Conditioning and expressing indifference with choice functions

Arthur Van Camp, Gert de Cooman and Erik Quaeghebeur

Ghent University, SYSTeMS   Centrum Wiskunde & Informatica, Amsterdam

Arthur.VanCamp@UGent.be, Gert.deCooman@UGent.be,
Erik.Quaeghebeur@{UGent.be,cwi.nl}

We want to broaden

probability theory

in order to deal with imprecision and indecision.

We want to broaden

probability theory

in order to deal with imprecision and indecision.

Sets of desirable gambles are very successful imprecise models.

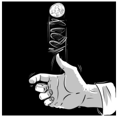
Working with them is simple and elegant.

Sets of desirable gambles allow for conservative inference.

They can be ordered according to an
“is not more conservative than”
relation.

We want to use identical ideas for choice functions.

Motivating example

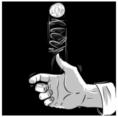


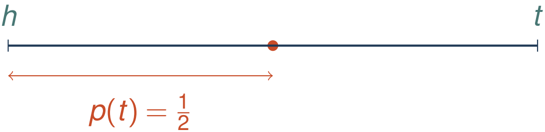
h t

$\mathscr{X} = \{h, t\}$

Motivating example

fair coin

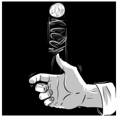




$\mathscr{X} = \{h, t\}$

Motivating example

coin with identical sides of unknown type

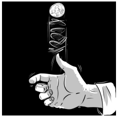


$\mathscr{X} = \{h, t\}$

$$p_h(x) = \begin{cases} 1 & \text{if } x = h \\ 0 & \text{if } x = t \end{cases}$$

$$p_t(x) = \begin{cases} 0 & \text{if } x = h \\ 1 & \text{if } x = t \end{cases}$$

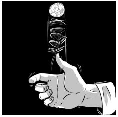
Motivating example



$\mathscr{X} = \{h, t\}$

Such an assessment cannot be modelled using sets of desirable gambles!

Motivating example



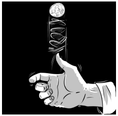
$\mathscr{X} = \{h, t\}$

Such an assessment cannot be modelled using sets of desirable gambles!

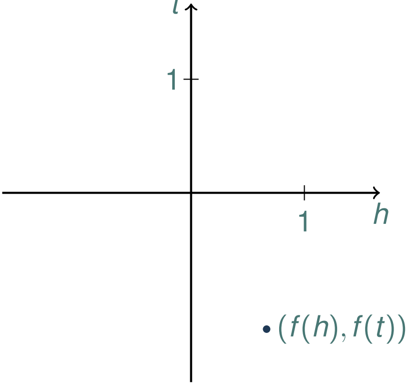
Sets of desirable gambles

Gambles

The random variable X takes values x in the possibility space $\mathscr{X}$.
A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$.
We collect them in $\mathscr{L}$ (or $\mathscr{L}(\mathscr{X})$).



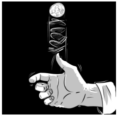
$\mathscr{X} = \{h, t\}$



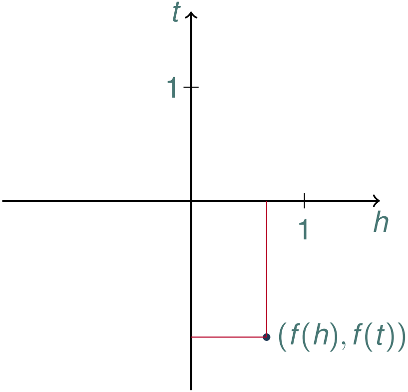
A set of desirable gambles $\mathscr{D}$ is a set of gambles that a subject strictly prefers to zero.

Gambles

The random variable X takes values x in the possibility space $\mathscr{X}$.
A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$.
We collect them in $\mathscr{L}$ (or $\mathscr{L}(\mathscr{X})$).



$\mathscr{X} = \{h, t\}$

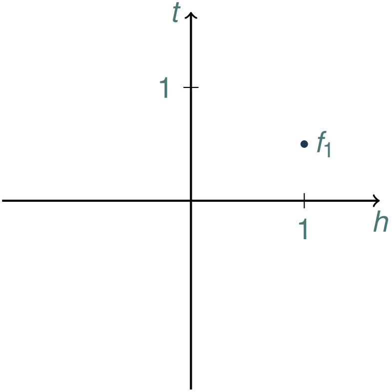


A set of desirable gambles $\mathscr{D}$ is a set of gambles that a subject strictly prefers to zero.

Coherence for a set of desirable gambles

An assessment can be given as follows:

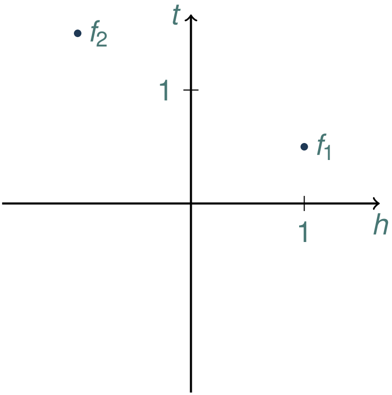
	h	t
f_1	1	$^1/_2$



Coherence for a set of desirable gambles

An assessment can be given as follows:

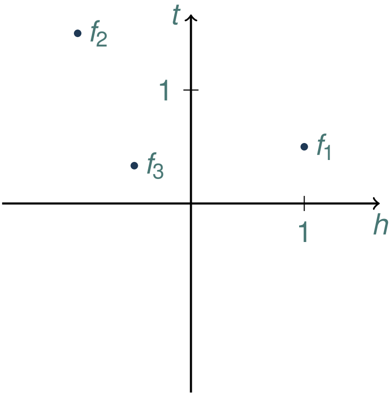
	h	t
f_1	1	$1/2$
f_2	-1	$3/2$



Coherence for a set of desirable gambles

An assessment can be given as follows:

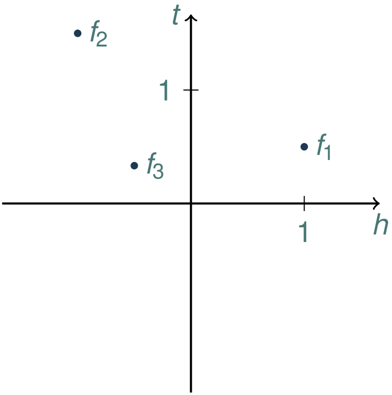
	h	t
f_1	1	$1/2$
f_2	-1	$3/2$
f_3	$-1/2$	$1/3$



Coherence for a set of desirable gambles

An assessment can be given as follows:

	h	t
f_1	1	$1/2$
f_2	-1	$3/2$
f_3	$-1/2$	$1/3$



A set of desirable gambles $\mathscr{D}$ is called coherent if

$D1$. if $f > 0$ then $f \in \mathscr{D}$,

$D2$. if $f \leq 0$ then $f \notin \mathscr{D}$,

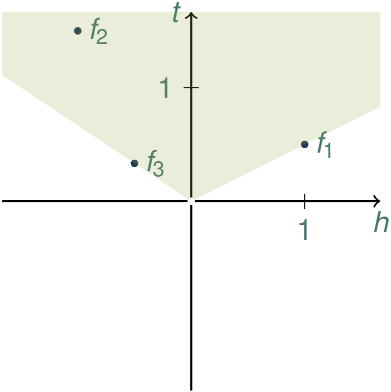
$D3$. if $f, g \in \mathscr{D}$ then $f + g \in \mathscr{D}$,

$D4$. if $f \in \mathscr{D}$ and $\lambda \in \mathbb{R}_{>0}$ then $\lambda f \in \mathscr{D}$.

Coherence for a set of desirable gambles

An assessment can be given as follows:

	h	t
f_1	1	$1/2$
f_2	-1	$3/2$
f_3	$-1/2$	$1/3$



A set of desirable gambles $\mathscr{D}$ is called coherent if

$D1$. if $f > 0$ then $f \in \mathscr{D}$,

$D2$. if $f \leq 0$ then $f \notin \mathscr{D}$,

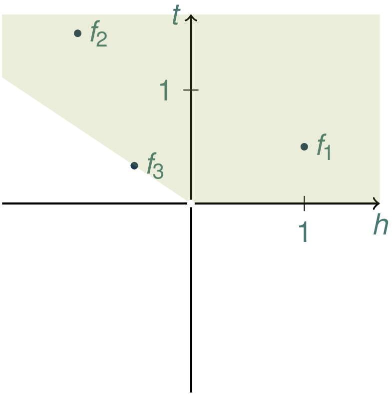
$D3$. if $f, g \in \mathscr{D}$ then $f + g \in \mathscr{D}$,

$D4$. if $f \in \mathscr{D}$ and $\lambda \in \mathbb{R}_{>0}$ then $\lambda f \in \mathscr{D}$.

Coherence for a set of desirable gambles

An assessment can be given as follows:

	h	t
f_1	1	$1/2$
f_2	-1	$3/2$
f_3	$-1/2$	$1/3$



A set of desirable gambles $\mathscr{D}$ is called coherent if

$D1$. if $f > 0$ then $f \in \mathscr{D}$,

$D2$. if $f \leq 0$ then $f \notin \mathscr{D}$,

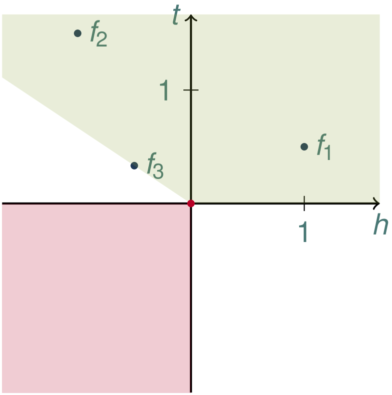
$D3$. if $f, g \in \mathscr{D}$ then $f + g \in \mathscr{D}$,

$D4$. if $f \in \mathscr{D}$ and $\lambda \in \mathbb{R}_{>0}$ then $\lambda f \in \mathscr{D}$.

Coherence for a set of desirable gambles

An assessment can be given as follows:

	h	t
f_1	1	$1/2$
f_2	-1	$3/2$
f_3	$-1/2$	$1/3$



A set of desirable gambles $\mathscr{D}$ is called coherent if

$D1$. if $f > 0$ then $f \in \mathscr{D}$,

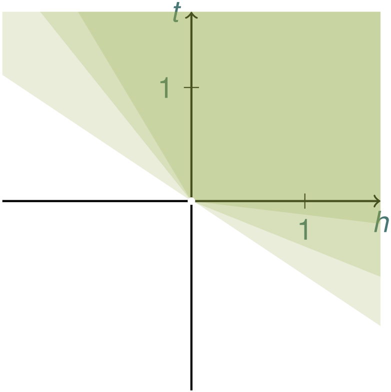
$D2$. if $f \leq 0$ then $f \notin \mathscr{D}$,

$D3$. if $f, g \in \mathscr{D}$ then $f + g \in \mathscr{D}$,

$D4$. if $f \in \mathscr{D}$ and $\lambda \in \mathbb{R}_{>0}$ then $\lambda f \in \mathscr{D}$.

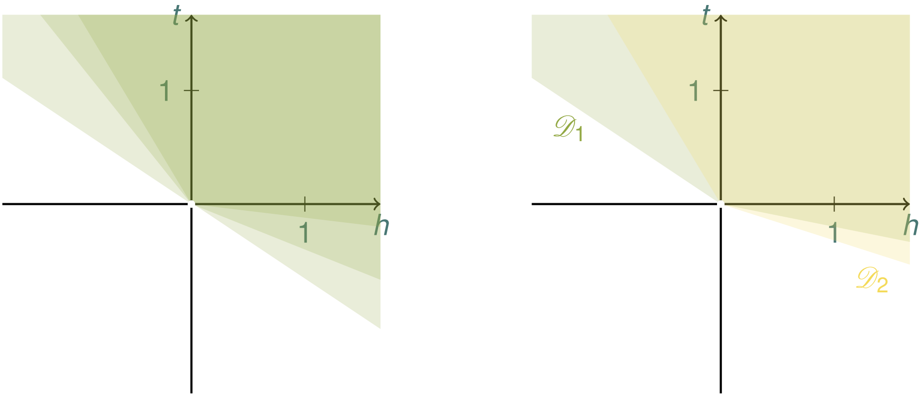
Natural extension

We call $\mathscr{D}_1$ not more informative than $\mathscr{D}_2$ if $\mathscr{D}_1 \subseteq \mathscr{D}_2$.



Natural extension

We call $\mathscr{D}_1$ not more informative than $\mathscr{D}_2$ if $\mathscr{D}_1 \subseteq \mathscr{D}_2$.



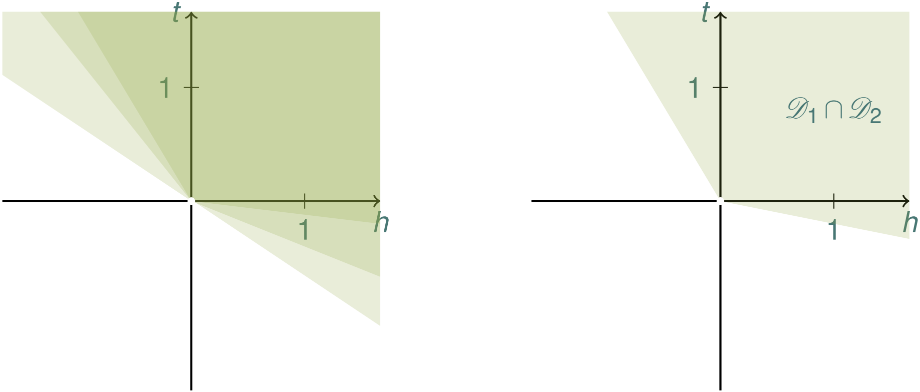
Given a collection $\mathbf{D} = \{\mathscr{D}_1, \mathscr{D}_2, \ldots\}$ of coherent sets of desirable gambles, then the infimum (under the relation $\subseteq$)

$$\inf \mathbf{D} = \bigcap \mathbf{D}$$

is a coherent set of desirable gambles.

Natural extension

We call $\mathscr{D}_1$ not more informative than $\mathscr{D}_2$ if $\mathscr{D}_1 \subseteq \mathscr{D}_2$.



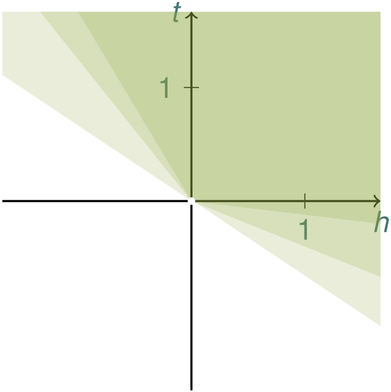
Given a collection $\mathbf{D} = \{\mathscr{D}_1, \mathscr{D}_2, \ldots\}$ of coherent sets of desirable gambles, then the infimum (under the relation $\subseteq$)

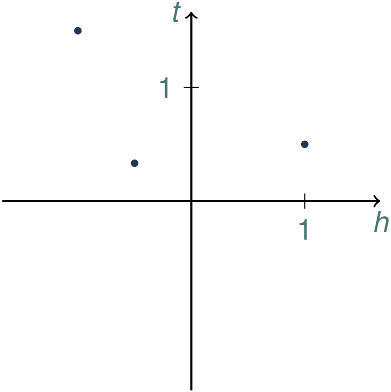
$$\inf \mathbf{D} = \bigcap \mathbf{D}$$

is a coherent set of desirable gambles.

Natural extension

We call $\mathscr{D}_1$ not more informative than $\mathscr{D}_2$ if $\mathscr{D}_1 \subseteq \mathscr{D}_2$.



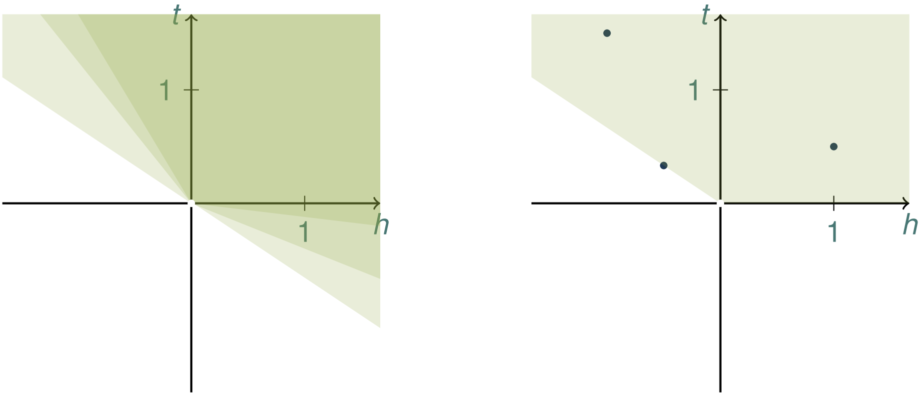


Given a desirability assessment $\mathscr{A}$, then define its natural extension as

$$\bigcap\{\mathscr{D} \text{ coherent} \colon \mathscr{A} \subseteq \mathscr{D}\}.$$

Natural extension

We call $\mathscr{D}_1$ not more informative than $\mathscr{D}_2$ if $\mathscr{D}_1 \subseteq \mathscr{D}_2$.



Given a desirability assessment $\mathscr{A}$, then define its natural extension as

$$\bigcap \{\mathscr{D} \text{ coherent} \colon \mathscr{A} \subseteq \mathscr{D}\}.$$

Alternative representation

With a coherent set of desirable gambles $\mathscr{D}$ there corresponds a binary relation (called preference relation) $\prec_{\mathscr{D}}$ on the set of gambles:

$$f \prec_{\mathscr{D}} g \Leftrightarrow g - f \in \mathscr{D}.$$

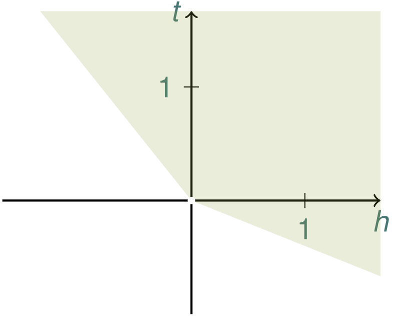
$\prec_{\mathscr{D}}$ is irreflexive, transitive, mix-independent and monotone.

Alternative representation

With a coherent set of desirable gambles $\mathscr{D}$ there corresponds a binary relation (called preference relation) $\prec_{\mathscr{D}}$ on the set of gambles:

$$f \prec_{\mathscr{D}} g \Leftrightarrow g - f \in \mathscr{D}.$$

$\prec_{\mathscr{D}}$ is irreflexive, transitive, mix-independent and monotone.

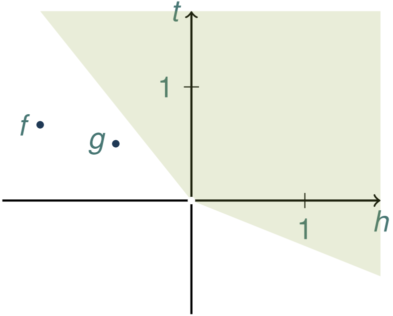


Alternative representation

With a coherent set of desirable gambles $\mathscr{D}$ there corresponds a binary relation (called preference relation) $\prec_{\mathscr{D}}$ on the set of gambles:

$$f \prec_{\mathscr{D}} g \Leftrightarrow g - f \in \mathscr{D}.$$

$\prec_{\mathscr{D}}$ is irreflexive, transitive, mix-independent and monotone.

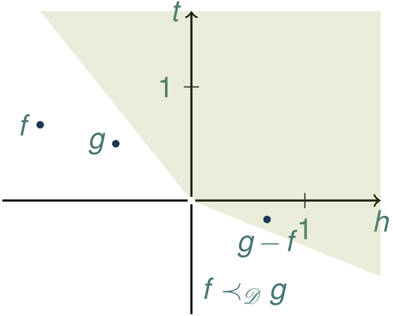


Alternative representation

With a coherent set of desirable gambles $\mathscr{D}$ there corresponds a binary relation (called preference relation) $\prec_{\mathscr{D}}$ on the set of gambles:

$$f \prec_{\mathscr{D}} g \Leftrightarrow g - f \in \mathscr{D}.$$

$\prec_{\mathscr{D}}$ is irreflexive, transitive, mix-independent and monotone.

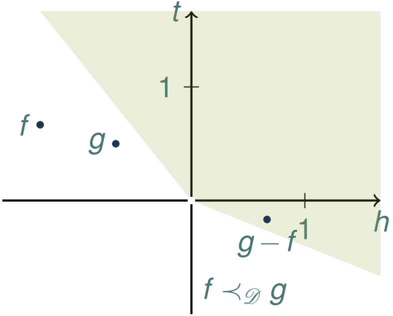


Alternative representation

With a coherent set of desirable gambles $\mathscr{D}$ there corresponds a binary relation (called preference relation) $\prec_{\mathscr{D}}$ on the set of gambles:

$$f \prec_{\mathscr{D}} g \Leftrightarrow g - f \in \mathscr{D}.$$

$\prec_{\mathscr{D}}$ is irreflexive, transitive, mix-independent and monotone.



Conversely, from a coherent preference relation $\prec$ on the gambles, define

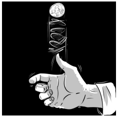
$$\mathscr{D}_{\prec} := \{f : 0 \prec f\}.$$

We can use these representations interchangeably:

$$\mathscr{D}_{\prec_{\mathscr{D}}} = \mathscr{D}.$$

Example: coin flip

Example



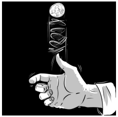
$\mathscr{X} = \{h, t\}$

$$p_h(x) = \begin{cases} 1 & \text{if } x = h \\ 0 & \text{if } x = t \end{cases} \qquad p_t(x) = \begin{cases} 0 & \text{if } x = h \\ 1 & \text{if } x = t \end{cases}$$

Define $f \prec_h g$ if $E_{p_h}(f) < E_{p_h}(g)$ (equivalently $f(h) < g(h)$),

and $f \prec_t g$ if $E_{p_t}(f) < E_{p_t}(g)$ (equivalently $f(t) < g(t)$).

Example

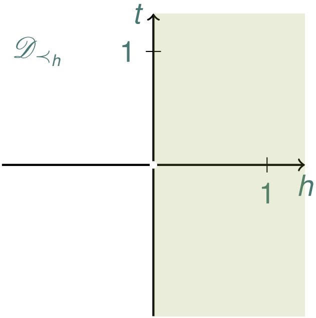


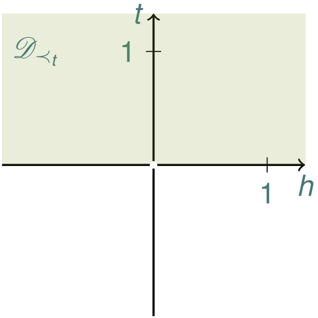
$\mathscr{X} = \{h, t\}$

$$p_h(x) = \begin{cases} 1 & \text{if } x = h \\ 0 & \text{if } x = t \end{cases} \qquad p_t(x) = \begin{cases} 0 & \text{if } x = h \\ 1 & \text{if } x = t \end{cases}$$

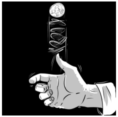
Define $f \prec_h g$ if $E_{p_h}(f) < E_{p_h}(g)$ (equivalently $f(h) < g(h)$),

and $f \prec_t g$ if $E_{p_t}(f) < E_{p_t}(g)$ (equivalently $f(t) < g(t)$).





Example

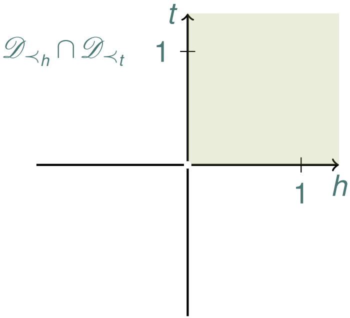


$\mathscr{X} = \{h, t\}$

$$p_h(x) = \begin{cases} 1 & \text{if } x = h \\ 0 & \text{if } x = t \end{cases} \qquad p_t(x) = \begin{cases} 0 & \text{if } x = h \\ 1 & \text{if } x = t \end{cases}$$

Define $f \prec_h g$ if $E_{p_h}(f) < E_{p_h}(g)$ (equivalently $f(h) < g(h)$),

and $f \prec_t g$ if $E_{p_t}(f) < E_{p_t}(g)$ (equivalently $f(t) < g(t)$).



No distinction between a “coin with identical sides” and a “vacuous coin”!

Choice functions

Choice functions

We call $\mathcal{Q}(\mathscr{L})$ the collection of all non-empty finite subsets of $\mathscr{L}$.

Choice functions

We call $\mathscr{Q}(\mathscr{L})$ the collection of all non-empty finite subsets of $\mathscr{L}$.

A choice function C is a map

$$C\colon \mathscr{Q}(\mathscr{L}) \to \mathscr{Q}(\mathscr{L}) \cup \{\emptyset\} \colon O \mapsto C(O) \text{ such that } C(O) \subseteq O.$$

As an equivalent representation, we define $R(O) \coloneqq O \setminus C(O)$ as the rejection function.

Choice relations

Another equivalent representation is the choice relation $<$ on $\mathscr{Q}(\mathscr{L})$:

$$O_1 <_R O_2 \Leftrightarrow O_1 \subseteq R(O_1 \cup O_2).$$

If R is coherent, the choice relation $<_R$ is a strict partial order.

Choice relations

Another equivalent representation is the choice relation $<$ on $\mathcal{Q}(\mathcal{L})$:

$$O_1 <_R O_2 \Leftrightarrow O_1 \subseteq R(O_1 \cup O_2).$$

If R is coherent, the choice relation $<_R$ is a strict partial order.

Given a choice relation $<$ we define the corresponding rejection function as

$$R_<(O) = \bigcup\{O' \subseteq O \colon O' < O\},$$

and we can use these representations interchangeably:

$$R_{<_R} = R.$$

Coherence for choice functions

A choice function C is called coherent if

1. $\emptyset \neq C(O)$,
2. if $g < f$ then $\{g\} < \{f\}$ (or equivalently, $g \notin C(\{f, g\})$),
3. 3.1 if $O_1 \subseteq R(O_2)$ and $O_2 \subseteq O_3$ then $O_1 \subseteq R(O_3)$,
   3.2 if $O_1 \subseteq R(O_2)$ and $O_3 \subseteq O_1$ then $O_1 \setminus O_3 \subseteq R(O_2 \setminus O_3)$,
4. 4.1 if $O_1 \subseteq R(O_2)$ then $O_1 + \{f\} := \{g + f \colon g \in O_1\} \subseteq R(O_2 + \{f\})$,
   4.2 if $O_1 \subseteq R(O_2)$ then $\lambda O_1 := \{\lambda f \colon f \in O_1\} \subseteq R(\lambda O_2)$,
5. if $f_1 \leq f_2$ and for all $g \in O_1 \setminus \{f_1, f_2\}$:
   5.1 if $f_2 \in O_1$ and $g \in R(O_1 \cup \{f_1\})$ then $g \in R(O_1)$,
   5.2 if $f_1 \in O_1$ and $g \in R(O_1)$ then $g \in R(\{f_2\} \cup O_1 \setminus \{f_1\})$,

for all O_1, O_2, $O_3 \in \mathscr{Q}(\mathscr{L})$, f, f_1, f_2, $g \in \mathscr{L}$ and $\lambda \in \mathbb{R}_{>0}$.

“not more informative” relation

Given two coherent choice functions C_1 and C_2, we call C_1 “not more informative than” C_2 if

$$C_1(O) \supseteq C_2(O) \text{ for all } O \in \mathcal{Q}(\mathscr{L}).$$

“not more informative” relation

Given two coherent choice functions C_1 and C_2, we call C_1 “not more informative than” C_2 if

$$C_1(O) \supseteq C_2(O) \text{ for all } O \in \mathscr{Q}(\mathscr{L}).$$

Given a collection $\mathbf{C} = \{C_1, C_2, \dots\}$ of coherent choice functions, its infimum (under the “not more informative than” relation)

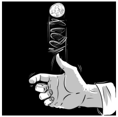
$$\inf \mathbf{C}(O) = \bigcup_{C \in \mathbf{C}} C(O) \text{ for all } O \in \mathscr{Q}(\mathscr{L})$$

is a coherent choice function as well.

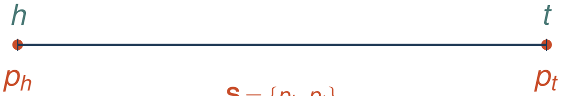
Example

Coin flip

The two sides of the coin are identical of unknown type.



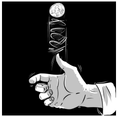
$\mathscr{X} = \{h, t\}$



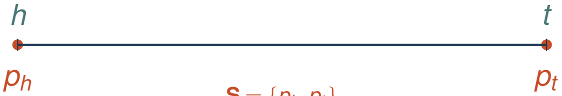
$\mathbf{S} = \{p_h, p_t\}$

Coin flip

The two sides of the coin are identical of unknown type.



$\mathscr{X} = \{h, t\}$

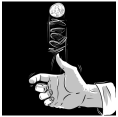


$\mathbf{S} = \{p_h, p_t\}$

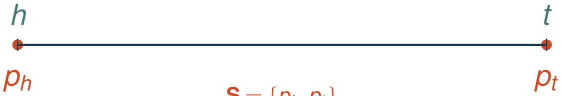
Define $C_{\mathbf{S}}(O)$ as those $f \in O$ for which there is a $p \in \mathbf{S}$ such that f maximises expected utility under p and f is undominated in O.

Coin flip

The two sides of the coin are identical of unknown type.



$\mathscr{X} = \{h, t\}$



$\mathbf{S} = \{p_h, p_t\}$

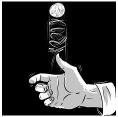
Define $C_{\mathbf{S}}(O)$ as those $f \in O$ for which

either $f(h) \geq g(h)$ for every $g \in O$ or $f(t) \geq g(t)$ for every $g \in O$

and f is undominated in O.

Coin flip

The two sides of the coin are identical of unknown type.



$\mathscr{X} = \{h, t\}$

h —— t

p_h —— p_t

$\mathbf{S} = \{p_h, p_t\}$

Define $C_{\mathbf{S}}(O)$ as those $f \in O$ for which

either $f(h) \geq g(h)$ for every $g \in O$ or $f(t) \geq g(t)$ for every $g \in O$

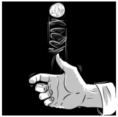
and f is undominated in O.

h —— t

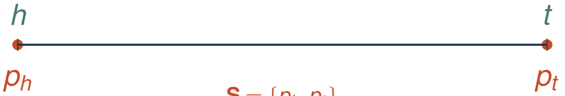
$\mathbf{S}'$

Coin flip

The two sides of the coin are identical of unknown type.



$\mathscr{X} = \{h, t\}$



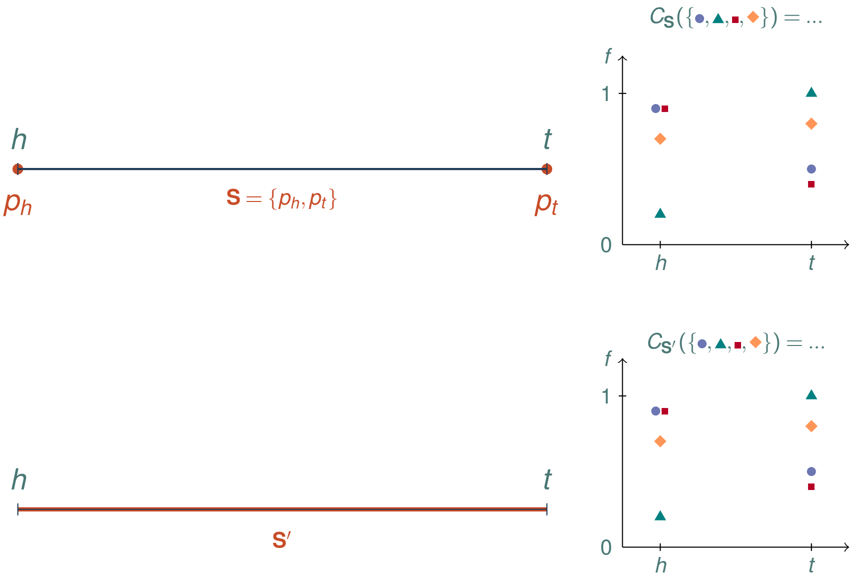
$\mathbf{S} = \{p_h, p_t\}$

Define $C_{\mathbf{S}}(O)$ as those $f \in O$ for which
either $f(h) \geq g(h)$ for every $g \in O$ or $f(t) \geq g(t)$ for every $g \in O$
and f is undominated in O.

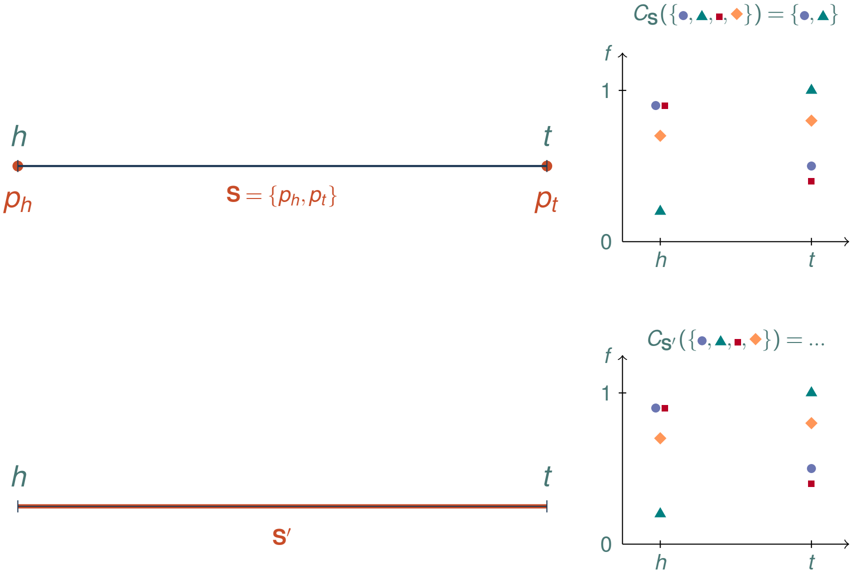
$\mathbf{S}'$

Define $C_{\mathbf{S}'}(O)$ as those $f \in O$ for which there is a $p \in \mathbf{S}'$ such that
f maximises expected utility under p and f is undominated in O.

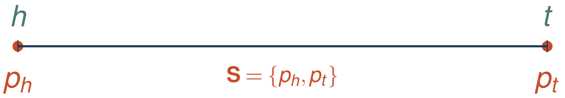
$C_{\mathbf{s}}$ and $C_{\mathbf{s}'}$ are different

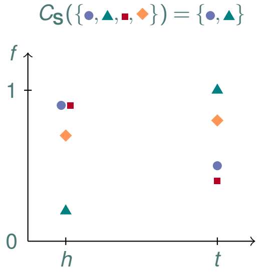


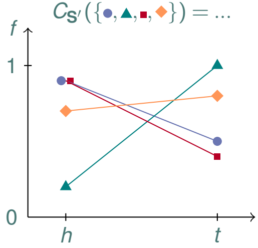
$C_{\mathbf{S}}$ and $C_{\mathbf{S}'}$ are different



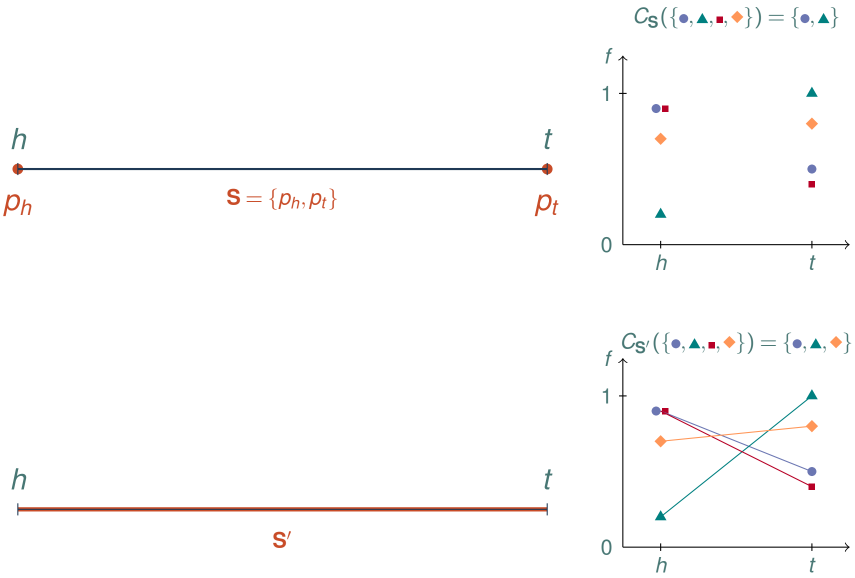
$C_{\mathbf{S}}$ and $C_{\mathbf{S}'}$ are different







$C_{\mathbf{S}}$ and $C_{\mathbf{S}'}$ are different



Connection between desirability and choice functions

Connection between $\mathscr{D}$ and C

From C to $\mathscr{D}$.

Let C be a coherent choice function.
Look at the behaviour of the choice relation $<_C$ on singletons.
We define the set of desirable gambles $\mathscr{D}_C$ as

$$\begin{aligned}\mathscr{D}_C &= \{f - g \colon \{g\} <_C \{f\}\}\\ &= \{f - g \colon \{f\} = C(\{f, g\}) \quad \text{and} \quad f \neq g\}.\end{aligned}$$

If C is coherent, then $\mathscr{D}_C$ is coherent as well.

Connection between $\mathscr{D}$ and C

From $\mathscr{D}$ to C.

Let $\mathscr{D}$ be a coherent set of desirable gambles.
Define the compatible choice functions $\mathbf{C}_{\mathscr{D}}$ as those choice functions that have the same binary relation as $\mathscr{D}$:

$$\mathbf{C}_{\mathscr{D}} = \{C\colon (\forall f, g \in \mathscr{L})\{f\} <_C \{g\} \Leftrightarrow g - f \in \mathscr{D}\}.$$

Connection between $\mathscr{D}$ and C

From $\mathscr{D}$ to C.

Let $\mathscr{D}$ be a coherent set of desirable gambles.
Define the compatible choice functions $\mathbf{C}_{\mathscr{D}}$ as those choice functions that have the same binary relation as $\mathscr{D}$:

$$\mathbf{C}_{\mathscr{D}} = \{C\colon (\forall f, g \in \mathscr{L})\{f\} <_C \{g\} \Leftrightarrow g - f \in \mathscr{D}\}.$$

We are looking for the infimum of $\mathbf{C}_{\mathscr{D}}$:

$$C_{\mathscr{D}}(O) \coloneqq \inf \mathbf{C}_{\mathscr{D}}(O) = \{f \in O\colon (\forall g \in O) g - f \notin \mathscr{D}\}$$

for all $O \in \mathscr{Q}(\mathscr{L})$.

Connection between $\mathscr{D}$ and C

From $\mathscr{D}$ to C.

Let $\mathscr{D}$ be a coherent set of desirable gambles.
Define the compatible choice functions $\mathbf{C}_{\mathscr{D}}$ as those choice functions that have the same binary relation as $\mathscr{D}$:

$$\mathbf{C}_{\mathscr{D}} = \{C\colon (\forall f, g \in \mathscr{L})\{f\} <_C \{g\} \Leftrightarrow g - f \in \mathscr{D}\}.$$

We are looking for the infimum of $\mathbf{C}_{\mathscr{D}}$:

$$C_{\mathscr{D}}(O) := \inf \mathbf{C}_{\mathscr{D}}(O) = \{f \in O\colon (\forall g \in O) g - f \notin \mathscr{D}\}$$

for all $O \in \mathscr{Q}(\mathscr{L})$.

Equivalently, in terms of choice and preference relations:

$$O_1 <_{C_{\mathscr{D}}} O_2 \Leftrightarrow (\forall f \in O_1)(\exists g \in O_2) f \prec_{\mathscr{D}} g$$

for all O_1, $O_2 \in \mathscr{Q}(\mathscr{L})$.

Some nice properties

When working with desirability, we can work with choice functions without losing information:

$$\mathscr{D}_{\inf\{C_{\mathscr{D}_1}, C_{\mathscr{D}_2}\}} = \inf\{\mathscr{D}_1, \mathscr{D}_2\} \quad \text{or} \quad \mathscr{D}_{C_{\mathscr{D}_1} \cup C_{\mathscr{D}_2}} = \mathscr{D}_1 \cap \mathscr{D}_2.$$

When working with choice functions, we cannot work with desirability in general without losing information:

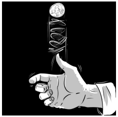
$$C_{\inf\{\mathscr{D}_{C_1}, \mathscr{D}_{C_2}\}}(O) \supseteq (\inf\{C_1, C_2\})(O) \text{ for all } O \text{ in } \mathscr{Q}(\mathscr{L})$$

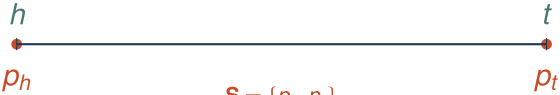
or

$$C_{\mathscr{D}_{C_1} \cap \mathscr{D}_{C_2}}(O) \supseteq (C_1 \cup C_2)(O) \text{ for all } O \text{ in } \mathscr{Q}(\mathscr{L})$$

Example

Coin flip

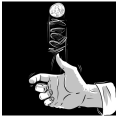


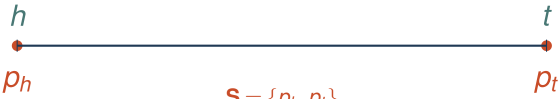


$\mathbf{S} = \{p_h, p_t\}$

$C_{\mathbf{S}}(O)$ are those $f \in O$ for which there is an $x \in \{h, t\}$ such that $f(x) \geq g(x)$ for every $g \in O$ and f is undominated in O.

Coin flip





$\mathbf{S} = \{p_h, p_t\}$

$C_{\mathbf{S}}(O)$ are those $f \in O$ for which there is an $x \in \{h, t\}$ such that $f(x) \geq g(x)$ for every $g \in O$ and f is undominated in O.

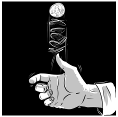
$C_{\mathbf{S}}(O) = \inf\{C_{p_h}, C_{p_t}\}$

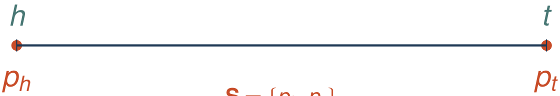
$C_{p_h}(O)$ are those $f \in O$ such that $f(h) \geq g(h)$ for every $g \in O$

$C_{p_t}(O)$ are those $f \in O$ such that $f(t) \geq g(t)$ for every $g \in O$

and undominated.

Coin flip





$\mathbf{S} = \{p_h, p_t\}$

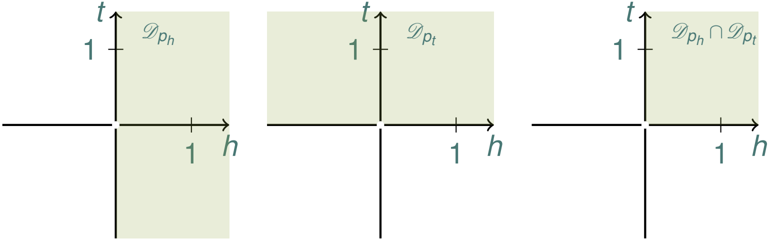
$C_{\mathbf{S}}(O)$ are those $f \in O$ for which there is an $x \in \{h, t\}$ such that $f(x) \geq g(x)$ for every $g \in O$ and f is undominated in O.

$C_{\mathbf{S}}(O) = \inf\{C_{p_h}, C_{p_t}\}$

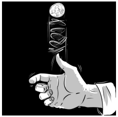
$C_{p_h}(O)$ are those $f \in O$ such that $f(h) \geq g(h)$ for every $g \in O$

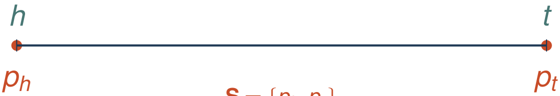
$C_{p_t}(O)$ are those $f \in O$ such that $f(t) \geq g(t)$ for every $g \in O$

and undominated.



Coin flip



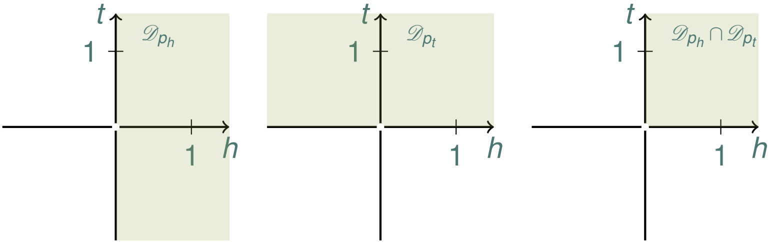


$\mathbf{S} = \{p_h, p_t\}$

$C_{\mathbf{S}}(O)$ are those $f \in O$ for which there is an $x \in \{h, t\}$ such that $f(x) \geq g(x)$ for every $g \in O$ and f is undominated in O.

$C_{\mathbf{S}}(O) = \inf\{C_{p_h}, C_{p_t}\}$

$C_{\mathscr{D}_{p_h} \cap \mathscr{D}_{p_t}}(O) \supseteq (\inf\{C_{p_h}, C_{p_t}\})(O)$ for all O in $\mathscr{Q}(\mathscr{L})$.



Conditioning

Conditioning with sets of desirable gambles

You have a coherent set of desirable gambles $\mathscr{D} \subseteq \mathscr{L}(\mathscr{X})$ and you have the only additional information that X belongs to some subset B of $\mathscr{X}$.

Conditioning with sets of desirable gambles

You have a coherent set of desirable gambles $\mathscr{D} \subseteq \mathscr{L}(\mathscr{X})$ and you have the only additional information that X belongs to some subset B of $\mathscr{X}$.

We define the set of desirable gambles conditional on B by

$$\mathscr{D}\rfloor B := \{f \in \mathscr{L}(B) \colon f\mathbb{I}_B \in \mathscr{D}\}.$$

Here, $\mathbb{I}_B \in \mathscr{L}(\mathscr{X})$ is the indicator of B:

$$\mathbb{I}_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

for all $x \in \mathscr{X}$.

Then

$$f \in \mathscr{D}\rfloor B \Leftrightarrow f\mathbb{I}_B \in \mathscr{D}.$$

If $B \neq \emptyset$, then $\mathscr{D}\rfloor B$ is a coherent set of desirable gambles on B.

Conditioning with choice functions

For a choice function C, we want a conditioning rule that leads to the same relation for $\mathscr{D}_C$:

$$\mathscr{D}_{C\rfloor B} = \mathscr{D}_C\rfloor B := \{f \in \mathscr{L}(B) \colon f\mathbb{I}_B \in \mathscr{D}_C\}.$$

Conditioning with choice functions

For a choice function C, we want a conditioning rule that leads to the same relation for $\mathscr{D}_C$:

$$\mathscr{D}_{C\rfloor B} = \mathscr{D}_C\rfloor B := \{f \in \mathscr{L}(B) \colon f\mathbb{I}_B \in \mathscr{D}_C\}.$$

We define for each option set $O \in \mathscr{Q}(\mathscr{L}(B))$ the sets

$$O\uparrow^f := \{g_1 \in \mathscr{L}(\mathscr{X}) \colon g_1\mathbb{I}_{B^c} = f\mathbb{I}_{B^c} \text{ and } (\exists g_2 \in O) g_1\mathbb{I}_B = g_2\mathbb{I}_B\} \in \mathscr{Q}(\mathscr{L}(\mathscr{X}))$$

for each $f \in \mathscr{L}(B^c)$ and $B \subseteq \mathscr{X}$, and for each option set $O \in \mathscr{Q}(\mathscr{L}(\mathscr{X}))$

$$O\downarrow_B := \{f \in \mathscr{L}(B) \colon (\exists g \in O) f\mathbb{I}_B = g\mathbb{I}_B\} \in \mathscr{Q}(\mathscr{L}(B))$$

for each $B \subseteq \mathscr{X}$.

Conditioning with choice functions

For a choice function C, we want a conditioning rule that leads to the same relation for $\mathscr{D}_C$:

$$\mathscr{D}_{C\rfloor B} = \mathscr{D}_C \rfloor B := \{f \in \mathscr{L}(B) \colon f\mathbb{I}_B \in \mathscr{D}_C\}.$$

We define for each option set $O \in \mathscr{Q}(\mathscr{L}(B))$ the sets

$$O\uparrow^f := \{g_1 \in \mathscr{L}(\mathscr{X}) \colon g_1\mathbb{I}_{B^c} = f\mathbb{I}_{B^c} \text{ and } (\exists g_2 \in O) g_1\mathbb{I}_B = g_2\mathbb{I}_B\} \in \mathscr{Q}(\mathscr{L}(\mathscr{X}))$$

for each $f \in \mathscr{L}(B^c)$ and $B \subseteq \mathscr{X}$, and for each option set $O \in \mathscr{Q}(\mathscr{L}(\mathscr{X}))$

$$O\downarrow_B := \{f \in \mathscr{L}(B) \colon (\exists g \in O) f\mathbb{I}_B = g\mathbb{I}_B\} \in \mathscr{Q}(\mathscr{L}(B))$$

for each $B \subseteq \mathscr{X}$.

Given a choice function C, we propose the following conditioning rule to obtain $C\rfloor B$:

$$C\rfloor B(O) = C(O\uparrow^f)\downarrow_B.$$

Conditioning with choice functions

Given a choice function C, we propose the following conditioning rule to obtain $C\rfloor B$.

$$C\rfloor B(O) = C(O\uparrow^{f})\downarrow_{B}$$

Proposition $C\rfloor B(O) = C(O\uparrow^{f})\downarrow_{B}$ does not depend on the choice of f: given f_1 and f_2 in $\mathscr{L}(B^c)$, then

$$C(O\uparrow^{f_1})\downarrow_{B} = C(O\uparrow^{f_2})\downarrow_{B} .$$

Conditioning with choice functions

Given a choice function C, we propose the following conditioning rule to obtain $C\rfloor B$.

$$C\rfloor B(O) = C(O\uparrow^f)\downarrow_B$$

Proposition $C\rfloor B(O) = C(O\uparrow^f)\downarrow_B$ does not depend on the choice of f: given f_1 and f_2 in $\mathscr{L}(B^c)$, then

$$C(O\uparrow^{f_1})\downarrow_B = C(O\uparrow^{f_2})\downarrow_B .$$

Proposition Given a coherent choice function C on $\mathscr{L}(\mathscr{X})$, then $C\rfloor B$ defined by $C\rfloor B(O) = C(O\uparrow^f)\downarrow_B$ is a coherent choice function on $\mathscr{Q}(\mathscr{L}(B))$.

Conditioning with choice functions

Given a choice function C, we propose the following conditioning rule to obtain $C\rfloor B$.

$$C\rfloor B(O) = C(O\uparrow^{f})\downarrow_B$$

Proposition $C\rfloor B(O) = C(O\uparrow^{f})\downarrow_B$ does not depend on the choice of f: given f_1 and f_2 in $\mathscr{L}(B^c)$, then

$$C(O\uparrow^{f_1})\downarrow_B = C(O\uparrow^{f_2})\downarrow_B .$$

Proposition Given a coherent choice function C on $\mathscr{L}(\mathscr{X})$, then $C\rfloor B$ defined by $C\rfloor B(O) = C(O\uparrow^{f})\downarrow_B$ is a coherent choice function on $\mathscr{Q}(\mathscr{L}(B))$.

Proposition Given a coherent choice function C, then $\mathscr{D}_{C\rfloor B} = \mathscr{D}_C\rfloor B$.

Question

Is there an intuitive interpretation for our conditioning rule

$$C\rfloor B(O) = C(O\uparrow^{f})\downarrow_{B}\ ?$$

Modelling indifference

Indifference with sets of desirable gambles

To model indifference, we need a second set of gambles: the set of indifferent gambles $\mathscr{I}$.
Two gambles f and g are called indifferent (we write $f \approx g$) if

$$\mathscr{D} + \mathscr{I} \subseteq \mathscr{D},$$

where

$$\mathscr{I} := \{\alpha(f-g) \colon \alpha \in \mathbb{R}\}$$

is the set of indifferent gambles.

Then $f \approx g \Leftrightarrow f - g \approx 0$.

Indifference with choice functions

There are two ideas. A coherent choice function C expresses indifference between f and g if:

Seamus Bradley

$$f \approx g \Leftrightarrow (\forall O \supseteq \{f,g\})(f \in C(O) \Leftrightarrow g \in C(O))$$

Gert de Cooman

$$f \approx g \Leftrightarrow (\forall O \in \mathscr{Q}(\mathscr{L}))C(O)_{f \leftrightarrow g} = C(O_{f \leftrightarrow g})$$

where $O_{f \leftrightarrow g}$ is obtained from O by "changing f for g or g for f":

$$O_{f \leftrightarrow g} := \begin{cases} O & \text{if } (f \notin O \text{ and } g \notin O) \text{ or } (f,g \in O) \\ \{f\} \cup O \setminus \{g\} & \text{if } f \notin O \text{ and } g \in O \\ \{g\} \cup O \setminus \{f\} & \text{if } f \in O \text{ and } g \notin O \end{cases}$$

Indifference with choice functions

Seamus Bradley

$$
\begin{aligned}
f \approx g &\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in C(O) \Leftrightarrow g \in C(O)) \\
&\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in R(O) \Leftrightarrow g \in R(O))
\end{aligned}
$$

Indifference with choice functions

Seamus Bradley

$$f \approx g \Leftrightarrow (\forall O \supseteq \{f,g\})(f \in C(O) \Leftrightarrow g \in C(O))$$
$$\Leftrightarrow (\forall O \supseteq \{f,g\})(f \in R(O) \Leftrightarrow g \in R(O))$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference:
$\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f-g) \colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Seamus Bradley?

Indifference with choice functions

Seamus Bradley

$$
\begin{aligned}
f \approx g &\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in C(O) \Leftrightarrow g \in C(O)) \\
&\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in R(O) \Leftrightarrow g \in R(O))
\end{aligned}
$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference: $\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f - g) \colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Seamus Bradley? Assume that $f \in R_{\mathscr{D}}(O)$, then

$$(\exists h_2 \in O) h_2 - f \in \mathscr{D}$$

Indifference with choice functions

Seamus Bradley

$$
\begin{aligned}
f \approx g &\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in C(O) \Leftrightarrow g \in C(O)) \\
&\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in R(O) \Leftrightarrow g \in R(O))
\end{aligned}
$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference: $\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f-g)\colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Seamus Bradley? Assume that $f \in R_{\mathscr{D}}(O)$, then

$$
\begin{aligned}
&(\exists h_2 \in O) h_2 - f \in \mathscr{D} \\
&(\exists h_2 \in O) h_2 - f + \alpha(f-g) \in \mathscr{D}
\end{aligned}
$$

Indifference with choice functions

Seamus Bradley

$$
\begin{aligned}
f \approx g &\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in C(O) \Leftrightarrow g \in C(O)) \\
&\Leftrightarrow (\forall O \supseteq \{f, g\})(f \in R(O) \Leftrightarrow g \in R(O))
\end{aligned}
$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference: $\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f - g) \colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Seamus Bradley? Assume that $f \in R_{\mathscr{D}}(O)$, then

$$
\begin{aligned}
&(\exists h_2 \in O) h_2 - f \in \mathscr{D} \\
&(\exists h_2 \in O) h_2 - f + \alpha(f - g) \in \mathscr{D} \\
&(\exists h_2 \in O) h_2 - g \in \mathscr{D}
\end{aligned}
$$

hence $g \in R_{\mathscr{D}}(O)$, so $R_{\mathscr{D}}$ fulfils Seamus Bradley.

Indifference with choice functions

Gert de Cooman

$$f \approx g \Leftrightarrow (\forall O \in \mathcal{Q}(\mathscr{L}))C(O)_{f\leftrightarrow g} = C(O_{f\leftrightarrow g})$$

Indifference with choice functions

Gert de Cooman

$$f \approx g \Leftrightarrow (\forall O \in \mathscr{Q}(\mathscr{L}))C(O)_{f\leftrightarrow g} = C(O_{f\leftrightarrow g})$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference:
$\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f-g) \colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Gert de Cooman?

Indifference with choice functions

Gert de Cooman

$$f \approx g \Leftrightarrow (\forall O \in \mathscr{Q}(\mathscr{L}))C(O)_{f\leftrightarrow g} = C(O_{f\leftrightarrow g})$$

Given a coherent set of desirable gambles $\mathscr{D}$ that expresses indifference: $\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$ with $\mathscr{I} = \{\alpha(f-g) \colon \alpha \in \mathbb{R}\}$.
Does $R_{\mathscr{D}}$ fulfil Gert de Cooman?

$R_{\mathscr{D}}$ fulfils Gert de Cooman.

Connection between Seamus Bradley and Gert de Cooman

Gert de Cooman implies Seamus Bradley:

$$(\forall O \supseteq \{f, g\})(f \in C(O) \Leftrightarrow g \in C(O)) \Rightarrow (\forall O \in \mathscr{Q}(\mathscr{L}))C(O)_{f \leftrightarrow g} = C(O_{f \leftrightarrow g})$$

Indifference from C to $\mathscr{D}_C$

Conversely, assume a coherent choice function C that “reflects indifference” between f and g. What properties need C in order for

$$\mathscr{D}_C + \mathscr{I} \subseteq \mathscr{D}_C$$

to hold?

Indifference from C to $\mathscr{D}_C$

Conversely, assume a coherent choice function C that “reflects indifference” between f and g. What properties need C in order for

$$\mathscr{D}_C + \mathscr{I} \subseteq \mathscr{D}_C$$

to hold?
Take arbitrary $h \in \mathscr{D}_C + \mathscr{I}$, then

$$
\begin{aligned}
(\exists h_1, h_2 \in \mathscr{L}, \alpha \in \mathbb{R}) h_1 \in R(\{h_1, h_2\}) \text{ and } h &= (h_2 - h_1) + \alpha(f - g) \\
h &= (h_2 + \alpha f) - (h_1 + \alpha g) \\
\Rightarrow (\exists h_1, h_2 \in \mathscr{L}) h_1 + \alpha g \in R(\{h_1 + \alpha g, h_2 + \alpha f\}) & \text{ Gert de Cooman} \\
\Rightarrow h \in \mathscr{D}_C
\end{aligned}
$$

whence Gert de Cooman is a sufficient property.

Question

Which of the two “rules” seems the most intuitive?

Does Seamus Bradley imply that $\mathscr{D}_C + \mathscr{I} \subseteq \mathscr{D}_C$?