Changing the eigenfrequency spectrum using passive vibration absorbers

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Abstract

Using a substructure coupling technique the problem of assigning anti resonances is reformulated as a pole placement problem which can be analyzed in a more general framework. A possible approach to create these points of zero vibration is by attaching passive undamped vibration absorbers. As this approach is sensitive to changes in the excitation frequency, a robustness measure is proposed. Based on this measure, a better understanding is provided regarding the attachment location of the absorber and the absorber mass.

1 Introduction

A common approach in vibration control is to extend the main system with a passive localized addition. The term passive implies that no extra power is added to the system. In its simplest form this is realized by attaching a single degree of freedom undamped spring-mass system as introduced by Frahm [1], i.e. the dynamic absorber. When dealing with harmonic excitations, this device succeeds in suppressing the vibration completely at the point of attachment. Further more, in a resonant condition the eigenfrequencies of the main spring-mass system are shifted away from the disturbing excitation frequency, thereby creating a spectral gap. In practice, damping is added to the absorber to make it robust against changes in the forcing frequency and against uncertainties with respect to the physical implementation. A popular optimization procedure is developed by Den Hartog [2]. Although nowadays Den Hartog's approach is still popular, new and more complex absorbers are investigated. Creating absorbers that allow for higher vibration reduction over a wider frequency band forms a challenging problem. In the present paper the problem of vibration absorption is addressed in a more general systems theoretical setting. The effect of passive undamped vibration absorbers on the entire eigenfrequency spectrum of the system is studied. Changing the eigenfrequency spectrum in a well-considered way allows to create anti resonances such that undesired excitation frequencies are completely blocked.

In Section 2 we define a general framework for the pole allocation problem. Section 3 explains via a substructure coupling technique that the creation of anti resonances is actually a pole allocation problem and shows the restrictions on the obtainable spectrum. Section 4 discusses the influence of the mass and the attachment location of the absorber on the robustness of the control technique. Section 5 contains an example. The paper ends with a conclusion in Section 6.

2 Vibration control and passive absorbers

The mass-elastic systems under consideration are lumped parameter systems described by linear time invariant (LTI) differential equations. A well established control technique is the allocation of poles. Therefore

structural modifications and local additions of absorbers are classed in the general framework of pole placement techniques for LTI systems. This is used among others by [3] and [4].

The lumped mass-elastic system to be controlled is modelled as

$$M\ddot{q} + C\dot{q} + Kq = f \tag{1}$$

with mass matrix $M = M' \in R^{n\times n} > 0$, stiffness matrix $K = K' \in R^{n\times n} \geq 0$, damping matrix $C = C' \in R^{n\times n}$ and external load f. $q \in R^n$ is the vector of generalized coordinates. Consider n_a absorber masses $m_{a_i},\ i = 1 \ldots n_a$. Let $M_a \in R^{n_a \times n_a}$ be the diagonal matrix containing all the absorber masses and $q_a \in R^{n_a}$ the vector of generalized coordinates which configures the absorber masses. All generalized coordinates are gathered in the vector $\hat{q} \stackrel{\triangle}{=} \left[\begin{array}{cc} q & q_a \end{array}\right]' \in R^{\hat{n}}$ with $\hat{n} = n + n_a$. We extend the matrices K and C respectively to $\hat{K} \in R^{\hat{n}\times\hat{n}}$ and $\hat{C} \in R^{\hat{n}\times\hat{n}}$ by adding rows and columns of zeros. A new block diagonal matrix $\hat{M} \in R^{\hat{n}\times\hat{n}}$ is defined as :

$$\hat{M} \stackrel{\triangle}{=} \left[\begin{array}{cc} M & \\ & M_a \end{array}\right]$$

We only allow interconnections between absorber masses and lumped masses of the main system. An interconnection is a combination of a spring and a damper with constants $k_{a_i} \geq 0$ and $c_{a_i} \geq 0$. As a consequence every counteracting force due to the effect of the absorber is proportional with relative displacements or relative velocities between the absorber and the main system. Absorber masses are not mutually connected. Let $K_a \in R^{\hat{n}\times\hat{n}}$ be the interconnecting stiffness matrix and $C_a \in R^{\hat{n}\times\hat{n}}$ the interconnecting damping matrix. The effect of the absorber can be formulated as a control force u :

$$u = -K_a\hat{q} - C_a\dot{\hat{q}} \tag{2}$$

We define the state vector x as :

$$x = \left[\begin{array}{c} \hat{q} \\ \dot{\hat{q}} \end{array}\right] \in R^{2\hat{n}\times 2\hat{n}}$$

The control force (2) can be rewritten as :

$$u = -BFC_s x \tag{3}$$

The output vector $y \stackrel{\triangle}{=} C_s x$ contains all relative displacements and relative velocities that appear in the control force. A row of the matrix F contains zeros and the spring and the damper constant of one interconnection such that one component of vector Fy equals the total force due to one interconnection. The matrix B has elements $B(i,j) \in \{-1,\ 0,\ 1\}$ such that every force acts on the correct mass with the correct sign. Both B and C_s follow from the geometry of the interconnections and the number of absorber masses n_a . We define the matrices

$$A_s = \left[\begin{array}{cc} 0 & I \\ -\hat{M}^{-1}\hat{K} & -\hat{M}^{-1}\hat{C} \end{array}\right] \quad ; \quad B_s = \left[\begin{array}{c} 0 \\ -\hat{M}^{-1}\hat{B} \end{array}\right] \tag{4}$$

with unity matrix $I \in R^{n\times n}$. The state space representation of the main system with absorber becomes :

$$\dot{x} = (A_s - B_s F C_s)x \tag{5}$$

This clearly shows that the design of vibration absorbers can be interpreted as the design of a feedback matrix F to allocate the poles of (5). Some optimization or performance measure can be defined to calculate F via a numerical routine as in [3]. However, the choice of producing counteracting forces via absorbers restricts the obtainable spectrum and the number of the poles to be assigned. The root loci of (5) can visualize these restrictions.

The final purpose of vibration control is to minimize excessive vibration amplitudes. A possible approach is the assignment of anti resonances, i.e. to create locations on the structure where the vibration is completely blocked for one well-defined excitation frequency. In the following a substructure coupling technique is used to reformulate the assignment of anti resonances to the problem of assigning resonances such that the general form (5) can be used.

3 Assignment of anti resonances

Assigning anti resonances can be accomplished by structural modification [5, 6], where mass and stiffness properties of the structure are altered. An alternative is the attachment of undamped vibration absorbers [7]. Here, a substructure coupling technique is used that allows to determine in a well-considered way the changes of the structure needed to assign certain anti resonances. An important part in this matter is a thorough understanding of the location and the amount of modification. The modification is restricted to attaching vibration absorbers, although it could be extended to cope with structural modification as well.

Consider a structure modelled as an undamped multi-degree-of-freedom lumped mass-elastic system excited with load vector f (Figure 1).

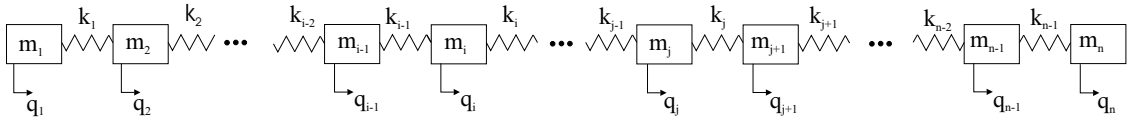

Figure 1: Main system

The equations of motion (1), become:

$$M\ddot{q} + Kq = f \tag{6}$$

Laplace transformation (zero initial conditions) yields:

$$\begin{aligned} Q(s) &= \left(Ms^2 + K\right)^{-1} F(s) && (7)\\ &= H(s)F(s) && (8) \end{aligned}$$

with H the transfer function matrix. For a harmonic load f the transfer function matrix becomes:

$$H(\omega) = (K - M\omega^2)^{-1} \tag{9}$$

The $(ij)^{th}$ element of this symmetric matrix is written as H_{ij} and denotes the transfer function between a harmonic load f_j applied at coordinate q_j (exciter point) and the response at coordinate q_i (receiver point). The zeros of H_{ij} are known as the anti resonances, the poles are known as the resonances or eigenfrequencies. While the resonances are the same for each H_{ij}, this is not true for the anti resonances.

3.1 Substructure coupling technique

The mass-elastic system given in Figure 1 is divided into three subsystems according to [8] (Figure 2). The choice of the subsystems depends on the exciter point E (coordinate q_j) and the receiver point R (coordinate q_i).

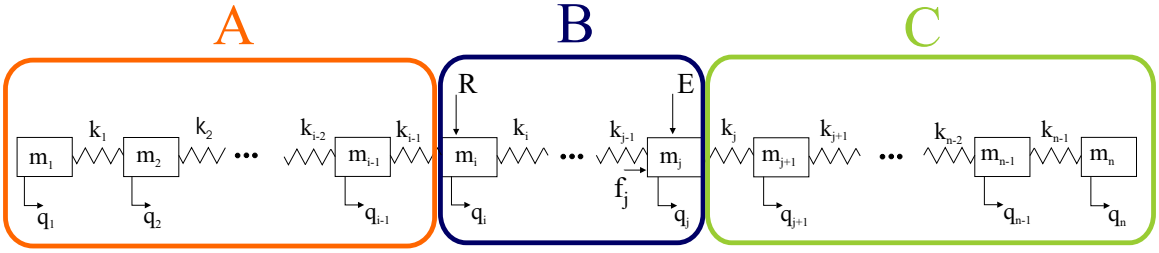

Figure 2: Main system divided into three subsystems

3.1.1 The multi-degree-of-freedom absorber

In [8] the main system is divided into three subsystems according to figure 2 where subsystems A and C are considered as multi-degree-of-freedom absorbers. It is shown that $H_{ij} \triangleq H_{RE}$ can be written as:

$$H_{ij}(\omega) \triangleq H_{RE}(\omega) = \alpha_{ij} \frac{\prod_{k=1}^{i-1}(\omega^2 - \hat{\mu}_k^2) \prod_{k=1}^{r}(\omega^2 - \tilde{\mu}_k^2)}{\prod_{k=1}^{n}(\omega^2 - \mu_k^2)} \tag{10}$$

In (10) α_{ij} is a constant, $\hat{\mu}_k$ are the eigenfrequencies of subsystem A when grounded at spring k_{i-1}, $\tilde{\mu}_k$ are the eigenfrequencies of subsystem C when grounded at spring k_j and μ_k are the eigenfrequencies of the overall system. Equation (10) shows that the anti resonances of H_{RE} are made up by the eigenfrequencies of the grounded subsystems A and C (as a consequence the number of anti resonances of H_{RE} decreases as the number of degrees of freedom in between R and E increases). Therefore, assigning anti resonances to H_{RE} is equivalent with assigning resonances to the grounded subsystems A and C.

3.1.2 Coupling technique exploring frequency response functions

A different substructure coupling technique is explained in [9]. Systems X and Y are coupled through a set of coupling coordinates yielding the assembled system Z (Figure 3).

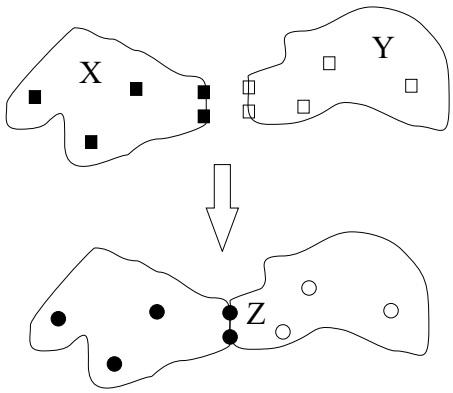

Figure 3: Coupling systems X and Y

This method is interesting as it denotes the relation between the frequency response functions of the two coupled systems X and Y and the frequency response functions of the overall system Z:

$$[H_Z]_{PP} = [H_X]_{PP} - [H_X]_{PS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_X]_{SP} \tag{11}$$
$$[H_Z]_{PP} = [H_X]_{PS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_Y]_{SS} \tag{12}$$
$$[H_Z]_{PT} = [H_X]_{PS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_Y]_{ST} \tag{13}$$
$$[H_Z]_{SS} = [H_X]_{SS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_Y]_{SS} \tag{14}$$
$$[H_Z]_{ST} = [H_X]_{SS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_Y]_{ST} \tag{15}$$
$$[H_Z]_{TT} = [H_Y]_{TT} - [H_Y]_{TS}([H_X]_{SS} + [H_Y]_{SS})^{-1}[H_Y]_{ST} \tag{16}$$

where S denotes the set of coupling coordinates, P and T denote the set of free coordinates corresponding to systems X and Y respectively.

In Section 3.1.3 we apply this coupling technique subsequently to the system of Figure 2.

3.1.3 Coupling of three systems

First consider the main system of figure 2 as the coupling of two systems, system A and system (B+C). As the right end of system A ends with a spring, we extend this system with a weightless attachment point W_A in order to make the coupling technique of Section 3.1.2 feasible (Figure 4).

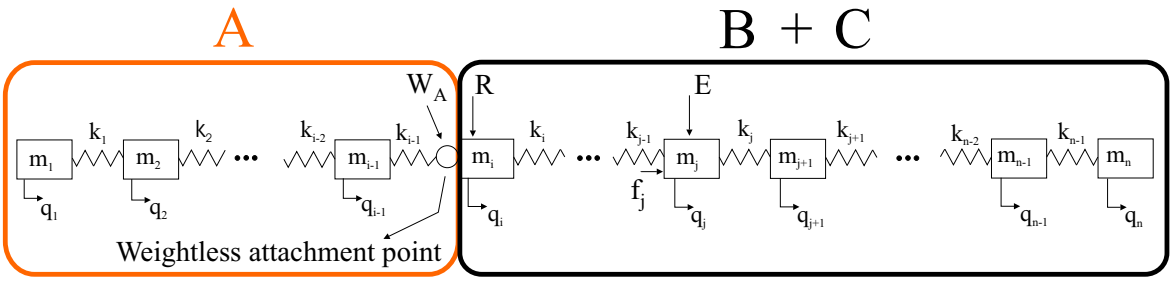

Figure 4: Coupling of subsystem A and subsystem (B+C)

H_{RE} can be written as (dependence on ω is omitted):

$$H_{RE} = [H_A]_{W_A W_A} \left\{ [H_A]_{W_A W_A} + [H_{(B+C)}]_{RR} \right\}^{-1} [H_{(B+C)}]_{RE} \tag{17}$$

where $[H_A]_{W_A W_A}$ denotes the element corresponding to point W_A in the frequency response function matrix $[H_A]$:

$$[H_A] = \begin{bmatrix} k_1 - m_1\omega^2 & -k_1 & & & \\ -k_1 & k_1 + k_2 - m_2\omega^2 & -k_2 & & \\ & & \ddots & & \\ & & -k_{i-2} & k_{i-2} + k_{i-1} - m_{i-1}\omega^2 & -k_{i-1} \\ & & & -k_{i-1} & k_{i-1} \end{bmatrix}^{-1} \tag{18}$$

$$\triangleq (K_A - M_A\omega^2)^{-1} \tag{19}$$

Next the system (B+C) is decoupled into B and C. Again a weightless attachment point W_C is added at the left of C (Figure 5).

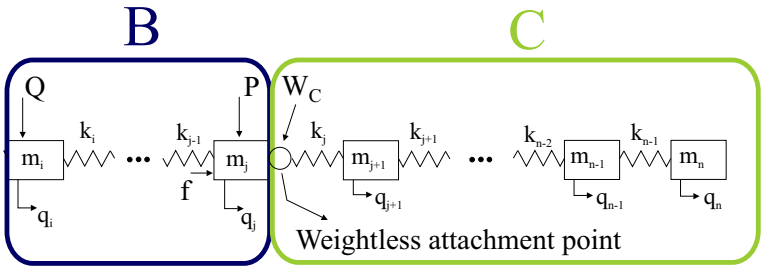

Figure 5: Coupling of subsystem B and subsystem C

$[H_{(B+C)}]_{RE}$ becomes:

$$[H_{(B+C)}]_{RE} = [H_B]_{RE} \left\{ [H_B]_{EE} + [H_C]_{W_C W_C} \right\}^{-1} [H_C]_{W_C W_C} \tag{20}$$

with

$$[H_B] = \begin{bmatrix} k_i - m_i\omega^2 & -k_i & & & \\ -k_i & k_i + k_{i+1} - m_{i+1}\omega^2 & -k_{i+1} & & \\ & & \ddots & & \\ & & -k_{j-2} & k_{j-2} + k_{j-1} - m_{j-1}\omega^2 & -k_{j-1} \\ & & & -k_{j-1} & k_{j-1} - m_j\omega^2 \end{bmatrix}^{-1} \quad (21)$$

$$\triangleq (K_B - M_B\omega^2)^{-1} \quad (22)$$

$$[H_C] = \begin{bmatrix} k_j & -k_j & & & \\ -k_j & k_j + k_{j+1} - m_{j+1}\omega^2 & -k_{j+1} & & \\ & & \ddots & & \\ & & -k_{n-2} & k_{n-2} + k_{n-1} - m_{n-1}\omega^2 & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1} - m_n\omega^2 \end{bmatrix}^{-1} \quad (23)$$

$$\triangleq (K_C - M_C\omega^2)^{-1} \quad (24)$$

By combining (17) and (20), H_{RE} becomes:

$$H_{RE} = \frac{[H_A]_{W_AW_A}[H_B]_{RE}[H_C]_{W_CW_C}}{\left\{[H_A]_{W_AW_A} + [H_{(B+C)}]_{RR}\right\}\left\{[H_B]_{EE} + [H_C]_{W_CW_C}\right\}} \quad (25)$$

Simplifying (25) shows that the anti resonances of H_{RE} are given by the anti resonances of $[H_A]_{W_AW_A}$, $[H_B]_{RE}$ and $[H_C]_{W_CW_C}$. This means the problem of assigning anti resonances to H_{RE} is reduced to assigning anti resonances to the subsystems A,B and C.

More precisely:

- The anti resonances of $[H_A]_{W_AW_A}$ are given by deleting both row and column corresponding to point W_A in $(K_A - M_A\omega^2)$ (19) and taking the determinant. Hence, the anti resonances of $[H_A]_{W_AW_A}$ are given by the resonances of subsystem A when grounded at spring k_{i-1}.
- The same analysis holds for subsystem C. The anti resonances of $[H_C]_{W_CW_C}$ are given by the resonances of subsystem C when grounded at spring k_j.
- As R and E denote the first and last coordinate of subsystem B, $[H_B]_{RE}$ has no anti resonances.

The same results as given in Section 3.1.1 are retrieved. However, in the subsequent analysis, H_{RE} given by (25) appears to be more useful than (10).

3.2 Assigning the anti resonances

As shown in Section 3.1, the analysis of assigning anti resonances to H_{RE} can be divided into three parts (25). As the analysis for subsystems A and C (Figure 2) appears quite similar, they are discussed together.

a) Subsystem B

Attaching an undamped vibration absorber anywhere along subsystem B (at coordinates q_i to q_j) creates an anti resonance for H_{RE} at the tuning frequency $\omega_a^2 = \frac{k_a}{m_a}$ of the absorber. This can be seen from (22). Assume without loss of generality a vibration absorber attached at coordinate $v'q_B = q_{i+1}$ with

$q_B = [q_i\ q_{i+1}\ \cdots\ q_j]'$ and $v = [0\ 1\ 0\ \cdots\ 0]'$. This extends B with one degree of freedom. Equation (22) becomes:

$$[H_B] = \begin{bmatrix} & & & 0 \\ & [K_B + vv'ka - M_B\omega^2] & & -k_a \\ & & & 0 \\ & & & 0 \\ 0 & -k_a & 0 \cdots 0 & k_a - m_a\omega^2 \end{bmatrix}^{-1} \tag{26}$$

The anti resonances are given by the roots of the numerator of element $[H_B]_{RE}$ in $[H_B]$:

$$(-k_i)(-k_{i+1}(-k_{i+2}) \cdots (-k_n - 1)(k_a - m_a\omega^2) = 0 \tag{27}$$

Hence, the absorber effectively blocks excitation signals with frequency $\omega = \omega_a$.

Remark As $[H_B]_{RE}$ in (25) has no anti resonances, changing the mass or stiffness properties of the structure (i.e. structural modification)in this area has no influence on the anti resonances of H_{RE}. However, it does change the resonances of the overall system. Therefore, vibration reduction can still be achieved by shifting the resonances towards the anti resonances and thereby creating a pole-zero cancellation [10]. This will not be discussed further on.

b) Subsystems A and C

Assigning anti resonances to $[H_A]_{W_AW_A}$ and $[H_C]_{W_CW_C}$ corresponds to assigning resonances to the grounded subsystems A_g and C_g (Figure 6).

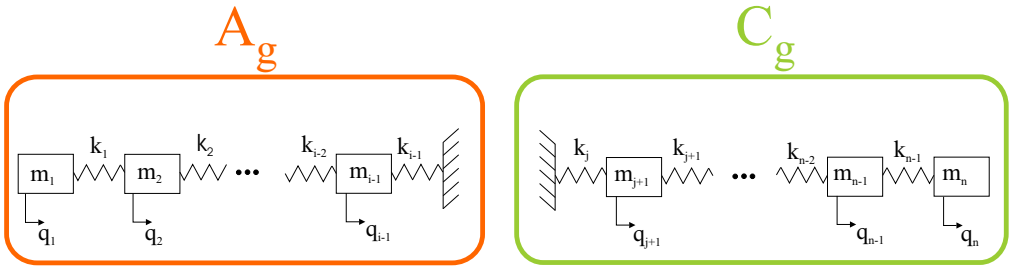

Figure 6: Grounded subsystems A and C

This means the resonances of A_g or C_g need to be shifted to the required anti resonances of H_{RE}. This defines a pole placement problem which can be analyzed through Section 2. Here we focus on the assignment of one resonance by attaching a single-degree-of-freedom undamped vibration absorber. Attaching a vibration absorber with coordinate q_a, mass m_a and stiffness k_a to the main system at location $v'q$ changes the equations of motion given in (6) as follows:

$$M\ddot{q} + Kq + vk_a(v'q - q_a) = f \tag{28}$$

$$m_a\ddot{q}_a + k_a(q_a - v'q) = 0 \tag{29}$$

Taking the Laplace transform (zero initial conditions) and eliminating q_a yields:

$$Q(s) = \hat{H}(s)F(s) \tag{30}$$

where

$$\hat{H}(s) = \left\{ Ms^2 + K + v\frac{m_ak_as^2}{m_as^2 + k_a}v' \right\}^{-1} \tag{31}$$

Using the Sherman-Morrison matrix inversion theorem [12] the transfer function matrix given in (31) can be rewritten as follows:

$$\hat{H}(s) = H(s) - H(s)v\left[v'H(s)v + \frac{m_a s^2 + k_a}{m_a k_a s^2}\right]^{-1} v'H(s) \tag{32}$$

Pre and post multiplying of Eq. (32) with v' and v respectively gives:

$$v'\hat{H}(s)v = v'H(s)v - v'H(s)v\left[v'H(s)v + \frac{m_a s^2 + k_a}{m_a k_a s^2}\right]^{-1} v'H(s)v \tag{33}$$

or simplified:

$$v'\hat{H}(s)v = \frac{v'H(s)v}{1 + \sigma(s)s^2 v'H(s)v} \tag{34}$$

where

$$\sigma(s) = \frac{m_a k_a}{m_a s^2 + k_a} \tag{35}$$

$v'\hat{H}v$ is the transfer function between the generalized force at coordinate $v'q$ and its response at the same coordinate. The resonances of the system with absorber are given by ($s = j\omega$):

$$1 - \frac{m_a k_a \omega^2}{k_a - m_a \omega^2} v'H(\omega)v = 0 \tag{36}$$

Solving for k_a, equation (36) becomes:

$$k_a = \frac{m_a \omega^2}{1 - m_a \omega^2 v'H(\omega)v} \tag{37}$$

As stated by [11], to obtain positive values for k_a, the following inequality should be satisfied:

$$v'H(\omega)v < \frac{1}{m_a \omega^2} \tag{38}$$

Given the attachment location $v'q$ and the absorber mass m_a, (38) shows that not every frequency can be assigned as a resonance. However, decreasing m_a relaxes the inequality and allows for a wider frequency range to assign. Moreover, in the limit $m_a \to 0$ every resonance can be assigned.

4 Robustness

In the previous section it is outlined how anti resonances can be created for a certain frequency response function H_{RE} with an undamped vibration absorber. The main problem with this design is the potential lack of robustness. A slight change of the excitation frequency could introduce a new resonant condition causing excessive vibration amplitudes. As our main concern is vibration reduction, we seek the absorber design with the highest robustness according to Proposition 4.1:

Proposition 4.1 *Increasing the robustness of the absorber design implies increasing the minimal spectral gap between the created anti resonance and its neighboring resonances resulting from (36).*

In other words: When an anti resonance is created by attaching an absorber, the distance in frequency between this anti resonance and the new resonances next to this anti resonance should be as large as possible. Where 'as large as possible' means with respect to the maximum allowable absorber mass and the available attachment locations.
We recall the following theorem of Arnold, (1991), p. 253. [13]:

Theorem 4.1 *Under an increase of rigidity of a system all characteristic frequencies are increased.*

and state Theorem 4.2:

Theorem 4.2 *Consider an undamped absorber attached at coordinate $v'q$ with tuning frequency ω_a. Increasing the absorber mass shifts the resonances away from ω_a in a monotone way. They can not be shifted beyond the neighboring anti resonances of $v'H(\omega)v$.*

An extended proof is omitted here. The proof is based on Theorem 4.1, equation (34) and interlacing properties of the poles and zeros of $v'\hat{H}(\omega)v$ and $v'H(\omega)v$. Figure 7 shows the pole-zero maps of both $v'H(\omega)v$ and $v'\hat{H}(\omega)v$ for a certain absorber mass m_a. The arrows illustrate the shift of the resonances as m_a increases.

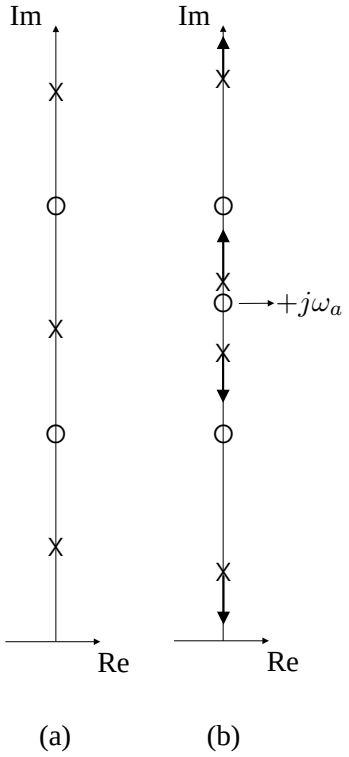

Figure 7: (a) Without absorber: poles and zeros of $v'H(\omega)v$; (b) With absorber: poles and zeros of $v'\hat{H}(\omega)v$ as a function of the absorber mass m_a.

The influence of both the absorber mass m_a and the attachment location $v'q$ is analyzed according to Proposition 4.1. We focus on resonant conditions, i.e. the excitation frequency equals one of the resonance frequencies ω_i of the main system. The analysis differs whether the absorber is attached at subsystem B or at subsystems A and C.

4.1 Subsystem B

To obtain an anti resonance at $\omega = \omega_i$, the tuning frequency ω_a of the absorber should be taken according to (27):

$$\omega_a = \omega_i \tag{39}$$

This reduces the absorber design to the determination of two parameters, the absorber mass m_a and the attachment location $v'q$. According to Theorem 4.2 increasing the absorber mass increases the robustness

of the design as the neighboring resonances are shifted further away. The achievable shift can be analyzed with the general framework explained in Section 2. Off course this increase is also limited due to practical considerations.

The determination of the attachment location is less straightforward. When dealing with a sufficiently low absorber mass and well separated resonances the single mode approach is valid [14]. This means the modal form of $v'H(\omega)v$:

$$v'H(\omega)v = v'(K - M\omega^2)^{-1}v \tag{40}$$

$$= \sum_{k=1}^{n} \frac{(v'e_k)^2}{\omega_k^2 - \omega^2} \tag{41}$$

with resonances ω_k ($k = 1, ..., n$) and mass normalized eigenvectors e_k ($k = 1, ..., n$) can be approximated by:

$$v'H(\omega)v \approx \frac{(v'e_i)^2}{\omega_i^2 - \omega^2} \tag{42}$$

From Eq. (36) and (42) it can be seen that for a given mass m_a, the shift of the resonances near ω_i will increase as $(v'e_i)^2$ increases. We recover the well known result that an anti nodal location, i.e. the location with the maximum value for $|v'e_i|$, yields the best location.

When the single mode approach is not valid, other modes have to be taken into consideration. As an anti resonance is to be created for ω_i, the neighboring modes ω_{i-1} and ω_{i+1} have the highest influence. Therefore, when determining the attachment location in this case, the mode activities $|v'e_{i-1}|$ and $|v'e_{i+1}|$ will become important. An alternative approach is looking at the neighboring anti resonances of ω_i as these define the limits for the shift of the new resonances (Theorem 4.2).

4.2 Subsystems A and C

As opposed to the analysis for subsystem B, the absorber's parameters (m_a, ω_a and $v'q$) need to be determined in an integrated way as they all influence each other. As shown by (38), choosing the attachment location $v'q$ a priori limits the assignable resonances of these subsystems (assignable anti resonances for the overall system) for a certain absorber mass m_a and vice versa. Therefore we determine the attachment location that leaves the widest freedom of choice for m_a and ω_a in assigning a certain resonance. Then m_a and ω_a are determined in order to maximize the robustness (Proposition 4.1). The analysis is equivalent for both subsystems A and C.

In the single mode approach the location with the maximum value for $|v'e_i|$, yields the best location. If this single mode approach is not valid (e.g. a large absorber mass) a different analysis should be used. Figure 8 shows the alternating poles and zeros along the imaginary axis of $v'_A H_A v_A$ corresponding to subsystem A ($v'_A q_A$ denotes a location on A).

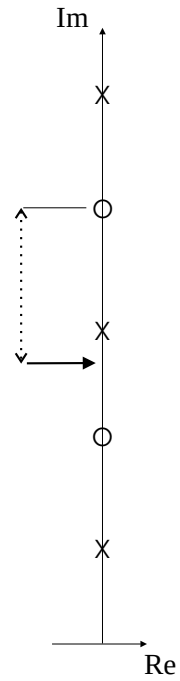

Figure 8: Possible tuning frequencies ω_a for the absorber to assign a certain frequency.

Assume a resonance frequency has to be assigned at the frequency indicated by the arrow. According to Theorem 4.2 an absorber with tuning frequency ω_a anywhere along the dotted line can assign this frequency. For both A and C we determine the attachment location that maximizes the length of this dotted line. The remaining parameters m_a and ω_a are chosen as follows. Increasing m_a increases the robustness of the design towards the overall system. The tuning frequency ω_a is derived from (37).

4.3 Discussion

For each subsystem the absorber design is determined that maximizes the robustness w.r.t. changes in the excitation frequency. The question remains where the absorber should be attached (subsystem A, B or C) to obtain the overall maximal robustness. Intuitively one would assume subsystem B as in this region the receiver point R and the exciter point E are located. However this is not clear a priori. An example regarding this problem is given in Section 5.

5 Example

Consider an undamped spring-mass system (8 DOF) (Fig. 9). The excitation frequency at point E is equal to ω_2 (Table 1). For this frequency an anti resonance has to be assigned at point R.

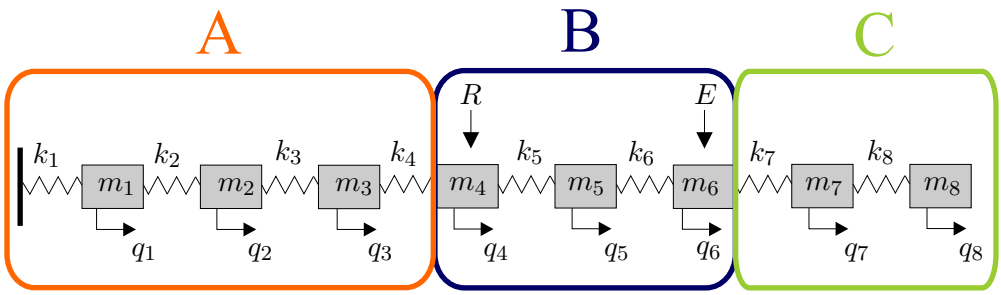

Figure 9: Spring-mass system with 8 degrees of freedom

Table 1: Modal parameters

Mass values (kg)	Spring values (N/m)	Resonances (rad/s)
$m_1 = 1$	$k_1 = 1$	$\omega_1 = 0.131$
$m_2 = 2$	$k_2 = 2$	$\omega_2 = 0.409$
$m_3 = 3$	$k_3 = 4$	$\omega_3 = 0.730$
$m_4 = 4$	$k_4 = 3$	$\omega_4 = 1.115$
$m_5 = 5$	$k_5 = 1$	$\omega_5 = 1.184$
$m_6 = 3$	$k_6 = 2$	$\omega_6 = 1.600$
$m_7 = 4$	$k_7 = 3$	$\omega_7 = 1.699$
$m_8 = 2$	$k_8 = 2$	$\omega_8 = 2.245$

The system is divided into three subsystems according to Section 3.1. For each subsystem the absorber parameters (attachment location and tuning frequency ω_a) are determined that maximize the robustness according to Proposition 4.1. For the absorber mass, m_a = 1 kg is taken. As we are dealing with a sufficiently low absorber mass and well separated resonances, the single mode approach is valid. The eigenvector e_2 (Table 2) corresponding to resonance ω_2 indicates the best attachment location.

Table 2: Eigenvector e_2

Location	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
	-0.192	-0.272	-0.289	-0.264	0.010	0.121	0.188	0.225

Table 3: Absorber parameters for subsystems A, B and C.

Subsystem	Attachment location	ω_a (rad/s)	Neighboring resonances (rad/s)	Spectral gap (rad/s)
A	q_3	$\omega_a = 0.422$	0.360-0.481	0.049
B	q_4	$\omega_a = 0.409$	0.360-0.467	0.049
	q_6	$\omega_a = 0.409$	0.389-0.438	0.020
C	q_8	$\omega_a = 0.465$	0.381-0.489	0.028

The results are summarized in Table 3. The robustness is determined by the minimal spectral gap between the anti resonance (0.409 rad/s) and its neighboring resonances. This is shown in the last column of Table 3. As could be expected, attaching the absorber at the receiver point yields the best result. However, the absorber attached at q_3 in subsystem A achieves the same robustness.

6 Conclusion

We discussed the ability of passive undamped vibration absorbers to create anti resonances for a mass elastic system as this is an important option in vibration control. A substructure coupling technique showed the equivalence between the assignment of anti resonances and the assignment of resonances. Therefore the design of passive vibration absorbers was fitted into the general pole allocation problem of linear time invariant systems.

Robustness was an important issue. It was defined as avoiding the situation where a desired anti resonance is very close to a neighboring resonance such that some variation on the excitation frequency can be allowed. The choice of the location of the absorber as well as the absorber mass were shown to have significant influence in this matter.

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