Gel'fand-Tsetlin procedure for the construction of orthogonal bases in Hermitean Clifford analysis

Fred Brackx[*], Roman Lávička[†], Hennie De Schepper[*] and Vladimír Souček[†]

[*]*Clifford Research Group, Faculty of Engineering, Ghent University, Galglaan 2, B-9000 Gent, Belgium*
[†]*Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha,Czech Republic*

Abstract. In this note, we describe the Gel'fand-Tsetlin procedure for the construction of an orthogonal basis in spaces of Hermitean monogenic polynomials of a fixed bidegree. The algorithm is based on the Cauchy-Kowalewski extension theorem and the Fischer decomposition in Hermitean Clifford analysis.

Keywords: Clifford analysis, orthogonal basis, Gel'fand-Tsetlin basis
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INTRODUCTION

The aim of the paper is to describe an algorithm for the construction of orthogonal bases in spaces of Hermitean monogenic polynomials of fixed bidegree. The construction of orthogonal bases in spaces of spherical monogenics in classical Clifford analysis has a long history; a general framework and further references may be found in [11], while explicit fomulae in lower dimension were developed in [9, 10, 1]. Here we consider the similar problem in Hermitean Clifford analysis using ideas stemming from representation theory, more particularly the well-known construction of the Gel'fand-Tsetlin (GT) basis for irreducible modules of classical Lie groups (see e.g. [12]). The spaces of Hermitean monogenic polynomials of fixed bidegree, taking values in a homogeneous part of spinor space, form irreducible representations of the group $U(n)$. By considering the GT basis for these particular modules we are able to obtain bases for these spaces, which, by construction, will be orthogonal w.r.t. any $U(n)$ invariant inner product.

THE GEL'FAND-TSETLIN BASIS IN THE CASE OF $U(n)$ MODULES

In Hermitean Clifford analysis, the main symmetry group is $U(n)$. According to the construction of a GT basis for an irreducible $U(n)$ module, we first have to choose a chain of subgroups $U(n) \supset U(n-1) \supset \ldots \supset U(1)$. We fix the embeddings such that in each step the last variable is preserved by the corresponding subgroup. Next we need to use the branching rules for $U(n)$ modules, which are expressed using the highest weights of the corresponding irreducible modules. Irreducible $U(n)$ modules are classified by their highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, where the integers λ_i satisfy the traditional condition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The branching rules then look as follows, see e.g. [13].

Theorem 1. *Under the restriction to $U(n-1)$ an irreducible $U(n)$ module V_λ with highest weight λ decomposes as*

$$V_\lambda = \oplus_{\mu, \lambda \succ \mu} V_\mu$$

where each of the summands appears with multiplicity one. Here the notation $\lambda = (\lambda_1, \ldots, \lambda_n) \succ \mu = (\mu_1, \ldots, \mu_{n-1})$ means that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$.

By induction we get the following Gel'fand-Tsetlin (GT) basis for V_λ.

Theorem 2. *The irreducible representation V_λ decomposes into a sum of one-dimensional subspaces $V_\lambda = \oplus_{\Lambda \in A} V_\Lambda$, where the set A of GT-labels Λ is given by $A = \left\{ \Lambda = (\lambda^n, \ldots, \lambda^1) \mid \lambda^n \succ \lambda^{n-1} \succ \ldots \succ \lambda^1; \lambda^j = (\lambda_1^j, \ldots, \lambda_j^j) \right\}$.*
If moreover V_λ is endowed with a $U(n)$ invariant scalar product, then the above decomposition is orthogonal.

We now want to apply this procedure to spaces of Hermitean monogenic polynomials. We will show that the required decomposition can be constructed step by step, using two well-known tools of Clifford analysis: the Fischer decomposition and the Cauchy-Kovalevskaya extension.

HERMITEAN MONOGENIC POLYNOMIALS

Let us consider the Euclidean vector space E of even dimension $2n$ endowed with a scalar product B, and let $\mathbb{C}_{2n}$ denote the corresponding complex Clifford algebra. Then we may consider the Dirac operator $\underline{\partial}$ acting on functions with values in the basic spinor representation $\mathbb{S}$ of $\mathbb{C}_{2n}$. Null solutions of $\underline{\partial}$ are called monogenic functions; the symmetry group for monogenicity is $O(2n)$.

Let $J \in SO(B)$ be a complex structure on E compatible with the scalar product B. We can always choose an orthonormal basis $e_1,\ldots,e_{2n}$ of E such that $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1,\ldots,n$. We define the isotropic basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $E \otimes \mathbb{C}$ by $\mathfrak{f}_j = \frac{1}{2}(e_j - i e_{n+j})$, $\mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + i e_{n+j})$, $j = 1,\ldots,n$. The Hermitean Clifford variables $\underline{z}$ and $\underline{z}^\dagger$ are given by

$$\underline{z} = \sum_{j=1}^n \mathfrak{f}_j z_j, \quad \underline{z}^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c \quad \text{with } z_j = x_j + iy_j \text{ and } z_j^c = x_j - iy_j$$

Finally, we define the Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ by

$$\partial_{\underline{z}}^\dagger = \sum_{j=1}^n \mathfrak{f}_j \partial_{z_j^c}, \quad \partial_{\underline{z}} = \sum_{j=1}^n \mathfrak{f}_j^\dagger \partial_{z_j}, \quad \text{with } \partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j}) \text{ and } \partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$$

A continuously differentiable function g in an open region Ω of $\mathbb{C}^n$ with values in $\mathbb{C}_{2n}$ is called (left) Hermitean monogenic in Ω if it satisfies in Ω the system $\partial_{\underline{z}} g = 0 = \partial_{\underline{z}}^\dagger g$. A Hermitean monogenic function is monogenic due to $\underline{\partial} = 2(\partial_{\underline{z}}^\dagger - \partial_{\underline{z}})$. The symmetry for the Hermitean monogenic equations breaks down to the group $U(n)$ leaving the complex structure invariant. The isotropic subspaces generated by z_i, resp. $\bar{z}_i$, form the defining representations of $U(n)$ and its dual. Spinor space $\mathbb{S}$ can be realized within the Clifford algebra as $\mathbb{S} = \mathbb{C}_{2n}I$ by means of a minimal idempotent I, here chosen to be $I = I_1 \ldots I_n$ with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger$, $j = 1,\ldots,n$. As $\mathfrak{f}_j I = 0$, we have that $\mathbb{S} \cong \bigwedge_n^{*\dagger} I$ where $\bigwedge_n^{*\dagger}$ denotes the complex Grassmann algebra generated by $\{\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_n^\dagger\}$. Hence spinor space $\mathbb{S}$ further decomposes into homogeneous parts as $\mathbb{S} = \bigoplus_{r=0}^n \mathbb{S}^{(r)}$ with $\mathbb{S}^{(r)} = (\bigwedge_n^\dagger)^{(r)} I$. For more details on Hermitean Clifford analysis, see [2, 3, 4, 7].

THE CAUCHY-KOVALSKEVSKAYA (CK) EXTENSION

For the cases $r = 0$, respectively $r = n$, the notion of Hermitean monogenicity coincides with the notion of antiholomorphy, respectively holomorphy, in n complex variables. Hence we will restrict the spinor homogeneity degree r to $1 \leq r \leq n-1$ from now on. The classical idea of the CK extension is to characterize solutions of suitable (systems of) PDE's by their restriction, sometimes together with the restrictions of some of their derivatives, to a submanifold of codimension one. In [5] we have done this for the Hermitean monogenic system, by restricting the solutions of the Hermitean Dirac operators (and some of their derivatives) to the vector subspace $\mathbb{C}^{n-1}$ of complex codimension 1.

The spaces of homogeneous polynomials of bidegree (a,b) on $\mathbb{C}^n$, with values in $\mathbb{S}^{(r)}$, will be denoted by $\mathscr{P}_{a,b}^r$, or, if we need to distinguish between dimensions, by $\mathscr{P}_{a,b}^r(\mathbb{C}^n)$. Furthermore we denote by $\mathscr{M}_{a,b}^r$ (or $\mathscr{M}_{a,b}^r(\mathbb{C}^n)$) the space of Hermitean monogenic homogeneous polynomials of bidegree (a,b), with values in $\mathbb{S}^{(r)}$. We single out the variables (z_n, z_n^c) and consider restrictions to $\mathbb{C}^{n-1} = \{z \in \mathbb{C}^n \,|\, z_n = z_n^c = 0\}$. We may then split the value space $\mathbb{S}^{(r)}$ as

$$\mathbb{S}^{(r)} \equiv (\textstyle\bigwedge_n^\dagger)^{(r)} I = (\textstyle\bigwedge_{n-1}^\dagger)^{(r)} I \bigoplus (\textstyle\bigwedge_{n-1}^\dagger)^{(r-1)} \mathfrak{f}_n^\dagger I$$

Hence any polynomial p with values in $(\bigwedge_n^\dagger)^{(r)} I$ can be split as $p = p^0 I + p^1 \mathfrak{f}_n^\dagger I$, where p^0 has values in $(\bigwedge_{n-1}^\dagger)^{(r)}$ and p^1 has values in $(\bigwedge_{n-1}^\dagger)^{(r-1)}$.

Now consider $M_{a,b} \in \mathscr{M}_{a,b}^r$ and its restricted derivatives

$$\frac{\partial^i M_{a,b}}{\partial z_1^i}\Big|_{\mathbb{C}^{n-1}} \equiv p_{a-i,b} = p_{a-i,b}^0 I + p_{a-i,b}^1 \mathfrak{f}_n^\dagger I, \qquad i = 0,\ldots,a \tag{1}$$

$$\frac{\partial^j M_{a,b}}{\partial z_1^{c\,j}}\Big|_{\mathbb{C}^{n-1}} \equiv p_{a,b-j} = p_{a,b-j}^0 I + p_{a,b-j}^1 \mathfrak{f}_n^\dagger I, \qquad j = 0,\ldots,b \tag{2}$$

Then the following theorem holds, see [5].

Theorem 3. *(The CK extension)*

(i) *Any $M_{a,b} \in \mathcal{M}_{a,b}^r$ is uniquely determined by its restrictions $p_{a,b-j}^0 I$, $j = 0, \ldots, b$ and $p_{a-i,b}^1 \mathfrak{f}_n^\dagger I$, $i = 0, \ldots, a$, defined in (1)-(2), henceforth called the "initial data" for $M_{a,b}$.*

(ii) *The initial data cannot be arbitrary, but are characterized as follows.*

 • *The polynomials $p_{a,b-j}^0 I \in \mathcal{P}_{a,b-j}^r(\mathbb{C}^{n-1})$ should belong to $\mathrm{Ker}\,(\partial_{\underline{z}})$ for $r < n-1$; for $r = n-1$ this condition is trivially fulfilled. The space of all such initial data will be denoted by*

$$\mathscr{A}_{a,b-j}^r = \mathrm{Ker}\,(\partial_{\underline{z}}) \cap \mathcal{P}_{a,b-j}^r(\mathbb{C}^{n-1})$$

 • *The polynomials $p_{a-i,b}^1 I \in \mathcal{P}_{a-i,b}^{r-1}(\mathbb{C}^{n-1})$ should belong to $\mathrm{Ker}\,(\partial_{\underline{z}}^\dagger)$ for $r > 1$; for $r = 1$ this condition is trivially fulfilled. The space of all corresponding initial data will be denoted by*

$$\mathscr{B}_{a-i,b}^r = \left\{ p_{a-i,b}^1 \mathfrak{f}_n^\dagger I \mid p_{a-i,b}^1 I \in \mathrm{Ker}\,(\partial_{\underline{z}}^\dagger) \cap \mathcal{P}_{a-i,b}^{r-1}(\mathbb{C}^{n-1}) \right\}$$

 • *The CK extension map from the space of initial data $\oplus_{j=0}^b \mathscr{A}_{a,b-j}^r \oplus \oplus_{i=0}^a \mathscr{B}_{a-i,b}^r$ to $\mathcal{M}_{a,b}^r$ is an isomorphism commuting with the action of $U(n-1)$, whence it yields a splitting of $\mathcal{M}_{a,b}^r$ into a direct sum of $U(n-1)$ invariant subspaces.*

FISCHER DECOMPOSITIONS FOR TWO KERNELS

The final step in the construction of the GT basis is the decomposition of both spaces of initial data $\mathscr{A}_{a,b-j}^r$ and $\mathscr{B}_{a-i,b}^r$ into irreducible components under the action of $U(n-1)$. The tool needed here is the Fischer decomposition of the kernels of the Hermitean Dirac operators (see [8]), which we will apply for spinor valued polynomials on $\mathbb{C}^{n-1}$. Hence in the formulation of [8] the dimension and the degree of the forms are adapted to the present situation.

Theorem 4. *Consider spinor valued functions on $\mathbb{C}^{n-1}$ and let $1 \le r \le n-2$. Then the following statements hold.*

(i) *Under the action of $U(n-1)$, the space $\mathrm{Ker}\,_{a,b}^r(\partial_{\underline{z}}) \equiv \mathrm{Ker}\,(\partial_{\underline{z}}) \cap \mathcal{P}_{a,b}^r(\mathbb{C}^{n-1}) \equiv \mathscr{A}_{a,b}^r$ has the multiplicity free irreducible decomposition*

$$\mathrm{Ker}\,_{a,b}^r(\partial_{\underline{z}}) = \mathcal{M}_{a,b}^r \oplus \bigoplus_{j=0}^{\min(a,b-1)} |\underline{z}|^{2j} \underline{z}^\dagger \mathcal{M}_{a-j,b-j-1}^{r-1} \oplus \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} (\underline{z}^\dagger \underline{z} + \frac{(a-j-1+r)}{(a+r)} \underline{z}\,\underline{z}^\dagger) \mathcal{M}_{a-j-1,b-j-1}^r$$

(ii) *Under the action of $U(n-1)$, the space $\mathrm{Ker}\,_{a,b}^{r-1}(\partial_{\underline{z}}^\dagger) \equiv \mathrm{Ker}\,(\partial_{\underline{z}}^\dagger) \cap \mathcal{P}_{a,b}^{r-1}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition*

$$\mathrm{Ker}\,_{a,b}^{r-1}(\partial_{\underline{z}}^\dagger) = \mathcal{M}_{a,b}^{r-1} \oplus \bigoplus_{j=0}^{\min(a-1,b)} |\underline{z}|^{2j} \underline{z} \mathcal{M}_{a-j-1,b-j}^r \oplus \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} (\underline{z}\underline{z}^\dagger + \frac{(b-j-1+n-r+1)}{(b+n-r+1)} \underline{z}\,\underline{z}^\dagger) \mathcal{M}_{a-j-1,b-j-1}^{r-1}$$

THE INDUCTION STEP

The main aim of this paper is to establish an inductive algorithm allowing to write the GT basis for the space of Hermitean monogenics in an explicit form.

The case $n = 1$ corresponds with complex valued functions on $\mathbb{C}$. The case $r = 0$ leads to antiholomorphic functions, the case $r = 1$ to holomorphic functions. In both cases, the GT basis is the standard basis for the Taylor series, i.e., it consists of the monomials $\bar{z}^j$, respectively z^j, $j = 0, 1, 2, \ldots$.

Now assume that we explicitly know the GT bases in dimension $n-1$ for all spaces $\mathcal{M}_{a',b'}^{r'}(\mathbb{C}^{n-1})$, $r' = 0, \ldots, n-1$ and $a', b' \in \mathbb{N}$. Then consider the space $\mathcal{M}_{a,b}^r(\mathbb{C}^n)$. Theorem 3 describes how to split it into $U(n-1)$ invariant subspaces and the proof of the theorem, see [5], contains explicit formulae for the CK extension mapping into $\mathcal{M}_{a,b}^r(\mathbb{C}^n)$ as a differential operator acting on the chosen initial data. Next, we use Theorem 4 to decompose in their turn the spaces

of initial data, obtained in Theorem 3, into $U(n-1)$ irreducible components. Due to the induction assumption, we can use the explicit form of the GT basis in the individual spaces of Hermitean monogenics in dimension $n-1$ to get an explicit form of the GT basis in the spaces of initial data. In such a way, we have constructed an algorithm for the computation of the elements in the GT basis in dimension n explicitly.

As a check, let us consider the branching rules explicitly for the space $\mathscr{M}_{a,b}^r(\mathbb{C}^n)$. We will only treat the general case $1 < r < n$, since for $r = 1$ or $r = n$, there are small variations in the form of the highest weights, whence these cases cannot be treated in the same uniform way. The highest weight of $\mathscr{M}_{a,b}^r$ is $\lambda = (a+1,1,\ldots,1,0,\ldots,0,-b)$, where the last component 1 is at the r-th place. We will denote this weight shortly by $[a,-b]_r$. Refering at Theorem 4, we see that the set of highest weights of components with values in $\mathbb{S}^{(r)}$ in the decomposition of $\text{Ker}\,_{a,b}^r(\partial_{\underline{z}})$ into irreducible components under the action of $U(n-1)$, is the string $[a,-b]_r, [a-1,-b+1]_r, [a-2,-b+2]_r, \ldots$ ending either with $[a-b,0]_r$ or with $[0,-b+a]_r$, while for the components with values in $\mathbb{S}^{r-1}$ we obtain the string $[a,-b+1]_{r-1}, [a-1,-b+2]_{r-1}, \ldots$ ending either with $[a-b+1,0]_{r-1}$ or with $[0,-b+a+1]_{r-1}$. Representing those labels as points in two rectangular grids, one for r and one for $r-1$, with vertices $(0,0), (a,0), (0,-b)$ and $(a,-b)$, we obtain two anti-diagonal segments. A similar observation holds for the decomposition of $\text{Ker}\,_{a,b}^{r-1}(\partial_{\underline{z}}^\dagger)$. Considering all corresponding summands in the decomposition of the initial data spaces for $\mathscr{M}_{a,b}^r$, the whole collection of labels (with multiplicity one) gives the full rectangles $[j,k]_r$ and $[j,k]_{r-1}$ with $j = 0,\ldots,a$ and $k = -b,\ldots,0$. Taking the homogeneity degrees r and $r-1$ together, we immediately see that we obtain the same result as predicted by the abstract branching rules in Theorem 1. This not only provides us with a confirmation of our analysis, but in fact even offers an independent proof, using only tools from Clifford analysis, of the branching rules for those representations which are realized as spaces of Hermitean monogenic functions.

The resulting explicit forms of the bases in complex dimension $n = 2$ can be found in [6].

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