# Discrete-Time Queues with Zero-Regenerative Arrivals: Moments and Examples 

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#### Abstract

In this paper we investigate a single-server discrete-time queueing system with single-slot service times. The stationary ergodic arrival process this queueing system is subject to, satisfies a regeneration property when there are no arrivals during a slot. Expressions for the mean and the variance of the queue content in steady state are obtained for this broad class which includes among others autoregressive arrival processes and $M / G / \infty$-input or train arrival processes. To illustrate our results, we then consider a number of numerical examples.


## Keywords

Queueing theory; $M / G / \infty$; Regenerative process

## 1. INTRODUCTION

The statistical characteristics of arrivals at a queue greatly contribute to the overall performance of queueing systems. Moments of both queue content and waiting times are significantly affected by the presence of time correlation in the arrival process. As such correlation is omnipresent in measured traffic on communication networks, there is a continuing interest in analytically tractable queuing models which can accurately model the correlation in the arrival process.
Most arrival models can be described in a Markovian framework. If the state-space of the arrival process is finite, this is e.g. the case for the discrete batch Markovian arrival model (DB-MAP) [1], efficient numerical algorithms can be devised to assess performance of the corresponding queueing systems. Such matrix analytical techniques [3] have been used to assess a multitude of queueing systems, trading in the availability of closed-form expressions of the performance measures of interest for modelling versatility.
As an alternative to the finite state-space models, Markovian arrival models with an infinite but structured state-

[^0]space have been proposed. Prime examples of the latter class include autoregressive arrival models $[7,8,9]$ as well as $M / G / \infty$-input - often referred to as train arrival or session models, see $[2,5,11]$.
For an order $N$ discrete autoregressive arrival model, the numbers of arrivals in slot $n$ equals the number of arrivals in slot $n-k$ with probability $p_{k}(k=1, \ldots, N)$ or equals an independent random variable with probability $1-\sum_{k} p_{k}$. Discrete-time autoregressive processes have been used to capture the main characteristics of VBR video traffic [4].
$M / G / \infty$ input models employ the queue content process of a discrete-time $M / G / \infty$ queue as arrival process. The $M / G / \infty$-input is two-level in the sense that the number of customer arrivals generated by this process equals the number of customers in a separate $M / G / \infty$ queue. In practice, this two-level structure is an abstraction of an upper and a lower layer in a network protocol stack. In the upper layer a connection or session is established that continues for a certain duration, equivalent to the service time of a customer in the separate $M / G / \infty$-queue. Each slot this session stays in the $M / G / \infty$-queue, it generates a packet to be transmitted by the lower layer in the protocol stack. Another fact about $M / G / \infty$-input that attracts a lot of attention is its versatility in capturing correlation in the arrival process with only two parameters: the arrival rate of the $M / G / \infty$-queue and the session length distribution. For more information and analysis on this type of arrival process we refer to [6].
In contrast to the algorithmic approach omnipresent in assessing arrival processes with finite state space, for particular types of autoregressive arrival models and train arrival models, closed-form expressions for the moments of the buffer occupancy and delay are available.
This paper identifies a class of arrival models for which such a closed-form expression is available. In particular, we consider the following Lindley-type recursion which describes the evolution of the queue content at slot boundaries of a single-server discrete-time queuing system with singleslot service times,
\[

$$
\begin{equation*}
U_{k+1}=\left(U_{k}-1\right)^{+}+A_{k} \tag{1}
\end{equation*}
$$

\]

Here, $\left\{A_{k}, k \in \mathbb{Z}\right\}$ is a stationary ergodic sequence of nonnegative integer random variables which adheres to the following assumptions.
[A.1] The arrival process regenerates when there are no ar-
rivals. That is,

$$
\begin{align*}
& \operatorname{Pr}\left[A_{k+1}=i_{k+1}, \ldots, A_{k+\ell}=i_{k+\ell} \mid A_{k}=0, \mathcal{F}_{-\infty}^{k}\right] \\
& \quad=\operatorname{Pr}\left[A_{k+1}=i_{k+1}, \ldots, A_{k+\ell}=i_{k+\ell} \mid A_{k}=0\right], \tag{2}
\end{align*}
$$

for $\ell \in \mathbb{N}^{*}=\{1,2, \ldots\}$ and $i_{j} \in \mathbb{N}$ for all $j \in \mathbb{Z}$ and where $\mathcal{F}_{-\infty}^{k}$ is the natural filtration of the arrival process $\left\{A_{k}\right\}$. Here and in the remainder, let $\mathcal{F}_{m}^{M}=$ $\sigma\left(A_{m}, A_{m+1}, \ldots, A_{M}\right)$ be the $\sigma$-algebra generated by the random variables $A_{k}, k=m, \ldots, M$.
[A.2] Let $f$ denote the regeneration time, i.e. the smallest positive integer for which the number of arrivals is 0 , given that $A_{0}=0$. We assume the existence of the third moment of the number of customers entering the system during a regeneration period,

$$
\mathrm{E}\left[\left(\sum_{n=1}^{f-1} A_{n}\right)^{4}\right]<\infty .
$$

In the section 3, it is shown that the regeneration property is key to the steady-state analysis of the recursion (1). Indeed, $U_{k+1}=0$ implies $A_{k}=0$ such that $\left(U_{k+1}, A_{k}\right)$ regenerates when $U_{k+1}=0$. Albeit somewhat artificial, many arrival processes adhere to the regeneration property, some examples are given in the next section. To illustrate our approach, performance of a queuing system with an elaborate arrival process is explored by some numerical examples in section 4. Finally conclusions are drawn in section 5 .

## 2. EXAMPLES

Many stochastic processes are zero-regenerative, here are a number of examples.

Example 1. A discrete autoregressive stochastic process of order 1 is a sequence $\left\{A_{k}\right\}$ for which the following holds.

$$
A_{k+1}=B_{k+1} A_{k}+\left(1-B_{k+1}\right) N_{k},
$$

in which $\left\{B_{k}\right\}$ is a sequence of i.i.d. Bernoulli distributed random variables and $\left\{N_{k}\right\}$ is a sequence of i.i.d. integervalued random variables. These processes are zero-regenerative ([7]) by construction. For such processes, the state of the arrival process equals the number of arrivals. Hence, when there are no arrivals, the arrival process is in a fixed state which implies zero-regeneration.

Example 2. $M / G / \infty$ input is zero regenerative, as discussed above ([6]).

Example 3. Discrete batch Markovian arrival processes are zero-regenerative if there exists a fixed state such that the arrival process returns to this state whenever there are no arrivals. Note that in order to be zero-regenerative these DBMAPs need not have a finite state space. A prime example of a finite DBMAP is the aggregation of a number of interrupted Bernoulli processes. [11].

Example 4. Consider the following generalised on-off process. During time, the arrival process alternates between an off-state and an on-state. When the arrival process is in the off-state, there are no arrivals. When the arrival process is in the on-state, there is at least one arrival in every slot. The off-times are assumed to constitute a sequence of independent and identically geometrically distributed random
variables. No further assumptions are imposed on the number of arrivals and the duration of the on-times. The lack of memory of the geometric distribution then guarantees zeroregeneration.

Example 5. The aggregation of two zero-regenerative processes is zero-regenerative. Whenever the aggregated process is zero, both constituting processes are zero as well. Hence, the resulting process regenerates.

## 3. QUEUEING ANALYSIS

By a standard Loynes argument [10], there exists a stationary ergodic process $U_{k}^{*}$ satisfying (1) for $\mathrm{E}\left[A_{0}\right]<1$. Moreover, for any initial $U_{0}$, we have $\left|U_{k}^{*}-U_{k}\right| \rightarrow 0$ for $k \rightarrow \infty$ almost surely. The purpose of this section is to find the first two moments of this stationary process, $\mathrm{E}\left[U_{0}^{*}\right]$ and $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$.

As the stationary solution $U_{n}^{*}$ satisfies (1), we have,

$$
U_{1}^{*}-U_{0}^{*}+\mathbb{1}_{\left\{U_{0}^{*}>0\right\}}=A_{0},
$$

such that,

$$
\begin{array}{r}
\mathrm{E}\left[U_{1}^{*}-U_{0}^{*}+\mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right]=\mathrm{E}\left[U_{1}^{*}-U_{0}^{*}\right]+\mathrm{E}\left[\mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right] \\
 \tag{3}\\
=\operatorname{Pr}\left[U_{0}^{*}>0\right]=\mathrm{E}\left[A_{0}\right] .
\end{array}
$$

The second step is a result of the stationary property of $U_{k}^{*}$. By squaring and taking expectancies of both sides of (1) and applying the above equality, we can find an expression for $\mathrm{E}\left[U_{0}^{*}\right]$. Here we used the fact that $U_{0}^{*} \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}=U_{0}^{*}$.

$$
\begin{equation*}
2 \mathrm{E}\left[\left(1-A_{0}\right) U_{0}^{*}\right]=\mathrm{E}\left[A_{0}^{2}\right]+\mathrm{E}\left[A_{0}\right]-2 \mathrm{E}\left[A_{0} \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right] \tag{4}
\end{equation*}
$$

There are two unknown expectancies in this equation $\mathrm{E}[(1-$ $\left.\left.A_{0}\right) U_{0}^{*}\right]$ and $\mathrm{E}\left[A_{0} \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right]$, which we shall derive below in a more general framework - this will be useful for obtaining $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$ later. Let therefore $B_{k}$ be some function of $\left(A_{k}, A_{k+1}, \ldots, A_{k+N}\right)$, with $N<\infty$. Applying Lindley's equation on $\mathrm{E}\left[B_{k} U_{0}^{*}\right]$ for $k \geq 0$ and using the stationary property of the arrival process and $U_{k}^{*}$ yields

$$
\begin{align*}
& \mathrm{E}\left[B_{k} U_{0}^{*}\right]=\mathrm{E}\left[B_{k+1}\left(U_{0}^{*}-\mathbb{1}_{\left\{U_{0}^{*}>0\right\}}+A_{0}\right)\right] \\
& \quad=\mathrm{E}\left[B_{k+1} U_{0}^{*}\right]+\mathrm{E}\left[B_{k+1} A_{0}\right]-\mathrm{E}\left[B_{k+1} \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right] \tag{5}
\end{align*}
$$

Then because of the regeneration property we can write

$$
\begin{array}{r}
\mathrm{E}\left[B_{k} \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right]=\mathrm{E}\left[B_{0}\right]-\mathrm{E}\left[B_{k} \mid U_{0}^{*}=0\right] \operatorname{Pr}\left[U_{0}^{*}=0\right] \\
=\mathrm{E}\left[B_{0}\right]-\mathrm{E}\left[B_{k+1} \mid A_{0}=0\right]\left(1-\mathrm{E}\left[A_{0}\right]\right) \tag{6}
\end{array}
$$

Hence by plugging (6) into (5), we have,

$$
\begin{align*}
\mathrm{E}\left[B_{k} U_{0}^{*}\right] & =\mathrm{E}\left[B_{k+1} U_{0}^{*}\right]+\left(\mathrm{E}\left[B_{k+1} A_{0}\right]-\mathrm{E}\left[B_{k+1}\right] \mathrm{E}\left[A_{0}\right]\right) \\
& +\left(\mathrm{E}\left[B_{k+2} \mid A_{0}=0\right]-\mathrm{E}\left[B_{0}\right]\right)\left(1-\mathrm{E}\left[A_{0}\right]\right) \\
& =\mathrm{E}\left[B_{k+1} U_{0}^{*}\right]+\beta_{k+2}\left(1-\mathrm{E}\left[A_{0}\right]\right)+\gamma_{k+1}, \tag{7}
\end{align*}
$$

with,

$$
\begin{aligned}
\beta_{k} & =\mathrm{E}\left[B_{k} \mid A_{0}=0\right]-\mathrm{E}\left[B_{0}\right], \\
\gamma_{k} & =\mathrm{E}\left[B_{k} A_{0}\right]-\mathrm{E}\left[B_{k}\right] \mathrm{E}\left[A_{0}\right] .
\end{aligned}
$$

By repeated application of (7), we further find,

$$
\mathrm{E}\left[B_{0} U_{0}^{*}\right]=\mathrm{E}\left[B_{n} U_{0}^{*}\right]+\left(1-\mathrm{E}\left[A_{0}\right]\right) \sum_{\ell=2}^{n+1} \beta_{\ell}+\sum_{\ell=1}^{n} \gamma_{\ell}
$$

Taking the limit of both sides for $n \rightarrow \infty$ we find,

$$
\begin{equation*}
\mathrm{E}\left[B_{0} U_{0}^{*}\right]=\mathrm{E}\left[B_{0}\right] \mathrm{E}\left[U_{0}^{*}\right]+\left(1-\mathrm{E}\left[A_{0}\right]\right) \sum_{\ell=2}^{\infty} \beta_{\ell}+\sum_{\ell=1}^{\infty} \gamma_{\ell}, \tag{8}
\end{equation*}
$$

where we used the fact that the arrival process (or any function of it) becomes independent of the buffer content at time 0 as $n$ goes to infinity - i.e. $\lim _{n \rightarrow \infty} \mathrm{E}\left[U_{0}^{*} B_{n}\right]=$ $\mathrm{E}\left[U_{0}^{*}\right] \mathrm{E}\left[B_{0}\right]$ - and this as a result of the regeneration property (A.1) and A.2. It can easily be seen for instance that for $B_{k}=1$ the above formula holds. Replacing $B_{k}$ by $A_{0}$ in (6) and by $1-A_{0}$ in (4), and applying the recursion in (7), yields after some tedious derivations the following result for $\mathrm{E}\left[U_{0}^{*}\right]$.

$$
\begin{align*}
& \mathrm{E}\left[U_{0}^{*}\right]=\mathrm{E}\left[A_{0}\right]+\sum_{m=1}^{\infty}\left(\mathrm{E}\left[A_{m} \mid A_{0}=0\right]-\mathrm{E}\left[A_{0}\right]\right) \\
& \quad+\frac{\mathrm{E}\left[\left(A_{0}\right)^{2}\right]-\mathrm{E}\left[A_{0}\right]}{2\left(1-\mathrm{E}\left[A_{0}\right]\right)}+\sum_{m=1}^{\infty} \frac{\mathrm{E}\left[A_{0} A_{m}\right]-\mathrm{E}\left[A_{0}\right]^{2}}{1-\mathrm{E}\left[A_{0}\right]} . \tag{9}
\end{align*}
$$

For the second moment $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$ we need to cube both sides of (1), yielding

$$
\begin{align*}
\left(U_{1}^{*}\right)^{3}= & \left(U_{0}^{*}\right)^{3}-\mathbb{1}_{\left\{U_{0}^{*}>0\right\}}+\left(A_{0}\right)^{3}+3\left(1+\left(A_{0}\right)^{2}\right) U_{0}^{*} \\
& -3\left(1-A_{0}\right)\left(U_{0}^{*}\right)^{2}+3 A_{0}\left(1-A_{0}\right) \mathbb{1}_{\left\{U_{0}^{*}>0\right\}} . \tag{10}
\end{align*}
$$

Taking expectancies on both sides and using the stationary property of $U_{k}^{*}$ then gives

$$
\begin{align*}
& 3 \mathrm{E}\left[\left(1-A_{0}\right)\left(U_{0}^{*}\right)^{2}\right]=\mathrm{E}\left[\left(A_{0}\right)^{3}\right]-\mathrm{E}\left[A_{0}\right] \\
& \quad+3 \mathrm{E}\left[\left(1+\left(A_{0}\right)^{2}\right) U_{0}^{*}\right]+3 \mathrm{E}\left[A_{0}\left(1-A_{0}\right) \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right] \tag{11}
\end{align*}
$$

Because of (6), this last expectation is straightforward to calculate.

$$
\begin{align*}
& \mathrm{E}\left[A_{0}\left(1-A_{0}\right) \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right]=\mathrm{E}\left[A_{0}\right]-\mathrm{E}\left[\left(A_{0}\right)^{2}\right] \\
&-\mathrm{E}\left[A_{1}\left(1-A_{1}\right) \mid A_{0}=0\right]\left(1-\mathrm{E}\left[A_{0}\right]\right) . \tag{12}
\end{align*}
$$

The second to last expectation in (11) can be obtained using (9) and (8). Taking $B_{n}=\left(A_{n}\right)^{2}$ in (8) yields the following equality.

$$
\begin{align*}
& \mathrm{E}\left[\left(A_{0}\right)^{2} U_{0}^{*}\right]=\left(1-\mathrm{E}\left[A_{0}\right]\right) \sum_{\ell=2}^{\infty}\left(\mathrm{E}\left[\left(A_{l}\right)^{2} \mid A_{0}=0\right]-\mathrm{E}\left[\left(A_{0}\right)^{2}\right]\right) \\
& +\sum_{\ell=1}^{\infty}\left(\mathrm{E}\left[\left(A_{l}\right)^{2} A_{0}\right]-\mathrm{E}\left[\left(A_{0}\right)^{2}\right] \mathrm{E}\left[A_{0}\right]\right)+\mathrm{E}\left[\left(A_{0}\right)^{2}\right] \mathrm{E}\left[U_{0}^{*}\right] \tag{13}
\end{align*}
$$

The only unknown expectation in (11) now, not equal to $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$, is $\mathrm{E}\left[\left(1-A_{0}\right)\left(U_{0}^{*}\right)^{2}\right]$ which still has to be written as a function of $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$, if we wish to obtain a formula for this second moment. As was the case in determining E[(1 $\left.A_{0}\right) U_{0}^{*}$ ] earlier, here again we shall adopt a more general framework, using $B_{k}$ - i.e. we try to determine $\mathrm{E}\left[B_{k} U_{0}^{*}\right]$, $k \geq 0$ and substitute $B_{k}=\left(1-A_{k}\right)$ and $k=0$. To do this, we need te square Lindley's equation (1).

$$
\begin{array}{r}
\mathrm{E}\left[B_{k}\left(U_{0}^{*}\right)^{2}\right]=\mathrm{E}\left[B_{k+1}\left(U_{0}^{*}\right)^{2}\right]+\mathrm{E}\left[B_{k+1}\left(1-2 A_{0}\right) \mathbb{1}_{\left\{U_{0}^{*}>0\right\}}\right] \\
+\mathrm{E}\left[B_{k+1}\left(A_{0}\right)^{2}\right]-2 \mathrm{E}\left[B_{k+1}\left(1-A_{0}\right) U_{0}^{*}\right] . \tag{14}
\end{array}
$$

The middle two terms of the left-hand side, after simplification using equation (6), become

$$
\mathrm{E}\left[B_{k+1}\left(1-A_{0}\right)^{2}\right]-\mathrm{E}\left[B_{k+2}\left(1-2 A_{1}\right) \mid A_{0}=0\right]\left(1-\mathrm{E}\left[A_{0}\right]\right)
$$

The last term in (14) can again be found using (8).

$$
\begin{align*}
& \mathrm{E}\left[B_{k+1}\left(1-A_{0}\right) U_{0}^{*}\right]=\mathrm{E}\left[B_{k+1}\left(1-A_{0}\right)\right] \mathrm{E}\left[U_{0}^{*}\right] \\
+ & \left(1-\mathrm{E}\left[A_{0}\right]\right) \sum_{\ell=2}^{\infty}\left(\mathrm{E}\left[B_{k+\ell+1}\left(1-A_{\ell}\right) \mid A_{0}=0\right]-\mathrm{E}\left[B_{k+1}\left(1-A_{0}\right)\right]\right) \\
+ & \sum_{\ell=1}^{\infty}\left(\mathrm{E}\left[B_{k+\ell+1}\left(1-A_{\ell}\right) A_{0}\right]-\mathrm{E}\left[B_{k+1}\left(1-A_{0}\right)\right] \mathrm{E}\left[A_{0}\right]\right) . \tag{15}
\end{align*}
$$

And so we can rewrite equation (14) in the following condensed form.

$$
\begin{equation*}
\mathrm{E}\left[B_{k}\left(U_{0}^{*}\right)^{2}\right]=\mathrm{E}\left[B_{k+1}\left(U_{0}^{*}\right)^{2}\right]+\delta_{k+1}, \tag{16}
\end{equation*}
$$

where $\delta_{k+1}$ is a known function of the arrival process. Repeating the above recursion, we further find,

$$
\begin{equation*}
\mathrm{E}\left[B_{0}\left(U_{0}^{*}\right)^{2}\right]=\mathrm{E}\left[B_{n}\left(U_{0}^{*}\right)^{2}\right]+\sum_{\ell=1}^{n} \delta_{\ell} . \tag{17}
\end{equation*}
$$

Analogous to the first moment, we take the limit of both sides for $n \rightarrow \infty$.

$$
\begin{equation*}
\mathrm{E}\left[B_{0}\left(U_{0}^{*}\right)^{2}\right]=\mathrm{E}\left[B_{0}\right] \mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]+\sum_{\ell=1}^{\infty} \delta_{\ell}, \tag{18}
\end{equation*}
$$

where again we used the regeneration property to ensure the independence between $B_{n}$ and $U_{0}^{*}$ for $n$ sufficiently large. With all parts of (11) known except for $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$ we can now solve it for $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$, yielding

$$
\begin{align*}
& \mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]=\frac{1+\mathrm{E}\left[\left(A_{0}\right)^{2}\right]}{1-\mathrm{E}\left[A_{0}\right]} \mathrm{E}\left[U_{0}^{*}\right]+\mathrm{E}\left[A_{1}\left(A_{1}-1\right) \mid A_{0}=0\right] \\
& + \\
& +\frac{1}{3} \frac{\mathrm{E}\left[A_{0}\left(A_{0}-1\right)\left(A_{0}-2\right)\right]}{1-\mathrm{E}\left[A_{0}\right]}+\sum_{l=2}^{\infty}\left(\mathrm{E}\left[\left(A_{l}\right)^{2} \mid A_{0}=0\right]-\mathrm{E}\left[\left(A_{0}\right)^{2}\right]\right) \\
& +\sum_{l=1}^{\infty} \frac{\mathrm{E}\left[\left(A_{l}\right)^{2} A_{0}\right]-\mathrm{E}\left[\left(A_{0}\right)^{2}\right] \mathrm{E}\left[A_{0}\right]}{1-\mathrm{E}\left[A_{0}\right]}+\sum_{l=1}^{\infty}\left\{\frac{\mathrm{E}\left[\left(A_{0}-1\right)^{2}\left(A_{l}-1\right)\right]}{1-\mathrm{E}\left[A_{0}\right]}\right. \\
& -\mathrm{E}\left[\left(2 A_{1}-1\right)\left(A_{l+1}-1\right) \mid A_{0}=0\right]+2 \mathrm{E}\left[U_{0}^{*}\right] \frac{\mathrm{E}\left[\left(A_{0}-1\right)\left(A_{l}-1\right)\right.}{1-\mathrm{E}\left[A_{0}\right]} \\
& \left.+2 \sum_{n=2}^{\infty} \mathrm{E}\left[\left(A_{n}-1\right)\left(A_{n+l}-1\right) \mid A_{0}=0\right]-\mathrm{E}\left[\left(A_{0}-1\right)\left(A_{l}-1\right)\right]\right)  \tag{19}\\
& \left.+2 \sum_{n=1}^{\infty} \frac{\mathrm{E}\left[A_{0}\left(A_{n}-1\right)\left(A_{n+l}-1\right)\right]-\mathrm{E}\left[\left(A_{0}-1\right)\left(A_{l}-1\right)\right] \mathrm{E}\left[A_{0}\right]}{1-\mathrm{E}\left[A_{0}\right]}\right\} .
\end{align*}
$$

## 4. NUMERICAL EXAMPLES

We now focus on a specific example which extends the train arrival model presented in [11], to illustrate our result.

## Train arrivals with autoregressive session arrivals.

Consider a train-arrival model where the number of new trains in consecutive slots constitutes a discrete autoregressive (DAR) process while the lengths of the trains constitutes a sequence of independent and identically distributed (i.i.d.) positive random variables. Such an arrival model
is characterised by (1) a sequence $\left\{B_{k}\right\}$ of independent and identically Bernoulli distributed random variables with $\mathrm{E}\left[B_{0}\right]=$ $p$ signifying whether or not the number of new train arrivals equals that of the previous slot, (2) an i.i.d. sequence $\left\{N_{k}\right\}$ of non-negative integer-valued random variables indicating the number of new train arrivals in slot $k$ if $B_{k}=0$, and (3) a doubly-indexed i.i.d. sequence $\left\{G_{k, n}\right\}$ of positive integervalued random variables, indicating the length of the $n$ 'th train arriving in slot $k$. Let $A_{k}$ denote the number of arrivals in slot $k$ and let $S_{k}$ denote the number of new train arrivals in this slot. These random variables are then expressed in terms of $B_{k}, N_{k}$ and $G_{k, n}$ as follows,

$$
\begin{align*}
A_{k} & =\sum_{m=0}^{\infty} \sum_{n=1}^{S_{k-m}} \mathbb{1}_{\left\{G_{k-m, n}>m\right\}}, \\
S_{k+1} & =B_{k+1} S_{k}+\left(1-B_{k+1}\right) N_{k+1} . \tag{20}
\end{align*}
$$

Note that this stationary ergodic arrival process satisfies A.1, the regeneration property. Depending on which choice is made for the three series of i.i.d. random variables $B_{k}$, $N_{k}$, and $G_{k, n}$ A. 2 will be satisfied. Assuming such random variables, our result allows us to calculate $\mathrm{E}\left[U_{0}^{*}\right]$ and $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$. Setting $g_{n}=\operatorname{Pr}[G>n]$, and taking expectation in the preceding equations, we find

$$
\begin{array}{r}
\mathrm{E}\left[S_{0}\right]=p \mathrm{E}\left[S_{0}\right]+(1-p) \mathrm{E}\left[N_{0}\right] \Rightarrow \mathrm{E}\left[S_{0}\right]=\mathrm{E}\left[N_{0}\right] \\
\mathrm{E}\left[A_{0}\right]=\sum_{m=0}^{\infty} \mathrm{E}\left[\sum_{n=1}^{S_{k-m}} g_{m}\right]=\mathrm{E}\left[S_{0}\right] \sum_{m=0}^{\infty} m \operatorname{Pr}[G=m] \\
=\mathrm{E}[N] \mathrm{E}[G] \tag{22}
\end{array}
$$

Here and in the remainder, we drop the indices of the random variables whenever possible. Moreover, by conditioning on the slot where the number of new trains changes, we find,

$$
\begin{align*}
\mathrm{E}\left[A_{n} \mid A_{0}=0\right] & =\sum_{\ell=0}^{n-1} \operatorname{Pr}\left[B_{1}=1, \ldots, B_{\ell}=1, B_{\ell+1}=0\right] \\
\times & \mathrm{E}\left[A_{n} \mid A_{0}=0, B_{1}=1, \ldots, B_{\ell}=1, B_{\ell+1}=0\right] \\
& =\sum_{\ell=0}^{n-1} p^{\ell}(1-p) \mathrm{E}[N] \sum_{m=0}^{n-1-\ell} g_{m} . \tag{23}
\end{align*}
$$

Further, substitution of (20) in $\mathrm{E}\left[A_{0} A_{n}\right]$ and accounting for correlations yields,

$$
\begin{align*}
& \mathrm{E}\left[A_{0} A_{n}\right]=\mathrm{E}[N]^{2} \mathrm{E}[G]^{2}+\mathrm{E}[N] \sum_{m=0}^{\infty}\left(1-g_{m}\right) g_{m+n} \\
&+\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]^{2}\right) \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} p^{|n-r+m|} g_{m} g_{r} \tag{24}
\end{align*}
$$

Substituting previous expressions into (9), and some rather tedious simplifications later, we find,

$$
\begin{align*}
& \mathrm{E}\left[U_{0}^{*}\right]= \frac{\mathrm{E}[N]^{2} \mathrm{E}[G] \mathrm{E}\left[G^{2}\right]-\mathrm{E}[N] \mathrm{E}[G]^{2}}{(1-\mathrm{E}[N] \mathrm{E}[G])} \\
&+\frac{\mathrm{E}[N] \mathrm{E}[G](1-2 p)-\mathrm{E}[N]^{2} \mathrm{E}[G]^{2}(1-2 p)}{(1-p)(1-\mathrm{E}[N] \mathrm{E}[G])} \\
&+\frac{\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]^{2}\right) \mathrm{E}[G]^{2}(1+p)}{2(1-p)(1-\mathrm{E}[N] \mathrm{E}[G])} \tag{25}
\end{align*}
$$

When $p=0$, the arrival model simplifies to one where the number of new trains is a sequence of i.i.d. random variables. The mean buffer content for this model can be found in [11], which focusses on this train arrival model. This mean can of course be obtained using the above formula:

$$
\begin{gather*}
\mathrm{E}\left[U_{0}^{*}\right]=\frac{\mathrm{E}[G] \mathrm{E}[S]^{2} E\left[G^{2}\right]+\mathrm{E}[G]^{2} \mathrm{E}\left[S^{2}\right]+2 \mathrm{E}[S] \mathrm{E}[G]}{2(1-\mathrm{E}[S] \mathrm{E}[G])} \\
-\frac{3 \mathrm{E}[S]^{2} \mathrm{E}[G]^{2}+\mathrm{E}[S] \mathrm{E}[G]^{2}}{2(1-\mathrm{E}[S] \mathrm{E}[G])} \tag{26}
\end{gather*}
$$

On the other hand, assuming single slot train-lengths this means $\mathrm{E}[G]=\mathrm{E}\left[G^{2}\right]=1$ - we obtain the single-server queuing system with discrete autoregressive arrivals studied in [9]. The expression of the mean queue content then simplifies to,

$$
\mathrm{E}\left[U_{0}^{*}\right]=\frac{\mathrm{E}[N](1-3 p)+\mathrm{E}\left[N^{2}\right](1+p)-2(1-p) \mathrm{E}[N]^{2}}{2(1-\mathrm{E}[N])(1-p)} .
$$

To obtain the second moment of the system with autoregressive session arrivals, one has to additionally calculate $\mathrm{E}\left[A_{k} A_{n} \mid A_{0}=0\right]$ and $\mathrm{E}\left[A_{0} A_{k} A_{n}\right], \forall k, n \geq 0$ (see equation (19)). Applying the same methods of conditioning as above we derive $\mathrm{E}\left[A_{k} A_{n} \mid A_{0}=0\right]$ for $k \leq n$ without loss of generality.

$$
\begin{array}{r}
\mathrm{E}\left[A_{k} A_{n} \mid A_{0}=0\right]=\sum_{l=1}^{k} p^{l-1}(1-p)\left\{\sum_{m=0}^{n-l} \sum_{r=0}^{k-l} g_{m} g_{r}\right. \\
\times\left(\mathrm{E}[N]^{2}+p^{|n-m-k+r|}\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]^{2}\right)\right) \\
\left.+\sum_{m=n-k}^{n-l} g_{m}\left[g_{m-n+k}\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]\right)+\mathrm{E}[N]\right]\right\} . \tag{27}
\end{array}
$$

Accounting for correlation we additionally obtain the third order cross product $\mathrm{E}\left[A_{0} A_{k} A_{n}\right]$ where $0 \leq k \leq n$.

$$
\begin{aligned}
& \mathrm{E}\left[A_{0} A_{k} A_{n}\right]=\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} g_{m} g_{r} g_{s}\left(\mathrm{E}\left[N^{3}\right] p^{z-x}\right. \\
& +\mathrm{E}\left[N^{2}\right] \mathrm{E}[N]\left(p^{y-x}\left(1-p^{z-y}\right)+\left(1-p^{y-x}\right) p^{z-y}\right) \\
& \left.\quad+\mathrm{E}[N]^{3}\left(1-p^{y-x}\right)\left(1-p^{z-y}\right)\right) \\
& +\sum_{m, s=0}^{\infty} g_{m+k} g_{s}\left(1-g_{m}\right)\left[\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]^{2}\right) p^{|m-s+n|}+\mathrm{E}[N]^{2}\right] \\
& +\sum_{m, r=0}^{\infty}\left(g_{m} g_{r+n-k}\left(1-g_{r}\right)+g_{r} g_{m+n}\left(1-g_{m}\right)\right) \\
& \quad \times\left[\left(\mathrm{E}\left[N^{2}\right]-\mathrm{E}[N]^{2}\right) p^{|m-r+k|}+\mathrm{E}[N]^{2}\right]
\end{aligned}
$$

$$
\begin{equation*}
+\mathrm{E}[N] \sum_{m=0}^{\infty} g_{m+n}\left(1-g_{m}-2 g_{m+k}\left(1-g_{m}\right)\right) \tag{28}
\end{equation*}
$$

where

$$
\begin{aligned}
& x=\min (-m, k-r, n-s) \\
& z=\max (-m, k-r, n-s) \\
& y=-m+(k-r)+(n-s)-(x+z) .
\end{aligned}
$$



Figure 1: With $\mathrm{E}[N] \mathrm{E}[G]=0.5$ constant, $p=0.9$.

Since $\mathrm{E}\left[U_{0}^{*}\right]$ was already obtained, an expression for the second moment $\mathrm{E}\left[\left(U_{0}^{*}\right)^{2}\right]$ can be derived from the above results, and hence also $\operatorname{Var}\left[U_{0}^{*}\right]$ and $\sigma_{U_{0}^{*}}$.

It is interesting to see what effect autoregressive arrivals can have on a queue fed by this type of $D A R / G / \infty$-input, when different parameters are tweaked. For instance in figure 1, we set $\rho=\mathrm{E}[N] \mathrm{E}[G]=0.5$ fixed, and as $\mathrm{E}[N]$ increases, the trains are shorter on average. In this example $N$ is geometrically distributed, and $G$ 's distribution is that of a shifted geometric distribution. Furthermore $p=0.9$, meaning the arrival process in the $D A R / G / \infty$-queue is highly correlated. Figure 1 shows $\mathrm{E}\left[U_{0}^{*}\right]$ as well as the standard deviation $\sigma_{U_{0}^{*}}$.

As $\mathrm{E}[N]$ approaches zero, or equivalently $\mathrm{E}[G]$ approaches $\infty$, as do the average queue content and standard variation. This is due to the higher correlation of arrivals during consecutive slots. The train lengths are i.i.d. but as not many enter the queue, this independence has little effect to reduce standard variation as well as average buffer content. The reverse is true for high values of $\mathrm{E}[N]$. Note however that $\mathrm{E}[N]$ can't increase indefinitely since $\mathrm{E}[G] \geq 1$ by definition.

In a second figure the effect of the autocorrelation parameter $p$ on the first two queue content moments is plotted. Figure 2 is obtained with the same parameters as the previous figure, only here $\mathrm{E}[N]=0.25$ is fixed and hence $\mathrm{E}[G]=2$. When $p=0$ train arrivals are i.i.d., and consequently buffer content and variation is lowest. It reaches $\infty$ for $p \rightarrow 1$. When $p=1$ one chooses $N>0$ with a finite probability resulting in an overflowing queue (see also [8]).

Lastly, we plot the autocorrelation function $\mathrm{E}\left[A_{0} A_{n}\right]$ for different values of $n$. The situation is again the same as before, with $p=0.9$. We observe that the graph in figure 3 decreases from $\mathrm{E}\left[\left(A_{0}\right)^{2}\right]$ for $n=0$ to $\mathrm{E}\left[A_{0}\right]^{2}$ for $n \rightarrow \infty$. This figure effectively shows the asymptotic independence of the the zero-regenerative arrival process.

## 5. CONCLUSIONS

This paper identified a versatile class of arrival processes


Figure 2: $\mathrm{E}[N]=0.25, \mathrm{E}[G]=2$, buffer content and standard variation as a function of $p$.


Figure 3: A glimpse of the autocorrelation function $\mathrm{E}\left[A_{0} A_{n}\right]$. Parameters are $\mathrm{E}[N]=0.25, \mathrm{E}[G]=2$, and $p=0.9$.
for which a closed-form expression for the first two moments can be obtained in closed form. Apart from stationarity and ergodicity, only a regeneration property when there are no arrivals is imposed. Numerous arrival processes adhere to this zero-regenerative property. To illustrate our results, some numerical examples are introduced which generalise various models in literature.

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