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Kernel reconstruction in a semilinear parabolic problem with integral overdetermination

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Abstract

A semilinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered. The missing kernel is recovered from an additional integral measurement. The existence, uniqueness and regularity of a weak solution is addressed. We design a numerical algorithm based on Rothe's method, derive a priori estimates and prove convergence of iterates towards the exact solution.

Key words: parabolic IBVP, convolution kernel, reconstruction, convergence, a priori estimates

1 Introduction

We want to determine the solution u and reconstructed a solely time-dependent convolution kernel K of the following nonlinear problem

$$\begin{cases} \partial_t u - \Delta u + K(t)h + (K * u)(t) = f(u, \nabla u), \quad \text{in } \Omega \times \Theta, \\ -\nabla u \cdot \mathbf{n} = g, \quad \text{on } \Gamma \times \Theta, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \end{cases} \tag{1}$$

where Ω is a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$, with $\partial\Omega = \Gamma$ and $\Theta = [0, T]$, $T > 0$, the time frame, when a global measurement

$$\int_\Omega u(\mathbf{x}, t)\mathrm{d}\mathbf{x} = m(t)$$

is known.

Such type of integro-differential problems arise for example elastoplasticity (cf. [1]) or in the theory of reactive contaminant transport. In [2] one considers the following differential equation

$$\partial_t C + \nabla \cdot (\mathbf{V}C) - \Delta C = \frac{-\rho_b}{n}\partial_t S$$

for the aqueous concentration C and sorbed concentration per unit mass of solid S with mass transformation rate in first order kinetics form of

$$\partial_t S = K_r(K_d C - S)$$

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with desorption rate K_r and equilibrium distribution coefficient K_d. This is indeed a problem of type (1) for $u = C$ with

$$K(t) = -\frac{\rho_b}{n} K_r^2 K_d e^{-K_r t}, \quad h(t) = -\frac{S_0}{K_r K_d} \quad \text{and} \quad f(x, \mathbf{r}) = \frac{-\rho_b}{n} K_r K_d x - \mathbf{V} \cdot \mathbf{r}.$$

We will prove the following existence and uniqueness result.

Theorem *Suppose f is bounded and Lipschitz continuous in all variables, $g \in \mathrm{C}^1(\Theta, \mathrm{L}^2(\Gamma))$, $h \in \mathrm{C}^0(\Theta, \mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(\Theta, \mathrm{L}^2(\Omega))$ and $\min_{t \in \Theta} |(h(t), 1)| \geq \omega > 0$, $m \in \mathrm{C}^2(\Theta, \mathbb{R})$ and $u_0 \in \mathrm{H}^2(\Omega)$. Then there exists a unique couple solutions $\langle u, K \rangle$ to (1), where $u \in \mathrm{C}(\Theta, \mathrm{H}^1(\Omega))$, $\partial_t u \in \mathrm{L}^\infty(\Theta, \mathrm{L}^2(\Omega))$ and $K \in \mathrm{C}(\Theta)$, $K' \in \mathrm{L}^2(\Theta)$.*

Moreover we construct a numerical model to solve this problem based on the variational formulation and Rothe's functions [3].

Algorithm: numerical scheme in pseudo code

input : $T > 0$, $n \in \mathbb{N}$ and functions f, g, h, m and u_0
output: kernel K and solution u at discrete time steps

$\tau \leftarrow T/n$;
$\theta \leftarrow [0 : \tau : T]$;
$\mathtt{K} \leftarrow \mathtt{zeros}(n+1)$;
$\mathtt{u} \leftarrow \mathtt{eval}(u_0, \theta)$;
$\mathtt{K}[0] \leftarrow \frac{1}{(h_0, 1)} \left((f(u_0, \nabla u_0), 1) - m'_0 - (g_0, 1)_\Gamma \right)$;
for $i = 1$ **to** n **do**
  $\mathtt{K}[i] \leftarrow \frac{1}{(h_i, 1) + m_0 \tau} \left((f_{i-1}, 1) - (g_i, 1)_\Gamma - \sum_{k=1}^{i-1} K_k m_{i-k} \tau - m'_i \right)$;
  $\mathtt{u}[i] \leftarrow \mathtt{solveEP}(B(u_i, \phi) = F_i(\phi))$;

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