New techniques for the two-sided quaternionic Fourier transform

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Abstract In this paper, it is shown that there exists a Hermite basis for the two-sided quaternionic Fourier transform. This basis is subsequently used to give an alternative proof for the inversion theorem and to give insight in translation and convolution for the quaternionic Fourier transform.

1 Introduction

The topic of hypercomplex Fourier transforms has recently lead to various new insights, in two more or less separated communities of mathematicians on the one hand and engineers on the other hand. For the mathematical point of view, we refer to e.g. [1, 4, 5, 6, 8]. For the engineering point of view, we refer to [2, 3, 10, 12, 14] and the references mentioned therein.

The goals in both communities are quite different as well. Whereas in mathematics one is in general interested in structural properties of such transforms and the underlying algebraic ideas, in engineering one really wants to construct and study transforms that are applicable in, say, image processing (with better results than the ordinary Fourier transform).

The main aim of the present paper is to bridge the gap between both approaches, at least for one particular transform, namely the two-sided quaternionic Fourier transform (qFT). For this particular transform (see e.g. [10, 12, 14]), which is very important in image processing, we will adapt the typical techniques used in the mathematical community. This will allow for a new proof of the inversion theorem and new insights in the concept of convolution for the qFT.

The paper is organized as follows. In section 2 we discuss the Hermite basis for the ordinary Fourier transform. In section 3 we define the qFT and generalize

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the Hermite basis. As a consequence, an inversion theorem is obtained. Finally, in section 4 we consider the translation operator for the qFT and discuss the impact on convolution.

2 Classical Fourier transform

The classical Fourier transform, defined over the real line, is given by

$$\mathscr{F}(f)(y) := (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-ixy} f(x) dx$$

for $f \in L_1(\mathbb{R})$.

Let us now assume that f is either in the Schwartz space of rapidly decreasing functions $\mathscr{S}(\mathbb{R})$ or in the space of square integrable functions $L_2(\mathbb{R})$. In the former case, the integral definition is still valid as $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$. In the latter case, the Fourier transform needs to be extended via a density argument. For both function spaces, an interesting basis exists, given by the Hermite functions. These functions are defined by

$$\psi_k := \left(x - \frac{d}{dx}\right)^k e^{-x^2/2}$$
$$= H_k(x) e^{-x^2/2}$$

for $k = 0, 1, 2, \ldots$ and are orthogonal with respect to the L_2 inner product. The normalization can be computed explicitly, but will not be used in the present paper. Note that $H_k(x)$ is a polynomial of degree k, the so-called Hermite polynomial (see e.g. [18]).

The functions ψ_k diagonalize the Fourier transform. In particular, one has

$$\mathscr{F}(\psi_k)(y) = (-i)^k \psi_k(y), \tag{1}$$

which follows immediately from the operational calculus. As arbitrary functions in $\mathscr{S}(\mathbb{R})$ or $L_2(\mathbb{R})$ can be expanded in terms of the Hermite basis:

$$f = \sum_{k=0}^{+\infty} a_k \psi_k, \qquad a_k \in \mathbb{C},$$

we immediately obtain the action of the Fourier transform on f via

$$\mathscr{F}(f) = \sum_{k=0}^{+\infty} (-i)^k a_k \psi_k \tag{2}$$

For numerical computations, it is in general not efficient to decompose a function into its Hermite components and apply formula (2). FFT methods will work faster.

Nevertheless, it is possible to generalize the Hermite basis to the quaternionic setting, where it allows to prove several theoretical results in a straight-forward way.

3 The two-sided quaternionic Fourier transform

3.1 The algebra of quaternions

The quaternion algebra $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfying

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

Every quaternion can be written explicitly as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

with $q_0, \ldots, q_3$ real numbers. The quaternionic conjugate is given by

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$

and satisfies

$$|q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

A pure quaternion $\mathbf{q}$ is a quaternion with real part $q_0 = 0$. It satisfies

$$\mathbf{q}^2 = -(q_1^2 + q_2^2 + q_3^2).$$

3.2 Definition of the qFT

Let $\mu, \nu \in \mathbb{H}$ be pure quaternions with $\mu^2 = \nu^2 = -1$. Then, following [14], we define the two-sided qFT as

$$\mathscr{F}^{\mu,\nu}(h)(y_1, y_2) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-\mu x_1 y_1} h(x_1, x_2) e^{-\nu x_2 y_2} dx_1 dx_2, \quad h \in L_1(\mathbb{R}^2; \mathbb{H})$$

where we have introduced a different normalization $(2\pi)^{-1}$. The first definition of this two-sided transform, with $\mu = \mathbf{j}$ and $\nu = \mathbf{k}$, was introduced in the Ph.D. thesis [10], see also [11]. In earlier work, a one sided version was given by Ernst et al. [13] and by Delsuc [9], although these authors use an adaptation of the quaternion algebra. The applicability of the qFT to color image processing was first demonstrated in [16] using a discrete version. At that point, the switch was made to general axes μ

and ν instead of **j** and **k**. Indeed, for color image processing there is an arbitrary but preferred axis of the grey-line in the color space, so the transform kernel axes are generally aligned to or perpendicular to this axis. At the same time ([17]), a change was again made to one-sided transforms, mostly driven by the complexity of the resulting operational formula when using the two-sided qFT definition. For a recent review on the use of the qFT in image processing, we refer the reader to [12].

In the sequel we will concentrate on the two-sided qFT, but the techniques developed are also applicable to one-sided versions.

Note that we will often use the shorthand $\underline{x} := (x_1, x_2)$, $\underline{y} := (y_1, y_2)$.

3.3 Basis and inversion theorem

We are now interested in the function spaces $\mathscr{S}(\mathbb{R}^2; \mathbb{H})$ and $L_2(\mathbb{R}^2; \mathbb{H})$, to which the qFT is immediately extended.

Again there exists an interesting basis for these spaces,

$$\left\{ \psi_{k,\ell} \right\}_{k,\ell \in \mathbb{N}}$$

with

$$\psi_{k,\ell} := H_k(x_1) H_\ell(x_2) e^{-x_1^2/2} e^{-x_2^2/2}$$

with H_k, H_ℓ the Hermite polynomial of degree k, resp. ℓ. This basis is the tensor product basis, as also used in [7]. Every function h in $\mathscr{S}(\mathbb{R}^2; \mathbb{H})$ or $L_2(\mathbb{R}^2; \mathbb{H})$ can now be decomposed as

$$h = \sum_{k,\ell=0}^{+\infty} a_{k,\ell} \psi_{k,\ell}, \qquad a_{k,\ell} \in \mathbb{H}.$$

It is important to note that, although the basis functions are real-valued, the coefficients in the expansion are quaternions.

Let us compute the qFT of the basis functions $\psi_{k,\ell}$. We obtain, using (1)

$$\mathscr{F}^{\mu,\nu}(\psi_{k,\ell})(y_1, y_2) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-\mu x_1 y_1} \psi_{k,\ell}(x_1, x_2) e^{-\nu x_2 y_2} dx_1 dx_2$$

$$= (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-\mu x_1 y_1} H_k(x_1) H_\ell(x_2) e^{-x_1^2/2} e^{-x_2^2/2} e^{-\nu x_2 y_2} dx_1 dx_2$$

$$= \left((2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-\mu x_1 y_1} H_k(x_1) e^{-x_1^2/2} dx_1 \right)$$

$$\times \left((2\pi)^{-1/2} \int_{-\infty}^{+\infty} H_\ell(x_2) e^{-x_2^2/2} e^{-\nu x_2 y_2} dx_2 \right)$$

$$= (-\mu)^k H_k(y_1) H_\ell(y_2) e^{-y_1^2/2} e^{-y_2^2/2} (-\nu)^\ell$$

$$= (-\mu)^k \psi_{k,\ell}(y_1, y_2)(-\nu)^\ell.$$

Hence we have obtained the following result:

Theorem 1. *The eigenfunctions of the qFT are given by the functions $\psi_{k,\ell}$ with following (quaternionic) eigenvalues*

$$\mathscr{F}^{\mu,\nu}(\psi_{k,\ell})(y_1,y_2) = (-\mu)^k(-\nu)^\ell \psi_{k,\ell}(y_1,y_2).$$

One has to be careful in applying this result to a function h expanded in the Hermite basis. In that case we obtain

$$\mathscr{F}^{\mu,\nu}(h)(y_1,y_2) = \sum_{k,\ell=0}^{+\infty} \mathscr{F}^{\mu,\nu}\left(a_{k,\ell}\psi_{k,\ell}\right) \tag{3}$$

$$= \sum_{k,\ell=0}^{+\infty} (-\mu)^k a_{k,\ell}(-\nu)^\ell \psi_{k,\ell},$$

as the expansion coefficients $a_{k,\ell}$ do not necessarily commute with μ and ν.

Using this result, we can give an alternative proof of the inversion theorem. The original result was obtained in [10], page 59.

Theorem 2. *The inverse transform for the qFT $\mathscr{F}^{\mu,\nu}$ acting on the function space $\mathscr{S}(\mathbb{R}^2;\mathbb{H})$ is given by $\mathscr{F}^{-\mu,-\nu}$.*

Proof. We need to show that

$$\mathscr{F}^{-\mu,-\nu}\left(\mathscr{F}^{\mu,\nu}(h)\right) = h$$

for all $h \in \mathscr{S}(\mathbb{R}^2;\mathbb{H})$. Now applying formula (3) we obtain

$$\mathscr{F}^{-\mu,-\nu}\left(\mathscr{F}^{\mu,\nu}(h)\right) = \mathscr{F}^{-\mu,-\nu}\left(\sum_{k,\ell=0}^{+\infty} (-\mu)^k a_{k,\ell}(-\nu)^\ell \psi_{k,\ell}\right)$$

$$= \sum_{k,\ell=0}^{+\infty} \mathscr{F}^{-\mu,-\nu}\left((-\mu)^k a_{k,\ell}(-\nu)^\ell \psi_{k,\ell}\right)$$

$$= \sum_{k,\ell=0}^{+\infty} \mu^k(-\mu)^k a_{k,\ell}(-\nu)^\ell \nu^\ell \psi_{k,\ell}$$

$$= \sum_{k,\ell=0}^{+\infty} a_{k,\ell}\psi_{k,\ell}$$

$$= h$$

yielding the desired result. $\square$

4 Translation and convolution for the qFT

Tying in with recent advances in pure mathematics (see e.g. [19] for the case of the so-called Dunkl transform), it is possible to connect a generalized translation operator to a hypercomplex Fourier transform. We will do this for the qFT and prove that for this particular transform, the generalized translation coincides with ordinary translation.

Let us first explain the concept for the ordinary FT. Define the translation over y of a function f by

$$\tau_y f(x) := f(x - y).$$

Then we compute

$$\mathcal{F}(\tau_z f)(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-ixy} f(x - z) dx$$

$$= e^{-izy} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-ixy} f(x) dx$$

$$= e^{-izy} \mathcal{F}(f)(y).$$

This means that the ordinary translation is recovered via

$$\tau_z f(u) = \mathcal{F}^{-1}\left(e^{-izy} \mathcal{F}(f)(y)\right). \tag{4}$$

This formula can immediately be generalized to any invertible hypercomplex Fourier transform. This has already been done for the Clifford-Fourier transform, see [8]. Now we want to apply this formula to the two-sided qFT. To that aim, we define $\widetilde{\tau}_z g$, with g a quaternionic function, via

$$\mathcal{F}^{\mu,\nu}(\widetilde{\tau}_z g)(\underline{y}) := e^{-\mu z_1 y_1} \mathcal{F}^{\mu,\nu}(g)(\underline{y}) e^{-\nu z_2 y_2}.$$

Applying the inverse qFT then yields:

$$\widetilde{\tau}_z g(\underline{u}) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{\mu u_1 y_1} e^{-\mu z_1 y_1} \mathcal{F}^{\mu,\nu}(g)(\underline{y}) e^{-\nu z_2 y_2} e^{\nu u_2 y_2} dy_1 dy_2.$$

Let us expand the right-hand side:

$$\widetilde{\tau}_z g(\underline{u}) = (2\pi)^{-2} \int_{\mathbb{R}^4} e^{\mu y_1 (u_1 - z_1 - x_1)} g(\underline{x}) e^{\nu y_2 (u_2 - z_2 - x_2)} dx_1 dx_2 dy_1 dy_2$$

$$= (2\pi)^{-2} \int_{\mathbb{R}^2} \left(\int_{-\infty}^{+\infty} e^{\mu y_1 (u_1 - z_1 - x_1)} dy_1\right) g(\underline{x}) \left(\int_{-\infty}^{+\infty} e^{\nu y_2 (u_2 - z_2 - x_2)} dy_2\right) dx_1 dx_2$$

$$= \int_{\mathbb{R}^2} \delta(u_1 - z_1 - x_1) g(\underline{x}) \delta(u_2 - z_2 - x_2) dx_1 dx_2$$

$$= g(u_1 - z_1, u_2 - z_2)$$

$$= \tau_z g(\underline{u}).$$

Here we have used the well-known integral representation of the Delta function:

$$\delta(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixy} dy.$$

We hence obtain that the generalized translation for the qFT coincides with the ordinary translation. The reader familiar with the work on generalized translations in e.g. [19, 8] may find this a surprising result. However, note that the qFT was originally designed in [10] precisely in order to satisfy this property.

As a consequence, we find that convolution for the qFT may be defined by

$$f * g(\underline{x}) := \int_{\mathbb{R}^2} f(\underline{y}) \tau_y g(\underline{x}) dy, \qquad \text{with} \quad \tau_y g(\underline{x}) := g(\underline{x} - \underline{y}). \tag{5}$$

This is of course nothing but ordinary convolution.

Note that this convolution product does not satisfy the typical convolution property, i.e.

$$\mathscr{F}^{\mu,\nu}(f * g) \neq \mathscr{F}^{\mu,\nu}(f)\mathscr{F}^{\mu,\nu}(g).$$

Alternatively, based on an idea of Mustard [15], one may define convolution for the qFT as follows

$$f *_q g(x) := \mathscr{F}^{-\mu,-\nu}\left(\mathscr{F}^{\mu,\nu}(f)\mathscr{F}^{\mu,\nu}(g)\right) \tag{6}$$

which clearly satisfies

$$\mathscr{F}^{\mu,\nu}(f *_q g) = \mathscr{F}^{\mu,\nu}(f)\mathscr{F}^{\mu,\nu}(g).$$

The differences and analogies between the two convolution products (5) and (6) can again be studied using the Hermite basis $\psi_{k,\ell}$. This problem will be tackled in a subsequent publication.

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