Affine embeddings of $(0, \alpha)$-geometries

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Abstract

It is the purpose of this research note to give an overview of the recent results on full embeddings of $(0, \alpha)$-geometries in affine spaces.

1 Definitions

A $(0, \alpha)$-*geometry* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a connected partial linear space of order (s, t) with the property that for every anti-flag (p, L) the number $\alpha(p, L)$ of lines of $\mathcal{S}$ through p intersecting L equals 0 or a constant α. This class of geometries generalizes the class of *partial geometries* introduced by Bose [1]. In a partial geometry, denoted as $\mathrm{pg}(s, t, \alpha)$, $\alpha(p, L) = \alpha$ for every antiflag (p, L). A $\mathrm{pg}(s, t, t)$ is also called a *(Bruck) net*, while a $\mathrm{pg}(s, t, 1)$ is a *generalized quadrangle* of order (s, t).

The point graph of a partial geometry is strongly regular. If the point graph of a $(0, \alpha)$-geometry $\mathcal{S}$ is strongly regular, then $\mathcal{S}$ is called a *semipartial geometry* $\mathrm{spg}(s, t, \alpha, \mu)$ [10]. Here μ is the number of vertices adjacent to two non-adjacent vertices in the point graph of $\mathcal{S}$. Every partial geometry is a semipartial geometry, but not vice versa. A semipartial geometry which is not a partial geometry is called a *proper* semipartial geometry.

A $(0, \alpha)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is said to be *fully embedded* (or, shortly, *embedded*) in an affine space $\mathrm{AG}(n, q)$ if the lines of $\mathcal{S}$ are lines of $\mathrm{AG}(n, q)$, if $\mathcal{P}$ is the set of all affine points on the lines of $\mathcal{S}$ and if I is as in $\mathrm{AG}(n, q)$. We also require that $\mathcal{P}$ spans $\mathrm{AG}(n, q)$. We say that $\mathcal{S}$ has a *planar net* if there is an affine plane such that the points and lines of $\mathcal{S}$ in it form a net. We denote by Π_∞ the space at infinity of $\mathrm{AG}(n, q)$. If $\alpha > 1$ it can be proved (see for example [3]) that if a plane π containing two intersecting lines of $\mathcal{S}$ does not contain a planar net of $\mathcal{S}$, then $\alpha = 2, q = 2^h$ and the lines of $\mathcal{S}$ in π form a dual oval with nucleus $\pi \cap \Pi_\infty$. If $\alpha = 1$ and if the plane π contains two intersecting lines of $\mathcal{S}$, then either π contains a planar net of $\mathcal{S}$ or $\mathcal{S}$ and π have γ concurrent lines in common, $2 \leq \gamma \leq q + 1$. Hence, in this case there is less structure and we think that a classification of $(0, 1)$-geometries fully embedded in an affine space is very difficult. For some extra information on this case we refer to [3].

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2 Motivation

The *linear representation* $T^*_{n-1}(\mathcal{K}_\infty)$ of a set $\mathcal{K}_\infty \subseteq \Pi_\infty$ is the partial linear space embedded in $\mathrm{AG}(n,q)$ which has as line set the set of all affine lines intersecting Π_∞ in a point of $\mathcal{K}_\infty$. If $\mathcal{K}_\infty$ is a set of type $\{0,1,\alpha+1\}$, i.e., a set intersecting every line in $0,1$ or $\alpha+1$ points, spanning Π_∞ then $T^*_{n-1}(K_\infty)$ is a $(0,\alpha)$-geometry. Every two intersecting lines of $T^*_{n-1}(\mathcal{K}_\infty)$ are contained in a (necessarily unique) planar net.

Thas [13] has classified the affine embeddings of partial geometries. In the case $\alpha > 1$ (that is, where the partial geometry is not a generalized quadrangle) only linear representations occur. In particular, when $\alpha > 1$, $\mathcal{K}_\infty$ is the complement of a hyperplane in Π_∞ or $n = 3$ and $\mathcal{K}_\infty$ is a maximal arc in Π_∞. Debroey and Thas [9] have classified the embeddings of semipartial geometries in $\mathrm{AG}(n,q)$ for $n \leq 3$. Again only linear representations occur when $\alpha > 1$. In particular, when $\alpha > 1$, and the geometry is a proper semipartial geometry then $n = 3$ and $\mathcal{K}_\infty$ is a unital or a Baer subplane of Π_∞. Recently De Winter [8] proved that if a linear representation $T^*_3(\mathcal{K}_\infty)$ in $\mathrm{AG}(4,q)$ is a proper semipartial geometry with $\alpha > 1$, then $\mathcal{K}_\infty$ is a Baer subspace of Π_∞.

However Hirschfeld and Thas [12] describe an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ $\mathrm{TQ}(4,q)$ which, for even q, is embedded in $\mathrm{AG}(4,q)$ but is not a linear representation. This geometry is obtained by intersecting the secant lines of a nonsingular elliptic quadric in $\mathrm{PG}(5,q)$ through a given external point with a hyperplane not containing that point. The intersection of $\mathrm{TQ}(4,q)$ with an affine hyperplane yields a $(0,2)$-geometry of order $(q-1,q)$ which we will denote as HT. This geometry is embedded in $\mathrm{AG}(3,q)$ and it is not a linear representation. Neither $\mathrm{TQ}(4,q)$ nor HT contains a planar net.

The existence of the geometries $\mathrm{TQ}(4,q)$ and HT motivates the characterization by De Clerck and Delanote [3] of the linear representations among the $(0,\alpha)$-geometries ($\alpha > 1$) embedded in $\mathrm{AG}(n,q)$. In particular, it follows from their result that if $\alpha > 2$ or if q is odd, then only linear representations can occur. Recently also the $(0,2)$-geometries embedded in $\mathrm{AG}(n,2^h)$ were classified by De Feyter [4, 5, 6, 7]. Two new constructions, found by De Feyter, of $(0,2)$-geometries arise which are not linear representations. We describe them in the next section.

3 The geometries $\mathcal{A}(O_\infty)$ and $\mathcal{I}(n,q,e)$

De Feyter has constructed two $(0,2)$-geometries with an affine embbedding. We give a short description, for more information we refer to the papers of De Feyter, [4, 5, 6, 7].

The $(0,2)$-geometry $\mathcal{A}(O_\infty)$ which is embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, is constructed as follows. Let O_∞ be an oval of Π_∞ with nucleus n_∞. Choose a basis such that Π_∞ : $X_3 = 0, n_\infty(1,0,0,0)$ and such that $(0,1,0,0)$, $(0,0,1,0)$, $(1,1,1,0) \in O_\infty$. Let f be the o-polynomial (see [11], section 8.4) such that $O_\infty = \{(f(\rho),\rho,1,0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(0,1,0,0)\}$ and for every affine point $p(x,y,z,1)$ let $O^p_\infty = \{(y+z\rho+f(\rho),\rho,1,0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(z,1,0,0)\}$. Let $\mathcal{L}_p$ be the set of lines through p and a point of O^p_∞. Let $\mathcal{P}$ be the point set of $\mathrm{AG}(3,q)$, $\mathcal{B} = \bigcup_{p\in\mathcal{P}} \mathcal{L}_p$, and let I be the natural incidence. If O_∞ is not a conic then $\mathcal{A}(O_\infty) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is connected and is indeed a $(0,2)$-geometry of order $(2^h - 1, 2^h)$ [5]. If O_∞ is a conic then the incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ consists of two connected components which are projectively equivalent with HT [5], and we let $\mathcal{A}(O_\infty)$ be any of these two components. Note that $\mathcal{A}(O_\infty)$ is not a semipartial geometry and has no planar nets.

Another $(0,2)$-geometry $\mathcal{I}(n,q,e)$ embedded in $\mathrm{AG}(n,q)$, $n \geq 2$, $q = 2^h$ is constructed as follows. Let U be a hyperplane of $\mathrm{AG}(n,q)$, and choose a basis such that $\Pi_\infty : X_n = 0$ and $U : X_{n-1} = 0$. Let $e \in \{1, 2, \ldots, h-1\}$ be such that $\gcd(e,h) = 1$, and let φ be the collineation of $\mathrm{PG}(n,q)$ mapping $p(x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x_n)$ to $p^\varphi(x_0^{2^e}, x_1^{2^e}, \ldots, x_{n-2}^{2^e}, x_n^{2^e}, x_{n-1}^{2^e})$. Put $U_\infty = U \cap \Pi_\infty$ and let $\mathcal{K}_\infty$ be the set of points of U_∞ fixed by φ. Then $\mathcal{K}_\infty$ is the point set of a projective geometry $\mathrm{PG}(n-2,2) \subseteq U_\infty$. Let $\mathcal{B}$ be the set of affine lines L such that either $L \subseteq U$ and $L \cap \Pi_\infty \in \mathcal{K}_\infty$, or L intersects U in an affine point p and $L \cap \Pi_\infty = p^\varphi$. Let $\mathcal{P}$ be the set of affine points on the lines of $\mathcal{B}$, and let I be the natural incidence. Then $\mathcal{I}(n,q,e) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a $(0,2)$-geometry of order $(2^h - 1, 2^{n-1} - 1)$ embedded in $\mathrm{AG}(n, 2^h)$ [6]. These geometries are not semipartial geometries and have planar nets as well as planes containing a dual oval of lines of $\mathcal{I}(n,q,e)$.

4 Results and applications

We recall that, given a $(0,\alpha)$-geometry $(\alpha > 1)$ embedded in $\mathrm{AG}(n,q)$, every intersection with a plane containing at least two intersecting lines of the geometry, is either a planar net or is a geometry whose lines form a dual oval with nucleus the line at infinity of the plane (in which case $\alpha = 2$ and $q = 2^h$). De Clerck and Delanote [3] have proved the following theorem.

Theorem 4.1 ([3]) *If $\mathcal{S}$ is a $(0,\alpha)$-geometry $(\alpha > 1)$ embedded in $\mathrm{AG}(n,q)$ such that every two intersecting lines of $\mathcal{S}$ are contained in a planar net of $\mathcal{S}$, then $\mathcal{S}$ is a linear representation of a set $\mathcal{K}_\infty$ in Π_∞. In particular this conclusion holds if $\alpha > 2$ or q is odd.*

A $(0,2)$-geometry of order $(1,t)$ is just a complete graph. The embedding of complete graphs in $\mathrm{AG}(n,2)$ is trivial, so we assume in the following theorem, due to De Feyter, that $q > 2$.

Theorem 4.2 ([4, 5, 6, 7]) *Let $\mathcal{S}$ be a $(0,2)$-geometry embedded in $\mathrm{AG}(n,q)$, $q = 2^h$, $h > 1$, of order $(q-1,t)$. Then one of the following cases occurs.*

1. *$\mathcal{S}$ is a linear representation of a set $\mathcal{K}_\infty$ in Π_∞ which spans Π_∞ and is of type $\{0,1,3\}$.*

2. *$\mathcal{S} = \mathcal{I}(n,q,e)$.*

3. *$n = 2$ and the lines of $\mathcal{S}$ together with Π_∞ form a dual hyperoval.*

4. *$n = 3$ and $\mathcal{S} = \mathcal{A}(O_\infty)$.*

5. *$n = 4$ and $\mathcal{S} = \mathrm{TQ}(4,q)$.*

The most difficult part of the proof of Theorem 4.2 is when $n = 3$. In [6] it is proved that when $n = 3$ and there is a planar net, then we are in case 1 or 2. In [5] it is proved that under the conditions $n = 3$, $t = q$ and such that there are no planar nets, case 4 is occuring. In [5, 4] it is assumed that $n = 3$, while $t \neq q$, and that there are no planar nets, and a contradiction is found. Finally in [7] the case $n > 3$ is treated. This final part of the proof uses mainly an induction argument on n. In particular we rely on the following fact. Let $\mathcal{S}$ be a $(0, \alpha)$-geometry ($\alpha > 1$) embedded in $\mathrm{AG}(n, q)$ and let $\mathrm{AG}(m, q)$ be an affine subspace. If a connected component of the incidence structure induced by $\mathcal{S}$ in $\mathrm{AG}(m, q)$ contains two intersecting lines, then it is again a $(0, \alpha)$-geometry, which is embedded in $\mathrm{AG}(m, q)$. Similar arguments do not hold however for $\alpha = 1$, making this case much more difficult. Note that Theorem 4.2 does not classify the sets $\mathcal{K}_\infty$ in case 1 as this would imply the unlikely classification of all sets of type $\{0, 1, 3\}$ in $\mathrm{PG}(n, 2^h)$.

Corollary 4.3 *Let $\mathcal{S}$ be a semipartial geometry $\mathrm{spg}(s, t, \alpha, \mu)$, $\alpha > 1$, embedded in $\mathrm{AG}(n, q)$, $q > 2$. Then one of the following cases occurs.*

1. *$\mathcal{S}$ is a linear representation of a set $\mathcal{K}_\infty$ of type $\{0, 1, \alpha + 1\}$ in Π_∞ such that through every point of Π_∞ not in $\mathcal{K}_\infty$ there are exactly $\frac{\mu}{\alpha(\alpha+1)}$ lines containing $\alpha + 1$ points of $\mathcal{K}_\infty$.*
2. *$n = 2$ and the lines of $\mathcal{S}$ together with Π_∞ form a dual hyperoval.*
3. *$n = 4$ and $\mathcal{S} = \mathrm{TQ}(4, q)$.*

We recall that for $n \leq 4$ the linear representations of semipartial geometries with $\alpha > 1$ are classified [9, 8]. The only known linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ with $n \geq 5$ that is a proper semipartial geometry with $\alpha > 1$ is $T^*_{n-1}(\mathcal{B}_\infty)$, where $\mathcal{B}_\infty$ is a Baer subspace of Π_∞.

Corollary 4.3 classifies the semipartial geometries with $\alpha > 1$ embedded in $\mathrm{AG}(n, q)$ which are not linear representations. This extends partially the result of Debroey and Thas [9] for $\mathrm{AG}(2, q)$ and $\mathrm{AG}(3, q)$, where no restriction is assumed (see also [3] for another proof). Corollary 4.3 also improves the characterization by Brown, De Clerck and Delanote [2] saying that $\mathrm{TQ}(4, q)$ is the only $\mathrm{spg}(q - 1, q^2, 2, 2q(q - 1))$ embedded in $\mathrm{AG}(4, q)$.

References

[1] R. C. Bose. Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.*, 13:389–419, 1963.

[2] M. R. Brown, F. De Clerck, and M. Delanote. Affine semipartial geometries and projections of quadrics. *J. Combin. Theory Ser. A*, 103(2):281—289, 2003.

[3] F. De Clerck and M. Delanote. On $(0, \alpha)$-geometries and dual semipartial geometries fully embedded in an affine space. *Des. Codes Cryptogr.*, 32:103–110, 2004.

[4] N. De Feyter. Classification of $(0, 2)$-geometries embedded in $\mathrm{AG}(3, q)$. Preprint.

[5] N. De Feyter. The embedding in $\mathrm{AG}(3,q)$ of $(0,2)$-geometries with no planar nets. To appear in *J. Combin. Theory Ser. A*.

[6] N. De Feyter. The embedding in $\mathrm{AG}(3,q)$ of $(0,2)$-geometries containing a planar net. Preprint.

[7] N. De Feyter. The embedding of $(0,2)$-geometries and semipartial geometries in $\mathrm{AG}(n,q)$. To appear in *Adv. Geom.*

[8] S. De Winter. Linear representations of semipartial geometries. Preprint.

[9] I. Debroey and J. A. Thas. Semipartial geometries in $\mathrm{AG}(2,q)$ and $\mathrm{AG}(3,q)$. *Simon Stevin*, 51(4):195–209, 1977/78.

[10] I. Debroey and J. A. Thas. On semipartial geometries. *J. Combin. Theory Ser. A*, 25(3):242–250, 1978.

[11] J. W. P. Hirschfeld. *Projective geometries over finite fields.* The Clarendon Press Oxford University Press, New York, 1979. Oxford Mathematical Monographs.

[12] J. W. P. Hirschfeld and J. A. Thas. Sets of type $(1,\, n,\, q+1)$ in $\mathrm{PG}(d,\, q)$. *Proc. London Math. Soc. (3)*, 41(2):254–278, 1980.

[13] J. A. Thas. Partial geometries in finite affine spaces. *Math. Z.*, 158(1):1–13, 1978.

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