Orthogonal bases of Hermitean monogenic polynomials: an explicit construction in complex dimension 2

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Abstract. In this contribution we construct an orthogonal basis of Hermitean monogenic polynomials for the specific case of two complex variables. The approach combines group representation theory, see [5], with a Fischer decomposition for the kernels of each of the considered Dirac operators, see [4], and a Cauchy-Kovalevskaya extension principle, see [3].

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BASICS OF HERMITEAN CLIFFORD ANALYSIS

Let $(e_1,\ldots,e_m)$ be an orthonormal basis of $\mathbb{R}^m$, then multiplication in the complex Clifford algebra $\mathbb{C}_m$ is governed by the rule $e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}$, $\alpha,\beta = 1,\ldots,m$, whence $\mathbb{C}_m$ is generated additively by the elements $e_A = e_{j_1}\ldots e_{j_h}$, where $A = \{j_1,\ldots,j_h\} \subset \{1,\ldots,m\}$, with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, and $e_\emptyset = 1$.

The framework for Hermitean Clifford analysis is introduced by means of a complex structure, i.e. an $\mathrm{SO}(m;\mathbb{R})$–element J with $J^2 = -\mathbf{1}$ (see [1, 2]). So, the dimension is forced to be even: $m = 2n$. Usually J is chosen to act upon the generators of $\mathbb{C}_{2n}$ as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1,\ldots,n$. By means of the projection operators $\pm\frac{1}{2}(\mathbf{1}\pm iJ)$ associated to J, first the Witt basis elements $(\mathfrak{f}_j,\mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained: $\mathfrak{f}_j = \frac{1}{2}(\mathbf{1}+iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j})$ and $\mathfrak{f}_j^\dagger = -\frac{1}{2}(\mathbf{1}-iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j})$, $j = 1,\ldots,n$, satisfying the relations $\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0$ and $\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}$, $j,k = 1,\ldots,n$. Next, a vector $(x_1,\ldots,x_n,y_1,\ldots,y_n) \in \mathbb{R}^{2n}$ is identified with $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$, producing the Hermitean variables $\underline{z} = \frac{1}{2}(\mathbf{1}+iJ)[\underline{X}] = \sum_{j=1}^n \mathfrak{f}_j z_j$ and $\underline{z}^\dagger = -\frac{1}{2}(\mathbf{1}-iJ)[\underline{X}] = \sum_{j=1}^n \mathfrak{f}_j^\dagger \bar{z}_j$, expressed in the complex variables $z_j = x_j + iy_j$ and their conjugates $\bar{z}_j = x_j - iy_j$, $j = 1,\ldots,n$. Finally, the Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^n (e_j\partial_{x_j} + e_{n+j}\partial_{y_j})$ gives rise to the Hermitean Dirac operators $\partial_{\underline{z}}^\dagger = \frac{1}{4}(\mathbf{1}+iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n \mathfrak{f}_j\partial_{\bar{z}_j}$ and $\partial_{\underline{z}} = -\frac{1}{4}(\mathbf{1}-iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n \mathfrak{f}_j^\dagger\partial_{z_j}$, involving the Cauchy–Riemann operators $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and their conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, $j = 1,\ldots,n$. The Hermitean variables and Dirac operators are isotropic, whence the Laplacian decomposes as $\Delta_{2n} = 4(\partial_{\underline{z}}\partial_{\underline{z}}^\dagger + \partial_{\underline{z}}^\dagger\partial_{\underline{z}})$, while also $\underline{z}\,\underline{z}^\dagger + \underline{z}^\dagger\underline{z} = |\underline{z}|^2$.

We take functions with values in an irreducible representation $\mathbb{S}_n$ of $\mathbb{C}_{2n}$, called spinor space, which is realized within $\mathbb{C}_{2n}$ using a primitive idempotent $I = I_1\ldots I_n$, with $I_j = \mathfrak{f}_j\mathfrak{f}_j^\dagger$, $j = 1,\ldots,n$. With that choice $\mathbb{S}_n \equiv \mathbb{C}_{2n}I \cong \bigwedge_n^\dagger I$, where $\bigwedge_n^\dagger$ denotes the Grassmann algebra generated by the $\mathfrak{f}_j^\dagger$'s, since $\mathfrak{f}_j I = 0$. Hence $\mathbb{S}_n$ decomposes into homogeneous parts as $\mathbb{S}_n = \bigoplus_{r=0}^n \mathbb{S}_n^{(r)} = \bigoplus_{r=0}^n (\bigwedge_n^\dagger)^{(r)} I$, with $(\bigwedge_n^\dagger)^{(r)} = \mathrm{span}_{\mathbb{C}}(\mathfrak{f}_{k_1}^\dagger \wedge \mathfrak{f}_{k_2}^\dagger \wedge \cdots \wedge \mathfrak{f}_{k_r}^\dagger : \{k_1,\ldots,k_r\} \subset \{1,\ldots,n\})$.

A continuously differentiable function g in an open region Ω of $\mathbb{R}^{2n}$, taking values in $\mathbb{S}_n$, then is called (left) Hermitean monogenic in Ω iff it satisfies in Ω the system $\partial_{\underline{z}} g = 0 = \partial_{\underline{z}}^\dagger g$. A major difference with Euclidean Clifford analysis concerns the underlying group invariance. Where $\partial_{\underline{X}}$ is invariant under the action of $\mathrm{SO}(m)$, the system invariance of $(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)$ breaks down to the group $\mathrm{U}(n)$, see e.g. [1, 2]. For this reason $\mathrm{U}(n)$ will play a fundamental role in the construction of an orthogonal basis of Hermitean monogenic polynomials, as explained in [5].

The spaces of homogeneous polynomials on $\mathbb{C}^n$ with bidegree of homogeneity (a,b) in $(\underline{z},\underline{z}^\dagger)$, taking values in $\mathbb{S}_n^{(r)}$, will be denoted by $\mathscr{P}_{a,b}^r(\mathbb{C}^n)$. By $\mathscr{M}_{a,b}(\mathbb{C}^n)$ we denote the space of Hermitean monogenic polynomials of bidegree (a,b) in $(\underline{z},\underline{z}^\dagger)$, and by $\mathscr{M}_{a,b}^r(\mathbb{C}^n)$ its subspace with values in $\mathbb{S}_n^{(r)}$; the latter may be further split as

$$\mathbb{S}_n^{(r)} \equiv (\textstyle\bigwedge_n^\dagger)^{(r)} I = (\bigwedge_{n-1}^\dagger)^{(r)}(\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_{n-1}^\dagger)\, I \bigoplus (\bigwedge_{n-1}^\dagger)^{(r-1)}(\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_{n-1}^\dagger)\,\mathfrak{f}_n^\dagger I$$

whence we can decompose polynomials in $\mathscr{M}^r_{a,b}(\mathbb{C}^n)$ as $p_{a,b} = p^0_{a,b} I + p^1_{a,b} \mathfrak{f}^\dagger_n I$, with $p^0_{a,b}$ taking values in $(\bigwedge^\dagger_{n-1})^{(r)}(\mathfrak{f}^\dagger_1,\ldots,\mathfrak{f}^\dagger_{n-1})$ and $p^1_{a,b}$ taking values in $(\bigwedge^\dagger_{n-1})^{(r-1)}(\mathfrak{f}^\dagger_1,\ldots,\mathfrak{f}^\dagger_{n-1})$. Note that for $r = 0$ or $r = n$ one of these components becomes trivial. In the same order of ideas we single out the variables $(z_n,\bar{z}_n)$ and rewrite the Hermitean variables as $\underline{z} = \widetilde{\underline{z}} + \mathfrak{f}_n z_n$ and $\underline{z}^\dagger = \widetilde{\underline{z}}^\dagger + \mathfrak{f}^\dagger_n \bar{z}_n$, and the Hermitean Dirac operators as $\partial_{\underline{z}} = \widetilde{\partial}_{\underline{z}} + \mathfrak{f}^\dagger_n \partial_{z_n}$ and $\partial^\dagger_{\underline{z}} = \widetilde{\partial}_{\underline{z}}^\dagger + \mathfrak{f}_n \partial_{\bar{z}_n}$. We will consider restrictions to $\{z_n = 0 = \bar{z}_n\}$, identified with $\mathbb{C}^{n-1}$. The following results were then proven in [3].

Proposition 1. *(i) Given the polynomial $p^0_{a,b-j} I \in Ker(\widetilde{\partial}_{\underline{z}})$ on $\mathbb{C}^{n-1}$ $(j = 0,\ldots,b)$, there exists a unique polynomial $M^0_{a,b,j} \in \mathscr{M}_{a,b}(\mathbb{C}^n)$, given by*

$$M^0_{a,b,j} = \bar{z}_n{}^j \left(\sum_{k=0}^{\min(2a+1,2(b-j))} \frac{1}{\lfloor \frac{k}{2} \rfloor !} \frac{1}{\lfloor \frac{k+1}{2} \rfloor !} \left(z_n \widetilde{\partial}_{\underline{z}} \mathfrak{f}_n + \bar{z}_n \widetilde{\partial}_{\underline{z}}^\dagger \mathfrak{f}^\dagger_n \right)^k p^0_{a,b-j} I \right)$$

such that $\partial^j_{\bar{z}_n} M^0_{a,b,j}|_{\mathbb{C}^{n-1}} = p^0_{a,b-j} I$ and all other derivatives w.r.t. $\bar{z}_n$ vanish in $\mathbb{C}^{n-1}$.

(ii) Given the polynomial $p^1_{a-i,b} \mathfrak{f}^\dagger_n I \in Ker(\widetilde{\partial}_{\underline{z}}^\dagger)$ on $\mathbb{C}^{n-1}$ $(i = 0,\ldots,a)$, there exists a unique polynomial $M^1_{a,b,i} \in \mathscr{M}_{a,b}(\mathbb{C}^n)$, given by

$$M^1_{a,b,i} = z_n{}^i \left(\sum_{k=0}^{\min(2a,2b+1)} \frac{1}{\lfloor \frac{k}{2} \rfloor !} \frac{1}{\lfloor \frac{k+1}{2} \rfloor !} \left(z_n \widetilde{\partial}_{\underline{z}} \mathfrak{f}_n + \bar{z}_n \widetilde{\partial}_{\underline{z}}^\dagger \mathfrak{f}^\dagger_n \right)^k p^1_{a-i,b} \mathfrak{f}^\dagger_n I \right)$$

such that $\partial^i_{z_n} M^1_{a,b,i}|_{\mathbb{C}^{n-1}} = p^1_{a-i,b} \mathfrak{f}^\dagger_n I$ and all other derivatives w.r.t. z_n vanish in $\mathbb{C}^{n-1}$.

The polynomial $M^0_{a,b,j}$ (respectively $M^1_{a,b,i}$) is called the Hermitean Cauchy-Kovalevskaya extension of the initial polynomial $p^0_{a,b-j} I$ (respectively the initial polynomial $p^1_{a-i,b} \mathfrak{f}_n I$). This CK extension will play an important role in the construction of the desired orthogonal basis. Indeed, introducing, as in [5], the following spaces of initial polynomials:

$$\begin{aligned} \mathscr{A}^r_{a,b-j} &= \left\{ p^0_{a-i,b} I \mid p^0_{a-i,b} I \in \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}) \cap \mathscr{P}^r_{a,b-j}(\mathbb{C}^{n-1}) \right\} \\ \mathscr{B}^r_{a-i,b} &= \left\{ p^1_{a-i,b} \mathfrak{f}^\dagger_n I \mid p^1_{a-i,b} I \in \mathrm{Ker}(\partial^\dagger_{\underline{z}}) \cap \mathscr{P}^{r-1}_{a-i,b}(\mathbb{C}^{n-1}) \right\} \end{aligned}$$

the CK extension map is an isomorphism from $\oplus^b_{j=0} \mathscr{A}^r_{a,b-j} \oplus \oplus^a_{i=0} \mathscr{B}^r_{a-i,b}$ to $\mathscr{M}^r_{a,b}$, commuting with the action of $\mathrm{U}(n-1)$, whence it yields a splitting of $\mathscr{M}^r_{a,b}$ into a direct sum of $\mathrm{U}(n-1)$ invariant subspaces. Since the initial polynomials on $\mathbb{C}^{n-1}$ for the CK extension have to be submit to the compatibility condition of being either in the kernel of $\widetilde{\partial}_{\underline{z}}$ or in the kernel of $\widetilde{\partial}_{\underline{z}}^\dagger$, the so-called Fischer decomposition of these kernels in terms of Hermitean monogenics will also be involved. Under the action of $\mathrm{U}(n-1)$, see [4], the space $\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}}) \equiv \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}) \cap \mathscr{P}^r_{a,b}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}}) = \mathscr{M}^r_{a,b} \bigoplus_{j=0}^{\min(a,b-1)} |\underline{z}|^{2j} \underline{z}^\dagger \mathscr{M}^{r-1}_{a-j,b-j-1} \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} \left(\underline{z}^\dagger \underline{z} + \frac{(a-j-1+r)}{(a+r)} \underline{z}\, \underline{z}^\dagger\right) \mathscr{M}^r_{a-j-1,b-j-1} \quad (1)$$

and the space $\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}_{\underline{z}}^\dagger) \equiv \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}^\dagger) \cap \mathscr{P}^{r-1}_{a,b}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}_{\underline{z}}^\dagger) = \mathscr{M}^{r-1}_{a,b} \bigoplus_{j=0}^{\min(a-1,b)} |\underline{z}|^{2j} \underline{z} \mathscr{M}^r_{a-j-1,b-j} \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} \left(\underline{z z}^\dagger + \frac{(b-j-1+n-r+1)}{(b+n-r+!)} \underline{z}\, \underline{z}^\dagger\right) \mathscr{M}^{r-1}_{a-j-1,b-j-1} \quad (2)$$

It now becomes clear that, once the desired bases have been constructed in dimension $n-1$, these results can been used as building blocks in the above Fischer decompositions, yielding bases for the spaces $\mathscr{A}^r_{a,b-j}$ and $\mathscr{B}^r_{a-i,b}$ of initial polynomials. Subsequent application of the CK extension procedure, will then produce a basis for the space $\mathscr{M}^r_{a,b}$ in dimension n, which, by construction, will be orthogonal w.r.t. any $\mathrm{U}(n)$ invariant inner product.

We will now follow this general procedure as explained above, and, in more detail, in [5], to explicitly obtain orthogonal bases for the spaces $\mathscr{M}^r_{a,b}(\mathbb{C}^2)$, $r = 0,1,2$, $(a,b) \in \mathbb{N}^2$. Since the procedure is inductive, we need however to start with the case $n = 1$.

THE CASE $n=1$

In this case we are considering polynomials $f(z_1,\bar{z}_1)$ defined in the complex plane and taking values in the spinor space $\mathbb{S}_1=\mathrm{span}_{\mathbb{C}}\{1,\mathfrak{f}_1^\dagger\}I$. The Hermitean Dirac operators are simply $\widetilde{\partial}_{\underline{z}}=\mathfrak{f}_1^\dagger\,\partial_{z_1}$ and $\widetilde{\partial}_{\underline{z}}^{\dagger}=\mathfrak{f}_1\,\partial_{\bar{z}_1}$, whence Hermitean monogenicity means nothing else but anti-holomorphy in the case $r=0$ and holomorphy in the case $r=1$. The symmetry group here is $\mathrm{U}(1)\simeq\mathrm{SO}(2)$.

For $r=0$ the U(1) modules $\widetilde{\mathscr{M}}^0_{0,b}$ are given by span$\left\{\frac{\bar{z}_1{}^b}{b!}I\right\}$, $b=0,1,2,\ldots$. They have highest weight $(-b)$.

For $r=1$ the U(1) modules $\widetilde{\mathscr{M}}^1_{a,0}$ are given by span$\left\{\frac{z_1{}^a}{a!}\mathfrak{f}_1^\dagger I\right\}$, $a=0,1,2,\ldots$. They have highest weight $(a+1)$.

THE CASE $n=2$

Now we consider polynomials $f(z_1,\bar{z}_1,z_2,\bar{z}_2)$ taking values in the spinor space $\mathbb{S}_2=\mathrm{span}_{\mathbb{C}}\{1,\mathfrak{f}_1^\dagger,\mathfrak{f}_2^\dagger,\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger\}I$. If $r=0$ or $r=2$ we are again confronted with (anti-)holomorphy, see [2], so we will focus on the interesting case $r=1$.

The dimension of the U(2) module $\mathscr{M}^1_{a,b}$ is $a+b+2$, see [3]. Each of the spaces of initial polynomials $\mathscr{A}^1_{a,b-j}$, $j=0,\ldots,b$ and $\mathscr{B}^1_{a-i,b}$, $i=0,\ldots,a$, is one-dimensional. The general theory of the CK extension procedure, see [3], predicts that the compatibility conditions imposed on these initial polynomials will be trivially fulfilled, so they simply are all homogeneous polynomials in the variables z_1 ans $\bar{z}_1$ of the appropriate bidegree, which is moreover confirmed by the Fischer decompositions (1)–(2):

$$\mathscr{A}^1_{a,b-j} = \mathrm{span}_{\mathbb{C}}\left\{(-1)^{b-j}\,\frac{z_1{}^a}{a!}\,\frac{\bar{z}_1{}^{b-j}}{(b-j)!}\,\mathfrak{f}_1^\dagger I\right\}, \qquad j=0,\ldots,b$$

$$\mathscr{B}^1_{a-i,b} = \mathrm{span}_{\mathbb{C}}\left\{(-1)^{b}\,\frac{z_1{}^{a-i}}{(a-i)!}\,\frac{\bar{z}_1{}^{b}}{b!}\,\mathfrak{f}_2^\dagger I\right\}, \qquad i=0,\ldots,a$$

By CK extension each of the spaces of initial polynomials thus gives rise to exactly one Hermitean monogenic basis polynomial, together yielding an orthogonal basis for $\mathscr{M}^1_{a,b}$, see [5]. These basis polynomials are respectively given by

$$\begin{aligned}
M^0_{a,b,j} &= \sum_{k=0}^{\min(a,b-j)}(-1)^{b-j-k}\,\frac{z_2{}^k}{k!}\,\frac{\bar{z}_2{}^{k+j}}{(k+j)!}\,\frac{z_1{}^{a-k}}{(a-k)!}\,\frac{\bar{z}_1{}^{b-j-k}}{(b-j-k)!}\,\mathfrak{f}_1^\dagger I \\
&+ \sum_{k=0}^{\min(a,b-j-1)}(-1)^{b-j-k-1}\,\frac{z_2{}^k}{k!}\,\frac{\bar{z}_2{}^{k+j+1}}{(k+j+1)!}\,\frac{z_1{}^{a-k}}{(a-k)!}\,\frac{\bar{z}_1{}^{b-j-k-1}}{(b-j-k-1)!}\,\mathfrak{f}_2^\dagger I, \qquad j=0,\ldots,b \\
M^1_{a,b,i} &= \sum_{k=0}^{\min(a-i,b)}(-1)^{b-k}\,\frac{z_2{}^{k+i}}{(k+i)!}\,\frac{\bar{z}_2{}^{k}}{k!}\,\frac{z_1{}^{a-i-k}}{(a-i-k)!}\,\frac{\bar{z}_1{}^{b-k}}{(b-k)!}\,\mathfrak{f}_2^\dagger I \\
&+ \sum_{k=0}^{\min(a-i-1,b)}(-1)^{b-k}\,\frac{z_2{}^{k+i+1}}{(k+i+1)!}\,\frac{\bar{z}_2{}^{k}}{k!}\,\frac{z_1{}^{a-i-k-1}}{(a-i-k-1)!}\,\frac{\bar{z}_1{}^{b-k}}{(b-k)!}\,\mathfrak{f}_1^\dagger I, \qquad i=0,\ldots,a
\end{aligned}$$

The following properties may then be verified right away.

Property 1. *Under derivation with respect to the "new" variables $(z_2,\bar{z}_2)$, the orthogonal basis polynomials of $\mathscr{M}^1_{a,b}$ act as follows:*

$$\begin{aligned}
&\partial_{z_2}M^1_{a,b,i}=M^1_{a-1,b,i-1} \quad && \partial_{\bar{z}_2}M^1_{a,b,i}=M^1_{a,b-1,i+1} \quad && i=1,\ldots,a\\
&\partial_{z_2}M^0_{a,b,j}=M^0_{a-1,b,j+1} && \partial_{\bar{z}_2}M^0_{a,b,j}=M^0_{a,b-1,j-1} && j=1,\ldots,b\\
&\partial_{z_2}M^1_{a,b,0}=M^0_{a-1,b,0} && \partial_{\bar{z}_2}M^0_{a,b,0}=M^1_{a,b-1,0} &&
\end{aligned}$$

Property 2. *Under derivation with respect to the "old" variables $(z_1,\bar{z}_1)$, the orthogonal basis polynomials of $\mathscr{M}^1_{a,b}$ act as follows:*

$$\begin{aligned}
&\partial_{z_1}M^1_{a,b,i}=M^1_{a-1,b,i} \quad && -\partial_{\bar{z}_1}M^1_{a,b,i}=M^1_{a,b-1,i} \quad && i=0,\ldots,a\\
&\partial_{z_1}M^0_{a,b,j}=M^0_{a-1,b,j} && -\partial_{\bar{z}_1}M^0_{a,b,j}=M^0_{a,b-1,j} && j=0,\ldots,b
\end{aligned}$$

Remark 1. *Property 1 holds in any dimension n, whereas Property 2 is specific for the case* $n = 2$.

Remark 2. *Since the orthogonal Hermitean monogenic basis polynomials are determined only up to a constant, the final expressions may always be normalized, according to some preferred behaviour or property. Here, we have in fact normalized all initial data by requiring that*

(i) if $p^0_{a,b} I \in \mathscr{A}^1_{a,b}$, *then* $\partial^a_{z_1}(-\partial_{\bar z_1})^b [p^0_{a,b} I] = \mathfrak{f}^\dagger_1 I$;

(ii) if $p^1_{a,b}\mathfrak{f}^\dagger_2 I \in \mathscr{B}^1_{a,b}$, *then* $\partial^a_{z_1}(-\partial_{\bar z_1})^b [p^1_{a,b}\mathfrak{f}^\dagger_2 I] = \mathfrak{f}^\dagger_2 I$.

These normalization conditions are reflected in the eventual orthogonal basis as follows:

$$\partial^a_{z_1}(-\partial_{\bar z_1})^{b-j}\left[M^0_{a,b,j}\right] = \frac{\bar z_2{}^j}{j!}\mathfrak{f}^\dagger_1 I, \qquad \partial^{a-i}_{z_1}(-\partial_{\bar z_1})^{b}\left[M^1_{a,b,i}\right] = \frac{z_2{}^i}{i!}\mathfrak{f}^\dagger_2 I$$

Finally let us check the branching rules for the space $\mathscr{M}^1_{a,b}$ with highest weight $\lambda = (a+1,-b)$. From group representation theory, see e.g. [6], we know that when restricting the symmetry to U(1), the irreducible U(2) module $\mathscr{M}^1_{a,b}$ decomposes into irreducible U(1) modules as

$$\mathscr{M}^1_{a,b} = \bigoplus_{\mu \succ \lambda} V_\mu = \bigoplus_{k=-b}^{a+1} V_k$$

where each summand appears with multiplicity one; this decomposition is orthogonal w.r.t. any U(2) invariant scalar product on $\mathscr{M}^1_{a,b}$. On the other hand the Fischer decompositions (1)–(2) produce the U(1) irreducible components of the spaces of initial data $\mathscr{A}^1_{a,b-j}$ $(j = 0,\dots,b)$ and $\mathscr{B}^1_{a-i,b}$ $(i = 0,\dots,a)$. Assuming that $a > b$ (the cases $a \le b$ being similar) we have in fact that $\mathscr{A}^1_{a,b-j}$ is a shifted version of the U(1) module $\widetilde{\mathscr{M}}^1_{a-b+j,0}$ with highest weight $(a-b+j+1)$, for all $j = 0,\dots,b$. Similarly, $\mathscr{B}^1_{a-i,b}$ is a shifted version of the U(1) module $\widetilde{\mathscr{M}}^1_{a-i-b-1,0}$ with highest weight $(a-i-b)$, for all $i = 0,\dots,a-b-1$, while $\mathscr{B}^1_{a-i,b}$ is a shifted version of $\widetilde{\mathscr{M}}^0_{0,b-a+i}$ with highest weight $(a-i-b)$, for all $i = a-b,\dots,a$. As the CK extension map is an isomorphism between the initial data space $\oplus^b_{j=0}\mathscr{A}^1_{a,b-j} \oplus \oplus^a_{i=0}\mathscr{B}^1_{a-i,b}$ and the space $\mathscr{M}^1_{a,b}$, which commutes with the action of U(1), our construction of the orthogonal basis of $\mathscr{M}^1_{a,b}$ exactly yields the above splitting of $\mathscr{M}^1_{a,b}$ into the direct sum of $a+b+2$ U(1) invariant subspaces V_k, $k = -b,\dots,a+1$.

Remark 3. *For completeness we mention here the cases* $r = 0$ *and* $r = 2$. *For* $r = 0$ *the orthogonal basis of* $\mathscr{M}^0_{0,b}$ *consists of all homogeneous anti-holomorphic polynomials in* $(\bar z_1, \bar z_2)$, *i.e.* $\frac{\bar z_1{}^{b-j}}{(b-j)!}\frac{\bar z_2{}^j}{j!}$, $j = 0,\dots,b$, *while for* $r = 2$, *the orthogonal basis of* $\mathscr{M}^2_{a,0}$ *consists of all homogeneous holomorphic polynomials in* (z_1,z_2), *i.e.* $\frac{z_1{}^{a-i}}{(a-i)!}\frac{z_2{}^i}{i!}$, $i = 0,\dots,a$.

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