Synchronization in a Population of Oscillators

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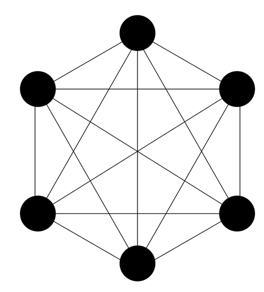
1 Description of the System

1. A cell $\equiv$ a dynamical system with one attracting limit cycle.
2. Each cell is therefore a self-sustaining periodic process with its own natural frequency.
3. A large number of cells are coupled to form a network.
4. The cell dynamics can be represented by a dynamics on the circle.



Synchronization

- Total Phase Locking $=$ the phases of all cells move around the circle with mutually constant phase differences.
- Synchronization $=$ the phases of all cells move around the circle with zero phase differences.
- Partial Synchronization $=$ the population splits into two groups: one totally phase locking group and one desynchronized group.

2 Kuramoto's Model: An Infinite Number of Cells

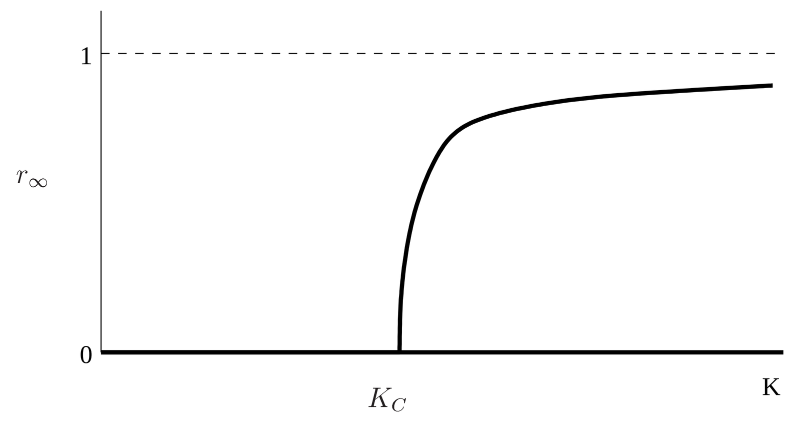
The system equations considered in this model are [2]:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i \in \{1, \cdots, N\}, \qquad (1)$$

with

- K: the strength of the mutual interaction,
- θ_i: the phase of the i-th cell,
- ω_i: the natural frequency of the i-th cell.

1. Consider a population of infinitely many cells of which the natural frequencies are drawn from some prescribed distribution (e.g. Gaussian).
2. Kuramoto investigated the properties of the sum of the phases of all cells:
$$r(t)e^{i\psi(t)} \triangleq \int_0^{2\pi} e^{i\theta} \rho(\theta, t) \mathrm{d}\theta,$$
with $\rho(\theta, t)$ the number density at phase θ at time t.
3. Simulations [1] show that after a transient $r(t)$ has a constant steady state for every value of K. Call this steady state value r_∞.
4. r_∞ depends on the value of K in the following way:



Observations

- If $K < K_C$ the only possible steady state solution is $r_\infty = 0$. The cells act as if they were uncoupled. It can be proven that this solution is stable for $K < K_C$.
- If $K > K_C$ there are two steady state solutions: the incoherent motion $r_\infty = 0$ which is unstable for $K > K_C$ and the partially synchronizing branch. The stability of the latter has not yet been proven.
- Only if the coupling strength is infinitely large, the whole population synchronizes: $\lim_{K\to\infty} r_\infty(K) = 1$.

3 Applying Kuramoto's Model to a Finite Number of Cells

Introduce the order parameter:

$$r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}.$$

- If the population exhibits Total Phase Locking, then $r(t)$ is constant in time. Call this constant value r_∞.
- It can be shown that r_∞ has to satisfy

$$r_\infty(K) = \frac{1}{N} \sum_{j=1}^{N} \pm \sqrt{1 - \left(\frac{\omega_j - \omega_m}{K r_\infty(K)}\right)^2},$$

with $\omega_m =: \frac{1}{N}\sum_{j=1}^{N} \omega_j$ the mean natural frequency.

- The value of r_∞ is independent of the state variables θ_i. Then the dynamics (1) can be rewritten as a system of N decoupled equations:

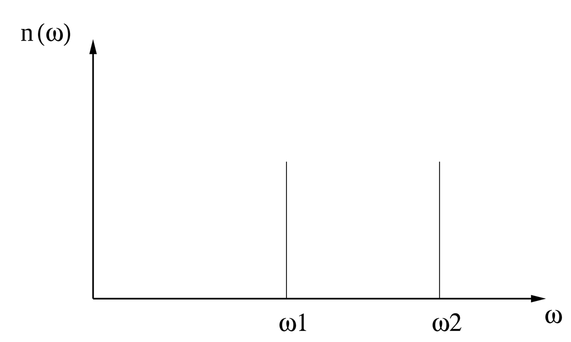
$$\dot{\theta}_i = \omega_i - K r_\infty \sin(\theta_i), \quad i \in \{1, \cdots, N\}. \qquad (2)$$

- If $K r_\infty > \omega_i, i \in \{1, \cdots, N\}$ each of the above equations has a stable and an unstable equilibrium, indicated respectively by θ_i^s and θ_i^u.
- Equilibria of system (2) are also equilibria of system (1) and vice versa.
- It can be proven that the state with $\theta_i = \theta_i^u$ for some $i \in \{1, \cdots, N\}$ and $\theta_j = \theta_j^s$ for the other oscillators, is unstable when regarded as an equilibrium state of system (1).
- The only equilibrium state that is possibly stable is the state with

$$\theta_i = \theta_i^s, \quad \forall i \in \{1, \cdots, N\}.$$

Example:

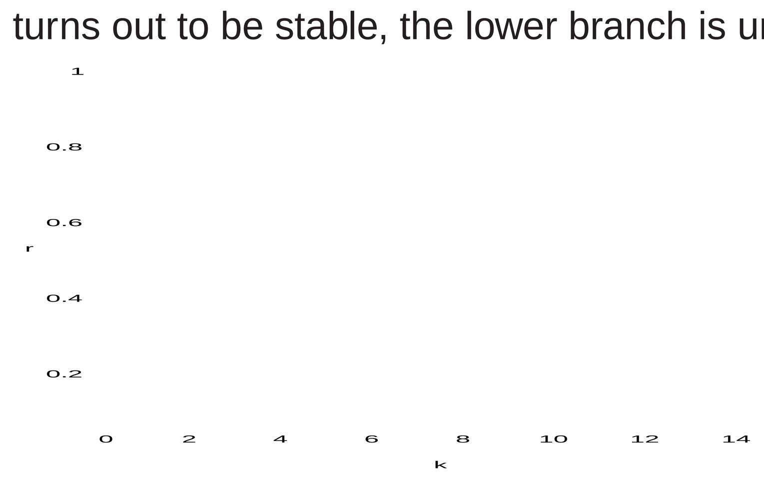
Consider the following population:



- There are 2 total phase locking solutions with $\theta_i = \theta_i^s, \quad \forall i \in \{1, \cdots, N\}$:

$$r_\infty = \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{\omega_x}{K}\right)^2}},$$

with $\omega_x =: (\omega_2 - \omega_1)/2$. The smallest value of K for which these solutions exist is $K \geq |\omega_2 - \omega_1|$.
- The upper branch turns out to be stable, the lower branch is unstable.



References

[1] Y. Kuramoto. Cooperative dynamics of oscillator community. *Prog. Theoret. Phys. Suppl.*, 79:223–240, 1984.

[2] S.H. Strogatz. From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143:1–20, 2000.