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HOW TO MATHEMATICALLY CLASSIFY PHYSICAL QUANTITIES?

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The mathematical structure classifying the physical quantities is presently *unknown*. We prove that classes of physical quantities that are expressed according to the SI convention are represented by integer lattice points of the seven dimensional integer lattice. The Chebyshev norm is the measure for the major classification. We demonstrate that the mathematical structure classifying the physical quantities is based on leader classes. A leader class is a constellation of integer lattice points, that are mathematically connected through a signed permutation of the integer lattice point coordinates. We assign to each leader class representative a Gödel number that creates an order between the representatives of the classes of physical quantities. The appendices contain a preliminary classification of common physical quantities and also numerical data useful as starting point for the further classification of new physical quantities.

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1. Introduction

We know from chemistry the celebrated table of Mendeleev that provides a structure and order in the chemical elements. That discovery was a major step in further development of chemistry. A similar structure is lacking in physics. This article reports about a mathematical structure that classifies the physical quantities. James Clerck Maxwell addressed partially the research question in his article "On the mathematical classification of physical quantities" [1] but didn't elaborate on the mathematical structure. We follow a bottom-up approach starting from the building blocks of the *physics language*, that are the *physical quantities*. Each physical quantity is represented by a symbol or label. Physical quantities are found in the form of scalars, vectors, multi-vectors, matrices and/or tensors. All the physical quantities are eventually *measured* through their respective components and thus we restrict our analysis to the *components* of physical quantities. The choice of a system of units [2,3,4,5] and the number of dimensions are open issues [5,6] amongst physicists. In the limit one thinks of dimensionless physics [5]. Throughout this article we will adopt the *convention* of the SI units and dimensions for the physical quantities.

1.1. *Outline of the paper*

Section 1 comprises the definitions and preliminaries that are needed to allow a mathematical classification of physical quantities. In section 2, we discuss the images of classes of physical quantities as integer lattice points of $\mathbb{Z}^7$. We propose in section 3 that the classification of classes of physical quantities is based on an equivalence relation applied to measure polytopes. Section 4 contains the future work and conclusion of the present research.

1.2. *Preliminaries*

A component of a physical quantity is a quantity that is used in the description of physical processes. Let a universal set of components of physical quantities be $\mathbb{U}_p$. We partition this set in equivalence classes with notation $[a]$ where a is the representative of the equivalence class. In the class energy $[E]$ we find physical quantities like potential energy, kinetic energy, work, heat, internal energy, ... which are all represented by the class $[E]$. A set of base quantities is a finite number of classes of physical quantities, which by *convention* are regarded as *dimensionally independent* in a system of physical quantities and equations defining the relationships between them. The *International System of Units (SI)* base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity. The set of classes of base physical quantities is called $\mathcal{B} \doteq \{[l],[m],[t],[i],[T],[n],[L]\}$. The base units are the set $\mathcal{U} \doteq \{u_i \mid u_1 = \mathrm{m},\ u_2 = \mathrm{kg},\ u_3 = \mathrm{s},\ u_4 = \mathrm{A},\ u_5 = \mathrm{K},\ u_6 = \mathrm{mol},\ u_7 = \mathrm{cd}\}$. The dimensional product is the expression of a class of a physical quantity as a product of powers of base quantities. Each class of a physical quantity has parameters X^i, called dimensional exponents. We write $[a]$ as function of the SI base units $u_i \in \mathcal{U}$ and the dimensional exponents $X^i \in \mathbb{Z}$,

$$[a] \doteq \{a_1\} \cdot \prod_{i=1}^{7} u_i^{X^i} \,, \tag{1}$$

where the physical quantity $[a]$ of the idealized physical system assumes a numerical real value $\{a_1\} \in \mathbb{R}$. It is known that some physical quantities (rms of a quantity, noise spectral density, specific detectivity, thermal inertia, thermal effusivity, ...) are defined as the *square root* of some product or quotient of other physical quantities. These physical quantities will have fractional exponents, where $X^i \in \mathbb{Q}$ and so will not comply with the above definition. Each of these physical quantities are, by a proper exponentiation, transformed to a physical quantity having integer exponents which then complies with the above definition.

2. Image of a class of physical quantities

Let the set of integer septuples $\mathbb{Z}^7 \doteq \{(X^1, \ldots, X^7) \mid X^i \in \mathbb{Z}\}$ be called the 7-dimensional integer lattice. Classes of physical quantities can be imaged on lattice points in the 7-dimensional integer lattice. A set of lattice points is called a *lattice constellation* [7]. The image of a class of physical quantities $[a]$ has the notation $\breve{a}$ which clearly indicates the distinction with physical quantities represented by scalars, vectors, multi-vector, matrices and/or tensors. The image of the class of dimensionless physical quantities $[\kappa]$ has the notation $\breve{o}$ which represents the origin of the integer lattice $\mathbb{Z}^7$. We will see further that there is a mathematical justification for this notation. The image of the class energy $[E]$ is $\breve{E}$. We denote the function dimensional exponent as 'dex'.

Definition 1. The function 'dex' is defined from $\mathbb{U}_p$ into $\mathbb{Z}^7$ and formally as dex : $\mathbb{U}_\mathrm{p} \to \mathbb{Z}^7 \mid \mathrm{dex}\,([a]) \doteq \breve{a} = (A^1, \ldots, A^7)$ where $A^i \in \mathbb{Z}$.

The A^is are the contravariant components of the lattice point $\breve{a}$. This means that the exponents of the units of a class of physical quantities, taken in the *correct order*, form the coordinates of a point in the integer lattice $\mathbb{Z}^7$. Every possible integer lattice point is the image of one class of physical quantities and so the mapping 'dex' is bijective from $\mathbb{U}_p$ on $\mathbb{Z}^7$ and expresses 'dex' as an isomorphism between $\mathbb{U}_p$ and $\mathbb{Z}^7$. The Abelian group $\mathbb{Z}^7$ is a $\mathbb{Z}$-module. The family $\{\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4, \mathbb{Z}^5, \mathbb{Z}^6\}$ are $\mathbb{Z}$-submodules of $\mathbb{Z}^7$. The $\mathbb{Z}$-module $\mathbb{Z}^7/\mathbb{Z}$ is called the quotient module of $\mathbb{Z}^7$ with respect to $\mathbb{Z}$. The prerequisite for the creation of a vector space is the existence of a field $\mathbb{F}$ for the scalars. The elements of the vector space are then vectors. This justifies the notation $\breve{a}$, indicating that the elements of $\mathbb{Z}^7, +, \cdot$ are *not* vectors $\boldsymbol{a}$. We select 7 linearly independent lattice points $\breve{e}_1, \ldots, \breve{e}_7$ of $\mathbb{Z}^7$. The $\breve{e}_i$s form a covariant basis [8] for the integer lattice in $\mathbb{Z}^7$. Every lattice point is expressed in a unique way as the linear combination: $\breve{x} = X^1\breve{e}_1 + \ldots + X^7\breve{e}_7$ where the coefficients X^i are called the contravariant components of $\breve{x}$. The inner product is defined as the expression: $\breve{x} \cdot \breve{y} = \sum_{i=1}^{7}\sum_{j=1}^{7} a_{ij}X^iY^j$ where $a_{ij} = a_{ji}$. Consider seven lattice points $\breve{e}^i$ satisfying the expression $\breve{e}^i = \sum_{k=1}^{7} a^{ik}\breve{e}_k$. This contravariant basis spans the space $\mathbb{Z}^7$ resulting in the equations $\sum_{i=1}^{7} a_{ij}\breve{e}^i = \sum_{i=1}^{7}\sum_{k=1}^{7} a_{ij}a^{ik}\breve{e}_k = \sum_{k=1}^{7} \delta_j^k\breve{e}_k = \breve{e}_j$. A lattice point $\breve{x}$ has covariant components X_i, such that $\breve{x} = \sum_{i=1}^{7} X_i\breve{e}^i$. These components are related to the contravariant components by the expressions: $X^j = \sum_{i=1}^{7} a^{ij}X_i$ and $X_i = \sum_{i=1}^{7} a_{ij}X^j$. With this notation the inner product is represented as $\breve{x} \cdot \breve{y} = \sum_{i=1}^{7} X^iY_i = \sum_{k=1}^{7} X_kY^k$. Observe that, since $\breve{e}^i \cdot \breve{e}_j = \sum_{i=1}^{7} a^{ik}\breve{e}_k \cdot \breve{e}_j = \sum_{i=1}^{7} a^{ik}a_{jk} = \delta_j^i$, each $\breve{e}^i$ is orthogonal to every $\breve{e}_j$ except $\breve{e}_i$. We obtain that $\breve{e}^i \cdot \breve{e}_j = 1$. We are free to select seven basis lattice points. These points will receive the agreed [9] symbol for the dimension. We define: $\breve{l} \doteq \breve{e}_1 = L = (1,0,0,0,0,0,0)$, $\breve{m} \doteq \breve{e}_2 = M = (0,1,0,0,0,0,0)$, $\breve{t} \doteq \breve{e}_3 = T = (0,0,1,0,0,0,0)$, $\breve{i} \doteq \breve{e}_4 = I = (0,0,0,1,0,0,0)$, $\breve{T} \doteq \breve{e}_5 = \Theta = (0,0,0,0,1,0,0)$, $\breve{n} \doteq \breve{e}_6 = N = (0,0,0,0,0,1,0)$, $\breve{L} \doteq \breve{e}_7 = J = (0,0,0,0,0,0,1)$, with $\breve{e}_i \in \mathbb{Z}^7$. This basis generates a *cubic lattice* [10] that is orthonormal. We claim without giving proofs of the following "dex" identities:

$$\forall [a],[b] \in \mathbb{U}_p \mid \mathrm{dex}\,([a][b]) = \mathrm{dex}\,(a) + \mathrm{dex}\,(b) \ , \tag{2a}$$

$$\forall [a],[b] \in \mathbb{U}_p \mid \mathrm{dex}\left(\frac{[a]}{[b]}\right) = \mathrm{dex}\,(a) - \mathrm{dex}\,(b) \ , \tag{2b}$$

$$\forall [a],[b],[c] \in \mathbb{U}_p \mid \mathrm{dex}\,([a][b][c]) = \mathrm{dex}\,([a]([b][c])) = \mathrm{dex}\,(([a][b])[c]) \ , \tag{2c}$$

$$\forall p \in \mathbb{Z} \mid \mathrm{dex}\,([a]^p) = p\,\mathrm{dex}\,(a) \ . \tag{2d}$$

Definition 2. The inverse of the "dex" function is a function of $\mathbb{Z}^7$ into $\mathbb{U}_p$, and defined as $\mathrm{dex}^{-1} : \forall \breve{\mathrm{a}} \in \mathbb{Z}^7,\ \exists [\mathrm{a}] \in \mathbb{U}_\mathrm{p} \mid \mathrm{dex}^{-1}(\breve{\mathrm{a}}) = [\mathrm{a}]$.

We claim without giving proofs of the following dex^{-1} identities:

$$\forall \breve{a},\breve{b} \in \mathbb{Z}^7 \mid [a][b] = \mathrm{dex}^{-1}\left(\breve{a} + \breve{b}\right) \ , \tag{3a}$$

$$\forall \breve{a},\breve{b} \in \mathbb{Z}^7 \mid \frac{[a]}{[b]} = \mathrm{dex}^{-1}\left(\breve{a} - \breve{b}\right) \ , \tag{3b}$$

$$\forall \breve{a},\breve{b},\breve{c} \in \mathbb{Z}^7 \mid \mathrm{dex}^{-1}\left(\breve{a} + \breve{b} + \breve{c}\right) = \mathrm{dex}^{-1}\left(\breve{a} + (\breve{b} + \breve{c})\right) = \mathrm{dex}^{-1}\left((\breve{a} + \breve{b}) + \breve{c}\right) \ , \tag{3c}$$

$$\forall p \in \mathbb{Z} \mid [a]^p = \mathrm{dex}^{-1}\,(p\,\breve{a}) \ . \tag{3d}$$

We call the expression $N(\breve{x}) \doteq \|\breve{x}\|_1 = \sum_{i=1}^{7}\sum_{k=1}^{7} a_{ik} X^i X^k$, the ℓ_1-norm of $\breve{x}$ in $\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_2 \doteq \sqrt{\sum_{i=1}^{7}\sum_{k=1}^{7} a_{ik} X^i X^k}$ the ℓ_2-norm or Euclidean norm of $\breve{x}$ in $\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_\infty = max\{|X^1|,\ldots,|X^7|\}$ the Chebyshev norm or infinity norm of $\breve{x}$ in $\mathbb{Z}^7$. Let $\breve{x},\breve{y}$ be lattice points of $\mathbb{Z}^7$. The ℓ_2-distance (Euclidean distance) between the points $\breve{x},\breve{y}$ is defined by: $d(\breve{x},\breve{y}) = \|\breve{x} - \breve{y}\|_2 = \sqrt{\sum_{i=1}^{7}(X_i - Y_i)(X^i - Y^i)}$ where $\breve{x} - \breve{y} = (X^1 - Y^1,\ldots,X^7 - Y^7)$ if $\breve{x} = (X^1,\ldots,X^7)$ and $\breve{y} = (Y^1,\ldots,Y^7)$. We call two integer lattice points *neighbours* if their ℓ_2-distance is 1. We assign to each lattice point $\breve{x}$ of $\mathbb{Z}^7$ a hyperplane $H_{\breve{x}}$. A set $H_{\breve{x}}$ in $\mathbb{Z}^7$ is a hyperplane [11] if and only if there exist scalars $C_0, C_1, \ldots, C_7$, where not all $C_1,\ldots,C_7$ are zero, such that $H_{\breve{x}} = \{(X^1,\ldots,X^7) \mid C_0 + C_1 X^1 + \ldots + C_7 X^7 = 0\}$. Consider now the lattice point $\breve{y} = (Y^1,\ldots,Y^7)$ and select its associated hyperplane $H_{\breve{y}}$ that contains the lattice point $\breve{o}$. The lattice point $\breve{x}$ is incident on the hyperplane

$H_{\breve{y}}$ when it satisfies the equation $\sum_{i=1}^{7} Y^i X_i = 0$. The distance between the lattice point $\breve{z}$ and the hyperplane $H_{\breve{y}}$, measured along the perpendicular, is the projection of $\breve{o}\breve{z}$ in the direction of $\breve{o}\breve{y}$ that is given by the equation $\dfrac{\breve{z}\cdot\breve{y}}{\|\breve{y}\|_2} = \dfrac{\sum_{i=1}^{7} Z_i Y^i}{\sqrt{\sum_{i=1}^{7} Y_i Y^i}}$. Let the lattice point $\breve{x}'$ be the image of $\breve{x}$ by reflection in the hyperplane $H_{\breve{y}}$. Consider the lattice point $\breve{z}$ satisfying $\breve{z} = \breve{x} - \breve{x}'$, then the line $\breve{o}\breve{z}$ is parallel to the line $\breve{o}\breve{y}$. We define now a general reflection [8] in the hyperplane $H_{\breve{y}}$ as $\breve{x} - \breve{x}' = 2\dfrac{\breve{x}\cdot\breve{y}}{\breve{y}\cdot\breve{y}}\breve{y}$. We call the lattice point $\breve{y}$ the *root* [12] of the reflecting hyperplane $H_{\breve{y}}$. The root system for the Lie algebra B_7 [13] has the basis $\breve{\alpha}_1, \ldots, \breve{\alpha}_7$ defined by $\breve{\alpha}_1 = \breve{e}_1 - \breve{e}_2,\ \breve{\alpha}_2 = \breve{e}_2 - \breve{e}_3,\ \ldots, \breve{\alpha}_6 = \breve{e}_6 - \breve{e}_7,\ \breve{\alpha}_7 = \breve{e}_7$. This root system generates the $\mathbb{Z}^7$ integer lattice as root lattice [12] by reflections in the hyperplanes associated with the roots. The reflections are characterized by *signed permutation matrices* [13].

Definition 3. Let the surjective function "psc", represent the *parity of the sum of coordinates* of a lattice point of $\mathbb{Z}^7$ and define:

$$\text{psc} : \mathbb{Z}^7 \to \{0,1\} \mid \text{psc}(\breve{x}) = |\sum_{i=1}^{7} X^i| \pmod{2}, X^i \in \mathbb{Z}.$$

The "psc" function is a 2-colouring function. We have an *evensum* lattice point when $\text{psc}(\breve{x}) = 0$ and an *oddsum* lattice point when $\text{psc}(\breve{x}) = 1$ where $\breve{x} \in \mathbb{Z}^7$. Observe that the lattice points $\breve{x}$ for which $\text{psc}(\breve{x}) = 0$ are elements of D_7 that is an indecomposable root lattice [14] defined as $D_7 = \{(X^1, \ldots, X^7) \in \mathbb{Z}^7 \mid \sum_{i=1}^{7} X^i \textit{ is even}\}$. The lattice D_7 has 84 minimal points, that are $\pm\breve{e}_j \pm \breve{e}_k$ where $(1 \le j < k \le 7)$. These 84 points form a simple basis derived from the canonical basis $\breve{e}_1, \ldots, \breve{e}_7$ of $\mathbb{Z}^7$. Consider a lattice point $\breve{x}_0$ and points $\breve{x}$, which have the property $\breve{x}_0 + \breve{x} \in A \Leftrightarrow \breve{x}_0 - \breve{x} \in A$ then we call A a centrally symmetric set. In the remainder of the article we will assume that $\breve{x}_0 = \breve{o}$ is the origin of $\mathbb{Z}^7$. An *integer lattice polytope* is the convex hull of a set of finitely many points in $\mathbb{Z}^d$. A measure polytope P_d^s of edge-length $2s$ is a subset of $\mathbb{Z}^d$ with the following property $P_d^s = \{\breve{x}(X^1, \ldots, X^d) \in \mathbb{Z}^d \mid \|\breve{x}\|_\infty = s\}$, where $X^i \in \mathbb{Z}$ and $(1 \le i \le d)$.

3. Classification of components of physical quantities

To classify the components of physical quantities we need to find a partitioning of the integer lattice $\mathbb{Z}^7$. It is known from linear vector quantization [15,16,17] that the ℓ_2-norm and the phase of a lattice point are used to partition a lattice. However, this norm and phase are not the correct classifiers for the physical quantities. If we use as classifier the ℓ_∞-norm we obtain equivalence classes in which the elements are related through a signed permutation.

3.1. *Measure polytope properties*

Theorem 1. *Let P_d^s be a centrally symmetric d-dimensional measure polytope of edge-length 2s then the cardinality of P_d^s is $(2s+1)^d$.*

Proof. *For* $d = 0$ *the result is trivial.*
For $d = 1$ *we have the set* $P_1^s = \{-s, \ldots, 0, \ldots, s\}$ *with edge-length* $2s$ *. Let us denote the cardinality of the set* S *by* $\#(S)$ *then* $\#(P_1^s) = 2s + 1$ *.*
For $d = 2$ *we have to increase the dimension d by 1, which corresponds to calculate the Cartesian product of the sets* $P_1^s \times P_1^s = P_2^s$ *.*
It is a property of cardinal numbers that: $\#(P_2^s) = \#(P_1^s) \times \#(P_1^s) = \#(P_1^s) \cdot \#(P_1^s) = (2s+1)^2$ *. Assume that* $\#\left(P_{d-1}^s\right) = (2s+1)^{d-1}$ *. Then* $\#(P_d^s) = \#\left(P_{d-1}^s\right) \cdot \#(P_1^s) = (2s+1)^{d-1} \cdot (2s+1) = (2s+1)^d$ *.* □

We distinguish the measure polytope P_d^s by the parameters d and s, where d represents the dimension of the integer lattice and s represents the edge length. We define a leader class of a measure polytope as: A leader class of a measure polytope is the set of lattice points of $\mathbb{Z}^7$ that are connected through a signed permutation. A leader class of a measure polytope of $\mathbb{Z}^7$ is noted as $[(X^1, \ldots, X^7)]$ where $(X^1, \ldots, X^7)$ are the coordinates of the representative lattice point. Each leader class forms a set of lattice points that are symmetric about the origin. The cardinality of a leader class of a measure polytope is calculated using elementary combinatorics. Let $A = \{0, 1, 2, \ldots, k\}$ be the alphabet of measure polytope with edge length $2k$. The representative of a leader class of a measure polytope is a word w constructed from the alphabet A . The words w have a length d that corresponds to the dimension of $\mathbb{Z}^7$. Let d_i be the number of characters of type i of the alphabet A . Suppose that the characters are subjected to permutation and change of sign, then using combinatorics the cardinality is given by the equation

$$\#(w) = 2^{d-d_0} \frac{d!}{d_0! d_1! d_2! \ldots d_k!} \ . \tag{4}$$

Observe that each measure polytope in $\mathbb{Z}^7$ represents a centrally symmetric lattice polytope [8,18,19,20]. The number of vertices in each leader class is equal to the cardinality of w . Observe also that the representative lattice point, in coding theory [15] called an *absolute leader*, has only coordinates that are *non-negative integers*. We define the total degree of a monomial as:

Definition 4. A monomial m in $u_1, u_2, \ldots, u_7$ is a product of the form:

$$m = \prod_{i=1}^{7} u_i^{X^i} \ , \tag{5}$$

where all the exponents $X^i \in \mathbb{Z}_+$ and $u_i \in \mathcal{U}$ (see section 1). The total degree deg of this monomial is the sum $X^1 + \ldots + X^7$.

From the 7-tuple of non-negative integer exponents $(X^1, \dots, X^7) \in \mathbb{Z}_+^7$ a monomial [21] is constructed one-to-one of the form $m = \prod_{i=1}^7 u_i^{X^i}$ that we compare with equation (1). It means that a lot of results known from the commutative module of monomials are applicable to the classification of the components of physical quantities. The number of classes of monomials (Table 1) with Chebyshev norm $\|\check{x}\|_\infty \leq s$ in $\mathbb{Z}^7$ is the result from application of lemma 4 [22].

Table 1: Properties of the measure polytopes P_7^s in $\mathbb{Z}^7$ for $s \leq 10$.

$\|\check{x}\|_\infty = s$	sum(# ([a]))	cumul(sum(# ([a])))	# (P_7^s)	cumul(# (P_7^s))
0	1	1	1	1
1	2186	2187	7	8
2	75938	78125	28	36
3	745418	823543	84	120
4	3959426	4782969	210	330
5	14704202	19487171	462	792
6	43261346	62748517	924	1716
7	108110858	170859375	1716	3432
8	239479298	410338673	3003	6435
9	483533066	893871739	5005	11440
10	907216802	1801088541	8008	19448

In (Table 1) the second column shows the number of vertices while the third column gives the cumulated number of vertices. The fourth and fifth columns have a similar meaning but are expressing the number of classes in each measure polytope P_7^s.

3.2. *Enumeration of the measure polytopes*

The enumeration table (Table 3) of the measure polytope P_7^3 consists of 8 columns. The second column is the row identifier. The third column gives the representative of the leader class. The fourth column contains the sum of the absolute value of the coordinates of the lattice points being elements of the leader class that is exclusively the total degree of the monomial associated with the leader class. The fifth column gives the parity of the representative of the leader class. The sixth column gives the ℓ_1-norm of the representative. The seventh column gives the cardinality of the leader class. The eighth column gives the Gödel number of the representative. The ordering of the classes is based on *graded reverse lex order* [21]. We derive from Table 1 that the measure polytopes P_7^s are partitioned in $\binom{7+s-1}{s}$ equivalence classes. The cardinality of the leader classes is related to the theta series of the integer lattice $\mathbb{Z}^7$. We find in the OEIS [23] the sequence A008451 given by $r_7(N) = 1$, 14, 84, 280, 574, 840, 1288, 2368, 3444, 3542, 4424, 7560, 9240, 8456, 11088, 16576, 18494, 17808, 19740, 27720, 34440, 29456, 31304, 49728, 52808, 43414, 52248, 68320, 74048, 68376, 71120, 99456, 110964, 89936, 94864, 136080 . . . . The sequence represents the number of ways of writing a positive integer N as a sum of *seven* integral squares and is defined by:

$$\Theta_{\mathbb{Z}^7}(z) = \sum_{N=0}^{\infty} r_7(N) q^N \ , \tag{6}$$

where $q = e^{\pi i z}$ and N is the norm of the lattice point [24]. The enumeration table (Table 4) gives the relation between the sequence A008451 and the partitioning of 7-spheres in leader classes of the measure polytopes. The common physical quantities (Table 5) which belong to the measure polytopes, where the variable $\|\breve{x}\|_\infty = s$ taking values from 0 to 10, are enumerated. Table 5 is far from exhaustive, but it highlights the sparse distribution of the common physical quantities when taking in consideration the cardinalities (Table 1) of classes and vertices.

3.3. *Gödel number of a leader class*

We encode each integer lattice point of $\mathbb{Z}^7_+$ by using a similar scheme to the Gödel encoding [25] applied to 7 non-negative integer variables. We define the *Gödel number* in $\mathbb{Z}^d_+$, where d is the dimension of the integer lattice:

$$\phi_d(X^1 \dots X^d) = \prod_{i=1}^{d} p_i^{X^i} \ , \tag{7}$$

where p_i is the i-th prime number, $\breve{x} = (X^1, \dots, X^d)$ and $X^i \in \mathbb{Z}_+$. $\phi_7(1110000) = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^0 \cdot 13^0 \cdot 17^0 = 30$ This encoding which we denote as ϕ_7 is injective between $\mathbb{Z}^7_+$ and $\mathbb{Z}_+$. The range of ϕ_7 is a subset Φ_7 of the non-negative integers $\mathbb{Z}_+$ because all the primes which are different from the first 7 primes are not images of lattice points of $\mathbb{Z}^7_+$, as well as all the composite numbers having divisors larger than 17. Observe that each of the base physical quantities of the set $\mathcal{B}$ are assigned to a prime number. The base physical quantities play the same role as the prime numbers, being the *atoms* in number theory [26]. An enumeration (Table 6) of the first 67 lattice points is given. Observe that the leader class representative has always the *smallest* Gödel number of the class.

4. Future work and conclusion

We construct the mathematical foundation for *the classification of physical quantities* . We define an isomorphism between the classes of physical quantities and the seven dimensional integer lattice points. We discover that the Chebyshev norm, generating leader classes, is at the basis of the structure S classifying the classes of physical quantities. We show that morphisms exists between leader classes and monomials. Assignment of a *Gödel number* to each leader class in $\mathbb{Z}^7_+$ generates an order relation between the leader classes. This research shows that our knowledge about the components of physical quantities and about their constellations is far from being understood and that large *hypervolumes* of $\mathbb{Z}^7$, are still to be explored.

The appendices contain a preliminary classification of common physical quantities based on the measure polytopes P_7^s. The appendices also contain numerical data useful as starting point for the further exploration of the constellations of physical quantities.

5. Tables

Tables should be inserted in the text as close to the point of reference as possible. Some space should be left above and below the table.

Tables should be numbered sequentially in the text in Arabic numerals. Captions are to be centralized above the tables. Typeset tables and captions in 8 pt roman with baselineskip of 10 pt.

Table 2. Comparison of acoustic for frequencies for piston-cylinder problem.

Piston mass	Analytical frequency (Rad/s)	TRIA6-S_1 model (Rad/s)	% Error
1.0	281.0	280.81	0.07
0.1	876.0	875.74	0.03
0.01	2441.0	2441.0	0.0
0.001	4130.0	4129.3	0.16

If tables need to extend over to a second page, the continuation of the table should be preceded by a caption, e.g., "*Table 2.* (*Continued*)".

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Appendix A. Measure polytopes

The enumeration table (Table 3) of measure polytopes P_7^4 consists of 8 columns. The second column is the row identifier. The third column gives the representative of the leader class. The fourth column contains the sum of the absolute value of the coordinates of the lattice points being elements of the leader class that is exclusively the total degree of the monomial associated with the leader class. The fifth column gives the parity of the representative of the leader class. The sixth column gives the ℓ_1-norm of the representative. The seventh column gives the cardinality of the leader class. The eighth column gives the Gödel number of the representative. Observe that for $\|\breve{x}\|_\infty = 1$ the representative lattice points of the leader classes generate the *successive minima* R_i of the lattice $\mathbb{Z}^7$ [27]. The successive minima R_i are given in the column 6 and correspond to the values of $N(\breve{z})$, the norm of the lattice point and thus the representative lattice points of the leader classes for $s = 1$ form a set of minimal points of the lattice $\mathbb{Z}^7$ [27]. Observe that the leader class $[2^2 10^4]$ contains 840 integer lattice points with the same geometrical properties as the physical quantity *energy.* The 7×7 signed permutation matrix $\mathsf{P}_{21-2,221}$ transforms all *energy constellations* to the leader class $[2^2 10^4]$:

$$\mathsf{P}_{21-2,221} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.1}$$

The representative of the leader class $[2^2 10^4]$ is a physical quantity that could be expressed as an integral of the form $\int \kappa(\lambda m_0)^2 dt$. This is the time integral of the square of the quantity with lattice point (1,1,0,0,0,0,0).

Table 3: Partitions of the measure polytope P_7^4

$\lVert\breve{x}\rVert_\infty = s$	Id	leader class	deg	psc $(\breve{z})$	$N(\breve{z})$	Number of vertices	Gödel number
0	1	$[0^7]$	0	0	0	1	1
1	1	$[10^6]$	1	1	1	14	2
1	2	$[1^2 0^5]$	2	0	2	84	6
1	3	$[1^3 0^4]$	3	1	3	280	30
1	4	$[1^4 0^3]$	4	0	4	560	210
1	5	$[1^5 0^2]$	5	1	5	672	2310
1	6	$[1^6 0]$	6	0	6	448	30030
1	7	$[1^7]$	7	1	7	128	510510
2	1	$[20^6]$	2	0	4	14	4
2	2	$[210^5]$	3	1	5	168	12
2	3	$[21^2 0^4]$	4	0	6	840	60
2	4	$[2^2 0^5]$	4	0	8	84	36
2	5	$[21^3 0^3]$	5	1	7	2240	420
2	6	$[2^2 10^4]$	5	1	9	840	180
2	7	$[21^4 0^2]$	6	0	8	3360	4620
2	8	$[2^2 1^2 0^3]$	6	0	10	3360	1260
2	9	$[2^3 0^4]$	6	0	12	280	900
2	10	$[21^5 0]$	7	1	9	2688	60060
2	11	$[2^2 1^3 0^2]$	7	1	11	6720	13860
2	12	$[2^3 10^3]$	7	1	13	2240	6300
2	13	$[21^6]$	8	0	10	896	1021020
2	14	$[2^2 1^4 0]$	8	0	12	6720	180180
...	...	...	...	...	...	...	...

$\|\breve{x}\|_\infty = s$	Id	leader class	deg	psc $(\breve{z})$	$N(\breve{z})$	Number of vertices	Gödel number
2	15	$[2^31^20^2]$	8	0	14	6720	69300
2	16	$[2^40^3]$	8	0	16	560	44100
2	17	$[2^21^5]$	9	1	13	2688	3063060
2	18	$[2^31^30]$	9	1	15	8960	900900
2	19	$[2^410^2]$	9	1	17	3360	485100
2	20	$[2^31^4]$	10	0	16	4480	15315300
2	21	$[2^41^20]$	10	0	18	6720	6306300
2	22	$[2^50^2]$	10	0	20	672	5336100
2	23	$[2^41^3]$	11	1	19	4480	107207100
2	24	$[2^510]$	11	1	21	2688	69369300
2	25	$[2^51^2]$	12	0	22	2688	1179278100
2	26	$[2^60]$	12	0	24	448	901800900
2	27	$[2^61]$	13	1	25	896	15330615300
2	28	$[2^7]$	14	0	28	128	260620460100
3	1	$[30^6]$	3	1	9	14	8
3	2	$[310^5]$	4	0	10	168	24
3	3	$[31^20^4]$	5	1	11	840	120
3	4	$[320^5]$	5	1	13	168	72
3	5	$[31^30^3]$	6	0	12	2240	840
3	6	$[3210^4]$	6	0	14	1680	360
3	7	$[3^20^5]$	6	0	18	84	216
3	8	$[31^40^2]$	7	1	13	3360	9240
3	9	$[321^20^3]$	7	1	15	6720	2520
3	10	$[32^20^4]$	7	1	17	840	1800
3	11	$[3^210^4]$	7	1	19	840	1080
3	12	$[31^50]$	8	0	14	2688	120120
3	13	$[321^30^2]$	8	0	16	13440	27720
3	14	$[32^210^3]$	8	0	18	6720	12600
3	15	$[3^21^20^3]$	8	0	20	3360	7560
3	16	$[3^220^4]$	8	0	22	840	5400
3	17	$[31^6]$	9	1	15	896	2042040
3	18	$[321^40]$	9	1	17	13440	360360
3	19	$[32^21^20^2]$	9	1	19	20160	138600
3	20	$[32^30^3]$	9	1	21	2240	88200
3	21	$[3^21^30^2]$	9	1	21	6720	83160
3	22	$[3^2210^3]$	9	1	23	6720	37800
3	23	$[3^30^4]$	9	1	27	280	27000
3	24	$[321^5]$	10	0	18	5376	6126120
3	25	$[32^21^30]$	10	0	20	26880	1801800
3	26	$[32^310^2]$	10	0	22	13440	970200
3	27	$[3^21^40]$	10	0	22	6720	1081080
3	28	$[3^221^20^2]$	10	0	24	20160	415800
3	29	$[3^22^20^3]$	10	0	26	3360	264600
3	30	$[3^310^3]$	10	0	28	2240	189000
...	...	...	...	...	...	...	...

$\|\breve{x}\|_\infty = s$	Id	leader class	deg	psc $(\breve{z})$	$N(\breve{z})$	Number of vertices	Gödel number
3	31	$[32^21^4]$	11	1	21	13440	30630600
3	32	$[32^31^20]$	11	1	23	26880	12612600
3	33	$[32^40^2]$	11	1	25	3360	10672200
3	34	$[3^21^5]$	11	1	23	2688	18378360
3	35	$[3^221^30]$	11	1	25	26880	5405400
3	36	$[3^22^210^2]$	11	1	27	20160	2910600
3	37	$[3^31^20^2]$	11	1	29	6720	2079000
3	38	$[3^320^3]$	11	1	31	2240	1323000
3	39	$[32^31^3]$	12	0	24	17920	214414200
3	40	$[32^410]$	12	0	26	13440	138738600
3	41	$[3^221^4]$	12	0	26	13440	91891800
3	42	$[3^22^21^20]$	12	0	28	40320	37837800
3	43	$[3^22^30^2]$	12	0	30	6720	32016600
3	44	$[3^31^30]$	12	0	30	8960	27027000
3	45	$[3^3210^2]$	12	0	32	13440	14553000
3	46	$[3^40^3]$	12	0	36	560	9261000
3	47	$[32^41^2]$	13	1	27	13440	2358556200
3	48	$[32^50]$	13	1	29	2688	1803601800
3	49	$[3^22^21^3]$	13	1	29	26880	643242600
3	50	$[3^22^310]$	13	1	31	26880	416215800
3	51	$[3^31^4]$	13	1	31	4480	459459000
3	52	$[3^321^20]$	13	1	33	26880	189189000
3	53	$[3^32^20^2]$	13	1	35	6720	160083000
3	54	$[3^410^2]$	13	1	37	3360	101871000
3	55	$[32^51]$	14	0	30	5376	30661260600
3	56	$[3^22^31^2]$	14	0	32	26880	7075668600
3	57	$[3^22^40]$	14	0	34	6720	5410805400
3	58	$[3^321^3]$	14	0	34	17920	3216213000
3	59	$[3^32^210]$	14	0	36	26880	2081079000
3	60	$[3^41^20]$	14	0	38	6720	1324323000
3	61	$[3^420^2]$	14	0	40	3360	1120581000
3	62	$[32^6]$	15	1	33	896	521240920200
3	63	$[3^22^41]$	15	1	35	13440	91983691800
3	64	$[3^32^21^2]$	15	1	37	26880	35378343000
3	65	$[3^32^30]$	15	1	39	8960	27054027000
3	66	$[3^41^3]$	15	1	39	4480	22513491000
3	67	$[3^4210]$	15	1	41	13440	14567553000
3	68	$[3^50^2]$	15	1	45	672	12326391000
3	69	$[3^22^5]$	16	0	38	2688	1563722760600
3	70	$[3^32^31]$	16	0	40	17920	459918459000
3	71	$[3^421^2]$	16	0	42	13440	247648401000
3	72	$[3^42^20]$	16	0	44	6720	189378189000
3	73	$[3^510]$	16	0	46	2688	160243083000
3	74	$[3^32^4]$	17	1	43	4480	7818613803000
…	…	…	…	…	…	…	…

$\|\breve{x}\|_\infty = s$	Id	leader class	deg	psc $(\breve{z})$	$N(\breve{z})$	Number of vertices	Gödel number
3	75	$[3^4 2^2 1]$	17	1	45	13440	3219429213000
3	76	$[3^5 1^2]$	17	1	47	2688	2724132411000
3	77	$[3^5 20]$	17	1	49	2688	2083160079000
3	78	$[3^4 2^3]$	18	0	48	4480	54730296621000
3	79	$[3^5 21]$	18	0	50	5376	35413721343000
3	80	$[3^6 0]$	18	0	54	448	27081081027000
3	81	$[3^5 2^2]$	19	1	53	2688	602033262831000
3	82	$[3^6 1]$	19	1	55	896	460378377459000
3	83	$[3^6 2]$	20	0	58	896	7826432416803000
3	84	$[3^7]$	21	1	63	128	133049351085651000

Appendix B. Relation between 7-spheres and the leader classes of measure polytopes

Table 4: Partitioning 7-spheres in leader classes of measure polytopes

N	disjunct union of leader classes of measure polytopes	$r_7(N)$
0	$[0^7]$	1
1	$[10^6]$	14
2	$[1^2 0^5]$	84
3	$[1^3 0^4]$	280
4	$[1^4 0^3] \cup [20^6]$	574
5	$[1^5 0^2] \cup [210^5]$	840
6	$[1^6 0] \cup [21^2 0^4]$	1288
7	$[1^7] \cup [21^3 0^3]$	2368
8	$[2^2 0^5] \cup [21^4 0^2]$	3444
9	$[2^2 10^4] \cup [21^5 0] \cup [30^6]$	3542
10	$[2^2 1^2 0^3] \cup [21^6] \cup [310^5]$	4424
11	$[2^2 1^3 0^2] \cup [31^2 0^4]$	7560
12	$[2^3 0^4] \cup [2^2 1^4 0] \cup [31^3 0^3]$	9240
13	$[2^3 10^3] \cup [2^2 1^5] \cup [320^5] \cup [31^4 0^2]$	8456
14	$[2^3 1^2 0^2] \cup [3210^4] \cup [31^5 0]$	11088
15	$[2^3 1^3 0] \cup [321^2 0^3] \cup [31^6]$	16576
16	$[2^4 0^3] \cup [2^3 1^4] \cup [321^3 0^2] \cup [40^6]$	18494
17	$[2^4 10^2] \cup [32^2 0^4] \cup [321^4 0] \cup [410^5]$	17808
18	$[2^4 1^2 0] \cup [3^2 0^5] \cup [32^2 10^3] \cup [321^5] \cup [41^2 0^4]$	19740
19	$[2^4 1^3] \cup [3^2 10^4] \cup [32^2 1^2 0^2] \cup [41^3 0^3]$	27720
20	$[2^5 0^2] \cup [3^2 1^2 0^3] \cup [32^2 1^3 0] \cup [41^4 0^2] \cup [420^5]$	34440
21	$[2^5 10] \cup [32^3 0^3] \cup [3^2 1^3 0^2] \cup [32^2 1^4] \cup [41^5 0] \cup [4210^4]$	29456
22	$[2^5 1^2] \cup [3^2 20^4] \cup [32^3 10^2] \cup [3^2 1^4 0] \cup [41^6] \cup [421^2 0^3]$	31304
23	$[3^2 210^3] \cup [32^3 1^2 0] \cup [3^2 1^5] \cup [421^3 0^2]$	49728
24	$[2^6 0] \cup [3^2 21^2 0^2] \cup [32^3 1^3] \cup [421^4 0] \cup [42^2 0^4]$	52808
25	$[2^6 1] \cup [32^4 0^2] \cup [3^2 21^3 0] \cup [421^5] \cup [42^2 10^3] \cup [430^5] \cup [50^6]$	43414
...	...	...

N	disjunct union of leader classes of measure polytopes	$r_7(N)$
26	$[3^22^20^3] \cup [32^410] \cup [3^221^4] \cup [42^21^20^2] \cup [4310^4] \cup [510^5]$	52248
27	$[3^30^4] \cup [3^22^210^2] \cup [32^41^2] \cup [42^21^30] \cup [431^20^3] \cup [51^20^4]$	68320
28	$[2^7] \cup [3^310^3] \cup [3^22^21^20] \cup [42^21^4] \cup [42^30^3] \cup [431^30^2] \cup [51^30^3]$	74048
29	$[3^31^20^2] \cup [32^50] \cup [3^22^21^3] \cup [42^310^2] \cup [431^40] \cup [4320^4] \cup [51^40^2] \cup [520^5]$	68376
30	$[3^22^30^2] \cup [3^31^30] \cup [32^51] \cup [42^31^20] \cup [431^5] \cup [43210^3] \cup [51^50] \cup [5210^4]$	71120
31	$[3^320^3] \cup [3^22^310] \cup [3^31^4] \cup [42^31^3] \cup [4321^20^2] \cup [51^6] \cup [521^20^3]$	99456
32	$[3^3210^2] \cup [3^22^31^2] \cup [42^40^2] \cup [4^20^5] \cup [4321^30] \cup [521^30^2]$	110964
33	$[3^321^20] \cup [32^6] \cup [42^410] \cup [4^210^4] \cup [4321^4] \cup [432^20^3] \cup [521^40] \cup [52^20^4]$	89936
34	$[3^22^40] \cup [3^321^3] \cup [42^41^2] \cup [432^210^2] \cup [43^20^4] \cup [4^21^20^3] \cup [521^5] \cup [52^210^3] \cup [530^5]$	94864
35	$[3^32^20^2] \cup [3^22^41] \cup [43^210^3] \cup [4^21^30^2] \cup [432^21^20] \cup [52^21^20^2] \cup [5310^4]$	136080

Appendix C. Classification of common physical quantities

Table 5 contains 5 columns. The first column represents the name of a common physical quantity. The second column indicates to which shell that the physical quantity belongs. The third column gives the *Id* of the leader class within the respective polytope shell. The fourth column lists the leader class that contains the physical quantity. The fifth column identifies the physical quantity by its integer lattice point in $\mathbb{Z}^7$.

Table 5: Classification of common physical quantities.

physical quantity	s	Id	leader class	vertex
plane angle	0	1	$[0^7]$	(0,0,0,0,0,0,0)
solid angle	0	1	$[0^7]$	(0,0,0,0,0,0,0)
linear strain	0	1	$[0^7]$	(0,0,0,0,0,0,0)
shear strain	0	1	$[0^7]$	(0,0,0,0,0,0,0)
bulk strain	0	1	$[0^7]$	(0,0,0,0,0,0,0)
relative elongation	0	1	$[0^7]$	(0,0,0,0,0,0,0)
refractive index	0	1	$[0^7]$	(0,0,0,0,0,0,0)
electric susceptibility	0	1	$[0^7]$	(0,0,0,0,0,0,0)
mass ratio	0	1	$[0^7]$	(0,0,0,0,0,0,0)
fine-structure constant (α_e)	0	1	$[0^7]$	(0,0,0,0,0,0,0)
(α_w)	0	1	$[0^7]$	(0,0,0,0,0,0,0)
(α_s)	0	1	$[0^7]$	(0,0,0,0,0,0,0)
(α_G)	0	1	$[0^7]$	(0,0,0,0,0,0,0)
redshift	0	1	$[0^7]$	(0,0,0,0,0,0,0)
Poisson's ratio	0	1	$[0^7]$	(0,0,0,0,0,0,0)
length	1	1	$[10^6]$	(1,0,0,0,0,0,0)
height	1	1	$[10^6]$	(1,0,0,0,0,0,0)
breadth	1	1	$[10^6]$	(1,0,0,0,0,0,0)
thickness	1	1	$[10^6]$	(1,0,0,0,0,0,0)
distance	1	1	$[10^6]$	(1,0,0,0,0,0,0)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
radius	1	1	$[10^6]$	(1,0,0,0,0,0,0)
diameter	1	1	$[10^6]$	(1,0,0,0,0,0,0)
path length	1	1	$[10^6]$	(1,0,0,0,0,0,0)
persistence length	1	1	$[10^6]$	(1,0,0,0,0,0,0)
length of arc	1	1	$[10^6]$	(1,0,0,0,0,0,0)
Planck length	1	1	$[10^6]$	(1,0,0,0,0,0,0)
wavelength	1	1	$[10^6]$	(1,0,0,0,0,0,0)
Compton wavelength	1	1	$[10^6]$	(1,0,0,0,0,0,0)
relaxation length	1	1	$[10^6]$	(1,0,0,0,0,0,0)
luminosity distance	1	1	$[10^6]$	(1,0,0,0,0,0,0)
mass	1	1	$[10^6]$	(0,1,0,0,0,0,0)
reduced mass	1	1	$[10^6]$	(0,1,0,0,0,0,0)
Planck mass	1	1	$[10^6]$	(0,1,0,0,0,0,0)
time	1	1	$[10^6]$	(0,0,1,0,0,0,0)
period	1	1	$[10^6]$	(0,0,1,0,0,0,0)
relaxation time	1	1	$[10^6]$	(0,0,1,0,0,0,0)
time constant	1	1	$[10^6]$	(0,0,1,0,0,0,0)
time interval	1	1	$[10^6]$	(0,0,1,0,0,0,0)
proper time	1	1	$[10^6]$	(0,0,1,0,0,0,0)
Planck time	1	1	$[10^6]$	(0,0,1,0,0,0,0)
half-life time	1	1	$[10^6]$	(0,0,1,0,0,0,0)
specific impulse	1	1	$[10^6]$	(0,0,1,0,0,0,0)
electric current	1	1	$[10^6]$	(0,0,0,1,0,0,0)
thermodynamic temperature	1	1	$[10^6]$	(0,0,0,0,1,0,0)
Planck temperature	1	1	$[10^6]$	(0,0,0,0,1,0,0)
thermal expansion coefficient	1	1	$[10^6]$	(0,0,0,0,-1,0,0)
amount of substance	1	1	$[10^6]$	(0,0,0,0,0,1,0)
luminous intensity	1	1	$[10^6]$	(0,0,0,0,0,0,1)
luminous flux	1	1	$[10^6]$	(0,0,0,0,0,0,1)
wave number	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
optical power	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
spatial frequency	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
absorption coefficient	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
laser gain	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
rotational constant	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
Rydberg constant	1	1	$[10^6]$	(-1,0,0,0,0,0,0)
frequency	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
angular frequency	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
circular frequency	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
activity	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
specific material permeability	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
angular velocity	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
decay constant	1	1	$[10^6]$	(0,0,-1,0,0,0,0)
Avogadro constant	1	1	$[10^6]$	(0,0,0,0,0,-1,0)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
velocity	1	2	$[1^2 0^5]$	(1,0,-1,0,0,0,0)
group velocity	1	2	$[1^2 0^5]$	(1,0,-1,0,0,0,0)
volumetric flux	1	2	$[1^2 0^5]$	(1,0,-1,0,0,0,0)
speed	1	2	$[1^2 0^5]$	(1,0,-1,0,0,0,0)
speed of light in vacuum	1	2	$[1^2 0^5]$	(1,0,-1,0,0,0,0)
magnetic field strength	1	2	$[1^2 0^5]$	(-1,0,0,1,0,0,0)
magnetisation	1	2	$[1^2 0^5]$	(-1,0,0,1,0,0,0)
temperature gradient	1	2	$[1^2 0^5]$	(-1,0,0,0,1,0,0)
electric charge	1	2	$[1^2 0^5]$	(0,0,1,1,0,0,0)
charge of the weak interaction (g_w)	1	2	$[1^2 0^5]$	(0,0,1,1,0,0,0)
charge of the strong interaction (g_s)	1	2	$[1^2 0^5]$	(0,0,1,1,0,0,0)
Yukawa constants	1	2	$[1^2 0^5]$	(0,0,1,1,0,0,0)
electric flux	1	2	$[1^2 0^5]$	(0,0,1,1,0,0,0)
catalytic activity	1	2	$[1^2 0^5]$	(0,0,-1,0,0,1,0)
molar mass	1	2	$[1^2 0^5]$	(0,1,0,0,0,-1,0)
second radiation constant	1	2	$[1^2 0^5]$	(1,0,0,0,1,0,0)
luminous energy	1	2	$[1^2 0^5]$	(0,0,1,0,0,0,1)
linear density	1	2	$[1^2 0^5]$	(-1,1,0,0,0,0,0)
mass flow rate	1	2	$[1^2 0^5]$	(0,1,-1,0,0,0,0)
electric dipole moment	1	3	$[1^3 0^4]$	(1,0,1,1,0,0,0)
linear momentum	1	3	$[1^3 0^4]$	(1,1,-1,0,0,0,0)
Faraday constant	1	3	$[1^3 0^4]$	(0,0,1,1,0,-1,0)
dynamic viscosity	1	3	$[1^3 0^4]$	(-1,1,-1,0,0,0,0)
fluidity	1	3	$[1^3 0^4]$	(1,-1,1,0,0,0,0)
magnetogyric ratio	1	3	$[1^3 0^4]$	(0,-1,1,1,0,0,0)
vacuum condensate of Higgs field (η)	1	3	$[1^3 0^4]$	(0,1,-1,-1,0,0,0)
area	2	1	$[20^6]$	(2,0,0,0,0,0,0)
elastic modulus	2	1	$[20^6]$	(2,0,0,0,0,0,0)
Thomson cross section	2	1	$[20^6]$	(2,0,0,0,0,0,0)
spacetime curvature	2	1	$[20^6]$	(-2,0,0,0,0,0,0)
angular acceleration	2	1	$[20^6]$	(0,0,-2,0,0,0,0)
acceleration	2	1	$[210^5]$	(1,0,-2,0,0,0,0)
areal velocity	2	2	$[210^5]$	(2,0,-1,0,0,0,0)
mass attenuation coefficient	2	2	$[210^5]$	(2,-1,0,0,0,0,0)
radiant exposure	2	2	$[210^5]$	(0,1,-2,0,0,0,0)
diffusion constant	2	2	$[210^5]$	(2,0,-1,0,0,0,0)
thermal diffusivity	2	2	$[210^5]$	(2,0,-1,0,0,0,0)
kinematic viscosity	2	2	$[210^5]$	(2,0,-1,0,0,0,0)
quantum of circulation	2	2	$[210^5]$	(2,0,-1,0,0,0,0)
electric current density	2	2	$[210^5]$	(-2,0,0,1,0,0,0)
luminance	2	2	$[210^5]$	(-2,0,0,0,0,0,1)
illuminance	2	2	$[210^5]$	(-2,0,0,0,0,0,1)
luminous emittance	2	2	$[210^5]$	(-2,0,0,0,0,0,1)
irradiance	2	2	$[210^5]$	(-2,0,0,0,0,0,1)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
magnetic dipole moment	2	2	$[210^5]$	(2,0,0,1,0,0,0)
Bohr magneton	2	2	$[210^5]$	(2,0,0,1,0,0,0)
surface density	2	2	$[210^5]$	(-2,1,0,0,0,0,0)
surface tension	2	2	$[210^5]$	(0,1,-2,0,0,0,0)
stiffness	2	2	$[210^5]$	(0,1,-2,0,0,0,0)
compliance	2	2	$[210^5]$	(0,-1,2,0,0,0,0)
moment of inertia	2	2	$[210^5]$	(2,1,0,0,0,0,0)
accelerator luminosity	2	2	$[210^5]$	(-2,0,-1,0,0,0,0)
force	2	3	$[21^20^4]$	(1,1,-2,0,0,0,0)
energy density	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
radiant energy density	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
sound energy density	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
toughness	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
pressure	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
modulus of elasticity	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
Young's modulus	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
shear modulus	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
compression modulus	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
normal stress	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
shear stress	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
energy momentum tensor	2	3	$[21^20^4]$	(-1,1,-2,0,0,0,0)
Planck constant	2	3	$[21^20^4]$	(2,1,-1,0,0,0,0)
angular momentum	2	3	$[21^20^4]$	(2,1,-1,0,0,0,0)
action	2	3	$[21^20^4]$	(2,1,-1,0,0,0,0)
spin	2	3	$[21^20^4]$	(2,1,-1,0,0,0,0)
acoustic impedance	2	3	$[21^20^4]$	(-2,1,-1,0,0,0,0)
mass flux	2	3	$[21^20^4]$	(-2,1,-1,0,0,0,0)
magnetic flux density	2	3	$[21^20^4]$	(0,1,-2,-1,0,0,0)
magnetic induction	2	3	$[21^20^4]$	(0,1,-2,-1,0,0,0)
surface charge density	2	3	$[21^20^4]$	(-2,0,1,1,0,0,0)
dielectric polarisation	2	3	$[21^20^4]$	(-2,0,1,1,0,0,0)
electrical displacement	2	3	$[21^20^4]$	(-2,0,1,1,0,0,0)
electrical quadrupole moment	2	3	$[21^20^4]$	(2,0,1,1,0,0,0)
luminous exposure	2	3	$[21^20^4]$	(-2,0,1,0,0,0,1)
absorbed dose	2	4	$[2^20^4]$	(2,0,-2,0,0,0,0)
dose equivalent	2	4	$[2^20^4]$	(2,0,-2,0,0,0,0)
specific energy	2	4	$[2^20^4]$	(2,0,-2,0,0,0,0)
gravitational potential	2	4	$[2^20^4]$	(2,0,-2,0,0,0,0)
molar Planck constant	2	5	$[21^30^3]$	(2,1,-1,0,0,-1,0)
magnetic vector potential	2	5	$[21^30^3]$	(1,1,-2,-1,0,0,0)
thermal conductivity	2	5	$[21^30^3]$	(1,1,-2,0,-1,0,0)
thermal resistivity	2	5	$[21^30^3]$	(-1,-1,2,0,1,0,0)
torque	2	6	$[2^210^4]$	(2,1,-2,0,0,0,0)
moment of a force	2	6	$[2^210^4]$	(2,1,-2,0,0,0,0)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
specific heat capacity	2	6	$[2^2 10^4]$	(2,0,-2,0,-1,0,0)
energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
potential energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
kinetic energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
work	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
Lagrange function	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
Hamilton function	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
Hartree energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
ionization energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
electron affinity	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
electronegativity	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
dissociation energy	2	6	$[2^2 10^4]$	(2,1,-2,0,0,0,0)
magnetic constant	2	8	$[2^2 1^2 0^3]$	(1,1,-2,-2,0,0,0)
permeability	2	8	$[2^2 1^2 0^3]$	(1,1,-2,-2,0,0,0)
magnetic flux	2	8	$[2^2 1^2 0^3]$	(2,1,-2,-1,0,0,0)
magnetic moment	2	8	$[2^2 1^2 0^3]$	(2,1,-2,-1,0,0,0)
entropy	2	8	$[2^2 1^2 0^3]$	(2,1,-2,0,-1,0,0)
specific heat	2	8	$[2^2 1^2 0^3]$	(2,1,-2,0,-1,0,0)
Boltzmann constant	2	8	$[2^2 1^2 0^3]$	(2,1,-2,0,-1,0,0)
Josephson constant	2	8	$[2^2 1^2 0^3]$	(-2,-1,2,1,0,0,0)
magnetic flux quantum	2	8	$[2^2 1^2 0^3]$	(2,1,-2,-1,0,0,0)
chemical potential	2	8	$[2^2 1^2 0^3]$	(2,1,-2,0,0,-1,0)
molar energy	2	8	$[2^2 1^2 0^3]$	(2,1,-2,0,0,-1,0)
molar heat capacity	2	8	$[2^2 1^3 0^2]$	(2,1,-2,0,-1,-1,0)
molar gas constant	2	11	$[2^2 1^3 0^2]$	(2,1,-2,0,-1,-1,0)
molar entropy	2	11	$[2^2 1^3 0^2]$	(2,1,-2,0,-1,-1,0)
inductance	2	12	$[2^3 10^3]$	(2,1,-2,-2,0,0,0)
self-inductance	2	12	$[2^3 10^3]$	(2,1,-2,-2,0,0,0)
mutual inductance	2	12	$[2^3 10^3]$	(2,1,-2,-2,0,0,0)
magnetisability	2	12	$[2^3 10^3]$	(2,-1,2,2,0,0,0)
volume	3	1	$[30^6]$	(3,0,0,0,0,0,0)
Loschmidt constant	3	1	$[30^6]$	(-3,0,0,0,0,0,0)
number density	3	1	$[30^6]$	(-3,0,0,0,0,0,0)
mass density	3	2	$[310^5]$	(-3,1,0,0,0,0,0)
specific volume	3	2	$[310^5]$	(3,-1,0,0,0,0,0)
amount of substance concentration	3	2	$[310^5]$	(-3,0,0,0,0,1,0)
molar volume	3	2	$[310^5]$	(3,0,0,0,0,-1,0)
heat flux density	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
Poynting vector	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
radiative flux	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
thermal emittance	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
sound intensity	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
radiance	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
irradiance	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
radiant exitance	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
radiant emittance	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
radiosity	3	2	$[310^5]$	(0,1,-3,0,0,0,0)
volume rate of flow	3	2	$[310^5]$	(3,0,-1,0,0,0,0)
jerk	3	2	$[310^5]$	(1,0,-3,0,0,0,0)
electric field gradient	3	3	$[31^20^4]$	(0,1,-3,-1,0,0,0)
electric charge density	3	3	$[31^20^4]$	(-3,0,1,1,0,0,0)
heat transfer coefficient	3	3	$[31^20^4]$	(0,1,-3,0,-1,0,0)
thermal insulance	3	3	$[31^20^4]$	(0,-1,3,0,1,0,0)
spectral exitance	3	3	$[31^20^4]$	(-1,1,-3,0,0,0,0)
spectral radiance	3	3	$[31^20^4]$	(-1,1,-3,0,0,0,0)
spectral irradiance	3	3	$[31^20^4]$	(-1,1,-3,0,0,0,0)
spectral power	3	3	$[31^20^4]$	(1,1,-3,0,0,0,0)
spectral intensity	3	3	$[31^20^4]$	(1,1,-3,0,0,0,0)
luminous energy density	3	3	$[31^20^4]$	(-3,0,1,0,0,0,1)
catalytic activity concentration	3	3	$[31^20^4]$	(-3,0,-1,0,0,1,0)
reaction rate	3	3	$[31^20^4]$	(-3,0,-1,0,0,1,0)
absorbed dose rate	3	4	$[320^5]$	(2,0,-3,0,0,0,0)
thermal conductivity	3	5	$[31^30^3]$	(1,1,-3,0,-1,0,0)
first hyper-susceptibility	3	5	$[31^30^3]$	(-1,-1,3,1,0,0,0)
electric field	3	5	$[31^30^3]$	(1,1,-3,-1,0,0,0)
radiant intensity	3	6	$[3210^4]$	(2,1,-3,0,0,0,0)
radiant flux	3	6	$[3210^4]$	(2,1,-3,0,0,0,0)
Newton constant of gravitation	3	6	$[3210^4]$	(3,-1,-2,0,0,0,0)
power	3	6	$[3210^4]$	(2,1,-3,0,0,0,0)
sound energy flux	3	6	$[3210^4]$	(2,1,-3,0,0,0,0)
bolometric luminosity	3	6	$[3210^4]$	(2,1,-3,0,0,0,0)
responsivity	3	6	$[321^20^3]$	(-2,-1,3,1,0,0,0)
electric potential difference	3	9	$[321^20^3]$	(2,1,-3,-1,0,0,0)
electric potential	3	9	$[321^20^3]$	(2,1,-3,-1,0,0,0)
thermal conductance	3	9	$[321^20^3]$	(2,1,-3,0,-1,0,0)
thermal resistance	3	9	$[321^20^3]$	(-2,-1,3,0,1,0,0)
electromotive force	3	9	$[321^20^3]$	(2,1,-3,-1,0,0,0)
luminous efficacy	3	9	$[321^20^3]$	(-2,1,3,0,0,0,1)
electrical resistance	3	14	$[32^210^3]$	(2,1,-3,-2,0,0,0)
reactance	3	14	$[32^210^3]$	(2,1,-3,-2,0,0,0)
impedance	3	14	$[32^210^3]$	(2,1,-3,-2,0,0,0)
conductance	3	14	$[32^210^3]$	(-2,-1,3,2,0,0,0)
admittance	3	14	$[32^210^3]$	(-2,-1,3,2,0,0,0)
susceptance	3	14	$[32^210^3]$	(-2,-1,3,2,0,0,0)
characteristic impedance of vacuum	3	14	$[32^210^3]$	(2,1,-3,-2,0,0,0)
von Klitzing constant	3	14	$[32^210^3]$	(2,1,-3,-2,0,0,0)
specific resistance	3	15	$[3^21^20^3]$	(3,1,-3,-1,0,0,0)
electrical resistivity	3	22	$[3^2210^3]$	(3,1,-3,-2,0,0,0)
...	...	...	...	...

physical quantity	s	Id	leader class	vertex
electrical conductivity	3	22	$[3^2210^3]$	(-3,-1,3,2,0,0,0)
second moment of area	4	1	$[40^6]$	(4,0,0,0,0,0,0)
jounce	4	2	$[410^5]$	(1,0,-4,0,0,0,0)
electric polarisability	4		$[4210^4]$	(0,-1,4,2,0,0,0)
Stefan-Boltzmann constant	4		$[4310^4]$	(0,1,-3,0,-4,0,0)
first radiation constant	4		$[4310^4]$	(4,1,-3,0,0,0,0)
electrical mobility	4		$[431^20^3]$	(3,1,-4,-1,0,0,0)
electric capacitance	4		$[42^210^3]$	(-2,-1,4,2,0,0,0)
electric constant	4		$[43210^3]$	(-3,-1,4,2,0,0,0)
permittivity	4		$[43210^3]$	(-3,-1,4,2,0,0,0)
second hyper-susceptibility	6		$[62^30^3]$	(-2,-2,6,2,0,0,0)
first hyper-polarisability	7		$[73210^3]$	(-1,-2,7,3,0,0,0)
second hyper-polarisability	10		$[(10)4320^3]$	(-2,-3,10,4,0,0,0)

Appendix D. Gödel number of leader classes

Table 6 contains in the first column the row identifier. In the second column we list the vertices in the order of appearance in the Gödel walk. The third column gives the value of the Gödel number up to the number 100. The fourth column shows the dimension d of $\mathbb{Z}^d \times \{0\}^{7-d}$ in which the lattice point is embedded. The fifth column indicates to which measure polytope P_7^s the lattice point belongs. The sixth column shows the leader class containing the lattice point.

Table 6: Successive Gödel numbers in $\mathbb{Z}^7$.

Id	vertex	Gödel number	dimension	$\|\check{x}\|_\infty = s$	leader class
1	(0,0,0,0,0,0,0)	1	0	0	$[0^7]$
2	(1,0,0,0,0,0,0)	2	1	1	$[10^6]$
3	(0,1,0,0,0,0,0)	3	2	1	$[10^6]$
4	(2,0,0,0,0,0,0)	4	1	2	$[20^6]$
5	(0,0,1,0,0,0,0)	5	3	1	$[10^6]$
6	(1,1,0,0,0,0,0)	6	2	1	$[1^20^5]$
7	(0,0,0,1,0,0,0)	7	4	1	$[10^6]$
8	(3,0,0,0,0,0,0)	8	1	3	$[30^6]$
9	(0,2,0,0,0,0,0)	9	2	2	$[20^6]$
10	(1,0,1,0,0,0,0)	10	3	1	$[1^20^5]$
11	(0,0,0,0,1,0,0)	11	5	1	$[10^6]$
12	(2,1,0,0,0,0,0)	12	2	2	$[210^5]$
13	(0,0,0,0,0,1,0)	13	6	1	$[10^6]$
14	(1,0,0,1,0,0,0)	14	4	1	$[1^20^5]$
15	(0,1,1,0,0,0,0)	15	3	1	$[1^20^5]$
16	(4,0,0,0,0,0,0)	16	1	4	$[40^6]$
17	(0,0,0,0,0,0,1)	17	7	1	$[10^6]$
18	(1,2,0,0,0,0,0)	18	2	2	$[210^5]$
...	...	...	...	...	...

Id	vertex	Gödel number	dimension	$\|\breve{x}\|_\infty = s$	leader class
19	(2,0,1,0,0,0,0)	20	3	2	$[210^5]$
20	(0,1,0,1,0,0,0)	21	4	1	$[1^2 0^5]$
21	(1,0,0,0,1,0,0)	22	5	1	$[1^2 0^5]$
22	(3,1,0,0,0,0,0)	24	2	3	$[310^5]$
23	(0,0,2,0,0,0,0)	25	3	2	$[20^6]$
24	(1,0,0,0,0,1,0)	26	6	1	$[1^2 0^5]$
25	(0,3,0,0,0,0,0)	27	2	3	$[30^6]$
26	(2,0,0,1,0,0,0)	28	4	2	$[210^5]$
27	(1,1,1,0,0,0,0)	30	3	1	$[1^3 0^4]$
28	(5,0,0,0,0,0,0)	32	1	5	$[50^6]$
29	(0,1,0,0,1,0,0)	33	5	1	$[1^2 0^5]$
30	(1,0,0,0,0,0,1)	34	7	1	$[1^2 0^5]$
31	(0,0,1,1,0,0,0)	35	4	1	$[1^2 0^5]$
32	(2,2,0,0,0,0,0)	36	2	2	$[2^2 0^5]$
33	(0,1,0,0,0,1,0)	39	6	1	$[1^2 0^5]$
34	(3,0,1,0,0,0,0)	40	3	3	$[310^5]$
35	(1,1,0,1,0,0,0)	42	4	1	$[1^3 0^4]$
36	(2,0,0,0,1,0,0)	44	5	2	$[210^5]$
37	(0,2,1,0,0,0,0)	45	3	2	$[210^5]$
38	(4,1,0,0,0,0,0)	48	2	4	$[410^5]$
39	(0,0,0,2,0,0,0)	49	4	2	$[20^6]$
40	(1,0,2,0,0,0,0)	50	3	2	$[210^5]$
41	(0,1,0,0,0,0,1)	51	7	1	$[1^2 0^5]$
42	(2,0,0,0,0,1,0)	52	6	2	$[210^5]$
43	(1,3,0,0,0,0,0)	54	2	3	$[310^5]$
44	(0,0,1,0,1,0,0)	55	5	1	$[1^2 0^5]$
45	(3,0,0,1,0,0,0)	56	4	3	$[310^5]$
46	(2,1,1,0,0,0,0)	60	3	2	$[21^2 0^4]$
47	(0,2,0,1,0,0,0)	63	4	2	$[210^5]$
48	(6,0,0,0,0,0,0)	64	1	6	$[60^6]$
49	(0,0,1,0,0,1,0)	65	6	1	$[1^2 0^5]$
50	(1,1,0,0,1,0,0)	66	5	1	$[1^3 0^4]$
51	(2,0,0,0,0,0,1)	68	7	2	$[210^5]$
52	(1,0,1,1,0,0,0)	70	4	1	$[1^3 0^4]$
53	(3,2,0,0,0,0,0)	72	2	3	$[320^5]$
54	(0,1,2,0,0,0,0)	75	3	2	$[210^5]$
55	(0,0,0,1,1,0,0)	77	5	1	$[1^2 0^5]$
56	(1,1,0,0,0,1,0)	78	6	1	$[1^3 0^4]$
57	(4,0,1,0,0,0,0)	80	3	4	$[410^5]$
58	(0,4,0,0,0,0,0)	81	2	4	$[40^6]$
59	(2,1,0,1,0,0,0)	84	4	2	$[21^2 0^4]$
60	(0,0,1,0,0,0,1)	85	7	1	$[1^2 0^5]$
61	(3,0,0,0,1,0,0)	88	5	3	$[310^5]$
62	(1,2,1,0,0,0,0)	90	3	2	$[21^2 0^4]$
...	...	...	...	...	...

Id	vertex	Gödel number	dimension	$\|\breve{x}\|_\infty = s$	leader class
63	(0,0,0,1,0,1,0)	91	6	1	$[1^2 0^5]$
64	(5,1,0,0,0,0,0)	96	2	5	$[510^5]$
65	(1,0,0,2,0,0,0)	98	4	2	$[210^5]$
66	(0,2,0,0,1,0,0)	99	5	2	$[210^5]$
67	(2,0,2,0,0,0,0)	100	3	2	$[2^2 0^5]$

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