DEVELOPMENT OF A CONTINUUM PLASTICITY MODEL FOR THE COMMERCIAL FINITE ELEMENT CODE ABAQUS

Mohsen Safaei, Wim De Waele

Ghent University, Laboratory Soete, Belgium

Abstract The present work relates to the development of computational material models for sheet metal forming simulations. In this specific study, an implicit scheme with consistent Jacobian is used for integration of large deformation formulation and plane stress elements. As a privilege to the explicit scheme, the implicit integration scheme is unconditionally stable. The backward Euler method is used to update trial stress values lying outside the yield surface by correcting them back to the yield surface at every time increment. In this study, the implicit integration of isotropic hardening with the von Mises yield criterion is discussed in detail. In future work it will be implemented into the commercial finite element code ABAQUS by means of a user material subroutine.

Keywords UMAT, Implicit, Backward Euler, Mises, Isotropic

1 INTRODUCTION

Advanced high strength steels (AHSS) are increasingly found in structural applications because of their high specific strength combined with reasonable to good formability. Numerical modeling of AHSS forming processes is confronted with several challenges, primarily because of their specific strain hardening behaviour and the anisotropic nature of its strength properties. In recent years several comprehensive constitutive models have been developed [1-3] and implemented in dedicated software. In the commercial finite element code ABAQUS [4] the user is limited to the Hill 48 model for anisotropic yielding. More advanced models have to be implemented by means of the user material (UMAT) subroutine. Unfortunately, the codes of such subroutines for advanced constitutive models are typically not available as open source.

Implementing material constitutive equations into finite element code can simply be explained as defining stress update algorithms in which an imposed strain increment gives rise to a corresponding stress increment. Describing this incremental relation is not always straightforward. This paper discusses on the fundamentals needed for the implementation in ABAQUS, by means of a user material subroutine UMAT, of von Mises plasticity with isotropic hardening using the implicit backward Euler method. As opposed to explicit integration, implicit integration used with consistent Jacobian overcomes the convergence problems encountered at the transition from elastic to plastic behaviour. As a result, overestimation of stress will not appear. Moreover, contrary to explicit integration which uses small time steps, implicit integration enables the use of larger time steps so CPU time can be highly reduced.

Note that a finite element method is referred to as an implicit finite element method when implicit schemes for the integration of the momentum balance, or equilibrium equations are employed. Therefore an implicit finite element method can use implicit or explicit integration for constitutive equations such as stress update algorithms [5].

2 FUNDAMENTAL CONCEPTS OF PLASTICITY

Throughout the following discussion, the strain tensor is adopted as the primary variable. This is in accordance with the approach used in [6], which considers the elastic-plastic behaviour as a strain-driven problem. In the following subsections, we summarize the governing equations of classical rate-independent plasticity within the context of the three-dimensional infinitesimal theory.

2.1 Strain tensor decomposition and stress-strain relation

When a material is deformed, it generally strain hardens. Contrary to perfect plasticity, the stress increases with increasing deformation. The strain tensor can be decomposed into its plastic and elastic components. Classical additive decomposition of the strain tensor can be written as:

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{1}$$

In which ε^e and ε^p denote elastic and plastic strains. Therefore Hooke's law for elastic stress can be written as:

$$\boldsymbol{\sigma} = \boldsymbol{C}:\boldsymbol{\varepsilon}^e = \boldsymbol{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{2}$$

With $\boldsymbol{C}$ the tensor of elastic moduli which is assumed constant.

The multi-axial stress tensor σ can be presented as [7]

$$\sigma = 2G\varepsilon^e + \lambda tr(\varepsilon^e)I$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{13}\\\sigma_{23}\end{bmatrix} = \begin{bmatrix}\lambda+2G & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\2\varepsilon_{12}\\2\varepsilon_{13}\\2\varepsilon_{23}\end{bmatrix} \tag{3}$$

In which λ and G are shear modulus and Lame constant, respectively.

$$\lambda = \frac{Ev}{(1-2v)(1+v)} \tag{4}$$

$$G = \frac{E}{2(1+v)} \tag{5}$$

For plane stress, in which $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$, the stress and strain tensors are written as

$$\sigma = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^T$$

$$\varepsilon = [\varepsilon_{11} \quad \varepsilon_{22} \quad 2\varepsilon_{12}]^T \tag{6}$$

Note that $\varepsilon_{33} \neq 0$, although it does not appear explicitly. The above equations take the following form:

$$\sigma = \frac{E(1-v)}{1-v^2}\varepsilon^e + \frac{Ev}{1-v^2}tr(\varepsilon^e)I$$

$$C = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix} \tag{7}$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\2\varepsilon_{12}\end{bmatrix}$$

2.2 Incompressibility

In metal plasticity, it is generally assumed that deformation occurs without volume change. This assumption can be written as:

$$\dot{\varepsilon}^p{}_{11} + \dot{\varepsilon}^p{}_{22} + \dot{\varepsilon}^p{}_{33} = 0 \tag{8}$$

or

$$tr(d\varepsilon^p) = 0$$

In which tr is the trace function.

2.3 Effective stress and effective plastic strain rate

The von Mises effective stress is defined as:

$$\sigma_{eff} = \left[\frac{3}{2}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2)\right]^{1/2} \tag{9}$$

where σ_{ij} denotes a component of the Cauchy stress tensor. This can be written more succinctly using the double dot product:

$$\sigma_{eff} = \left(\frac{3}{2}\boldsymbol{\sigma}':\boldsymbol{\sigma}'\right)^{1/2} = \sqrt{\frac{3}{2}}||\boldsymbol{\sigma}'|| \tag{10}$$

Where ||(.)|| is the norm of tensor (.) and σ' is the deviator of the tensorial stress defined by:

$$\sigma' \equiv s = \sigma - \frac{1}{3}tr(\sigma)I \tag{11}$$

with I a third order identity matrix. The hydrostatic stress does not contribute to deformation.

This procedure does not conform for plane stress conditions. For these conditions a mapping matrix $\bar{P}$ is used to relate the stress and its deviator

$s = \bar{P}\sigma$

In vector notation, the deviator is given by

$s = [s_{11} \quad s_{22} \quad s_{12}]^T$

and the mapping matrix is written as (12)

$$\bar{P} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The effective plastic strain rate $\dot{p}$ is defined as:

$$\dot{p} = (\frac{2}{3}\dot{\varepsilon}^p:\dot{\varepsilon}^p)^{1/2} = \sqrt{\frac{2}{3}}||\dot{\varepsilon}^p|| \tag{13}$$

Considering the plane stress case, the effective plastic strain rate can be written as:

$$\dot{p} = \gamma\sqrt{\frac{2}{3}\eta^T P \eta} \tag{14}$$

Where γ is called the plastic multiplier (see also section 2.6).

To address the factor 2 of the shear strain component (see Eqn(6)$_2$) when converted to vector notation rather than matrix notation, the mapping matrix $\bar{P}$ is changed to P and called the projection matrix.

$$P = \frac{1}{3}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{15}$$

Further, η is equal to (16)

$\eta = \sigma - \tilde{\beta}$

in which $\tilde{\beta}$ is the back stress tensor that defines the location of the centre of the yield surface. In many metals subjected to cyclic loading, it has been experimentally observed that the yield surface undergoes a global translation in the direction of plastic flow. This phenomenon is called kinematic hardening. The back stress is omitted in this study, since only isotropic hardening will be considered.

2.4 Yield criterion

The yield function f defines a locus for which $f < 0$ and $f = 0$ means elastic and plastic deformation, respectively. The von Mises yield criterion is given by:

$$f(\sigma,p) = \sigma_{eff} - k(p) = \left(\frac{3}{2}\sigma':\sigma'\right)^{1/2} - k(p) \le 0 \tag{17}$$

This can be transformed to

$$f(\sigma,p) = ||\sigma'|| - \sqrt{\frac{2}{3}}k(p) \leq 0 \tag{18}$$

In these equations $k(p)$ is the flow stress corresponding to the equivalent plastic strain p.

For the case of plane stress, the yield criterion is written as

$$f(\sigma,p) = \sqrt{\eta^T P \eta} - \sqrt{\frac{2}{3}}k(p) \leq 0 \tag{19}$$

2.5 Loading/un-loading and consistency conditions

The Kuhn-Tucker unilateral constraints (also called complementarity conditions) provide the most convenient formulation of the loading/unloading conditions for classical plasticity. Stresses must be admissible and plastic strain can only take place on the yield surface, therefore γ and σ sigma are restricted by the following unilateral constraints [8]:

$$\gamma \geq 0, \quad f(\sigma,p) \leq 0$$
$$\gamma f(\sigma,p) = 0 \tag{20}$$

In addition $\gamma \geq 0$ must satisfy the consistency requirement. This hypothesis describes that when the material hardens, the load point remains on the yield surface, and can be mathematically expressed as:

$$\gamma \dot{f}(\sigma,p) = 0 \tag{21}$$

These conditions imply that in case of elastic deformation or $f(\sigma,p) < 0$ it follows that $\gamma = 0$. The plastic strain rate is thus equal to zero and the response is instantaneously elastic:

$$\dot{\sigma} = C:\varepsilon = C:(\dot{\varepsilon} - \dot{\varepsilon}^p) \tag{22}$$

In the case that $f(\sigma,p) = 0$ the Kuhn-Tucker constraints are automatically satisfied and γ can be positive or zero. If $\dot{f}(\sigma,p) < 0$ than it can be concluded that $\gamma = 0$, which corresponds to (elastic) unloading from a plastic state. If $\dot{f}(\sigma,p) = 0$ than $\gamma = 0$ is termed neutral loading and $\gamma > 0$ is a state of plastic loading.

In case $\gamma > 0$, the consistency condition Eqn (21) can be written as:

$$\dot{f}(\sigma,p) = \frac{\partial f}{\partial \sigma}:\dot{\sigma} + \frac{\partial f}{\partial p}\dot{p} = 0 \tag{23}$$

Note that when working in principal space, the double contracted product is substituted with dot product. Therefore Eqn **Error! Reference source not found.** results in

$$(\partial_\sigma f).\dot{\sigma} + (\partial_p f)\dot{p} = 0 \tag{24}$$

In which $\partial_\sigma f$ is the partial derivative of f to σ.

2.6 Normality hypothesis

The normality hypothesis implies that the increment of plastic strain is normal to the yield surface at the load point. This can be written as:

$$\dot{\varepsilon}^p = \gamma \frac{\partial f}{\partial \sigma} \tag{25}$$

In which $\frac{\partial f}{\partial \sigma}$ and γ express the direction of the incremental plastic strain rate and its magnitude, respectively.

Considering the tri-axial von Mises yield function:

$$\frac{\partial f}{\partial \sigma} = \frac{\boldsymbol{\sigma}'}{||\boldsymbol{\sigma}'||} \tag{26}$$

Therefore Eqn (25) can be written as:

$$\dot{\varepsilon}^p = \gamma \frac{\boldsymbol{\sigma}'}{||\boldsymbol{\sigma}'||} \tag{27}$$

This can be rewritten for the case of plane stress as

$$\dot{\varepsilon}^p = \gamma P \eta \tag{28}$$

3 IMPLICIT INTEGRATION

One of the advantages of implicit over explicit integration is its unconditional stability in the sense that stress never drifts away from the yield surface, using an elastic (trial) stress and a plastic corrector. For an imposed strain increment, an elastic trial stress (also called predicator stress), which falls outside the yield surface, is calculated. Using a plastic corrector, the stress is corrected to be back on the yield surface. The stress is updated by means of the mentioned plastic corrector in such a way that consistency is satisfied. This concept is schematized in Figure 1. Figure 2 illustrates the corresponding algorithm of the user material subroutine for implementation of isotropic hardening and von Mises plasticity in ABAQUS.

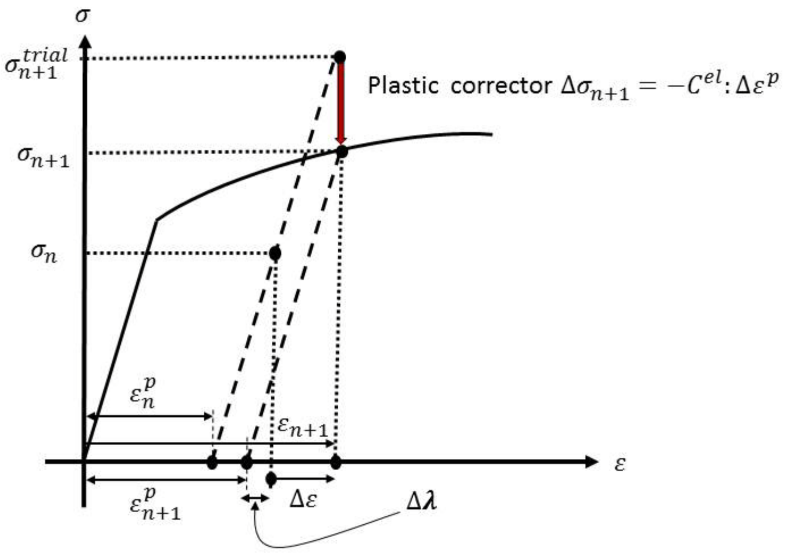

Figure 1 Backward Euler stress update scheme using trial stress and plastic corrector

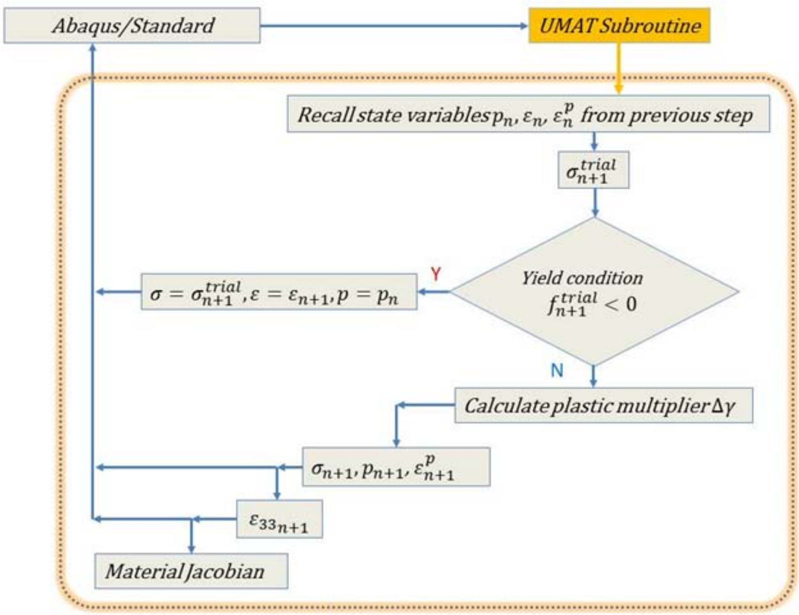

Figure 2: Schematic illustration of the return mapping algorithm for the ABAQUS UMAT.

Assume that the total and plastic strain fields and the equivalent plastic strain (includes the hardening parameters) at a time t_n are known $\{\varepsilon_n, \varepsilon_n^p, p_n\}$. The elastic strain and stress tensors are dependent variables and can be calculated based on these 'driving' variables (see section 2.1).

At the start of the algorithm, an increment $\Delta\varepsilon$ (determined from a given displacement field) is applied and the total strain at time $t_{n+1} = t + \Delta t$ is updated as

$$\boldsymbol{\varepsilon}_{n+1} = \varepsilon_n + \Delta\varepsilon \tag{29}$$

First consider a purely elastic step (trial state), which in general will not correspond to an actual state. As depicted in Figure 1 it is concluded that

$$\sigma_{n+1}^{trial} = C\varepsilon_{n+1} - C\varepsilon_n^p = \sigma_n + C\Delta\varepsilon \tag{30}$$

with $\boldsymbol{\varepsilon}_{n+1}^{p\ \ trial} = \varepsilon_n^p$.

In the following step, the yield condition is checked by calculating

$$f_{n+1}^{trial} = \left\|\xi^{trial}\right\| - \sqrt{\frac{2}{3}}k(p_n) \tag{31}$$

in which

$$\xi_{n+1}^{trial} = \sigma_{n+1}^{trial} - \tilde{\beta}_n \tag{32}$$

As stated higher, the variable $\tilde{\beta}$ is omitted in this study since only isotropic hardening is considered. If $f_{n+1}^{trial} < 0$ then the Kuhn-Tucker conditions imply that $\Delta\gamma = 0$ and the step is elastic. The basic variables are updated as

$$\sigma_{n+1} = \xi_{n+1}^{trial}$$

$$\varepsilon = \varepsilon_{n+1} \tag{33}$$

In case $f_{n+1}^{trial} > 0$ the trial (elastic) state cannot be a solution since the constraint condition is violated. The process is thus incrementally plastic, which requires:

$$f(\sigma_{n+1}, p_{n+1}) = 0 \text{ and } \Delta\gamma > 0 \tag{34}$$

The update process depends parametrically on the multiplier $\Delta\gamma$ which is determined by enforcing the consistency condition Eqn (23) at time t_{n+1}. After some numerical treatment, this results in

$$f^2(\Delta\gamma) = \frac{1}{2}\bar{f}^2{}_{n+1} - \frac{1}{3}\left[k\left(p_n + \sqrt{\frac{2}{3}}\Delta\gamma\bar{f}_{n+1}\right)\right]^2 = \tag{35}$$

In which $\bar{f}_{n+1}$ is written as:

$$\bar{f}_{n+1} = \sqrt{\xi_{n+1}{}^T P \xi_{n+1}} \tag{36}$$

whith ξ_{n+1} the updated stress at the current step

$$\xi_{n+1} = \sigma_{n+1}$$

Then it follows:

$$\bar{f}_{n+1}^2(\Delta\gamma) = \frac{1}{2}\frac{\frac{1}{3}(\eta_{n+1,11}^{trial})^2}{\left\{1 + (\frac{E}{3(1-\upsilon)})\Delta\gamma\right\}^2} + \frac{(\eta_{n+1,22}^{trial})^2 + 2(\eta_{n+1,12}^{trial})^2}{[1 + 2G\Delta\gamma]^2} \tag{37}$$

in which

$$\eta_{n+1}^{trial} = Q^T \xi_{n+1}^{trial} \tag{38}$$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Eqn (37) should be solved by Newton iteration, for which:

$$\Delta\gamma_{k+1} = \Delta\gamma_k - \frac{f(\Delta\gamma_k)}{\partial_{\Delta\gamma} f(\Delta\gamma_k)} \tag{39}$$

This iteration results in the converged $\Delta\gamma$.

The function Ξ is called modified elastic tangent modulus which in case of isotropic hardening is written as:

$$\Xi = [C^{-1} + \Delta\gamma P]^{-1} \tag{40}$$

So far most of the parameters required are found and the update procedure is done as follows:

$$\sigma_{n+1} = \xi_{n+1} = \Xi(\Delta\gamma) C^{-1} \xi_{n+1}^{trial} \tag{41}$$

$$p_{n+1} = p_n + \sqrt{\frac{2}{3}} \Delta\gamma \bar{f}_{n+1}(\Delta\gamma) \tag{42}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma P \xi_{n+1} \tag{43}$$

The strain in the thickness direction ε_{33} is updated as follows:

$$\varepsilon_{33_{n+1}} = \frac{-\upsilon}{E}\left(\sigma_{n+1,11} \quad + \sigma_{n+1,22} \quad\right) - (\varepsilon_{n+1,11}^p \quad + \varepsilon_{n+1,22}^p \quad) \tag{44}$$

3.1 Material Jacobian

Implementation of plasticity constitutive equations into ABAQUS using implicit integration, demands the definition of the tangent stiffness matrix or material Jacobian ($^{\partial\Delta\sigma}/_{\partial\Delta\varepsilon}$) which highly depends on material behavior. It should be noted that the material Jacobian does not affect the accuracy of the solution but the rate of the convergence of the solution. In case of isotropic elasticity the material Jacobian is the same as the elastic tangent stiffness (C^{el}).

$$^{\partial\Delta\sigma}/_{\partial\Delta\varepsilon}\,|_{n+1} = \Xi - \frac{[\Xi P \xi_{n+1}][\Xi P \xi_{n+1}]^T}{\xi_{n+1}^T P \Xi P \xi_{n+1} + \bar{\beta}_{n+1}}$$

$$\bar{\beta}_{n+1} = \frac{2}{3\theta_2}(\partial_p k_{n+1}) \xi_{n+1}^T P \xi_{n+1} \tag{45}$$

$$\theta_2 = 1 - \frac{2}{3}\partial_p k_{n+1} \Delta\gamma$$

4 CONCLUSIONS

This paper presents elastic-plastic integration of material constitutive law for case of plane stress has been presented. The terminology and method used in this paper owes to the work of Simo and Hoghes [6]. The key parameter in this is the plastic multiplier γ for which the Newton's iteration method has been used. Effective plastic strain and plastic strain are stored at the end of the subroutine and are recalled at the next iteration. Stress update for thickness direction is dealt with in a different manner than the one that is used for tri-axial and plane stress cases. Finally, material Jacobian is returned to the program even though that it has no effect on the accuracy of the solution but the rate that it converges.

5 NOMENCLATURE

ε_{ii}	Component of in-plane strain
σ_{ij}	Component of Cauchy stress tensor
σ_0	Initial yield stress
σ_{eff}	Effective stress
υ	Poisson's ratio
E	young's modulus
$tr[\varepsilon]$	$\sum_{i=1}^{3}\varepsilon_{ii}$
σ_e^{trial}	Trial stress
$\sigma^{trial'}$	Deviator trial stress
r	Isotropic hardening function
Δ	Increment
:	Double contracted product
$\partial\delta\sigma/\partial\delta\varepsilon$	Material Jacobian
G	Shear modulus
γ	Plastic multiplier
Ξ	Modified elastic tangent moduli
$k(p)$	Isotropic yield stress
p	Effective plastic strain
P	Projection matric for plane stress
ξ	Stress
η	Stress
C^{el}	Elastic tangent stiffness
n	Iteration number
$\dot{()}$	Time differentiation
$\partial_\sigma f$	Partial derivative of f to σ

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