Evaluating the value of structural heath monitoring with longitudinal performance indicators and hazard functions using Bayesian dynamic predictions

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Objectives, abstract and conclusions

The objective of this paper is to illustrate the evaluation of the value of Structural Health Monitoring (SHM) in the framework of pre-posterior analysis, elaborating the joint modeling of the time-varying structural performance and hazard function in inspection/repair planning and expected total life cycle cost calculation. The hazard function in the joint model is updated by the assumed monitoring outcome, which results in a change of inspection/repair plan and expected Total Life Cycle Cost (TLCC). The difference of expected TLCC is defined as the value of SHM which changes as decision parameter $h(t)^T$ related to the inspection time changes within the maximum acceptable value h_{max}.

Technical information

The joint modeling of the time-varying structural performance and hazard function is introduced in the framework of Bayesian decision making. This framework is taken as basis to update the inspection/repair planning, expected TLCC and correspondingly the value of SHM.

Abstract: In this article, we present a framework for evaluating the value of structural heath monitoring (SHM) in order to optimize the SHM implementation and to estimate the inspection/repair investment. A joint model consisting of a time-varying structural Performance Indicator (PI) prediction and a hazard function is proposed. Structural condition data used to estimate and update predictions of PI can be obtained from monitoring and a Bayesian dynamic model is proposed to forecast structural performance. For the survival process the time-dependent PI values and their changing rates are incorporated in the hazard function, in which the risk of structural failure depends on both the current value of PI and the slope of its trajectory at time t. A Bayesian approach using the Markov-chain Monte Carlo method can be adopted for the parameter estimation. The advantageous feature of these predictions is that they are dynamically updated as extra monitoring data are collected, providing real time risk assessment using all recorded information. Furthermore, inspections, with a probability of detecting damage followed by a probability of repair action, are planned when the hazard function at time *t* crosses a threshold value. Subsequently, the problem of optimizing the strategy of monitoring and maintenance can be solved by minimizing the expected total cost during the expected service life of the structure. Finally, the Value of SHM (VoSHM) is obtained by calculating the difference between the expected Total Life Cycle Costs (TLCC) of the inspection/repair plans with and without SHM implemented.

5 Introduction

Civil engineering structures are subjected to time-dependent degradation processes which require considerations of a wide range of uncertainties. When it is required to forecast or make decisions under uncertainty and risk, gathering further information prior to making the decision is often crucial. Such information reduces the uncertainty and thus facilitates improved decision making. As Structural Health Monitoring (SHM) provides a way for collecting information and reducing uncertainty, it has received a lot of attention and is also widely implemented in practice. Since monitoring systems and inspection methods are due to costs limited in time, the decisions should be based on the evaluation of its benefit, which should be done prior to its installation. The general decision can be characterized by whether or not to apply SHM, which strategy to apply, and when to apply it if it's a short term monitoring strategy. Among these decisions, whether or not to apply SHM plays a fundamental role, based on which also decisions related to which kind of SHM and when to apply it can be investigated as well as their characteristics (e.g. precision, time intervals). The value of a certain SHM strategy is quantified by the value of information it provides, in monetary form. This can be calculated in the framework of decision theory introduced in Raiffa and Schlaifer (1961) as the difference between the expected life-cycle cost, or expected benefits, of performing SHM or not, as also presented by Faber and Thöns (2014). In order to calculate the expected TLCC, a threshold-based inspection/repair planning should be decided prior to further analysis of e.g. inspection times, repair rules and related costs. It requires a decision tree describing all possible events and the calculation of probabilities for each event. To do that, a joint modeling of the time-varying structural Performance Indicator (PI) and the hazard function is proposed first to calculate the failure rate (cf. infra). A threshold approach, as described in Straub and Faber (2006), is then adopted for planning the inspection time based on maximum failure rate and pre-posterior analysis for calculating the expected total life-cycle cost. The effect of SHM is accounted for by incorporating the monitoring outcome, including its uncertainty, to update the probability distribution of parameters related to the time-varying PIs, which in turn results in a change of inspection planning and expected total life-cycle cost. The VoSHM is then evaluated as the difference between the expected TLCC with and without SHM implemented. In the next parts, the process of this joint modeling and the parameter estimation methodology are first introduced. Then a framework for hazard based maintenance planning and determination of the VoSHM is elaborated.

6 Joint model of longitudinal data and hazard function

The joint modeling of longitudinal data, or in other words time sequent performance data, and the time to the occurrence of a particular event is an active area of research in statistics, mainly applied in medical related research, where two types of outcomes are recorded: (1) time sequent response measurements and (2) the time to an event of interest, such as time to death. These two types of outcomes are often analyzed using joint modeling of time sequent and time-to-event data, facilitating a prognostic tool for estimation and evaluation of risk rate for patients subjected to a certain disease, see Baghfalaki et al. (2014). Statistical methods for estimating and evaluating risk scores using reference data have been extensively studied in this field. It is clear that a similar procedure might be of interest for civil engineering structures. The joint modeling procedure and parameter estimation method are presented in the following section. Figure 1 shows an illustrative example of the joint modeling.

6.1 Process for modeling the time dependent structural performance:

The time dependent PIs can be expressed as:

$$y(t) = m(t) + \epsilon(t) \tag{1}$$

where $y(t)$ denotes the value of the time dependent observation outcome at any particular time point *t*, $m(t)$ is the underlying structural state which is a function of t and random effects denoted by b_i, and $\epsilon(t)$ the error terms that are assumed independent of the random effects. We assume that $\epsilon(t) \sim N(0, \sigma_1^2)$ and Cov. $[\epsilon(t), \epsilon(t')] = 0$ for $t \neq t'$.

6.2 Survival process defining the hazard function:

$$h(t) = \lim_{s \to 0} \frac{Pr(t < T < t + s | T > t)}{s} = h_0(t) \exp\{\alpha_1 m(t) + \alpha_2 m'(t) + \alpha_3\} \tag{2}$$

Where $h_0(t)$ is the baseline hazard function. This formulation postulates that the risk for a failure event at time t is associated with parameters α_1 and α_2 which quantify the strength of connection between the value of $m(t)$, its derivative over time $m'(t)$ and the failure rate for an event (failure according to a limit state function) at the same time point. There are several association structures between $m(t)$ and $h(t)$ for an event. For the functional forms one can refer to Rizopoulos et al. (2014). Here we formulate the joint model of the two process based on the assumption that the failure risk depends on both the current value of the $m(t)$ and its changing rate $m'(t)$, which is often the case for degrading structures. For purpose of illustration, see Figure 2. For a certain limit of PI, the curve with higher $m(t)$ value and lower decreasing rate $m'(t)$ at time t leads to a lower failure rate h(t), as for PI_1(t) (as indicated by the red curve in Figure 2). From comparing PI_2(t) and PI_3(t) with similar deceasing rate $m'(t)$ at time t, it is observed that higher $m(t)$ leads to a lower risk of failure. α_3 is a another regression coefficient. Under the assumption that the hazard of the failure event is mainly based on the value of $m(t)$ and its decreasing rate $m'(t)$, α_3 can be predefined to a certain value.

More specifically, for the survival process, we consider a parametric proportional hazard model with Weibull baseline hazard, as in equation (3). The use of the Weibull proportional hazard model has the advantage that it is the only one that is a hazard model proportional to the baseline hazard as well as an accelerated failure time model. This is suitable for modeling degrading structures, as the failure risk accelerates with time, especially in the end period of the service life.

$$h(t) = \sigma_2 t^{\sigma_2 - 1} \exp\{\alpha_1 m(t) + \alpha_2 m'(t) + \alpha_3\} \tag{3}$$

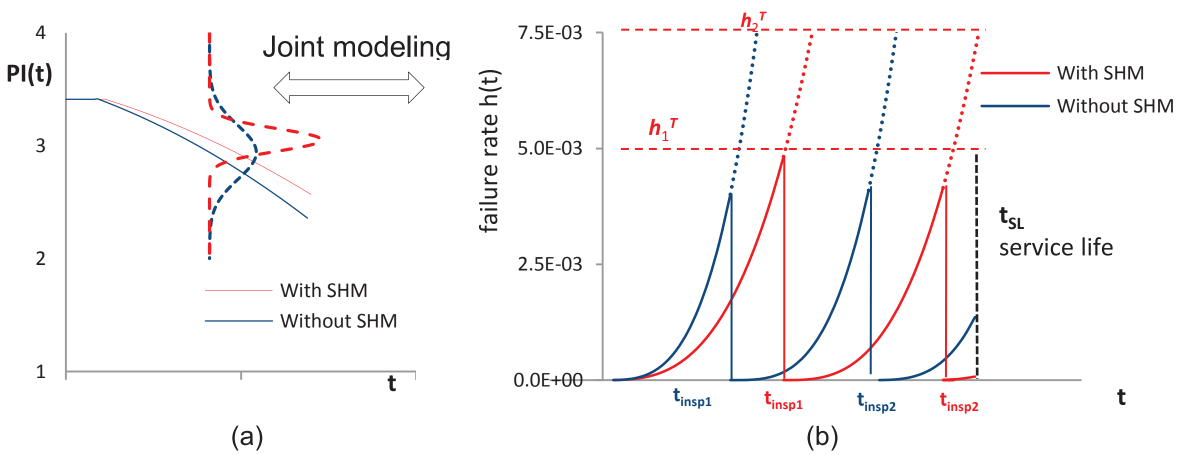

Figure 1. Joint modeling for (a) time-dependent structural performance and (b) hazard function

Details for the estimation of parameters in the two processes are provided in the Appendix. As soon as the evaluated value of the parameters are available, the failure rate h(t) at a time point and the failure probability during a period of time can be calculated and used for inspection time planning.

Under the Bayesian specification of the joint model, we can derive estimations and predictions for the hazard rate for a subject based on the observed time-dependent structural performance outcomes. Although the model fitting process can require considerable calculation and simulation work, it is very efficient for use once the joint model is formulated. Moreover, the hazard value h(t) can be updated when new monitoring information is recorded for updating the performance indicator, thus proceeding in a time-dynamic manner.

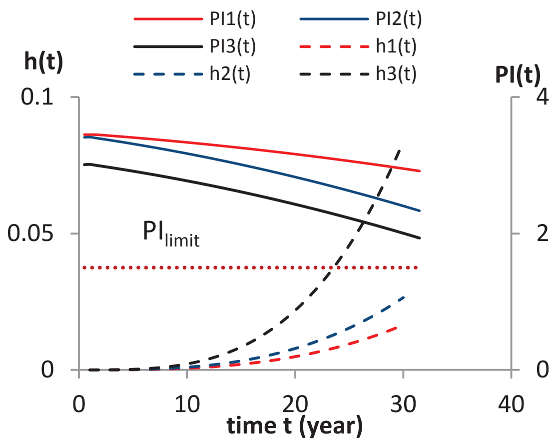

Figure 2 Illustration of interaction of PI(t) and PI'(t) with hazard function

One advantage of applying the joint modeling in civil engineering is that the functional form of the time-dependent feature of the structural performance, i.e. PI(t), is often available. Simulations of reference datasets can be performed by using a prior pdf assumption, which afterwards can be implemented into the joint modeling of the time-dependent performance and survival process. In this paper, the simulation based parameter estimation is not treated in detail and focus is given to the quantification of value of SHM, which is presented in the following part.

7 Hazard based maintenance planning and value of SHM

7.1 Value of SHM

An event tree model of a structure (e.g. the beam of a bridge) with or without monitoring is presented in Figure 3. For the time being, it is assumed that the structure returns to its original state after repair. This assumption makes it convenient to calculate the probabilities of occurrence of each branch in the event tree; for details refer to section 3.2.

As illustrated previously in Figure 1, the implementation of SHM can provide more information of the structure which would lead to a significant change of inspection/maintenance planning and as a result, the TLCC. Before making the decision of whether or not to implementing a certain SHM system, a prior estimation of the value of information that can be provided by the SHM is essential. Since the structural state and monitoring results are both uncertain, the decision problem can be described in terms of the following notations and events in a pre-posterior framework for Bayesian decision theory as developed in Raiffa and Schlaifer (1961):

Θ: Time-dependent structural state with prior pdf $f'_{\Theta}(\theta)$. Its time-dependent evolution is described by the joint model introduction in section 2;

Z: The inspection outcome which has an influence on the probability of detection and repair;

e: The inspection decision (i.e. inspection date, type of inspection, etc.). Inspection decision *e* varies according to the value of threshold $h(t)^T$ which is applied ;

a: The maintenance action determined by the decision rule d and as a function of the inspection outcome Z and inspection decision e, i.e. *a=d(e, z)*;

X: The monitoring result variable which leads to an updating of the probability distribution of Θ to $f''_{\Theta}(\theta)$. We denote M_0 for the case without taking SHM, and M_1 for undertaking a certain monitoring strategy, as in the first node illustrated in Figure 3.

Analysis is then carried out for determining the expected TLCC (cf. infra).

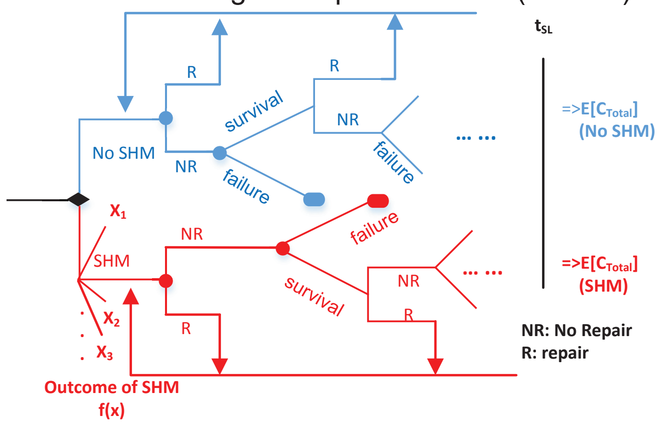

Figure 3 Decision tree model for inspection/repair planning with and without monitoring

For M_0, the true structural state Θ is described by a prior probability density function $f'_{\Theta}(\theta)$ and the expected TLCC is:

$$E'_{\Theta}[C_T(e,d,\theta)] = \int_{\Theta} C_T(e,d,\theta) f'_{\Theta}(\theta) d\theta \tag{4}$$

For M_1, the analysis is in principle identical, except that new monitoring information is available and taken into account. Based on the additional monitoring information, the prior pdf $f'_{\Theta}(\theta)$ is updated to the posterior $f''_{\Theta}(\theta)$ leading to a change of the inspection and repair decisions. The expected TLCC is thus:

$$E''_{\Theta}[C_T(x,e,d,\theta)] = \int_{\Theta} C_T(x,e,d,\theta) f''_{\Theta}(\theta|x) d\theta \tag{5}$$

In addition, based on the prior pdf of Θ and the monitoring strategy M, the probability of occurrence of the possible monitoring outcomes can be evaluated as:

$$f_X(x|M) = E_{\Theta}[f_X(x|M,\theta)] \tag{6}$$

The monitoring data enable to determine the equivalent stress ranges, of course, accounting for the measurement uncertainties.The value of information obtained by one monitoring outcome X, denoted as Condition Value of Sample Information (CVSI), has therefore the value

$$v(x) = CVSI(x) = E'_{\Theta}[C_T(e,d,\theta)] - E''_{\Theta}[C_T(x,e,d,\theta)] \tag{7}$$

The expected value of this SHM strategies can, therefore, be determined by the Expected Value of Sample Information (EVSI) as

$$EVSI = E_X\,[CVSI(x)] = \int_X v(x)\, f_X(x) dx = E'_{\Theta}[C_T(e,d,\theta)] - \int_X E''_{\Theta}[C_T(x,e,d,\theta)]\, f_X(x) dx \tag{8}$$

By assigning different pdfs for the monitoring result X, different monitoring strategies can be defined and the optimal decision on the monitoring strategy is the one with maximum value $\max_X EVSI(x)$.

In order to be able to evaluate the expected TLCC (see section 3.5), we need first to calculate the probabilities related to the decision tree based on a risk acceptance criterion h_{max} and decision rules.

7.2 Probabilities related to the decision tree

For a certain monitoring strategy (M_0 or M_1), calculation of the expected TLCC is needed based on the branches after the first node. This requires again pre-posterior analysis since the inspection results are also uncertain prior to being carried out. Hazard (failure rate) based inspection planning is applied in this contribution. The probability of occurrence of the branches after the first node in the decision tree is calculated based on the probabilities of detection of a certain deterioration state P_{det} and probabilities of taking repair action P_{rep} given a detected deterioration state with no repair before t as in equation (9) and (10) in accordance with Frangopol et al. (1997), Kim et al. (2011) and Barone (2013) .

$$P_{det} = \Phi\left(\frac{\delta(t) - \delta_{0.5}}{\sigma_{0.5}}\right) \tag{9}$$

$$P_{rep} = \left(\frac{\delta(t)}{\delta_{max}}\right)^{r_a} \qquad \delta(t) \leq \delta_{max} \tag{10}$$

in which Φ = standard Gaussian cumulative distribution function. $\delta(t)$ and δ_{max} are damage intensity at time t and maximum acceptable damage intensity; $\delta_{0.5}$ and $\sigma_{0.5}$ are parameters describing the quality of the inspection procedure, representing the damage intensity corresponding to a 50% probability of damage detection, and its standard deviation; r_a is a model parameter reflecting the attitude of the decision maker towards a repair action.

The branch of a failure event in the event tree requires the calculation of a failure probability $p_F(t)$ during time t given no repair before t. It can be calculated as soon as the hazard function is determined.

7.3 Risk acceptance criterion

The risk acceptance criterion used in this contribution is denoted by h_{max}, the maximum acceptable annual failure rate. It is related to the failure consequences of a structure and can be obtained from the Probabilistic Model Code, JCSS (2002), where the target reliability index as a function of the consequence of failure and the risk reduction cost is defined. For existing structures of which the relative cost for increasing the safety is generally large, the acceptance criterion can be lowered as also suggested in JCSS (2002), i.e. $\Delta p_f^{max} = 10^{-4}\, yr^{-1}$for large consequences, $\Delta p_f^{max} = 5 \times 10^{-4}\, yr^{-1}$ for moderate consequences and $\Delta p_f^{max} = 10^{-3}\, yr^{-1}$ for minor consequences, where Δp_f^{max} is the maximum acceptable annual probability of failure, Straub (2004). Of course decision makers can decide to use other values than the latter ones. It can be noticed from the definition of the hazard function in section 2 that the value of hazard h(t) in this paper corresponds to the annual probability of failure Δp_f. Similarly, the maximum acceptable hazard value h_{max} can be specified in accordance with Δp_f^{max}.

7.4 Decision rule

As the probabilities of detecting a deterioration state and the corresponding repair are both function of the structural state which is a time dependent variable described by PI(t), the planning of inspection times will have a large influence on the probability of occurrences of each branch in the decision tree. The threshold approach introduced in Straub (2006) is thereby implemented in such a way that inspection is carried out in the year before the threshold of failure rate $h(t)^T$ is crossed.

The h(t)^T is a decision parameter that can be changed as long as it remains lower than h_{max}. For decisions on repair, it requires the specification of the parameters used in equation (10) :

$$\begin{cases} h(t)^T \leq h_{max} & \text{for inspection} \\ \delta_{max} = \delta_0, r_a = r_0 & \text{for repair} \end{cases} \tag{11}$$

In which δ_0 and r_0 are values assigned by the decision makers. The decision for repair needs a direct indication of unacceptable defect sizes for a specific spot, (e.g. max pit depth of a rebar or maximum allowable crack depth). After an inspection was carried out, the repair decision is made based on the inpection result, detected structural states and repair attitude of the decision maker.

7.5 Expected TLCC

The expect TLCC for an inspection plan is calculated based on the occurrence of each branch in the decision tree as well as the cost of the basic events, i.e. the expected cost of failure, inspection, repair and monitoring if undertaken. The expected cost for each branch is calculated as the sum of all the events that happened in the branch. The expected TLCC for this inspection plan then takes weighted sums of the costs for all branches based on the occurrence probability of each branch introduced in section 3.2.

Specifically, the expected TLCC of a structural component during its design service life t_{SL} consists of the cost of failure, inspection, repair and monitoring (if undertaken):

$$E[C_T(e,d,t_{SL})] = E[C_F(e,d,t_{SL})] + E[C_I(e,d,t_{SL})] + E[C_R(e,d,t_{SL})] + E[C_M(e,d,t_{SL})] \tag{12}$$

Where $E[C_T(e,d,t_{SL})]$ is the expected TLCC, and $E[C_F(e,d,t_{SL})]$, $E[C_I(e,d,t_{SL})]$, $E[C_R(e,d,t_{SL})]$, $E[C_M(e,d,t_{SL})]$ are the expected cost of failure, inspection, repair and monitoring respectively, which can be calculated in accordance with Straub (2004) and Faber and Thöns (2014).

$$E[C_F(e,d,t_{SL})] = \sum_{t=1}^{t_{SL}} \left[\left(1 - \sum_{i=1}^{t-1} p_R(e,d,i)\right) \frac{1}{(1+r)^t} \left(h(e,d,t)\big(1 - p_F(e,d,t-1)\big)C_F + p_R(e,d,t)E[C_F(e,d,t_{SL}-t)]\right) \right] \tag{13}$$

$$E[C_I(e,d,t_{SL})] = \sum_{t=t_1}^{t_{nInsp}} \left[\big(1 - p_F(e,d,t)\big) \left(1 - \sum_{i=1}^{t-1} p_R(e,d,i)\right) \frac{1}{(1+r)^t} \cdot \left(C_{Insp} + p_R(e,d,t)E[C_I(e,d,t_{SL}-t)]\right) \right] \tag{14}$$

$$E[C_R(e,d,t_{SL})] = \sum_{t=t_1}^{t_{nInsp}} \left[\big(1 - p_F(e,d,t)\big) \left(1 - \sum_{i=1}^{t-1} p_R(e,d,i)\right) \frac{1}{(1+r)^t} \cdot p_R(e,d,t) \cdot \left(C_R + E[C_R(e,d,t_{SL}-t)]\right) \right] \tag{15}$$

$$E[C_M(e,d,t_{SL})] = C_{M_{Iv}} + C_{M_{Is}} + \big(1 - p_F(e,d,t)\big) \cdot C_{M_{Op}} \cdot \frac{1}{(1+r)^t} \tag{16}$$

Where t_{nInsp} is the time for the nth planned inspection, r is the discount rate. $C_F, C_{Insp}, C_R, \boldsymbol{C_M} = (C_{M_{Iv}}, C_{M_{Is}}, C_{M_{Op}})$ are the expected cost of failure, cost of inspection, cost of repair and cost of monitoring consisting of system investment $C_{M_{Iv}}$, installation $C_{M_{Is}}$ and operation $C_{M_{Op}}$ respectively, Faber and Thöns (2014). For details, we refer to Straub (2004) and Thöns (2012).

It should be mentioned that with a different value of $h(t)^T$, the planning of inspection times changes, leading to a change of the expect TLCC. Similarly, given a certain $h(t)^T$, the expected TLCC can also be different for M_0 and M_1, since the planned inspection times are likely to be different in case the monitoring outcome leads to a different joint model. Therefore, the maximum VoSHM can be found by changing $h(t)^T$, considering the risk acceptance criterion, to maximize the EVSI. The evaluation process is illustrated in the following flow chart in Figure 4:

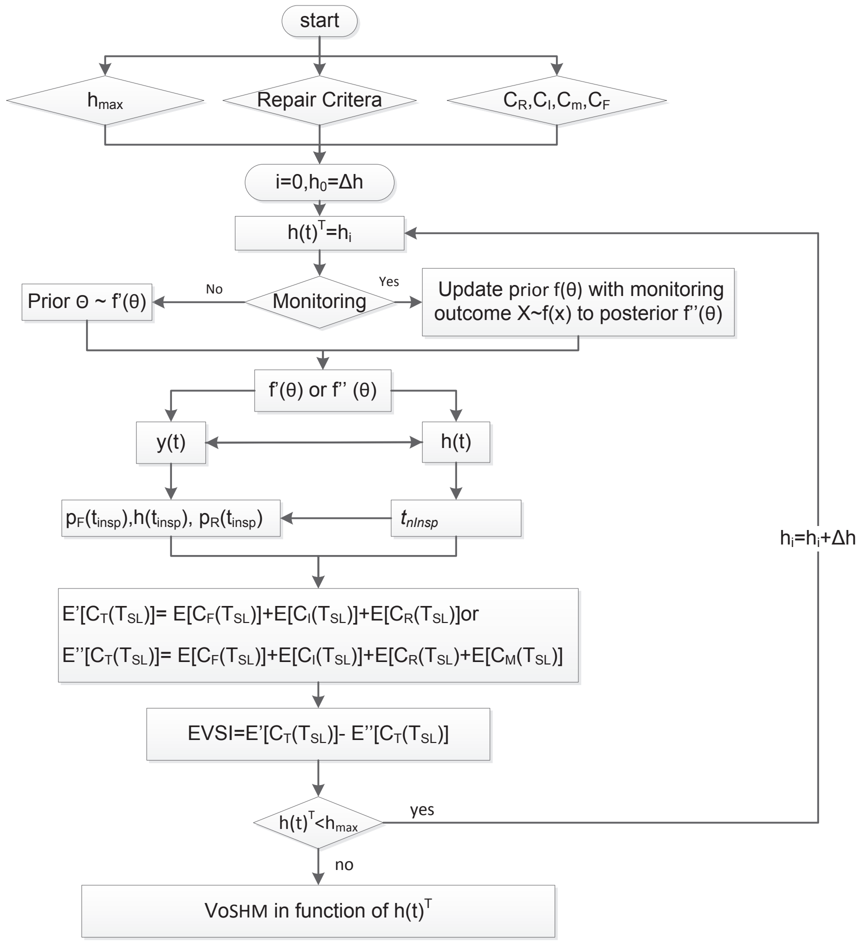

Figure 4. Flow chart for evaluating the value of SHM

8 Conclusive remarks

The paper proposes a framework for evaluating the VoSHM based on pre-posterior analysis incorporating Bayesian decision theory. A joint modeling of a time dependent structural performance function and a hazard function is first introduced and the derived hazard function is used as a tool for determining optimal inspection/repair plans for deteriorating structures. Consequently, also the expected TLCC is calculated. The effects of uncertainties related to the quality of monitoring outcomes are considered and incorporated in the joint model which leads to an updated the inspection/repair planning and expected TLCC. The difference between the prior and posterior expected TLCC is defined as the VoSHM and its dependency on the failure rate threshold is explained. Further research on the parameters estimation of joint modeling and identification of an optimal SHM strategy among others are needed.

9 Acknowledgements

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11 Appendix

For the joint model, we assume a reference dataset consisting of N samples each with its performance trajectory and time-to-event information available. The time-to-event (or time to failure) is defined by the structural performance crossing a limit value. Let T_i denote the observed lifetime for the i^{th} individual, i =1, 2, ..., N, which is taken as the minimum of the true failure time T_i^* and the censoring time C_i (e.g. didn't fail up to the time when the experiment ends), that is, $T_i = \min(T_i^*, C_i)$. We define a censoring indicator, $\delta_i = I(T_i^* \le C_i)$, which takes the value 1 when $T_i^* \le C_i$, and 0 otherwise. Therefore, the time to failure dataset consists of the pairs $\{(T_i, \delta_i), i = 1, 2, ..., N\}$. Furthermore, we let $\boldsymbol{y_i}$ denotes the time dependent PI data for the ith subject, with element y_{il} denoting the value of the PI outcome observed at time point t_{il}, $l = 1, . . . , n_i$.

One way to estimate the joint model's parameters is based on the Markov Chain Monte Carlo algorithm (MCMC). The likelihood of the model is derived under the assumption that the vector of time-independent random effects *b*, included in m(t) accounts for all interdependencies between the observed outcomes. That is, given the random effects, the longitudinal and survival process are assumed independent, and in addition, the longitudinal responses of each subject are assumed independent. Formally we have:

$$p(y_i, T_i, \delta_i | b_i, \theta) = p(y_i | b_i, \theta) p(T_i, \delta_i | b_i, \theta) \quad \text{A(1)}$$

$$p(y_i | b_i, \theta) = \prod_l p(y_{il} | b_i, \theta) \quad \text{A(2)}$$

In which the parameters included in the two processes of the joint model is denoted by θ =[α_1, α_2, α_3, σ_1, σ_2, D), where D is the parameter characterizing the random effects *b.* Thus, the likelihood contribution for the i[th] subject conditional on the parameters and random effects takes the form:

$$
\begin{aligned}
& p(y_i, T_i, \delta_i | b_i, \theta) \\
& = \prod_{l=1}^{n_i} p\left(y_{il} | b_i; \theta_y\right) \, p(T_i, \delta_i | b_i; \theta_t) \, p(b_i; \theta_b) \\
& \propto \left[\begin{array}{c} (\sigma_1^2)^{-\frac{n_i}{2}} \exp\{-\sum_l \left(y_{il} - m_i(l)\right)^2 / 2\sigma_1{}^2\} \times [\sigma_2 t^{\sigma_2 - 1} \exp\{\alpha_1 m_i(T_i) + \alpha_2 m_i{}'(T_i) + \alpha_3\}]^{\delta_i} \\ \times \exp\left[-\int_0^{T_i} \sigma_2 t^{\sigma_2 - 1} \exp\{\alpha_1 m_i(s) + \alpha_2 m_i{}'(s) + \alpha_3\} ds\right] \\ \times p(b_i; \theta_b) \end{array} \right]
\end{aligned}
\quad \text{A(3)}
$$

The parameter vector **θ** is divided into 3 parts, where θ_t denotes the parameters for the failure time outcome, θ_y the parameters for the longitudinal outcomes, and θ_b the unique parameters of the variance of the random effects, and p(·) denotes an appropriate probability density function, which is a normal distribution for $p\left(y_{il} | b_i; \theta_y\right)$, and for $p(T_i, \delta_i | b_i; \theta_t)$ derived from the hazard function.

Therefore, the likelihood function for the complete data ***D*a**=(***y***, ***T, δ***) is given by:

$$L(\boldsymbol{D_a} | \theta) = \prod_{i=1}^{N} [p(y_i, T_i, \delta_i | b_i, \theta)] \quad \text{A(4)}$$

Bayesian specification of the model needs to consider prior distributions for all the unknown parameters **θ**. In the situation where no prior information is available, non-informative prior distributions for the parameters should be adopted. Assuming elements of the parameter vector to be independent, we can take some traditional prior distributions. In particular, for the regression parameters of the survival model α_1, α_2 and α_3 we use independent univariate normal priors. For σ_1, σ_2 and D we take inverse-Gamma priors. In general, we denote the prior distribution of the parameters as $\pi(\theta)$.

Based on the prior distribution and likelihood function, the joint posterior density of parameters **θ**, $\pi''(\theta)$, is given by combining A(4) and $\pi(\theta)$:

$$\pi''(\theta) \propto L(\boldsymbol{D_a} | \theta) \pi(\theta) = \prod_{i=1}^{N} [p(y_i, T_i, \delta_i | b_i, \theta)] \, \pi(\theta) \quad \text{A(5)}$$

MCMC methods such as the Gibbs sampler and Metropolis–Hastings algorithm can be used to draw samples, from which characteristics of the marginal posterior distribution of interest can be inferred. As a result of the simulation, the parameters of **θ** are estimated and the hazard function is determined.

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