# A Perfectly Matched Layer Based Modeling Technique for 1-D Periodic Microstrip Structures

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**Abstract:** For the simulation of one-dimensional (1-D) periodic microstrip structures, a Mixed Potential Integral Equation (MPIE) technique combined with the Method of Moments (MoM) is presented, solving for the unknown current density flowing within a unit cell of the periodic structure. At the crux of the formalism are the pertinent 1-D periodic Green's functions. These are obtained by invoking the Perfectly Matched Layer (PML)-paradigm. The proposed formalism is illustrated and validated computing the currents flowing within a representative unit cell of a 1-D periodic antenna array.

**Keywords:** 1-D periodic microstrip structures, Mixed Potential Integral Equation, Method of Moments, Perfectly Matched Layer

## 1 Introduction

The study of one-dimensional (1-D) periodic microstrip configurations has been a research topic for many years. The range of applications comprises antenna arrays, electromagnetic bandgap structures, frequency selective surfaces, leaky-wave antennas, etc. The periodic character of these structure can be exploited by using the Floquet-Bloch theorem, allowing to restrict the analysis of the complete structure (with infinite extent) to one representative unit cell.

Often, such simulation methods for 1-D periodic microstrip structures rely on a Mixed Potential Integral Equation (MPIE) combined with the Method of Moments (MoM), solving for the unknown current density flowing within a unit cell of the periodic structure [1]. This is also the case in this contribution. At the crux of the formalism are the pertinent 1-D periodic Green's functions. Unfortunately, these Green's functions are not (immediately) available in closed-form. In [1] the 1-D periodic Green's functions in the spatial domain are obtained through an efficient sum of inverse Fourier transforms. Here, a different approach is adopted, based on the Perfectly Matched Layer (PML)-paradigm [2].

In the sequel, all sources and fields are assumed to be time-harmonic with angular frequency $\omega$ and time dependencies $e^{j\omega t}$ are suppressed. Also, transverse to $z$ restrictions of vectors $\mathbf{v}$ are denoted $\check{\mathbf{v}} \equiv v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} = -\hat{\mathbf{z}} \times [\hat{\mathbf{z}} \times \mathbf{v}]$; here $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit Cartesian vectors.

## 2 Description of the formalism

Consider the 1-D periodic microstrip geometry of Fig. 1. A dielectric layer of thickness $d$ and with relative permittivity $\epsilon_r$ and relative permeability $\mu_r$ is placed on a perfect electrically conducting (PEC) ground plane. A 1-D periodic PEC metallization $\mathcal{M}$ resides at the substrate-air interface $z = d$. This metallization $\mathcal{M}$ is periodic along the $x$-direction and the period is $b$. In Fig 1, one unit cell consists of three square patches. A plane wave $\mathbf{E}^{\mathrm{PW}}(\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \mathbf{E}_0 e^{jk_0 z}$, with wavenumber $k_0 = \omega/c$

and where $c$ is the speed of light, impinges perpendicularly upon the substrate. This plane wave causes an incident field $\mathbf{E}^{\mathrm{i}}(\mathbf{r})$, which in turn induces unknown current densities $\check{\mathbf{J}}(\boldsymbol{\rho} \equiv x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + d\hat{\boldsymbol{z}})$ flowing on the metallization.

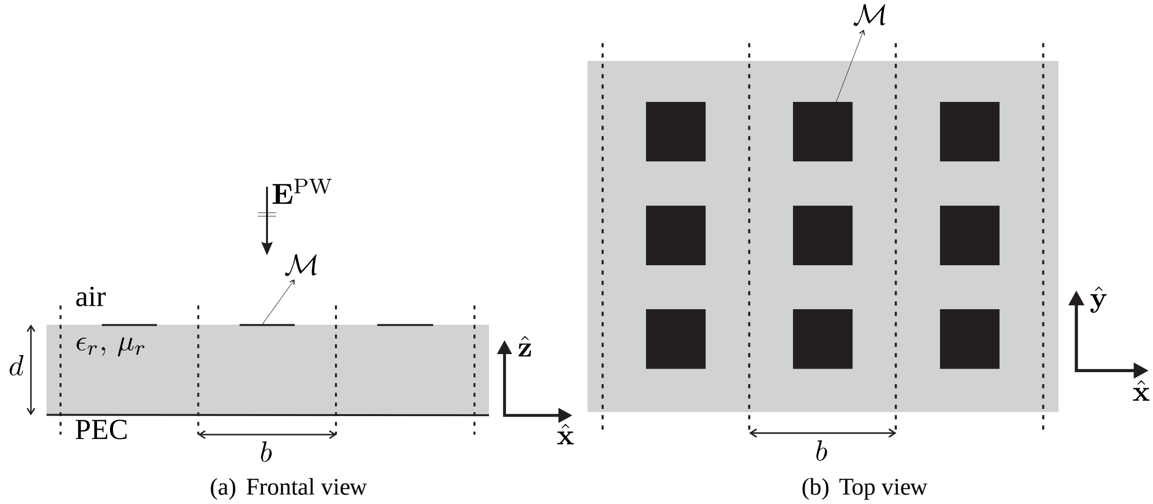



Fig. 1: Geometry of a 1-D periodic microstrip array.

Demanding that the total electric field tangential to the metallization $\mathcal{M}$ vanishes, results in the MPIE

$$\check{\mathbf{E}}^{\mathrm{i}}(\boldsymbol{\rho}) = j\omega \iint_{\mathcal{M}^m} G_A^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}')\check{\mathbf{J}}(\boldsymbol{\rho}')\, d\boldsymbol{\rho}' - \frac{1}{j\omega}\check{\boldsymbol{\nabla}} \iint_{\mathcal{M}^m} G_V^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}') \left(\check{\boldsymbol{\nabla}}' \cdot \check{\mathbf{J}}(\boldsymbol{\rho}')\right)\, d\boldsymbol{\rho}', \quad \forall \boldsymbol{\rho} \in \mathcal{M}^m, \tag{1}$$

with $\check{\boldsymbol{\nabla}} = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}$ and where $\mathcal{M}^m$ is that part of the metallization that lies within one unit cell $S_{\mathrm{uc}}^m = \{\boldsymbol{\rho} \equiv x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + d\hat{\boldsymbol{z}} : mb \leq x < (m+1)b, -\infty < y < \infty\},\ m \in \mathbb{Z}$. The integral equation (1) can be solved by the MoM [3], yielding an $N \times N$ linear system in $N$ unknown current expansion coefficients,

$$\mathbf{V} = \overline{\mathbf{Z}} \cdot \mathbf{I}. \tag{2}$$

which is solved using direct or iterative solution schemes.

At the crux of the above formalism are the pertinent 1-D periodic layered medium Green's function $G_A^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ for the magnetic vector potential and $G_V^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ for the electric scalar potential. These Green's functions are determined by the application of the PML-paradigm, described in [2]. To this end, the semi-infinite layer of air $z > d$ is closed by a PEC-backed PML, which corresponds to placing a PEC-plate at a complex distance $z = d + \mathcal{D}$ above the microstrip configuration [4]. This formalism yields the following series expansions for the Green's functions:

$$G_A^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}') = -\frac{j}{2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty} \frac{H_0^{(2)}\left(\beta_{\mathrm{TE},n}\sqrt{(x-x'-mb)^2+(y-y')^2}\right)}{M^{\mathrm{TE}}(\beta_{\mathrm{TE},n})}, \tag{3}$$

$$G_V^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}') = -\frac{j\omega^2}{2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty} \frac{H_0^{(2)}\left(\beta_{\mathrm{TE},n}\sqrt{(x-x'-mb)^2+(y-y')^2}\right)}{\beta_{\mathrm{TE},n}^2 M^{\mathrm{TE}}(\beta_{\mathrm{TE},n})} + \frac{j}{2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty} \frac{H_0^{(2)}\left(\beta_{\mathrm{TM},n}\sqrt{(x-x'-mb)^2+(y-y')^2}\right)}{\beta_{\mathrm{TM},n}^2 M^{\mathrm{TM}}(\beta_{\mathrm{TM},n})}. \tag{4}$$

Here, $H_0^{(2)}(\cdot)$ is the zeroth-order Hankel function of the second kind and

$$M^{\mathrm{TE}}(\beta) = \frac{d}{\mu_1}\frac{1}{\sin^2\gamma_1 d} - \frac{\cot\gamma_1 d}{\mu_1\gamma_1} + \frac{\mathcal{D}}{\mu_0}\frac{1}{\sin^2\gamma_0\mathcal{D}} - \frac{\cot\gamma_0\mathcal{D}}{\mu_0\gamma_0}, \tag{5}$$

$$M^{\mathrm{TM}}(\beta) = \frac{\epsilon_1\cot\gamma_1 d}{\gamma_1^3} + \frac{\epsilon_1 d}{\gamma_1^2\sin^2\gamma_1 d} + \frac{\epsilon_0\cot\gamma_0\mathcal{D}}{\gamma_0^3} + \frac{\epsilon_0\mathcal{D}}{\gamma_0^2\sin^2\gamma_0\mathcal{D}}, \tag{6}$$

with $\gamma_0 = \sqrt{k_0^2-\beta^2}$ and $\gamma_1 = \sqrt{k_0^2\epsilon_r\mu_r-\beta^2}$. The $\beta_{\mathrm{TE},n}$ and $\beta_{\mathrm{TM},n}$ represent the propagation constants of the eigenmodes of the PML-closed waveguide. All these propagation constants have a negative imaginary part, and hence, only a limited set of modes needs to be retained in the summations. Applying the Poisson summation to (3) and (4) yields

$$G_A^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}') = -\frac{j}{b}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\frac{e^{-j\frac{2\pi m}{b}(x-x')}e^{-j\sqrt{\beta_{\mathrm{TE},n}^2-\left(\frac{2\pi m}{b}\right)^2}|y-y'|}}{M^{\mathrm{TE}}(\beta_{\mathrm{TE},n})\sqrt{\beta_{\mathrm{TE},n}^2-\left(\frac{2\pi m}{b}\right)^2}} \tag{7}$$

$$\begin{aligned}G_V^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}') = &-\frac{j\omega^2}{b}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\frac{e^{-j\frac{2\pi m}{b}(x-x')}e^{-j\sqrt{\beta_{\mathrm{TE},n}^2-\left(\frac{2\pi m}{b}\right)^2}|y-y'|}}{\beta_{\mathrm{TE},n}^2 M^{\mathrm{TE}}(\beta_{\mathrm{TE},n})\sqrt{\beta_{\mathrm{TE},n}^2-\left(\frac{2\pi m}{b}\right)^2}}\\ &+\frac{j}{b}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\frac{e^{-j\frac{2\pi m}{b}(x-x')}e^{-j\sqrt{\beta_{\mathrm{TM},n}^2-\left(\frac{2\pi m}{b}\right)^2}|y-y'|}}{\beta_{\mathrm{TM},n}^2 M^{\mathrm{TM}}(\beta_{\mathrm{TM},n})\sqrt{\beta_{\mathrm{TM},n}^2-\left(\frac{2\pi m}{b}\right)^2}}.\end{aligned} \tag{8}$$

The convergence of series (3) and (7) for $G_A^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}')$, and (4) and (8) for $G_V^{\mathrm{per}}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ is thoroughly discussed in [2]. Both kinds of series are used in the MPIE-MoM, presented in this contribution. For $m = 0$, the series (3) and (4) reduce to the PML-based series for non-periodic microstrip substrates [5] and it is known that these series become impractical for $|\boldsymbol{\rho}-\boldsymbol{\rho}'| \to 0$. Alternative series for small $|\boldsymbol{\rho}-\boldsymbol{\rho}'|$, using the Sommerfeld-integral based non-periodic Green's function when $m = 0$, have been proposed in [2]. These series are also used in this contribution, i.e. for the calculation of the self-patch integrals.

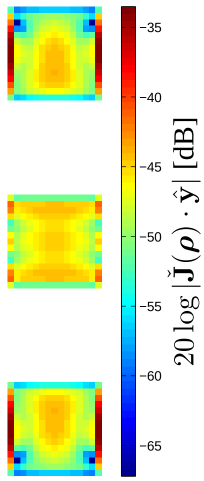
## 3 Numerical validation

The above described MPIE-MoM is implemented and validated by considering a representative example. Consider a microstrip substrate of thickness $d = 3.17$ mm, relative permittivity $\epsilon_r = 11.7$, and relative permeability $\mu_r = 1$. The metallization is the one shown in Fig. 1. One unit cell consist of three patches of dimension $10\ \mathrm{mm} \times 10\ \mathrm{mm}$. The period is $b = 40$ mm and the spacing between the patches along the $y$-direction is 10 mm. The angular frequency is $\omega = 2\pi 10$ GHz. The structure is illuminated by a plane wave $\mathbf{E}^{\mathrm{PW}}(\mathbf{r}) = e^{jk_0z}\hat{\mathbf{y}}$. The $y$-oriented current is compared with the solution of a reference program, the SVD-PML-MLMFA [6]. With the SVD-PML-MLMFA a finite array of $7\times 3$ patches, i.e. seven unit cells, is simulated and the current flowing on the central unit cell is used as a reference result. Both currents are plotted in Fig. 2. Although only seven periods have been used in the reference program, good agreement can already be observed, as such validating the approach described in this contribution.

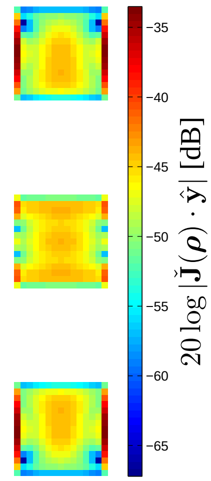
The results are obtained using a Linux-based 64-bit AMD Opteron 270 computer with 8 GB of RAM running at 2 GHz. A BiCGstab iterative solver was used to solve linear system (2). The SVD-PML-MLFMA and the iterative solver are set to reach 6 digits of accuracy.

## 4 Conclusion

Application of the PML-paradigm leads to closed-form 1-D periodic Green's functions for microstrip configurations. These Green's functions form the basis of an MPIE-MoM formalism for the determination




(a) Results obtained with the 1-D periodic MPIE-MoM as described in this contribution




(b) Results obtained with the SVD-PML-MLMFA as described in [6] (only the three central patches are shown)

Fig. 2: Comparison of the $y$-oriented current flowing within one unit cell of an array of patches.

of the currents flowing on 1-D periodic metallizations residing on a microstrip substrate. The method is implemented and tested by comparing it with the currents flowing on a finite array of patches, showing good agreement.

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