First steps towards Little's Law with imprecise probabilities

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Abstract

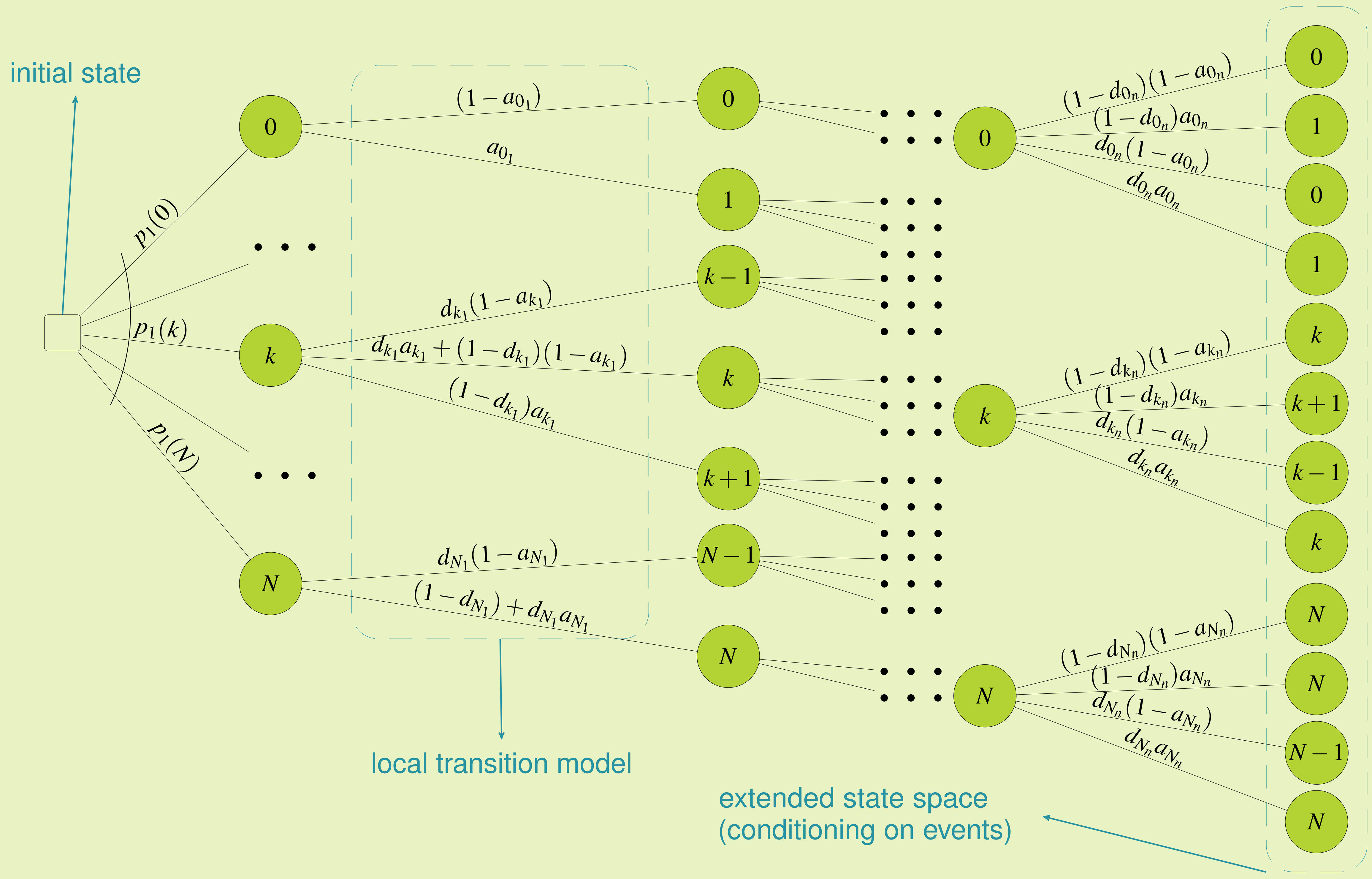
Little's Law Little's Law is described by the equation ***L*=*λW***, where ***L*** is the average number of items in a queuing system, ***λ*** is the average number of arriving items per unit time and ***W*** is the average waiting time of an item.

Our queuing setting We examine the law for a discrete-time, single-server queue where the *arrivals* and the servicing (*departures*) happen according to imprecise Bernoulli processes: *forward irrelevant* arrivals occur at each discrete time point with probability interval $[\underline{a},\overline{a}]$ and, similarly, *forward irrelevant* departures occur at each discrete time point with probability interval $[\underline{d},\overline{d}]$. Arrivals and departures are assumed to be strongly independent. We make two additional assumptions regarding the properties of the queue. The first one is that upon arriving, an item needs to remain in the queue till served. Secondly, departure is characterised by the FIFO (first in first out) principle.

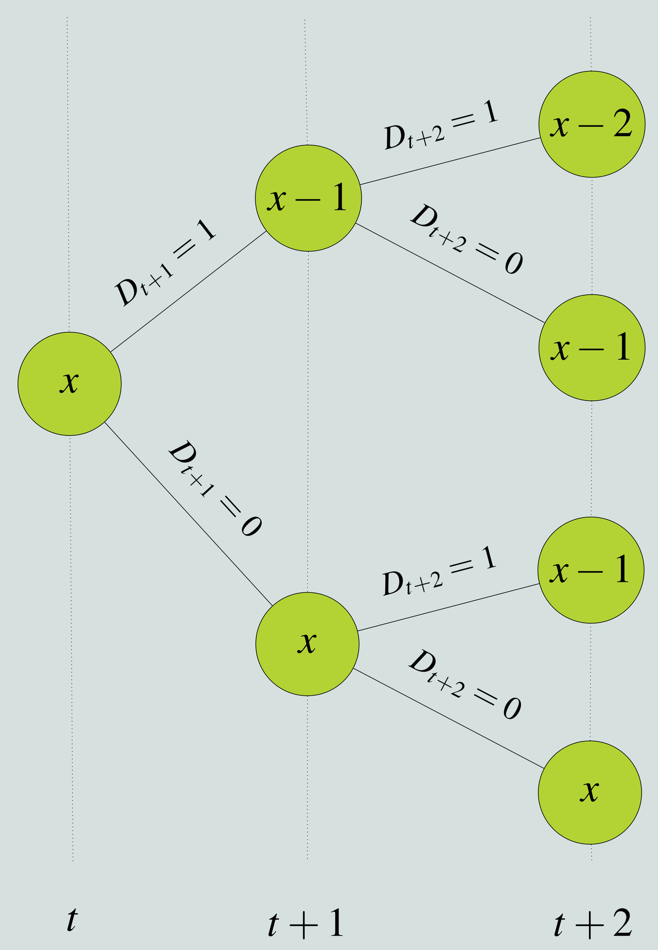
Our results Using the framework of *coherent lower (and upper) previsions*, our main result is a relation between the lower (and upper) prevision of the remaining waiting time of the last item in the queue (W_t) and the lower (and upper) prevision of the number of items in the queue (X_t) at any given time point t. More specifically, we prove that $\underline{P}(X_t)=\overline{d}\underline{P}(W_t)$ and $\overline{P}(X_t)=\underline{d}\overline{P}(W_t)$.

The queue we consider

We consider a queue with maximum capacity N, where, at each discrete time point, departure is assumed to occur **prior** to arrival. We can treat this queuing system as an Imprecise Markov Chain. Due to forward irrelevance, its credal set is very intuitive. It consists of all the precise models that can be constructed as depicted below. Arrivals occur with probability $a_{k_t}\in[\underline{a},\overline{a}]$, which can differ between states and time points. The same holds for departures ($d_{k_t}\in[\underline{d},\overline{d}]$). In these expressions, k_t corresponds to the number of items at time point t. Finally, p_1 is the initial probability distribution, taking values in some credal set $\mathscr{M}_1$.



Heuristic proof of the main result



Due to the FIFO principle and the assumption that an item stays in the queue till served, the waiting time (W_t) of the last item of the queue, at any time point t, will depend on the number of items that exist in the queue at that time (X_t) and on the sequence of departures from $t+1\to\infty$.

Due to forward irrelevance, the lower prevision of W_t can be obtained by means of the following expression, which closely resembles the law of iterated expectation, as used in precise-probabilistic approaches.

$$\underline{P}(W_t)=\underline{P}\left(\sum_{x\in\mathscr{X}}\mathbb{I}_{\{x\}}(X_t)\underline{P}(W_t|X_t=x)\right). \qquad (1)$$

In this expression, $\mathbb{I}_x$ is the so-called indicator of x.

The next step consists in realising that $\underline{P}(W_t|X_t=x)$ depends on the departure process only. It is the time until x departures have occurred. We will use $\underline{P}(W(x))$ as an alternative, shorthand notation for $\underline{P}(W_t|X_t=x)$. Then, $\underline{P}(W(x))$, similarly to (1), can be written as follows

$$\underline{P}(\mathbb{I}_{d^c}(D_{t+1})\underline{P}(W(x)|D_{t+1}=0)+\mathbb{I}_d(D_{t+1})\underline{P}(W(x)|D_{t+1}=1)). \qquad (2)$$

We also know that the remaining waiting time $W(x)$ of the item under study, at time t, is actually the remaining waiting time at time $t+1$ increased by one. This means that the two components of (2) become

$$\underline{P}(W(x)|D_{t+1}=0)=\underline{P}(W_t|X_t=x,D_{t+1}=0)=\underline{P}(W_{t+1}|X_{t+1}=x)+1 \qquad (3a)$$

$$\underline{P}(W(x)|D_{t+1}=1)=\underline{P}(W_t|X_t=x,D_{t+1}=1)=\underline{P}(W_{t+1}|X_{t+1}=x-1)+1 \qquad (3b)$$

Taking into account that the probability tree is infinite with respect to its length ($t\to\infty$), the item under study at $t+1$ waits until x departures occur if $D_{t+1}=0$, otherwise ($D_{t+1}=1$) it waits until $x-1$ departures occur. Hence (3a) and (3b) become

$$\underline{P}(W(x)|D_{t+1}=0)=\underline{P}(W(x))+1\ ,\ \underline{P}(W(x)|D_{t+1}=1)=\underline{P}(W(x-1))+1 \qquad (4)$$

where $\underline{P}(W(0))=0$.

We understand also that the following inequality holds

$$\underline{P}(W(x))\geq\underline{P}(W(x-1)). \qquad (5)$$

Now (2) can be reformulated as follows

$$\begin{aligned}\underline{P}(W(x))&=\underline{P}(\mathbb{I}_d(D_{t+1})\underline{P}(W(x)|D_{t+1}=1)+\mathbb{I}_{d^c}(D_t)\underline{P}(W(x)|D_{t+1}=0))=\\&=\underline{P}(\mathbb{I}_d(D_{t+1})\underline{P}(W(x-1))+\mathbb{I}_{d^c}(D_{t+1})\underline{P}(W(x)))+1=\\&=\overline{d}\underline{P}(W(x-1))+(1-\overline{d})\underline{P}(W(x))+1\end{aligned} \qquad (6)$$

where the second equality is derived from (4) and the third from (5).

Therefore, (6) results in

$$\underline{P}(W(x))=\overline{d}\underline{P}(W(x-1))+(1-\overline{d})\underline{P}(W(x))+1\Rightarrow\underline{P}(W(x))=\frac{x}{\overline{d}}. \qquad (7)$$

By plugging (7) into (1) we have that

$$\begin{aligned}\underline{P}(W_t)&=\underline{P}\left(\sum_{x\in\mathscr{X}}\mathbb{I}_{\{x\}}(X_t)\underline{P}(W_t|X_t=x)\right)=\underline{P}\left(\sum_{x\in\mathscr{X}}\mathbb{I}_{\{x\}}(X_t)\frac{x}{\overline{d}}\right)\\&=\underline{P}\left(\sum_{x\in\mathscr{X}}\mathbb{I}_{\{x\}}(X_t)x\right)\frac{1}{\overline{d}}=\frac{\underline{P}(X_t)}{\overline{d}}\Rightarrow\underline{P}(X_t)=\overline{d}\underline{P}(W_t).\end{aligned}$$

Similarly we can prove that $\overline{P}(X_t)=\underline{d}\overline{P}(W_t)$.

Additional results

Different types of independence More stringent independence assumptions can be imposed on the arrival and departure processes, such as epistemic independence or strong independence. With sensitivity analysis, we still get the same result as for forward irrelevance. $\frac{P(X_t)}{d}$ is minimized for $\underline{a}$ and $\overline{d}$, meaning that the expected remaining waiting time is minimized when we take the lowest arrival probability and the highest respective one for departure. This leads to the same bounds under either epistemic independence or strong independence. Also, strong independence between arrivals and departures can be weakened to epistemic independence.

Experimental work We also provide some experimental results regarding the lower and upper prevision of the number of items in the queue ($\overline{P}(X_t)$), the number of items in the queue conditional on an entrance ($\overline{P}(X_t|\text{entr})$) and the remaining waiting time of the last item conditional on an entrance ($\overline{P}(W_t|\text{entr})$), as $t\to\infty$. The $\overline{P}(W_t|\text{entr})$ is our approach to studying the *imprecise average waiting time*. We used a queue of maximum capacity 5, departures occur prior to arrivals, where the probability interval of arrival is $[0.6,0.8]$ and the respective one of departure is $[0.5,0.7]$. Additionally, we chose the *vacuous model* to be our initial model.