The Cauchy-Kovalevskaya Extension Theorem in discrete Clifford analysis

Hilde De Ridder

Supervisor(s): Hennie De Schepper, Frank Sommen

I. Introduction

In various engineering disciplines, the need is felt for a discrete higher dimensional function theory, to deal with boundary value problems. In discrete Clifford analysis we develop such a theory for functions of a discrete vector variable, meanwhile generalizing features of holomorphic functions in the complex plane. In this framework, the Cauchy–Kovalevskaya (CK) extension theorem is an important tool to generate bases for our function spaces, which is essential for numerical methods.

II. The CK Extension Theorem

The idea behind the CK–extension [1] is to characterize solutions of partial differential equations by their restriction to a space of one dimension less. For the Cauchy–Riemann operator in the complex plane, the theorem states that a holomorphic function is completely determined by its restriction to the real axis. More precisely, given a real-analytic function $f_0(x)$ in $|x| < a$,

$$F(z) = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} f_0^{(k)}(x)$$

is holomorphic in $|z| < a$ and $F(z)|_{\mathbb{R}} = f_0(x)$. This result has been generalized to higher dimension in the framework of continuous Clifford analysis and was used to generate orthogonal bases of so-called monogenic functions, the higher dimensional analogues of holomorphic functions.

Dept. of Math. Analysis, Ghent University. The author acknowledges support by the grant no. B/10675/02 of Ghent University (BOF). E-mail: hdr@cage.ugent.be.

III. Discrete Clifford analysis

Now, we consider functions defined on the equidistant lattice $\mathbb{Z}_h^m$ with mesh width h and taking values in an appropriate Clifford algebra. Discrete monogenic functions are null solutions of a discrete Dirac operator D factorizing the star Laplacian.

As a consequence of the discrete framework, more involved discrete homogeneous polynomials $\xi_j^k[1]$ of degree k, i.e. eigenfunctions with eigenvalue k of a discrete Euler operator E, will replace the classical homogeneous powers x_j^k. This is also reflected in the structure of the CK–extension theorem, where, given $f(x_2, \ldots, x_m)$ defined on $\mathbb{Z}_h^{m-1}$, a unique discrete monogenic function $\mathrm{CK}[f]$ on $\mathbb{Z}_h^m$, with $\mathrm{CK}[f]|_{x_1=0} = f$, can be constructed. With D' being the restriction of the Dirac operator D to the hyperplane $x_1 = 0$, we explicitly have

$$\mathrm{CK}[f] = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_m)$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} D' f_k$; convergence is ensured by properties of the $\xi_j^k[1]$.

IV. Conclusions

A CK–extension theorem was established in a discrete setting, allowing to generate discrete monogenic functions out of any given discrete function. The next step in the research programme thus will consist in constructing orthogonal bases of discrete monogenic functions, which are essential in applications.

References

[1] R. Cooke, The Cauchy–Kovalevskaya Theorem (preprint, available online: `http://www.cems.uvm.edu/~cooke/ckthm.pdf`).