On the first place antitonicity in QL-implications

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Abstract

In order to obtain a demanded fuzzy implication, a number of properties have been proposed, among which the first place antitonicity, the second place monotonicity and the boundary conditions are the most important ones. The three classes of fuzzy implications derived from the implication in binary logic, S-, R- and QL-implications all satisfy the second place monotonicity and the boundary conditions. However, not all the QL-implications satisfy the first place antitonicity as S- and R- implications do. In this paper we study the QL-implications satisfying the first place antitonicity. First we study the relationship between the first place antitonicity and other required properties of QL-implications. And then we work on the conditions under which a QL-implication generated by different combinations of a t-conorm S, a t-norm T and a strong fuzzy negation n will satisfy the first place antitonicity, especially on the cases that both S and T are continuous. We also investigate the interrelationships between S- and R-implications on one hand and QL-implications satisfying the first place antitonicity on the other.

Keywords: Fuzzy implication, QL-implication, the First place antitonicity.

1 Introduction

A fuzzy implication is a fuzzy connective that has played important roles in different fuzzy domains [5, 6, 7, 13, 14]. There are several different definitions of a fuzzy implication, e.g., [1, 3, 4, 9]. In this paper we define a fuzzy implication as a $[0,1]^2 \to [0,1]$ mapping that satisfies the boundary conditions:

I0. I(0,0)=I(0,1)=I(1,1)=1, I(1,0)=0

One of the fuzzy inference methods is the generalized modus ponens. In fuzzy logic, the generalized modus ponens is realized through IF-THEN rules. Let X and Y be two linguistic variables on the universe of discourses U and V respectively. Moreover, let A and $A^{'}$ be two fuzzy sets on U and let B and $B^{'}$ be two fuzzy sets on V. A, $A^{'}$, B and $B^{'}$ may refer to linguistic concepts. An IF-THEN rule is expressed by:

IF X is A, THEN Y is B
X is $A^{'}$

Y is $B^{'}$
where $B^{'}$ is obtained through Zadeh's compositional rule of inference:

$$B^{'}(v) = \sup_{u \in U} T(A^{'}(u), I(A(u), B(v)))$$

In this formula, T is a t-norm and I is a fuzzy implication. In order to obtain a suitable conclusion of the fuzzy inference, a number of properties have been proposed for the fuzzy implication I [4, 10, 11, 15, 17], among which the most important ones are:

I1. the first place antitonicity:
$x_1 < x_2 \Rightarrow I(x_1, y) \geq I(x_2, y)$,
for all x_1, x_2, $y \in [0,1]$;

I2. the second place monotonicity:
$y_1 < y_2 \Rightarrow I(x, y_1) \leq I(x, y_2)$,
for all x, y_1, $y_2 \in [0,1]$;

I3. the neutrality property:
$I(1, x) = x$, for all $x \in [0,1]$;

I4. the exchange principle:
$I(x, I(y, z)) = I(y, I(x, z))$, for all x, y, $z \in [0,1]$;

I5. the ordering property:
$x \leq y \Leftrightarrow I(x, y) = 1$, for all x, $y \in [0,1]$;

I6. the contrapositive principle:
$I(x, y) = I(n(y), n(x))$, for all x, $y \in [0,1]$, w.r.t. a strong fuzzy negation n;

I7. the continuity.

I1 and I2 are the most important properties. Some authors even define a fuzzy implication I as a $[0,1]^2 \to [0,1]$ mapping that satisfies I1, I2 and

I9 $I(0,x) = 1$, for all $x \in [0,1]$;

I10 $I(x,1) = 1$, for all $x \in [0,1]$;

I11 $I(1,0) = 0$.

Notice that if I satisfies I0, I1 and I2, then I9, I10 and I11 will be satisfied immediately.
There are three important classes of fuzzy implications derived from the implication in binary logic ([11], Chapter 11):

1. Strong implication (S-implication),
$I(x,y) = S(n(x),y)$, where S is a t-conorm and n is a strong fuzzy negation;

2. Residuated implication (R-implication),
$I(x,y) = \sup\{t \in [0,1] | T(x,t) \le y\}$, where T is a t-norm;

3. Quantum logic implication (QL-implication),
$I(x,y) = S(n(x),T(x,y))$, where S is a t-conorm, T is a t-norm and n is a strong fuzzy negation.

All these three classes of fuzzy implications satisfy I0 and I2. However, although all S-implications and R-implications satisfy I1 ([10], Definition 1.15), not all the QL-implications do.
Some work on whether a QL-implication satisfies I1 or not has been done in [8], [16] and [12]. In [8], the conditions under which a QL-implication I_{QL} and a t-norm T_* satisfy the residuation property: $T_*(x,z) \le y \Leftrightarrow z \le I_{QL}(x,y)$, for all x, y, $z \in [0,1]$ are found. This means that I_{QL} is an R-implication as well ([8], Example 4.5). Hence I_{QL} satisfies I1 provided these conditions are fulfilled. However, being an R-implication is sufficient but not necessary for a QL-implication to satisfy I1 (see Remark 4, 5, 6 in this paper). In [16], the authors have worked out how a QL-implication satisfies I1 as well as I4 ([16], Definition 1, Theorem 7, Theorem 11). It is proved that such a QL-implication is an S-implication as well. Again, being an S-implication is sufficient but not necessary for a QL-implication to satisfy I1 (see Remark 5, 6 in this paper). And in [12], the authors have worked out for a group of QL-implications the conditions under which they satisfy I1. They restrict the relationship between the t-conorm and the strong fuzzy negation which construct the QL-implications ([12], Proposition 9).
In this paper, we study the QL-implications generated by a t-conorm S, a t-norm T and a strong fuzzy negation n that satisfy I1, especially for the cases that both S and T are continuous. First the relationship between I1 and the other properties of QL-implications is studied in Section 3.1. And then the conditions under which a QL-implication satisfies I1 are obtained in Section 3.2. Some QL-implications that satisfy I1 are equivalent to S-implications or R-implications while others are not. We denote these facts in Remark 4 to 7. Preliminaries are given in Section 2 and conclusions are given in Section 4 respectively.

2 Preliminaries

Definition 1. *An automorphism of the interval $[a,b] \subset \boldsymbol{R}$ is a continuous, strictly increasing mapping φ from $[a,b]$ to $[a,b]$ with boundary conditions $\varphi(a) = a$ and $\varphi(b) = b$ ([4], Definition 0).*

Lemma 1. *If ϕ is an automorphism of the unit interval, then ϕ^{-1} is also an automorphism of the unit interval.*

Lemma 2 (The chain rule)**.** *The composition of two automorphisms of the unit interval is again an automorphism.*

Definition 2. *Two mappings F, G: $[0,1]^n \to [0,1]$ are conjugated to each other, if there exists an automorphism ϕ of the unit interval such that $G = F_\phi$, where $F_\phi(x_1,x_2,\cdots,x_n)$
$= \phi^{-1}(F(\phi(x_1),\phi(x_2),\cdots,\phi(x_n)))$,
$x_1,x_2,\cdots,x_n \in [0,1]$ ([2], Definition 2).*

It is easy to see that $G = F_\phi \Leftrightarrow F = G_{\phi^{-1}}$.

Definition 3. *A mapping n: $[0,1] \to [0,1]$ is a fuzzy negation if it is decreasing and satisfies: $n(0) = 1$, $n(1) = 0$.*

Definition 4. *A fuzzy negation that satisfies $n(n(x)) = x$, for all $x \in [0,1]$ is called a strong fuzzy negation.*

We denote the standard strong fuzzy negation as n_0, i.e., $n_0(x) = 1 - x$, for all $x \in [0,1]$.

Definition 5. *Let ϕ be an automorphism of the unit interval. Then n_ϕ denotes the strong fuzzy negation that is conjugated to n_0, i.e.,
$n_\phi(x) = \phi^{-1}(1 - \phi(x))$, for all $x \in [0,1]$.*

Definition 6. *A mapping T: $[0,1]^2 \to [0,1]$ is a triangular norm (t-norm for short) if for all $x,y,z \in [0,1]$ it satisfies:*

T1. boundary condition: $T(x,1) = x$;

T2. monotonicity: $y \le z \Rightarrow T(x,y) \le T(x,z)$;

T3. commutativity: $T(x,y) = T(y,x)$;

T4. associativity: $T(x, T(y, z)) = T(T(x, y), z)$.

Three important continuous t-norms are:

1. $T_M(x, y) = \min(x, y)$, (minimum)
2. $T_P(x, y) = xy$, (product)
3. $T_{\text{Ł}}(x, y) = \max(x + y - 1, 0)$, (bounded product)

Definition 7. *Let ϕ be an automorphism of the unit interval and T be a t-norm. Then T_ϕ denotes the t-norm that is conjugated to T, i.e., $T_\phi(x, y) = \phi^{-1}(T(\phi(x), \phi(y)))$, for all $(x, y) \in [0, 1]^2$.*

T_M is conjugated to itself, i.e., if ϕ is an automorphism of the unit interval, then $T_M = T_{M\phi}$.

Definition 8. *Let $\{[a_m, b_m]\}$ be a non-empty family of non-overlapping, closed, proper subintervals of $[0, 1]$ and $\{\phi_m\}$ be a family of automorphisms of the unit interval. Then a continuous t-norm T_o is called an ordinal sum of $\{[a_m, b_m], T_m\}$, where $T_m = T_{P\phi m}$ or $T_m = T_{\text{Ł}\phi m}$, if $T_o(x, y)$ is equal to:*

$$\begin{cases} a_m + (b_m - a_m)T_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}), (x, y) \in [a_m, b_m]^2 \\ T_M(x, y), \quad \textit{otherwise} \end{cases}$$

If there exists only one subinterval $[a_1, b_1]$ of $[0, 1]$ with $a_1 = 0$, $b_1 = 1$ and ϕ_1 being the automorphism of the unit interval, then $T_o = T_{P\phi1}$ or $T_o = T_{\text{Ł}\phi1}$. In this paper, as to 'an ordinal sum of $\{[a_m, b_m], T_m\}$', we mean that there exists at least one subinterval $[a_k, b_k]$ such that $a_k \neq 0$ or $b_k \neq 1$.
It is stated in ([10], Section 1.3.4) that a continuous t-norm is either T_M, or conjugated to T_P, or conjugated to $T_{\text{Ł}}$, or an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$ with T_m being conjugated to T_P or $T_{\text{Ł}}$.

Definition 9. *A mapping S: $[0, 1]^2 \to [0, 1]$ is a triangular conorm (t-conorm for short) if for all $x, y, z \in [0, 1]$ it satisfies:*

S1. boundary condition: $S(x, 0) = x$;

S2. monotonicity: $y \leq z \Rightarrow S(x, y) \leq S(x, z)$;

S3. commutativity: $S(x, y) = S(y, x)$;

S4. associativity: $S(x, S(y, z)) = S(S(x, y), z)$.

Three important continuous t-conorms are:

1. $S_M(x, y) = \max(x, y)$, (maximum)
2. $S_P(x, y) = x + y - xy$, (probabilistic sum)
3. $S_{\text{Ł}}(x, y) = \min(x + y, 1)$, (bounded sum)

Definition 10. *Let ϕ be an automorphism of the unit interval and S be a t-conorm. Then S_ϕ denotes the t-conorm that is conjugated to S, i.e., $S_\phi(x, y) = \phi^{-1}(S(\phi(x), \phi(y)))$, for all $(x, y) \in [0, 1]^2$.*

Definition 11. *Let ϕ be an automorphism of the unit interval and I be a fuzzy implication. Then I_ϕ denotes the fuzzy implication that is conjugated to I, i.e., $I_\phi(x, y) = \phi^{-1}(I(\phi(x), \phi(y)))$, for all $(x, y) \in [0, 1]^2$.*

3 QL-implications and the first place antitonicity

First we give two propositions and three lemmas that will play important roles in this section.

Proposition 1. *A necessary condition for a QL-implication generated by a t-conorm S, a t-norm T and a strong fuzzy negation n to satisfy I1, I4, I5 or I6 is $S(n(x), x) = 1$, for all $x \in [0, 1]$.*

For the case that the t-conorm S is continuous, Proposition 1 can be further expressed by next proposition, according to [4] and [12].

Proposition 2. *A necessary condition for a QL-implication generated by a continuous t-conorm S, a t-norm T and a strong fuzzy negation n to satisfy I1, I4, I5 or I6 is that there exists an automorphism ϕ of the unit interval such that $S = S_{\text{Ł}\phi}$ and n satisfies $n(x) \geq n_\phi(x)$, for all $x \in [0, 1]$.*

Lemma 3. *([10], Theorem 1.13) A fuzzy implication is an S-implication if and only if it satisfies I3, I4 and I6.*

Lemma 4. *([10], Theorem 1.14) A fuzzy implication is an R-implication if and only if it satisfies I2, I4 and I5.*

Lemma 5. *Let ϕ be an automorphism of the unit interval. Then a QL-implication I_{QL} satisfies I1 iff $I_{QL\phi}$ satisfies I1.*

3.1 Relationship between the first place antitonicity and the other potential properties of QL-implications

As stated in the Introduction, all QL-implications satisfy I0 and I2. It is also easy to see that each QL-implication satisfies I3. Moreover, a QL-implication generated by a t-conorm S, a t-norm T and a strong fuzzy negation is continuous if both S and T are continuous. Thus we will only consider how a QL-implication I_{QL} satisfies I1, I4, I5 or I6 and the interrelationship between the I_{QL}'s satisfying them.

Theorem 1. *A QL-implication I_{QL} satisfies I4 iff I_{QL} is also an S-implication.*

Proof. $\Longleftarrow$: Directly from Lemma 3.
$\Longrightarrow$: According to ([12], Remark 2), if I_{QL} satisfies I4, then I_{QL} also satisfies I6. Since I_{QL} always satisfies I3, according to Lemma 3, I_{QL} is also an S-implication. □

Remark 1. As stated in the Introduction, an S-implication always satisfies I1. Thus if a QL-implication satisfies I4, then it also satisfies I1.

Next theorem is for the case that the t-conorm S which constructs the QL-implication is continuous, i.e, according to Proposition 2, there exists an automorphism ϕ of the unit interval such that $S = S_{\text{Ł}\phi}$.

Theorem 2. *Let ϕ be an automorphism of the unit interval. A QL-implication I_{QL} generated by the t-conorm $S_{\text{Ł}\phi}$, a t-norm T and the strong fuzzy negation n_ϕ satisfies I4 iff there exists $s \in [0,+\infty]$ such that $T = T^s_\phi$, where T^s is a Frank t-norm, defined as:*

$$T^s(x,y) = \begin{cases} T_{\text{Ł}}(x,y), & s = 0 \\ T_P(x,y), & s = 1 \\ T_M(x,y), & s = +\infty \\ log_s(1 + \frac{(s^x-1)(s^y-1)}{s-1}), & otherwise \end{cases} \tag{1}$$

Proof. $\Longleftarrow$: According to the proof of ([12], Corollary 1), such a QL-implication I_{QL} is also an S-implication. Thus according to Lemma 3, I_{QL} satisfies I4.
$\Longrightarrow$: According to ([12], Remark 2), if I_{QL} satisfies I4, then I_{QL} also satisfies I6. Moreover, since I_{QL} satisfies I2, according to ([4], Lemma 1 (ii)), I_{QL} satisfies I1. Thus according to ([12], Proposition 9) and ([12], Corollary 1), $T_{\phi^{-1}}$ is a Frank t-norm, i.e., $T = T^s_\phi$. □

A QL-implication satisfies I4 implies that it satisfies I6, but not the reverse. Comparing next theorem and Theorem 2, we can see that there exist QL-implications that satisfy I6 but not I4.

Theorem 3. *([12], Proposition 11) Let ϕ be an automorphism of the unit interval. A QL-implication I_{QL} generated by the t-conorm $S_{\text{Ł}\phi}$, a t-norm T and the strong fuzzy negation n_ϕ satisfies I6 iff $T_{\phi^{-1}}$ is an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$, where T_m are Frank t-norms defined in (1) with the parameter $s \in [0,+\infty[$.*

Remark 2. Since a QL-implication I_{QL} always satisfies I2, according to ([4], Lemma 1 (ii)), if I_{QL} satisfies I6, then it also satisfies I1.

Now we consider the conditions under which a QL-implication satisfies I5. Next theorem is for the case that both the t-conorm and the t-norm which construct the QL-implication are continuous.

Theorem 4. *Let ϕ be an automorphism of the unit interval. A QL-implication I_{QL} generated by the t-conorm $S_{\text{Ł}\phi}$, a continuous t-norm T and a strong fuzzy negation n satisfies I5 iff for all $y \in [0,1]$:*

i) $T(y,y) = n_\phi(n(y))$ and

ii) $T(x,y) \geq n_\phi(n(x))$, for all $x \in [0,y]$ and

iii) $T(x,y) < n_\phi(n(x))$, for all $x \in [y,1]$.

Proof. I_{QL} satisfies I5 iff
$I_{QL}(x,y) = \phi^{-1}(\min(\phi(n(x)) + \phi(T(x,y)), 1)) = 1 \Leftrightarrow x \leq y$, which means $\phi(n(x))+\phi(T(x,y)) \geq 1 \Leftrightarrow x \leq y$. Define for all $y \in [0,1]$, $F_y(x) = \phi(n(x)) + \phi(T(x,y))$, for all $x \in [0,1]$. Then $F_y(x) \geq 1$, i.e.,
$T(x,y) \geq n_\phi(n(x))$ iff $x \in [0,y]$ and $F_y(x) < 1$, i.e.,
$T(x,y) < n_\phi(n(x))$ iff $x \in]y,1]$. Moreover, since T, n and ϕ are all continuous, F_y is continuous. Thus $F_y(y) = 1$, i.e., $T(y,y) = n_\phi(n(y))$. □

Example 1. Let I_{QL} be the QL-implication defined in Theorem 4 with $T(y,y) = n_\phi(n(y))$, for all $y \in [0,1]$. Since T_M is the one and only the one t-norm that satisfies $T(y,y) = y$, for all $y \in [0,1]$, we have that if $T = T_M$, then $n = n_\phi$ and that if $n = n_\phi$, then $T = T_M$. Actually I_{QL} generated by $S_{\text{Ł}\phi}$, T_M and n_ϕ is an R-implication, i.e., $I_{QL}(x,y) = \sup\{t \in [0,1] | T_{\text{Ł}\phi}(x,t) \leq y\}$. According to Lemma 4, I_{QL} satisfies I4 and I5. Moreover, as stated in the Introduction, I_{QL} satisfies I1.

Remark 3. The QL-implication I_{QL} defined in Example 1 is also an S-implication, i.e.,
$I_{QL}(x,y) = S_{\text{Ł}\phi}(n_\phi(x), y)$. Thus according to Lemma 3, I_{QL} also satisfies I6.

3.2 QL-implications that satisfy the first place antitonicity

In this section we will focus on the characterizations of QL-implications satisfying I1. We mainly focus on the continuous cases. We will also indicate whether a QL-implication satisfying I1 is also an S-implication or an R-implication.

Theorem 5. *A QL-implication I_{QL} generated by the t-conorm $S_{\text{Ł}}$, the t-norm T_M and a strong fuzzy negation n satisfies I1 iff $n(x) \geq n_0(x)$, for all $x \in [0,1]$.*

Proof. $\Longrightarrow$: Straightforward from Proposition 2.
$\Longleftarrow$: For all x_1, x_2 and $y \in [0,1]$, assume $x_1 < x_2$. If $x_1 \leq y$, then $I_{QL}(x_1,y)= S_{\text{Ł}}(n(x_1), \min(x_1,y))$
$= S_{\text{Ł}}(n(x_1), x_1)$. Since $n(x_1) \geq n_0(x_1)$,
$I_{QL}(x_1,y) = 1 \geq I_{QL}(x_2,y)$. Thus we need only consider the situation that $y < x_1 < x_2$. In this case,

$I_{QL}(x_1, y) = S_Ł(n(x_1), \min(x_1, y)) = S_Ł(n(x_1), y)$ and $I_{QL}(x_2, y) = S_Ł(n(x_2), y)$. Since $S_Ł(\cdot, y)$ is increasing and n is decreasing, we have $I_{QL}(x_1, y) \geq I_{QL}(x_2, y)$. Thus for all x_1, x_2 and $y \in [0, 1]$, $x_1 < x_2$ implies: $I_{QL}(x_1, y) \geq I_{QL}(x_2, y)$, i.e., I_{QL} satisfies I1. □

Corollary 1. *Let ϕ and φ denote two automorphisms of the unit interval. Then a QL-implication I_{QL} generated by the t-conorm $S_{Ł\phi}$, the t-norm T_M and a strong fuzzy negation n_φ satisfies I1 iff $n_\varphi(x) \geq n_\phi(x)$, for all $x \in [0, 1]$.*

Proof. $I_{QL}(x, y) = S_{Ł\phi}(n_\varphi(x), T_M(x, y))$
$= \phi^{-1}(S_Ł(\phi(n_\varphi(x)), T_M(\phi(x), \phi(y))))$.
Putting $\gamma = \varphi \circ \phi^{-1}$, then $\phi(n_\varphi(x)) = n_\gamma(\phi(x))$.
According to Lemma 1 and Lemma 2, γ is also an automorphism of the unit interval. So n_γ is a strong fuzzy negation. Thus
$I_{QL}(x, y) = \phi^{-1}(S_Ł(n_\gamma(\phi(x)), T_M(\phi(x), \phi(y))))$
$= \phi^{-1}(I'_{QL}(\phi(x), \phi(y)))$, where
$I'_{QL}(x, y) = S_Ł(n_\gamma(x), T_M(x, y))$. According to Theorem 5, I'_{QL} satisfies I1 iff $n_\gamma(x) \geq n_0(x)$, for all $x \in [0, 1]$. And according to Lemma 5, I_{QL} satisfies I1 iff I'_{QL} satisfies I1. Thus I_{QL} satisfies I1 iff $n_\gamma(x) \geq n_0(x)$, which leads to $n_\gamma(\phi(x)) \geq 1 - \phi(x)$, which means $\phi(n_\varphi(x)) \geq 1 - \phi(x)$, i.e.,
$n_\varphi(x) \geq \phi^{-1}(1 - \phi(x)) = n_\phi(x)$, for all $x \in [0, 1]$. □

Remark 4. According to Example 1 and Remark 3, for the QL-implication I_{QL} defined in Theorem 5 with $n \geq n_0$, if $n = n_0$, then I_{QL} is equivalent to both an S-implication and an R-implication.
On the contrary, we suppose that there exists $x_0 \in \,]0, 1[$ such that $n(x_0) > n_0(x_0)$. Then there exists y_0 such that $x_0 > y_0 \geq 1 - n(x_0)$, which leads to $n(x_0) + y_0 \geq 1$, which means $I_{QL}(x_0, y_0) = 1$ provided $x_0 > y_0$. Thus I_{QL} does not satisfy I5. Therefore according to Lemma 4, I_{QL} is not an R-implication. But I_{QL} is an S-implication, i.e., $I_{QL}(x, y) = S_Ł(n(x), y)$.
Similarly, for the QL-implication I_{QL} defined in Corollary 1 with $n_\varphi \geq n_\phi$, if $n_\varphi = n_\phi$, then I_{QL} is equivalent to both an S-implication and an R-implication. If on the contrary $n_\varphi \neq n_\phi$, then I_{QL} is not an R-implication but an S-implication, i.e., $I_{QL}(x, y) = S_{Ł\phi}(n_\varphi(x), y)$.

According to Proposition 2, a necessary condition for a QL-implication I_{QL} generated by a continuous t-conorm S, a t-norm T and a strong fuzzy negation n to satisfy I1 is that there exists an automorphism ϕ of the unit interval such that $S = S_{Ł\phi}$ and $n \geq n_\phi$. The authors of [12] have done the work for the special case that $n = n_\phi$. Next theorem gives the sufficient and necessary condition for I_{QL} of such case to satisfy I1.

Theorem 6. *([12], Proposition 9) Let ϕ be an automorphism of the unit interval. A QL-implication I_{QL} generated by the t-conorm $S_{Ł\phi}$, a t-norm T and the strong fuzzy negation n_ϕ satisfies I1 iff $T_{\phi^{-1}}$ satisfies the Lipschitz condition, i.e., for all $x_1, x_2, y \in [0, 1]$,*

$$x_1 \leq x_2 \Rightarrow T_{\phi^{-1}}(x_2, y) - T_{\phi^{-1}}(x_1, y) \leq x_2 - x_1, \quad (2)$$

There are t-norms sufficient to fulfill the Lipschitz condition (2), here we give examples:

Example 2 According to (1), a Frank t-norm $T^s = T_P$ if $s = 1$ and $T^s = T_Ł$ if $s = 0$. It has been stated in ([12], Remark 4) that a t-norm T which is a Frank t-norm or an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$, where T_m are Frank t-norms, always satisfies the Lipschitz condition (2). Thus T_P, $T_Ł$ and T_o which is an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$, where $T_m = T_P$ or $T_m = T_Ł$ all satisfy the Lipschitz condition (2). Hence according to Theorem 6, a QL-implication generated by the t-conorm $S_{Ł\phi}$, the t-norm $T_{P\phi}$, $T_{Ł\phi}$ or $T_{o\phi}$ and the strong fuzzy negation n_ϕ satisfies I1.

Remark 5 Let I_{QL} be a QL-implication generated by the t-conorm $S_{Ł\phi}$, a t-norm T that $T_{\phi^{-1}}$ satisfies the Lipschitz condition (2) and the strong fuzzy negation n_ϕ. Then according to ([12], Corollary 1), I_{QL} satisfies I4 iff $T_{\phi^{-1}}$ is a Frank t-norm defined in (1). Thus according to Theorem 1, I_{QL} is also an S-implication as soon as $T_{\phi^{-1}}$ is a Frank t-norm. Moreover, if $T_{\phi^{-1}}$ is not a Frank t-norm, eg., an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$, where $T_m = T_P$ or $T_m = T_Ł$, then I_{QL} does not satisfy I4. Thus according to Lemma 3 and Lemma 4, it is neither an S-implication nor an R-implication.

Besides the QL-implications generated by the t-conorm $S_{Ł\phi}$, a t-norm T and the strong fuzzy negation n_ϕ, which we discussed above, there exist other combinations of a t-conorm S, a t-norm T and a strong fuzzy negation n to generate a QL-implication I_{QL} which satisfies I1. It is sufficient but not necessary for n to be n_ϕ while $S = S_{Ł\phi}$. Next we discuss the cases that provided both S and T are continuous, what conditions should n fulfill to make I_{QL} satisfy I1. Since T is either T_M, or conjugated to T_P or conjugated to $T_Ł$, or an ordinal sum the non-empty family $\{[a_m, b_m], T_m\}$ with T_m being conjugated to T_P or $T_Ł$, we have the next theorems and corollaries. First we consider the cases that $T = T_P$ or T is conjugated to T_P.

Theorem 7. *Let n be a strong fuzzy negation and define a mapping f as $f(x) = \frac{1-n(x)}{x}$, for all $x \in]0,1]$. Then a QL-implication I_{QL} generated by the t-conorm $S_{Ł}$, the t-norm T_P and n satisfies I1 iff f is increasing.*

Proof. For all x_1, x_2 and $y \in [0,1]$, assume $x_1 < x_2$.
$\Longrightarrow$: In order for I_{QL} to satisfy I1, it is necessary that $I_{QL}(x_2, y) = 1$ implies $I_{QL}(x_1, y) = 1$, namely, $y \geq \frac{1-n(x_2)}{x_2}$ implies $y \geq \frac{1-n(x_1)}{x_1}$. Thus $f(x_2) = \frac{1-n(x_2)}{x_2} \geq \frac{1-n(x_1)}{x_1} = f(x_1)$, for all $x_1 < x_2$, i.e., f is increasing.
$\Longleftarrow$: If $I_{QL}(x_1, y) = 1$, then it is always greater than $I_{QL}(x_2, y)$. If $I_{QL}(x_2, y) = 1$, then since f is increasing, according to the proof above, $I_{QL}(x_1, y) = 1 = I_{QL}(x_2, y)$. Thus we need only consider the situation that $I_{QL}(x_1, y) = n(x_1) + x_1 y < 1$ and $I_{QL}(x_2, y) = n(x_2) + x_2 y < 1$, i.e., $x_1 > 0$, $x_2 > 0$, $y < \frac{1-n(x_1)}{x_1}$ and $y < \frac{1-n(x_2)}{x_2}$. Since f is increasing, $\frac{1-n(x_2)}{x_2} \geq \frac{1-n(x_1)}{x_1}$. Thus:
$n(x_1)x_1 - n(x_2)x_1 \geq x_2 - x_2 n(x_1) - x_1 + n(x_1)x_1$, which leads to $\frac{n(x_1)-n(x_2)}{x_2-x_1} \geq \frac{1-n(x_1)}{x_1} > y$. Therefore $n(x_1) + x_1 y > n(x_2) + x_2 y$, i.e., $I_{QL}(x_1, y) \geq I_{QL}(x_2, y)$. Hence I_{QL} satisfies I1. □

Example 3. Let $\phi(x) = x^2$. Then $n(x) = \sqrt{1-x^2}$ and $f(x) = \frac{1-n(x)}{x} = \frac{1-\sqrt{1-x^2}}{x}$, for all $x \in]0,1]$. Since $\frac{df(x)}{dx} = \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}x^2} \geq 0$, f is increasing. Thus I_{QL} defined by $I_{QL}(x,y) = S_{Ł}(\sqrt{1-x^2}, xy)$ satisfies I1.

Corollary 2. *Let ϕ, φ and γ be three automorphisms of the unit interval, where $\gamma = \varphi \circ \phi^{-1}$. And define a mapping f as $f(x) = \frac{1-n_\gamma(x)}{x}$, for all $x \in]0,1]$. Then a QL-implication I_{QL} generated by the t-conorm $S_{Ł\phi}$, the t-norm $T_{P\phi}$ and n_φ satisfies I1 iff f is increasing.*

Proof. $I_{QL}(x,y) = S_{Ł\phi}(n_\varphi(x), T_{P\phi}(x,y))$
$= \phi^{-1}(S_{Ł}(\phi(n_\varphi(x)), T_P(\phi(x), \phi(y))))$.
Since $\gamma = \varphi \circ \phi^{-1}$,
$I_{QL}(x,y) = \phi^{-1}(S_{Ł}(n_\gamma(\phi(x)), T_P(\phi(x), \phi(y))))$.
According to Lemma 1 and Lemma 2, γ is also an automorphism of the unit interval. So n_γ is a strong fuzzy negation. Thus
$I_{QL}(x,y) = \phi^{-1}(I'_{QL}(\phi(x), \phi(y)))$, where
$I'_{QL}(x,y) = S_{Ł}(n_\gamma(x), T_P(x,y))$. According to Theorem 7, I'_{QL} satisfies I1 iff f is increasing. And according to Lemma 5, I_{QL} satisfies I1 iff I'_{QL} satisfies I1. Thus I_{QL} satisfies I1 iff f is increasing. □

Remark 6. Let I_{QL} be the QL-implication and f be the mapping defined in Theorem 7 with f being increasing. Then $f(x) \leq f(1) = 1$, for all $x \in]0,1]$, which leads to $n(x) \geq n_0(x)$, for all $x \in]0,1]$. Since $n(0) = n_0(0)$, we have $n(x) \geq n_0(x)$, for all $x \in [0,1]$. If $n = n_0$, then according to Example 2 and Remark 5, I_{QL} is also an S-implication. If on the contrary $n \neq n_0$, then consider:
$I_{QL}(x, I_{QL}(y,z))$
$= \min(n(x) + x \cdot \min(n(y) + yz, 1), 1)$,
which is equivalent to:

i) $n(x) + x(n(y) + yz)$, if $n(y) + yz < 1$ and $n(x) + x(n(y) + yz) < 1$;
ii) 1 otherwise,

and $I_{QL}(y, I_{QL}(x,z))$
$= \min(n(y) + y \cdot \min(n(x) + xz, 1), 1)$,
which is equivalent to:

i) $n(y) + y(n(x) + xz)$, if $n(x) + xz < 1$ and $n(y) + y(n(x) + xz) < 1$;
ii) 1 otherwise.

Since n is continuous and since $g(x) = \frac{1-x}{n(x)}$, $x \in [0,1[$ cannot be constant, there exist x_0, $y_0 \in [0,1[$ such that $x_0 \neq y_0$ and $g(x_0) \neq g(y_0)$. Let $g(x_0) > g(y_0)$. Then we have $\frac{\frac{1-n(y_0)}{y_0} - n(x_0)}{x_0} < \frac{\frac{1-n(x_0)}{x_0} - n(y_0)}{y_0}$. Thus there exists z_0 such that
$\frac{\frac{1-n(y_0)}{y_0} - n(x_0)}{x_0} \leq z_0 < \frac{\frac{1-n(x_0)}{x_0} - n(y_0)}{y_0} \leq \frac{1-n(y_0)}{y_0}$.
Therefore x_0, y_0, z_0 satisfy $n(y_0) + y_0 z_0 < 1$ and $n(x_0) + x_0(n(y_0) + y_0 z_0) < 1$ and $n(y_0) + y_0(n(x_0) + x_0 z_0) \geq 1$, which means $I_{QL}(y_0, I_{QL}(x_0, z_0)) = 1$ while $I_{QL}(x_0, I_{QL}(y_0, z_0)) < 1$. Thus I_{QL} does not satisfy I4. According to Lemma 3 and Lemma 4, I_{QL} is neither an S-implication nor an R-implication.
Similarly, let I_{QL} be the QL-implication and f be the mapping defined in Corollary 2 with f being increasing, we have $n_\varphi(x) \geq n_\phi(x)$, for all $x \in [0,1]$. If $n_\varphi = n_\phi$, then I_{QL} is also an S-implication. If on the contrary $n_\varphi \neq n_\phi$, then I_{QL} is neither an S-implication nor an R-implication.

Theorem 8. *A QL-implication I_{QL} generated by the t-conorm $S_{Ł}$, the t-norm $T_{Ł}$ and a strong fuzzy negation n satisfies I1 iff $n = n_0$.*

Proof. $\Longleftarrow$: Straightforward from Theorem 6 and Example 2.
$\Longrightarrow$: Take $0 < x_1 < x_2 < 1$. Since $1 - x_2 < 1 - x_1 < 2 - n(x_1) - x_1$, for all $x_1 \in]0,1[$, there exists y_0 such that $1 - x_1 < y_0 < 2 - n(x_1) - x_1$ and $1 - x_2 < y_0$. Thus $I_{QL}(x_1, y_0) = n(x_1) + x_1 + y_0 - 1 < 1$ and $I_{QL}(x_2, y_0) = \min(n(x_2) + x_2 + y_0 - 1, 1)$. If I_{QL} satisfies I1, then it is necessary that $I_{QL}(x_2, y_0) < 1$, i.e.,

$n(x_2) + x_2 + y_0 - 1 < 1$. Namely, $y_0 < 2 - n(x_1) - x_1$ implies $y_0 < 2 - n(x_2) - x_2$. Thus define f as $f(x) = n(x) + x$, f must be decreasing for all $x \in]0, 1[$. Since n is continuous, f is continuous. Thus f must be decreasing for all $x \in [0, 1]$. Moreover, we have $f(0) = f(1) = 1$. Therefore $f(x) = 1$, for all $x \in [0, 1]$, which means $n(x) = 1 - x$, for all $x \in [0, 1]$, i.e., $n = n_0$. □

Corollary 3. *Let ϕ and φ be two automorphisms of the unit interval. Then a QL-implication I_{QL} generated by the t-conorm $S_{Ł\phi}$, the t-norm $T_{Ł\phi}$ and a strong fuzzy negation n_φ satisfies I1 iff $n_\varphi = n_\phi$.*

Proof. $I_{QL}(x, y) = S_{Ł\phi}(n_\varphi(x), T_{Ł\phi}(x, y))$
$= \phi^{-1}(S_Ł(\phi(n_\varphi(x)), T_Ł(\phi(x), \phi(y))))$.
Putting $\gamma = \varphi \circ \phi^{-1}$, then $\phi(n_\varphi(x)) = n_\gamma(\phi(x))$.
According to Lemma 1 and Lemma 2, γ is also an automorphism of the unit interval. So n_γ is a strong fuzzy negation. Thus $I_{QL}(x, y)$
$= \phi^{-1}(S_Ł(n_\gamma(\phi(x)), T_Ł(\phi(x), \phi(y))))$
$= \phi^{-1}(I'_{QL}(\phi(x), \phi(y)))$,
where $I'_{QL}(x, y) = S_Ł(n_\gamma(x), T_Ł(x, y))$.
According to Theorem 8, I'_{QL} satisfies I1 iff $n_\gamma = n_0$. And according to Lemma 5, I_{QL} satisfies I1 iff I'_{QL} satisfies I1. Thus I_{QL} satisfies I1 iff $n_\gamma = n_0$. So $n_\gamma(x) = 1 - x$, $n_\gamma(\phi(x)) = 1 - \phi(x)$, being equivalent to $\phi(n_\varphi(x)) = 1 - \phi(x)$, which means $n_\varphi(x) = \phi^{-1}(1 - \phi(x))$, for all $x \in [0, 1]$. Hence $n_\varphi = n_\phi$. □

Remark 7. According to Example 2 and Remark 5, the QL-implication defined in Theorem 8 with $n = n_0$ and the QL-implication defined in Corollary 3 with $n_\varphi = n_\phi$ are equivalent to S-implications.

Next we consider the t-norm T which constructs the QL-implication to be an ordinal sum of the non-empty family $\{[a_m, b_m], T_m\}$, where $T_m = T_P$ or $T_m = T_Ł$. Because of space limitation, we omit the proofs of the theorems and corollaries below.

Theorem 9. *Let $\{[a_m, b_m]\}$ be a non-empty family of non-overlapping, closed, proper subintervals of $[0, 1]$ and T_o be an ordinal sum of $\{[a_m, b_m], T_m\}$, where $T_m = T_P$. Moreover, let n be a strong fuzzy negation and define $f_m(x) = \frac{1-n(x)-a_m}{x-a_m}$, for all m and $x \in]a_m, b_m]$. Then a QL-implication I_{QL} generated by the t-conorm $S_Ł$, T_o and n satisfies I1 iff $n(x) \geq n_0(x)$, for all $x \in [0, 1]$ and f_m is increasing, for all m.*

Theorem 10. *Let $\{[a_m, b_m]\}$ be a non-empty family of non-overlapping, closed, proper subintervals of $[0, 1]$ and T_o be an ordinal sum of $\{[a_m, b_m], T_m\}$, where $T_m = T_Ł$. Then a QL-implication I_{QL} generated by the t-conorm $S_Ł$, T_o and a strong fuzzy negation n satisfies I1 iff*

i) $n(x) = n_0(x)$, if $x \in [a_m, b_m]$ and

ii) $n(x) \geq n_0(x)$, otherwise.

Synthesize Theorem 9 and Theorem 10, we have the next corollary.

Corollary 4. *Let $\{[a_i, b_i]\}$ and $\{[a_j, b_j]\}$ be two non-empty families of non-overlapping, closed, proper subintervals of $[0, 1]$ and $\{[a_m, b_m]\} = \{[a_i, b_i]\} \cup \{[a_j, b_j]\}$. T_o is an ordinal sum of $\{[a_m, b_m], T_m\}$, which is defined as:*

$$\begin{cases} a_i + (b_i - a_i)T_P(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & (x, y) \in [a_i, b_i]^2 \\ a_j + (b_j - a_j)T_Ł(\frac{x-a_j}{b_j-a_j}, \frac{y-a_j}{b_j-a_j}), & (x, y) \in [a_j, b_j]^2 \\ T_M(x, y), & \textit{otherwise} \end{cases} \tag{3}$$

Moreover, let n be a strong fuzzy negation and define $f_i(x) = \frac{1-n(x)-a_i}{x-a_i}$, for all i and $x \in]a_i, b_i]$. Then a QL-implication I_{QL} generated by the t-conorm $S_Ł$, T_o and n satisfies I1 iff

i) f_i is increasing, for all i, and

ii) $n(x) = n_0(x)$, for all $x \in [a_j, b_j]$, and

iii) $n(x) \geq n_0(x)$, for all $x \notin [a_i, b_i]$ and all $x \notin [a_j, b_j]$.

For the QL-implication being conjugated to the one defined in Corollary 4, we have the next corollary.

Corollary 5. *Let T_o be a t-norm defined by (3), n be a strong fuzzy negation and ϕ be an automorphism of the unit interval. Define $f_i(x) = \frac{1-\phi(n(\phi^{-1}(x)))-a_i}{x-a_i}$, for all i and $x \in]a_i, b_i]$. Then a QL-implication I_{QL} generated by the t-conorm $S_{Ł\phi}$, $T_{o\phi}$ and n satisfies I1 iff*

i) f_i is increasing, for all i, and

ii) $n(x) = n_\phi(x)$, for all $x \in [a_j, b_j]$, and

iii) $n(x) \geq n_\phi(x)$, for all $x \notin [a_i, b_i]$ and all $x \notin [a_j, b_j]$.

Proof. Suppose $n'(x) = \phi(n(\phi^{-1}(x)))$. According to Lemma 5, I_{QL} satisfies I1 iff $I_{QL\phi^{-1}}$, which is expressed as $I_{QL\phi^{-1}}(x, y) = S_Ł(n'(x), T_o(x, y))$, satisfies I1. According to Corollary 5, $I_{QL\phi^{-1}}$ satisfies I1 iff

i) f'_i is increasing, for all i, and

ii) $n'(x) = n_0(x)$, for all $x \in [a_j, b_j]$, and

iii) $n^{'}(x) \geq n_0(x)$, for all $x \notin [a_i, b_i]$ and all $x \notin [a_j, b_j]$,

where $f_i^{'}(x) = \frac{1-n^{'}(x)-a_i}{x-a_i}$, for all $x \in \,]a_i, b_i]$. The three conditions are equivalent to

i) f_i is increasing, for all i, and

ii) $n(x) = n_\phi(x)$, for all $x \in [a_j, b_j]$, and

iii) $n(x) \geq n_\phi(x)$, for all $x \notin [a_i, b_i]$ and all $x \notin [a_j, b_j]$.

□

4 Conclusions

In this paper, we have studied the QL-implications generated by a t-conorm, a t-norm and a strong fuzzy negation that satisfy the properties which are required to obtain a suitable conclusion in fuzzy inference. Especially the first place antitonicity (property I1 in the Introduction) of QL-implications has been studied. Theorem 1 to Theorem 4 state the general relationship between a QL-implication satisfying the first place antitonicity and the other properties. Moreover, Theorem 5 to Theorem 10 together with Corollary 1 to Corollary 5 state sufficient and necessary conditions for the QL-implications generated by different combinations of a t-conorm, a t-norm and a strong fuzzy negation to satisfy the first place antitonicity. Whether the QL-implications which satisfy the first place antitonicity are equivalent to S- or R- implications have been illustrated in Remarks 4 to 7.

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