MODELING AND SOLVING THE MULTI-PERIOD INVENTORY ROUTING PROBLEM WITH CONSTANT DEMAND RATES

Yiqing ZHONG, El-Houssaine AGHEZZAF

Department of Industrial Management / Ghent University
Technologiepark 903
9052-Zwijnaarde - Belgium
Yiqing.zhong@ugent.be, Elhoussaine.aghezzaf@ugent.be

ABSTRACT: *The inventory routing problem (IRP) is one of the challenging optimization problems in supply chain logistics. It combines inventory control and vehicle routing optimization. The main purpose of the IRP is to determine optimal delivery times and quantities to be delivered to customers, as well as optimal vehicle routes to distribute these quantities. The IRP is an underlying logistical optimization problem for supply chains implementing vendor-managed inventory (VMI) policies, in which the supplier takes responsibility for the management of the customers' inventory. In this paper, we consider a multi-period inventory routing problem assuming constant demand rates (MP-CIRP). The proposed model is formulated as a linear mixed-integer program and solved with a Lagrangian relaxation method. The solution obtained by the Lagrangian relaxation method is then used to generate a close to optimal feasible solution of the MP-CIRP by solving a series of assignment problems. The numerical experiments carried out so far show that the proposed Lagrangian relaxation approach finds quite good solutions for the MP-CIRP and in reasonable computation times.*

KEYWORDS: *Multi-period, Inventory-routing, Lagrangian relaxation, Mixed-integer programming*

1 INTRODUCTION

The inventory routing problem (IRP) is one of the important and challenging optimization problems in the management of supply chain logistics. The problem involves the integration of inventory management and vehicle routing optimization. The typical main objective in an IRP is to determine an optimal distribution policy, consisting of a set of vehicle routes and delivery quantities that minimize the total inventory holding and transportation costs. This problem arises in distribution systems implementing a 'Vendor Managed Inventory' (VMI) policy. In practice, implementing policies such as VMI has proven to considerably improve the overall performance of the supply network.

VMI is an agreement between a supplier and his regular customers according to which customers agree to the alternative that the supplier decides the timing and size of the deliveries. This agreement grants the supplier the full authority to manage inventories at his customers'. This allows the supplier to act proactively and take responsibility for the inventory management of his regular customers, instead of reacting to the orders placed by these customers.

The IRP is then an underlying optimization model for the VMI policy where inventory and distribution decisions have to be made simultaneously.

2 A BRIEF LITERATURE REVIEW

Since Bell *et al.* (1983) first investigated the integrated inventory management and vehicle scheduling, various versions of the inventory routing problems (IRPs) have been extensively studied. A large variety of solution approaches have also been proposed for the solution of these problems. Inventory routing problems can be modeled and approached in different ways depending on the characteristics of its parameters. Different models can be obtained for example, when customers consume the product at a stable or at a variable rate; when the planning horizon is finite or infinite, and so on. Ferdergruen and Zipkin (1984) address a single period IRP with stochastic demands and a fixed fleet vehicle size. Dror and Ball (1987) decompose a multi-period IRP into series of single period problem. They study the problem with constant demands and then propose and compare two solution approaches for the resulting single period problem. Trudeau and Dror (1992) solve the similar problem for uncertain demands. Campbell *et*

al. (2002) and Campbell and Savelsbergh (2004) also worked on the multi-period IRPs where the decisions are executed over a finite horizon. For recent research devoted to the multi-period IRPs, we refer to e.g. Lei *et al.* (2006), Archetti *et al.* (2007), Yu *et al.* (2008), Boudia *et al.* (2009) and Taarit *et al.* (2010). Other fundamental contributions to this body of research are that of Anily and Federgruen (1990), Gallego and Simchi-Levi (1990) and Hall (1992). Anily and Federgruen (1990) consider a distribution system with one warehouse and multiple retailers. The authors take only transportation costs into account and use fixed partition policies to derive their solution approach. A recent thorough review of the literatures on the IRPs can be found in Andersson *et al.* (2010).

In a previous work, we focused on the single-vehicle inventory routing problem (SV-CIRP) (see for example Aghezzaf *et al.* 2011 and Zhong and Aghezzaf 2011). This problem belongs to the class of infinite planning horizon inventory routing problems. For this class of problems the appropriate objective function to be minimized is the total long-run average transportation and inventory costs. In this paper, we consider a multi-period inventory routing problem (MP-CIRP) where the customers consume the product at a constant rate. This multi-period IRP is concerned with a distribution system using a fleet of homogeneous vehicles to distribute a product from a single depot to a set of customers having stable demands. The considered distribution policies are executed over a given finite horizon, for example on a set T of consecutive periods (or days). The objective is to determine the quantities to be delivered to the customers, the delivery time, and to design the vehicle delivery routes, so that the total distribution and inventory costs are minimized. The resulting distribution plan must prevent stockouts from occurring at all customers during the planning horizon. Based on the formulation of the SV-CIRP given in (Aghezzaf *et al.* 2011, Zhong and Aghezzaf 2011) and the distribution pattern of 'multi-tour', i.e. a vehicle can make a set of different tours when it is used (see e.g. Aghezzaf *et al.* 2006), we build up a practical linear mixed-integer model for this MP-CIRP. This proposed model considers the vehicle fleet size as part of the optimization problem and has to be determined. Also, the initial inventory levels at the customers have to be determined in this problem, instead of predefined amounts as done in some other works (see for example Yu *et al.* 2008, Taarit *et al.* 2010 and references therein). In addition, a Lagrangian relaxation method to solve this MP-CIRP is developed and thoroughly discussed. Numerical experiments demonstrate the effectiveness of this proposed approach.

The remainder of this paper is organized as follows. In Section 2, a linear mixed-integer formulation for the MP-CIRP is presented. In Section 3, a Lagrangian relaxation based approach proposed to decompose and solve the considered MP-CIRP is thoroughly discussed. In Section 4, some computational results are presented and compared with the results obtained by the CPLEX solver. Finally, some concluding remarks are provided in Section 5.

3 FORMULATION FOR THE MP-CIRP

As already mentioned above, the MP-CIRP, discussed in this paper, consists of a single distribution center r using a fleet of homogeneous vehicles to distribute a single product to a set of geographically dispersed customers S over a given planning horizon. It is assumed that customer-demand rates and travel times are stable over time. Thus, the objective of this MP-CIRP is to determine the quantities to be delivered to the customers, the delivery time, and to design the vehicle delivery routes, so that the total distribution and inventory costs is minimized while preventing stockouts from occurring at all customers during the whole planning horizon.

To build up our model for the MP-CIRP, some main assumptions are made below:

- The time necessary for loading and unloading a vehicle is neglected in the model;

- Inventory capacities at the depot and the customers are assumed to be large enough so that the corresponding capacity constraints can be omitted in the model;

- Transportation costs are assumed to be proportional to travel times;

- Split deliveries are not allowed, each customer is always completely replenished by one vehicle, in the same tour in each period of the planning horizon.

A more formal description and a proposed linear mixed-integer formulation of the MP-CIRP are given in the following paragraphs:

Let $H = \{1, 2, ..., T\}$ be the planning horizon set of consecutive periods indexed by t. Let π be the size in time unit of one period, for example 8 working hours. Let S be the set of customers indexed by i and j; and $S^+ = S \cup \{0\}$, where 0 represents the depot. A homogeneous fleet of vehicles V is used to serve these customers. The other necessary parameters of the model are given below:

- ψ^v: the fixed operating cost of vehicle $v \in V$ (in euro per vehicle);

- κ^v: the capacity of vehicle $v \in V$ (in ton);

- η_{jt}: the per unit per period holding cost of the product at customer $j \in S$ (in euro per ton per period);

- η_{j0}: the initial per unit holding cost of product at customer $j \in S$ (in euro per ton);

- D_{jt}: the demand at customer $j \in S$ in period $t \in H$ (in ton), i.e. $D_{jt} = d_{jt}\pi$ for $t \in H$, where d_{jt} represents the demand rate at customer j (in ton per hour);

- δ: travel cost of vehicle (in euro per km);

- ν: vehicle average speed (in km per hour);

- tr_{ij}: duration of a trip from customer $i \in S^+$ to customer $j \in S^+$ (in hour);

- φ_{jt}: the handling cost per delivery at customer $j \in S$ in period $t \in H$ (in euro per delivery).

The variables of the model are defined as follows:

- I_{jt}: the inventory level at customer $j \in S$ by the end of period $t \in H$ (in ton);

- I_{j0}: the initial inventory level at customer $j \in S$ (in ton);

- Q_{ijt}^v: the quantity of product remaining in vehicle $v \in V$ when it travels directly to the customer $j \in S^+$ from customer $i \in S^+$ in period $t \in H$. This quantity equals zero when the trip (i,j) is not on any tour made by vehicle $v \in V$ in period t (in ton);

- q_{jt}: the quantity that is delivered to customer $j \in S$ in period $t \in H$, and 0 otherwise (in ton);

- x_{ijt}^v: a binary variable sets to 1 if customer $j \in S^+$ is visited immediately after customer $i \in S^+$ by vehicle $v \in V$ in period $t \in H$, and 0 otherwise;

- y^v: a binary variable sets to 1 if vehicle $v \in V$ is being used, and 0 otherwise;

Thus, the linear mixed-integer formulation for the multi-period IRP is given as follows:

(MP-CIRP) Minimize

$$CV = \sum_{v \in V} \sum_{t \in H} \sum_{i \in S^+} \sum_{j \in S^+} (\delta\nu tr_{ij} + \varphi_{jt}) x_{ijt}^v$$

$$+ \sum_{v \in V} \psi^v y^v + \sum_{t \in H} \sum_{j \in S} \eta_{jt} I_{jt} + \sum_{j \in S} \eta_{j0} I_{j0} \qquad (1)$$

Subject to:

$$\sum_{v \in V} \sum_{i \in S^+} x_{ijt}^v \leq 1, \ \forall j \in S, t \in H, \qquad (2)$$

$$\sum_{i \in S^+} x_{ijt}^v - \sum_{k \in S^+} x_{jkt}^v = 0, \ \forall j \in S^+, t \in H, v \in V, \ (3)$$

$$I_{j1} = I_{j0} + q_{j1} - D_{j1}, \ \forall j \in S, \qquad (4)$$

$$I_{jt} = I_{j,t-1} + q_{jt} - D_{jt}, \ \forall j \in S, t \in H \text{ and } t \geq 2, \ (5)$$

$$\sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v - \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v = q_{jt}, \ \forall j \in S, t \in H, (6)$$

$$Q_{ijt}^v \leq \kappa^v x_{ijt}^v, \ \forall i,j \in S^+, t \in H, v \in V, \qquad (7)$$

$$\sum_{i \in S^+} \sum_{j \in S^+} tr_{ij} x_{ijt}^v \leq \pi, \ \forall t \in H, v \in V, \qquad (8)$$

$$I_{j0} \leq I_{jt} \ \forall j \in S, t \in H \text{ and } t = T, \qquad (9)$$

$$\sum_{i \in S^+} x_{ijt}^v \leq y^v, \ \forall j \in S, t \in H, v \in V, \qquad (10)$$

$$x_{ijt}^v, y^v \in \{0,1\}, I_{j0}, I_{jt} \geq 0, Q_{ijt}^v \geq 0, q_{jt} \geq 0,$$

$$\forall i,j \in S^+, t \in H, v \in V. \qquad (11)$$

The objective function (1) is similar to the one considered for SV-CIRP (see e.g. Aghezzaf *et al.* 2011). Here also, the same four cost components are considered. The total fixed operating cost of using the vehicle(s) given by $\sum_v \psi^v y^v$. The total transportation cost given by $\sum_v \sum_t \sum_i \sum_j \delta\nu tr_{ij} x_{ijt}^v$. The total delivery handling cost given by $\sum_v \sum_t \sum_i \sum_j \varphi_{jt} x_{ijt}^v$. The total inventory holding cost, including the initial inventory holding cost and the inventory holding cost at every end of period $t \in H$, given by $\sum_t \sum_j (\eta_{jt} I_{jt} + \eta_{j0} I_{j0})$.

For the restrictions, constraints (2) guarantee that each customer is visited by each vehicle at most once in period t. Constraints (3) are the usual flow conservation constraints assuring that if a vehicle arrives at a customer, it must leave after it served this customer to a next customer or to the depot. Constraints (4), (5) are the inventory balance constraints of each customer. Constraints (6) are the delivered load balance constraints. These constraints eliminate possible formation of sub-tours. Constraints (7) ensure that the quantity carried by a vehicle should not exceed

the vehicle's maximum capacity. Constraints (8) indicate that the total travel time of a vehicle should not exceed the considered horizon length in each period. Constraints (9) indicate that the final inventory level at customer j at the end of period T shall cover its initial inventory. Constraints (10) indicate that at every period, a vehicle cannot be used to serve any customer unless this vehicle is already selected. Constraints (11) are the integrality and sign constraints to be imposed on the variables.

4 LAGRANGIAN RELAXATION FOR MP-CIRP

The MP-CIRP is an NP-complete problem as it contains the vehicle routing problem (VRP), known to be NP complete, as a sub-problem in each period. Large instances of the MP-CIRP are therefore hard to solve to optimality in a reasonable computational time. This section discusses a Lagrangian relaxation approach to decompose and possibly solve or generate some lower and upper bounds for the problem.

In the proposed Lagrangian relaxation procedure for MP-CIRP, constraints (6) are assumed to be the complicating restrictions and are relaxed. Along the same lines as in Taarit *et al.* 2010, the resulting problem decomposes into an inventory allocation sub-problem (denoted by IA-P) and a vehicle routing sub-problem (denoted by RT-P). These sub-problems involve fewer variables and constraints respectively and they can be solved more efficiently by some standard optimization MIP-solver. Afterwards, a sub-gradient algorithm (Fisher 1981, Shor 1985) is applied to update the Lagrangian multipliers and derive a lower bound on the optimal solution of the original problem. The Lagrangian relaxation approach is extensively and successfully used to solve complicated mixed-integer programs (see e.g. Yu *et al.* 2008, Li *et al.* 2009).

4.1 Relaxation and decomposition

Reconsider the formulation of MP-CIRP, as mentioned above, the constraints that complicate this model, constraints (6), combine inventory allocation variables q and flow variables Q. These constraints are relaxed and incorporated in the objective function with unrestricted Lagrangian multipliers μ_{jt} for all $j \in S$ and $t \in H$. The resulting relaxed problem (denoted by RP-MPIRP) is then stated as follows:

(RP-MPIRP) Minimize

$$CV_{LR} = \sum_{v \in V} \sum_{t \in H} \sum_{i \in S^+} \sum_{j \in S^+} \left(\delta \nu tr_{ij} + \varphi_j \right) x_{ijt}^v$$

$$+ \sum_{v \in V} \psi^v y^v + \sum_{t \in H} \sum_{j \in S} \eta_{jt} I_{jt} + \sum_{j \in S} \eta_{j0} I_{j0}$$

$$+ \sum_{t \in H} \sum_{j \in S} \mu_{jt} \left(q_{jt} + \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v \right) (12)$$

Subject to (2)-(5) and (7)-(11).

As in Taarit *et al.* 2010, the relaxed problem (RP-MPIRP) can then be decomposed into two sub-problems: an inventory allocation sub-problem (IA-P) and a vehicle routing sub-problem (RT-P), stated below respectively:

The inventory allocation sub-problem (IA-P):
(IA-P) Minimize

$$CV_{IA-P} = \sum_{t \in H} \sum_{j \in S} \eta_{jt} I_{jt} + \sum_{j \in S} \eta_{j0} I_{j0} + \sum_{t \in H} \sum_{j \in S} \mu_{jt} q_{jt} (13)$$

Subject to (4) - (5), (9) and

$$\sum_{t \in H} D_{jt} - I_{j0} \leq \sum_{t \in H} q_{jt} \text{ for all } j \in S, \tag{14}$$

$$I_{j0} \geq 0, I_{jt} \geq 0, q_{jt} \geq 0, \forall j \in S, \forall t \in H. \tag{15}$$

Note that the sub-problem IA-P can be further decomposed into independent sub-problems associated with each customer j ($\forall j \in S$). Moreover, a set of valid constraints (14) are appended to the formulation of IA-P in order to enhance the relaxed sub-problem. These constraints prevent stockouts at each customer over the planning horizon.

Similarly, to enhance the relaxed sub-problem RT-P, the following valid constraints need to be appended to the model (Taarit *et al.* 2010):

$$D_{j1} - I_{j0} \leq \sum_{v \in V} \sum_{i \in S^+} Q_{ij1}^v \text{ for all } j \in S \tag{16}$$

The above inequalities prevent stockouts at each customer during the first period of the planning horizon. Therefore, the sub-problem RT-P is presented as follows:

(RT-P) Minimize

$$CV_{RT-P} = \sum_{v \in V} \sum_{t \in H} \sum_{i \in S^+} \sum_{j \in S^+} \left(\delta \nu tr_{ij} + \varphi_j \right) x_{ijt}^v$$

$$+ \sum_{v \in V} \psi^v y^v$$

$$+ \sum_{t \in H} \sum_{j \in S} \mu_{jt} \left(\sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v \right) \tag{17}$$

Subject to (2) - (3), (7) - (8), (10) and (16),

$$x_{ijt}^v, y^v \in \{0, 1\}, Q_{ijt}^v \geq 0, \forall i, j \in S^+, t \in H, v \in V \quad (18)$$

Also note that the sub-problem RT-P can be further decomposed into independent sub-problems associated with every period t of the planning horizon H. The sub-problem RT-P is a mixed-integer program with fewer variables and constraints than the original routing problem, and it can be solved more efficiently by the standard Branch-and-Bound.

4.2 Lagrangian procedure

Based on the above decomposition of the original MP-CIRP, a lower bound to the optimal solution of the original problem can be generated for any given Lagrangian multipliers μ. The best lower bound can be obtained by the optimal Lagrangian multipliers, solution of the following Lagrangian dual problem of LR-MPIRP (denoted by DP-MPIRP):

(DP-MPIRP) Maximize $D(\mu_{jt})$

where $D(\mu_{jt}) =$

$$\text{Min} \sum_{v \in V} \psi^v y^v + \sum_{v \in V} \sum_{t \in H} \sum_{i \in S^+} \sum_{j \in S^+} (\delta \nu tr_{ij} + \varphi_j) x_{ijt}^v$$

$$+ \sum_{t \in H} \sum_{j \in S} \eta_{jt} I_{jt} + \sum_{j \in S} \eta_{j0} I_{j0}$$

$$+ \sum_{t \in H} \sum_{j \in S} \mu_{jt} \left(q_{jt} + \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v \right) \quad (19)$$

is the solution value of the relaxed problem LR-MPIRP.

To solve the problem DP-MPIRP, the corresponding sub-problems IA-P and RT-P have to be solved, and then a sub-gradient algorithm is used to improve the value of $D(\mu_{jt})$. In addition, a Lagrangian heuristic method is developed to provide a feasible solution of the MP-CIRP. Thus, the sub-gradient optimization procedure generates lower bounds and upper bounds iteratively and tries to update the best lower bound and upper bound of the problem. The main procedure is summarized in the following algorithm.

Recall the formulation of MP-CIRP, let $X = (x_{ijt}^v, y^v)$ be the binary variables of the model, $Z = (Q_{ijt}^v, q_{jt}, I_{jt}, I_{j0})$ be the continuous variables, let $P = \{(X, Z) : (2), ..., (11)\}$ be the set of feasible solutions determined by the system of the constraints (2) - (11). Thus, an instance of the MP-CIRP can be stated as $\text{Min}_{(X,Z) \in P} CV(X, Z)$, where CV is the cost function (1). In addition, let $g_{jt} (\forall j \in S, \forall t \in H)$

denote the corresponding subgradients in the optimization procedure, where

$$g_{jt} = q_{jt} + \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v. \quad (20)$$

Thus, the proposed algorithm is presented as below:

Algorithm 1 *(The Lagrangian procedure for MP-CIRP)*

Step 0. (Initialization):

Let LB be the best lower bound, UB be the best upper bound, and (X^*, Z^*) be the optimal solution found so far. Let σ be the subgradient agility and k be the iteration number. Initialize $LB = 0$, $k = 1$, initialize the values for the Lagrangian multipliers μ and the value for $\sigma \in (0, 1)$.

Step 1. (Initializing the first UB):

Generate a feasible solution (X_0, Z_0) for the MP-CIRP by supposing each customer $j \in S$ is served by one vehicle separately, and then solve the corresponding inventory allocation problems to obtain the objective value $CV(X_0, Z_0)$ of the MP-CIRP. Let the current best upper bound $UB := CV(X_0, Z_0)$, and update $(X^*, Z^*) := (X_0, Z_0)$.

Step 2. (Lower bound computation):

Solve the sub-problems IA-P(μ_k) and RT-P(μ_k) respectively. Let C_{IA-P}^k and C_{RT-P}^k denote the corresponding objective values, then the new lower bound value is $C_{LR}^k := C_{IA-P}^k + C_{RT-P}^k$. If this new lower bound is greater than LB, then set $LB := C_{LR}^k$; otherwise set $\sigma := \sigma/2$.

Step 3. (Upper bound computation):

Based on the solutions of IA-P(μ_k) and RT-P(μ_k), the Lagrangian heuristic method (see the algorithm in Section 4.3) is called to derive a feasible solution (X_k, Z_k). If this solution improves the current best upper bound, then set $UB := CV(X_k, Z_k)$, and update $(X^*, Z^*) := (X_k, Z_k)$.

Step 4. (Updating μ):

Set step size s_k by $s_k := \sigma \left(UB - C_{LR}^k \right) / \|g_k\|^2$, where g_k are the current subgradients, determined by (20). Update the Lagrangian multipliers in iteration $k + 1$: $\mu_{k+1} := \mu_k + s_k g_k$.

Step 5. (Stopping rule):

If (1) k exceeds the maximal number of iterations, or (2) C_{LR}^k is not improved for a given number of iterations, then output the LB, UB and the current

optimal solution (X^*, Z^*), and stop; otherwise set $k := k + 1$, and then go to Step 2.

4.3 Lagrangian heuristic method

Solving the sub-problems of IA-P and RT-P generates a lower bound for the MP-CIRP. However due to the relaxation procedure, solutions obtained by solving IA-P and RT-P at each iteration are not feasible for the original problem (MP-CIRP). Therefore, referring to the IRP heuristic method developed in Aghezzaf *et al.* (2006) and the VRP heuristic method developed in Clarke and Wright (1964), we propose a saving-based heuristic approach that exploits the Lagrangian information, to derive a feasible solution for the MP-CIRP at each iteration.

In brief, the optimal solution of sub-problem IA-P provides the information on the quantities that have to be delivered to each customer at each period of the planning horizon. These quantities can be considered as customer orders and a vehicle routing problem is then solved for each period separately. The resulting solution is feasible if the corresponding constraints, such as the vehicle capacity constraints and the traveling time constraints, are not violated. Detailed steps of this algorithm are presented in the following paragraphs.

For each period $t \in H$, we assume that q_j is the quantity to be delivered to the customer $j \in S$. Let R_t be the set of served customers in period t, i.e. $R_t = \{j : q_j > 0, \forall j \in S\}, \forall t \in H$. Note that R_t is possibly a void set for some $t \in H$. In this case, it can be neglected and no vehicle route needs to be generated. The proposed saving-based heuristic method is then outlined as below:

Algorithm 2 *(The Lagrangian heuristic algorithm for MP-CIRP)*

Step 0. (Initialization):

Suppose one vehicle is available for serving the customers (i.e. though the traveling time constraint is satisfied by each tour, it is ignored at first at the level of multi tour). A temporary multi-tour is initiated with the basic multi-tours, each serving one of the customers by the vehicle. There are thus as many tours in the initial temporary multi-tour as there are customers in the set R_t.

Step 1. (Savings):

The core of the saving-based heuristic is the process of combining two tours into one multi-tour to achieve some cost saving. This is implemented as follows:

- *Suppose the current multi-tour L^* makes n tours.*

These n tours are put into a single list of tours $C^1, ..., C^n$. We then calculate the cost values $CV^{C^1}, ..., CV^{C^n}$ for each tour.

- *For all $1 \leq i < j \leq n$, combine tours C^i and C^j into one tour, denoted by C^+ (by finding the TSP through all customers covered by both tours plus the depot. If this tour is infeasible (that is it doesn't satisfy the travel time constraint) then it is disregarded and generate a new multi-tour that makes tours $C^1, ..., C^{i-1}, C^{i+1}, ..., C^{j-1}, C^{j+1}, ..., C^n$ and C^+. If the cost value CV^{C^+} is smaller than the sum of cost values of C^i and C^j, then we have achieved a saving $SV := CV^{C^i} + CV^{C^j} - CV^{C^+}$.*

- *Calculate all combinations of two tours from the list and the best feasible combination is kept, i.e. the one which results in the largest saving. This best feasible combination is then added to the tour list and the two combined tours are removed from the list.*

Step 2. (Stopping rule):

Repeat Step 1 until no further feasible combination resulting in a positive saving can be found. Calculate the total traveling time T_{min} for the current multi-tour L^*, if $T_{min} > \pi$, then calculate the vehicle number $VN := \lfloor T_{min}/\pi \rfloor + 1$, and add the corresponding additional fixed using costs of the vehicles to the multi-tour cost. Finally output the best feasible multi-tour and its cost value.

Through calling the Algorithm 2, we find a feasible solution for each period $t \in H (R_t \neq \phi)$ separately. As a consequence, at each iteration of the Lagrangian relaxation approach, we generate a feasible solution that is an upper bound for the MP-CIRP using the above saving-based heuristic algorithm. The best upper bound is updated iteratively as long as a better delivery schedule is obtained. To improve the upper bound of the MP-CIRP further, we apply an adjustment procedure to the best feasible solution of the MP-CIRP found by the Lagrangian relaxation approach. The main goal of this adjustment is to eliminate the unnecessary deliveries to the customers during the planning horizon as many as possible, and to diminish the potential transportation costs in the meantime.

Roughly speaking, the adjustment procedure is to try to combine two or more deliveries at customer j ($j \in S$) into one delivery. As long as a better feasible solution of the MP-CIRP is found, then this adjustment is called an effective adjustment and it is kept. Details of the adjustment procedure are presented as follows:

Algorithm 3 *(The adjustment procedure for MP-CIRP)*

Step 0. (Initialization):

Let UB be the best upper bound and (X^*, Z^*) be the optimal solution found so far. Let $q_{jt}^* \in Z^*$ $(\forall j \in S, \forall t \in H)$ be the current best delivery schedule and $I_{j0}^* \in Z^*$ $(\forall j \in S)$ be the current best initial inventory level. Let W be the set of the customers that have more than one delivery during the planning horizon, i.e. $W = \left\{ j : q_{jm}^* > 0 \text{ and } q_{jn}^* > 0, \text{ for all } m, n \in T, \text{ and } m \neq n \right\}$. Initialize the iteration number $k = 0$.

Step 1. (Adjustment):

For all remaining customers in the set W, do the adjustment as follows:

(1-a): *Select a customer j from W, and delete it from the set W. For this customer, starting from the final period during which the delivery takes place, do the delivery combination. Let q_{jt}^k be the current delivery schedule, the combination is executed as $q_{jm}^k := \left(q_{jm}^* + \sum_{m < n \leq T} q_{jn}^* \right) \leq \kappa$ and then set $\sum_{m < n \leq T} q_{jn}^k = 0$, for $q_{jm}^* > 0$, $q_{jn}^* > 0$ and $1 \leq m < n \leq T$, where κ is the vehicle capacity. In the same time, adjust the corresponding inventory levels of customer j. For this new delivery schedule, call the saving-based heuristic approach (Algorithm 2) to find a new feasible solution (X_k, Z_k). If this new solution gives a better objective value such that $CV(X_k, Z_k) < UB$, then update the current best upper bound by $UB := CV(X_k, Z_k)$, and update the current optimal solution by $(X^*, Z^*) := (X_k, Z_k)$.*

(1-b): *For customer j, according to the current best delivery schedule q_{jt}^*, if there still exist $q_{jm}^* > 0$ and $q_{jn}^* > 0$, for $1 \leq m < n \leq T$, then absorb the delivered quantities at period m into the initial inventory. Let I_{j0}^k be the current initial inventory level, the absorbing is executed as $I_{j0}^k := \left(I_{j0}^* + \sum_{1 \leq m < n} q_{jm}^* \right)$ and $q_{jn}^k := \left(q_{jn}^* + \sum_{1 \leq m < n} q_{jm}^* \right) \leq \kappa$, and then set $\sum_{1 \leq m < n} q_{jm}^k = 0$, for $q_{jm}^* > 0$, $q_{jn}^* > 0$ and $1 \leq m < n \leq T$. At the same time, adjust the corresponding inventory levels of customer j. Then call Algorithm 2 for the new delivery schedule to find a new feasible solution (X_k, Z_k). If a better feasible solution is obtained, then update the current best upper bound by $UB := CV(X_k, Z_k)$, and update the current optimal solution by $(X^*, Z^*) := (X_k, Z_k)$.*

Step 2. (Stopping rule):

If there exist remaining customers in set W, then set $k := k + 1$ and goto Step 1. Otherwise output UB and the optimal solution (X^*, Z^*), and stop.

Through the above adjustment procedure (Algorithm 3), we try to eliminate those unnecessary deliveries during the planning horizon as many as possible, and meanwhile to diminish the potential transportation costs. Thus, the upper bound of the MP-CIRP could be improved as far as possible.

5 NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments to evaluate the performance of the proposed Lagrangian relaxation approach for the MP-CIRP, using some randomly generated instances according to the generation scheme proposed by Yu *et al.* 2008. We consider different sets of problem instances with different customer and planning horizon sizes. Each problem set is identified by the number of customers N and time horizons T. The four tested problem sets are considered: $(N = 15, T = 3)$, $(N = 15, T = 6)$, $(N = 25, T = 3)$ and $(N = 25, T = 6)$.

More specially, for the set of the instances consisting of 15 customers (denoted by A15-x-T-y, where 'x' is the index of instances and 'y' is the index of time horizons), the customers are distributed randomly and uniformly over a square of 30 by 30 km, and the distribution center is always put in the center of the square. Demand rates of customers are generated randomly and uniformly between 1 and 3 ton per hour. The inventory holding costs are generated randomly and uniformly between 0.1 and 0.15 (in euro per ton per period). Fixed delivery handling costs are the same for all customers, which are 25 euro per delivery. A fleet of homogeneous vehicles with capacity 100 ton is used to serve these customers. The fixed using cost for each vehicle is 50 euro. The vehicles can travel up to 50 km per hour, and the travel cost of vehicle is 1 euro per km per hour. The time unit of one period π is set to 8 hours in all instances. Whereas for the set of the instances consisting of 25 customers (denoted by A25-x-T-y), the customers are distributed randomly over a square of 100 by 100 km (in clusters), and the depot is always put in the center of the square. Demand rates of customers are generated randomly and uniformly between 0.1 and 3 ton per hour. The inventory holding costs are also generated randomly and uniformly between 0.1 and 0.15 (in euro per ton per period) and fixed delivery handling costs are 10 euro per delivery. Again, a fleet of homogeneous vehicles with capacity 100 ton is used to serve the customers. The vehicles can also travel up to 50 km per hour, and the travel cost of vehicle is 1 euro per km. The fixed using cost of one vehicle is 30 euro. Also, the time unit of one period π is still set to 8 hours in all instances. The proposed approach for the MPRIP is implemented in MATLAB R14 7.0.1 with the callable library of TOMLAB (Holmstrom *et al.* 2010) and all instances are tested on a laptop

with Intel Core(TM)2 CPU T5500 @1.66GHz, 1.0GB RAM.

To compare our solution values obtained from the Lagrangian relaxation approach, we also solve the generated instances using AMPL with CPLEX 11.2. All instances are executed on a Dell PC with Pentium(R) 4 CPU @3.00GHz, 1.0GB RAM. Due to a large number of variables and constraints in the problem, solving the instances with AMPL is quite time-consuming, even for the small-size problems. Therefore, we preset the time limitation of running an instance is 8 hours, 10 hours, 12 hours and 15 hours for the problem sets of $(N = 15, T = 3)$, $(N = 15, T = 6)$, $(N = 25, T = 3)$ and $(N = 25, T = 6)$ respectively. The computational results of these different problem sets are shown in the following tables.

Instances	LB(eur)	UB(eur)	CPU(s)	Gap1
A15-0-T-3	676.08	712.67	49.933	5.134
A15-1-T-3	678.35	724.14	52.760	4.878
A15-2-T-3	678.22	715.43	57.609	5.202
A15-3-T-3	655.71	701.92	51.688	6.584
A15-4-T-3	690.22	741.07	58.437	6.862
A15-5-T-3	677.78	730.91	50.781	7.263
A15-6-T-3	624.16	644.12	52.422	3.097
A15-7-T-3	552.91	562.69	52.140	1.737
A15-8-T-3	646.87	678.77	49.985	4.699
A15-9-T-3	648.28	676.18	53.547	4.125
Average			52.930	4.958

Table 1: Results of the instances with (N=15,T=3)

Instances	LB(eur)	UB(eur)	CPU(s)	Gap1
A15-0-T-6	1006.93	1258.50	127.341	19.989
A15-1-T-6	997.57	1221.10	98.672	18.305
A15-2-T-6	955.81	1178.00	95.656	18.862
A15-3-T-6	956.75	1289.40	100.110	25.796
A15-4-T-6	1004.76	1264.70	139.090	20.554
A15-5-T-6	1007.98	1268.30	127.400	20.525
A15-6-T-6	888.60	1053.80	110.271	15.676
A15-7-T-6	866.07	1074.20	110.020	19.374
A15-8-T-6	924.78	1175.40	122.733	21.326
A15-9-T-6	947.51	1202.90	132.672	21.228
Average			116.388	20.163

Table 2: Results of the instances with (N=15,T=6)

For different problem sets, Table 1- 2 and Table 5 give the lower bounds and upper bounds found by the Lagrangian relaxation approach, and the gaps between lower bounds and upper bounds, i.e. *Gap1*, where it is determined by

$$Gap1 = \frac{UB - LB}{UB} \times 100\%$$

Instances	UB-LR(eur)	UB-AMPL(eur)	Gap2
A15-0-T-3	712.67	711.39	0.179
A15-1-T-3	724.14	723.18	0.132
A15-2-T-3	715.43	712.34	0.432
A15-3-T-3	701.92	697.48	0.632
A15-4-T-3	741.07	727.05	1.831
A15-5-T-3	730.91	720.82	1.397
A15-6-T-3	644.12	644.12	0
A15-7-T-3	562.68	562.68	0
A15-8-T-3	678.77	670.94	1.153
A15-9-T-3	676.18	676.18	0
Average			0.575

Table 3: Solution comparisons of the Lagrangian relaxation approach and AMPL: (N=15, T=3)

Instances	UB-LR(eur)	UB-AMPL(eur)	Gap2
A15-0-T-6	1258.50	1231.40	2.153
A15-1-T-6	1221.10	1174.77	3.794
A15-2-T-6	1178.00	1130.27	4.052
A15-3-T-6	1289.40	1210.03	6.134
A15-4-T-6	1264.70	1227.70	2.925
A15-5-T-6	1268.30	1222.04	3.621
A15-6-T-6	1053.80	1043.52	0.976
A15-7-T-6	1074.20	1031.13	4.009
A15-8-T-6	1175.40	1132.42	3.656
A15-9-T-6	1202.90	1158.95	3.653
Average			3.496

Table 4: Solution comparisons of the Lagrangian relaxation approach and AMPL: (N=15, T=6)

In addition, in Table 3- 4 and Table 6, the comparisons between the results obtained from the Lagrangian relaxation approach and AMPL are presented, and the values of *Gap2* illustrate the difference in the upper bounds, where

$$Gap2 = \frac{UB_{LR} - UB_{AMPL}}{UB_{LR}} \times 100\%$$

Note that for the problem sets $(N = 15, T = 3)$ and $(N = 15, T = 6)$, in the limitation of running time, the most of the instances are solved to optimality with AMPL. The results shown in the tables demonstrate the effectiveness of the proposed Lagrangian relaxation approach. Near optimal solutions are found for the problem sets $(N = 15, T = 3)$ and $(N = 15, T = 6)$ by the Lagrangian relaxation approach. For these two problem sets, the gaps between the solutions obtained from the Lagrangian relaxation approach and AMPL, in average, are 0.575% and 3.496% respectively, and the worst case is 1.831% and 6.134% respectively. The averaged gaps between the upper bound and lower bound for these two prob-

Instances	LB(eur)	UB(eur)	CPU(s)	Gap1
A25-0-T-3	884.74	1097.50	479.44	19.388
A25-1-T-3	1043.31	1356.80	657.19	23.104
A25-2-T-3	855.28	1094.31	570.58	21.840
A25-3-T-3	806.88	1030.50	600.23	21.788
A25-4-T-3	1066.41	1348.49	644.11	20.918
Average			590.30	21.407
A25-0-T-6	1410.46	1882.60	1042.90	25.080
A25-1-T-6	1782.30	2307.03	1159.56	22.751
A25-2-T-6	1371.52	1907.90	1141.33	28.118
A25-3-T-6	1234.55	1697.10	954.22	26.078
A25-4-T-6	1765.40	2314.35	1109.80	23.722
Average			1081.56	25.147

Table 5: Results of the instances with (N=25,T=3) and (N=25,T=6)

Instances	UB-LR(eur)	UB-AMPL(eur)	Gap2
A25-0-T-3	1097.50	1087.62	0.900
A25-1-T-3	1356.80	1280.59	5.616
A25-2-T-3	1094.31	1068.06	2.394
A25-3-T-3	1030.50	1017.63	1.249
A25-4-T-3	1348.49	1290.88	4.273
Average			2.886
A25-0-T-6	1882.60	1790.51	4.886
A25-1-T-6	2307.03	2254.43	2.280
A25-2-T-6	1907.90	1813.72	4.932
A25-3-T-6	1697.10	1630.73	3.910
A25-4-T-6	2314.35	2196.17	5.106
Average			4.221

Table 6: Solution comparisons of the LR approach and AMPL: (N=25, T=3) and (N=25, T=6)

lem sets are 4.958% and 20.163% respectively. Compared with the running time of AMPL, the averaged computation time is only 1 or 2 minutes. On the other hand, for the problem sets ($N = 25, T = 3$) and ($N = 25, T = 6$), the averaged values of *Gap1* are 21.407% and 25.147% respectively, whereas the averaged values of *Gap2* are 2.886% and 4.221% respectively. We observe that the MP-CIRP may become more flexible to make a tradeoff between the distribution costs and the inventory costs in multiple periods, with the increase of the planning horizon from $T = 3$ to $T = 6$. Thus the averaged values of *Gap2* rise to 3.496% and 4.221% for the problems sets ($N = 15, T = 6$) and ($N = 25, T = 6$) respectively, when they are compared with the corresponding values of problem sets with $T = 3$ (see Table 4 and Table 6). In terms of the computation times, high quality solutions can be obtained by the proposed Lagrangian relaxation approach.

6   CONCLUSION

The considered multi-period inventory routing problem (MP-CIRP) consists in a single depot distributing a single product to a set of customers having stationary demands, using a fleet of homogeneous vehicles over a given finite horizon. The objective is to determine the quantities to be delivered to the customers, the delivery time, and to design the vehicle delivery routes, so that the total distribution and inventory costs are minimized. In this paper, this MP-CIRP is formulated as a linear mixed-integer program with some side contraints. A Lagrangian relaxation approach is proposed to decompose the MP-CIRP and to derive both lower bound and upper bound for the problem. Computational results on some medium size instances demonstrate the effectiveness of the proposed Lagrangian relaxation approach. It can find high quality solutions for the MP-CIRP in a reasonable computational time. Numerical experiments on the large-scale problems are currently in preparation, and other extensions to the MP-CIRP and its solution approaches are also under investigation.

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