An alternative definition of the notion valuation in the theory of near polygons

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Abstract

Valuations of dense near polygons were introduced in [9]. A valuation of a dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a map f from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the set $\mathbb{N}$ of nonnegative integers satisfying very nice properties with respect to the set of convex subspaces of $\mathcal{S}$. In the present paper, we give an alternative definition of the notion valuation and prove that both definitions are equivalent. In the case of dual polar spaces and many other known dense near polygons, this alternative definition can be significantly simplified.

1 Introduction

1.1 Basic definitions

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point on L nearest to p. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called *generalized quadrangles* (Payne and Thas [11]).

If X_1 and X_2 are two nonempty sets of points of a near polygon $\mathcal{S}$, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If $X_1 = \{x_1\}$, we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. For every nonempty set X of points of $\mathcal{S}$ and every $i \in \mathbb{N}$, $\Gamma_i(X)$ denotes the set of all points y of $\mathcal{S}$ for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

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A nonempty set X of points of a near polygon $\mathcal{S}$ is called a *subspace* if every line meeting X in at least two points has all its points in X. A subspace X is called *convex* if every point on a shortest path between two points of X is also contained in X. Having a subspace X, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are contained in X. If X is a convex subspace, then $\mathcal{S}_X$ is a sub-near-polygon of $\mathcal{S}$. If $X_1, X_2, \ldots, X_k$ are objects of $\mathcal{S}$ (like points, and nonempty sets of points), then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace of $\mathcal{S}$ containing $X_1, X_2, \ldots, X_k$. Obviously, $\langle X_1, X_2, \ldots, X_k \rangle$ is the intersection of all convex subspaces containing $X_1, X_2, \ldots, X_k$. The maximal distance between two points of a convex subspace F of $\mathcal{S}$ is called the *diameter* of F.

A near polygon $\mathcal{S}$ is called *dense* if every line of $\mathcal{S}$ is incident with at least three points and if every two points of $\mathcal{S}$ at distance 2 have at least two common neighbours. By Theorem 4 of Brouwer and Wilbrink [3], every two points of a dense near polygon at distance δ from each other are contained in a unique convex sub-2δ-gon. These convex sub-2δ-gons are called *quads* if $\delta = 2$ and *hexes* if $\delta = 3$. The existence of quads in dense near polygons was already shown in Shult and Yanushka [12, Proposition 2.5]. With every point x of a dense near polygon $\mathcal{S}$, there is associated a linear space $\mathcal{L}(\mathcal{S}, x)$ which is called the *local space at* x. The points, respectively lines, of $\mathcal{L}(\mathcal{S}, x)$ are the lines, respectively quads, through x, and incidence is containment.

1.2 The main results

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon. A function f from $\mathcal{P}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties (we call $f(x)$ the *value* of x):

(V1) there exists at least one point with value 0;

(V2) every line L of $\mathcal{S}$ contains a unique point x_L with smallest value and $f(x) = f(x_L)+1$ for every point x of L different from x_L;

(V3) every point x of $\mathcal{S}$ is contained in a (necessarily unique) convex subspace F_x which satisfies the following properties:

- (i) $f(y) \leq f(x)$ for every point y of F_x.
- (ii) every point z of $\mathcal{S}$ which is collinear with a point y of F_x and which satisfies $f(z) = f(y) - 1$ also belongs to F_x.

Examples. (1) Let x be a given point of $\mathcal{S}$ and define $f(y) := \mathrm{d}(x, y)$ for every $y \in \mathcal{P}$. Then f is a valuation of $\mathcal{S}$. We call f a *classical valuation* of $\mathcal{S}$.

(2) Let O be an ovoid of $\mathcal{S}$, i.e. a set of points of $\mathcal{S}$ meeting each line in a unique point. Then define $f(x) := 0$ if $x \in O$ and $f(x) := 1$ if $x \in \mathcal{P} \setminus O$. Then f is a valuation of $\mathcal{S}$. We call f an *ovoidal valuation of* $\mathcal{S}$.

Consider the following property for a function $f : \mathcal{P} \to \mathbb{N}$:

($V3'$) Through every point x of $\mathcal{S}$, there exists a convex subspace F_x of $\mathcal{S}$ such that the lines through x contained in F_x are precisely the lines through x containing a point with value $f(x) - 1$.

If Property (V3') is satisfied, then the convex subspace F_x through x is uniquely determined: if $\mathcal{L}_x$ denotes the set of lines through x containing a point with value $f(x) - 1$, then F_x coincides with the smallest convex subspace of $\mathcal{S}$ containing all lines of $\mathcal{L}_x$. (By Brouwer and Wilbrink [3], see also [7, Theorem 2.14], a convex subspace F of $\mathcal{S}$ is completely determined by the set of lines of F through one of its points.)

The following theorem provides an alternative definition of the notion valuation.

Theorem 1.1 (Section 2) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon and let f be a map from $\mathcal{P}$ to $\mathbb{N}$. Then f is a valuation of $\mathcal{S}$ if and only if f satisfies Properties (V1), (V2) and (V3').*

It will turn out that in many dense near polygons, Property (V3') is a consequence of Property (V2). We first observe the following.

Theorem 1.2 (Section 2) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon and let f be a map from $\mathcal{P}$ to $\mathbb{N}$ satisfying Property ($V2$). Then for every point x of $\mathcal{S}$, the set of lines through x containing a point with value $f(x) - 1$ is a subspace of the local space $\mathcal{L}(\mathcal{S}, x)$.*

Definition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon and let x be a point of $\mathcal{S}$. We say that the local space $\mathcal{L}(\mathcal{S}, x)$ at x is *regular* if for every subspace S of $\mathcal{L}(\mathcal{S}, x)$, there exists a convex subspace F_S through x such that the lines through x contained in F_S are precisely the elements of S.

In Sections 3 and 4, we will give a description of the known examples of dense near polygons. Among other examples, we will discuss there the class of the dual polar spaces and two near hexagons which we will denote by $\mathbb{E}_1$ and $\mathbb{E}_2$. We will prove the following:

Theorem 1.3 (Section 3) (a) *All local spaces of a thick dual polar space are regular.*

(b) *Let $\mathcal{S}$ be a known dense near polygon not containing hexes isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$. Then every local space of $\mathcal{S}$ is regular.*

Remarks. (1) Every local space of the near hexagon $\mathbb{E}_1$ is isomorphic to the complete graph of 12 vertices (regarded as linear space). No such local space is regular: subspaces containing $i \in \{3, 4, \ldots, 11\}$ points do not correspond with convex subspaces.

(2) Every local space of the near hexagon $\mathbb{E}_2$ is isomorphic to $\mathrm{PG}(3, 2)$ (regarded as linear space). No such local space is regular: subspaces carrying the structure of a $\mathrm{PG}(2, 2)$ do not correspond with convex subspaces.

By Theorems 1.1, 1.2 and 1.3, we have

Corollary 1.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon every local space of which is regular. Then a map $f : \mathcal{P} \to \mathbb{N}$ is a valuation of $\mathcal{S}$ if and only if it satisfies Properties $(V1)$ and $(V2)$. In particular, this holds if $\mathcal{S}$ is a thick dual polar space or a known dense near polygon without hexes isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$.*

The following theorem shows that the conclusion of Corollary 1.4 is not necessarily true in case there are hexes isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$.

Theorem 1.5 (Section 4) *Let $\mathcal{S}$ be a near hexagon isomorphic to either $\mathbb{E}_1$ or $\mathbb{E}_2$. Then there exists a map $f : \mathcal{P} \to \mathbb{N}$ which satisfies Properties (V1) and (V2) and which is not a valuation of $\mathcal{S}$.*

2 Proofs of Theorems 1.1 and 1.2

In this section $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon and f is a map from $\mathcal{P}$ to the set $\mathbb{N}$ of nonnegative integers.

Lemma 2.1 *If f is a valuation of $\mathcal{S}$, then f satisfies Property (V3').*

Proof. Let x be an arbitrary point of $\mathcal{S}$ and let F_x denote the necessarily unique convex subspace through x for which Property (V3) is satisfied. Let L be an arbitrary line through x.

If $L \subseteq F_x$, then by (V3,i), $f(y) \leq f(x)$ for every $y \in L$. Hence, L contains a unique point with value $f(x) - 1$ by (V2).

Conversely, suppose that L contains a point with value $f(x) - 1$. Then $L \subseteq F_x$ by (V3,ii).

So, $F'_x := F_x$ is the unique convex subspace of $\mathcal{S}$ through x such that the lines through x contained in F'_x are precisely the lines through x containing a point with value $f(x) - 1$. ■

Lemma 2.2 *Suppose f satisfies Property (V2) and let Q be a quad of $\mathcal{S}$. Then precisely one of the following two cases occurs:*

(i) there exists a point $x^* \in Q$ such that $f(x) = f(x^*) + d(x^*, x)$ for every $x \in Q$;

(ii) there exists an ovoid O of Q and an $m^* \in \mathbb{N}$ such that $f(x) = m^* + d(x, O)$ for every $x \in Q$.

Proof. It is well-known that for every point x of Q, the set $\{x\} \cup (\Gamma_1(x) \cap Q)$ is a maximal subspace of Q. This implies that the set $\Gamma_2(x) \cap Q$ is connected. We can distinguish the following two cases:

(i) There exist points $x^*, y_1 \in Q$ such that $f(y_1) - f(x^*) \geq 2$. By Property (V2), $\mathrm{d}(y_1, x^*) \geq 2$. Hence, $\mathrm{d}(y_1, x^*) = 2$ and $f(y_1) - f(x^*) = 2$. Every point collinear with x^* and y_1 has value $f(x^*) + 1$ by (V2). Hence, also every point of $\Gamma_1(x^*) \cap Q$ has value $f(x^*) + 1$ by (V2).

Now, suppose y_2 is a point of $Q \cap \Gamma_2(x^*)$ at distance 1 from y_1. Then $f(y_1) = f(x^*)+2$ and the unique point on the line y_1y_2 collinear with x^* has value $f(x^*) + 1$. By (V2) applied to the line y_1y_2, it follows that $f(y_2) = f(x^*) + 2$.

Now, invoking the connectedness of $\Gamma_2(x^*) \cap Q$, we see that every point of $\Gamma_2(x^*) \cap Q$ has value $f(x^*) + 2$.

Summarizing, we have that $f(x) = f(x^*) + \mathrm{d}(x^*, x)$ for every $x \in Q$.

(ii) $|f(x_1) - f(x_2)| \leq 1$ for any two points x_1 and x_2 of Q. Put $m^* := \min\{f(x) \,|\, x \in Q\}$. Then $f(x) \in \{m^*, m^* + 1\}$. Since every line of Q contains a unique point with smallest value, the set of points of Q with value m^* is an ovoid of Q. Hence, $f(x) = m^* + \mathrm{d}(x, O)$ for every $x \in Q$. ■

The following lemma is precisely Theorem 1.2.

Lemma 2.3 *If f satisfies Property (V2), then for every point x of $\mathcal{S}$, the set of lines through x containing a point with value $f(x) - 1$ is a subspace of the local space $\mathcal{L}(\mathcal{S}, x)$.*

Proof. Let L_1 and L_2 be two lines through x containing a (unique) point with value $f(x) - 1$ and let Q denote the unique quad through L_1 and L_2. We need to show that every line of Q through x contains a point with value $f(x) - 1$. By Lemma 2.2, one of the following two cases occurs:

(1) There exists a point $x^* \in Q$ such that $f(u) = f(x^*) + \mathrm{d}(x^*, u)$ for every $u \in Q$. Since there are at least two lines of Q through x containing a point with value $f(x) - 1$, we necessarily have $\mathrm{d}(x^*, x) = 2$. But then every line of Q through x contains a unique point with value $f(x) - 1 = f(x^*) + 1$, namely the unique point on that line collinear with x^*.

(2) There exists an ovoid O of Q and an $m^* \in \mathbb{N}$ such that $f(u) = m^* + \mathrm{d}(u, O)$ for every $u \in Q$. Since L_1 and L_2 contain points with value $f(x) - 1$, x does not belong to O. Clearly, every line of Q through x contains a unique point with value $f(x) - 1$, namely the unique point on that line belonging to O. ■

Lemma 2.4 *Suppose f satisfies Property (V2). Let F be a convex subspace of $\mathcal{S}$ and put $M := \max\{f(x) \,|\, x \in F\}$. If x and y are two points of F such that $f(x) = f(y) = M$, then x and y are connected by a path which entirely consists of points of F with value equal to M.*

Proof. We will prove the lemma by induction on $\mathrm{d}(x, y)$. The lemma trivially holds if $\mathrm{d}(x, y) \leq 1$. So, suppose $\mathrm{d}(x, y) \geq 2$. Let L_x be an arbitrary line through x contained in the convex subspace $\langle x, y\rangle \subseteq F$ and let u denote the unique point on L_x at distance $\mathrm{d}(x, y) - 1$ from y. Let L_y denote a line of $\langle x, y\rangle$ through y not contained in $\langle u, y\rangle$. Then every point of L_x has distance $\mathrm{d}(x, y)-1$ from a unique point of L_y. Now, since (V2) holds, the lines L_x and L_y contain unique points with value $M-1$. So, since $|L_x|, |L_y| \geq 3$, there exist points $x' \in L_x$ and $y' \in L_y$ such that $\mathrm{d}(x', y') = \mathrm{d}(x, y) - 1$ and $f(x') = f(y') = M$. By the induction hypothesis, x' and y' are connected by a path which entirely consists of points of F with value equal to M. Hence, also x and y are connected by a path which entirely consists of points of F with value equal to M. ■

Lemma 2.5 *Suppose f satisfies Property (V2). Let Q be a quad of $\mathcal{S}$, let x and y be two distinct collinear points of Q such that $f(x) = f(y)$ and let L_x and L_y be two lines of Q different from xy such that $x \in L_x$ and $y \in L_y$. Then the following holds: if L_x contains a point with value $f(x) - 1$, then L_y contains a point with value $f(y) - 1$.*

Proof. By Lemma 2.2, we can distinguish two possibilities:

(1) There exists a point $x^* \in Q$ such that $f(u) = f(x^*) + \mathrm{d}(x^*, u)$ for every point $u \in Q$. Since $f(x) = f(y)$, either x^*, x, y are contained on a line or $x, y \in Q \cap \Gamma_2(x^*)$. In the former case, no line of Q through x distinct from xy contains a point with value $f(x) - 1$. In the latter case, every line of Q through x contains a unique point with value $f(x) - 1$. But in this case, also every line of Q through y contains a unique point with value $f(y) - 1$.

(2) There exists an ovoid O of Q and an $m^* \in \mathbb{N}$ such that $f(u) = m^* + \mathrm{d}(u, O)$ for every $u \in Q$. Then $x, y \notin O$. In this case, every line of Q through x contains a unique point with value $f(x) - 1$ and every line of Q through y contains a unique point with value $f(y) - 1$. ∎

Definition. If f satisfies Property (V3'), then for every point x of $\mathcal{S}$, we denote by F_x the unique convex subspace of $\mathcal{S}$ through x such that the lines through x contained in F_x are precisely the lines through x containing a point with value $f(x) - 1$.

Lemma 2.6 *If f satisfies Properties (V2) and (V3'), then f also satisfies Property (V3,i) with respect to the convex subspaces F_x, $x \in \mathcal{P}$.*

Proof. Let x be a point of $\mathcal{S}$. We need to show that $f(y) \leq f(x)$ for every point y of F_x. Suppose the contrary holds. Then choose a $y \in F_x$ such that $f(y) > f(x)$ with $\mathrm{d}(x, y)$ as small as possible. Let y_1 be a point of F_x collinear with y at distance $\mathrm{d}(x, y) - 1$ from x. Then since $\mathrm{d}(x, y_1) < \mathrm{d}(x, y)$, $f(y_1) \leq f(x)$. Hence, $f(y) = f(x) + 1$ and $f(y_1) = f(x)$. By Lemma 2.4, there now exists a path $y_1, y_2, \ldots, y_k = x$ in $\langle x, y_1 \rangle$ connecting y_1 with x such that $f(y_i) = f(x)$ for every $i \in \{1, \ldots, k\}$. (Since $\mathrm{d}(u, x) \leq \mathrm{d}(x, y_1) < \mathrm{d}(x, y)$, we have $f(u) \leq f(x)$ for every $u \in \langle x, y_1 \rangle$.) We now inductively define a line L_i, $i \in \{1, \ldots, k\}$, of F_x through y_i and show that this line contains a point with value $f(x) + 1$. Put $L_1 := y_1 y \subseteq F_x$. As remarked above y has value $f(x) + 1$. Suppose now that for a certain $i \in \{1, \ldots, k-1\}$, we have defined the line L_i. Since L_i contains a point with value $f(x)+1$ and $y_i y_{i+1}$ contains a point with value $f(x) - 1$ (recall (V2)), we have $L_i \neq y_i y_{i+1}$. Now, let Q denote the unique quad through L_i and $y_i y_{i+1}$ and let L_{i+1} be a line of Q through y_{i+1} distinct from $y_i y_{i+1}$. Since $f(y_{i+1}) = f(x)$, there are two possibilities by Property (V2). Either L_{i+1} contains a point with value $f(x) - 1$ or a point with value $f(x) + 1$. In the former case, it would follow from Lemma 2.5, that also L_i would contain a point with value $f(x) - 1$, a contradiction. Hence, L_{i+1} contains a point with value $f(x) + 1$. Also, since L_i and $y_i y_{i+1}$ are contained in F_x, the quad Q is contained in F_x and hence $L_{i+1} \subseteq F_x$.

A contradiction is now readily obtained. The line L_k through $y_k = x$ is contained in F_x and contains a point with value $f(x) + 1$. But by (V3'), we would also have that L_k

contains a point with value $f(x)-1$. So, our assumption was wrong and $f(y) \leq f(x)$ for every point y of F_x. ∎

Lemma 2.7 *Suppose f satisfies Properties (V2) and (V3'). Then $F_y = F_x$ for every point x of $\mathcal{S}$ and every $y \in F_x$ with $f(y) = f(x)$.*

Proof. By Lemmas 2.4 and 2.6, there exists a path $y = y_1, y_2, \dots, y_k = x$ which entirely consists of points of F_x with value $f(x)$. We show the following by downwards induction on $i \in \{1, 2, \dots, k\}$:

- If L is a line through y_i containing a point with value $f(x)-1$, then $L \subseteq F_x$.
- If L is a line through y_i containing a point with value $f(x)+1$, then L is not contained in F_x.

Obviously, this claim holds if $i = k$. So, suppose $i < k$ and that the claim holds for the number $i+1$.

Let L be a line through y_i containing a point with value $f(x)-1$. If $L = y_iy_{i+1}$, then $L \subseteq F_x$. So, suppose $L \neq y_iy_{i+1}$. Let L' be a line through y_{i+1} distinct from y_iy_{i+1} contained in the quad $\langle L, y_iy_{i+1}\rangle$. By Lemma 2.5, L' contains a point with value $f(x)-1$. Hence, $L' \subseteq F_x$ by the induction hypothesis. Since L' and y_iy_{i+1} are contained in F_x, the quad $\langle L', y_iy_{i+1}\rangle = \langle L, y_iy_{i+1}\rangle$ is contained in F_x. Hence, $L \subseteq F_x$.

Let L be a line through y_i containing a point with value $f(x)+1$. Then L cannot be contained in F_x by Lemma 2.6.

Hence, the above claim holds for every $i \in \{1, 2, \dots, k\}$. The fact that it holds for $i = 1$ implies that $F_y = F_x$. ∎

Lemma 2.8 *Suppose f satisfies Properties (V2) and (V3'). Then f also satisfies Property (V3,ii) with respect to the convex subspaces F_x, $x \in \mathcal{P}$.*

Proof. Let x be an arbitrary point of $\mathcal{S}$, let y be a point of F_x and let z be a point collinear with y for which $f(z) = f(y)-1$. We need to show that $z \in F_x$. We will prove this by induction on the number $f(x)-f(y)$ (which is nonnegative by Lemma 2.6). If $f(x) = f(y)$, then we have that $z \in F_y = F_x$ by Lemma 2.7. So, suppose $f(x) > f(y)$. Then $x \notin F_y$ by Lemma 2.6. So, $F_x \not\subseteq F_y$ and there exists a line L through y contained in F_x but not in F_y. Let u be an arbitrary point of $L \setminus \{y\}$. Then $f(u) = f(y)+1$ and hence $\mathrm{d}(u,z) = 2$ by (V2). Let v denote a common neighbour of u and z distinct from y. Then $f(v) = f(y)$. The line uv is a line through $u \in F_x$ containing a point with value $f(u)-1$, namely the point v. By the induction hypothesis, $uv \subseteq F_x$. Since also $L \subseteq F_x$, the quad $\langle uv, L\rangle = \langle u, z\rangle$ is contained in F_x. Hence, $z \in F_x$. ∎

Theorem 1.1 is an immediate corollary of Lemmas 2.1, 2.6 and 2.8.

3 Proof of Theorem 1.3

Every known dense near polygon without hexes isomorphic to $\mathbb{E}_1$ and $\mathbb{E}_2$ is up to isomorphism either a line, a thick dual polar space of rank $n \geq 2$, a near polygon $\mathbb{I}_n$ for some $n \geq 2$, a near polygon $\mathbb{H}_n$ for some $n \geq 2$, a near $2n$-gon $\mathbb{G}_n$ for some $n \geq 2$, the near hexagon $\mathbb{E}_3$, or is obtained from these near polygons by successive application of the product and glueing constructions. So, in order to prove Theorem 1.3, it suffices to verify the following: (I) all local spaces of a thick dual polar space of rank $n \geq 2$ are regular; (II) all local spaces of the near $2n$-gon $\mathbb{I}_n$, $n \geq 2$, are regular; (III) all local spaces of the near $2n$-gon $\mathbb{H}_n$, $n \geq 2$, are regular; (IV) all local spaces of the near $2n$-gon $\mathbb{G}_n$, $n \geq 2$, are regular; (V) all local spaces of $\mathbb{E}_3$ are regular; (VI) if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two dense near polygons of diameter at least 1 such that every local space of $\mathcal{A}_i$, $i \in \{1, 2\}$, is regular, then also every local space of the product near polygon $\mathcal{A}_1 \times \mathcal{A}_2$ is regular; (VII) if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two dense near polygons of diameter at least 2 such that every local space of $\mathcal{A}_i$, $i \in \{1, 2\}$, is regular, then also every local space of any glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ is regular.

(I) Let Π be a nondegenerate thick polar space of rank $n \geq 2$. With Π there is associated a dual polar space Δ whose points are the maximal (i.e. $(n-1)$-dimensional) totally singular subspaces of Π and whose lines are the $(n-2)$-dimensional totally singular subspaces of Π (natural incidence). If γ is an $(n-1-k)$-dimensional totally singular subspace of Π, then the set of all maximal singular subspaces of Π containing γ is a convex subspace F_γ of Δ. Conversely, every convex subspace of Δ is obtained in this way. Now, let α be an arbitrary point of Δ. Then α can be regarded as an $(n-1)$-dimensional projective space. From this point of view, the local space $\mathcal{L}(\Delta, \alpha)$ at α is nothing else than the dual projective space associated with α. If S is a subspace of $\mathcal{L}(\Delta, \alpha)$, then S consists of all hyperplanes of α which contain a given subspace β of α. Then S, regarded as set of lines of Δ, consists of all lines of Δ through α contained in F_β. This proves that every local space of Δ is regular.

(II) Consider a nonsingular parabolic quadric $Q(2n, 2)$, $n \geq 2$, in $\mathrm{PG}(2n, 2)$ and a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$. Let $\mathbb{I}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry: (i) $\mathcal{P}$ is the set of all maximal subspaces (of dimension $n-1$) of $Q(2n, 2)$ not contained in $Q^+(2n-1, 2)$; (ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $Q(2n, 2)$ not contained in $Q^+(2n-1, 2)$; (iii) incidence is reverse containment. Then by Brouwer et al. [2], $\mathbb{I}_n$ is a dense near $2n$-gon. If γ is an $(n-1-k)$-dimensional subspace of $Q(2n, 2)$ which is not contained in $Q^+(2n-1, 2)$ if $k \in \{0, 1\}$, then the set F_γ of all maximal subspaces of $Q(2n, 2)$ containing γ is a convex subspace F_γ of $\mathbb{I}_n$. Conversely, every convex subspace of $\mathbb{I}_n$ is obtained in this way. It follows that every local space of $\mathbb{I}_n$ is isomorphic to the projective space $\mathrm{PG}(n-1, 2)$ (regarded as linear space) in which a point has been removed. Specifically, if α is a point of $\mathbb{I}_n$, then $\mathcal{L}(\mathbb{I}_n, \alpha)$ is the dual projective space associated with $\alpha \cong \mathrm{PG}(n-1, 2)$ in which the point $\alpha \cap Q^+(2n-1, 2)$ has been removed. Now, let S be a subspace of $\mathcal{L}(\mathbb{I}_n, \alpha)$. Then there exists a subspace β in α distinct from $\alpha \cap Q^+(2n-1, 2)$ such that S consists of all hyperplanes of α through β distinct from $\alpha \cap Q^+(2n-1,2)$. The following obviously holds: the lines through α contained in the convex subspace F_β are precisely the elements of S. This proves that every local space of $\mathbb{I}_n$ is regular. More information on the convex subpolygons of the dense near $2n$-gon $\mathbb{I}_n$ can be found in [7, Section 6.4].

(III) Let A be a set of size $2n+2$, $n \geq 2$. Let $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry: (i) $\mathcal{P}$ is the set of all partitions of A in $n+1$ subsets of size 2; (ii) $\mathcal{L}$ is the set of all partitions of A in $n-1$ subsets of size 2 and one subset of size 4; (iii) a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition defined by p is a refinement of the partition defined by L. By Brouwer et al. [2], $\mathbb{H}_n$ is a dense near $2n$-gon. If $\mathcal{M}_n$ denotes the partial linear space whose points, respectively lines, are the subsets of size 2, respectively size 3, of the set $\{A_1, A_2, \ldots, A_{n+1}\}$ (natural incidence), then every local space of $\mathbb{H}_n$ is isomorphic to the linear space $\mathcal{L}_n$ obtained from $\mathcal{M}_n$ by adding lines of size 2. In fact, for every point x of $\mathbb{H}_n$, we can construct the following explicit isomorphism ϕ_x between $\mathcal{L}(\mathbb{H}_n, x)$ and $\mathcal{L}_n$. Recall that the point x is a partition $\{A_1, A_2, \ldots, A_{n+1}\}$ of A in $n+1$ subsets of size 2. Then for every line L of $\mathbb{H}_n$, put $\phi_x(L) := \{A_i, A_j\}$ where A_i and A_j are the unique elements of $\{A_1, A_2, \ldots, A_{n+1}\}$ such that $A_i \cup A_j$ is contained in the partition defined by L.

Now, let x be an arbitrary point of $\mathbb{H}_n$ and let S be an arbitrary subspace of $\mathcal{L}(\mathbb{H}_n, x)$. As before, let $\{A_1, \ldots, A_{n+1}\}$ be the partition of A corresponding with x and let ϕ_x be the isomorphism between $\mathcal{L}(\mathbb{H}_n, x)$ and $\mathcal{L}_n$ as defined above. Then $\phi_x(S)$ is a subspace of $\mathcal{L}_n$. So, there exist mutually disjoint subsets $\alpha_1, \ldots, \alpha_k$ ($k \geq 0$) of size at least 2 of $\{A_1, A_2, \ldots, A_{n+1}\}$ such that the points of $\phi_x(S)$ are precisely the pairs of $\{A_1, \ldots, A_n\}$ which are contained in α_i for some $i \in \{1, \ldots, k\}$. Now, for every $i \in \{1, \ldots, k\}$, put $B_k := \bigcup_{C \in \alpha_i} C$ and let $B_{k+1}, \ldots, B_l$ denote those elements of $\{A_1, A_2, \ldots, A_{n+1}\}$ which are not contained in $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$. Then $\{B_1, B_2, \ldots, B_l\}$ is a partition of A in subsets of even size. By Theorem 6.15 of [7], the set of points of $\mathbb{H}_n$ which regarded as partitions of A are refinements of $\{B_1, B_2, \ldots, B_l\}$ is a convex subspace F_S of $\mathbb{H}_n$. The lines of $\mathbb{H}_n$ through x contained in F_S are precisely the elements of S. This proves that all local spaces of $\mathbb{H}_n$ are regular. More information on the convex subspaces of the dense near $2n$-gon $\mathbb{H}_n$ can be found in [7, Chapter 6.2].

(IV) Let $H(2n-1,4)$, $n \geq 2$, denote the Hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $\mathrm{PG}(2n-1,4)$ (with respect to a given reference system). If p is a point of $\mathrm{PG}(2n-1,4)$, then the number of nonzero coordinates of p (with respect to the same reference system) is called the *weight* of p. The set of all $i \in \{0, 1, \ldots, 2n-1\}$ such that the i-th coordinate of p is nonzero is called the *support* of p. The Hermitian variety $H(2n-1,4)$ consists of all points of $\mathrm{PG}(2n-1,4)$ of even weight.

Let $\mathbb{G}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry: (i) $\mathcal{P}$ is the set of all maximal subspaces of $H(2n-1,4)$ generated by n points of weight 2 whose supports are mutually disjoint; (ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $H(2n-1,4)$ which contain $n-2$ points of weight 2 whose supports are mutually disjoint; (iii) incidence is reverse containment. By De Bruyn [6], $\mathbb{G}_n$ is a dense near $2n$-gon. Every line L of $\mathbb{G}_n$ is generated by a unique set of $n-1$ points whose supports are mutually disjoint. This set either consists of $n-1$ points of weight 2 or $n-2$ points of weight 2 and 1 point of weight 4. If γ is a subspace of $H(2n-1,4)$ generated by points (of even weight) whose supports are mutually disjoint, then the set of all generators of $H(2n-1,4)$ containing γ is a convex subspace F_γ of $\mathbb{G}_n$ ([7, Theorem 6.27]). Conversely, every convex subspace of $\mathbb{G}_n$ is obtained in this way.

Now, consider a reference system in the projective space $\mathrm{PG}(n-1,4)$ and let U_i, $i \geq 1$, be the set of points of $\mathrm{PG}(n-1,4)$ of weight i (with respect to that reference system). Let $\mathcal{L}_n$ denote the linear space induced on $U_1 \cup U_2$ by the lines of $\mathrm{PG}(n-1,4)$. Then every local space of $\mathbb{G}_n$ is isomorphic to $\mathcal{L}_n$. In fact for every point x of $\mathbb{G}_n$, we can construct the following explicit isomorphism ϕ_x between $\mathcal{L}(\mathbb{G}_n, x)$ and $\mathcal{L}_n$. Recall that a point x of $\mathbb{G}_n$ is generated by n points $p_1, p_2, \ldots, p_n$ of weight 2 whose supports are mutually disjoint. Put $\mathrm{PG}(n-1,4) = \langle p_1, p_2, \ldots, p_n \rangle$. If L is a line through x, then one of the following two cases occurs: (1) there exists a unique $i \in \{1, \ldots, n\}$ such that $L = \langle \{p_1, \ldots, p_n\} \setminus \{p_i\} \rangle$; (2) there exists a unique pair $\{i, j\} \subseteq \{1, \ldots, n\}$ such that $L = \langle (\{p_1, \ldots, p_n\} \setminus \{p_i, p_j\}) \cup \{r\} \rangle$, where r is some point (of weight 4) of $p_i p_j \setminus \{p_i, p_j\}$. In the former case, we define $\phi_x(L) := p_i$ and in the latter case, $\phi_x(L) := r$.

Now, let x be an arbitrary point of $\mathbb{G}_n$. Then we know that $x = \langle p_1, p_2, \ldots, p_n \rangle$ where $p_1, p_2, \ldots, p_n$ are points of weight 2 whose supports are mutually disjoint. For every $p \in \langle p_1, p_2, \ldots, p_n \rangle$, let X_p be the smallest subset of $\{1, \ldots, n\}$ such that $p \in \langle p_i \,|\, i \in X_p \rangle$. For all $i, j \in X_p$ with $i \neq j$, let $p_{\{i,j\}}$ denote the unique point in the singleton $\langle p_i, p_j \rangle \cap \langle p, \{p_k \,|\, k \in X_p \setminus \{i, j\}\} \rangle$.

Now, let S be an arbitrary subspace of $\mathcal{L}(\mathbb{G}_n, x)$. Since $\phi_x(S)$ is a subspace of $\mathcal{L}_n$, we can find a subset $A = \{p_{i_1}, \ldots, p_{i_k}\} \subseteq \{p_1, \ldots, p_n\}$ $(k \geq 0)$ and points $q_1, \ldots, q_l \in \langle p_1, \ldots, p_n \rangle$ $(l \geq 0)$ such that: (i) $|X_{q_1}|, \ldots, |X_{q_l}| \geq 2$; (ii) the sets $\{p_{i_1}\}, \ldots, \{p_{i_k}\}, \{p_i \,|\, i \in X_{q_1}\}, \ldots, \{p_i \,|\, i \in X_{q_l}\}$ are mutually disjoint; (iii) a point of $\mathcal{L}_n$ belongs to $\phi_x(S)$ if and only if it belongs to $\langle p_{i_1}, \ldots, p_{i_k} \rangle$ or is of the form $(q_i)_{\{j,k\}}$ for some $i \in \{1, \ldots, l\}$ and some $j, k \in X_{q_i}$ with $j \neq k$. Let $q_{l+1}, \ldots, q_m$ denote those points of $\{p_1, \ldots, p_n\}$ which are not contained in $\{p_{i_1}, \ldots, p_{i_k}\} \cup \{p_i \,|\, i \in X_{q_1}\} \cup \cdots \cup \{p_i \,|\, x \in X_{q_l}\}$. Then the supports of the points $q_1, \ldots, q_m$ of $\mathrm{PG}(2n-1,4)$ are mutually disjoint. This means that there is a convex subspace F_β of $\mathbb{G}_n$ associated with the subspace $\beta = \langle q_1, \ldots, q_m \rangle$ of $H(2n-1,4)$. Now, a line L of $\mathbb{G}_n$ through x belongs to F_β if and only if $L \in S$. This proves that every local space of $\mathbb{G}_n$ is regular. More information on the convex subspaces of the near $2n$-gon $\mathbb{G}_n$ can be found in [7, Section 6.3].

(V) Consider in $\mathrm{PG}(6,3)$ a nonsingular parabolic quadric $Q(6,3)$ and a nontangent hyperplane π intersecting $Q(6,3)$ in a nonsingular elliptic quadric $Q^-(5,3)$. There is a polarity associated with $Q(6,3)$ and we call two points of $\mathrm{PG}(6,3)$ orthogonal when one of them is contained in the polar hyperplane of the other. Let N denote the set of 126 points of π for which the corresponding polar hyperplane intersects $Q(6,3)$ in a nonsingular elliptic quadric. Let $\mathbb{E}_3 = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry: (i) the elements of $\mathbb{E}_3$ are the 6-tuples of mutually orthogonal points of N; (ii) the elements of $\mathbb{E}_3$ are the pairs of mutually orthogonal points of N; (iii) incidence is reverse containment. By Brouwer and Wilbrink [3], $\mathbb{E}_3$ is a dense near hexagon. The first construction of this near hexagon is due to Aschbacher [1]. Every local space of $\mathbb{E}_3$ is isomorphic to the linear space $\overline{W(2)}$ obtained from the generalized quadrangle $W(2)$ of order 2 by adding its ovoids as extra lines (see [7, Theorem 6.98]). Since every subspace of $\overline{W(2)}$ is either the empty set, a singleton, a line or the whole space, every local space of $\mathbb{E}_3$ is regular.

(VI) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two dense near polygons of diameter at least 1. Then a product near polygon $\mathcal{A}_1 \times \mathcal{A}_2$ can be defined, see [7, Section 1.6]. Let x be an arbitrary point of $\mathcal{A}_1 \times \mathcal{A}_2$ and let $\mathcal{L}$ be a set of lines through x forming a subspace of $\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2, x)$. Through x there are convex subspaces F_1 and F_2 such that: (i) $F_1 \cong \mathcal{A}_1$, $F_2 \cong \mathcal{A}_2$; (ii) $F_1 \cap F_2 = \{x\}$; (iii) every line through x is contained in either F_1 or F_2. Let $\mathcal{L}_i$, $i \in \{1, 2\}$, denote the set of lines through x contained in F_i. Then $\mathcal{L}_i \cap \mathcal{L}$ is a subspace of $\mathcal{L}(F_i, x)$. Since $\mathcal{L}(F_i, x)$ is regular, there exists a unique convex subspace G_i of F_i through x such that the lines through x contained in G_i are precisely the lines of $\mathcal{L}_i \cap \mathcal{L}$. Now, by [7, Section 4.6], the convex subspace $\langle G_1, G_2\rangle$ intersects F_1 in G_1 and F_2 in G_2. Hence, the set of lines through x contained in $\langle G_1, G_2\rangle$ coincides with $\mathcal{L}$. This proves that all local spaces of $\mathcal{A}_1 \times \mathcal{A}_2$ are regular.

(VII) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two dense near polygons of diameter at least 2. If $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy certain nice conditions (see [7, Theorem 4.11]) then a so-called glued near polygon $\mathcal{A}_1 \otimes \mathcal{A}_2$ can be derived from $\mathcal{A}_1$ and $\mathcal{A}_2$. Let x be an arbitrary point of $\mathcal{A}_1 \otimes \mathcal{A}_2$ and let $\mathcal{L}$ be a set of lines through x forming a subspace of $\mathcal{L}(\mathcal{A}_1 \otimes \mathcal{A}_2, x)$. Through x there are convex subspaces F_1 and F_2 such that: (i) $F_1 \cong \mathcal{A}_1$, $F_2 \cong \mathcal{A}_2$; (ii) $F_1 \cap F_2$ is a line L; (iii) every line through x distinct from L is contained in either F_1 or F_2. Let $\mathcal{L}_i$, $i \in \{1, 2\}$, denote the set of lines through x contained in F_i. Then $\mathcal{L}_i \cap \mathcal{L}$ is a subspace of $\mathcal{L}(F_i, x)$. Since $\mathcal{L}(F_i, x)$ is regular, there exists a unique convex subspace G_i of F_i through x such that the lines through x contained in G_i are precisely the lines of $\mathcal{L}_i \cap \mathcal{L}$. Now, by [7, Section 4.6], the convex subspace $\langle G_1, G_2\rangle$ intersects F_1 in G_1 and F_2 in G_2. Hence, the set of lines through x contained in $\langle G_1, G_2\rangle$ coincides with $\mathcal{L}$. This proves that all local spaces of $\mathcal{A}_1 \otimes \mathcal{A}_2$ are regular. For an extensive discussion of glued near polygons and their properties, we refer to [7, Chapter 4].

4 Proof of Theorem 1.5

4.1 The case of the near hexagon $\mathbb{E}_1$

Let M denote the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

The 12 columns of the matrix M define a set $\mathcal{K}$ of 12 points in $\mathrm{PG}(5,3)$. This set of 12 points, which was first discovered by Coxeter [4], has several nice properties, see e.g. Lemma 4.1 below. For every point x of $\mathrm{PG}(5,3)$, define the *generating index* $i_{\mathcal{K}}(x)$ of x as the minimal number of points of $\mathcal{K}$ which are necessary to generate a subspace containing x.

Lemma 4.1 ([4], [8]) (a) *The maximal index of a point of* $\mathrm{PG}(5,3)$ *is equal to* 3.

(b) *If L is a line of* $\mathrm{PG}(5,3)$ *through a point x of $\mathcal{K}$, then $L \setminus \{x\}$ contains a unique point with smallest index.*

(c) *Every $i \in \{1,2,3,4,5\}$ distinct points of $\mathcal{K}$ generate an $(i-1)$-dimensional subspace of* $\mathrm{PG}(5,3)$. *The 4-dimensional subspace generated by 5 distinct points of $\mathcal{K}$ contains precisely 6 points of $\mathcal{K}$.*

(d) *The group of automorphisms of* $\mathrm{PG}(5,3)$ *stabilizing $\mathcal{K}$ acts 5-transitively on the set of points of $\mathcal{K}$.*

Now, embed $\mathrm{PG}(5,3)$ as a hyperplane in the projective space $\mathrm{PG}(6,3)$. Let $\mathbb{E}_1$ be the following point-line geometry: (i) the points of $\mathbb{E}_1$ are the points of $\mathrm{PG}(6,3) \setminus \mathrm{PG}(5,3)$; (ii) the lines of $\mathbb{E}_1$ are the lines of $\mathrm{PG}(6,3)$, not contained in $\mathrm{PG}(5,3)$, which contain a point of $\mathcal{K}$; (iii) incidence is derived from the one of $\mathrm{PG}(6,3)$. Then by De Bruyn and De Clerck [8], $\mathbb{E}_1$ is a dense near hexagon. The first construction of $\mathbb{E}_1$ (using cosets of the extended ternary Golay code) is due to Shult and Yanushka [12, Section 2.5].

Let $\mathcal{P}$ denote the point-set of $\mathbb{E}_1$. In this subsection, we will prove that there exists a map $f : \mathcal{P} \to \mathbb{N}$ satisfying properties (V1) and (V2) such that $\max\{f(x) \,|\, x \in \mathcal{P}\} = 2$. In view of the fact that every valuation of $\mathbb{E}_1$ is either classical or ovoidal (see [10, Theorem 1]), this implies that f is not a valuation of $\mathbb{E}_1$. This proves that Theorem 1.5 holds in the case the near hexagon is isomorphic to $\mathbb{E}_1$.

Let x be an arbitrary point of $\mathrm{PG}(5,3)$ with index 3. For every $y_1 \in \mathcal{K}$, the line xy_1 contains a unique point y_1' with index 2 (Lemma 4.1(b)). By Lemma 4.1(c), there exist unique points y_2 and y_3 of $\mathcal{K}$ such that $y_1' \in \langle y_2, y_3 \rangle$. Define $A_{y_1} := \{y_1, y_2, y_3\}$ and $\alpha_{y_1} := \langle y_1, y_2, y_3 \rangle$. By Lemma 4.1(c), α_{y_1} is a plane which intersects $\mathcal{K}$ in the set A_{y_1}. Obviously, $A_{y_1} = A_{y_2} = A_{y_3}$ and $x \in \alpha_{y_1} = \alpha_{y_2} = \alpha_{y_3}$. So, the set $U := \{\alpha_y \,|\, y \in \mathcal{K}\}$ has size $\frac{|\mathcal{K}|}{3} = 4$. By Lemma 4.1(c), any two distinct planes of U intersect in the point x. Put $U = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then for every $i \in \{1,2,3,4\}$, there exists a unique line $L_i \subseteq \alpha_i$ through x disjoint from $\alpha_i \cap \mathcal{K}$. Let α be the subspace of $\mathrm{PG}(5,3)$ generated by the lines L_1, L_2, L_3 and L_4.

We claim that α is a plane of $\mathrm{PG}(5,3)$, i.e. the lines of α through x are precisely the lines L_1, L_2, L_3 and L_4. By Lemma 4.1(d), we may without loss of generality suppose that the points $(1,0,0,0,0,0)$, $(0,1,0,0,0,0)$ and $(0,0,1,0,0,0)$ belong to α_1 and that $(0,0,0,1,0,0)$ and $(0,0,0,0,1,0)$ belong to α_2. Then the sixth point of $\mathcal{K}$ in the subspace $\langle (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0) \rangle$ is the unique point of $\alpha_2 \cap \mathcal{K}$ distinct from $(0,0,0,1,0,0)$ and $(0,0,0,0,1,0)$. This point is equal to $(1,1,-1,-1,1,0)$. It follows that the point x is equal to $(1,1,-1,0,0,0)$. The remaining planes of U are (up to transposition) $\alpha_3 = \langle(0,0,0,0,0,1),(1,0,1,-1,-1,1),(0,-1,-1,-1,-1,-1)\rangle$ and $\alpha_4 = \langle(1,1,0,1,-1,-1),(1,-1,1,0,1,-1),(1,-1,-1,1,0,1)\rangle$. We can now easily calculate the lines L_1, L_2, L_3 and L_4:

$$\begin{aligned} L_1 &= \langle(1,1,-1,0,0,0),(1,-1,0,0,0,0)\rangle, \\ L_2 &= \langle(1,1,-1,0,0,0),(0,0,0,1,1,0)\rangle, \\ L_3 &= \langle(1,1,-1,0,0,0),(1,-1,0,1,1,0)\rangle, \\ L_4 &= \langle(1,1,-1,0,0,0),(1,-1,0,-1,-1,0)\rangle. \end{aligned}$$

Hence, $\alpha = \langle L_1, L_2, L_3, L_4\rangle$ is a plane of $\mathrm{PG}(5,3)$. Now, let B be an arbitrary 3-space of $\mathrm{PG}(6,3)$ through α not contained in $\mathrm{PG}(5,3)$. Let $\mathcal{A}$ denote the projective plane obtained by taking the quotient space of $\mathrm{PG}(6,3)$ over the subspace B. The 13 points $A_1, A_2, \ldots, A_{13}$ of $\mathcal{A}$ are the 13 4-spaces of $\mathrm{PG}(6,3)$ containing B. Without loss of generality, we may suppose that $A_i = \langle B, \alpha_i\rangle$ for every $i \in \{1,2,3,4\}$. By Lemma 4.1(c), $\{A_1, A_2, A_3, A_4\}$ is a set of 4 points of $\mathcal{A}$, no three of which are collinear.

Now, for every map $\mu : \{A_1, \ldots, A_{13}\} \to \mathbb{N}$ satisfying $\mu(A_1) = \mu(A_2) = \mu(A_3) = \mu(A_4) = 1$, let f_μ be the following map from $\mathcal{P}$ to $\mathbb{N}$: if $y \in B \setminus \mathrm{PG}(5,3)$, then $f_\mu(y) = 0$; if $y \in A_i \setminus (\mathrm{PG}(5,3) \cup B)$ for a certain $i \in \{1, \ldots, 13\}$, then $f_\mu(y) = \mu(A_i)$. The function f_μ satisfies properties (V1) and (V2) if and only if

> $(*)$ for every line χ of $\mathcal{A}$ containing A_i, $i \in \{1,2,3,4\}$, there exists a unique point $z_{\chi,i} \in \chi \setminus \{A_i\}$ such that $\mu(z) = \mu(z_{\chi,i}) + 1$ for every $z \in \chi \setminus \{A_i, z_{\chi,i}\}$.

We show that there exists a function $\mu : \{A_1, \ldots, A_{13}\} \to \mathbb{N}$ satisfying Property $(*)$. Let A_5, A_6 and A_7 be those points of $\mathcal{A}$ such that $\{A_1, A_2, A_5, A_7\}$ and $\{A_3, A_4, A_6, A_7\}$ are lines of $\mathcal{A}$. Then define $\mu(A_1) = \mu(A_2) = \mu(A_3) = \mu(A_4) = \mu(A_5) = \mu(A_6) = 1$, $\mu(A_7) = 0$ and $\mu(A) = 2$ for every point A of $\mathcal{A}$ not contained in $\{A_1, \ldots, A_7\}$. Then μ satisfies Property $(*)$. Hence, f_μ is a map satisfying (V1) and (V2). Since the maximal value attained by f_μ is equal to 2, f_μ is not a valuation of $\mathbb{E}_1$. (Recall that every valuation of $\mathbb{E}_1$ is either classical or ovoidal.)

4.2 The case of the near hexagon $\mathbb{E}_2$

Let $\mathcal{D}$ denote the unique Steiner system $S(5,8,24)$. (Recall that there are 24 points in such a Steiner system, each block contains 8 points and every five distinct points are contained in a unique block.) If B_1 and B_2 are two distinct blocks of $S(5,8,24)$, then $|B_1 \cap B_2| \in \{0,2,4\}$. Moreover, if $|B_1 \cap B_2| = 0$, then the complement of $B_1 \cup B_2$ is again a block. From $S(5,8,24)$, we can construct the following incidence structure $\mathbb{E}_2$: (i) the points of $\mathbb{E}_2$ are the blocks of $S(5,8,24)$; (ii) the lines of $\mathbb{E}_2$ are the triples of mutually disjoint blocks; (iii) incidence is containment. Then $\mathbb{E}_2$ is a dense near hexagon by Shult and Yanushka [12, p. 40] (see also [7, Section 6.6]).

Now, let x and y be two given distinct points of $\mathcal{D}$. If B is a block of $\mathcal{D}$, then we define $f(B) := 0$ if $x, y \in B$, $f(B) := 1$ if $x, y \notin B$ and $f(B) := 2$ if $|\{x,y\} \cap B| = 1$. Clearly, f satisfies Properties (V1) and (V2). The map f cannot be a classical valuation of $\mathbb{E}_2$ (since $f(B) \neq 3$ for any block B of $\mathcal{D}$), nor an ovoidal valuation of $\mathbb{E}_2$ (there exists a block B with $f(B) = 2$). Now, by [10, Theorem 2] (see also [7, Theorem 6.81]), every valuation of $\mathbb{E}_2$ is either classical or ovoidal. Hence, f is not a valuation.

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