On flocks of infinite quadratic cones

F. De Clerck *H. Van Maldeghem*[*]

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

We extend some of the theory of flocks of a finite quadratic cone to the infinite case and give some examples. One of the results we prove is that a generalized quadrangle is coming from a flock if and only if all derivations of the flock are well defined.

1 Introduction

J. A. Thas has joined together the worlds of *flocks* and *generalized quadrangles* in his celebrated paper [11]. Since then a lot of people investigate this relationship and the theory has grown to unexpected heights, most of the breakthroughs being established by J. A. Thas himself (constructions, characterizations, derivations, BLT-sets, . . .). In this paper dedicated to J. A. Thas, we make a modest contribution by deleting the finiteness condition. In fact, it turns out that also for the infinite case many examples exist. We will in fact mimic a lot of ideas originally from J. A. Thas, but the methods will not be completely the same. The way we want to play the game is with coordinatization (for projective planes see Hughes and Piper [5], for generalized quadrangles see Hanssens and Van Maldeghem [4]). At the same time, it offers an alternative proof for the results of Thas [11] in the finite case.

The paper is organized as follows. In section 2 we discuss the existence of flocks of quadratic cones, of generalized quadrangles related to those flocks (we will call them *flock quadrangles*), and of projective planes related to flocks (we will call them

[*]The second author is a Senior Research Associate of the National Fund for Scientific Research (Belgium)

Received by the editors in February 1994

AMS Mathematics Subject Classification: Primary 51E12, Secondary 51M30

Keywords: conical flock, generalized quadrangles, translation planes.

flock planes). The existence conditions for these three structures appear to be very related. In this way we extend Thas' result [11] to the general case. Moreover, we will show that a generalized quadrangle arises from a flock if and only if all derivations of the flock are well defined. Section 3 is devoted to examples. Firstly we give an example of a flock over the reals with a specific topological property. Secondly we generalize the flocks associated to the generalized quadrangles $K(q)$ discovered by Kantor [6]. We also show its relationship to the split Cayley hexagon $H(\mathbb{K})$ over some field $\mathbb{K}$.

2 Basic definitions and results

2.1 Generalized quadrangles

A generalized quadrangle (GQ) (introduced by Tits [13]) is an incidence structure $\mathcal{Q} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines respectively, and for which I is a symmetric point-line incidence relation satisfying the following properties :

GQ1 Two distinct points are incident with at most one line.

GQ2 If (x, L) is a non-incident point-line pair, then there is a unique point-line pair (y, M) for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$.

We will always assume that $\mathcal{Q}$ is thick, i.e., every line is incident with at least 3 points, while every point is incident with at least 3 lines. If the GQ is finite, i.e., both $\mathcal{P}$ and $\mathcal{B}$ are finite sets, then we assume that all lines are incident with $s+1$ points, while all points are incident with $t+1$ lines, and we say that $\mathcal{Q}$ is of *order* (s, t). Note that it is conjectured that if a thick generalized quadrangle has one of its parameters s or t finite, then the other one also has to be finite. The conjecture has been proved in case one of the parameters is at most 4 (see [2] for more details and references).

For any point x of $\mathcal{Q}$, the set of all points lying on the lines through x is denoted by $x^\perp$. In the sequel, we will assume that the reader is familiar with the basic theory and definitions of finite generalized quadrangles and for the terminology and the notations we refer to [10].

2.2 Coordinatization of generalized quadrangles

We also will assume that the reader is familiar with coordinatization techniques of projective planes (see for instance [5]). As for generalized quadrangles, we recall the coordinatization as it was introduced by Hanssens and Van Maldeghem [4].

Theorem 2.1 *Let R_1 and R_2 be two sets both containing at least two elements and sharing some element* 0*. Suppose $\infty \notin R_1 \cup R_2$. Let*

$$\Psi_1 : R_1 \times R_2 \times R_1 \times R_2 \to R_1,$$

$$\Psi_2 : R_1 \times R_2 \times R_1 \times R_2 \to R_2$$

be two quaternary operations. Define the following incidence point-line geometry $\mathcal{Q} = Q(R_1, R_2, \Psi_1, \Psi_2)$*. The points of* $\mathcal{Q}$ *are a symbol* (∞)*, the elements* (a) *with* $a \in R_1$*, the elements* (k, b) *with* $k \in R_2$ *and* $b \in R_1$*, and the elements* (a, l, a') *with* $a, a' \in R_1$ *and* $l \in R_2$*. The lines of* $\mathcal{Q}$ *are the symbol* $[\infty]$*, the elements* $[k]$ *with* $k \in R_2$*, the elements* $[a, l]$ *with* $a \in R_1$ *and* $l \in R_2$*, and the elements* $[k, b, k']$ *with* $k, k' \in R_2$ *and* $b \in R_1$*. Incidence (denoted by* I*) is given as follows. For all* $a, a', b \in R_1$ *and all* $k, k', l \in R_2$ *we have* $[k, b, k']\mathrm{I}(k, b)\mathrm{I}[k]\mathrm{I}(\infty)\mathrm{I}[\infty]\mathrm{I}(a)\mathrm{I}[a, l]\mathrm{I}(a, l, a')$*, no other incidences occur except that* $(a, l, a')\mathrm{I}[k, b, k']$ *if and only if*

$$\begin{cases} \Psi_1(a, k, b, k') = a', \\ \Psi_2(a, k, b, k') = l. \end{cases}$$

Then $\mathcal{Q}$ *is a generalized quadrangle if and only if the following properties hold.*

(0) $\Psi_1(0, k, b, 0) = b = \Psi_1(a, 0, b, k')$*, for all* $a, b \in R_1$ *and all* $k, k' \in R_2$*,*
$\Psi_2(0, k, b, k') = k' = \Psi_2(a, 0, 0, k')$*, for all* $a, b \in R_1$ *and all* $k, k' \in R_2$*,*

(A) *(i) for all* $a, a' \in R_1$ *and all* $k, l \in R_2$*, there exist unique* $b^* \in R_1$ *and* $k^* \in R_2$ *such that*

$$\begin{cases} \Psi_1(a, k, b^*, k^*) = a', \\ \Psi_2(a, k, b^*, k^*) = l, \end{cases}$$

(ii) for all $a, b \in R_1$ *and all* $k, l \in R_2$*, there exists a unique* $k^* \in R_2$ *such that* $\Psi_2(a, k, b, k^*) = l$*,*

(B) *(i) for all* $a, a', b \in R_1$ *and all* $k, l \in R_2$*,* $a \neq b$*, there exist unique* $b^* \in R_1$ *and* $k^*, l^* \in R_2$ *such that*

$$\begin{cases} a' &= \Psi_1(a, k^*, b^*, l^*) \\ l &= \Psi_2(a, k^*, b^*, l^*) \\ k &= \Psi_2(b, k^*, b^*, l^*), \end{cases}$$

(ii) for all $a, b \in R_1$ *and all* $k, k', l \in R_2$*,* $k \neq l$*, there exist unique* $a^* \in R_1$ *and* $l^* \in R_2$ *such that*

$$\begin{cases} \Psi_1(a^*, k, b, k') &= \Psi_1(a^*, l, a, l^*) \\ \Psi_2(a^*, k, b, k') &= \Psi_2(a^*, l, a, l^*), \end{cases}$$

(C) *for all* $a, a', b \in R_1$ *and all* $k, k', l \in R_2$*, the system*

$$\begin{cases} \Psi_1(a, k^*, b^*, l^*) &= a' \\ \Psi_1(a^*, k^*, b^*, l^*) &= \Psi_1(a^*, k, b, k') \\ \Psi_2(a, k^*, b^*, l^*) &= l \\ \Psi_2(a^*, k^*, b^*, l^*) &= \Psi_2(a^*, k, b, k') \end{cases}$$

has

(i) *exactly one solution* $(a^*, b^*, k^*, l^*) \in R_1^2 \times R_2^2$ *if* $l \neq \Psi_2(a,k,b,k')$ *and* $a' \neq \Psi_1(a,k,b,k'')$, *where* $k'' \in R_2$ *is defined by* $\Psi_2(a,k,b,k'') = l$ *(see* (A)*(ii)),*

(ii) *no solution in* $(a^*, b^*, k^*, l^*) \in R_1^2 \times R_2^2$ *if either* $l = \Psi_2(a,k,b,k')$ *and* $a' \neq \Psi_1(a,k,b,k'')$, *where* $k'' \in R_2$ *is defined by* $\Psi_2(a,k,b,k'') = l$ *(see (A)(ii)), or* $l \neq \Psi_2(a,k,b,k')$ *and* $a' = \Psi_1(a,k,b,k'')$, *where* $k'' \in R_2$ *is defined by* $\Psi_2(a,k,b,k'') = l$.

Remark. The properties (A), (B) and (C) are the algebraic interpretation of axiom (GQ2). Note that every generalized quadrangle can be obtained in such an algebraic way.

If $Q(R_1, R_2, \Psi_1, \Psi_2)$ is a generalized quadrangle, then $\mathcal{R} = (R_1, R_2, \Psi_1, \Psi_2)$ is called a *quadratic quaternary ring.*

2.3 Elation generalized quadrangles

Let $\mathcal{Q} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized quadrangle and let p be some point of $\mathcal{Q}$. If there exists a group of automorphisms of $\mathcal{Q}$ fixing every line through p and acting regularly on the set of points opposite p, then $\mathcal{Q}$ is called an *elation generalized quadrangle* with *elation point* p. There is a general group-theoretical method to construct finite elation generalized quadrangles, due to Kantor [7]. Details of this construction can also be found in [10].

2.4 Flock quadrangles

A *conical flock* or briefly a *flock* is a partition of the points on the cone (minus the vertex) into disjoint plane irreducible conics. It has been proved by Thas [11], that a finite conical flock is equivalent to a special class of elation generalized quadrangles and to a special class of translation planes. For an overview of the models and some characterization theorems we refer to [12].

Our (alternative) point of view will be a little bit different from the standard one, used in the finite case. The theory of coordinatization seems to be very suitable for this purpose and allows us to skip the group-theoretical background.

Let $\mathbb{K}$ be any field and let $x : \mathbb{K} \to \mathbb{K}$, $y : \mathbb{K} \to \mathbb{K}$ and $z : \mathbb{K} \to \mathbb{K}$ be three arbitrary maps with the only restriction that they all map 0 to 0. We will denote the image of an element $k \in \mathbb{K}$ under the map x by x_k, and similarly for y and z. We define the following quaternary ring $\mathcal{R}_{x,y,z} = (\mathbb{K} \times \mathbb{K}, \mathbb{K}, \Psi_1, \Psi_2)$.

$$\begin{cases} \Psi_1(a,k,b,k') = (b_0 + 2a_0x_k + a_1y_k, b_1 + a_0y_k + 2a_1z_k), \\ \Psi_2(a,k,b,k') = k' + a_0^2x_k + a_0a_1y_k + a_1^2z_k + a_0b_0 + a_1b_1, \end{cases} \tag{1}$$

with $a = (a_0, a_1)$, $b = (b_0, b_1)$.

If $\mathbb{K}$ has even characteristic, we denote by $\mathcal{C}_1(\mathbb{K})$, the set of elements $k \in \mathbb{K}$ such that the equation $X^2 + X + k = 0$ has no solutions in $\mathbb{K}$. Moreover, given a quadratic form $aX^2 + bX + c$ in a field $\mathbb{K}$ of characteristic 2, we will call acb^{-2} the *discriminant* of the quadratic form. If the field has odd characteristic or has characteristic 0 then the *discriminant* of the quadratic form is the element $b^2 - 4ac$.

Theorem 2.2 *The quaternary ring $\mathcal{R}_{x,y,z}$ defined in* (1) *is a quadratic quaternary ring if and only if the following two conditions are satisfied:*

(i) for all $r \in \mathbb{K} \cup \{\infty\}$, the mapping $\mathbb{K} \to \mathbb{K} : k \mapsto x_k r^2 + y_k r + z_k$ is a bijection (for $r = \infty$, this means that $k \mapsto x_k$ is a bijection),

(ii) for all $r \in \mathbb{K} \cup \{\infty\}, l \in \mathbb{K}$, the mapping

$$\mathbb{K} \setminus \{l\} \to \mathbb{K} \setminus \{0\} : k \mapsto \frac{(x_k - x_l)r^2 + (y_k - y_l)r + (z_k - z_l)}{(y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)}$$

is a bijection.

Proof. We have to prove that conditions *(i)* and *(ii)* are equivalent to conditions (0), (A), (B) and (C) of theorem 2.1.

We first note that condition *(i)* implies immediately

(i') for every $k, l \in \mathbb{K}$, $k \neq l$, the element $(y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)$ differs from 0.

Suppose first that *(i)* and *(ii)* hold. Conditions (0) and (A) are trivially satisfied (i.e. without the assumptions *(i)* or *(ii)*). Consider condition (B)*(i)*. We can obviously eliminate b^* and l^* from the system of equations. We then find that the equation

$$m = (a_0 - b_0)^2 x_{k^*} + (a_0 - b_0)(a_1 - b_1) y_{k^*} + (a_1 - b_1)^2 z_{k^*}$$

with

$$m = k - l - (b_0 - a_0)a_0' - (b_1 - a_1)a_1',$$

must have a unique solution k^* for every $a, b, a' \in \mathbb{K} \times \mathbb{K}$, $a \neq b$, and every $k, l \in \mathbb{K}$. By condition *(i)*, there exists a unique $k^* \in \mathbb{K}$ such that

$$\frac{m}{(a_1 - b_1)^2} = \left(\frac{a_0 - b_0}{a_1 - b_1}\right)^2 x_{k^*} + \frac{a_0 - b_0}{a_1 - b_1} y_{k^*} + z_{k^*},$$

showing (B)*(i)*.

Consider condition (B)*(ii)*. Once a^* is determined by the first equation, the second will give us l^*. The first equation is equivalent to the following system of equations in a^*.

$$\left\{ \begin{array}{rcl} 2(x_l - x_k)a_0^* + (y_l - y_k)a_1^* & = & b_0 - a_0, \\ (y_l - y_k)a_0^* + 2(z_l - z_k)a_1^* & = & b_1 - a_1, \end{array} \right.$$

with $k, l \in \mathbb{K}$, $k \neq l$ and $a, b \in \mathbb{K} \times \mathbb{K}$. This has a unique solution if and only if the determinant is not equal to 0. But this follows immediately from condition *(i')*.

We will now check condition (C).

The system of equations reads

$$\left\{\begin{array}{l} b_0^* + 2a_0 x_{k^*} + a_1 y_{k^*} = a_0' \\ b_1^* + a_0 y_{k^*} + 2a_1 z_{k^*} = a_1' \\ b_0^* + 2a_0^* x_{k^*} + a_1^* y_{k^*} = b_0 + 2a_0^* x_k + a_1^* y_k \\ b_1^* + a_0^* y_{k^*} + 2a_1^* z_{k^*} = b_1 + a_0^* y_k + 2a_1^* z_k \\ l^* + a_0^2 x_{k^*} + a_0 a_1 y_{k^*} + a_1^2 z_{k^*} + a_0 b_0^* + a_1 b_1^* = l \\ l^* + a_0^{*2} x_{k^*} + a_0^* a_1^* y_{k^*} + a_1^{*2} z_{k^*} + a_0^* b_0^* + a_1^* b_1^* = \\ \qquad k' + a_0^{*2} x_k + a_0^* a_1^* y_k + a_1^{*2} z_k + a_0^* b_0 + a_1^* b_1. \end{array}\right. \tag{2}$$

We eliminate b_0^* from the first and third equation and obtain

$$2(a_0^* - a_0)x_{k^*} + (a_1^* - a_1)y_{k^*} = b_0 - a_0' + 2a_0^* x_k + a_1^* y_k, \tag{3}$$

similarly, by eliminating b_1^* from the second and fourth equation we obtain

$$2(a_1^* - a_1)z_{k^*} + (a_0^* - a_0)y_{k^*} = b_1 - a_1' + 2a_1^* z_k + a_0^* y_k. \tag{4}$$

One easily checks that

$$\begin{array}{rcl} b_0^* & = & b_0 + 2a_0^*(x_k - x_{k^*}) + a_1^*(y_k - y_{k^*}) \\ b_1^* & = & b_1 + 2a_1^*(z_k - z_{k^*}) + a_0^*(y_k - y_{k^*}) \end{array}$$

We eliminate l^* from the last two equations in (2) and obtain

$$\begin{array}{l} (a_0^{*2} - a_0^2)x_{k^*} + (a_0^* a_1^* - a_0 a_1)y_{k^*} + (a_1^{*2} - a_1^2)z_{k^*} \\ \quad +(a_0^* - a_0)b_0^* + (a_1^* - a_1)b_1^* = \\ \qquad k' - l + a_0^{*2} x_k + a_0^* a_1^* y_k + a_1^{*2} z_k + a_0^* b_0 + a_1^* b_1. \end{array} \tag{5}$$

We substitute b_0^* and b_1^* in this equation. If we finally denote $X = x_{k^*} - x_k$, $Y = y_{k^*} - y_k$, $Z = z_{k^*} - z_k$, $A_0 = a_0^* - a_0$ and $A_1 = a_1^* - a_1$, then the equations (3), (4) and (5) can be written (after an elementary calculation) as

$$2XA_0 + YA_1 = B_0 \tag{6}$$

$$YA_0 + 2ZA_1 = B_1 \tag{7}$$

$$A_0^2 X + A_0 A_1 Y + A_1^2 Z = L \tag{8}$$

where

$$\left\{\begin{array}{rcl} B_0 & = & b_0 - a_0' + 2x_k a_0 + y_k a_1 \\ B_1 & = & b_1 - a_1' + 2z_k a_1 + y_k a_0 \\ L & = & l - k' - a_0^2 x_k - a_0 a_1 y_k - a_1^2 z_k - a_0 b_0 - a_1 b_1. \end{array}\right.$$

Note by the way that

$$B = (B_0, B_1) = \Psi_1(a, k, b, k') - a',$$

while

$$L = l - \Psi_2(a, k, b, k').$$

The system of equations (6) and (7) can be solved for (A_0, A_1) because the determinant of the matrix of the system equals $4XZ - Y^2$ and this is always non-zero by condition *(i')*. If we substitute the solution in equation (8), then we obtain after multiplying everything by $4XZ - Y^2$:

$$(B_1^2 X - B_0 B_1 Y + B_0^2 Z) = L(4XZ - Y^2). \tag{9}$$

Hence, if $(B_0, B_1) = (0, 0)$, then $L = 0$. Conversely, if $L = 0$, then from (8) and using condition *(i)* (put $A_0/A_1 = r$), it follows that $k^* = k$, i.e., that $X = Y = Z = 0$ and so from (6) and (7) one may conclude that $B = (B_0, B_1) = (0, 0)$. This shows that the situation in condition (C)*(ii)* never occurs. On the other hand, condition (C)*(i)* is an immediate consequence of (9) and of *(ii)* (divide (9) by $Y^2 - 4XZ$ and put $r = -B_1/B_0$).

With the above calculations it is now an easy exercise to show that conversely *(i)* and *(ii)* follow from (B) and (C). □

Definition

We call a field $\mathbb{K}$ of characteristic 2 *full* if the sum of 2 elements of $\mathcal{C}_1$ is never in $\mathcal{C}_1$. If the characteristic of the field is different from 2 then the field is called *full* if the product of two non-squares is a square. For full fields theorem 2.2 can be reformulated as in the next 2 theorems.

Theorem 2.3 *If $\mathbb{K}$ is full and the characteristic of $\mathbb{K}$ is not 2, then $\mathcal{R}_{x,y,z}$ defined in* (1) *is a quadratic quaternary ring if and only if the following three conditions hold:*

(a) for every $k, l \in \mathbb{K}$, $k \neq l$, the element $(y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)$ is not a square in $\mathbb{K}$,

(b) for every $r \in \mathbb{K}$, the mapping $\mathbb{K} \to \mathbb{K} : k \mapsto x_k r^2 + y_k r + z_k$ is surjective,

(c) for all $r, l \in \mathbb{K}$, the mapping

$$\mathbb{K} \setminus \{l\} \to \mathbb{K} \setminus \{0\} : k \mapsto \frac{(x_k - x_l)r^2 + (y_k - y_l)r + (z_k - z_l)}{(y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)}$$

is surjective.

Proof. It is straightforward to see that the injectivity of the map under *(i)* implies condition *(a)*, and hence *(a)*, *(b)* and *(c)* follow easily from *(i)* and *(ii)*.

Conversely, *(a)* implies the injectivity of the mapping in *(i)* and so *(ii)* follows from *(a)* and *(b)*. There remains to show that the map in *(c)* is injective, assuming $\mathbb{K}$ is full. Therefore we have to show that the discriminant $D(k, l, m)$ of the quadratic form

$$\left(\frac{x_k - x_l}{D(k,l)} - \frac{x_m - x_l}{D(m,l)}\right) r^2 + \left(\frac{y_k - y_l}{D(k,l)} - \frac{y_m - y_l}{D(m,l)}\right) r + \left(\frac{z_k - z_l}{D(k,l)} - \frac{z_m - z_l}{D(m,l)}\right) \tag{10}$$

with

$$D(i,l) = (y_i - y_l)^2 - 4(x_i - x_l)(z_i - z_l), \qquad i \in \{k, m\},$$

is a non-square. One easily computes (and the calculations are completely the same as on top of page 165 of Bader, Lunardon and Thas [1]) that

$$D(k,l,m) = \frac{D(k,m)}{D(k,l)D(m,l)},$$

which is a non-square since $D(k,l)D(m,l)$ is a square as $\mathbb{K}$ is full. $\square$

Theorem 2.4 *If $\mathbb{K}$ is a full field of characteristic 2, then $\mathcal{R}_{x,y,z}$ defined in (1) is a quadratic quaternary ring if and only if the following three conditions hold:*

(a') for every $k, l \in \mathbb{K}$, $k \neq l$, we have $y_k \neq y_l$, and the element $(x_k + x_l)(z_k + z_l)(y_k + y_l)^{-2}$ belongs to $\mathcal{C}_1(\mathbb{K})$.

(b') for every $r \in \mathbb{K}$, the mapping $\mathbb{K} \to \mathbb{K} : k \mapsto x_k r^2 + y_k r + z_k$ is surjective,

(c') for all $r, l \in \mathbb{K}$, the mapping

$$\mathbb{K} \setminus \{l\} \to \mathbb{K} \setminus \{0\} : k \mapsto \frac{(x_k + x_l)r^2 + (y_k + y_l)r + (z_k + z_l)}{(y_k + y_l)^2}$$

is surjective.

Proof. The proof is similar to the one of the former theorem. We now have to prove that the discriminant of the quadratic form (10) belongs to $\mathcal{C}_1(\mathbb{K})$. An easy computation shows that in this case

$$D(k,l,m) = \frac{(x_k + x_l)(z_k + z_l)}{(y_k + y_l)^2} + \frac{(x_l + x_m)(z_l + z_m)}{(y_l + y_m)^2} + \frac{(x_m + x_k)(z_m + z_k)}{(y_m + y_k)^2}.$$

Since $\mathbb{K}$ is full, $D(k,l,m) \in \mathcal{C}_1(\mathbb{K})$. $\square$

Corollary

Assume $\mathbb{K}$ is a finite field, then $\mathcal{R}_{x,y,z}$ defined in (1) is a quadratic quaternary ring if and only if *(a)* holds in case the characteristic is odd, or *(a')* holds in case the characteristic is even. Indeed, first of all a finite field is always full, moreover we have seen that the injectivity of the mappings in *(i)* and *(ii)* is a consequence of *(a)* or *(a')*. But in the finite case an injective map from a set to itself is also surjective.

Theorem 2.5 *Let x, y and z again be maps from $\mathbb{K}$ to itself, as in the beginning of paragraph 2.4. Consider in $\mathrm{PG}(3,\mathbb{K})$ the set $\mathcal{F}$ of planes with equation $x_k X_0 + z_k X_1 + y_k X_2 + X_3 = 0$. Then the intersection of these planes with the quadratic cone $\mathcal{C}$ with equation $X_0 X_1 = X_2^2$ defines a flock if and only if condition (i) of theorem 2.2 holds. If the characteristic of the field is not equal to 2, then this is equivalent to condition (a) and (b). For a field of characteristic 2, this is equivalent to (a') and (b'). If $\mathbb{K}$ is finite, then this is equivalent to condition (a) or (a').*

Proof. A general point p (different from the vertex of the cone) of $\mathcal{C}$ has coordinates $(r^2, 1, r, s)$, $r \in \mathbb{K} \cup \{\infty\}$, $s \in \mathbb{K}$. Exactly one member of $\mathcal{F}$ contains p if and only if there exists a unique $k \in \mathbb{K}$ such that

$$x_k r^2 + y_k r + z_k = -s.$$

This is clearly equivalent to condition *(i)*. The rest of the theorem easily follows from the proofs of theorems 2.3 and 2.4. □

Definition

If the maps x, y, z define a flock as in the previous theorem, then we call (as in the finite case) (x, y, z) a $\mathbb{K}$-*clan*. The following result is an immediate consequence of this definition, and theorems 2.2 and 2.5.

Theorem 2.6 *The maps x, y, z as above define a quadratic quaternary ring $\mathcal{R}_{x,y,z}$ and hence a generalized quadrangle if and only if for every $l \in \mathbb{K}$ the trio $({}^lX, {}^lY, {}^lZ)$, with*

$${}^lX_k = \frac{x_k - x_l}{D(k,l)}, \qquad {}^lY_k = \frac{y_k - y_l}{D(k,l)}, \qquad {}^lZ_k = \frac{z_k - z_l}{D(k,l)},$$

for $k \neq l$ (and with $D(k,l) = (y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)$) and ${}^lX_l = {}^lY_l = {}^lZ_l = 0$, is a $\mathbb{K}$-clan.

If $\mathbb{K}$ is a finite field of odd characteristic, then the flocks corresponding with $({}^lX, {}^lY, {}^lZ)$ are precisely the derivations of the original flock as defined in [1] in a geometrical way (yielding BLT-sets). For even characteristic we refer for instance to [9] Hence, for a general field we will also call the flocks corresponding with $({}^lX, {}^lY, {}^lZ)$ the *derivations* of the original one. We can reformulate theorem 2.6 as follows: *a generalized quadrangle arises from a flock if and only if all derivations of the flock are well defined.* So it is natural to call the quadrangles arizing from flocks *flock quadrangles*.

Remarks

In the case where the characteristic of a finite field $\mathbb{K}$ is odd, Knarr [8] has given a simple geometric construction of the generalized quadrangle from the BLT-set. This construction holds without any change in the infinite case, provided all derivations of the flock are well defined, hence we will not repeat this construction here.

The construction of a translation plane with dimension at most two over the kernel, via the Klein correspondence also holds in the infinite case without any change. Henceforth we will not repeat this here. Instead, we will present an algebraic way to relate these planes by establishing their coordinatizing field.

2.5 Flock planes and quadrangles

Let $\mathbb{K}$ be a field and $x, y, z : \mathbb{K} \to \mathbb{K}$ three arbitrary maps with the only restriction that they map 0 to 0. We define the following ternary ring $F_{x,y,z} = (\mathbb{K} \times \mathbb{K}, T)$.

$$T((k,t),(k_1,t_1),(k_2,t_2)) = (k', t'),$$

where

$$\begin{aligned} k' &= k_2 + kk_1 - z_t x_{t_1}, \\ x_{t'} &= x_{t_2} + k_1 x_t + k x_{t_1} + y_t x_{t_1}. \end{aligned}$$

We have the following result.

Theorem 2.7 *The ternary ring $F_{x,y,z}$ is a planar ternary ring (i.e. it coordinatizes a projective plane) if and only if (x, y, z) is a $\mathbb{K}$-clan. In this case, it is a left quasifield and the corresponding projective plane is a translation plane.*

Proof. One can easily verify that the equation

$$T(m, a, b) = T(m, a', b'), \quad a \neq a',$$

in m has a unique solution for all a, a', b, b', $a \neq a'$, if and only if condition *(i)* of theorem 2.2 is satisfied. Similarly, the system of equations

$$\left\{ \begin{array}{rcl} T(m, a, b) &=& k \\ T(m', a, b) &=& k' \end{array} \right. \qquad m \neq m',$$

in a and b has a unique solution for all m, m', k, k', $m \neq m'$, if and only if condition *(i')* of the proof of theorem 2.2 is satisfied.

The second part of the theorem follows from an easy calculation. □

A projective plane arising as in the previous theorem will be called a *flock plane.* Hence flocks and flock planes are essentially equivalent objects.

Note that the desarguesian projective plane over $\mathbb{K}$ is a subplane of the flock plane associated with $F_{x,y,z}$. This can be seen by viewing $\mathbb{K}$ as $\mathbb{K} \times \{(0, 0)\} \subseteq \mathbb{K} \times \mathbb{K}$.

Throughout the rest of this section, we denote $a.b = a_0 b_0 + a_1 b_1$, for $a, b \in K \times K$.

Theorem 2.8 *Every flock quadrangle is an elation generalized quadrangle.*

Proof. We show that the flock quadrangle $\mathcal{Q}$ coordinatized by the quadratic quaternary ring $(\mathbb{K} \times \mathbb{K}, \mathbb{K}, \Psi_1, \Psi_2)$ and defined by the functions x, y and z is an elation generalized quadrangle with elation point (∞). Therefore, we define the map

$$\begin{array}{rcl} \theta : \Gamma \to \Gamma : \ (a, l, a') &\mapsto& (a + A, l + L - a.A', a' + A') \\ {[k, b, k']} &\mapsto& [k, b + A' - (2x_k A_0 + y_k A_1, y_k A_0 + 2z_k A_1), \\ && k' + b.A + A'.A - (x_k A_0^2 + y_k A_0 A_1 + z_k A_1^2)], \end{array}$$

where the image of a point or a line with zero, one or two coordinates is obtained by restriction. Clearly, θ preserves the incidence relation (defined by the quaternary operations Ψ_1 and Ψ_2). The set of all such collineations θ is a group which apparently acts regularly on the set of points opposite (∞) and which fixes all lines through (∞). This proves the assertion. □

2.6 The dual setting

The notion of a conical flock can be dualized as follows. With the generators of the cone $\mathcal{C}$ there corresponds an irreducible dual conic C^* in a plane π of $\mathrm{PG}(3, \mathbb{K})$. With the planes of the flock $\mathcal{F}$ there corresponds a set of points $\mathcal{F}^*$ of $\mathrm{PG}(3, \mathbb{K}) \setminus \pi$ with the property that any line L that has 2 points in common with $\mathcal{F}^*$ intersects π in a point not lying on any of the lines of C^*. If the characteristic of $\mathbb{K}$ is not 2, this means that this point is an interior point of the (point) conic C, defined by C^*.

3 Examples

3.1 Smooth flock quadrangles over the reals

We were not able to prove in general that for any given flock all derivations are well defined. We are not aware either of any counterexample. However, over the reals, we may define a *smooth flock* as a flock having smooth maps x, y, z, i.e. the maps x, y, z have continuous derivatives. In this case, the map

$$f(r, l) : \mathbb{R} \to \mathbb{R} : k \mapsto \frac{x_k - x_l}{D(k, l)} r^2 + \frac{y_k - y_l}{D(k, l)} r + \frac{z_k - z_l}{D(k, l)}$$

is a bijection onto $\mathbb{R} \setminus \{0\}$ if and only if its derivative has a constant sign, the limits in $+\infty$ and $-\infty$ are 0 and the limits in l are $\pm\infty$. Since the derivative can be viewed as a quadratic form in r, it has always constant sign whenever the discriminant is non-positive. The same is true of course for the function

$$g(r) : \mathbb{R} \to \mathbb{R} : k \mapsto x_k r^2 + y_k r + z_k,$$

except that it is continuous everywhere and the limits in $\pm\infty$ must be $\pm\infty$. This means (denoting the derivative with a prime) that ${y'_k}^2 - 4x'_k z'_k \leq 0$. It takes an elementary calculation to prove that the discriminant of $f(r, l)'$ equals

$$\frac{{y'_k}^2 - 4x'_k z'_k}{D(k, l)^2}.$$

Hence smooth flocks over the reals have always well defined derivations whenever the above limits are correct. In other words, if the limits are correct, a smooth flock over $\mathbb{R}$ defines a flock quadrangle. We give an example

The helical line as a dual flock

Since in euclidean 3-space, the slope r_0 of the lines tangent to a helical line $\mathcal{H}$ with an axis parallel to the Z-axis is constant and smaller than the slope of any line meeting $\mathcal{H}$ in at least two points, the helical line wil be an example of a dual flock if we consider in the plane π_∞ at infinity a conic obtained by intersecting π_∞ with the cone obtained by revolving about the Z-axis a line through the origin with a slope $r \leq r_0$. As an extremal case, one can take $r = r_0$. For instance, we can take as equations for $\mathcal{H}$:

$$X = \sin k, \qquad Y = \cos k - 1, \qquad Z = k.$$

The equation of the conic C at infinity is

$$X^2 + Y^2 = Z^2.$$

With some elementary coordinate transformations, we can apply the theory on the dual setting and obtain as maps x, y, z:

$$\begin{cases} x_k = \frac{k+\sin k}{2} \\ y_k = \cos k - 1 \\ z_k = \frac{k-\sin k}{2} \end{cases}$$

Calculating $y_t'^2 - 4x_t'z_t'$, we obtain 0 and so we obtain indeed a flock (which we already knew from the dual setting). It is easy to check that the requirements on the limits are fulfilled here and so all derivations are well defined, hence there is a flock quadrangle associated with it.

Now we look at these derivations.

By theorem 2.6 we know that the derivation is given by

$$\begin{cases} {}^lX_k = \frac{k-l+\sin k-\sin l}{2D(k,l)} \\ {}^lY_k = \frac{\cos k-\cos l}{D(k,l)} \\ {}^lZ_k = \frac{k-l-\sin k+\sin l}{2D(k,l)}, \end{cases}$$

with

$$D(k,l) = 4\sin^2\frac{k-l}{2} - (k-l)^2.$$

Applying the inverse coordinate transformations, we find the dual of this derived flock:

$$\begin{cases} X = \frac{\sin k-\sin l}{D(k,l)} \\ Y = \frac{\cos k-\cos l}{D(k,l)} \\ Z = \frac{k-l}{D(k,l)} \end{cases}$$

This dual flock intersects the plane π_∞ at the point $(\cos l, \sin l, 1) \in C$ and hence cannot be isomorphic to the original one, since $\mathcal{H}$ intersects π_∞ at $(0,0,1) \notin C$. On the other hand all the derived flocks are isomorphic, as they can be mapped onto each other by a suitable rotation.

3.2 A generalization of Kantors' quadrangles

Usually, the functions x, y and z are written in a matrix

$$A_k = \begin{pmatrix} x_k & y_k \\ 0 & z_k \end{pmatrix}.$$

We now give some known examples, presenting them without the original finiteness restriction. We put, as before, $D(k,l) = (y_k - y_l)^2 - 4(x_k - x_l)(z_k - z_l)$ (the *discriminant*), and

$$f(r,l) : \mathbb{R} \to \mathbb{R} : k \mapsto \frac{x_k - x_l}{D(k,l)} r^2 + \frac{y_k - y_l}{D(k,l)} r + \frac{z_k - z_l}{D(k,l)}.$$

We also put $g(r)(k) = x_k r^2 + y_k r + z_k$.

Some examples

1. Let $X^2 + bX + c$ be an irreducible polynomial over $\mathbb{K}$. Then put

$$A_k = \begin{pmatrix} k & bk \\ 0 & ck \end{pmatrix}.$$

This flock quadrangle is in fact isomorphic to the orthogonal quadrangle defined by the quadric $\mathcal{Q}$ in $\mathrm{PG}(5,\mathbb{K})$ with equation

$$X_0^2 + bX_0X_1 + cX_1^2 = X_2X_3 + X_4X_5.$$

The proof of this fact is given in [10] for the finite case, but it can easily be extended to the general case. The corresponding flock is called the *linear flock*.

2. Let $m \in \mathbb{K}$ and let σ be a field automorphism of $\mathbb{K}$. Put

$$A_k = \begin{pmatrix} k & 0 \\ 0 & -mk^\sigma \end{pmatrix}.$$

For each field $\mathbb{K}$, one can seek conditions on m and σ such that the condition *(i)* of theorem 2.2 (or the equivalent conditions) are satisfied. For a finite field, one must have that m is a non-square. Of course, when σ is trivial, then we are back to example 1, so we cannot take $\mathbb{K} \cong \mathbb{R}$. Let us give another example. Let k be any field and put $\mathbb{K} = k((t))$, the field of all Laurent series over k in the indeterminate t. Let σ be an automorphism of $\mathbb{K}$ preserving the natural valuation on $\mathbb{K}$. Put $m = t$. Then the completeness of $\mathbb{K}$ with respect to the valuation ensures us that condition *(i)* of theorem 2.2 is satisfied. In fact, we can generalize this example to any field with a complete valuation, taking for m an element with odd valuation (and σ an automorphism of the field preserving the valuation).
The finite examples are due to Kantor [7]. The infinite examples were constructed by Van Maldeghem [14] in connection with affine buildings of type $\tilde{C}_2$.

3. Note that no flock, not even a linear one, can ever live over the complex numbers $\mathbb{C}$ or any other algebraically closed field.

4. Let $\mathbb{K}$ be a field of characteristic different from 3 in which raising to the third power is a bijection, e.g. $\mathbb{K} \cong \mathrm{GF}(q)$, $q \equiv 2 \bmod 3$, or $\mathbb{K} \cong \mathbb{R}$, etc. Then put

$$A_k = \begin{pmatrix} k & 3k^2 \\ 0 & 3k^3 \end{pmatrix}.$$

Here, $D(k,l) = -3(k-l)^4$ and $g(r)(k) = \frac{1}{9}[(3k+r)^3 - r^3]$. Clearly, this defines a flock. We now show that also all derivations are well defined. Let $r, l, a \in \mathbb{K}$ be arbitrary. We have to show that there exists a unique $k \in \mathbb{K}$ such that $f(r,l)(k) = a$. Using the expression for $g(r)(k)$ and $D(k,l)$ above, one readily finds

$$f(r,l)(k) = \frac{(l - \frac{r}{3})^3 - (k - \frac{r}{3})^3}{(k-l)^4}.$$

Substituting $K = a(k - \frac{r}{3})$ and $L = a(l - \frac{r}{3})$, the equation $f(r,l)(k) = a$ is equivalent to (K is the unknown)

$$K^3 - L^3 = (K - L)^4.$$

Note that, if $L = 0$, then $K = 1$ is the unique solution, since $K \neq L$. So we may assume $L \neq 0$. Putting $x = K - L$, this further simplifies to (since $K \neq L$, so $x \neq 0$)

$$3\frac{L^2}{x^3} + 3\frac{L}{x^2} + \frac{1}{x} = 1,$$

which is, putting $y = \frac{1}{x}$, multiplying everything by $9L$ and then adding 1 to both sides, equivalent to

$$(3Ly + 1)^3 = 9L + 1,$$

and this has a unique solution in y since $L \neq 0$. This shows our claim.

These examples (in the finite case) are due to Kantor [6]. In the literature, they are usually denoted by $K(q)$ for a finite field $GF(q)$. So we will adopt the notation $K(\mathbb{K})$ here.

In the finite case all the derived flocks are isomorphic to the original one, this is also true in the infinite case as one can easily check.

Other (finite) examples are enumerated in [12].

3.3 Quadrangles arising from hexagons

In this paragraph, we give another construction, of the flock quadrangles under number 4 above (due to Kantor [6] in the finite case). We approach the problem again via coordinatization.

Consider the split Cayley hexagon $H(\mathbb{K})$ over some field $\mathbb{K}$ of characteristic different from 3. We can define this as follows. The points of $H(\mathbb{K})$ are the 0-, 1-, 2-, 3-, 4- and 5-tuples in $\mathbb{K}$ (denoted with round parentheses and a 0-tuple is just a the symbol (∞)); the lines are the same things, but denoted with square brackets; incidence is given as follows:

$$(a,l,a',l',a'')\mathrm{I}[a,l,a',l']\mathrm{I}\dots\mathrm{I}(a)\mathrm{I}[\infty]\mathrm{I}(\infty)\mathrm{I}[k]\mathrm{I}\dots(k,b,k',b')\mathrm{I}[k,b,k',b',k'']$$

and $(a,l,a',l',a'')\mathrm{I}[k,b,k',b',k'']$ if and only if the following conditions are satisfied (see [3])

$$\begin{cases} a'' = ak+b, \\ a' = a^2k+b'+2ab, \\ k'' = a^3k+l-3a''a^2+3aa', \\ k' = a^3k^2+l'-kl-3a^2a''k-3a'a''+3aa''^2, \end{cases}$$

Now fix a line L in $H(\mathbb{K})$. Let us, following Kantor [6], define a geometry $\Gamma(L)$ as follows. The points of $\Gamma(L)$ are the line L, the points at distance 3 from L and the lines opposite L; the lines are the points incident with L and the lines at distance 4 from L; incidence is the old incidence together with the old collinearity. We now give a description of $\Gamma(L)$ in terms of the coordinates, putting $L=[\infty]$.

First, we remark that $\Gamma(L)$ does not contain triangles through its point L. This makes it possible to coordinatize $\Gamma(L)$ as if it were a generalized quadrangle, choosing $R_1 = \mathbb{K}\times\mathbb{K}$ and $R_2=\mathbb{K}$ and denoting $(0,0)$ by 0, as before. We denote the coordinates in $\Gamma(L)$ with a subscript L and make the following labelling (following the general rules about coordinatization of a generalized quadrangle, see [4]):

$$\begin{array}{rcl@{\qquad\qquad}rcl} [\infty] & \to & (\infty)_L & (\infty) & \to & [\infty]_L \\ (-a_1,-a_0) & \to & ((a_0,a_1))_L & (3k) & \to & [k]_L \\ (0,9b_1,3b_0)) & \to & (0,(b_0,b_1))_L & [0,0,9l] & \to & [0,l]_L \end{array}$$

The factors 3 and 9, and the minus signs are not essential and can be deleted. We have introduced them to make the identification of $\Gamma(L)$ later on easier. This labelling defines coordinates for all elements of $\Gamma(L)$ and after a few calculations, one finds:

$$\begin{array}{rcl} (3k,9b_1-9kb_0,3b_0) & \to & (k,(b_0,b_1))_L \\ [-a_1,-a_0,9l] & \to & [(a_0,a_1),l]_L \\ [-a_1,-a_0,9l,3a_0',9a_1'] & \to & ((a_0,a_1),l,(a_0',a_1'))_L \\ (3k,9b_1-9kb_0,3b_0,9k') & \to & [k,(b_0,b_1),k']_L \end{array}$$

Incidence between points and lines not both having three coordinates is the usual one. Using the incidence conditions in $H(\mathbb{K})$, one computes the condition under which a point $(a, l, a')_L$ (with $a = (a_0, a_1)$, etc. as before) is incident with a line $[k, l, k']_L$. This condition is

$$\begin{cases} (a'_0, a'_1) &= (b_0 + 2a_0k + 3a_1k^2, b_1 + 3a_0k^2 + 6a_1k^3) \\ l &= k' + a_0^2k + 3a_0a_1k^2 + 3a_1^2k^3 + a_0b_0 + a_1b_1. \end{cases}$$

The right hand sides are exactly the quaternary operations of the ring $\mathcal{R}_{x,y,z}$ with x, y, z the maps corresponding to example 4 above. Hence we conclude:

Theorem 3.1 *The geometry Γ_L derived from the split Cayley hexagon as defined above, is a generalized quadrangle if and only if $\mathbb{K}$ is a field of characteristic different from 3 in which raising to the power 3 is a bijection in K. In this case, $\Gamma(L)$ is isomorphic to the flock quadrangle $K(\mathbb{K})$.*

Proof. To prove this theorem, we only have to show that $\Gamma(L)$ is not a generalized quadrangle for fields $\mathbb{K}$ of characteristic 3. But this can easily be done using the same method as above. We only needed the characteristic to be different from 3 in order to be able to introduce the factors 3 and 9. Deleting them gives us a quaternary ring which can never be quadratic. We leave the details of the computations to the reader. □

Remark

The non-existence of triangles in $\Gamma(L)$ implies the non-existence of four apartments in $H(\mathbb{K})$ two by two sharing a chain of length 4 starting with a line and for which the middle element of one of these chains is L (and let us call the latter configuration a *Kantor configuration* based at L). This is just expressing that lines opposite L, viewed as points of $\Gamma(L)$ cannot form a triangle. But apparently, no other triangles can arise by the properties of a generalized hexagon. Hence the non-existence of a Kantor configuration based at L is equivalent to the non-existence of triangles in $\Gamma(L)$. In the finite case, this is equivalent to $\Gamma(L)$ being a generalized quadrangle (because it has the right order). This extends to every finite generalized hexagon for which $s = t$. In the infinite classical non-characteristic 3 case, the absence of any Kantor configuration based at L is equivalent to raising to the third power being injective in $\mathbb{K}$, but not surjective in general. Examples of these arise in Laurent series fields, e.g. over finite fields of order $q \equiv 2 \bmod 3$.

Bader and Lunardon construct $H(q)$, $q \equiv 2 \bmod 3$ geometrically using the BLT-set of the flock corresponding to $K(q)$. Exactly the same construction applies in the infinite case, still under the assumption that raising to the third power is a bijection in $\mathbb{K}$. We will not repeat the construction here.

References

[1] **L. Bader, G. Lunardon, and J. A. Thas**. Derivation of flocks of quadratic cones. *Forum Math.*, 2, pp. 163–174, 1990.

[2] **A. E. Brouwer**. Generalized quadrangles with line size 4. In J. W. P. Hirschfeld, D. R. Hughes, and J. A. Thas, editors, *Advances in Finite Geometries and Designs*, pages 47–49. Oxford University Press, 1991.

[3] **V. De Smet and H. Van Maldeghem**. The finite Moufang hexagons coordinatized. *Contr. Alg. Geom*, 34, pp. 217–232, 1993.

[4] **G. Hanssens and H. Van Maldeghem**. Coordinatization of generalized quadrangles. *Ann. Discrete Math.*, 37, pp. 195–208, 1988.

[5] **D. R. Hughes and F. C. Piper**. *Projective Planes.* Springer Verlag, 1973.

[6] **W. M. Kantor**. Generalized quadrangles associated with $G_2(q)$. *J. Combin. Theory Ser. A*, 29, pp. 212–219, 1980.

[7] **W. M. Kantor**. Some generalized quadrangles with parameters (q^2, q). *Math. Z.*, 192, pp. 45–50, 1986.

[8] **N. Knarr**. A geometric construction of generalized quadrangles from polar spaces of rank three. *Resultate Math.*, 21, pp. 332–344, 1992.

[9] **S. E. Payne**. Collineations of the Subiaco generalized quadrangles. *Bull. Belgian Math. Soc., Simon Stevin*, 1, pp. 127–138, 1994.

[10] **S. E. Payne and J. A. Thas**. *Finite Generalized Quadrangles*, volume 104 of *Research Notes in Mathematics.* Pitman, 1984.

[11] **J. A. Thas**. Generalized quadrangles and flocks of cones. *Europ. J. Combinatorics*, 8, pp. 441–452, 1987.

[12] **J. A. Thas**. Generalized polygons. In F. Buekenhout, editor, *Handbook of Incidence Geometry.* North-Holland, to appear.

[13] **J. Tits**. Sur la trialité et certains groupes qui s'en déduisent. *Publ. Math. IHES*, 2, pp. 14–60, 1959.

[14] **H. Van Maldeghem**. Quadratic quaternary rings with valuation and affine buildings of type $\tilde{C}_2$. *Mitt. Math. Sem. Giessen*, 189, pp. 1–159, 1989.

F. De Clerck and **H. Van Maldeghem**
University of Ghent
Department of Pure Mathematics and Computer Algebra
Galglaan 2
B-9000 Gent
Belgium