A Higher Order Space-Time Galerkin Discretization for the Time Domain PMCHWT Equation

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Abstract: Broadband scattering by piecewise homogeneous media can efficiently be modeled using retarded potential boundary integral equations (RPBIEs). These equations can be solved using the marching-on-in-time algorithm, which is usually obtained by applying a collocation-in-time discretization to the RPBIE. In this contribution, however, a Galerkin-in-time discretization scheme is applied to the Poggio-Miller-Chan-Harrington-Wu-Tsai (PMCHWT) equation, describing scattering by penetrable media. Numerical results testify to the superior accuracy of this approach.

Keywords: Time Domain Analysis, Poggio-Miller-Chan-Harrington-Wu-Tsai Equation, Boundary Integral Equations, Accurate Discretization

1. Introduction

Retarded potential boundary integral equations (RPBIEs) efficiently model broadband scattering by piecewise homogeneous objects. Just like the frequency domain boundary integral equations (FD-BIEs), they reduce the problem domain to the surface of the scatterers. In contrast to FD-BIEs, however, they are not restricted to time-harmonic regime solutions, but are able to model transient and broadband phenomena.

Transient scattering by perfect electrically conducting (PEC) objects is modeled by the time domain electric, magnetic and combined field integral equations (TD-EFIE, TD-MFIE, and TD-CFIE, respectively). Transient scattering by penetrable objects on the other hand is described by the time domain Poggio-Miller-Chan-Harrington-Wu-Tsai (TD-PMCHWT) equation (see e.g. [1, 2] and references therein).

All of these RPBIEs can be solved numerically using the marching-on-in-time (MoT) algorithm. The standard approach is to subsequently evaluate the RPBIE at equidistant time steps (collocation-in-time), and to solve the resulting equation for the unknown current at these time steps using a spatial Galerkin method. Previous research into this algorithm has mainly focussed on its stability and computational cost, leading to a multitude of stabilization and acceleration techniques which have extended the range of scattering problems that can be simulated using MoT, see e.g. [3–7].

Alternatively, it is possible to discretize an RPBIE using a Galerkin-in-time procedure. The Galerkin-in-time discretization of the TD-PMCHWT equation has not been used up to now because the most straightforward schemes yield a set of linear systems that is not discretely causal, i.e. is not amenable to marching-on-in-time. In this contribution, a suitable Galerkin-in-time discretization procedure for the TD-PMCHWT equation is introduced, that does yield a discretely causal set of linear systems.

This approach has several advantages over the collocation-in-time scheme. First, it is theoretically better understood, as it fits within the finite element framework. Second, it is possible to extend this scheme to higher order temporal basis functions. Third, since the discretization scheme introduced in this contribution is a Galerkin scheme, the method is expected to be more accurate than its collocation-in-time counterparts. Indeed, similar discretization schemes for the TD-CFIE have been shown to be up to an order of magnitude more accurate and exhibit higher order convergence. [1]

In the next section, the TD-PMCHWT equation and its classic collocation-in-time discretization are revisited. In section 3, the new Galerkin-in-time discretization is presented. Finally, numerical results that testify to the superior accuracy of the new scheme are presented in section 4.

2. The Time Domain PMCHWT Equation and its Collocation-in-Time Discretization

Consider a closed domain Ω, with boundary Γ and exterior normal vector $\hat{\boldsymbol{n}}$. The interior of Γ is filled with a homogeneous non-dispersive medium with permittivity ϵ_1, permeability μ_1 and characteristic impedance $\eta_1 = \sqrt{\mu_1/\epsilon_1}$. The exterior of Γ is filled with a homogeneous non-dispersive medium with permittivity ϵ_0, permeability μ_0 and characteristic impedance $\eta_0 = \sqrt{\mu_0/\epsilon_0}$. For $t > 0$, Ω is illuminated by an incident electromagnetic field $\boldsymbol{e}^i, \boldsymbol{h}^i$. The tangential components of the total electromagnetic fields on Γ, $\boldsymbol{j} = \hat{\boldsymbol{n}} \times \boldsymbol{h}$ and $\boldsymbol{m} = \boldsymbol{e} \times \hat{\boldsymbol{n}}$, then obey the TD-PMCHWT equation:

$$\begin{pmatrix} \eta_0 \mathcal{T}_0 + \eta_1 \mathcal{T}_1 & -\mathcal{K}_0 - \mathcal{K}_1 \\ \mathcal{K}_0 + \mathcal{K}_1 & \frac{1}{\eta_0}\mathcal{T}_0 + \frac{1}{\eta_1}\mathcal{T}_1 \end{pmatrix} \begin{pmatrix} \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} + \begin{pmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{e}^i \\ \hat{\boldsymbol{n}} \times \boldsymbol{h}^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$\forall t > 0$, where $\mathcal{T}_i$ and $\mathcal{K}_i$ are the EFIE and MFIE operators in medium i, respectively, as defined in [6]. Applying the EFIE operator involves a temporal integration that complicates the implementation of the numerical solution method. For this reason, the temporal derivative of the TD-PMCHWT equation is often preferred over (1).

The TD-PMCHWT equation (1) (or its temporal derivative) can be discretized by the widely used collocation-in-time / Galerkin-in-space method: first, $\boldsymbol{j}$ and $\boldsymbol{m}$ are expanded in a set of N_T temporal basis functions $T_i(t) = T(t - i\Delta t)$, usually chosen to be Lagrange interpolants [2], and a set of N_S spatial basis functions $\boldsymbol{f}_m(\boldsymbol{r})$, usually chosen to be the Rao-Wilton-Glisson (RWG) functions.

These expansions are inserted into (1), which is then evaluated at subsequent time steps $t = j\Delta t$ (collocation-in-time), and spatially tested using the rotated RWG functions $\hat{\boldsymbol{n}} \times \boldsymbol{f}_n$. This results in a matrix equation of the following form:

$$-\begin{pmatrix} \mathsf{Z}_{mj}^{(0)} & \mathsf{Z}_{mm}^{(0)} \\ \mathsf{Z}_{jj}^{(0)} & \mathsf{Z}_{jm}^{(0)} \end{pmatrix} \begin{pmatrix} \mathsf{j}^{(j)} \\ \mathsf{m}^{(j)} \end{pmatrix} = \sum_{i=0}^{j-1} \begin{pmatrix} \mathsf{Z}_{mj}^{(j-i)} & \mathsf{Z}_{mm}^{(j-i)} \\ \mathsf{Z}_{jj}^{(j-i)} & \mathsf{Z}_{jm}^{(j-i)} \end{pmatrix} \begin{pmatrix} \mathsf{j}^{(i)} \\ \mathsf{m}^{(i)} \end{pmatrix} + \begin{pmatrix} \mathsf{e}_j^i \\ \mathsf{h}_j^i \end{pmatrix}, \tag{2}$$

for $j = 1, 2, ..., N_T$, where $\mathsf{Z}_{mj}^{(j)}$, $\mathsf{Z}_{mm}^{(j)}$, $\mathsf{Z}_{jj}^{(j)}$ and $\mathsf{Z}_{jm}^{(j)}$ are $N_S \times N_S$ matrices. $\mathsf{j}^{(j)}$ and $\mathsf{m}^{(j)}$ contain the RWG expansion coefficients of $\boldsymbol{j}(\boldsymbol{r}, j\Delta t)$ and $\boldsymbol{m}(\boldsymbol{r}, j\Delta t)$, respectively. Solving (2) for $\mathsf{j}^{(j)}$ and $\mathsf{m}^{(j)}$ for successive time steps j is called the MoT algorithm.

For the undifferentiated TD-PMCHWT equation, the first order Lagrange interpolants (also known as hat functions) yield usable results. While higher-order Lagrange interpolants are expected to yield more accurate results, it has been reported that they are detrimental to the stability of the MoT scheme. [2]

For the time-differentiated TD-PMCHWT, the first order Lagrange interpolants are not suitable because they are unable to represent the second temporal derivative of the currents. Higher-order Lagrange interpolants yield increasingly more accurate solutions, and do not appear to deteriorate the stability of the scheme.

3. The Temporal Galerkin Method for the Time-Differentiated PMCHWT Equation

Alternatively to the collocation-in-time scheme, in which the TD-PMCHWT equation is evaluated at subsequent time steps $t = k\Delta t$, it is also possible to discretize the equation using a Galerkin-in-time method. For most choices of basis and testing functions, this does not yield a discretely causal set of linear systems. If the basis and testing functions are carefully tailored, however, discrete causality can be obtained and a numerical solution can be constructed using the MoT algorithm. A lowest order Galerkin-in-time scheme has previously been applied to acoustic RPBIEs [8] and the TD-EFIE [9]. In [1], a higher order Galerkin-in-time scheme was applied to the TD-CFIE, and it was shown that this leads to significantly more accurate solutions. We now introduce a first and a higher order Galerkin-in-time discretization for the TD-PMCHWT equation.

In a general p-th order Galerkin scheme, the time axis is divided in N_T segments of equal size. To each of the nodes in this segmentation, p locally polynomial, globally continuous temporal basis functions $T_i^{(\alpha)}(t) = T^{(\alpha)}(t - i\Delta t)$, $\alpha = 1, 2, ..., p$, are associated. Thus, there are $2pN_TN_S$ degrees of freedom. In order to obtain a set of $2pN_TN_S$ linear equations, p locally polynomial but not globally continuous temporal testing functions $S_i^{(\alpha)}(t) = S^{(\alpha)}(t - i\Delta t)$, $\alpha = 1, 2, ..., p$, are associated to each segment of the time axis. Next, the equation is spatially tested with the rotated RWG functions. The temporal basis functions and testing functions are chosen such that the resulting set of linear equations is discretely causal and thus amenable to marching-on-in-time. The resulting MoT system is of the same form as (2), with $\mathbf{Z}_{mj}^{(j)}$, $\mathbf{Z}_{mm}^{(j)}$, $\mathbf{Z}_{jj}^{(j)}$ and $\mathbf{Z}_{jm}^{(j)}$ being $pN_S \times pN_S$ matrices.

Explicitly, the temporal basis and testing functions of orders 1 and 2 are those introduced in [1] and included here for completeness: for order 1:

$$T^{(1)}(t) = \begin{cases} 1 + t/\Delta t & -\Delta t \le t < 0 \\ 1 - t/\Delta t & 0 \le t \le \Delta t \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

$$S^{(1)}(t) = \begin{cases} 1 & -\Delta t \le t < 0 \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

and for order 2:

$$T^{(1)}(t) = \begin{cases} -4\frac{t}{\Delta t}\left(\frac{t}{\Delta t} + 1\right) & -\Delta t < t < 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

$$T^{(2)}(t) = \begin{cases} 2\left(\frac{t}{\Delta t} + 1\right)\left(\frac{t}{\Delta t} + \frac{1}{2}\right) & -\Delta t < t < 0 \\ 2\left(\frac{t}{\Delta t} - 1\right)\left(\frac{t}{\Delta t} - \frac{1}{2}\right) & 0 \le t < \Delta t \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

$$S^{(1)}(t) = \begin{cases} -\frac{t}{\Delta t} & -\Delta t < t < 0 \\ 0 & \text{otherwise} \end{cases}, \tag{7}$$

$$S^{(2)}(t) = \begin{cases} \frac{t}{\Delta t} + 1 & -\Delta t < t < 0 \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

4. Numerical Examples

Consider a sphere with radius $1\ m$, permittivity $2\epsilon_0$ and permeability μ_0. It is imbedded in vacuum, and illuminated by a gaussian pulse:

$$e^i(\boldsymbol{r},t) = \frac{4A}{w\sqrt{\pi}}\hat{\boldsymbol{p}}\exp\left(-\left(\frac{4}{w}\left(c(t-t_0)-\hat{\boldsymbol{k}}\cdot\boldsymbol{r}\right)\right)^2\right), \tag{9}$$

where the pulse's width $w = 16.7$ ns, the amplitude $A = 1$ V/m, the polarization $\hat{\boldsymbol{p}} = (1,0,0)$ and the propagation direction $\hat{\boldsymbol{k}} = (0,0,1)$. The sphere is discretized using an equilateral triangle mesh, on which $N_S = 999$ RWG functions are defined.

The accuracy of the following discretization schemes for the time-differentiated TD-PMCHWT equation is investigated:

(a) collocation-in-time using second degree Lagrange interpolants, $\Delta t = 0.63$ ns;
(b) collocation-in-time using third degree Lagrange interpolants, $\Delta t = 0.63$ ns;
(c) first order Galerkin-in-time, $\Delta t = 0.63$ ns;
(d) second order Galerkin-in-time, $\Delta t = 1.26$ ns.

By choosing the time step in (d) twice as large as in (a), (b) and (c), simulations with the same number of degrees of freedom per unit of time are compared. The accuracy is assessed by Fourier transforming the time domain currents, and comparing them to the frequency domain Mie series, which is the analytical solution to this scattering problem. Specifically, the relative $H_{\text{div}}^{-1/2}$ norm (which is equivalent to the relative energy norm) of the error is estimated by using the method described in [1].

In Fig. 1, the relative errors are plotted and compared to those achieved by frequency domain PMCHWT simulations ("FD"). The Galerkin-in-time schemes are significantly more accurate than the collocation-in-time schemes. Furthermore, the second order Galerkin-in-time scheme achieves the accuracy of the frequency-domain simulation (in which the harmonic time dependence is treated analytically, thus suggesting that the remaining error is completely due to the spatial discretization) over a broad frequency range, whereas the other schemes do not.

5. Conclusions

In this contribution, a higher-order Galerkin-in-time discretization scheme for the TD-PMCHWT equation has been introduced. By employing suitable temporal basis and testing functions, a discretely causal system is obtained, which is amenable to marching-on-in-time. Numerical examples show that this Galerkin-in-time discretization scheme yields significantly more accurate solutions than the classic collocation-in-time scheme.

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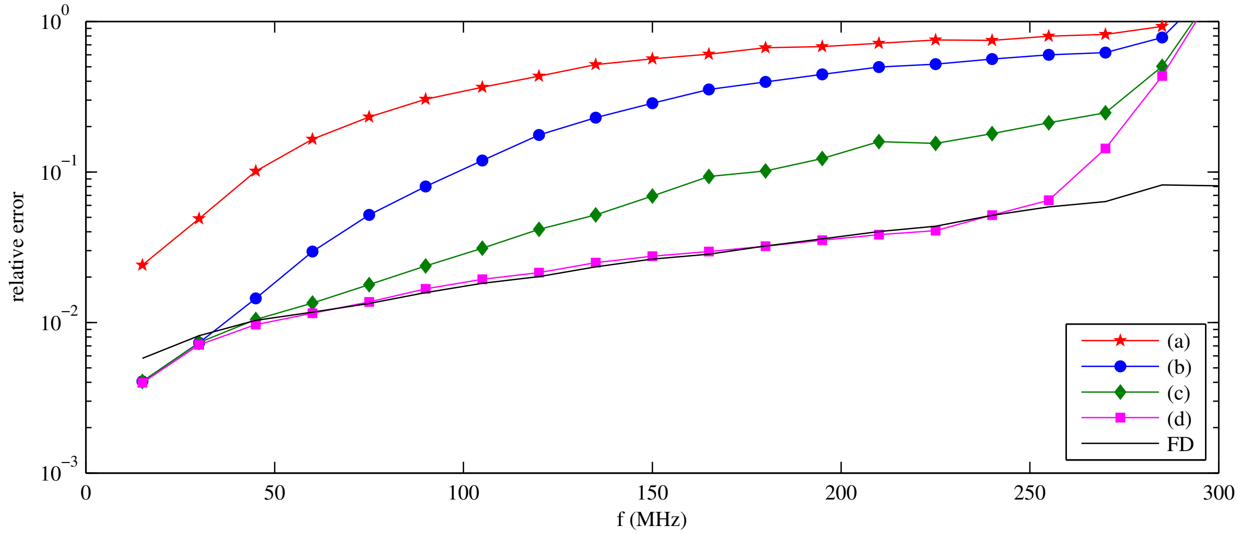

Figure 1. The relative $H_{\text{div}}^{-1/2}$ norm of the error for (a): collocation-in-time using second degree Lagrange interpolants, (b): collocation-in-time using third degree Lagrange interpolants, (c): first order Galerkin-in-time, (d): second order Galerkin-in-time, and FD: frequency domain PMCHWT.

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