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A class of Poisson–Nijenhuis structures on a tangent bundle

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Abstract
Equipping the tangent bundle TQ of a manifold with a symplectic form coming from a regular Lagrangian L, we explore how to obtain a Poisson–Nijenhuis structure from a given type $(1, 1)$ tensor field J on Q. It is argued that the complete lift J^c of J is not the natural candidate for a Nijenhuis tensor on TQ, but plays a crucial role in the construction of a different tensor R, which appears to be the pullback under the Legendre transform of the lift of J to T^*Q. We show how this tangent bundle view brings new insights and is capable also of producing all important results which are known from previous studies on the cotangent bundle, in the case when Q is equipped with a Riemannian metric. The present approach further paves the way for future generalizations.

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1. Introduction

There is a well-established theory of bi-Hamiltonian systems on the cotangent bundle T^*Q of a manifold Q or, more generally, on a symplectic or Poisson manifold. In particular, there is a link between separability of the Hamilton–Jacobi equation and certain classes of bi-Hamiltonian or quasi-bi-Hamiltonian systems, in which more specifically Poisson–Nijenhuis structures [17] play a prominent role. The immediate source of inspiration for the present work is a series of recent papers in this general field in which relations have been explored between such things as bi-differential calculi, complete integrability, Stäckel systems, compatible Poisson structures on an extended space, Gelfand–Zakharevich systems, so-called special conformal Killing tensors and so on. See [3, 4, 7, 9–13, 15, 16, 18, 21, 27] for a non-exhaustive list of

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recent contributions. In the more direct applications of these theoretical developments, the dominant geometrical space is a cotangent bundle, which of course comes equipped with its canonical symplectic or Poisson structure, and on which a compatible Poisson structure is obtained via the lift $\tilde{J}$ of a type (1, 1) tensor field J on the base manifold, thus giving birth to a Poisson–Nijenhuis structure. Recall that two Poisson structures P_1 and P_2 are said to be compatible if the pencil $P_1 + tP_2$ is Poisson for all t. The link with a class of Stäckel systems requires the availability of a metric g on Q, with respect to which J has the property of being a special conformal Killing tensor (a concept which is intimately linked with what is called Benenti tensor [4, 5], after the extensive work of Benenti on Hamilton–Jacobi separability (see, e.g., [1, 2])).

The point we want to emphasize now, however, is that some of these applications clearly come from meaningful questions about dynamical systems living on a tangent bundle TQ. This is, for example, the case with the theory of cofactor pair systems, as developed originally by Lundmark on a Euclidean space [18, 19] and generalized to (pseudo-)Riemannian manifolds and in more geometrical terms by Crampin and Sarlet [9]. The physical background for these systems is a kinetic energy type Lagrangian on TQ for which (at least in one possible interpretation) admissible non-conservative forces are being sought, in the sense that the resulting Newtonian system admits two quadratic first integrals, which in turn can generate a whole family of integrals in involution. The cofactors of the Killing tensors coming from these integrals then determine the special conformal Killing tensors which give rise to Poisson–Nijenhuis structures; we are then looking at examples of so-called bi-quasi-Hamiltonian systems [10].

A type (1, 1) tensor field J on Q also has a natural lift to TQ; it is usually called the *complete lift* and we will denote it by J^c (see [33] which is a standard reference for lift constructions or the defining relations (3) below). The difference with T^*Q of course is that TQ does not carry a canonical Poisson structure. However, a symplectic form is available (and can be constructed by pure tangent bundle techniques) as soon as a regular Lagrangian is given, which could for example be the kinetic energy Lagrangian coming from a metric on Q. It does, therefore, perfectly make sense to explore the possibility of obtaining Poisson–Nijenhuis structures on TQ by natural tangent bundle constructions. One of the goals we have in mind for the future is to arrive at a generalization of the theory of special conformal Killing tensors from Riemannian to Finsler spaces. The primary objective of this paper, however, is to set the ground for future developments by trying to understand in detail how the results one is by now familiar with in a cotangent bundle environment, can be obtained in a natural way by pure tangent bundle techniques. We shall see that this different way of approaching the subject offers new insights anyway. In fact, some of the preliminary considerations lead to results which are valid for arbitrary Lagrangians, not just kinetic energy type ones. In this respect, we are to some extent joining the interest in Poisson structures on a tangent bundle which is also present in the recent work by Vaisman [25, 31, 32].

In section 2, starting from a given Lagrangian L on TQ and a type (1, 1) tensor field J on Q, we show that J^cS determines an alternative almost tangent structure and use this to construct another type (1, 1) tensor R on TQ. The full characterization of R is developed in section 3, making use of the connection provided by the second-order equation field Γ, coming from the Lagrange equations. The eigenspace structure of R is discussed in appendix B, whereas further properties of general interest are derived in section 4. In section 5, we specialize to the particular case of a Riemannian manifold and the associated kinetic energy Lagrangian and explore how various known properties make their appearance within such a tangent bundle approach. Some unexpected new features come forward which are further discussed in appendix A. The road map to future developments is sketched in the final section.

2. Generalities

Suppose we are given a regular Lagrangian L on TQ and a type (1, 1) tensor field J on Q. L comes with its associated symplectic form ω_L and J determines a tensor field J^c, its *complete lift*, on TQ. One may wonder whether these data can give rise, under some circumstances, to a compatible Poisson structure. For that, J^c should be symmetric with respect to ω_L, the so-called Magri–Morosi concomitant must vanish (see [28, 20]), and also the Nijenhuis torsion of J^c must be zero (which is equivalent to the torsion of J being zero). It seems to us, however, that this is not the most interesting path to pursue. Experience in a variety of applications (see, for example, [8, 26, 29]) has shown that interesting type (1, 1) tensor fields R on a symplectic manifold (M, ω) arise from the construction of a second 2-form ω_1 and the determining formula:

$$i_{R(\xi)}\omega = i_\xi\omega_1 \qquad \forall\xi \in \mathcal{X}(TQ). \tag{1}$$

A direct advantage of such tensor fields is, for example, that they automatically have the required symmetry property

$$\omega(R\xi, \eta) = \omega(\xi, R\eta). \tag{2}$$

Furthermore, vanishing of the Magri–Morosi concomitant then is equivalent to $d\omega_1 = 0$ [11], after which we are left with the condition $N_R = 0$. Although the canonical lift $\tilde{J}$ of J to T^*Q can be defined (see [8]) via a relation like (1), this does not seem to be the case for J^c on TQ. We therefore start our investigation with an exploration of possible natural ways of constructing a second closed 2-form ω_1 from the given data on TQ.

For completeness, we list a number of useful properties of J^c, starting with defining relations with respect to the action on complete and vertical lifts of vector fields on Q (see [6] for more background on this): for $X, Y \in \mathcal{X}(Q)$,

$$J^c(X^c) = (JX)^c \qquad J^c(X^V) = (JX)^V. \tag{3}$$

Using the bracket relations

$$[X^V, Y^V] = 0 \qquad [X^V, Y^c] = [X, Y]^V \qquad [X^c, Y^c] = [X, Y]^c$$

it easily follows that

$$\mathcal{L}_{X^c}J^c = (\mathcal{L}_X J)^c \qquad \mathcal{L}_{X^V}J^c = (\mathcal{L}_X J)^V. \tag{4}$$

Also, for the Nijenhuis torsion we have

$$N_{J^c}(X^V, Y^V) = 0$$
$$N_{J^c}(X^c, Y^c) = (N_J(X, Y))^c$$
$$N_{J^c}(X^V, Y^c) = N_{J^c}(X^c, Y^V) = (N_J(X, Y))^V$$

from which it follows that $N_{J^c} = 0 \Longleftrightarrow N_J = 0$.

It is imperative to relate J^c to the canonical type (1, 1) tensor field S on TQ (the so-called vertical endomorphism), which satisfies $S(X^c) = X^V$, $S(X^V) = 0$. It is easy to see that J^c commutes with S, in the sense of endomorphisms on $\mathcal{X}(TQ)$, but also in the sense of the Nijenhuis bracket:

$$[J^c, S] = 0. \tag{5}$$

It follows that also the corresponding degree 1 derivations commute, meaning that

$$d_S\, d_{J^c} = -d_{J^c}\, d_S. \tag{6}$$

Finally, it is easy to verify that for any J,

$$N_{J^cS} = 0. \tag{7}$$

In fact, we can make the following more complete statement in that respect, which is trivial to prove, and essentially says that J^cS has almost the same properties as S.

Lemma 1. *We have $(J^cS)^2 = 0$ and $N_{J^cS} = 0$. Furthermore, if J is non-singular, J^cS determines an integrable almost tangent structure.*

Now recall the role which the vertical endomorphism plays in the definition of the Poincaré–Cartan 2-form ω_L: we have

$$\omega_L = \mathrm{d}(S(\mathrm{d}L)) = \mathrm{d}\,\mathrm{d}_S L.$$

(We make no notational distinction between the action of a type (1, 1) tensor field on vector fields and its dual action on 1-forms, but one should keep in mind that the order of composition of such action changes in passing from one interpretation to the other.) It then looks perfectly natural, given J and the sort of alternative integrable almost tangent structure which it creates, to consider the closed 2-from ω_1, defined (with various ways of writing the same expression) by

$$\omega_1 = \mathrm{d}(SJ^c(\mathrm{d}L)) = \mathrm{d}(S(\mathrm{d}_{J^c}L)) = \mathrm{d}(J^c(\mathrm{d}_S L)) = \mathrm{d}\,\mathrm{d}_{J^cS}L. \tag{8}$$

And so, with L and J as data, the type (1, 1) tensor field R which will carry our attention is defined by

$$i_{R(\xi)}\,\mathrm{d}\,\mathrm{d}_S L = i_\xi\,\mathrm{d}\,\mathrm{d}_{J^cS}L. \tag{9}$$

We know that it will define a Poisson–Nijenhuis structure if and only if $N_R = 0$.

The first objective now must be to obtain a reasonably practical description of R, for example, by recognizing its action on complete and vertical lifts. In fact, we believe that it is better for general purposes, to make use of horizontal and vertical lifts, rather than complete and vertical lifts. For that, of course, one needs a connection, but there is one available, namely the nonlinear connection associated with the Euler–Lagrange equations of L (being second-order differential equations on TQ).

3. Making use of a connection

As is well known, every second-order equation field

$$\Gamma = u^i\frac{\partial}{\partial q^i} + f^i(q,u)\frac{\partial}{\partial u^i} \tag{10}$$

in particular the one coming from the regular Lagrangian L, defines a horizontal distribution, with connection coefficients

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial u^j}. \tag{11}$$

As a result, every vector field ξ on TQ has a unique decomposition of the form $\xi = X^H + Y^V$, where X, Y are vector fields along the tangent bundle projection $\tau : TQ \to Q$. In coordinates, if $X = X^i(q,u)\partial/\partial q^i$, then $X^V = X^i\partial/\partial u^i$ and $X^H = X^i H_i$, with $H_i = \partial/\partial q^i - \Gamma^j_i\partial/\partial u^j$. The $C^\infty(TQ)$ module of such fields is denoted by $\mathcal{X}(\tau)$. An extensive calculus along the projection τ was developed in [22, 23]. We recall here some basic features of this calculus, which will be needed in what follows.

Interesting derivations and tensorial objects along τ are discovered by looking at the decomposition of Lie brackets of vector fields on TQ. We have, for example, that

$$[X^V, Y^V] = ([X,Y]_V)^V \tag{12}$$

$$[X^H, Y^V] = \left(\mathrm{D}_X^H Y\right)^V - \left(\mathrm{D}_Y^V X\right)^H \tag{13}$$

$$[X^H, Y^H] = ([X, Y]_H)^H + \mathcal{R}(X, Y)^V. \tag{14}$$

Here, $\mathcal{R}$ is the curvature of the nonlinear connection, a vector-valued 2-form along τ, D_X^H and D_X^V are the horizontal and vertical covariant derivative operators, which act on functions $F \in C^\infty(TQ)$ as

$$\mathrm{D}_X^H F = X^H(F) \qquad \mathrm{D}_X^V F = X^V(F) \tag{15}$$

and are further fully determined by the following action on basic vector fields

$$\mathrm{D}_X^H \frac{\partial}{\partial q^i} = X^j \Gamma_{ji}^k \frac{\partial}{\partial q^k} \qquad \mathrm{D}_X^V \frac{\partial}{\partial q^i} = 0 \tag{16}$$

the corresponding action on 1-forms being defined by duality. The horizontal and vertical brackets of elements of $\mathcal{X}(\tau)$ (in the present situation of a connection which has no torsion) are given by

$$[X, Y]_V = \mathrm{D}_X^V Y - \mathrm{D}_Y^V X \qquad [X, Y]_H = \mathrm{D}_X^H Y - \mathrm{D}_Y^H X. \tag{17}$$

Other brackets of interest are

$$[\Gamma, X^V] = -X^H + (\nabla X)^V \qquad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V. \tag{18}$$

Here Φ, a type $(1, 1)$ tensor along τ, is called the *Jacobi endomorphism*, and ∇ is the *dynamical covariant derivative*, which on functions acts like Γ and further satisfies $\nabla(\partial/\partial q^i) = \Gamma_i^j \partial/\partial q^j$.

There exist lift operations on many other tensorial objects. We mention two more constructions of interest now. First, there are the horizontal and vertical lifts of a type $(1, 1)$ tensor field U along τ, determined by

$$U^H(X^V) = U(X)^V \qquad U^H(X^H) = U(X)^H \tag{19}$$

$$U^V(X^V) = 0 \qquad U^V(X^H) = U(X)^V. \tag{20}$$

Next, if g is a symmetric type $(0, 2)$ tensor field along τ, its Kähler lift is a 2-form on TQ determined by

$$g^K(X^H, Y^H) = g^K(X^V, Y^V) = 0 \tag{21}$$

$$g^K(X^V, Y^H) = -g^K(X^H, Y^V) = g(X, Y). \tag{22}$$

In fact, the Poincaré–Cartan form ω_L is precisely the Kähler lift of the Hessian of L, defined intrinsically by

$$g = \mathrm{D}^V \mathrm{D}^V L \tag{23}$$

where D^V is the vertical covariant differential defined (on any tensor T) by $\mathrm{D}^V T(X, \ldots) = \mathrm{D}_X^V T(\cdots)$. Observe that the complete lift J^c of a $(1, 1)$ tensor J on Q, can be written as

$$J^c = J^H + (\nabla J)^V. \tag{24}$$

Other more specific properties of interest in the calculus along τ will be recalled when appropriate, but we should at least refer here also to the existence of a canonical vector field along τ, the *total time derivative* operator $\mathbf{T} = u^i \partial/\partial q^i$, whose vertical lift is the Liouville vector field Δ, whereas its horizontal lift is a second-order equation field, which need not be the one we started from; the two coincide for a quadratic spray.

We can now start the computation of the structure of the tensor field R defined by (9). To begin with, observe that from (19), (20) and (24), it easily follows that

$$J^c(X^V) = (JX)^V \tag{25}$$

$$J^c(X^H) = (JX)^H + \nabla J(X)^V. \tag{26}$$

Using the standard notation θ_L for $\mathrm{d}_S L$, we have

$$\theta_L(X^V) = 0 \qquad \theta_L(X^H) = \mathrm{d}L(X^V) = X^V(L) = \mathrm{D}^V_X L = \mathrm{d}^V L(X). \tag{27}$$

Note that in the second of these relations, a computation on TQ in the end is replaced by one involving a vector field and 1-form along τ. The point is that θ_L, being semi-basic, can be regarded as a 1-form along τ as well; the defining relation then reads $\theta_L = \mathrm{d}^V L$, where d^V is the vertical exterior derivative. The latter is completely determined by the following action on functions F and 1-forms α along τ:

$$\mathrm{d}^V F(X) = \mathrm{D}^V_X F \qquad \mathrm{d}^V\alpha(X, Y) = \mathrm{D}^V_X \alpha(Y) - \mathrm{D}^V_Y \alpha(X). \tag{28}$$

Similar relations hold for the horizontal exterior derivative d^H. The action of these exterior derivatives extends to vector-valued forms as well; it suffices for our purposes to know that for d^V or d^H acting on a type $(1, 1)$ tensor field along τ, the defining relation is formally the same as in the second of equations (28).

It is easy to see that $\omega_1 = \mathrm{d}(J^c\theta_L)$ gives zero when evaluated on two vertical vector fields. Next we have, passing as before from θ_L, regarded as 1-form on TQ, to its interpretation as 1-form along τ,

$$\begin{aligned}\omega_1(X^V, Y^H) &= \mathcal{L}_{X^V}(\theta_L((JY)^H)) - \theta_L(J^c([X^V, Y^H]))\\ &= \mathrm{D}^V_X(\theta_L(JY)) - \theta_L\big(J\big(\mathrm{D}^V_X Y\big)\big) = \mathrm{D}^V_X\theta_L(JY)\end{aligned}$$

from which it follows in view of (23) that

$$\omega_1(X^V, Y^H) = g(X, JY). \tag{29}$$

Proceeding in the same way, we get

$$\begin{aligned}\omega_1(X^H, Y^H) &= \mathcal{L}_{X^H}(\theta_L((JY)^H)) - \mathcal{L}_{Y^H}(\theta_L((JX)^H)) - \theta_L(J^c([X^H, Y^H]))\\ &= \mathrm{D}^H_X(\theta_L(JY)) - \mathrm{D}^H_Y(\theta_L(JX)) - \theta_L(J([X, Y]_H))\\ &= \mathrm{D}^H_X(J\theta_L(Y)) - \mathrm{D}^H_Y(J\theta_L(X)) - \theta_L\big(J\big(\mathrm{D}^H_X Y - \mathrm{D}^H_Y X\big)\big)\\ &= \mathrm{D}^H_X(J\theta_L)(Y) - \mathrm{D}^H_Y(J\theta_L)(X).\end{aligned}$$

It follows that

$$\omega_1(X^H, Y^H) = \mathrm{d}^H(J\theta_L)(X, Y). \tag{30}$$

Proposition 1. *The type* $(1, 1)$ *tensor field* R*, defined by (9), has the following structure:*

$$R(X^V) = (\overline{J}X)^V \tag{31}$$

$$R(X^H) = (JX)^H + (UX)^V \tag{32}$$

where $\overline{J}$ *is the transpose of* J *with respect to* $g = \mathrm{D}^V\mathrm{D}^V L$*, i.e.* $g(JX, Y) = g(X, \overline{J}Y)$*, and* U *is the tensor field along* τ*, determined by*

$$g(UX, Y) = \mathrm{d}^H(J\theta_L)(X, Y). \tag{33}$$

Proof. It is sufficient to take horizontal and vertical lifts of basic vector fields for finding the tensorial structure of R. We have $\omega_L(R(X^V), Y^V) = 0$ and

$$\omega_L(R(X^V), Y^H) = \omega_1(X^V, Y^H) = g(JY, X) = g(Y, \overline{J}X) = g^K((\overline{J}X)^V, Y^H)$$

from which (31) follows. Likewise

$$\omega_L(R(X^H), Y^V) = -g(JX, Y) = g^K((JX)^H, Y^V)$$

from which it follows that $R(X^H) = (JX)^H + (UX)^V$, for some U. Subsequently, using (22) and (30),

$$\omega_L(R(X^H), Y^H) = \omega_L((UX)^V, Y^H) = g(UX, Y) = \mathrm{d}^H(J\theta_L)(X, Y)$$

which completes the proof. □

Note that it follows from the skew-symmetry of the right-hand side in (33) that $\overline{U} = -U$. In appendix B, we investigate the eigenspace structure of R and the explicit construction of so-called Darboux–Nijenhuis coordinates.

4. Further properties of the tensor field *R*

Proposition 2. *We have* $R = J^c \Longleftrightarrow J = \overline{J}$ *and* $U = \nabla J$.

Proof. The result follows immediately from comparison of (31), (32) with (25), (26). □

A natural question which arises is whether R, in general, could commute with S, just as J^c does, either in the algebraic sense or with respect to the Nijenhuis bracket.

Proposition 3. $RS = SR \Longleftrightarrow J = \overline{J}$.

Proof. Using $S(X^V) = 0$ and $S(X^H) = X^V$, the result follows immediately from the characterization of R in proposition 1. □

Recall that the Nijenhuis bracket is defined by

$$\begin{aligned}[R, S](\xi, \eta) &= [R\xi, S\eta] + [S\xi, R\eta] + (RS + SR)([\xi, \eta]) \\ &\quad - R([S\xi, \eta] + [\xi, S\eta]) - S([R\xi, \eta] + [\xi, R\eta]).\end{aligned}$$

Proposition 4. *Assuming* $J = \overline{J}$*, so that* $RS = SR$*, we have* $[R, S] = 0 \Longleftrightarrow \mathrm{d}^V U = \mathrm{d}^H J$.

Proof. That $[R, S]$ vanishes on two vertical lifts is trivial. Again, it suffices for such calculations to consider lifts of basic vector fields (vector fields on Q), rather than vector fields along τ. Since J is basic as well, one then easily verifies, making use of the bracket relations (12) and (13), that also $[R, S](X^V, Y^H) = 0$. Next, we have

$$\begin{aligned}[R, S](X^H, Y^H) &= [(JX)^H + (UX)^V, Y^V] + [X^V, (JY)^H + (UY)^V] \\ &\quad + 2RS\big([X, Y]_H^H\big) - R([X^V, Y^H] + [X^H, Y^V]) \\ &\quad - S([(JX)^H + (UX)^V, Y^H] + [X^H, (JY)^H + (UY)^V]).\end{aligned}$$

Using the bracket relations (12)–(14), plus the fact that J, and by assumption also X and Y, are basic, this readily reduces to

$$\begin{aligned}[R, S](X^H, Y^H) &= \big(\mathrm{D}^V_X U(Y) - \mathrm{D}^V_Y U(X)\big)^V \\ &\quad + \big(J\big(\mathrm{D}^H_X Y - \mathrm{D}^H_Y X\big)\big)^V + \big(\mathrm{D}^H_Y(JX) - \mathrm{D}^H_X(JY)\big)^V\end{aligned}$$

which in view of properties such as (28) for type (1, 1) tensors, can be written as

$$[R, S](X^H, Y^H) = (\mathrm{d}^V U(X, Y) - \mathrm{d}^H J(X, Y))^V.$$

The result now follows. □

Another obvious question is under which circumstances R is truly a recursion operator for symmetries of Γ. For that, we compute $\mathcal{L}_\Gamma R$. Taking the Lie derivative with respect to Γ of the defining relations (31), (32) and also making use of the properties (18), it is fairly straightforward to verify that

$$\mathcal{L}_\Gamma R(X^V) = (J - \overline{J})(X)^H + (U + \nabla\overline{J})(X)^V, \tag{34}$$

$$\mathcal{L}_\Gamma R(X^H) = (\nabla J - U)(X)^H + (\nabla U + \Phi J - \overline{J}\Phi)(X)^V. \tag{35}$$

The following, therefore, is an interesting immediate result.

Proposition 5. $\mathcal{L}_\Gamma R = 0 \Longleftrightarrow J = \overline{J}, U = \nabla J = 0, \Phi J = J\Phi$.

Next, we address the question of recognizing the conditions under which R has zero Nijenhuis torsion. A direct computation of N_R, through its action on horizontal and vertical lifts, is extremely tedious and therefore not worth the effort, since we can actually rely on existing results concerning the complete lift $\tilde{J}$ of J to the cotangent bundle T^*Q. For the reader's convenience, we recall that the coordinate expression of $\tilde{J}$ is given by

$$\tilde{J} = J^i_j \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^j + \frac{\partial}{\partial p_j} \otimes \mathrm{d}p_i \right) + p_k \left(\frac{\partial J^k_i}{\partial q^j} - \frac{\partial J^k_j}{\partial q^i} \right) \frac{\partial}{\partial p_i} \otimes \mathrm{d}q^j. \tag{36}$$

Let $\mathrm{Leg} : TQ \to T^*Q$ denote the Legendre transform coming from the regular Lagrangian L.

Proposition 6. *The tensor field R defined by (9) is directly related to the complete lift $\tilde{J}$ on T^*Q: as a matter of fact we have $R = \mathrm{Leg}^*\tilde{J}$.*

Proof. The defining relation of $\tilde{J}$, as described in [8], reads (where ξ this time denotes an arbitrary vector field on T^*Q)

$$i_{\tilde{J}\xi}\, \mathrm{d}\theta = i_\xi \mathcal{L}_{J^v}\, \mathrm{d}\theta \tag{37}$$

and clearly has the same structure as (9). Here, $J^v = J^i_j p_i \partial/\partial p_j$ is the vertical lift of J, which is a vector field on T^*Q (cf [33]) and θ of course is the canonical 1-form on T^*Q. It is easy to see from the coordinate expression that $i_{J^v}\, \mathrm{d}\theta = \tilde{J}\theta$, so that (37) implies $i_{\tilde{J}\xi}\, \mathrm{d}\theta = i_\xi\, \mathrm{d}\tilde{J}\theta$. But it is equally trivial to verify in coordinates that $\mathrm{Leg}^*\tilde{J}\theta = J^c\theta_L = \mathrm{d}_{J^c S}L$. The result then immediately follows from taking the pullback under Leg of the new representation of (37). □

As an immediate consequence of the fact that $N_{\tilde{J}} = 0 \Leftrightarrow N_J = 0$, which incidentally requires also a fairly tedious calculation (cf [8]), we now come to the following conclusion.

Proposition 7. $N_R = 0 \Longleftrightarrow N_J = 0$.

It is of some interest to recall here the following characterization of $N_J = 0$.

Lemma 2. $N_J = 0$ *if and only if for all basic vector fields X, Y, we have*

$$\mathrm{D}^H_{JX} J(Y) - J\big(\mathrm{D}^H_X J(Y)\big) = \mathrm{D}^H_{JY} J(X) - J\big(\mathrm{D}^H_Y J(X)\big). \tag{38}$$

Proof. Taking into account that $[X, Y] = [X, Y]_H$ for basic vector fields, we have

$$\begin{aligned}[] [JX, JY] + J^2[X, Y] - J([JX, Y] + [X, JY]) &= \mathrm{D}^H_{JX}(JY) - \mathrm{D}^H_{JY}(JX) \\ &\quad + J^2\big(\mathrm{D}^H_X Y - \mathrm{D}^H_Y X\big) - J\big(\mathrm{D}^H_{JX} Y - \mathrm{D}^H_Y (JX) + \mathrm{D}^H_X (JY) - \mathrm{D}^H_{JY}(X)\big) \\ &= \mathrm{D}^H_{JX} J(Y) - J\big(\mathrm{D}^H_X J(Y)\big) - \mathrm{D}^H_{JY} J(X) + J\big(\mathrm{D}^H_Y J(X)\big) \end{aligned}$$

which gives the desired result. □

To conclude this section, we look at an interesting case of non-vanishing $\mathcal{L}_\Gamma R$. Observe first that Γ, by construction, is the Hamiltonian vector field associated with the energy function $E_L = \Delta(L) - L$, with respect to the symplectic form $\omega_L = \mathrm{d}\theta_L$, i.e. we have

$$i_\Gamma \, \mathrm{d}\theta_L = -\mathrm{d}E_L. \tag{39}$$

A general theorem proved in [11] then implies that in the present context, the fact that the 2-form ω_1 (8) is closed, is equivalent to stating that

$$i_{\mathcal{L}_\Gamma R} \, \mathrm{d}\theta_L = -2 \, \mathrm{d} \, \mathrm{d}_R E_L. \tag{40}$$

Since R is symmetric with respect to ω_L, the same is true for $\mathcal{L}_\Gamma R$, so that invariance of R is equivalent to having $\mathrm{d}\,\mathrm{d}_R E_L = 0$. The latter was the starting point for an application of a bi-differential calculus in [11], to which we shall return in the next section. The more general situation of a gauged bi-differential calculus in [11] corresponds to the assumption that for some basic function f,

$$\mathrm{d}\,\mathrm{d}_R E_L = \mathrm{d}f \wedge \mathrm{d}E_L. \tag{41}$$

Via the equality (40), this assumption is equivalent to stating that (cf proposition 5.3 in [11])

$$\mathcal{L}_\Gamma R = \Gamma \otimes \mathrm{d}f - \xi_f \otimes \mathrm{d}E_L \tag{42}$$

where ξ_f is the Hamiltonian vector field associated with f. It easily follows from $i_{S\xi_f}\omega_L = -S\big(i_{\xi_f}\omega_L\big) = S(\mathrm{d}f) = 0$ that ξ_f is vertical, say $\xi_f = X_f^V$, for some X_f along τ. Then,

$$i_{\xi_f}\omega_L(Y^H) = g^K\big(X_f^V, Y^H\big) = g(X_f, Y) = -Y^H(f).$$

Hence, in terms of fields along the projection τ, X_f is defined by

$$g(X_f, Y) = -Y^H(f) = -\mathrm{d}^H f(Y) \qquad \text{or} \qquad X_f \rfloor g = -\mathrm{d}^H f. \tag{43}$$

In the next section, we specialize to the case that the Lagrangian comes from a (pseudo) Riemannian metric on Q, and will focus most of the attention on the characterization of so-called special conformal Killing tensors in their tangent bundle manifestation.

5. The Riemannian case

Let g be a symmetric, non-singular type $(0, 2)$ tensor field on Q and put $L = \frac{1}{2} g_{ij} u^i u^j$. The nonlinear connection defined by the Euler–Lagrange equations then is the (linear) Levi-Civita connection of g, i.e. the connection coefficients (11) are of the form

$$\Gamma^i_j = \Gamma^i_{jk} u^k \tag{44}$$

where Γ^i_{jk} are the classical Christoffel symbols.

It is important to understand first how the fundamental covariant derivative operators of the calculus along τ, as referred to in the previous section, relate to classical tensor calculus in this case. For example, if J is a type $(1, 1)$ tensor field on Q, then both the dynamical covariant derivative ∇J and horizontal covariant derivatives such as $\mathrm{D}^H_X J$ relate to the classical covariant derivative $J^i_{j|k}$ as follows:

$$(\nabla J)^i_j = J^i_{j|k} u^k \qquad \big(\mathrm{D}^H_X J\big)^i_j = J^i_{j|k} X^k. \tag{45}$$

In fact, in this situation we have $\nabla = \mathrm{D}^H_{\mathbf{T}}$. It follows that, in particular, $\nabla g = 0$ and $\mathrm{D}^H_X g = 0, \forall X$ (and of course also $\mathrm{D}^V_X g = 0$ because g is basic). Another specific property of the case of a quadratic spray is that the so-called *deviation*, which in the language of the calculus along τ is $\nabla\mathbf{T}$, is zero. It is actually of interest to list all covariant derivatives of $\mathbf{T}$ here:

$$\nabla\mathbf{T} = 0 \qquad \mathrm{D}^V_X\mathbf{T} = X \qquad \mathrm{D}^H_X\mathbf{T} = 0. \tag{46}$$

These properties are easy to verify in coordinates, but let us take the opportunity to mention also the general commutator identity (see [23])

$$\left[\nabla, \mathrm{D}^V_X\right] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X \tag{47}$$

which can be used to show in a coordinate free way that the first two relations (46) imply the third.

We now look at the characterization of R in this context, more particularly the specification of the tensor field U. Note first that everything should now be easily expressible in terms of the metric since we have, for example, that

$$\theta_L(X) = g(\mathbf{T}, X) \qquad \text{or} \qquad \theta_L = \mathbf{T}\rfloor g. \tag{48}$$

We should keep in mind also that $L = E_L$ is a first integral, so that $\nabla L = \Gamma(L) = 0$, and that it further follows from (48), using the commutator property $[\nabla, \mathrm{d}^V] = -\mathrm{d}^H$, that

$$0 = \nabla\theta_L = \nabla\,\mathrm{d}^V L = -\mathrm{d}^H L. \tag{49}$$

Now, concerning the determination of U, we have

$$\begin{aligned}\mathrm{d}^H(J\theta_L)(X, Y) &= \mathrm{D}^H_X(J\theta_L)(Y) - \mathrm{D}^H_Y(J\theta_L)(X)\\ &= \mathrm{D}^H_X(g(\mathbf{T}, JY)) - g\big(\mathbf{T}, J\big(\mathrm{D}^H_X Y\big)\big) - \mathrm{D}^H_Y(g(\mathbf{T}, JX)) + g\big(\mathbf{T}, J\big(\mathrm{D}^H_Y X\big)\big).\end{aligned}$$

Taking into account that $\mathrm{D}^H_X g = 0$ and $\mathrm{D}^H_X\mathbf{T} = 0$, we conclude that U is determined by

$$g(UX, Y) = g\big(\mathbf{T}, \mathrm{D}^H_X J(Y) - \mathrm{D}^H_Y J(X)\big) = g(\mathbf{T}, \mathrm{d}^H J(X, Y)). \tag{50}$$

In coordinates, we of course work with the adapted frame of horizontal and vertical vector fields on TQ and their dual 1-forms, which are

$$\left\{H_i = \frac{\partial}{\partial q^i} - \Gamma^k_i\frac{\partial}{\partial u^k}, V_i = \frac{\partial}{\partial u^i}\right\} \qquad \left\{\mathrm{d}q^i, \eta^j = \mathrm{d}u^j + \Gamma^j_k\,\mathrm{d}q^k\right\}. \tag{51}$$

The coordinate expression for R then becomes

$$R = J^i_j H_i \otimes \mathrm{d}q^j + \overline{J}^i_{\ j} V_i \otimes \eta^j + U^i_j V_i \otimes \mathrm{d}q^j \tag{52}$$

where

$$U^i_j = g^{ik}\big(J^m_{k|j} - J^m_{j|k}\big)g_{ml}u^l. \tag{53}$$

It is interesting to have intrinsic expressions also, which implicitly determine the vertical and horizontal covariant derivatives of U.

Proposition 8. *For $Z \in \mathcal{X}(\tau)$, $\mathrm{D}^V_Z U$ and $\mathrm{D}^H_Z U$ are determined by*

$$g\big(\mathrm{D}^V_Z U(X), Y\big) = g\big(Z, \mathrm{D}^H_X J(Y) - \mathrm{D}^H_Y J(X)\big) \tag{54}$$

$$g\big(\mathrm{D}^H_Z U(X), Y\big) = g(\mathbf{T}, \mathrm{D}^H\mathrm{D}^H J(Z, X, Y) - \mathrm{D}^H\mathrm{D}^H J(Z, Y, X)). \tag{55}$$

Proof. The proof is a straightforward computation, which starts from (50) and takes the properties (46) into account, remembering further that

$$\mathrm{D}^H\mathrm{D}^H J(Z, X, Y) = \mathrm{D}^H_Z\mathrm{D}^H_X J(Y) - \mathrm{D}^H_{\mathrm{D}^H_Z X} J(Y). \qquad \square$$

Let us come back in this case of particular interest to the invariance of R, or the more general assumption (42). If $h = \frac{1}{2}g^{ij}p_i p_j$ is the corresponding Hamiltonian, we know from the results of the preceding section that $\mathcal{L}_\Gamma R = 0$ is equivalent to $\mathrm{d}\,\mathrm{d}_{\bar{J}} h = 0$. It was shown by a coordinate calculation in [11] that the latter condition is equivalent to $J = \overline{J}$ and $\nabla J = 0$. This is somewhat surprising now, since proposition 5, in general, imposes more conditions for having $\mathcal{L}_\Gamma R = 0$. Of course, if $\nabla J = 0$, it follows from (45) that also $\mathrm{D}^H_X J = 0$ and thus from (50) that also $U = 0$. This implies in particular, from proposition 2, that when R is invariant, it must be equal to J^c. But the more interesting information which follows from comparison with proposition 5 is that apparently, when J is symmetric and parallel in the Riemannian case, it will automatically commute with the Jacobi endomorphism Φ. This is not a trivial property to recognize. We therefore propose to verify in appendix A by a direct calculation that it is indeed a correct statement. That direct proof is of interest in its own right, because it illustrates how one can proceed with an integrability analysis in this context.

Now assume that (42) holds for some $f \in C^\infty(Q)$ and where $\xi_f = X_f^V$, with X_f defined by (43). In the present situation, we further have $E_L = L$ and $\Gamma = \mathbf{T}^H$ (since $\nabla \mathbf{T} = 0$).

Theorem 1. *Under the present circumstances, the tensor field $\mathcal{L}_\Gamma R$ is of the form (42) if and only if $J = \overline{J}$ and further satisfies*

$$\nabla J = \tfrac{1}{2}(\mathbf{T} \otimes \mathrm{d}^H f - X_f \otimes \theta_L). \tag{56}$$

In addition, U then is of the form

$$U = -\tfrac{1}{2}(\mathbf{T} \otimes \mathrm{d}^H f + X_f \otimes \theta_L) \tag{57}$$

and J further has the property

$$\Phi J - J\Phi = \tfrac{1}{2}(\mathbf{T} \otimes \nabla \mathrm{d}^H f + \nabla X_f \otimes \theta_L). \tag{58}$$

Finally, the tensor field R itself then is given by

$$R = J^c - \Delta \otimes \mathrm{d}f. \tag{59}$$

Proof. The right-hand side of (42), when evaluated on some X^V, results in $-X^V(E_L)X_f^V = -\theta_L(X)X_f^V$. Comparison with (34) shows that this requires J to be symmetric, plus the condition that

$$U = -\nabla J - X_f \otimes \theta_L.$$

Proceeding in the same way for an arbitrary horizontal argument X^H, comparison with (35) reveals the requirements

$$U = \nabla J - \mathbf{T} \otimes \mathrm{d}^H f$$

and

$$\nabla U + \Phi J - J\Phi = -X_f \otimes \mathrm{d}^H L = 0.$$

Compatibility of the two expressions for U above immediately leads to the conclusions (56) and (57). The first of these puts a restriction on J, while the second in fact is then automatically satisfied. To see this, we compute $\mathrm{d}^V \nabla J$ from (56). Remember (see [22]) that for a vector-valued 1-form along τ, of the form $\alpha \otimes X$, an exterior derivative such as d^V is computed via the rule: $\mathrm{d}^V(\alpha \otimes X) = \mathrm{d}^V\alpha \otimes X - \alpha \wedge \mathrm{d}^V X$. Now $\mathrm{d}^V \mathrm{d}^H f = -\mathrm{d}^H \mathrm{d}^V f = 0$, $\mathrm{d}^V \mathbf{T} = I$ (the identity tensor), $\mathrm{d}^V \mathrm{d}^V L = 0$ and finally also $\mathrm{d}^V X_f = 0$ from (43). It follows that

$$\mathrm{d}^H J = \mathrm{d}^V \nabla J = -\tfrac{1}{2}\, \mathrm{d}^H f \wedge I. \tag{60}$$

The defining relation (50) for U then easily leads to (57) and can in fact also be rewritten as

$$g(UX, Y) = -\tfrac{1}{2}\, \mathrm{d}^H f \wedge \theta_L(X, Y). \tag{61}$$

The final requirement that ∇U should be equal to $J\Phi - \Phi J$ leads immediately, from (57), to (58), or can equivalently, from (61), be expressed as

$$(\Phi J - J\Phi)\rfloor\, g = \tfrac{1}{2}\nabla\, \mathrm{d}^H f \wedge \theta_L. \tag{62}$$

The point is, however, that this again is not an extra condition, but a consequence of the fundamental condition (56). To see this, recall that (42) is equivalent to (41), which in turn, when translated into the corresponding cotangent bundle property, reads $\mathrm{d}\,\mathrm{d}_{\bar{J}} h = \mathrm{d}f \wedge \mathrm{d}h$. It was shown by a direct coordinate calculation in [11] that this condition requires the symmetric J to be a so-called *special conformal Killing tensor*. We shall verify in coordinates below that this is exactly the condition (56). Hence, (58) must be a corollary and one could obtain it in a direct way by following the pattern of the integrability analysis in appendix A.

The final statement (59) about R follows directly from comparison between (25), (26) and (31), (32), knowing that $J = \overline{J}$ and using (56) and (57), with $\mathbf{T}^V = \Delta$. □

In coordinates, the condition (56) reads

$$J^i_{j|k} = \frac{1}{2}\left(\delta^i_k \frac{\partial f}{\partial q^j} + g^{il}\frac{\partial f}{\partial q^l} g_{jk}\right). \tag{63}$$

The more elegant coordinate expression is obtained by lowering an index and reads

$$J_{lj|k} = \frac{1}{2}\left(g_{lk}\frac{\partial f}{\partial q^j} + g_{jk}\frac{\partial f}{\partial q^l}\right) \tag{64}$$

which is indeed the defining relation for a special conformal Killing tensor as used in previous work (see, e.g., [11, 9]). An advantage of the present framework is that we do obtain an easy to handle and elegant, intrinsic expression also for the condition on the tensor J in its type (1, 1) appearance, which is after all the way in which J is originally conceived.

We finish this overview of the Riemannian case by briefly rederiving the most important properties of special conformal Killing tensors from (56). They were obtained by coordinate calculations in [11, 9].

Theorem 2. *If J is symmetric and satisfies (56) for some function $f \in C^\infty(Q)$, then $N_J = 0$ and $f = \mathrm{tr}J$; moreover, if J is non-singular, then its cofactor tensor A is a Killing tensor.*

Proof. Acting with D^V_X on (56), and knowing that $\mathrm{D}^V_X X_f = 0$, it follows from the commutator property (47) that

$$\mathrm{D}^H_X J = \tfrac{1}{2}\big(X \otimes \mathrm{d}^H f - X_f \otimes \mathrm{D}^V_X \theta_L\big). \tag{65}$$

We then get

$$\begin{aligned}\mathrm{D}^H_{JX} J(Y) &- J\big(\mathrm{D}^H_X J(Y)\big) - \mathrm{D}^H_{JY} J(X) + J\big(\mathrm{D}^H_Y J(X)\big)\\ &= \tfrac{1}{2}\big(\mathrm{D}^V_{JY}\theta_L(X) - \mathrm{D}^V_{JX}\theta_L(Y)\big)X_f + \tfrac{1}{2}\big(\mathrm{D}^V_X\theta_L(Y) - \mathrm{D}^V_Y\theta_L(X)\big)JX_f.\end{aligned}$$

The second term manifestly vanishes because $\mathrm{d}^V\theta_L = \mathrm{d}^V\mathrm{d}^V L = 0$. Taking X and Y to be basic for simplicity, the coefficient of X_f can be rewritten as

$$\mathrm{D}^V_{JY}(\theta_L(X)) - \mathrm{D}^V_{JX}(\theta_L(Y)) = \mathrm{D}^V_{JY}(g(\mathbf{T}, X)) - \mathrm{D}^V_{JX}(g(\mathbf{T}, Y)) = g(JY, X) - g(JX, Y)$$

which is zero in view of the symmetry of J. Lemma 2 now implies $N_J = 0$.

From (56), we get

$$\begin{aligned}\nabla(\mathrm{tr}J) &= \tfrac{1}{2}(\langle \mathbf{T}, \mathrm{d}^H f\rangle - \langle X_f, \theta_L\rangle)\\ &= \tfrac{1}{2}(\nabla f - g(\mathbf{T}, X_f)) = \nabla f.\end{aligned}$$

Hence, $f = \mathrm{tr}\, J$ (up to a constant, which is irrelevant).

Finally, if A is the cofactor of J, meaning that $JA = (\det J)I$, we have

$$\big(\mathrm{D}^H_X J\big)A = -J\big(\mathrm{D}^H_X A\big) + X^H(\det J)I. \tag{66}$$

Again, it suffices to let X, Y, Z in what follows be basic vector fields, so that, for example $\mathrm{D}^V_X\theta_L(Z) = \mathrm{D}^V_X(\theta_L(Z)) = g(X, Z)$. For the sake of uniformity, we keep using the operators of the calculus along τ, although everything here of course happens on the base space Q (and expressions such as $X^H(f)$ mean simply $X(f)$). From (65), it follows that

$$\mathrm{D}^H_X J(AY) = \tfrac{1}{2}((AY)^H(f)X - g(X, AY)X_f).$$

Using this to compute $g\big(\mathrm{D}^H_X J(AY), AZ\big)$ and taking a cyclic sum over X, Y, Z (indicated by an ordinary summation symbol), we readily obtain, knowing that also A is symmetric,

$$\sum g\big(\mathrm{D}^H_X J(AY), AZ\big) = \sum (AX)^H(f)g(Y, AZ).$$

Next, using this to compute $\sum g\big(J\big(\mathrm{D}^H_X A\big)Y, AZ\big) = (\det J)\sum g\big(\mathrm{D}^H_X A(Y), Z\big)$ via (66), we arrive at

$$(\det J)\sum g\big(\mathrm{D}^H_X A(Y), Z\big) = \sum(X^H(\det J) - (AX)^H(f))g(Y, AZ).$$

But we know that $N_J = 0$ implies that $\mathrm{d}_J(\det J) = (\det J)\,\mathrm{d}(\mathrm{tr}\, J)$ (see, e.g., [5, 10]), which can be written as $\mathrm{d}^H(\det J)(JX') = (\det J)\,\mathrm{d}^H f(X')$, for all X'. Taking $X' = AX$, it follows that

$$X^H(\det J) = (AX)^H(f) \tag{67}$$

which in turn leads to

$$\sum g\big(\mathrm{D}^H_X A(Y), Z\big) = 0. \tag{68}$$

This is the way to express that A, as type $(1, 1)$ tensor, is a Killing tensor. The more familiar way is to look at the type $(0, 2)$ tensor field $\tilde{A}$, obtained by lowering an index, so that $J \lrcorner \tilde{A} = (\det J)g$. The cyclic sum condition

$$\sum \mathrm{D}^H_X \tilde{A}(Y, Z) = 0 \tag{69}$$

then is equivalent to (68). □

6. An outlook for further study

As stated in the introduction, our goal is to develop generalizations of the classical cases of Hamilton–Jacobi separable systems, or completely integrable systems, of which we have not given any examples here, because such examples can abundantly be found in the cited literature. There are reasons to believe that a tangent bundle approach will then have advantages over a cotangent bundle framework. The present study is a preliminary investigation about understanding how things work on a tangent bundle. Even so, we have already obtained in sections 3 and 4 some general results relating to an arbitrary Lagrangian function (not necessarily one of 'mechanical type'). But for a full generalization, also more general type $(1, 1)$ tensors J should be allowed, with components depending on coordinates and velocities (or coordinates and momenta). For such a J, the notion of complete lift is lost, so how to proceed? The point now is that we indeed have an idea of how to proceed in the tangent bundle set-up. It suffices to look at the expression (24) for J^c and to observe that the right-hand side is perfectly defined also for a tensor field J along the projection τ. In fact, we are then talking about a more general lifting procedure, which has been fully developed already in [22, 23] and is sometimes referred to as the Γ-lift. For a given J along τ which is not basic, the formula

$$\mathcal{J}_\Gamma J = J^H + (\nabla J)^V \tag{70}$$

indeed defines a type (1, 1) tensor field on TQ, which depends on a given second-order equation field Γ. A number of the calculations which will be involved in such a generalization start off in exactly the same way as in the present paper, but of course without the simplifications coming from certain objects being basic.

A particular case of interest which could already significantly generalize the well-known Riemannian situation of the previous section is to let g be the metric along τ coming from a Lagrangian which is the square of a Finsler function F. It is then appropriate to start from a J along τ (with the zero section of TQ excluded) which is homogeneous of degree zero in the velocities. Thus, one can use the lifting procedure above, where Γ is the Euler–Lagrange field of F^2. Work along these lines is in progress.

Appendix A. Aspects of integrability analysis

Starting from an arbitrary J on Q and defining R on TQ by (9), the conditions for having $\mathcal{L}_\Gamma R = 0$ are, according to proposition 5, that J is symmetric and parallel, and further commutes with the Jacobi endomorphism Φ. But we have argued indirectly in section 5 that in the Riemannian case, the third condition must be an automatic consequence of the first two. An explicit verification of this fact can only come from an integrability analysis on the partial differential equations satisfied by J.

The assumption is that $\nabla J = 0$, and since J is basic, also $\mathrm{D}^V_X J = 0$ for all X, so that (47) implies that also $\mathrm{D}^H_X J$ will be zero for all X (hardly a surprise of course in view of (45). The next interesting commutator to look at here is $\left[\nabla, \mathrm{D}^H_X\right]$, or more generally $\left[\mathrm{D}^H_X, \mathrm{D}^H_Y\right]$. Its general expression reads (see [23])

$$\left[\mathrm{D}^H_X, \mathrm{D}^H_Y\right] = \mathrm{D}^H_{[X,Y]_H} + \mathrm{D}^V_{\mathcal{R}(X,Y)} + \mu_{\mathrm{Rie}(X,Y)}. \tag{A.1}$$

Clearly, only the last term matters here; it is a derivation which, when acting on a vector field Z along τ, is given by

$$\mu_{\mathrm{Rie}(X,Y)} Z = \mathrm{Rie}(X, Y)Z = -\mathrm{D}^V_Z \mathcal{R}(X, Y).$$

In fact, Rie is here simply the classical Riemann tensor. Recall also that we have the following relations linking Φ and $\mathcal{R}$:

$$\mathrm{d}^V\Phi = 3\mathcal{R} \qquad \Phi(X) = \mathcal{R}(\mathbf{T}, X) \tag{A.2}$$

which implies, for example, that $\Phi(\mathbf{T}) = 0$.

Now, to express that $\left[\nabla, \mathrm{D}^H_X\right]J$ must be zero, we compute

$$\left[\nabla, \mathrm{D}^H_X\right](JY) - J\left(\left[\nabla, \mathrm{D}^H_X\right]Y\right) = \mathrm{Rie}(\mathbf{T}, X)(JY) - J(\mathrm{Rie}(\mathbf{T}, X)Y).$$

Using the second of (A.2) and the property $\mathrm{D}^V_Y\mathbf{T} = Y$, we can write $\mathrm{D}^V_Y\mathcal{R}(\mathbf{T}, X) = \mathrm{D}^V_Y\Phi(X) + \mathcal{R}(X, Y)$, by which the above expression in the end reduces to

$$A(X, Y) := -\mathrm{D}^V_{JY}\Phi(X) - \mathcal{R}(X, JY) + J\left(\mathrm{D}^V_Y\Phi(X)\right) + J(\mathcal{R}(X, Y)) = 0. \tag{A.3}$$

In particular, knowing that Φ here is quadratic in the velocities so that $\mathrm{D}^V_{\mathbf{T}}\Phi = 2\Phi$, it follows that

$$A(X, \mathbf{T}) := -\mathrm{D}^V_{J\mathbf{T}}\Phi(X) - \mathcal{R}(X, J\mathbf{T}) + J(\Phi(X)) = 0 \tag{A.4}$$

from which we further obtain that

$$0 = g(A(X, \mathbf{T}), \mathbf{T}) = -g(\mathcal{R}(X, J\mathbf{T}), \mathbf{T}) - g\left(\mathrm{D}^V_{J\mathbf{T}}\Phi(X), \mathbf{T}\right) + g(J\Phi(X), \mathbf{T}). \tag{A.5}$$

Now we recall from the study of the inverse problem of the calculus of variations that also Φ is symmetric with respect to g (see, e.g., theorem 8.1 in [23]). It then follows from the Bianchi identity

$$\sum g(\mathcal{R}(X, Y), Z) = 0 \tag{A.6}$$

applied with arguments X, JY, $\mathbf{T}$, that

$$g(\mathcal{R}(X, JY), \mathbf{T}) = g(\Phi J(Y), X) - g(\Phi X, JY) = 0. \tag{A.7}$$

So the first term on the right-hand side of (A.5) vanishes. Moreover, also $\mathrm{D}^V_X\Phi$ is symmetric with respect to g for all X, which implies that the second term can be rewritten as $g\big(X, \mathrm{D}^V_{J\mathbf{T}}\Phi(\mathbf{T})\big) = g\big(X, \Phi\big(\mathrm{D}^V_{J\mathbf{T}}\mathbf{T}\big)\big) = g(X, \Phi J(\mathbf{T}))$. The conclusion from (A.5) therefore is that $g(J\Phi(X), \mathbf{T}) = 0$. Computing the D^V_Y derivative of this result and subtracting the same expression with X and Y interchanged, it follows by using the first property in (A.2) and the symmetry of J and Φ that

$$3g(\mathcal{R}(X, Y), J\mathbf{T}) = g((J\Phi - \Phi J)X, Y). \tag{A.8}$$

Next, we use the Bianchi identity again to write

$$g(\mathcal{R}(X, Y), J\mathbf{T}) = g(\mathcal{R}(X, J\mathbf{T}), Y) - g(\mathcal{R}(Y, J\mathbf{T}), X)$$

and make use of (A.4) to compute the right-hand side. Taking X and Y to be basic for simplicity, we can write $g\big(\mathrm{D}^V_{J\mathbf{T}}\Phi(X), Y\big) = \mathrm{D}^V_{J\mathbf{T}}(g(\Phi X, Y))$ and likewise for the term with X and Y interchanged. It then readily follows that $g(\mathcal{R}(X, Y), J\mathbf{T}) = g((J\Phi - \Phi J)X, Y)$. Comparison with (A.8) leads to the conclusion that both sides must be zero, for arbitrary X, Y. Hence, we have shown in a direct way that in the Riemannian case,

$$J = \overline{J} \qquad \text{and} \qquad \nabla J = 0 \quad \Rightarrow \quad \Phi J = J\Phi. \tag{A.9}$$

We had an indirect proof of this fact in section 5. It is of interest to illustrate that it is indeed a non-trivial property by looking at coordinate expressions. We have

$$\Phi^i_j = \mathcal{R}^i_{kj}u^k \qquad \mathcal{R}^i_{kj} = R^i_{ljk}u^l \qquad \text{and thus} \qquad \Phi^i_j = R^i_{ljk}u^k u^l \tag{A.10}$$

where R^i_{ljk} are the components of the Riemann tensor, and are skew-symmetric in the last two subscripts. Now, from $J^i_{j|k} = 0$, taking a further covariant derivative, swapping indices and using the Ricci identities, the property which immediately follows is $J^i_j R^j_{kml} = R^i_{jml} J^j_k$. But the commutation of J and Φ is a different property and says that $J^i_j\big(R^j_{kml} + R^j_{lmk}\big) = \big(R^i_{kjl} + R^i_{ljk}\big)J^j_m$.

Appendix B. Darboux–Nijenhuis coordinates

It is well known that on a general (regular) Poisson–Nijenhuis manifold of dimension $2n$, if the recursion operator R has n distinct eigenvalues, there exist so-called Darboux–Nijenhuis coordinates, which diagonalize R and are at the same time Darboux coordinates for the symplectic form (see, e.g., [30, 13]). This will apply in particular to the general situation on TQ, described in sections 3 and 4. We wish to investigate here in some detail what is the structure of the eigenspaces of our R and how the construction of Darboux–Nijenhuis coordinates works when the eigenvalues are maximally distinct.

We begin by establishing results which are valid without special assumptions on the type $(1, 1)$ tensor J on Q, except that we will only consider real eigenvalues.

Lemma 3. *If $\xi = X^H + Y^V$ is an eigenvector of R, corresponding to the eigenvalue λ, then*

$$JX = \lambda X \qquad \text{and} \qquad UX + \overline{J}Y = \lambda Y. \tag{B.1}$$

It follows in particular that X is an eigenvector of J.

Proof. Using the characterization of R as described by (31), (32), it is immediate to see that $R\xi = \lambda\xi$ is equivalent to the two relations (B.1). □

Lemma 4. *J and $\overline{J}$ have the same eigenvalues. In fact, if X is an eigenvector of J, then $X \rfloor g$ is an eigenform of $\overline{J}$ with the same eigenvalue.*

Proof. We have $\overline{J}{}^l_j = g^{lk} J^i_k g_{ij}$, and therefore

$$\overline{J}{}^l_j - \lambda\delta^l_j = g^{lk}\big(J^i_k - \lambda\delta^i_k\big)g_{ij}.$$

Both statements now easily follow. □

Lemma 5. *Suppose that J is non-degenerate and has n distinct eigenvalues (which then are nonzero). Then, if $JX = \lambda X$, there exists a vector field Y along τ, such that $\overline{J}Y = \lambda Y - UX$.*

Proof. From $g(JX, Y) = g(X, \overline{J}Y) = \lambda g(X, Y)$, it follows that $g(X, \overline{J}Y - \lambda Y) = 0, \forall Y$. Extending $X = X_1$ to an orthogonal frame $\{X_1, \ldots, X_n\} = \{X_1, X_\alpha\}$ for g, and putting $Y = a^i X_i$, it follows that $a^i(\overline{J}X_i - \lambda X_i) \in \mathrm{sp}\{X_\alpha\}, \forall a^i$, which implies that

$$\overline{J}X_i = \lambda X_i + b^\alpha_i X_\alpha \qquad i = 1, \ldots, n$$

for some functions b^α_i. We know that λ is an eigenvalue of $\overline{J}$ as well, and that its eigenvalues are distinct. Hence, there exists a unique vector field of the form $X_1 + c^\alpha X_\alpha$ which spans the kernel of $\overline{J} - \lambda I$. But $(\overline{J} - \lambda I)(X_1 + c^\alpha X_\alpha) = \big(b^\alpha_1 + c^\beta b^\alpha_\beta\big)X_\alpha$, so the fact that unique functions c^β exist which make this zero implies that $\det\big(b^\alpha_\beta\big) \neq 0$. Now consider the equation $\overline{J}Y = \lambda Y - UX$ for the unknown $Y = a^i X_i \in \mathcal{X}(\tau)$. Since $g(UX, Y)$ is skew-symmetric in X, Y, we know that $g(UX_1, X_1) = 0$ and thus $UX_1 = d^\alpha X_\alpha$ for some functions d^α. The equation for Y can now be written in the form $a^\beta b^\alpha_\beta = -d^\alpha - a^1 b^\alpha_1$ and clearly has a unique solution for the a^β for each arbitrary choice of a^1. □

Proposition 9. *Let J be diagonalizable with distinct nonzero eigenvalues. Then a complete set of eigenvectors of R can be constructed as follows: (i) let X_i denote the eigenvector of J with eigenvalue λ_i and Z_i the eigenvector of $\overline{J}$ with the same eigenvalue; (ii) for each X_i, construct a vector Y_i such that $\overline{J}Y_i = \lambda_i Y_i - UX_i$. Then ${Z_i}^V$ and ${X_i}^H + {Y_i}^V$ are eigenvectors of R, corresponding to the eigenvalue λ_i.*

Proof. We have

$$R\big({Z_i}^V\big) = (\overline{J}Z_i)^V = \lambda_i {Z_i}^V$$
$$R\big(X_i^H + Y_i^V\big) = (JX_i)^H + (UX_i)^V + (\overline{J}Y_i)^V = \lambda_i\big(X_i^H + Y_i^V\big)$$

from which the result follows. □

That is about the purely algebraic aspects. Now let us further assume that $N_J = 0$. Darboux–Nijenhuis coordinates in fact should do three things at the same time: not only diagonalize R in coordinates, but also separate it, and bring the symplectic form into canonical form. It was proved in the fundamental paper of Frölicher and Nijenhuis [14] that if J is (algebraically) diagonalizable and the eigenvalues have constant multiplicity, then the *necessary and sufficient condition for diagonalizability in coordinates* is that $\mathcal{H}_J = 0$, where the *Haantjes tensor* $\mathcal{H}_J$ can be defined by

$$\mathcal{H}_J(X, Y) = J^2 N_J(X, Y) + N_J(JX, JY) - JN_J(JX, Y) - JN_J(X, JY). \tag{B.2}$$

Obviously, $N_J = 0$ implies $\mathcal{H}_J = 0$, but evaluating N_J on eigenvectors X and Y belonging to different eigenvalues, λ, μ say, further gives

$$0 = N_J(X, Y) = (\lambda - \mu)(X(\mu)Y + Y(\lambda)X)$$

so that $X(\mu) = Y(\lambda) = 0$. Hence, in coordinates which diagonalize J, the eigenvalues will only depend on the coordinates of the corresponding eigendistribution, which is the meaning of saying that J is separable in coordinates. Conversely, if J is separable, one can verify in such coordinates that $N_J = 0$. In other words, $N_J = 0$ (for a J which has the algebraic properties stated above) is the *necessary and sufficient condition for separability in coordinates*. Note in passing that the tools for studying such issues when J would more generally be a tensor field along τ have been developed in [24].

To understand what happens with R on TQ now, we need to look at the expression of R in a coordinate basis, rather than in the adapted frame as in (52); it reads (still for the general situation described by proposition 1)

$$R = J^i_j \frac{\partial}{\partial q^i} \otimes \mathrm{d}q^j + \overline{J}^i{}_j \frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j + \left(U^i_j + \overline{J}^i{}_k \Gamma^k_j - J^k_j \Gamma^i_k\right) \frac{\partial}{\partial u^i} \otimes \mathrm{d}q^j. \tag{B.3}$$

The following procedure now will lead to Darboux–Nijenhuis coordinates. First, perform the Legendre transform $(q, u) \rightarrow (q, p = \partial L/\partial u)$. Even though this is to be regarded here as a change of coordinates on TQ, the result will be that R acquires the form of the complete lift $\tilde{J}$ on T^*Q as given by (36). The 2-form ω_L meanwhile will already take its canonical form in the (x, p) coordinates. Now, assuming that J has distinct eigenvalues and zero Nijenhuis torsion, we know that there exists a coordinate change on Q which will diagonalize the expression $J^i_j(\partial/\partial q^i \otimes \mathrm{d}q^j)$ in such a way that the eigenvalues depend on at most one new coordinate q'. The resulting point transformation $(q, u) \rightarrow (q', u')$ on TQ, when expressed in the non-tangent bundle variables (q, p), formally is a 'canonical transformation' $(q, p) \rightarrow (q', p')$, i.e. it defines another Darboux chart for the symplectic form ω_L and it will have the additional effect of diagonalizing (and separating) R.

From a tangent bundle point of view, the first step in this procedure is rather unnatural, because it is not a tangent bundle change of coordinates. At first sight, it may look like one should nevertheless not change the order of the operations, because even though J and $\overline{J}$ have the same eigenvalues, a coordinate transformation which diagonalizes J will generally not at the same time diagonalize $\overline{J}$. However, the two coordinate changes under consideration here are of course of a quite special type: a Legendre transformation which does not change q but changes the fibre coordinate, and a point transformation. It is clear that such coordinate changes commute, so one can just as well diagonalize J first and then the subsequent Legendre transform will not destroy the diagonal form of J, will bring ω_L in canonical form, and at the same time will take care of the diagonalization of $\overline{J}$.

That the reversed procedure is somewhat more natural for the tangent bundle set-up may become clear in the special case when J is symmetric. It then follows from $g_{ij} J^i_k = g_{ik} J^i_j$ that in coordinates which diagonalize J, we will have $g_{kj}(\lambda^{(k)} - \lambda^{(j)}) = 0$ and thus $g_{kj} = 0$ for $j \neq k$. This gives useful information also when there is no urge to pass to Darboux–Nijenhuis coordinates: it means that in coordinates which diagonalize J, the given Lagrangian will separate with respect to the velocity variables, i.e. it will become of the form $L = \sum_i L^i(q, u^i)$, where L^i depends on u^i only.

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