; they are so called Grassmann K-vectors or pure K-vectors and using the Clifford vector representation $x \to \underline{x}$, the wedge product $x_1 \wedge \ldots \wedge x_K$ corresponds to a varying element of a cone inside the space of K-vectors of which the manifold of rays is the Grassmann manifold $G_{m,K}(R)$. In other words, the equation $x_{\langle R \rangle} = x_1 \wedge \ldots \wedge x_K$ may represent any such cone; it is called a "formal multivector manifold" which exists on the canonical level of radial algebra in the form of an equation. By now using the super-vector representation for $R(S) : x \to \grave{\underline{x}} + \underline{x}$, this formal manifold is mapped onto the set of solutions to the equation

$$x_{\langle R \rangle} = (\grave{\underline{x}}_1 + \underline{x}_1) \wedge \ldots \wedge (\grave{\underline{x}}_K + \underline{x}_K),$$

which is a super-surface of conical type inside the super-space $R_{2n,m;K}$ and the supermanifold of super-rays is the super-Grassmannian. In other words special super manifolds may be produced as a result of applying the super-multivector representation on radial algebra; they are in fact the image of the formal multivector manifolds which are defined as equations on the abstract level of radial algebra. Hence radial algebra is also a good basis for super-manifold theory in the sense that it produces the canonical examples. All this is still in an early stage of development and many ideas are to be expected. We therefore think we can suffice by listing a few examples of formal multivector manifolds.

(i) the sphere: given by the equation $x^2 = -1,\ x$ being an abstract vector variable. This abstract sphere projects down to all spheres and super-spheres.

(ii) the nullcone: given by the equation $x^2 = 0,\ x$ being an abstract vector variable.

(iii) the manifold of unit K-frames $(x_1, \ldots, x_K)$ (Stiefel manifold) equations: $x_1^2 = \ldots = x_K^2 = -1,\ x_i\ x_j = -x_j\ x_i$ for $j \neq i$ this leads to a definition of super-Stiefel manifolds and the manifold of wedge produces $x_1 \wedge \ldots \wedge x_K$ is another way to introduce formal and super-Grassmannians.

(iv) the manifold of nullframes $(x_1, \ldots, x_K)$ with equations $x_1^2 = \ldots = x_K^2 = 0,\ x_i\ x_j = -x_j\ x_i$. This leads to super-nullframes and in particular to super-twistor-space.

(v) Let B be a formal bivector and $[\cdot]_o$ denote the scalar part; then the equation $B^2 = [B^2]_o$ or $[B^2]_4 = 0$ is another equation for the manifold of pure bivectors $x_1 \wedge x_2$.

This shows that there is a huge class of formal and super-manifolds waiting to be studied.
We finish this paper with the definition of the multivector derivative in the setting of super-symmetry, thus generalizing the Dirac operator.
It is done simply by assigning to the K-vector variable $x = x_{\langle R\rangle}$ the Fischer adjoint $\partial_x = \partial_{x_{\langle R\rangle}}$ which is obtained by replacing

(i) coordinates $x_{b_1 \ldots b_l; a_1 \ldots a_k}$ by coordinate derivatives $\partial_{x_{b_1} \ldots b_l; a_1 \ldots a_k}$

(ii) $e_{b_1}` \wedge \ldots \wedge e_{b_l}` \wedge e_{a_1} \wedge \ldots \wedge e_{a_k}$ by $\bar{e}_{a_k} \wedge \ldots \wedge \bar{e}_{a_1} \wedge \bar{e}_{b_l}` \wedge \ldots \wedge \bar{e}_{b_1}`$

i.e. if we denote $x_{B;A} = x_{b_1 \ldots b_l; a_1 \ldots a_k},\ e_{B;A} = e_{b_1}` \wedge \ldots \wedge e_{b_l}` e_{a_1} \wedge \ldots \wedge e_{a_k}$ for short, then we put $x = x_{\langle R\rangle} = \sum_{|B|+|A|=K} x_{B;A}\ e_{B;A}$ and we have that

$$\partial_x = \partial_{x_{\langle R\rangle}} = \sum_{|A|+|B|=K} \partial_{x_{B;A}}\ \bar{e}_{B;A}$$

In the usual case, the number $\partial_x[x]$ is the dimension of K-vector space. In the super-symmetry case this number $\partial_x[x]$ can be an integer which is still thought of as the formal dimension of $R_{2n,m;K}$.
One can now go on developing super Clifford analysis parallell to Clifford analysis on multivector space.

References

[Be] F.A. Berezin, Introduction to algebra and analysis with anti-commuting variables, (Moscow University, Moscow, 1983); English transl.: Introduction to superanalysis, (Reidel, Dordrecht, 1987).

[Cr] A. Crumeyrolle, Orthogonal and symplectic Clifford algebras, Mathematics and Its Applications 57, (Kluwer Acad. Publ., Dordrecht 1990).

[CRS] W. Chan, G.C. Rota, J.A. Stein, The power of positive thinking, preprint.

[DSS] R. Delanghe, F. Sommen, V. Souček, Clifford algebra and spinor valued functions: a function theory for the Dirac operator, Math. and Its Appl 53 (Kluwer Acad. Publ., Dordrecht, 1992).

[Ha] K. Habermann, The Dirac Operator on Symplectic Spinors, preprint.

[HS] D. Hestenes, G. Sobczyk, Clifford algebra to geometric calculus, (D. Reidel, Dordrecht, 1985).

[Pa] V.P. Palamodov, Cogitations over Berezin's integral, Preprint series No. 14, Univ. of Oslo, 1994.

[So1] F. Sommen, An extension of Clifford analysis towards super-symmetry, to appear.

[So2] F. Sommen, An algebra of abstract vector variables, Portugaliae Math. Vol. 54, Fasc. 3 (1997) 287-310.

[So3] F. Sommen, The problem of defining abstract bivectors, Result. Math. 31 (1997) 148-160.

[VV] V.S. Vladimirov, I.V. Volovich, Superanalysis, Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 59, No. 1 (1984) 3-27.