# On the discrete geometry of physical quantities

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**Abstract**

The mathematical structure $S$ classifying the physical quantities is presently *unknown*. We prove that classes of physical quantities are represented by integer lattice points and that n-ary relations between physical quantities are represented by constellations of lattice points path-wise connected in the *seven* dimensional integer lattice $\mathbb{Z}^7$. The distribution of the path lengths of the n-ary relations, displays frequencies with a value $f = 1$ that indicates the existence of *unique constellations* between physical quantities. The most famous equation of physics $E = \gamma m_0 c^2$ is an element of the set of constellations that have frequency $f = 1$. We discover that the unique constellations representing *energy* are all embedded in a hyperplane of the integer lattice $\mathbb{Z}^7$. The measure polytopes $P_7^s$ with edge length $2s$, where $s = \ell_\infty$ is the Chebyshev norm, are the framework for the classification. We demonstrate that the mathematical structure $S$ classifying the physical quantities is based on *leader classes* which are distinct constellations of integer lattice points, that are related through a signed permutation of the integer lattice point coordinates. We assign to each leader class representative, that is an integer lattice point of $\mathbb{Z}^7_+$, a Gödel number. We relate the partitioning of a physical quantity to the factorization of the Gödel number of the leader class representative in distinct non-negative integer factors. We find that the physical quantity *energy* is uniquely defined by 17 lattice points forming its constellation in $\mathbb{Z}^7$. We define the *Gödel walk* as a unique walk of length $n$ in $\mathbb{Z}^k_+$ with $n, k \in \mathbb{Z}_+$. We prove that ternary relations between physical quantities are classified in 4 distinct 2-colouring patterns of $\mathbb{Z}^7$. Orthogonality and linear independence properties of the pairwise physical quantities result in classifying the ternary equations in 6 distinct types. We find that each physical quantity can be orthogonally decomposed in a *finite* number of constellations that are *rectangles*. The orthogonal decomposition results in the equation of a 7-sphere. The appendices contain a preliminary classification of common physical quantities based on the measure polytope $P_7^{10}$ and also numerical data useful as starting point for the further exploration of the discrete geometry of physical quantities.

*Keywords:* centrally symmetric polytope, lattice polytope, isoperimeter, 7-dimensional integer lattice
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## 1. Introduction

Scientists along centuries, have tried to understand nature, more specifically *the laws of nature* by capturing a subsystem of nature and encoding it into a formal mathematical system [1, 2]. Wigner

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postulated that the laws of nature form a *sharply defined set* [3] and that the laws of nature are the *correlations between events* [4] where the events are modeled by points in the spacetime continuum as defined by Einstein [5]. Feynman suggested that the best way to proceed for describing the physical reality is *to guess equations and disregard physical models or descriptions* [6, 7]. Tegmark proposes the mathematical universe hypothesis (MUH) [9, 8] as hypothetical framework for the *laws of the universe*. Lange [10] formulates the question *Must the fundamental laws of physics be complete?* while Barrow [11] expands the problem to the impact of Gödel's incompleteness theorem on nature and physics. Rickles [12] elaborates on the interpretation of the mathematical frameworks in their relation to nature. Recently, a conference was organized at the Perimeter Institute for Theoretical Physics [13] where mathematicians, philosophers of physics and physicists debated on *Laws of Nature: Their Nature and Knowability*. Fundamental questions as *What is a law of nature? How many laws are there?* were discussed amongst others. In this article we elaborate further on these topics, but focus on the *mathematical relations* [14] between physical quantities without giving these mathematical relations the status of *law of physics* and without giving an interpretation to the mathematical constructions. We will not study *spacetime models* but construct a *discrete geometric model* $M = (\Gamma, \Omega)$. Paraphrasing Rickles, we have $\Gamma = \mathbb{U}_p$ representing the possible components of physical quantities in a universe and $\Omega = \mathbb{Z}^7$ representing the geometric structure associated to the classes of physical quantities of a universe [15]. We further restrict the subject of this article to n-ary operations between physical quantities resulting in equations of the type $[x_n] = [x_0][x_1] \dots [x_{n-1}]$, where $[x_i], i \in [0, \dots, n]$ represent classes of physical quantities. A common way to express the relations between physical quantities is through their algebraic equations e.g. $E = \gamma m_0 c^2$. We show that there is an alternative representation that is based on *discrete geometry* and that this geometric representation results in the discovery of a mathematical structure $S$ that classifies the physical quantities. We follow a bottom-up approach starting from the building blocks of the *physics language*, that are the *physical quantities*. Each physical quantity is represented by a symbol or label. Physical quantities are found in the form of scalars, vectors, multi-vectors, matrices and/or tensors. All the physical quantities are eventually *measured* through their respective components and thus we restrict our analysis to the *components* of physical quantities. The choice of a system of units [16, 17, 18, 19] and the number of dimensions are open issues [19, 20] amongst physicists. In the limit one thinks of dimensionless physics [19]. Throughout this article we will adopt the *convention* of the SI units and dimensions. We use as mathematical framework a 7-dimensional integer lattice $\mathbb{Z}^7$. The basis of the integer lattice represents the 7 base units of the SI. We will demonstrate that *dimensionless physics* is not in contradiction with the use of a 7-dimensional integer lattice. On the contrary of dimensional exploration [21], we strongly rely on geometric properties related to regular systems of points [22] to study the geometric properties of the components of physical quantities. We prove that the properties *uniqueness*, *orthogonality* and *linear independence* between pairs of physical quantities result in a sharply defined set of relations between physical quantities, as hoped by Wigner [3]. We (re)discover the *complete* set of compatible physical quantities in the form of the set of decompositions of a vertex in pairwise orthogonal vertices. Maxwell addressed partially the research questions in his presentation "On the mathematical classification of physical quantities" [23]. In the footsteps of Maxwell, we study the factoring of *energy* in detail and find a *discrete* value distribution. The distribution represents the frequency of isoperimeters of the ternary operations representing *energy* constellations. The frequency of the pathlengths expresses the incidence of integer lattice points on families of 7-dimensional ellipsoids determined by the lattice point representing the physical quantity $[z]$. A signed permutation of the coordinates of each physical quantity of the leader class has the same path length distribution as the leader lattice point and so these physical quantities are all mathematically *equivalent* to the physical quantity represented by the leader class lattice point and thus express an automorphism of the leader class. Measure polytopes can be partitioned, based on the $\ell_\infty$-norm, in finite sets of leader classes. Assignment of a *Gödel number* to each physical quantity in $\mathbb{Z}_+^7$ reveals the existence of a unique *Gödel walk* of length 100 in $\mathbb{Z}_+^{25}$. The factorization of the Gödel number associated to the leader class representative results in the enumeration of all distinct forms of n-ary relations of the physical quantity. The existence of rules that have to be respected by the *laws of nature*, has been proposed by Wigner and Feynman [10]. We elaborate on this problem by proving *one* of these rules applicable for ternary operations between physical quantities resulting in equations of the type $[z] = [\kappa][x][y]$. One of these rules is related to the 2-colouring of 4-cycles. We list the cardinality of sets of pairwise orthogonal lattice points resulting in the representative vertex of leader class $[\check{z}]$.

### *1.1. Outline of the paper*

Section 1 comprises the definitions and preliminaries that are needed to allow a mathematical elaboration of the discrete geometry of the relations between physical quantities. In section 2, we discuss the images of classes of physical quantities as integer lattice points of $\mathbb{Z}^7$. We demonstrate in section 3 that equations of the type $[z] = [\kappa][x][y]$, where $[\kappa], [x], [y], [z]$ represent classes of physical quantities, have a geometric representation in the integer lattice $\mathbb{Z}^7$. In section 4, we discuss the cardinality of the isoperimetric distribution. We propose in section 5 that the classification of classes of physical quantities is based on an equivalence relation applied to measure polytopes. Graph properties related to relations between physical quantities are discussed in section 6. In section 7 we analyse the properties of linear independence and orthogonality for integer lattice points and discuss the complete set of decompositions of a vertex in pairwise orthogonal vertices. The properties of compatible physical quantities are discussed in section 8. Section 9 contains the future work and conclusion of the present research.

### *1.2. Preliminaries*

A component of a physical quantity is a quantity that is used in the description of physical processes. Let a universal set of components of physical quantities be $\mathbb{U}_p$. We partition this set in equivalence classes with notation $[a]$ where $a$ is the representative of the equivalence class. In the class energy $[E]$ we find physical quantities like potential energy, kinetic energy, work, heat, internal energy, ... which are all represented by the class $[E]$. A set of base quantities is a finite number of classes of physical quantities, which by *convention* are regarded as *dimensionally independent* in a system of physical quantities and equations defining the relationships between them. The *International System of Units (SI)* base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity. The set of classes of base physical quantities is called $\mathcal{B} \doteq \{[l], [m], [t], [i], [T], [n], [L]\}$. The base units are the set $\mathcal{U} \doteq \{u_i \mid u_1 = \mathrm{m},\ u_2 = \mathrm{kg},\ u_3 = \mathrm{s},\ u_4 = \mathrm{A},\ u_5 = \mathrm{K},\ u_6 = \mathrm{mol},\ u_7 = \mathrm{cd}\}$. The dimensional product is the expression of a class of a physical quantity as a product of powers of base quantities. Each class of a physical quantity has parameters $X^i$, called dimensional exponents. We write $[a]$ as function of the SI base units $u_i \in \mathcal{U}$ and the dimensional exponents $X^i \in \mathbb{Z}$,

$$[a] \doteq \{a_1\} \cdot \prod_{i=1}^{7} u_i^{X^i} \ , \tag{1}$$

where the physical quantity $[a]$ of the idealized physical system assumes a numerical value $\{a_1\}$. It is known that some physical quantities (rms of a quantity, noise spectral density, specific detectivity, thermal inertia, thermal effusivity, ...) are defined as the *square root* of some product or quotient of other physical quantities. These physical quantities will have fractional exponents, where $X^i \in \mathbb{Q}$ and so will not comply with the above definition. Each of these physical quantities are, by a proper exponentiation, transformed to a physical quantity having integer exponents which then complies with the above definition.

## 2. Image of a class of physical quantities

Let the set of integer septuples $\mathbb{Z}^7 \doteq \{(X^1, \ldots, X^7) \mid X^i \in \mathbb{Z}\}$ be called the 7-dimensional integer lattice. Classes of physical quantities can be imaged on lattice points in the 7-dimensional integer lattice. A set of lattice points is called a *lattice constellation* [25]. The image of a class of physical quantities $[a]$ has the notation $\breve{a}$ which clearly indicates the distinction with physical quantities represented by scalars, vectors, multi-vector, matrices and/or tensors. The image of the class of dimensionless physical quantities $[\kappa]$ has the notation $\breve{o}$ which represents the origin of the integer lattice $\mathbb{Z}^7$. We will see further that there is a mathematical justification for this notation. The image of the class energy $[E]$ is $\breve{E}$.

**Definition 1.** The function 'dex' is defined from $\mathbb{U}_p$ into $\mathbb{Z}^7$ and formally as $\text{dex} : \mathbb{U}_\text{p} \to \mathbb{Z}^7 \mid \text{dex}([a]) \doteq \breve{a} = (A^1, \ldots, A^7)$ where $A^i \in \mathbb{Z}$.

The $A^i$s are the contravariant components of the lattice point $\breve{a}$. This means that the exponents of the units of a class of physical quantities, taken in the *correct order*, form the coordinates of a point in the integer lattice $\mathbb{Z}^7$. Every possible integer lattice point is the image of one class of physical quantities and so the mapping 'dex' is bijective from $\mathbb{U}_p$ on $\mathbb{Z}^7$ and expresses 'dex' as an isomorphism between $\mathbb{U}_p$ and $\mathbb{Z}^7$. The Abelian group $\mathbb{Z}^7$ [26] is a $\mathbb{Z}$-module. The family $\{\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4, \mathbb{Z}^5, \mathbb{Z}^6\}$ are $\mathbb{Z}$-submodules of $\mathbb{Z}^7$. The $\mathbb{Z}$-module $\mathbb{Z}^7/\mathbb{Z}$ is called the quotient module of $\mathbb{Z}^7$ with respect to $\mathbb{Z}$. The prerequisite for the creation of a vector space is the existence of a field $\mathbb{F}$ for the scalars. The elements of the vector space are then vectors. This justifies the notation $\breve{a}$, indicating that the elements of $\mathbb{Z}^7, +, \cdot$ are *not* vectors $\boldsymbol{a}$. We select 7 linearly independent lattice points $\breve{e}_1, \ldots, \breve{e}_7$ of $\mathbb{Z}^7$. The $\breve{e}_i$s form a covariant basis [27] for the integer lattice in $\mathbb{Z}^7$. Every lattice point is expressed in a unique way as the linear combination: $\breve{x} = X^1\breve{e}_1 + \ldots + X^7\breve{e}_7$ where the coefficients $X^i$ are called the contravariant components of $\breve{x}$. The inner product is defined as the expression: $\breve{x} \cdot \breve{y} = \sum_{i=1}^{7}\sum_{j=1}^{7} a_{ij}X^iY^j$ where $a_{ij} = a_{ji}$. Consider seven lattice points $\breve{e}^i$ satisfying the expression $\breve{e}^i = \sum_{k=1}^{7} a^{ik}\breve{e}_k$. This contravariant basis spans the space $\mathbb{Z}^7$ resulting in the equations $\sum_{i=1}^{7} a_{ij}\breve{e}^i = \sum_{i=1}^{7}\sum_{k=1}^{7} a_{ij}a^{ik}\breve{e}_k = \sum_{k=1}^{7} \delta_j^k\breve{e}_k = \breve{e}_j$. A lattice point $\breve{x}$ has covariant components $X_i$, such that $\breve{x} = \sum_{i=1}^{7} X_i\breve{e}^i$. These components are related to the contravariant components by the expressions: $X^j = \sum_{i=1}^{7} a^{ij}X_i$ and $X_i = \sum_{i=1}^{7} a_{ij}X^j$. With this notation the inner product is represented as $\breve{x}\cdot\breve{y} = \sum_{i=1}^{7} X^iY_i = \sum_{k=1}^{7} X_kY^k$. Observe that, since $\breve{e}^i\cdot\breve{e}_j = \sum_{i=1}^{7} a^{ik}\breve{e}_k\cdot\breve{e}_j = \sum_{i=1}^{7} a^{ik}a_{jk} = \delta_j^i$, each $\breve{e}^i$ is orthogonal to every $\breve{e}_j$ except $\breve{e}_i$. We obtain that $\breve{e}^i\cdot\breve{e}_j = 1$. We are free to select seven basis lattice points. These points will receive the agreed [28] symbol for the dimension. We define:

$\breve{l} \doteq \breve{e}_1 = L = (1,0,0,0,0,0,0)$, $\breve{m} \doteq \breve{e}_2 = M = (0,1,0,0,0,0,0)$, $\breve{t} \doteq \breve{e}_3 = T = (0,0,1,0,0,0,0)$, $\breve{i} \doteq \breve{e}_4 = I = (0,0,0,1,0,0,0)$, $\breve{T} \doteq \breve{e}_5 = \Theta = (0,0,0,0,1,0,0)$, $\breve{n} \doteq \breve{e}_6 = N = (0,0,0,0,0,1,0)$, $\breve{L} \doteq \breve{e}_7 = J = (0,0,0,0,0,0,1)$, with $\breve{e}_i \in \mathbb{Z}^7$. This basis generates a *cubic lattice* [29] that is orthonormal. We claim without giving proofs of the following "dex" identities:

$$\forall [a],[b] \in \mathbb{U}_p \mid \operatorname{dex}([a][b]) = \operatorname{dex}(a) + \operatorname{dex}(b) \ , \tag{2a}$$

$$\forall [a],[b] \in \mathbb{U}_p \mid \operatorname{dex}\left(\frac{[a]}{[b]}\right) = \operatorname{dex}(a) - \operatorname{dex}(b) \ , \tag{2b}$$

$$\forall [a],[b],[c] \in \mathbb{U}_p \mid \operatorname{dex}([a][b][c]) = \operatorname{dex}([a]([b][c])) = \operatorname{dex}(([a][b])[c]) \ , \tag{2c}$$

$$\forall p \in \mathbb{Z} \mid \operatorname{dex}([a]^p) = p \operatorname{dex}(a) \ . \tag{2d}$$

**Definition 2.** The inverse of the "dex" function is a function of $\mathbb{Z}^7$ into $\mathbb{U}_p$, and defined as $\mathrm{dex}^{-1} : \forall \breve{\mathrm{a}} \in \mathbb{Z}^7, \exists [\mathrm{a}] \in \mathbb{U}_\mathrm{p} \mid \mathrm{dex}^{-1}(\breve{\mathrm{a}}) = [\mathrm{a}]$.

We claim without giving proofs of the following $\operatorname{dex}^{-1}$ identities:

$$\forall \breve{a}, \breve{b} \in \mathbb{Z}^7 \mid [a][b] = \operatorname{dex}^{-1}\left(\breve{a} + \breve{b}\right) \ , \tag{3a}$$

$$\forall \breve{a}, \breve{b} \in \mathbb{Z}^7 \mid \frac{[a]}{[b]} = \operatorname{dex}^{-1}\left(\breve{a} - \breve{b}\right) \ , \tag{3b}$$

$$\forall \breve{a}, \breve{b}, \breve{c} \in \mathbb{Z}^7 \mid \operatorname{dex}^{-1}\left(\breve{a} + \breve{b} + \breve{c}\right) = \operatorname{dex}^{-1}\left(\breve{a} + (\breve{b} + \breve{c})\right) = \operatorname{dex}^{-1}\left((\breve{a} + \breve{b}) + \breve{c}\right) \ , \tag{3c}$$

$$\forall p \in \mathbb{Z} \mid [a]^p = \operatorname{dex}^{-1}(p\, \breve{a}) \ . \tag{3d}$$

We call the expression $N(\breve{x}) \doteq \|\breve{x}\|_1 = \sum\limits_{i=1}^{7}\sum\limits_{k=1}^{7} a_{ik} X^i X^k$, the $\ell_1$-norm of $\breve{x}$ in $\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_2 \doteq \sqrt{\sum\limits_{i=1}^{7}\sum\limits_{k=1}^{7} a_{ik} X^i X^k}$ the $\ell_2$-norm or Euclidean norm of $\breve{x}$ in $\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_\infty = max\{|X^1|, \ldots, |X^7|\}$ the Chebyshev norm or infinity norm of $\breve{x}$ in $\mathbb{Z}^7$. Let $\breve{x}, \breve{y}$ be lattice points of $\mathbb{Z}^7$. The $\ell_2$-distance (Euclidean distance) between the points $\breve{x}, \breve{y}$ is defined by: $d(\breve{x}, \breve{y}) = \|\breve{x} - \breve{y}\|_2 = \sqrt{\sum\limits_{i=1}^{7}(X_i - Y_i)(X^i - Y^i)}$ where $\breve{x} - \breve{y} = (X^1 - Y^1, \ldots, X^7 - Y^7)$ if $\breve{x} = (X^1, \ldots, X^7)$ and $\breve{y} = (Y^1, \ldots, Y^7)$. We call two integer lattice points *neighbours* if their $\ell_2$-distance is 1. We assign to each lattice point $\breve{x}$ of $\mathbb{Z}^7$ a hyperplane $H_{\breve{x}}$. A set $H_{\breve{x}}$ in $\mathbb{Z}^7$ is a hyperplane [30] if and only if there exist scalars $C_0, C_1, \ldots, C_7$, where not all $C_1, \ldots, C_7$ are zero, such that $H_{\breve{x}} = \{(X^1, \ldots, X^7) \mid C_0 + C_1 X^1 + \ldots + C_7 X^7 = 0\}$. Consider now the lattice point $\breve{y} = (Y^1, \ldots, Y^7)$ and select its associated hyperplane $H_{\breve{y}}$ that contains the lattice point $\breve{o}$. The lattice point $\breve{x}$ is incident on the hyperplane $H_{\breve{y}}$ when it satisfies the equation $\sum\limits_{i=1}^{7} Y^i X_i = 0$. The distance between the lattice point $\breve{z}$ and the hyperplane $H_{\breve{y}}$, measured along the perpendicular, is the projection of $\breve{o}\breve{z}$ in the direction of $\breve{o}\breve{y}$ that is given by the equation $\dfrac{\breve{z} \cdot \breve{y}}{\|\breve{y}\|_2} = \dfrac{\sum\limits_{i=1}^{7} Z_i Y^i}{\sqrt{\sum\limits_{i=1}^{7} Y_i Y^i}}$. Let the lattice point $\breve{x}'$ be the image of $\breve{x}$ by reflection in the hyperplane $H_{\breve{y}}$. Consider the lattice point $\breve{z}$ satisfying $\breve{z} = \breve{x} - \breve{x}'$, then the line $\breve{o}\breve{z}$ is parallel to the line $\breve{o}\breve{y}$. We define now a general reflection [27] in the hyperplane $H_{\breve{y}}$ as $\breve{x} - \breve{x}' = 2\frac{\breve{x} \cdot \breve{y}}{\breve{y} \cdot \breve{y}}\breve{y}$. We call the lattice point $\breve{y}$ the *root* [31] of the reflecting hyperplane $H_{\breve{y}}$. The root system for the Lie algebra $B_7$ [32] has the basis $\breve{\alpha}_1, \dots, \breve{\alpha}_7$ defined by $\breve{\alpha}_1 = \breve{e}_1 - \breve{e}_2,\ \breve{\alpha}_2 = \breve{e}_2 - \breve{e}_3,\ \dots, \breve{\alpha}_6 = \breve{e}_6 - \breve{e}_7,\ \breve{\alpha}_7 = \breve{e}_7$. This root system generates the $\mathbb{Z}^7$ integer lattice as root lattice [31] by reflections in the hyperplanes associated with the roots. The reflections are characterized by *signed permutation matrices* [32]. As we will connect points in the integer lattice, we use the term *path* from graph theory [33], where a $k$-path is a simple graph of length $k$, i.e., consisting of $k$ vertices and $k$ edges and represented by a sequence of consecutive vertices $\breve{x}_0 \dots \breve{x}_{k-1}$ [33]. The Euclidean dimension of a graph $G$ is the smallest integer $p$ such that the vertices of $G$ can be represented by points in the Euclidean space $\mathbb{Z}^p$ with two points being 1 unit distance apart if and only if they represent adjacent vertices [34]. For undefined terms from graph theory see [33].

**Definition 3.** Let the surjective function "psc", represent the *parity of the sum of coordinates* of a lattice point of $\mathbb{Z}^7$ and define:

$$\mathrm{psc} : \mathbb{Z}^7 \to \{0,1\} \ | \ \mathrm{psc}\,(\breve{x}) = |\sum_{i=1}^{7} X^i| \pmod{2}\,, X^i \in \mathbb{Z}\,.$$

The "psc" function is a 2-colouring function. We have an *evensum* lattice point when $\mathrm{psc}\,(\breve{x}) = 0$ and an *oddsum* lattice point when $\mathrm{psc}\,(\breve{x}) = 1$ where $\breve{x} \in \mathbb{Z}^7$. Observe that the lattice points $\breve{x}$ for which $\mathrm{psc}\,(\breve{x}) = 0$ are elements of $D_7$ that is an indecomposable root lattice [35] defined as $D_7 = \{(X^1, \dots, X^7) \in \mathbb{Z}^7 \mid \sum_{i=1}^{7} X^i \ \textit{is even}\}$. The lattice $D_7$ has 84 minimal points, that are $\pm\breve{e}_j \pm \breve{e}_k$ where $(1 \le j < k \le 7)$. These 84 points form a simple basis derived from the canonical basis $\breve{e}_1, \dots, \breve{e}_7$ of $\mathbb{Z}^7$. Consider a lattice point $\breve{x}_0$ and points $\breve{x}$, which have the property $\breve{x}_0 + \breve{x} \in A \Leftrightarrow \breve{x}_0 - \breve{x} \in A$ then we call $A$ a centrally symmetric set. In the remainder of the article we will assume that $\breve{x}_0 = \breve{o}$ is the origin of $\mathbb{Z}^7$. An *integer lattice polytope* is the convex hull of a set of finitely many points in $\mathbb{Z}^d$. A measure polytope $P_d^s$ of edge-length $2s$ is a subset of $\mathbb{Z}^d$ with the following property $P_d^s = \{\breve{x}(X^1, \dots, X^d) \in \mathbb{Z}^d \mid \|\breve{x}\|_\infty = s\}$, where $X^i \in \mathbb{Z}$ and $(1 \le i \le d)$.

## 3. Geometric representation of equations between physical quantities

A relationship between $n$ components of physical quantities, represented by individual symbols, which may be used to describe a phenomenon, without exception, is a $n$-ary operation between physical quantities. The symbols are called *terms*. A finite sequence of terms is called a *formula*. Formulas can be combined with the relational operator "=" to generate *equations*. First we investigate the most basic $n$-ary operation where $n = 3$. The ternary operations under study, result in equations of the type $[z] = [\kappa][x][y]$ and the ternary operator is the multiplication operator.

**Theorem 1.** *The equation $[z] = [\kappa][x][y]$ is physically valid, with $[\kappa], [x], [y], [z]$ distinct classes of physical quantities obeying the properties:*

$$\mathrm{dex}^{-1}\,(\mathrm{dex}\,([z])) = [z], \qquad \mathrm{dex}^{-1}\,(\mathrm{dex}\,([\kappa])) = [\kappa],$$
$$\mathrm{dex}^{-1}\,(\mathrm{dex}\,([x])) = [x], \qquad \mathrm{dex}^{-1}\,(\mathrm{dex}\,([y])) = [y]\,,$$

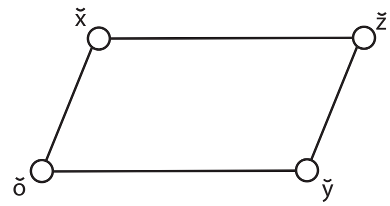


Figure 1: Parallelogram $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ representing the equation $[z] = [\kappa][x][y]$ in $\mathbb{Z}^7$ .

*if and only if, the 4-cycle* $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ *is a parallelogram in the integer lattice* $\mathbb{Z}^7$ *and* $\operatorname{dex}([x]) = \breve{x}$, $\operatorname{dex}([y]) = \breve{y}$, $\operatorname{dex}([z]) = \breve{z}$, $\operatorname{dex}([\kappa]) = \breve{o}$ *are distinct integer lattice points with* $\breve{o}$ *being the origin of the integer lattice* $\mathbb{Z}^7$ *.*

**Proof.** The proof is of the 'if and only if'–type where it is split in a necessary and sufficient condition. We aim to prove that a ternary operation between physical quantities is equivalent with a 4-cycle, being a parallelogram and vice-versa.

**Condition 1** (Necessary)**.** Let $[\kappa], [x], [y], [z] \in \mathbb{U}_p$ be distinct classes of physical quantities and $\operatorname{dex}([\kappa]) = \breve{o}$ be a dimensionless quantity. Suppose that the equation $[z] = [\kappa][x][y]$ is physically valid. By the "dex" identity (2a) we obtain $\operatorname{dex}([z]) = \operatorname{dex}([\kappa]) + \operatorname{dex}([x][y]) = \operatorname{dex}([\kappa]) + \operatorname{dex}([x]) + \operatorname{dex}([y])$ . By the definition of "dex", see 1, one writes

$$\breve{z} = \breve{o} + \breve{x} + \breve{y} \ , \tag{4}$$

where the addition is performed component-wise. The coordinates $(X^1, \ldots, X^7)$ of $\breve{x}$ , $(Y^1, \ldots, Y^7)$ of $\breve{y}$ and the origin $\breve{o}$ determine uniquely the coordinates of a lattice point $\breve{z}$ according to the above equation (4). As no degree of freedom is left over for the coordinates of $\breve{z}$, we claim that a parallelogram (Fig. 1) represented by the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ has been constructed in $\mathbb{Z}^7$ .

**Condition 2** (Sufficient)**.** Let the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ be a parallelogram (Fig. 1) with as diagonals the lines $\breve{o}\breve{z}$ and $\breve{x}\breve{y}$ . Let $\breve{o}$ be, without loss of generality, the origin of the integer lattice $\mathbb{Z}^7$ . By the definition of a 4-cycle one writes $\breve{o} = \breve{z} - \breve{x} - \breve{y}$ . This equation is rewritten as $\breve{z} = \breve{o} + \breve{x} + \breve{y}$ . We apply on both sides of the equation the function $\operatorname{dex}^{-1}$ , see 2, and obtain the equation $\operatorname{dex}^{-1}(\breve{z}) = \operatorname{dex}^{-1}(\breve{o} + \breve{x} + \breve{y})$ . By the definition, of $\operatorname{dex}^{-1}$ identity (3a) we obtain

$$\operatorname{dex}^{-1}(\operatorname{dex}(z)) = \operatorname{dex}^{-1}(\operatorname{dex}(\kappa)) \cdot \operatorname{dex}^{-1}(\operatorname{dex}(x)) \cdot \operatorname{dex}^{-1}(\operatorname{dex}(y)) \ . \tag{5}$$

As the product function $(\operatorname{dex}^{-1} \circ \operatorname{dex})$ results in the identity function we claim that there exists a set $\{[\kappa], [x], [y], [z]\} \subset \mathbb{U}_p$ for which

$$\begin{aligned} \operatorname{dex}^{-1}(\operatorname{dex}([z])) &= [z], & \operatorname{dex}^{-1}(\operatorname{dex}([\kappa])) &= [\kappa], \\ \operatorname{dex}^{-1}(\operatorname{dex}([x])) &= [x], & \operatorname{dex}^{-1}(\operatorname{dex}([y])) &= [y] \,. \end{aligned}$$

So, one obtains from the equation (5) the equation $[z] = [\kappa][x][y]$ that is to be considered as physically valid.

$\square$

*Remark.* The 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ is equal to $circuit(\breve{o}\breve{y}\breve{z}\breve{x}\breve{o})$ where the constellation *oyzxo* describes a hamiltonian circuit. Let the parallelogram *oyzxo* represent a directed graph on the vertices $1, \ldots, 4$ and let the variable $u_i$ denote the vertex that follows vertex $i$ in the sequence. The set of values that $u_i$ can take is the set of integers $j$ for which $(i, j)$ is an edge of the parallelogram *oyzxo*. The constraint $circuit(u_1, u_2, u_3, u_4)$ requires that $u = (u_1, u_2, u_3, u_4)$ describes a hamiltonian circuit, and thus $u$ is a circuit if $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$, where $\pi_1 = 1$ and $\pi_{i+1} = u_{\pi_i}$ for $i = 1, \ldots, (n-1)$ [36]. Thus $\pi_1, \ldots, \pi_n$ indicates the order in which the vertices are visited. Dimensionless physics is obtained in the integer lattice $\mathbb{Z}^7$ by considering dimensionless products of physical quantities as *k-cycles* containing the origin of the integer lattice $\mathbb{Z}^7$ and having *parallelogram* properties between its vertices.

## 4. Cardinality of isoperimetric parallelograms

Based on theorem 1 we explore the integer lattice and search for *new relations between physical quantities* by selecting 2 points $\breve{z}$ and $\breve{x}$ and create a constellation where the elements of the constellation are forming a parallelogram. *Distance from* $\breve{z}$ *to the origin* $\breve{o}$ was first studied but without success. Inspired by concepts of random walk from the origin and back to the origin, the walk length through the lattice was considered an interesting parameter to be studied. The followed approach was to select a fixed point $\breve{z}$ and to vary the point $\breve{x}$. For ease of calculation perimeters of triangles $p_t$ instead of parallelograms $p_p$ were calculated and then converted. The fixed point to start the survey through the integer lattice was selected to be $\breve{z} = \breve{E}$, representing *energy*. The question became now more specific: Which lattice points are generating triangles representing an *energy* constellation between physical quantities and how many of these triangles have the same perimeter? Two polygons are called *isoperimetric* [37] if they have the same perimeter. A program in MATLAB® was first created, but rapidly computational/memory problems occurred due to the large amount of data to be processed. The program was adapted and written in the programming language *C#*. The algorithm is given in appendix A. The absolute frequency of occurrence of these parallelogram perimeters $p_p$ are tabulated as a sequence of non-negative integers and represented graphically for $\breve{z} = \breve{E}$, as a *discrete value distribution*[38]. We observed that the constellations representing energy are connected through the discrete value distribution in such a way that the frequency $f$ is identical to the order $n$ of a graph $G$ of *vertices* representing *relations between physical quantities* and *edges* representing *a connection between relations of physical quantities.* This approach is similar to the one followed by Wigner where the *laws of nature* are the entities to which *the symmetry laws* apply [4].

### *4.1. Perimeter of a triangle*

Let $p_t$ be the perimeter of the triangle formed by the 3-cycle $\breve{o}\breve{z}\breve{x}\breve{o}$. The value of the perimeter $p_t$ is obtained by the formula $p_t = \sqrt{u} + \sqrt{v} + \sqrt{w}$ with $u, v, w \in \mathbb{Z}_+$ and expressed through the following equations:

$$u = \sum_{i=1}^{7} x_i^2 \qquad v = \sum_{i=1}^{7} (x_i - z_i)^2 \qquad w = \sum_{i=1}^{7} z_i^2 \,.$$

### 4.2. Area of a triangle

Let $A_t$ be the enclosed area of a triangle formed by the 3-cycle $\breve{o}\breve{z}\breve{x}\breve{o}$. The area of the triangle defined by the lattice points $\breve{o}\breve{z}\breve{x}$ is given by the equation, see Abramowitz and Stegun [39], $A_t = \frac{1}{2}hb$ , where $h$ is the height of the triangle which correponds to the distance from the lattice point $\breve{x}$ to the axis $\breve{o}\breve{z}$ and $b$ is the base of the triangle and corresponds to $\|\breve{z}\|_2$. We call $\phi$ the angle between $\breve{z}$ and $\breve{x}$ . From elementary goniometry, see Abramowitz and Stegun [39], we have:

$$cos^2(\phi) + sin^2(\phi) = 1 = \frac{(\sum_{i=1}^{7} z_i x_i)^2 + h^2 \|\breve{z}\|_2^2}{\|\breve{x}\|_2^2 \|\breve{z}\|_2^2} = \frac{(\sum_{i=1}^{7} z_i x_i)^2 + 4A_t^2}{\|\breve{x}\|_2^2 \|\breve{z}\|_2^2} \tag{6}$$

We rewrite the equation (6) to a quadratic form $Q(\breve{x})$

$$Q(\breve{x}) = (\sum_{i=1}^{7} z_i^2)(\sum_{i=1}^{7} x_i^2) - (\sum_{i=1}^{7} z_i x_i)^2 = 4A_t^2 \ ,$$

which is easily transformed to a matrix equation given by:

$$Q(\mathsf{X}) = \mathsf{X}^{tr}\mathsf{M}\mathsf{X} = 4A_t^2 = A_p^2 \ ,$$

and where $\mathsf{X}^{tr}$ is a $1 \times 7$ matrix, $\mathsf{M}$ is a symmetric $7 \times 7$ matrix and $\mathsf{X}$ is a $7 \times 1$ matrix. The term $A_p^2$ is the square of the area of the parallelogram formed by the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$. Observe that the quadratic form $Q(\mathsf{X})$ represents positive integers. The square of the area of the parallelogram has the property $A_p^2 \geq 1$, when degenerated parallelograms are excluded. The parallelogram for which $A_p^2 = 1$ is a fundamental parallelogram of $\mathbb{Z}^7$. Observe that the parallelograms have the lattice points $\breve{o}$ and $\breve{z}$ as foci of an ellipse $E_a$ that has the lattice points $\breve{x}$ and $\breve{y}$ incident of it. From the definition of an ellipse we have $2a = \sqrt{u} + \sqrt{v}$ .

### 4.3. Case study for the physical quantity energy

The lattice point $\breve{z} = (2, 1, -2, 0, 0, 0, 0) = \breve{E}$ represents the physical quantity *energy*. The graphical representation (Fig. 2) of the discrete value distribution of parallelogram perimeters $p_p$ for parallelograms representing equations between physical quantities in $\mathbb{Z}^7$ resulting in the physical quantity *energy* shows a rich structure. It reveals the *distribution of energy constellations*. The enumeration as class 6 (Table F.12) of the first 50 frequencies is not found in the OEIS database [40]. Observe that the lowest frequency $f_{min}$ in Fig. 2 for the *non-degenerated* parallelograms is $f_{min} = 1$ with exception of the point with perimeter $p_p = 6$, that is a degenerated parallelogram. This isoperimetric distribution shows that *unique* non-degenerated parallelograms exist, that form *unique constellations* between physical quantities. At perimeter $p_p = 7,657$ we find the well-known equation $E = \gamma m_0 c^2$ represented in its generic form as $E = \kappa_3 m_0 v^2$ (Table 1). Observe (Table 1) that the parity of the sum of the coordinates of the lattice points $\breve{x}$ are *odd* while those of the lattice points $\breve{y}$ are *even*. The components of physical quantities which are unknown to the author are marked $U_i$ in the equations of components of physical quantities resulting in the physical quantity energy. The first row represents a degenerated parallelogram. The dimensionless quantity $\kappa_1$ is associated to the dimensionless quantity $\gamma$ from the special relativity theory. The second row is recognized as the product of the linear momentum and the velocity. The third row is recognized as the kinetic energy and if $v = c$, as the famous equation $E = \gamma m_0 c^2$ . The fourth column gives the inner product of $\breve{x}$ and $\breve{y}$. Observe that the lattice points $\breve{x}$ and $\breve{y}$ are *orthogonal* for $E(1,3,1)$. Thus, the well-known equation $E = \gamma m_0 c^2$ is a *rectangle*. We show later in section 7 the importance of this property of a parallelogram. The fourth row is a well-known form appearing as a term in a Hamiltonian. The other rows express constellations between physical quantities that are unknown to the author.

*4.3.1. Graphs of order n=1*

We use the notation $E(n,g,v)$ for identifying the separate energy constellations. The index $n$ represents the order of the graph, the index $g$ identifies the graph and the index $v$ represents the vertex number in the graph.

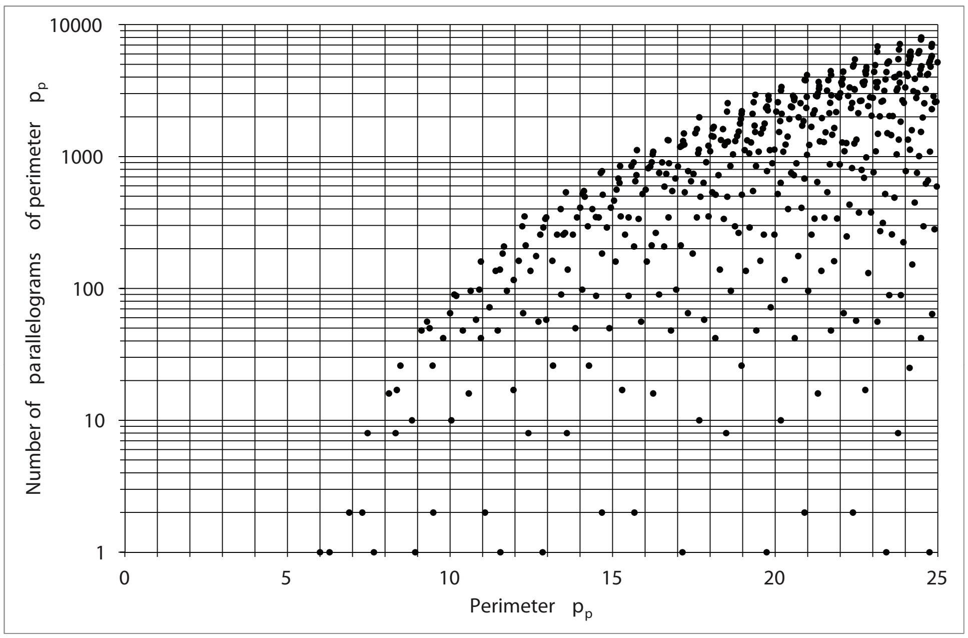


Figure 2: Discrete value distribution of parallelogram perimeters $p_p$ in $\mathbb{Z}^7$ resulting in the physical quantity *energy*.

Table 1: Graphs of order $n = 1$ for *energy*.

| $n$ | $g$ | $v$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x}\cdot\breve{y}$ | form |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 6,000 | (2,1,-2,0,0,0,0) | (0,0,0,0,0,0,0) | 0 | $E(1,1,1) = \kappa_1 E_0$ |
| 1 | 2 | 1 | 6,293 | (1,1,-1,0,0,0,0) | (1,0,-1,0,0,0,0) | 2 | $E(1,2,1) = \kappa_2 \boldsymbol{p}\cdot\boldsymbol{v}$ |
| 1 | 3 | 1 | 7,657 | (0,1,0,0,0,0,0) | (2,0,-2,0,0,0,0) | 0 | $E(1,3,1) = \kappa_3 m_0 v^2$ |
| 1 | 4 | 1 | 8,928 | (0,-1,0,0,0,0,0) | (2,2,-2,0,0,0,0) | -2 | $E(1,4,1) = \kappa_4 \dfrac{p^2}{m_0}$ |
| ... | ... | ... | ... | ... | ... | ... | ... |

| $n$ | $g$ | $v$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x}\cdot\breve{y}$ | form |
|---|---|---|---|---|---|---|---|
| 1 | 5 | 1 | 11,546 | (3,1,-3,0,0,0,0) | (-1,0,1,0,0,0,0) | -6 | $E(1,5,1)=\kappa_5 U_1 U_2$ |
| 1 | 6 | 1 | 12,845 | (-1,-1,1,0,0,0,0) | (3,2,-3,0,0,0,0) | -8 | $E(1,6,1)=\kappa_6 U_3 U_4$ |
| 1 | 7 | 1 | 17,146 | (4,1,-4,0,0,0,0) | (-2,0,2,0,0,0,0) | -16 | $E(1,7,1)=\kappa_7 \dfrac{U_5}{v^2}$ |
| 1 | 8 | 1 | 19,734 | (4,3,-4,0,0,0,0) | (-2,-2,2,0,0,0,0) | -22 | $E(1,8,1)=\kappa_8 \dfrac{U_6}{p^2}$ |
| 1 | 9 | 1 | 23,415 | (-3,-1,3,0,0,0,0) | (5,2,-5,0,0,0,0) | -32 | $E(1,9,1)=\kappa_9 U_7 U_8$ |
| 1 | 10 | 1 | 24,743 | (5,3,-5,0,0,0,0) | (-3,-2,3,0,0,0,0) | -36 | $E(1,10,1)=\kappa_{10} U_9 U_{10}$ |

The distribution in Fig. 2 is truncated at $p_p = 25$ due to edge effects at the hypercube surface. The edge effects are related to the memory capacity of the author's computer. The computation of the distribution in $\mathbb{Z}^7$ was performed for a Chebyshev norm $\|\breve{x}\|_\infty = 5$ . The analysis covers 524287 parallelograms. The connectivity of the graphs is represented by $n = f = 1$ that is a single vertex having a loop. The loop, which is an edge, is represented by a $7\times 7$ signed permutation matrix that transforms the relations in itself and so we find for the permutation matrix $\mathsf{P}_{11} = \mathsf{I}_7$ . The signed permutation matrices $\pi$ are $\mathbb{Z}$-linear maps for which $\pi\breve{o} = \breve{o}$ and $\pi(-\breve{a}) = -\pi\breve{a}$ for all $\breve{a} \in \mathbb{Z}^7$ . Observe in Fig. 3 that all the unique *energy* equations are embedded in $\mathbb{Z}^3 \times \{0\}^4$ and localized in the hyperplane $H_{\breve{a}} = \{(X^1,\ldots,X^7) \mid X^1 + X^3 = 0\}$ with $\breve{a} = (1,0,1,0,0,0,0)$ that represents the product of *length* and *time*. We know that this product is a relativistic invariant in the special relativity theory. Observe in Fig. 3 the symmetry axes determined by the line containing *origin* and *energy* and the line containing *velocity* and *linear momentum*. We calculated the squared area $A_p^2$ of each parallelogram and find for the graphs of order $n = f = 1$ the equation $\log_2(A_p^2) = 2k+1$ with $k \in \mathbb{Z}_+$ .

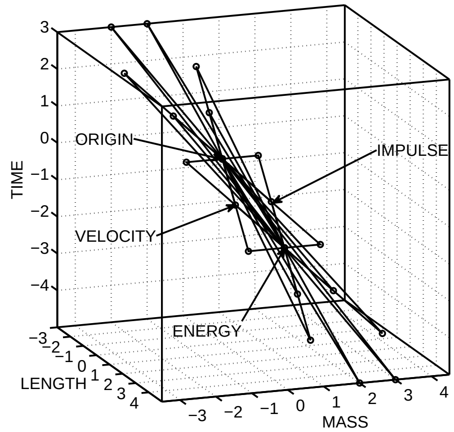


Figure 3: Unique parallelograms resulting in the physical quantity *energy*.

Let $x_i = \dfrac{1}{\kappa_i}$ with i = 1 to 10. The set $X = \{x_1, \ldots, x_{10}\}$ represents 10 dimensionless physical variables $x_i \in \mathbb{R}$ that are constructed from graphs of order 1.

**Example 4.1.** $x_1 = \dfrac{E_0}{E},\ x_2 = \dfrac{\boldsymbol{p} \cdot \boldsymbol{v}}{E},\ x_3 = \dfrac{m_0 v^2}{E},\ x_4 = \dfrac{p^2}{m_0 E}, \ldots$

We define a monomial $x^\alpha = \prod_{i=1}^{10} x_i^{\alpha_i}$ where $\alpha = (\alpha_1, \ldots, \alpha_{10})$ is a 10-tuple of non-negative integers. We form a finite linear combination of monomials $x^\alpha$ to obtain a multivariate polynomial $f$. The set of all multivariate polynomials in $x_1, \ldots, x_{10}$ with coefficients in $\mathbb{R}$ is denoted $\mathbb{R}[x_1, \ldots, x_{10}]$ [41]. Let $f_1, \ldots, f_s$ be multivariate polynomials in $\mathbb{R}[x_1, \ldots, x_{10}]$ then we define $\mathbf{V}(f_1, \ldots, f_s) = \{(x_1, \ldots, x_{10}) \in \mathbb{R}^{10} \mid f_i(x_1, \ldots, x_{10}) = 0\}$ for all $1 \leq i \leq s$. We call $\mathbf{V}(f_1, \ldots, f_s)$ the affine variety defined by $f_1, \ldots, f_s$ [41].

**Example 4.2.** $\mathbf{V}(x_2^2 + x_3^2 - 1)$, that describes a unit circle in $\mathbb{R}^2$, is the variety that represents the states of a free particle of rest mass $m_0$, after the assignment $v = c$ in the dimensionless variable $x_3$.

The further elaboration on the construction of other varieties based on dimensionless variables is beyond the scope of the present article.

*4.3.2. Graphs of order n=2*

We analyse the parallelograms in Fig. 2 having frequency $f = 2$. The result is given in the Table 2. Components of physical quantities which are *unknown* to the author are marked $U_j$. The first and second row in Table 2 represent two equations. The first equation is recognized as Planck's equation $E = \kappa_1 h\nu$, when the angular momentum $J = h$. The second equation $W = \kappa_2 \int \boldsymbol{F} \cdot d\boldsymbol{s}$ represents the work done by the force $\boldsymbol{F}$. Both equations are combined to a new constellation described by the form $\kappa_2 \int \boldsymbol{F} \cdot d\boldsymbol{s} = \kappa_1 h\nu$.

Table 2: Graphs of order $n = 2$ for *energy*.

| $n$ | $g$ | $v$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x} \cdot \breve{y}$ | form |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 6,899 | (0,0,-1,0,0,0,0) | (2,1,-1,0,0,0,0) | 1 | $E(2,1,1) = \kappa_1 J\omega$ |
| 2 | 1 | 2 | 6,899 | (1,0,0,0,0,0,0) | (1,1,-2,0,0,0,0) | 1 | $E(2,1,2) = \kappa_2 Fs$ |
| 2 | 2 | 1 | 7,301 | (2,0,-1,0,0,0,0) | (0,1,-1,0,0,0,0) | 1 | $E(2,2,1) = \kappa_3 \dfrac{\partial A}{\partial t} \dfrac{\partial m}{\partial t}$ |
| 2 | 2 | 2 | 7,301 | (1,0,-2,0,0,0,0) | (1,1,0,0,0,0,0) | 1 | $E(2,2,2) = \kappa_4 a U_1$ |
| 2 | 3 | 1 | 9,483 | (-1,0,0,0,0,0,0) | (3,1,-2,0,0,0,0) | -3 | $E(2,3,1) = \kappa_5 \dfrac{U_2}{r}$ |
| 2 | 3 | 2 | 9,483 | (0,0,1,0,0,0,0) | (2,1,-3,0,0,0,0) | -3 | $E(2,3,2) = \kappa_6 Pt$ |
| 2 | 4 | 1 | 11,075 | (3,2,-2,0,0,0,0) | (-1,-1,0,0,0,0,0) | -5 | $E(2,4,1) = \kappa_7 U_3 U_4$ |
| 2 | 4 | 2 | 11,075 | (2,2,-3,0,0,0,0) | (0,-1,1,0,0,0,0) | -5 | $E(2,4,2) = \kappa_8 \dfrac{U_5}{\frac{m}{t}}$ |

The signed $7 \times 7$ permutation matrix $\mathsf{P}_{211,212}$ that transforms *all* the relations of the graphs of order 2 is:

$$\mathsf{P}_{211,212} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix $\mathsf{P}_{211,212}$ has the property of being a symmetric matrix. Observe that the permutation matrix $\mathsf{P}_{211,212}$ has a block diagonal structure:

$$\mathsf{P}_{211,212} = \begin{bmatrix} \mathsf{S} & \mathsf{0}_4 \\ \mathsf{0}_4 & \mathsf{I}_4 \end{bmatrix} \qquad \mathsf{S} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Observe that the permutation matrices for the graphs of order 2 have a $4 \times 4$ identity matrix in the last bottom block matrix and so are acting only in $\mathbb{Z}^3 \times 0^4$. The third row of Table 2 represents the equation $E = \kappa_3 \dfrac{\partial A}{\partial t} \dfrac{\partial m}{\partial t}$, where $A$ represents an area. The factor $\dfrac{\partial A}{\partial t}$ represents a diffusion constant $D$ or a flux of vorticity. The fourth row of Table 2 represents the equation $E = \kappa_4 amr$ where $a$ represents the acceleration and where $E$ is recognized as potential energy when $a = g$ with $g$ the acceleration of the Earth gravitation. Both constellations combine to $\kappa_3 \dfrac{\partial A}{\partial t} \dfrac{\partial m}{\partial t} = \kappa_4 amr$. We anticipate a first order partial differential equation:

$$\kappa_3 D \frac{\partial m}{\partial t} - \kappa_4 amr = 0 \,.$$

The combinations of the constellations could also generate the following partial differential equation:

$$\kappa_3 \frac{\partial r^2}{\partial t} \frac{\partial m}{\partial t} - \kappa_4 amr = r(2\kappa_3 \frac{\partial r}{\partial t} \frac{\partial m}{\partial t} - \kappa_4 am) = r(2\kappa_3 v \frac{\partial m}{\partial t} - \kappa_4 \frac{\partial v}{\partial t} m) = 0 \,.$$

We see that the form and the combination of constellations is not uniquely defining one equation but a set of equations.

*4.3.3. Graphs of order n=8*

We analyse the parallelograms in Fig. 2 having frequency $f = n = 8$. The first graph of order 8 corresponds with a parallelogram having the perimeter $p_p = 7,464$ and the second graph of order 8 has a perimeter $p_p = 8,325$. The result for the first and second graphs are given in the Table 3. The components of physical quantities which are *unknown* to the author are marked $U_j$. The author is not aware if these equations are known to the physics community. Observe that the constellations of graph $g = 1$ are all related to $E(1,2,1) = \kappa_2 \boldsymbol{p} \cdot \boldsymbol{v}$ which is a graph of order $n = 1$.

Table 3: Graphs of order $n = 8$ for *energy*.

| $n$ | $g$ | $v$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x} \cdot \breve{y}$ | form |
|---|---|---|---|---|---|---|---|
| 8 | 1 | 1 | 7,464 | (1,0,-1,-1,0,0,0) | (1,1,-1,1,0,0,0) | 1 | $E(8,1,1) = \kappa_1 \frac{v}{I} U_1$ |
| 8 | 1 | 2 | 7,464 | (1,0,-1,0,-1,0,0) | (1,1,-1,0,1,0,0) | 1 | $E(8,1,2) = \kappa_2 \frac{v}{T} U_2$ |
| 8 | 1 | 3 | 7,464 | (1,0,-1,0,0,-1,0) | (1,1,-1,0,0,1,0) | 1 | $E(8,1,3) = \kappa_3 \frac{v}{n} U_3$ |
| 8 | 1 | 4 | 7,464 | (1,0,-1,0,0,0,-1) | (1,1,-1,0,0,0,1) | 1 | $E(8,1,4) = \kappa_4 \frac{v}{L} U_4$ |
| 8 | 1 | 5 | 7,464 | (1,0,-1,0,0,0,1) | (1,1,-1,0,0,0,-1) | 1 | $E(8,1,5) = \kappa_5 U_5 \frac{p}{L}$ |
| 8 | 1 | 6 | 7,464 | (1,0,-1,0,0,1,0) | (1,1,-1,0,0,-1,0) | 1 | $E(8,1,6) = \kappa_6 U_6 \frac{p}{n}$ |
| 8 | 1 | 7 | 7,464 | (1,0,-1,0,1,0,0) | (1,1,-1,0,-1,0,0) | 1 | $E(8,1,7) = \kappa_7 U_7 \frac{p}{T}$ |
| 8 | 1 | 8 | 7,464 | (1,0,-1,1,0,0,0) | (1,1,-1,-1,0,0,0) | 1 | $E(8,1,8) = \kappa_8 U_8 \frac{p}{I}$ |
| 8 | 2 | 1 | 8,325 | (0,0,0,-1,0,0,0) | (2,1,-2,1,0,0,0) | -1 | $E(8,2,1) = \kappa_9 \frac{1}{I} U_1$ |
| 8 | 2 | 2 | 8,325 | (0,0,0,0,-1,0,0) | (2,1,-2,0,1,0,0) | -1 | $E(8,2,2) = \kappa_{10} \frac{1}{T} U_2$ |
| 8 | 2 | 3 | 8,325 | (0,0,0,0,0,-1,0) | (2,1,-2,0,0,1,0) | -1 | $E(8,2,3) = \kappa_{11} \frac{1}{n} U_3$ |
| 8 | 2 | 4 | 8,325 | (0,0,0,0,0,0,-1) | (2,1,-2,0,0,0,1) | -1 | $E(8,2,4) = \kappa_{12} \frac{1}{L} U_4$ |
| 8 | 2 | 5 | 8,325 | (0,0,0,0,0,0,1) | (2,1,-2,0,0,0,-1) | -1 | $E(8,2,5) = \kappa_{13} L \frac{E}{L}$ |
| 8 | 2 | 6 | 8,325 | (0,0,0,0,0,1,0) | (2,1,-2,0,0,-1,0) | -1 | $E(8,2,6) = \kappa_{14} n \frac{E}{n}$ |
| 8 | 2 | 7 | 8,325 | (0,0,0,0,1,0,0) | (2,1,-2,0,-1,0,0) | -1 | $E(8,2,7) = \kappa_{15} T \frac{E}{T}$ |
| 8 | 2 | 8 | 8,325 | (0,0,0,1,0,0,0) | (2,1,-2,-1,0,0,0) | -1 | $E(8,2,8) = \kappa_{16} I \frac{E}{I}$ |

The number of signed permutation matrices for graphs of order 8 is $\binom{8}{2} = 28$. The permutation matrix $\mathsf{P}_{811,812}$ that transforms the constellation $E(8,1,1)$ in $E(8,1,2)$ is:

$$\mathsf{P}_{811,812} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and is one of the 28 permutation matrices describing the connectivity between these 8 constellations. It is obvious that this matrix $\mathsf{P}_{811,812}$ is not symmetric. The permutation matrix $\mathsf{P}_{821,822}$ that transforms the constellation $E(8,2,1)$ in $E(8,2,2)$ is identical to $\mathsf{P}_{811,812}$. Observe that the permutation matrix $\mathsf{P}_{811,812}$ has a block structure:

$$\mathsf{P}_{811,812} = \begin{bmatrix} \mathsf{I}_3 & \mathsf{O}_4 \\ \mathsf{O}_4 & \mathsf{T} \end{bmatrix} \qquad \mathsf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the permutation matrices for the graphs of order 8 have a $3 \times 3$ identity matrix in the first upper block matrix and so are not transforming the $\mathbb{Z}^3 \times 0^4$. Consider the $\mathbb{Z}$-module $\mathbb{Z}^7$ and the $\mathbb{Z}$-submodule $\mathbb{Z}^3$ then there exist a canonical $\mathbb{Z}$-linear map from $\mathbb{Z}^7$ to the factor group $\mathbb{Z}^7/\mathbb{Z}^3$ that sends a lattice point $\breve{x} \in \mathbb{Z}^7$ to the element $\breve{x} + \mathbb{Z}^3$. We study the dependency of the isoperimeter distribution for the physical quantity *energy* as function of the dimension $d$ of the integer lattice when $3 \leq d \leq 8$. The results (Table 4) show that the frequency $f$ in the isoperimeter distribution for the physical quantity *energy* is *uncorrelated* with the dimension $d$ of the integer lattice when $d \geq 5$ and $f = 1$ or $f = 2$.

Table 4: Variation of the frequency $f$ of the isoperimeter distribution for the physical quantity *energy* as a function of the dimension $d$ of the $\mathbb{Z}$-modules denoted as $\mathbb{Z}^d$ when $3 \leq d \leq 8$.

| Id | $p_p$ | $\mathbb{Z}^3$ | $\mathbb{Z}^4$ | $\mathbb{Z}^5$ | $\mathbb{Z}^6$ | $\mathbb{Z}^7$ | $\mathbb{Z}^8$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 6,146 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 6,449 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4 | 6,650 | 2 | 2 | 2 | 2 | 2 | 2 |
| 5 | 6,732 | 0 | 2 | 4 | 6 | 8 | 10 |
| 6 | 6,828 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 7,059 | 0 | 4 | 8 | 12 | 16 | 20 |
| 8 | 7,162 | 0 | 2 | 4 | 6 | 8 | 10 |
| 9 | 7,181 | 1 | 5 | 9 | 13 | 17 | 21 |
| 10 | 7,236 | 2 | 2 | 6 | 14 | 26 | 42 |
| 11 | 7,414 | 2 | 4 | 6 | 8 | 10 | 12 |
| 12 | 7,464 | 1 | 1 | 1 | 1 | 1 | 1 |

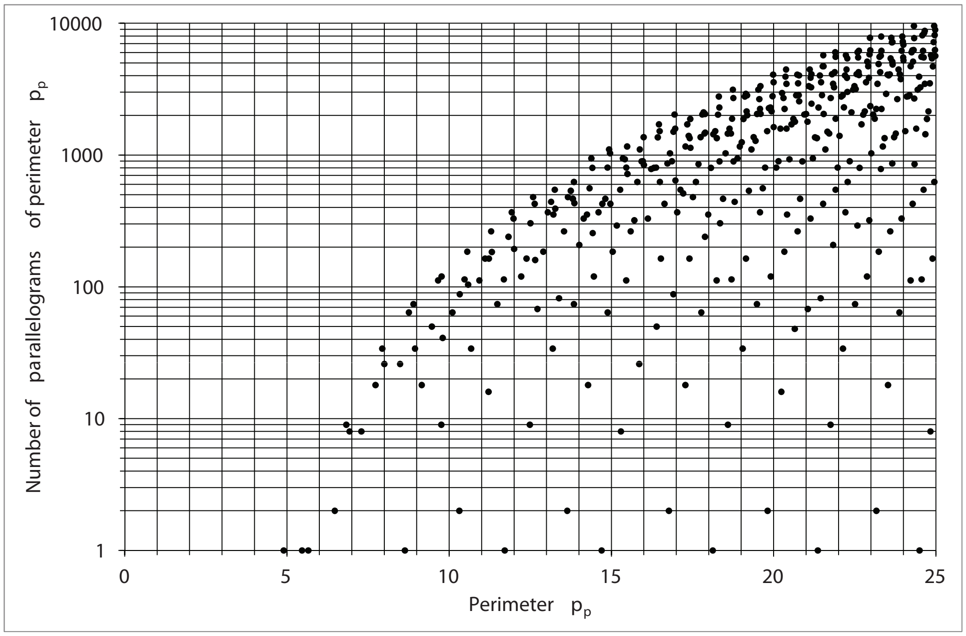


Figure 4: Discrete value distribution of parallelogram perimeters $p_p$ in $\mathbb{Z}^7$ resulting in the physical quantity *force*.

### *4.4. Case study for the physical quantity force*

The lattice point $\breve{z} = (1, 1, -2, 0, 0, 0, 0) = \breve{F}$ represents the physical quantity *force*. The graphical representation (Fig. 4) of the discrete value distribution of parallelogram perimeters $p_p$ for parallelograms representing equations between physical quantities in $\mathbb{Z}^7$ resulting in the physical quantity *force* shows also a rich structure. It reveals the *force* constellations. Observe that the lowest frequency $f_{min}$ in Fig. 4 is $f_{min} = 1$ . Detailed analysis of the collinearity of $\breve{x}$ and $\breve{y}$ indicates that the points with perimeter $p_p = 4,899$ and $p_p = 14,697$ are degenerated parallelograms. Observe (Table 5) that the parity of the sum of the coordinates of the lattice points $\breve{x}$ and $\breve{y}$ are always equal. The components of physical quantities which are *unknown* to the author are marked $U_j$ in the equations of components of physical quantities resulting in the physical quantity force.

Table 5: Unique parallelograms in $\mathbb{Z}^7$ for the physical quantity *force*.

| $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x} \cdot \breve{y}$ | form |
|---|---|---|---|---|
| 4,899 | (1,1,-2,0,0,0,0) | (0,0,0,0,0,0,0) | 0 | $\boldsymbol{F} = \kappa_1 \boldsymbol{F}_0$ |
| 5,464 | (1,1,-1,0,0,0,0) | (0,0,-1,0,0,0,0) | 1 | $\boldsymbol{F} = \kappa_2 \dfrac{\boldsymbol{dp}}{dt}$ |
| 5,657 | (1,0,-1,0,0,0,0) | (0,1,-1,0,0,0,0) | 1 | $\boldsymbol{F} = \kappa_3 \boldsymbol{v} \dfrac{\partial m}{\partial t}$ |
| 8,633 | (0,0,1,0,0,0,0) | (1,1,-3,0,0,0,0) | -3 | $\boldsymbol{F} = \kappa_4 t \dfrac{\boldsymbol{dF}}{dt}$ |
| … | … | … | … | … |

| $p_p$ | $\breve{x}$ | $\breve{y}$ | $\breve{x}\cdot\breve{y}$ | form |
|---|---|---|---|---|
| 11,710 | (-1,-1,1,0,0,0,0) | (2,2,-3,0,0,0,0) | -7 | $\boldsymbol{F} = \kappa_5 U_1 \dfrac{dp^2}{dt}$ |
| 14,697 | (-1,-1,2,0,0,0,0) | (2,2,-4,0,0,0,0) | -12 | $\boldsymbol{F} = \kappa_6 U_2 F^2$ |
| 18,122 | (-1,-1,3,0,0,0,0) | (2,2,-5,0,0,0,0) | -19 | $\boldsymbol{F} = \kappa_7 U_3 \dfrac{dF^2}{dt}$ |
| 21,361 | (-2,-2,3,0,0,0,0) | (3,3,-5,0,0,0,0) | -27 | $\boldsymbol{F} = \kappa_8 \dfrac{1}{(\dfrac{dp^2}{dt})}(\dfrac{dp}{dt})^2\boldsymbol{p}$ |

Wilczek [42, 43, 44] elaborated on Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ . We observe that this form of constellation is not appearing in the list of unique parallelograms. We don't find the lattice points (0,1,0,0,0,0,0) and (1,0,-2,0,0,0,0) as vertices of *unique* parallelograms, which is in correspondence with Wilczek's arguments. What we observe in the detailed data of the discrete value distribution is the occurrence of $\boldsymbol{F} = m\boldsymbol{a}$ in a constellation with the form $\kappa_8 \boldsymbol{r} \dfrac{\partial^2 m}{\partial t^2} = \kappa_9 m\boldsymbol{a}$, having frequency $f = 2$ for a perimeter $p_p = 6,472$ . The second row is the basic form where the force is expressed as the time derivative of the linear momentum [42]. The relation between *force* and *energy*, where a force is expressed as the space derivative of the energy [42] is found in the discrete value distribution at perimeter $p_p = 8$ and has frequency $f = 26$ . At perimeter $p_p = 10,312$ we find another constellation form $\kappa_{10} \dfrac{\partial E}{\partial t}\dfrac{\partial m}{\partial t} = \kappa_{11} v U_4$ with frequency $f = 2$ . The list (Table 5) of vertices, as well as the complete distribution is derived purely mathematically without *a priori* knowledge of physics using an algorithm Appendix A based on discrete geometry. Observe in Fig. 5 that all the unique *force* constellations are embedded in $\mathbb{Z}^3 \times 0^4$ and localized in the hyperplane $H_{\breve{b}} = \{(X^1, \ldots, X^7) \mid X^1 - X^2 = 0\}$ with $\breve{b} = (1, -1, 0, 0, 0, 0, 0)$ that represents the reciprocal of the *linear density*. One exception is observed for the equation $\boldsymbol{F} = \kappa_2 \boldsymbol{v} \dfrac{\partial m}{\partial t}$ that forms a parallelogram orthogonal to the hyperplane $H_{\breve{b}}$ . Observe in Fig. 5 the symmetry axes determined by the line containing *origin* and *force* and the line containing the *time derivative* and *impulse*.

### *4.5. Invariance of the isoperimetric distribution*

**Theorem 2.** *The isoperimetric distribution, for parallelograms containing the integer lattice points $\breve{o}$ and $\breve{z}$, is invariant when the coordinates of the integer lattice point $\breve{z}$ are subjected to a signed permutation.*

**Proof.** The isometric property of the above mapping and mapping combinations is the origin of the invariance in the isoperimetric distribution [45]. The perimeter of the parallelogram is based on the Euclidean distance ($\ell_2$-distance) between the lattice points and so neither a permutation of the coordinates nor a change in the sign of the coordinates will modify the value of the distance between the lattice points. □

For $n$-ary equations where $n \geq 4$ we have not a parallelogram, however the isometry properties remain valid when considering the path length of the path connecting the $n + 1$ lattice points of the constellation. The automorphism group of the 7-dimensional cubic lattice $\mathrm{Aut}(\mathbb{Z}^7)$ contains all

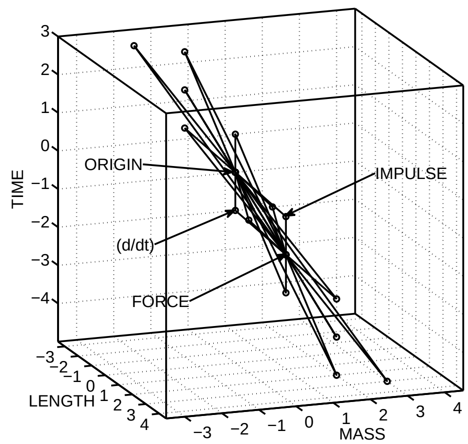


Figure 5: Unique parallelograms resulting in the physical quantity *force*.

permutations and sign changes of the 7 coordinates and has order $2^7\,7! = 645120$. Each signed permutation matrix is an orthogonal matrix [45].

**Example 4.3.** The components of the physical quantity *force*, represented by (1, 1, -2, 0, 0, 0, 0), and the components of the physical quantity *angular momentum*, represented by (2, 1, -1, 0, 0, 0, 0), have the same isoperimetric distribution.

**Example 4.4.** The components of the physical quantity *mass*, represented by (0, 1, 0, 0, 0, 0, 0), and the components of the physical quantity *frequency*, represented by (0, 0, -1, 0, 0, 0, 0), have the same isoperimetric distribution.

The fact that some physical quantities are related through a signed permutation implies that these physical quantities are *qualitatively* indistinguishable [46]. Feynman remarks that *"the fundamental laws of physics, when discovered, can appear in so many different forms that are not apparently identical at first, but with a little mathematical fiddling you can show the relationship"* [7]. These many different forms are what we define as the *constellations* of the physical quantity and the graphs of order $n$ express the relationship between these geometrical forms.

## 5. Classification of components of physical quantities

To classify the components of physical quantities we need to find a partitioning of the integer lattice $\mathbb{Z}^7$. It is known from linear vector quantization [47, 48, 49] that the $\ell_2$-norm and the phase of a lattice point are used to partition a lattice. However, this norm and phase are not the correct classifiers for the physical quantities. If we use as classifier the $\ell_\infty$-norm we obtain equivalence classes for which the elements of the class have the *same* isoperimetric distribution.

### *5.1. Measure polytope properties*

**Theorem 3.** *Let $P_d^s$ be a centrally symmetric d-dimensional measure polytope of edge-length 2s then the cardinality of $P_d^s$ is $(2s+1)^d$.*

**Proof.** For $d = 0$ the result is trivial.
For $d = 1$ we have the set $P_1^s = \{-s, \ldots, 0, \ldots, s\}$ with edge-length $2s$. Let us denote the cardinality of the set $S$ by $\#(S)$ then $\#(P_1^s) = 2s + 1$.
For $d = 2$ we have to increase the dimension $d$ by 1, which corresponds to calculate the Cartesian product of the sets $P_1^s \times P_1^s = P_2^s$.
It is a property of cardinal numbers [50] that: $\#(P_2^s) = \#(P_1^s) \times \#(P_1^s) = \#(P_1^s) \cdot \#(P_1^s) = (2s+1)^2$. Assume that $\#\left(P_{d-1}^s\right) = (2s+1)^{d-1}$. Then $\#(P_d^s) = \#\left(P_{d-1}^s\right) \cdot \#(P_1^s) = (2s+1)^{d-1} \cdot (2s+1) = (2s+1)^d$. □

We distinguish the measure polytope $P_d^s$ by the parameters $d$ and $s$, where $d$ represents the dimension of the integer lattice and $s$ represents the edge length. We define a leader class of a measure polytope as:

**Definition 4.** A leader class of a measure polytope is the set of lattice points of $\mathbb{Z}^7$ that have the same isoperimetric distribution.

A leader class of a measure polytope of $\mathbb{Z}^7$ is noted as $[(X^1, \ldots, X^7)]$ where $(X^1, \ldots, X^7)$ are the coordinates of the representative lattice point. Each leader class forms a set of lattice points that are symmetric about the origin. The cardinality of a leader class of a measure polytope is calculated using elementary combinatorics. Let $A = \{0, 1, 2, \ldots, k\}$ be the alphabet of measure polytope with edge length $2k$. The representative of a leader class of a measure polytope is a word $w$ constructed from the alphabet $A$. The words $w$ have a length $d$ that corresponds to the dimension of $\mathbb{Z}^7$. Let $d_i$ be the number of characters of type $i$ of the alphabet $A$. Suppose that the characters are subjected to permutation and change of sign, then using combinatorics the cardinality is given by the equation

$$\#(w) = 2^{d-d_0} \frac{d!}{d_0! d_1! d_2! \ldots d_k!} . \tag{7}$$

Observe that each measure polytope in $\mathbb{Z}^7$ represents a centrally symmetric lattice polytope [27, 51, 52, 53]. The number of vertices in each leader class is equal to the cardinality of $w$. Observe also that the representative lattice point, in coding theory [47] called an *absolute leader*, has only coordinates that are *non-negative integers*. We define the total degree of a monomial as:

**Definition 5.** A monomial $m$ in $u_1, u_2, \ldots, u_7$ is a product of the form:

$$m = \prod_{i=1}^{7} u_i^{X^i} , \tag{8}$$

where all the exponents $X^i \in \mathbb{Z}_+$ and $u_i \in \mathcal{U}$ (see section 1). The total degree deg of this monomial is the sum $X^1 + \ldots + X^7$.

From the 7-tuple of non-negative integer exponents $(X^1, \ldots, X^7) \in \mathbb{Z}_+^7$ a monomial [54] is constructed one-to-one of the form $m = \prod_{i=1}^{7} u_i^{X^i}$ that we compare with equation (1). It means that a lot of results known from the commutative module of monomials are applicable to the classification of the components of physical quantities. The number of classes of monomials (Table 6) with Chebyshev norm $\|\breve{x}\|_\infty \leq s$ in $\mathbb{Z}^7$ is the result from application of lemma 4 [55].

Table 6: Properties of the measure polytopes $P_7^s$ in $\mathbb{Z}^7$ for $s \leq 10$.

| $\|\breve{x}\|_\infty = s$ | sum(# ([a])) | cumul(sum(# ([a]))) | # ($P_7^s$) | cumul(# ($P_7^s$)) |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 2186 | 2187 | 7 | 8 |
| 2 | 75938 | 78125 | 28 | 36 |
| 3 | 745418 | 823543 | 84 | 120 |
| 4 | 3959426 | 4782969 | 210 | 330 |
| 5 | 14704202 | 19487171 | 462 | 792 |
| 6 | 43261346 | 62748517 | 924 | 1716 |
| 7 | 108110858 | 170859375 | 1716 | 3432 |
| 8 | 239479298 | 410338673 | 3003 | 6435 |
| 9 | 483533066 | 893871739 | 5005 | 11440 |
| 10 | 907216802 | 1801088541 | 8008 | 19448 |

In (Table 6) the second column shows the number of vertices while the third column gives the cumulated number of vertices. The fourth and fifth columns have a similar meaning but are expressing the number of classes in each measure polytope $P_7^s$ .

*5.2. Enumeration of the measure polytopes*

The enumeration table (Table C.9) of the measure polytope $P_7^3$ consists of 8 columns. The second column is the row identifier. The third column gives the representative of the leader class. The fourth column contains the sum of the absolute value of the coordinates of the lattice points being elements of the leader class that is exclusively the total degree of the monomial associated with the leader class. The fifth column gives the parity of the representative of the leader class. The sixth column gives the $\ell_1$-norm of the representative. The seventh column gives the cardinality of the leader class. The eighth column gives the Gödel number of the representative. The ordering of the classes is based on *graded reverse lex order* [54]. We derive from Table 6 that the measure polytopes $P_7^s$ are partitioned in $\binom{7+s-1}{s}$ equivalence classes. The cardinality of the leader classes is related to the theta series of the integer lattice $\mathbb{Z}^7$. We find in the OEIS [40] the sequence A008451 given by $r_7(N) = 1$, 14, 84, 280, 574, 840, 1288, 2368, 3444, 3542, 4424, 7560, 9240, 8456, 11088, 16576, 18494, 17808, 19740, 27720, 34440, 29456, 31304, 49728, 52808, 43414, 52248, 68320, 74048, 68376, 71120, 99456, 110964, 89936, 94864, 136080 . . . . The sequence represents the number of ways of writing a positive integer $N$ as a sum of *seven* integral squares and is defined by:

$$\Theta_{\mathbb{Z}^7}(z) = \sum_{N=0}^{\infty} r_7(N) q^N \ , \tag{9}$$

where $q = e^{\pi i z}$ and $N$ is the norm of the lattice point [56]. The enumeration table (Table D.10) gives the relation between the sequence A008451 and the partitioning of 7-spheres in leader classes of the measure polytopes. The common physical quantities (Table H.14) which belong to the measure polytopes, where the variable $\|\breve{x}\|_\infty = s$ taking values from 0 to 10, are enumerated. Table H.14 is far from exhaustive, but it highlights the sparse distribution of the common physical quantities when taking in consideration the cardinalities (Table 6 ) of classes and vertices.

### 6. Paths, walks and cycles in a 7-dimensional integer lattice

A path in $\mathbb{Z}^7$ is a non-empty graph $P = (V, E)$ of the form $V = \{\breve{x}_0, \dots, \breve{x}_k\}$ and $E = \{\breve{x}_0\breve{x}_1, \dots, \breve{x}_{k-1}\breve{x}_k\}$ where the $\breve{x}_i$ are all distinct [33]. As we will connect points in the integer lattice forming parallelograms, we use the term *k-cycle* from graph theory [33], where the $k$-cycle is a simple graph of length $k$, i.e., consisting of $k$ vertices and $k$ edges and represented by a sequence of consecutive vertices $\breve{x}_0 \dots \breve{x}_{k-1}\breve{x}_0$. Equations between physical quantities are represented by paths in $\mathbb{Z}^7$. Dimensional products are represented by cycles in $\mathbb{Z}^7$. A *walk of length* $k$ in $\mathbb{Z}^7$ is a non-empty alternating sequence $\breve{v}_0 e_0 \breve{v}_1 e_1 \dots e_{k-1}\breve{v}_k$ of vertices $\breve{v}_i$ and edges $e_i$ in $\mathbb{Z}^7$ such that $e_i = \{\breve{v}_i, \breve{v}_{i+1}\}$ for all $i < k$.

#### *6.1. Gödel walk in a 7-dimensional integer lattice*

We encode each integer lattice point of $\mathbb{Z}^7_+$ by using a similar scheme to the Gödel encoding [57] applied to 7 non-negative integer variables. We define the *Gödel number* in $\mathbb{Z}^d_+$, where $d$ is the dimension of the integer lattice:

$$\phi_d(X^1 \dots X^d) = \prod_{i=1}^{d} p_i^{X^i} , \tag{10}$$

where $p_i$ is the $i$-th prime number, $\breve{x} = (X^1, \dots, X^d)$ and $X^i \in \mathbb{Z}_+$.

**Example 6.1.** $\phi_7(1110000) = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^0 \cdot 13^0 \cdot 17^0 = 30$

This encoding which we denote as $\phi_7$ is injective between $\mathbb{Z}^7_+$ and $\mathbb{Z}_+$. The range of $\phi_7$ is a subset $\Phi_7$ of the non-negative integers $\mathbb{Z}_+$ because all the primes which are different from the first 7 primes are not images of lattice points of $\mathbb{Z}^7_+$, as well as all the composite numbers having divisors larger than 17. Observe that each of the base physical quantities of the set $\mathcal{B}$ are assigned to a prime number. The base physical quantities play the same role as the prime numbers, being the *atoms* in number theory [58]. If we walk through the integer sublattice $\mathbb{Z}^7_+$ respecting the ordering created by the Gödel encoding, then we generate a series of segments in $\mathbb{Z}^7_+$. We call this walk a *Gödel walk* through the integer sublattice $\mathbb{Z}^7_+$. The segments are known in number theory as the prime gaps $g(p) = n$ of gap length $n$. All the leader class representatives are located on the Gödel walk. When the Gödel walk is performed in $\mathbb{Z}^{25}_+$ then all the first 100 non-negative integers will be visited (Fig. 6). When restricting the dimension to $d = 7$ we find 67 non-negative integers that will be visited. An enumeration (Table I.15) of the first 67 lattice points shows also the crossings of the Gödel walk with the measure polytopes $P^s_7$. The *successive* lattice points of the Gödel walk are *orthogonal* when calculated for the first 100 lattice points in the integer lattice $\mathbb{Z}^{25}_+$. Observe that the Gödel walk represents a unique walk in $\mathbb{Z}^k_+$, where $k \in \mathbb{Z}_+$ because it requires orthogonality between successive lattice points and because it *minimizes* the function $\phi_k$ at each lattice point. There are 23 segments in $\mathbb{Z}^7_+$ and 28 segments in $\mathbb{Z}^3_+$ for the first 100 non-negative integers. The orthogonality between successive lattice points remains valid within the segments that have more than 1 lattice point. This walk encodes all the physical quantities of $\mathbb{Z}^7_+$ up to a signed permutation. Observe that the leader class representative has always the *smallest* Gödel number of the class. Physicists represent *correlations* between physical quantities graphically in the form of *cubes* that contain the respective physical quantities as the axes of the cube. The Gödel walk presents a *natural* way of selecting mutually orthogonal sequential physical quantities. Inspection of the list (Table I.15) results in cubes $C(i, j, k)$, where $i, j, k$ are successive Gödel numbers. We find 7 cubes $C(3, 4, 5) = \{M, L^2, T\}$, $C(5, 6, 7) = \{T, ML, I\}$, $C(7, 8, 9) = \{I, L^3, M^2\}$, $C(9, 10, 11) = \{M^2, LT, \Theta\}$, $C(11, 12, 13) =$

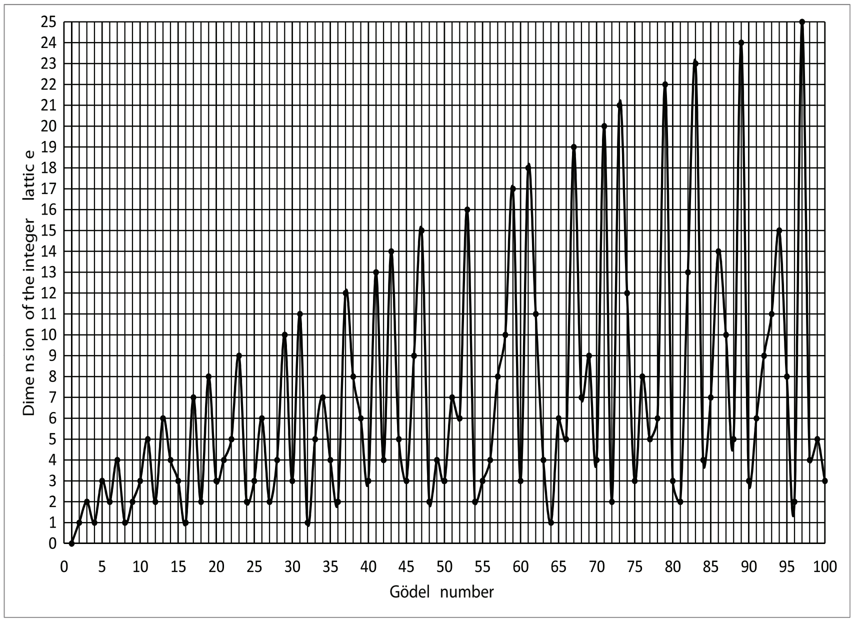


Figure 6: Gödel walk in $\mathbb{Z}_+^{25}$.

$\{\Theta, ML^2, N\}$, $C(13,14,15) = \{N, LI, MT\}$, $C(15,16,17) = \{MT, L^4, J\}$ where we use the agreed [28] symbol for the dimensions.

**Example 6.2.** The quantity $ML$ in the cube $C(5,6,7) = \{T, ML, I\}$ could be expressed as function of $\dfrac{\hbar}{c}$ and the product $T \times I$ is nothing else than the electric charge. The Compton effect for an electron can be represented by a volume $\dfrac{e\hbar}{c}$ in the cube $C(5,6,7)$.

The mutual orthogonality in the 7 cubes is *invariant* when the integer lattice points representing the cube axes are subject to a signed permutation. We transform the set $\{M, L^2, T\}$ in $\{M, L^2, T^{-1}\}$ and observe that the volume of the new cube represents the *angular momentum.* The set $\{M^2, LT, \Theta\}$ can be transformed to $\{M^2, LT^{-1}, \Theta\}$ representing a cube with on the x-axis the *mass squared*, on the y-axis the *speed* and on the z-axis the *thermodynamic temperature.*

### *6.2. Additive partitioning of leader classes*

The encoding of the leader classes with a Gödel number allows the factorization of the Gödel number in distinct factors. Richard J. Mathar (http://home.strw.leidenuniv.nl/ mathar/) has listed in the OEIS [40] the integer series A045778 that gives the factorization of non-negative integers up to n=1500. In the present article we focussed on the most elementary constellation of lattice points that form a parallelogram. The leader class is the representative for all the physical quantities which are vertices of a partition of a measure polytope $P_7^s$. A signed permutation can be found that maps the factored equations to equivalent equations of the desired physical quantity that is an element of the leader class. We show the method for the physical quantity *energy.*

**Example 6.3.** The leader class for *energy* is $[2^2 1 0^4]$. It has Gödel number $\phi_7(2210000) = 180$. From the OEIS[40] A045778 series we find as factorizations:

$$180 = 2 \times 3 \times 5 \times 6$$

The 4-factoring results in 1 equation that represents a 5-ary equation. By applying the Gödel decoding on the 4-factoring of $\phi_7(2210000) = 180$, we find the additive partitioning of the leader class representative $(2, 2, 1, 0, 0, 0, 0)$ in a 5-ary equation:

(i) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (0, 0, 1, 0, 0, 0, 0) + (1, 1, 0, 0, 0, 0, 0)$;

$180 = 2\times3\times30 = 2\times5\times18 = 2\times6\times15 = 2\times9\times10 = 3\times4\times15 = 3\times5\times12 = 3\times6\times10 = 4\times5\times9$

The 3-factoring results in 8 equations that represent each a 4-ary equation. The additive partitioning of the leader class representative (2,2,1,0,0,0,0) in 4-ary equations are:

(i) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (1, 1, 1, 0, 0, 0, 0)$

(ii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (0, 0, 1, 0, 0, 0, 0) + (1, 2, 0, 0, 0, 0, 0)$

(iii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (1, 1, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 0, 0, 0)$

(iv) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (0, 2, 0, 0, 0, 0, 0) + (1, 0, 1, 0, 0, 0, 0)$

(v) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (2, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 0, 0, 0)$

(vi) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (0, 0, 1, 0, 0, 0, 0) + (2, 1, 0, 0, 0, 0, 0)$

(vii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (1, 1, 0, 0, 0, 0, 0) + (1, 0, 1, 0, 0, 0, 0)$

(viii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (2, 0, 0, 0, 0, 0, 0) + (0, 0, 1, 0, 0, 0, 0) + (0, 2, 0, 0, 0, 0, 0)$

$$180 = 2 \times 90 = 3 \times 60 = 4 \times 45 = 5 \times 36 = 6 \times 30 = 9 \times 20 = 10 \times 18 = 12 \times 15$$

The 2-factoring results also in 8 equations that represent each a ternary equation. The additive partitioning of the leader class representative $(2, 2, 1, 0, 0, 0, 0)$ in 3-ary equations are:

(i) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 0, 0, 0, 0, 0) + (1, 2, 1, 0, 0, 0, 0)$

(ii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0) + (2, 1, 1, 0, 0, 0, 0)$

(iii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (2, 0, 0, 0, 0, 0, 0) + (0, 2, 1, 0, 0, 0, 0)$

(iv) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 0, 1, 0, 0, 0, 0) + (2, 2, 0, 0, 0, 0, 0)$

(v) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 1, 0, 0, 0, 0, 0) + (1, 1, 1, 0, 0, 0, 0)$

(vi) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (0, 2, 0, 0, 0, 0, 0) + (2, 0, 1, 0, 0, 0, 0)$

(vii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (1, 0, 1, 0, 0, 0, 0) + (1, 2, 0, 0, 0, 0, 0)$

(viii) $(2, 2, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) + (2, 1, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 0, 0, 0)$

We conclude that the leader class representative $(2,2,1,0,0,0,0)$ can be partitioned in 17 distinct terms. As this leader class is representative for the physical quantity energy we conclude to the existence of *17 distinct forms* of equations representing the physical quantity energy. Generalisation of this methodology will reveal the generic constellations for the leader class representatives.
The signed permutation matrix $\mathsf{P}_{\mathsf{energy}}$ transforms the leader class representative $(2,2,1,0,0,0,0)$ in the lattice point $(2,1,-2,0,0,0,0)$ and is given below:

$$\mathsf{P}_{\mathsf{energy}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

We apply the matrix $\mathsf{P}_{\mathsf{energy}}$ on the seventeen equations that represent the additive partitions of the leader class representative (2,2,1,0,0,0,0) and find the *energy* equations given in Table 7 . The columns marked $\breve{i}$, $\breve{j}$, $\breve{k}$, $\breve{l}$ and $\breve{m}$ contain the 17 lattice points in $\mathbb{Z}^7$ forming the *energy* constellation.

Table 7: Complete set of generic equations for the quantity *energy*.

| $\breve{i}$ | $\breve{j}$ | $\breve{k}$ | $\breve{l}$ | $\breve{m}$ | form |
|---|---|---|---|---|---|
| $(0^7)$ | $(1,0^6)$ | $(0,0,-1,0^4)$ | $(0,1,0^5)$ | $(1,0,-1,0^4)$ | $E_1 = \kappa_1 x\nu mv$ |
| $(0^7)$ | $(1,0^6)$ | $(0,0,-1,0^4)$ | $(1,1,-1,0^4)$ | $(0^7)$ | $E_2 = \kappa_2 x\nu p$ |
| $(0^7)$ | $(1,0^6)$ | $(0,1,0^5)$ | $(1,0,-2,0^4)$ | $(0^7)$ | $E_3 = \kappa_3 xma$ |
| $(0^7)$ | $(1,0^6)$ | $(1,0,-1,0^4)$ | $(0,1,-1,0^4)$ | $(0^7)$ | $E_4 = \kappa_4 xv\dfrac{\partial m}{\partial t}$ |
| $(0^7)$ | $(1,0^6)$ | $(0,0,-2,0^4)$ | $(1,1,0^5)$ | $(0^7)$ | $E_5 = \kappa_5 x\nu^2 \int mdx$ |
| $(0^7)$ | $(0,0,-1,0^4)$ | $(2,0^6)$ | $(0,1,-1,0^4)$ | $(0^7)$ | $E_6 = \kappa_6 \nu x^2 \dfrac{\partial m}{\partial t}$ |
| $(0^7)$ | $(0,0,-1,0^4)$ | $(0,1,0^5)$ | $(2,0,-1,0^4)$ | $(0^7)$ | $E_7 = \kappa_7 \nu m\dfrac{\partial A}{\partial t}$ |
| $(0^7)$ | $(0,0,-1,0^4)$ | $(1,0,-1,0^4)$ | $(1,1,0^5)$ | $(0^7)$ | $E_8 = \kappa_8 \nu v \int mdx$ |
| $(0^7)$ | $(2,0^6)$ | $(0,1,0^5)$ | $(0,0,-2,0^4)$ | $(0^7)$ | $E_9 = \kappa_9 x^2 m\nu^2$ |
| $(0^7)$ | $(1,0^6)$ | $(1,1,-2,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{10} = \kappa_{10} xF$ |
| $(0^7)$ | $(0,0,-1,0^4)$ | $(2,1,-1,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{11} = \kappa_{11} \nu J$ |
| $(0^7)$ | $(2,0^6)$ | $(0,1,-2,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{12} = \kappa_{12} x^2 \dfrac{\partial^2 m}{\partial t^2}$ |
| ... | ... | ... | ... | ... | ... |

| $\breve{i}$ | $\breve{j}$ | $\breve{k}$ | $\breve{l}$ | $\breve{m}$ | form |
|---|---|---|---|---|---|
| $(0^7)$ | $(0,1,0^5)$ | $(2,0,-2,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{13}=\kappa_{13}mv^2$ |
| $(0^7)$ | $(1,0,-1,0^4)$ | $(1,1,-1,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{14}=\kappa_{14}vp$ |
| $(0^7)$ | $(0,0,-2,0^4)$ | $(2,1,0^5)$ | $(0^7)$ | $(0^7)$ | $E_{15}=\kappa_{15}\nu^2\int\int mdA$ |
| $(0^7)$ | $(1,1,0^5)$ | $(1,0,-2,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{16}=\kappa_{16}a\int mdx$ |
| $(0^7)$ | $(2,0,-1,0^4)$ | $(0,1,-1,0^4)$ | $(0^7)$ | $(0^7)$ | $E_{17}=\kappa_{17}\dfrac{\partial A}{\partial t}\dfrac{\partial m}{\partial t}$ |

The symbols used in the column *form* have the following interpretation: $E_i$: energy; $x$: position, distance; $t$: time; $\nu$: frequency; $m$: mass; $A$: area; $v$: speed; $F$: force; $J$: angular momentum; $p$: linear momentum; $a$: acceleration; $\kappa_i$: dimensionless variable. The same methodology, as shown for the physical quantity energy, can be applied to any physical quantity. This will then generate for that physical quantity its complete set of generic equations. Table K.18 enumerates for leader classes with Gödel number $\leq 1500$ the factorization of the Gödel number in $i$ distinct factors. The number of distinct factors is found in the repective columns $Fi$ where $i \in [2,\ldots,5]$ . We conclude that there is a finite number of distinct equations for each physical quantity.

*6.3. Bicolouring of a 4-cycle representing an equation between physical quantities*

The hypothesis of the existence of *rules* that have to be respected by the *laws of physics*, has been proposed by Wigner and Feynman, see Lange[10]. We elaborate on this problem by proving *one* of these rules applicable for ternary equations $[z]=[\kappa][x][y]$ between the distinct physical quantities $[\kappa],[x],[y],[z]$ . The rule constraints the bicolouring of 4-cycles [59, 60, 61] in $\mathbb{Z}^7$ .

**Theorem 4.** *Any ternary equation $[z]=[\kappa][x][y]$ between distinct physical quantities $[\kappa],[x],[y],[z]$ represents a distinct colouring pattern* $(\mathrm{psc}(\breve{o}),\mathrm{psc}(\breve{x}),\mathrm{psc}(\breve{y}),\mathrm{psc}(\breve{z}))$ *that is an element of the set of colouring patterns* $\{(0,0,0,0),(0,0,1,1),(0,1,0,1),(0,1,1,0)\}$ .

**Proof.** We will use the method *proof by exhaustion* for this theorem. Let the four distinct integer lattice points $\breve{o},\breve{y},\breve{z},\breve{y}$ be the vertices of a *parallelogram*, represented by the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$. The parallelogram is the representation of the ternary equation $[z]=[\kappa][x][y]$ in the integer lattice $\mathbb{Z}^7$ . Let the colouring pattern be defined by the 4-tuple $(\mathrm{psc}(\breve{o}),\mathrm{psc}(\breve{x}),\mathrm{psc}(\breve{y}),\mathrm{psc}(\breve{z}))$ in which $\breve{o}$ is the origin of $\mathbb{Z}^7$ . By convention $\mathrm{psc}(\breve{o})$ is placed as the first element and $\mathrm{psc}(\breve{z})$ as the last element in the colouring patterns. By the definition of the "psc" function we obtain $\mathrm{psc}(\breve{o})=0$ . A 4-tuple having only two characters {0,1} has in total $2^4=16$ combinations of 4-tuples. So, we will review the 16 cases. The condition that the first element of the 4-tuple has to be 0 reduces the number of combinations to $2^3=8$ being the set of colouring patterns {(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1),(0,1,1,0), (0,1,1,1)}. The four distinct integer lattice points $\breve{o},\breve{x},\breve{y},\breve{z}$ of the parallelogram have the property $\breve{x}+\breve{y}=\breve{z}$ , see Fig. 1. From elementary number theory [62], it is known that:

(i) even $\pm$ even = even

(ii) odd $\pm$ odd = even

(iii) even $\pm$ odd = odd

The function "psc" is binary-valued on $\mathbb{Z}^7$ satisfying $\operatorname{psc}(\breve{x}+\breve{y}) = \operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y})$ for all $\breve{x}, \breve{y} \in \mathbb{Z}^7$. Thus the 4-tuples have the form $(0, \operatorname{psc}(\breve{x}), \operatorname{psc}(\breve{y}), \operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}))$ resulting in the following cases:

*Case* 1. (0,0,0,0)
If $\operatorname{psc}(\breve{x}) = \operatorname{psc}(\breve{y}) = 0$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 0$. The colouring pattern **(0,0,0,0)** satisfies the above property and is a *valid* colouring pattern. This colouring pattern is called monochromatic.

*Case* 2. (0,0,0,1)
If $\operatorname{psc}(\breve{x}) = \operatorname{psc}(\breve{y}) = 0$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 0$. The colouring pattern (0,0,0,1) violates the above property and is a *forbidden* colouring pattern.

*Case* 3. (0,0,1,0)
If $\operatorname{psc}(\breve{x}) = 0$ and $\operatorname{psc}(\breve{y}) = 1$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 1$. The colouring pattern (0,0,1,0) violates the above property and is a *forbidden* colouring pattern.

*Case* 4. (0,0,1,1)
If $\operatorname{psc}(\breve{x}) = 0$ and $\operatorname{psc}(\breve{y}) = 1$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 1$. The colouring pattern **(0,0,1,1)** satisfies the above property and is a *valid* colouring pattern. This colouring pattern is called two-coloured of pattern 2+2.

*Case* 5. (0,1,0,0)
If $\operatorname{psc}(\breve{x}) = 1$ and $\operatorname{psc}(\breve{y}) = 0$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 1$. The colouring pattern (0,1,0,0) violates the above property and is a *forbidden* colouring pattern.

*Case* 6. (0,1,0,1)
If $\operatorname{psc}(\breve{x}) = 1$ and $\operatorname{psc}(\breve{y}) = 0$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 1$. The colouring pattern **(0,1,0,1)** satisfies the above property and is a *valid* colouring pattern. This colouring pattern is called mixed two-coloured.

*Case* 7. (0,1,1,0)
If $\operatorname{psc}(\breve{x}) = 1$ and $\operatorname{psc}(\breve{y}) = 1$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 0$. The colouring pattern **(0,1,1,0)** satisfies the above property and is a *valid* colouring pattern. This colouring pattern is called two-coloured of pattern 1+2+1.

*Case* 8. (0,1,1,1)
If $\operatorname{psc}(\breve{x}) = 1$ and $\operatorname{psc}(\breve{y}) = 1$ then by number theory $\operatorname{psc}(\breve{x}) + \operatorname{psc}(\breve{y}) = 0$. The colouring pattern (0,1,1,1) violates the above property and is a *forbidden* colouring pattern.

We obtain as *valid* colouring patterns: $(0,0,0,0)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,0)$. $\square$

**Corollary 1.** *If* $\operatorname{psc}(\breve{z}) = 0$ *then* $\operatorname{psc}(\breve{x}) = \operatorname{psc}(\breve{y})$. *If* $\operatorname{psc}(\breve{z}) = 1$ *then* $\operatorname{psc}(\breve{x})$ *is the opposite of* $\operatorname{psc}(\breve{y})$.

## 7. Linear independence and orthogonality between classes of physical quantities

The representation of a class of physical quantities in $\mathbb{Z}^7$ gives the means to study the linear independence and the orthogonality between classes of physical quantities. Consider $\breve{z} = \breve{o} + \breve{x} + \breve{y}$ and form the inner product $\breve{z} \cdot \breve{y} = \breve{o} \cdot \breve{y} + \breve{x} \cdot \breve{y} + \breve{y} \cdot \breve{y}$. The classes $[x]$ and $[y]$ are orthogonal when $\breve{x} \cdot \breve{y} = 0$. We find $\breve{z} \cdot \breve{y} = \breve{y} \cdot \breve{y}$ which shows a linear relationship between $\|\breve{z}\|_2$ and $\|\breve{y}\|_2$. So, an equation $[z] = [\kappa][x][y]$ in which the classes $[x]$ and $[y]$ are orthogonal expresses a linear relationship between $[z]$ and $[x]$ or between $[z]$ and $[y]$ . We underline the difference between *linearly independent physical quantities* and *orthogonal physical quantities* [63]. From these properties we define 6 types of pairwise combinations of $[x]$ and $[y]$ . We give examples of each of the types. Consider the representation of distance by the lattice point $\breve{r} = (1, 0, 0, 0, 0, 0, 0)$ and the representation of the linear momentum by the lattice point $\breve{p} = (1, 1, -1, 0, 0, 0, 0)$ . We consider the $2 \times 7$ matrix formed by the coordinates of $\breve{r}$ and $\breve{p}$ and obtain the rank = 2 for this matrix which means that $\breve{r}$ and $\breve{p}$ are *linearly independent*. For the inner product we find $\breve{r} \cdot \breve{p} = 1 \neq 0$ and so $\breve{r}$ and $\breve{p}$ are *not orthogonal*. Consider the product of length and time with representation $\breve{lt} = (1, 0, 1, 0, 0, 0, 0)$ and energy represented by the lattice point $\breve{E} = (2, 1, -2, 0, 0, 0, 0)$, we find that $\breve{lt}$ and $\breve{E}$ are *linearly independent* and *orthogonal*. Consider the distance representation $\breve{r} = (1, 0, 0, 0, 0, 0, 0)$ and the wave vector representation $\breve{k} = (-1, 0, 0, 0, 0, 0, 0)$, we find that $\breve{r}$ and $\breve{k}$ are *linearly dependent* and *not orthogonal*. Consider the velocity representation $\breve{v} = (1, 0, -1, 0, 0, 0, 0)$ and the reciprocal velocity representation $\breve{v_r} = (-1, 0, 1, 0, 0, 0, 0)$, we find that $\breve{v}$ and $\breve{v_r}$ are *linearly dependent* and *orthogonal*. We conclude that ternary equations $[z] = [\kappa][x][y]$ of physical quantities are only one of the six following cases:

(i) $\breve{x} \cdot \breve{y} > 0$ and $2 \times 7$ matrix rank = 2 (not orthogonal with positive inner product, linearly independent)

(ii) $\breve{x} \cdot \breve{y} = 0$ and $2 \times 7$ matrix rank = 2 (orthogonal, linearly independent)

(iii) $\breve{x} \cdot \breve{y} < 0$ and $2 \times 7$ matrix rank = 2 (not orthogonal with negative inner product, linearly independent)

(iv) $\breve{x} \cdot \breve{y} > 0$ and $2 \times 7$ matrix rank $< 2$ (not orthogonal with positive inner product, linearly dependent)

(v) $\breve{x} \cdot \breve{y} = 0$ and $2 \times 7$ matrix rank $< 2$ (orthogonal, linearly dependent)

(vi) $\breve{x} \cdot \breve{y} < 0$ and $2 \times 7$ matrix rank $< 2$ (not orthogonal with negative inner product, linearly dependent)

*7.1. Decompositions of a vertex in pairwise orthogonal vertices*

The decomposition of a vertex $\breve{z}$ in two pairwise orthogonal vertices $\breve{x}$ and $\breve{y}$ assumes the existence of a system of Diophantine equations:

$$\breve{x} + \breve{y} - \breve{z} = 0 \ , \tag{13a}$$

$$\breve{x} \cdot \breve{y} = 0 \ , \tag{13b}$$

where $\breve{x}, \breve{y}, \breve{z} \in \mathbb{Z}^7$ . We eliminate $\breve{y}$ from the equation (13b) and find:

$$\breve{x} \cdot \breve{x} - \breve{x} \cdot \breve{z} = 0 \ . \tag{14}$$

We apply the method of "completing the square" and write equation (14) as:

$$(\breve{x} - \frac{\breve{z}}{2})^2 = (\frac{\breve{z}}{2})^2 \ , \tag{15}$$

that represents a seven-dimensional hypersphere with center at $\frac{\breve{z}}{2}$ with radius $\|\frac{\breve{z}}{2}\|_2$ . We note that the hyper-surface area of a *unit* radius hypersphere reaches a *maximum* in a 7-dimensional space

[64]. The center of the 7-sphere is only a lattice point of $\mathbb{Z}^7$ if all the coordinates of $\breve{z}$ are even. The solution set of the equation (15) are the integer lattice points incident on the 7-sphere and thus is a *finite* set. It is obvious that a bijection exists between the physical quantity having the vertex $\breve{z}$ and the 7-sphere with equation (15). A closed form for the solution set is not known to the author. We use the *brute force* method and list the vertices of 524287 parallelograms $\breve{o}\breve{x}\breve{z}\breve{y}$ embedded in $\mathbb{Z}^7$ representing equations $[z] = [\kappa][x][y]$ . From this listing, we find parallelograms that have the property of being a rectangle. Let $n_d = \#(O_d)$ represent the cardinality of the set of pairwise orthogonal vertices in $\mathbb{Z}^d \times \{0\}^{7-d}$ with dimension $d \in \mathbb{N}$ where $2 \leq d \leq 7$ . Table J.16 contains the cardinalities of the commonly known leader classes.

**Example 7.1.** We solve the equation (15) for the leader class $[2^2 10^4]$, that represents the class *energy*. Table K.17 enumerates the 60 pairs of orthogonal vertices of $\mathbb{Z}^7$ resulting in the vertex $\breve{E} = \breve{z} = (2, 1, -2, 0, 0, 0, 0)$ . The orthogonality analysis of the 524287 parallelograms spans a range of perimeters from $p_p = 6$ to $p_p = 23,832$ . Table 8 lists the 4 pairs having in column 1 the respective indices 1, 26, 35 and 36 that are embedded in $\mathbb{Z}^3 \times \{0\}^4$ . We find that the rectangles in the 7-sphere have the perimeter values $7,657$ $8,363$ and $8,472$ . The perimeter distribution indicates that the frequency of the *rectangle* perimeters is respectively 1, 17 and 26. Column 5 of Table 8 gives the $2 \times 7$ matrix rank. We observe that the 4 orthogonal pairs have rank 2 and thus are linearly independent. We find that the pair with index $Id = 1$ is the only rectangle having also frequency 1. This rectangle emphasizes the uniqueness of the form $E = \beta_1 m v^2$ that is best known as the equation $E = \gamma m_0 c^2$ .

Table 8: Equations of orthogonal lattice points for *energy* in $\mathbb{Z}^3 \times \{0\}^4$ .

| $Id$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $2 \times 7$ matrix rank | Form | Proposal |
|---|---|---|---|---|---|---|
| 1 | 7,657 | (0,1,0,0,0,0,0) | (2,0,-2,0,0,0,0) | 2 | $E = \beta_1 m v^2$ | $E = \gamma m_0 c^2$ |
| 26 | 8,363 | (1,-1,-1,0,0,0,0) | (1,2,-1,0,0,0,0) | 2 | $E = \beta_2 \dfrac{v}{m} \dfrac{p^2}{v}$ | $E = \gamma_2 \dfrac{p^2}{2m}$ |
| 35 | 8,472 | (2,1,0,0,0,0,0) | (0,0,-2,0,0,0,0) | 2 | $E = \beta_3 m A \nu^2$ | $E = \beta_3 m x^2 \omega^2$ |
| 36 | 8,472 | (0,1,-2,0,0,0,0) | (2,0,0,0,0,0,0) | 2 | $E = \beta_4 A \dfrac{m}{t^2}$ | $E = \beta_4 A \dfrac{\partial^2 m}{\partial t^2}$ |

The 4 rectangles representing ternary *energy* equations in $\mathbb{Z}^3 \times \{0\}^4$ are shown in (Fig. 7).

## 8. Compatible physical quantities

Two physical quantities $[x]$ and $[y]$ are called by Schwinger *compatible* [24] when the measurement of $[x]$ does not destroy the knowledge gained by the prior measurement of $[y]$ . The property of compatibility of physical quantities is related to the orthogonality of $[x]$ and $[y]$ . Pairwise orthogonal physical quantities are incident on unique 7-spheres (15) that are forming finite sets. So, each leader class representative has a unique *leader* hypersphere associated to it. A complete set of compatible physical quantities $\{[x_1], [x_2], \ldots, [x_k]\}$ is a set for which every pair of these physical quantities is *compatible* and that no other physical quantities exist apart from functions of the set that are compatible with every member of this set [24]. We know already one of these sets which is $\mathcal{B} \doteq \{[l], [m], [t], [i], [T], [n], [L]\}$ . Each of the complete sets will have to comply with

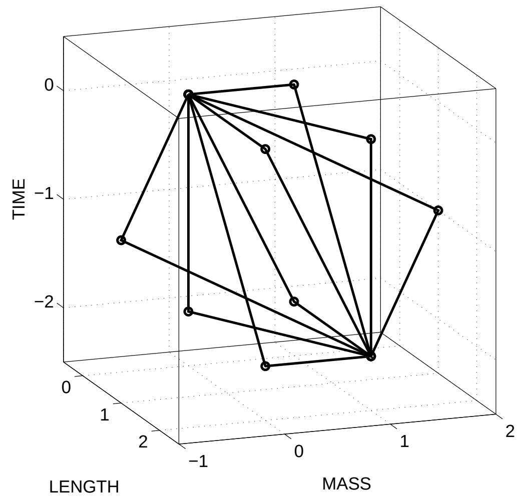


Figure 7: Rectangles embedded in $\mathbb{Z}^3 \times \{0\}^4$ representing ternary equations of *energy*

$\sum_{i=1}^{7} i = 28$ orthogonality conditions expressed between the physical quantities of the complete set. Observe that the orthogonality property between physical quantities in $\mathbb{Z}^7$ is related to the property of commuting operators in quantum physics. Two incompatible physical quantities will generate Heisenberg type inequalities [65]. The Heisenberg relation $\Delta x \Delta p \geq \hbar$ expresses the degree of correlation between *position* and *linear momentum* of a particle. When the mean in the correlation formula becomes zero then the correlation formula is reduced to the orthogonality formula [63]. The lattice points $\breve{x}$ and $\breve{p}$, that represent respectively *position* and *linear momentum* are non-orthogonal, while the sum of the vertices results in the vertex with coordinates $(2, 1, -1, 0, 0, 0, 0)$ that represents the angular momentum $\breve{J}$ in $\mathbb{Z}^7$. Observe in Table J.16 that the leader class $[21^2 0^4]$ has *only* 19 pairwise orthogonal physical quantities in $\mathbb{Z}^7$. We realize that the Table J.16 gives the cardinality of *compatible physical quantities*. We calculate the inner product of the physical quantities representing the reciprocal speed of light $\frac{1}{c}$, the Newton constant of gravitation $G$ and the Planck constant $h$ and find $\breve{\frac{1}{c}} \cdot \breve{G} = -5$, $\breve{\frac{1}{c}} \cdot \breve{h} = -3$ and $\breve{G} \cdot \breve{h} = 7$, that are all non-zero. It indicates that the fundamental constants {G,h,1/c} [66] are *incompatible* physical quantities and thus should *not* be considered as the variables of the "cube of physical theories" [66, 20, 67], because there is no *freedom* in choosing a point in this cube. We enumerated 7 *natural* cubes in the subsection 6.1 that are constructed from *compatible* physical quantities. However, we note that the physical quantities of the set {G,h,1/c} are *linearly independent*. Planck [68] proposed the physical quantities: Planck length $l_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}}$, the Planck mass $m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}}$ and the Planck time $t_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^5}}$, that form a set of *compatible* physical quantities. We conclude that pairwise orthogonal physical quantities are compatible and could generate operators that are commuting while pairwise non-orthogonal physical quantities are incompatible and could generate operators that are not commuting. The majority of pairs of physical quantities, that can generate operators, will result in *non-commuting* operators.

### *8.1. Physical quantities compatible with energy*

The values for the physical quantity energy range theoretically for an elementary particle from almost 0 to the Planck energy $E_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G}}$. The largest particle accelerators are exploring a very tiny part of this range. It is therefore difficult to make any verifiable statement about a physical quantity that is in some way depending on the physical quantity energy, when taken in account this vast range of energy values. The majority of the physical quantities are non-compatible with energy with the exception of those that fulfill the equation $\breve{x} \cdot \breve{E} = 0$. The solutions $\breve{x} \in \mathbb{Z}^7$ are lattice points embedded in the hyperplane $H_{\breve{x}} = \{(X^1, \ldots, X^7) \mid 2X^1 + X^2 - 2X^3 = 0\}$. These solutions represent physical quantities that combine to ternary relations valid up to the Planck energy.

**Example 8.1.** Consider the lattice points $\breve{x} = (-2, 2, -1, 0, 0, 0, 0)$ and $\breve{y} = (-3, 2, -2, 0, 0, 0, 0)$ that are orthogonal to the lattice point representing *energy*. We form the ternary relation and obtain $\breve{z} = (-5, 4, -3, 0, 0, 0, 0)$. We interpret the physical quantity $z$ as $z = \beta \frac{M^4}{r^5} \omega^3$ where $M$ is the mass of the system, $r$ is a characteristic length of the system and $\omega$ is the angular frequency of the system. This ternary relation $[z] = [\beta][x][y]$ is valid up to the Planck energy because the lattice points $\breve{x}, \breve{y}, \breve{z}$ are orthogonal to the lattice point $\breve{E}$.

## 9. Future work and conclusion

We construct the mathematical foundation for *the discrete geometry of physical quantities*. We prove that ternary operations between components of physical quantities are equivalent to a *parallelogram* in the integer lattice $\mathbb{Z}^7$. This equivalence is the basis for a *computer search for relations between physical quantities* based on geometric properties between the integer lattice points of $\mathbb{Z}^7$, which are the representatives of components of physical quantities. We develop an algorithm that creates a listing of the equations of the type $[z] = [\kappa][x][y]$ where $[\kappa], [x], [y], [z]$ represents components of physical quantities. We find that ternary relations between physical quantities are classified in 4 distinct 2-colouring patterns of $\mathbb{Z}^7$. Application of the algorithm for the case where $[z]$ is representing the class *energy*, results in a discrete value distribution that is characteristic for the leader class $[2^2 10^4]$. The analysis of the discrete value distribution for the physical quantity *energy* indicates the existence of *unique constellations* between physical quantities. We discover that the unique constellations representing *energy* are all embedded in a hyperplane of the integer lattice $\mathbb{Z}^7$. We observed that the frequency of some constellations is *not depending* on the dimension $d$ of the integer lattice. The algorithm that was applied for *energy* and also for *force* is applicable to any other component of a physical quantity resulting in the discovery of new constellations between physical quantities. The compilation of the listings generated by the algorithm, will result in a catalog of equations of the type $[z] = [\kappa][x][y]$. The equivalence relation $z_1$ *has the same isoperimetric distribution as* $z_2$ applied on a finite set, representing a measure polytope of $\mathbb{Z}^7$, results in the classification of physical quantities. We show that morphisms exists between these equivalence classes and monomials. Assignment of a *Gödel number* to each physical quantity in $\mathbb{Z}^7_+$ reveals the existence of a unique *Gödel walk* in $\mathbb{Z}^k_+$. A scheme is described for analyzing $n$-ary operations based on the factorization of the *Gödel number* of leader class representatives in distinct integer factors, that will allow the exploration of more complex constellations than parallelograms. The $n$-ary operations between physical quantities are representing *paths* connecting the lattice points of the constellation representing the physical quantity under study. Orthogonality and linear independence properties of the pairs of vertices $\breve{x}$ and $\breve{y}$ result in classifying the ternary equations $[z] = [\kappa][x][y]$ in 6 distinct types. We find that each vertex $\breve{z}$ can be decomposed in a finite number of pairwise *orthogonal* vertices incident on a *unique* 7-sphere. The discrete geometry of physical quantities provides inherently a predictive property for finding the form of equations between physical quantities that are yet to be discovered. This research shows that our knowledge about the components of physical quantities and about their constellations is far from being understood and that large *hypervolumes* of $\mathbb{Z}^7$, are still to be explored. The appendices contain a preliminary classification of common physical quantities based on the measure polytopes $P_7^s$. The appendices also contain numerical data useful as starting point for the further exploration of the discrete geometry of physical quantities.

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## Appendix A. 3-cycle isoperimetric distribution algorithm

**Algorithm.** *Calculate for each integer lattice point $\breve{x}$ of a 7-dimensional lattice the following:*

*(i)* $d(\breve{o}, \breve{z})$*, the Euclidean distance from $\breve{o}$ to the lattice point $\breve{z}$, representing a component of a physical quantity with coordinates $(Z^1, \ldots, Z^7)$ ,*

*(ii)* $d(\breve{x}, \breve{o})$ *, the Euclidean distance from $\breve{x}$ to the origin $\breve{o}$ ,*

*(iii) the cosine of the angle between $\breve{x}$ and $\breve{z}$ ,*

*(iv)* $2a = d(\breve{z}, \breve{x}) + d(\breve{x}, \breve{o})$ *, that is a characteristic of an ellipse,*

*(v) the perimeter of the 3-cycle* $p_t = d(\breve{o}, \breve{z}) + d(\breve{z}, \breve{x}) + d(\breve{x}, \breve{o})$ *,*

*(vi) store these results in a data structure allowing sorting by perimeter,*

*(vii) query the data structure to obtain the number of lattice points $\breve{x}$ generating the same triangle perimeter,*

*(viii) find for each triangle perimeter $p_t$ the number of points corresponding to this triangle perimeter and record the discrete value distribution,*

*(ix) select the set of vertices having the same perimeter starting with the shortest 3-cycle perimeter,*

*(x) calculate for each of these vertices the complementary vertices and write them in adjacent rows creating a listing of increasing perimeter.*

## Appendix B. Algorithm for finding all the n-ary operations of a physical quantity

**Algorithm.** *Execute the following steps:*

*(i)* *Identify to which class the physical quantity belongs;*

*(ii)* *apply the function "dex" on the class of the physical quantity and identify the lattice point $\breve{z}$, representing a component of a physical quantity with coordinates $(Z^1, \ldots, Z^7)$ ;*

*(iii)* *associate to the coordinates $(Z^1, \ldots, Z^7)$ its leader class representative;*

*(iv)* *calculate using the function $\phi_7()$ the Gödel number;*

*(v)* *if the Gödel number is $\leq 1500$ then;*

*(vi)* *open lookup table OEIS A045778 and identify the row correponding to the Gödel number and record the correponding factorization;*

*(vii)* *else*

*(viii)* *perform the factorization of the Gödel number in distinct integer factors;*

*(ix)* *calculate using the inverse Gödel encoding the additive partitions of the leader class representative;*

*(x)* *apply the appropriate signed permutation to transform the leader class representative in the physical quantity under investigation;*

*(xi)* *generate a table of forms of equations for the physical quantity under study.*

## Appendix C. Measure polytopes

The enumeration table (Table C.9) of measure polytopes $P_7^4$ consists of 8 columns. The second column is the row identifier. The third column gives the representative of the leader class. The fourth column contains the sum of the absolute value of the coordinates of the lattice points being elements of the leader class that is exclusively the total degree of the monomial associated with the leader class. The fifth column gives the parity of the representative of the leader class. The sixth column gives the $\ell_1$-norm of the representative. The seventh column gives the cardinality of the leader class. The eighth column gives the Gödel number of the representative. Observe that for $\|\breve{x}\|_\infty = 1$ the representative lattice points of the leader classes generate the *successive minima* $R_i$ of the lattice $\mathbb{Z}^7$ [69] . The successive minima $R_i$ are given in the column 6 and correspond to the values of $N(\breve{z})$, the norm of the lattice point and thus the representative lattice points of the leader classes for $s = 1$ form a set of minimal points of the lattice $\mathbb{Z}^7$ [69]. Observe that the leader class $[2^2 10^4]$ contains 840 integer lattice points with the same geometrical properties as the physical quantity *energy*. The $7 \times 7$ signed permutation matrix $\mathsf{P}_{21-2,221}$ transforms all *energy constellations* to the leader class $[2^2 10^4]$:

$$\mathsf{P}_{21-2,221} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{C.1}$$

The representative of the leader class $[2^2 10^4]$ is a physical quantity that could be expressed as an integral of the form $\int \kappa(\lambda m_0)^2 dt$. This is the time integral of the square of the quantity with lattice point (1,1,0,0,0,0,0).

Table C.9: Partitions of the measure polytope $P_7^4$

| $\lVert\breve{x}\rVert_\infty = s$ | Id | leader class | deg | psc $(\breve{z})$ | $N(\breve{z})$ | Number of vertices | Gödel number |
|---|---|---|---|---|---|---|---|
| 0 | 1 | $[0^7]$ | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | $[10^6]$ | 1 | 1 | 1 | 14 | 2 |
| 1 | 2 | $[1^2 0^5]$ | 2 | 0 | 2 | 84 | 6 |
| 1 | 3 | $[1^3 0^4]$ | 3 | 1 | 3 | 280 | 30 |
| 1 | 4 | $[1^4 0^3]$ | 4 | 0 | 4 | 560 | 210 |
| 1 | 5 | $[1^5 0^2]$ | 5 | 1 | 5 | 672 | 2310 |
| 1 | 6 | $[1^6 0]$ | 6 | 0 | 6 | 448 | 30030 |
| 1 | 7 | $[1^7]$ | 7 | 1 | 7 | 128 | 510510 |
| 2 | 1 | $[20^6]$ | 2 | 0 | 4 | 14 | 4 |
| 2 | 2 | $[210^5]$ | 3 | 1 | 5 | 168 | 12 |
| 2 | 3 | $[21^2 0^4]$ | 4 | 0 | 6 | 840 | 60 |
| 2 | 4 | $[2^2 0^5]$ | 4 | 0 | 8 | 84 | 36 |
| 2 | 5 | $[21^3 0^3]$ | 5 | 1 | 7 | 2240 | 420 |
| 2 | 6 | $[2^2 10^4]$ | 5 | 1 | 9 | 840 | 180 |
| 2 | 7 | $[21^4 0^2]$ | 6 | 0 | 8 | 3360 | 4620 |
| 2 | 8 | $[2^2 1^2 0^3]$ | 6 | 0 | 10 | 3360 | 1260 |
| 2 | 9 | $[2^3 0^4]$ | 6 | 0 | 12 | 280 | 900 |
| 2 | 10 | $[21^5 0]$ | 7 | 1 | 9 | 2688 | 60060 |
| 2 | 11 | $[2^2 1^3 0^2]$ | 7 | 1 | 11 | 6720 | 13860 |
| 2 | 12 | $[2^3 10^3]$ | 7 | 1 | 13 | 2240 | 6300 |
| 2 | 13 | $[21^6]$ | 8 | 0 | 10 | 896 | 1021020 |
| 2 | 14 | $[2^2 1^4 0]$ | 8 | 0 | 12 | 6720 | 180180 |
| 2 | 15 | $[2^3 1^2 0^2]$ | 8 | 0 | 14 | 6720 | 69300 |
| 2 | 16 | $[2^4 0^3]$ | 8 | 0 | 16 | 560 | 44100 |
| 2 | 17 | $[2^2 1^5]$ | 9 | 1 | 13 | 2688 | 3063060 |
| 2 | 18 | $[2^3 1^3 0]$ | 9 | 1 | 15 | 8960 | 900900 |
| 2 | 19 | $[2^4 10^2]$ | 9 | 1 | 17 | 3360 | 485100 |
| 2 | 20 | $[2^3 1^4]$ | 10 | 0 | 16 | 4480 | 15315300 |
| 2 | 21 | $[2^4 1^2 0]$ | 10 | 0 | 18 | 6720 | 6306300 |
| 2 | 22 | $[2^5 0^2]$ | 10 | 0 | 20 | 672 | 5336100 |
| 2 | 23 | $[2^4 1^3]$ | 11 | 1 | 19 | 4480 | 107207100 |
| ... | ... | ... | ... | ... | ... | ... | ... |

| $\|\check{x}\|_\infty = s$ | Id | leader class | deg | psc $(\check{z})$ | $N(\check{z})$ | Number of vertices | Gödel number |
|---|---|---|---|---|---|---|---|
| 2 | 24 | $[2^5 10]$ | 11 | 1 | 21 | 2688 | 69369300 |
| 2 | 25 | $[2^5 1^2]$ | 12 | 0 | 22 | 2688 | 1179278100 |
| 2 | 26 | $[2^6 0]$ | 12 | 0 | 24 | 448 | 901800900 |
| 2 | 27 | $[2^6 1]$ | 13 | 1 | 25 | 896 | 15330615300 |
| 2 | 28 | $[2^7]$ | 14 | 0 | 28 | 128 | 260620460100 |
| 3 | 1 | $[30^6]$ | 3 | 1 | 9 | 14 | 8 |
| 3 | 2 | $[310^5]$ | 4 | 0 | 10 | 168 | 24 |
| 3 | 3 | $[31^2 0^4]$ | 5 | 1 | 11 | 840 | 120 |
| 3 | 4 | $[320^5]$ | 5 | 1 | 13 | 168 | 72 |
| 3 | 5 | $[31^3 0^3]$ | 6 | 0 | 12 | 2240 | 840 |
| 3 | 6 | $[3210^4]$ | 6 | 0 | 14 | 1680 | 360 |
| 3 | 7 | $[3^2 0^5]$ | 6 | 0 | 18 | 84 | 216 |
| 3 | 8 | $[31^4 0^2]$ | 7 | 1 | 13 | 3360 | 9240 |
| 3 | 9 | $[321^2 0^3]$ | 7 | 1 | 15 | 6720 | 2520 |
| 3 | 10 | $[32^2 0^4]$ | 7 | 1 | 17 | 840 | 1800 |
| 3 | 11 | $[3^2 10^4]$ | 7 | 1 | 19 | 840 | 1080 |
| 3 | 12 | $[31^5 0]$ | 8 | 0 | 14 | 2688 | 120120 |
| 3 | 13 | $[321^3 0^2]$ | 8 | 0 | 16 | 13440 | 27720 |
| 3 | 14 | $[32^2 10^3]$ | 8 | 0 | 18 | 6720 | 12600 |
| 3 | 15 | $[3^2 1^2 0^3]$ | 8 | 0 | 20 | 3360 | 7560 |
| 3 | 16 | $[3^2 20^4]$ | 8 | 0 | 22 | 840 | 5400 |
| 3 | 17 | $[31^6]$ | 9 | 1 | 15 | 896 | 2042040 |
| 3 | 18 | $[321^4 0]$ | 9 | 1 | 17 | 13440 | 360360 |
| 3 | 19 | $[32^2 1^2 0^2]$ | 9 | 1 | 19 | 20160 | 138600 |
| 3 | 20 | $[32^3 0^3]$ | 9 | 1 | 21 | 2240 | 88200 |
| 3 | 21 | $[3^2 1^3 0^2]$ | 9 | 1 | 21 | 6720 | 83160 |
| 3 | 22 | $[3^2 210^3]$ | 9 | 1 | 23 | 6720 | 37800 |
| 3 | 23 | $[3^3 0^4]$ | 9 | 1 | 27 | 280 | 27000 |
| 3 | 24 | $[321^5]$ | 10 | 0 | 18 | 5376 | 6126120 |
| 3 | 25 | $[32^2 1^3 0]$ | 10 | 0 | 20 | 26880 | 1801800 |
| 3 | 26 | $[32^3 10^2]$ | 10 | 0 | 22 | 13440 | 970200 |
| 3 | 27 | $[3^2 1^4 0]$ | 10 | 0 | 22 | 6720 | 1081080 |
| 3 | 28 | $[3^2 21^2 0^2]$ | 10 | 0 | 24 | 20160 | 415800 |
| 3 | 29 | $[3^2 2^2 0^3]$ | 10 | 0 | 26 | 3360 | 264600 |
| 3 | 30 | $[3^3 10^3]$ | 10 | 0 | 28 | 2240 | 189000 |
| 3 | 31 | $[32^2 1^4]$ | 11 | 1 | 21 | 13440 | 30630600 |
| 3 | 32 | $[32^3 1^2 0]$ | 11 | 1 | 23 | 26880 | 12612600 |
| 3 | 33 | $[32^4 0^2]$ | 11 | 1 | 25 | 3360 | 10672200 |
| 3 | 34 | $[3^2 1^5]$ | 11 | 1 | 23 | 2688 | 18378360 |
| 3 | 35 | $[3^2 21^3 0]$ | 11 | 1 | 25 | 26880 | 5405400 |
| 3 | 36 | $[3^2 2^2 10^2]$ | 11 | 1 | 27 | 20160 | 2910600 |
| 3 | 37 | $[3^3 1^2 0^2]$ | 11 | 1 | 29 | 6720 | 2079000 |
| 3 | 38 | $[3^3 20^3]$ | 11 | 1 | 31 | 2240 | 1323000 |
| 3 | 39 | $[32^3 1^3]$ | 12 | 0 | 24 | 17920 | 214414200 |
| 3 | 40 | $[32^4 10]$ | 12 | 0 | 26 | 13440 | 138738600 |
| 3 | 41 | $[3^2 21^4]$ | 12 | 0 | 26 | 13440 | 91891800 |
| 3 | 42 | $[3^2 2^2 1^2 0]$ | 12 | 0 | 28 | 40320 | 37837800 |
| ... | ... | ... | ... | ... | ... | ... | ... |

| $\|\check{x}\|_\infty = s$ | Id | leader class | deg | psc $(\check{z})$ | $N(\check{z})$ | Number of vertices | Gödel number |
|---:|---:|---:|---:|---:|---:|---:|---:|
| 3 | 43 | $[3^2 2^3 0^2]$ | 12 | 0 | 30 | 6720 | 32016600 |
| 3 | 44 | $[3^3 1^3 0]$ | 12 | 0 | 30 | 8960 | 27027000 |
| 3 | 45 | $[3^3 210^2]$ | 12 | 0 | 32 | 13440 | 14553000 |
| 3 | 46 | $[3^4 0^3]$ | 12 | 0 | 36 | 560 | 9261000 |
| 3 | 47 | $[32^4 1^2]$ | 13 | 1 | 27 | 13440 | 2358556200 |
| 3 | 48 | $[32^5 0]$ | 13 | 1 | 29 | 2688 | 1803601800 |
| 3 | 49 | $[3^2 2^2 1^3]$ | 13 | 1 | 29 | 26880 | 643242600 |
| 3 | 50 | $[3^2 2^3 10]$ | 13 | 1 | 31 | 26880 | 416215800 |
| 3 | 51 | $[3^3 1^4]$ | 13 | 1 | 31 | 4480 | 459459000 |
| 3 | 52 | $[3^3 21^2 0]$ | 13 | 1 | 33 | 26880 | 189189000 |
| 3 | 53 | $[3^3 2^2 0^2]$ | 13 | 1 | 35 | 6720 | 160083000 |
| 3 | 54 | $[3^4 10^2]$ | 13 | 1 | 37 | 3360 | 101871000 |
| 3 | 55 | $[32^5 1]$ | 14 | 0 | 30 | 5376 | 30661260600 |
| 3 | 56 | $[3^2 2^3 1^2]$ | 14 | 0 | 32 | 26880 | 7075668600 |
| 3 | 57 | $[3^2 2^4 0]$ | 14 | 0 | 34 | 6720 | 5410805400 |
| 3 | 58 | $[3^3 21^3]$ | 14 | 0 | 34 | 17920 | 3216213000 |
| 3 | 59 | $[3^3 2^2 10]$ | 14 | 0 | 36 | 26880 | 2081079000 |
| 3 | 60 | $[3^4 1^2 0]$ | 14 | 0 | 38 | 6720 | 1324323000 |
| 3 | 61 | $[3^4 20^2]$ | 14 | 0 | 40 | 3360 | 1120581000 |
| 3 | 62 | $[32^6]$ | 15 | 1 | 33 | 896 | 521240920200 |
| 3 | 63 | $[3^2 2^4 1]$ | 15 | 1 | 35 | 13440 | 91983691800 |
| 3 | 64 | $[3^3 2^2 1^2]$ | 15 | 1 | 37 | 26880 | 35378343000 |
| 3 | 65 | $[3^3 2^3 0]$ | 15 | 1 | 39 | 8960 | 27054027000 |
| 3 | 66 | $[3^4 1^3]$ | 15 | 1 | 39 | 4480 | 22513491000 |
| 3 | 67 | $[3^4 210]$ | 15 | 1 | 41 | 13440 | 14567553000 |
| 3 | 68 | $[3^5 0^2]$ | 15 | 1 | 45 | 672 | 12326391000 |
| 3 | 69 | $[3^2 2^5]$ | 16 | 0 | 38 | 2688 | 1563722760600 |
| 3 | 70 | $[3^3 2^3 1]$ | 16 | 0 | 40 | 17920 | 459918459000 |
| 3 | 71 | $[3^4 21^2]$ | 16 | 0 | 42 | 13440 | 247648401000 |
| 3 | 72 | $[3^4 2^2 0]$ | 16 | 0 | 44 | 6720 | 189378189000 |
| 3 | 73 | $[3^5 10]$ | 16 | 0 | 46 | 2688 | 160243083000 |
| 3 | 74 | $[3^3 2^4]$ | 17 | 1 | 43 | 4480 | 7818613803000 |
| 3 | 75 | $[3^4 2^2 1]$ | 17 | 1 | 45 | 13440 | 3219429213000 |
| 3 | 76 | $[3^5 1^2]$ | 17 | 1 | 47 | 2688 | 2724132411000 |
| 3 | 77 | $[3^5 20]$ | 17 | 1 | 49 | 2688 | 2083160079000 |
| 3 | 78 | $[3^4 2^3]$ | 18 | 0 | 48 | 4480 | 54730296621000 |
| 3 | 79 | $[3^5 21]$ | 18 | 0 | 50 | 5376 | 35413721343000 |
| 3 | 80 | $[3^6 0]$ | 18 | 0 | 54 | 448 | 27081081027000 |
| 3 | 81 | $[3^5 2^2]$ | 19 | 1 | 53 | 2688 | 602033262831000 |
| 3 | 82 | $[3^6 1]$ | 19 | 1 | 55 | 896 | 460378377459000 |
| 3 | 83 | $[3^6 2]$ | 20 | 0 | 58 | 896 | 7826432416803000 |
| 3 | 84 | $[3^7]$ | 21 | 1 | 63 | 128 | 133049351085651000 |

**Appendix D. Relation between 7-spheres and the leader classes of measure polytopes**

Table D.10: Partitioning 7-spheres in leader classes of measure polytopes

| $N$ | disjunct union of leader classes of measure polytopes | $r_7(N)$ |
|---|---|---|
| 0 | $[0^7]$ | 1 |
| 1 | $[10^6]$ | 14 |
| 2 | $[1^2 0^5]$ | 84 |
| 3 | $[1^3 0^4]$ | 280 |
| 4 | $[1^4 0^3] \cup [20^6]$ | 574 |
| 5 | $[1^5 0^2] \cup [210^5]$ | 840 |
| 6 | $[1^6 0] \cup [21^2 0^4]$ | 1288 |
| 7 | $[1^7] \cup [21^3 0^3]$ | 2368 |
| 8 | $[2^2 0^5] \cup [21^4 0^2]$ | 3444 |
| 9 | $[2^2 10^4] \cup [21^5 0] \cup [30^6]$ | 3542 |
| 10 | $[2^2 1^2 0^3] \cup [21^6] \cup [310^5]$ | 4424 |
| 11 | $[2^2 1^3 0^2] \cup [31^2 0^4]$ | 7560 |
| 12 | $[2^3 0^4] \cup [2^2 1^4 0] \cup [31^3 0^3]$ | 9240 |
| 13 | $[2^3 10^3] \cup [2^2 1^5] \cup [320^5] \cup [31^4 0^2]$ | 8456 |
| 14 | $[2^3 1^2 0^2] \cup [3210^4] \cup [31^5 0]$ | 11088 |
| 15 | $[2^3 1^3 0] \cup [321^2 0^3] \cup [31^6]$ | 16576 |
| 16 | $[2^4 0^3] \cup [2^3 1^4] \cup [321^3 0^2] \cup [40^6]$ | 18494 |
| 17 | $[2^4 10^2] \cup [32^2 0^4] \cup [321^4 0] \cup [410^5]$ | 17808 |
| 18 | $[2^4 1^2 0] \cup [3^2 0^5] \cup [32^2 10^3] \cup [321^5] \cup [41^2 0^4]$ | 19740 |
| 19 | $[2^4 1^3] \cup [3^2 10^4] \cup [32^2 1^2 0^2] \cup [41^3 0^3]$ | 27720 |
| 20 | $[2^5 0^2] \cup [3^2 1^2 0^3] \cup [32^2 1^3 0] \cup [41^4 0^2] \cup [420^5]$ | 34440 |
| 21 | $[2^5 10] \cup [32^3 0^3] \cup [3^2 1^3 0^2] \cup [32^2 1^4] \cup [41^5 0] \cup [4210^4]$ | 29456 |
| 22 | $[2^5 1^2] \cup [3^2 20^4] \cup [32^3 10^2] \cup [3^2 1^4 0] \cup [41^6] \cup [421^2 0^3]$ | 31304 |
| 23 | $[3^2 210^3] \cup [32^3 1^2 0] \cup [3^2 1^5] \cup [421^3 0^2]$ | 49728 |
| 24 | $[2^6 0] \cup [3^2 21^2 0^2] \cup [32^3 1^3] \cup [421^4 0] \cup [42^2 0^4]$ | 52808 |
| 25 | $[2^6 1] \cup [32^4 0^2] \cup [3^2 21^3 0] \cup [421^5] \cup [42^2 10^3] \cup [430^5] \cup [50^6]$ | 43414 |
| 26 | $[3^2 2^2 0^3] \cup [32^4 10] \cup [3^2 21^4] \cup [42^2 1^2 0^2] \cup [4310^4] \cup [510^5]$ | 52248 |
| 27 | $[3^3 0^4] \cup [3^2 2^2 10^2] \cup [32^4 1^2] \cup [42^2 1^3 0] \cup [431^2 0^3] \cup [51^2 0^4]$ | 68320 |
| 28 | $[2^7] \cup [3^3 10^3] \cup [3^2 2^2 1^2 0] \cup [42^2 1^4] \cup [42^3 0^3] \cup [431^3 0^2] \cup [51^3 0^3]$ | 74048 |
| 29 | $[3^3 1^2 0^2] \cup [32^5 0] \cup [3^2 2^2 1^3] \cup [42^3 10^2] \cup [431^4 0] \cup [4320^4] \cup [51^4 0^2] \cup [520^5]$ | 68376 |
| 30 | $[3^2 2^3 0^2] \cup [3^3 1^3 0] \cup [32^5 1] \cup [42^3 1^2 0] \cup [431^5] \cup [43210^3] \cup [51^5 0] \cup [5210^4]$ | 71120 |
| 31 | $[3^3 20^3] \cup [3^2 2^3 10] \cup [3^3 1^4] \cup [42^3 1^3] \cup [4321^2 0^2] \cup [51^6] \cup [521^2 0^3]$ | 99456 |
| 32 | $[3^3 210^2] \cup [3^2 2^3 1^2] \cup [42^4 0^2] \cup [4^2 0^5] \cup [4321^3 0] \cup [521^3 0^2]$ | 110964 |
| 33 | $[3^3 21^2 0] \cup [32^6] \cup [42^4 10] \cup [4^2 10^4] \cup [4321^4] \cup [432^2 0^3] \cup [521^4 0] \cup [52^2 0^4]$ | 89936 |
| 34 | $[3^2 2^4 0] \cup [3^3 21^3] \cup [42^4 1^2] \cup [432^2 10^2] \cup [43^2 0^4] \cup [4^2 1^2 0^3] \cup [521^5] \cup [52^2 10^3] \cup [530^5]$ | 94864 |
| 35 | $[3^3 2^2 0^2] \cup [3^2 2^4 1] \cup [43^2 10^3] \cup [4^2 1^3 0^2] \cup [432^2 1^2 0] \cup [52^2 1^2 0^2] \cup [5310^4]$ | 136080 |

## Appendix E. Isoperimetric distributions of classes of the first polytope shell

Table E.11 consists of 8 columns and represents the isoperimetric distributions of leader classes of polytope shell $P_7^1 \setminus P_7^0$ . The first column is the index of the integer sequence of frequencies of the respective isoperimetric distributions. The other columns contain each the first 50 frequencies of the isoperimetric distribution corresponding to the leader classes containing known physical quantities from the measure polytope $P_7^1$ . Study of the minimum frequencies $f_{min}$ in the 7 distributions and the corresponding vertices results in finding the classes that have *unique* ternary operations. The results for the measure polytope $P_7^1$ are that *only* leader class 2 contains *unique* parallelograms.

The ternary operation for leader class 2 is represented by a physical quantity that is expressed as $length \times mass$. Observe that the frequencies in the sequence of leader class 1 also appear in the OEIS [40] sequence A000141 given by $r_6(m) = 1, 12, 60, 160, 252, 312, 544, 960 \ldots$. The sequence represents the number of ways of writing a positive integer $m$ as a sum of *six* integral squares. It is known that this OEIS sequence A000141 is related to the theta function [56].

Table E.11: Truncated ($n \leq 50$) integer sequences of the frequencies of the isoperimetric distributions of leader classes of the measure polytope $P_7^1 \setminus P_7^0$.

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl7$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 12 | 1 | 3 | 4 | 5 | 6 | 7 |
| 3 | 1 | 10 | 8 | 3 | 10 | 15 | 21 |
| 4 | 60 | 10 | 24 | 6 | 4 | 10 | 35 |
| 5 | 12 | 2 | 3 | 24 | 20 | 2 | 7 |
| 6 | 160 | 42 | 30 | 18 | 40 | 12 | 42 |
| 7 | 60 | 40 | 75 | 4 | 5 | 30 | 105 |
| 8 | 252 | 20 | 24 | 24 | 24 | 26 | 147 |
| 9 | 160 | 100 | 80 | 60 | 50 | 30 | 147 |
| 10 | 312 | 80 | 120 | 40 | 65 | 60 | 21 |
| 11 | 1 | 1 | 3 | 24 | 20 | 66 | 105 |
| 12 | 252 | 80 | 75 | 80 | 80 | 30 | 210 |
| 13 | 544 | 170 | 168 | 104 | 120 | 12 | 252 |
| 14 | 12 | 91 | 150 | 48 | 100 | 60 | 315 |
| 15 | 312 | 10 | 24 | 6 | 10 | 120 | 441 |
| 16 | 960 | 160 | 120 | 60 | 50 | 15 | 35 |
| 17 | 60 | 272 | 240 | 156 | 114 | 132 | 147 |
| 18 | 544 | 122 | 288 | 180 | 170 | 60 | 252 |
| 19 | 1020 | 42 | 1 | 78 | 200 | 60 | 350 |
| 20 | 160 | 182 | 75 | 36 | 40 | 92 | 595 |
| 21 | 960 | 420 | 150 | 104 | 120 | 102 | 735 |
| 22 | 876 | 280 | 246 | 156 | 128 | 165 | 574 |
| 23 | 252 | 100 | 504 | 264 | 160 | 110 | 35 |
| 24 | 1020 | 244 | 8 | 176 | 10 | 30 | 147 |
| 25 | 1560 | 544 | 120 | 4 | 320 | 120 | 315 |
| 26 | 312 | 400 | 288 | 80 | 65 | 180 | 595 |
| 27 | 876 | 2 | 400 | 180 | 170 | 20 | 882 |
| 28 | 2400 | 170 | 528 | 192 | 260 | 180 | 840 |
| 29 | 1 | 560 | 30 | 328 | 320 | 270 | 854 |
| 30 | 544 | 682 | 150 | 240 | 375 | 180 | 1260 |
| 31 | 1560 | 290 | 504 | 24 | 40 | 66 | 21 |
| 32 | 2080 | 20 | 750 | 96 | 100 | 102 | 147 |
| 33 | 12 | 272 | 510 | 264 | 160 | 200 | 441 |
| 34 | 960 | 800 | 80 | 480 | 400 | 360 | 735 |
| 35 | 2400 | 910 | 288 | 480 | 5 | 342 | 840 |
| 36 | 2040 | 362 | 528 | 193 | 560 | 166 | 1050 |
| 37 | 60 | 80 | 728 | 60 | 340 | 132 | 1575 |
| 38 | 1020 | 420 | 840 | 156 | 65 | 180 | 1785 |
| ... | ... | ... | ... | ... | ... | ... | ... |

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl7$ |
|---|---|---|---|---|---|---|---|
| 39 | 2080 | 580 | 3 | 328 | 200 | 15 | 1470 |
| 40 | 3264 | 1040 | 168 | 636 | 320 | 280 | 7 |
| 41 | 160 | 800 | 504 | 624 | 424 | 480 | 147 |
| 42 | 876 | 160 | 510 | 219 | 520 | 420 | 441 |
| 43 | 2040 | 544 | 576 | 6 | 20 | 132 | 574 |
| 44 | 4160 | 724 | 1227 | 104 | 530 | 60 | 854 |
| 45 | 252 | 1220 | 24 | 352 | 100 | 165 | 1575 |
| 46 | 1560 | 880 | 240 | 480 | 320 | 360 | 1750 |
| 47 | 3264 | 1 | 528 | 438 | 560 | 450 | 1533 |
| 48 | 4092 | 182 | 840 | 680 | 1 | 30 | 1932 |
| 49 | 312 | 682 | 1200 | 468 | 484 | 390 | 2387 |
| 50 | 2400 | 1600 | 1200 | 24 | 500 | 570 | 1 |

## Appendix F. Isoperimetric distributions of leader classes of the second polytope shell

Table F.12 consists of 11 columns and represents the isoperimetric distributions of leader classes of polytope shell $P_7^2 \setminus P_7^1$. The first column is the index of the integer sequence of frequencies of the respective isoperimetric distributions. The other columns contain each the first 50 frequencies of the isoperimetric distribution corresponding to the leader classes with $s = 2$ and respective $Id$ from the measure polytope $P_7^3$. Observe that minimum frequencies $f_{min} = 1$ are present in the distributions. Listing the vertices that correspond to those frequency minima results in finding the leader classes that have *unique* ternary operations. The leader class with $s = 2$ and $Id = 6$ (see C.9) has been studied in detail.

Table F.12: Truncated ($n \leq 50$) integer sequences of the frequencies of the isoperimetric distributions of leader classes of the polytope shell $P_7^2 \setminus P_7^1$.

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl7$ | $cl8$ | $cl11$ | $cl12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 6 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 12 | 1 | 1 | 5 | 3 | 2 | 4 | 1 | 3 | 3 |
| 4 | 30 | 10 | 2 | 20 | 3 | 2 | 3 | 2 | 2 | 3 |
| 5 | 60 | 10 | 9 | 31 | 9 | 8 | 4 | 4 | 6 | 3 |
| 6 | 81 | 11 | 8 | 80 | 19 | 1 | 10 | 8 | 3 | 6 |
| 7 | 160 | 40 | 8 | 50 | 6 | 16 | 20 | 8 | 10 | 3 |
| 8 | 126 | 1 | 18 | 42 | 21 | 8 | 17 | 13 | 14 | 1 |
| 9 | 12 | 40 | 34 | 2 | 3 | 17 | 20 | 6 | 11 | 18 |
| 10 | 252 | 1 | 26 | 160 | 36 | 26 | 4 | 26 | 4 | 19 |
| 11 | 156 | 50 | 26 | 85 | 45 | 10 | 40 | 28 | 28 | 18 |
| 12 | 60 | 81 | 1 | 100 | 18 | 1 | 44 | 14 | 36 | 18 |
| 13 | 312 | 11 | 64 | 20 | 1 | 48 | 20 | 16 | 29 | 6 |
| 14 | 272 | 80 | 74 | 182 | 57 | 56 | 16 | 2 | 18 | 21 |
| 15 | 160 | 120 | 34 | 136 | 83 | 50 | 1 | 34 | 3 | 40 |
| 16 | 544 | 100 | 18 | 170 | 63 | 26 | 44 | 60 | 32 | 9 |
| 17 | 480 | 10 | 50 | 80 | 21 | 2 | 80 | 2 | 48 | 45 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl7$ | $cl8$ | $cl11$ | $cl12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 18 | 252 | 50 | 112 | 244 | 50 | 42 | 20 | 16 | 12 | 47 |
| 19 | 960 | 90 | 9 | 211 | 82 | 65 | 80 | 60 | 62 | 39 |
| 20 | 511 | 1 | 120 | 272 | 9 | 10 | 32 | 24 | 62 | 3 |
| 21 | 312 | 170 | 41 | 560 | 120 | 90 | 60 | 52 | 45 | 18 |
| 22 | 1020 | 152 | 64 | 432 | 122 | 88 | 10 | 16 | 18 | 57 |
| 23 | 438 | 40 | 2 | 10 | 57 | 48 | 80 | 57 | 72 | 45 |
| 24 | 12 | 120 | 88 | 420 | 3 | 16 | 91 | 62 | 57 | 60 |
| 25 | 544 | 114 | 114 | 800 | 114 | 96 | 140 | 98 | 75 | 36 |
| 26 | 876 | 202 | 185 | 341 | 108 | 58 | 88 | 55 | 44 | 96 |
| 27 | 780 | 10 | 104 | 182 | 135 | 98 | 44 | 36 | 132 | 9 |
| 28 | 60 | 320 | 34 | 42 | 36 | 42 | 4 | 13 | 11 | 43 |
| 29 | 960 | 81 | 112 | 544 | 249 | 160 | 106 | 88 | 68 | 81 |
| 30 | 1560 | 170 | 164 | 580 | 82 | 2 | 140 | 100 | 106 | 44 |
| 31 | 1200 | 260 | 16 | 455 | 150 | 72 | 40 | 52 | 45 | 78 |
| 32 | 160 | 352 | 164 | 244 | 19 | 136 | 122 | 84 | 134 | 18 |
| 33 | 1020 | 411 | 264 | 100 | 210 | 48 | 184 | 144 | 6 | 104 |
| 34 | 2400 | 40 | 184 | 682 | 219 | 139 | 130 | 98 | 140 | 111 |
| 35 | 1040 | 100 | 74 | 724 | 276 | 1 | 80 | 82 | 160 | 83 |
| 36 | 252 | 202 | 114 | 520 | 83 | 184 | 96 | 94 | 96 | 36 |
| 37 | 876 | 400 | 1 | 560 | 3 | 208 | 20 | 34 | 32 | 66 |
| 38 | 2080 | 1 | 240 | 910 | 108 | 96 | 184 | 1 | 93 | 3 |
| 39 | 1020 | 560 | 368 | 170 | 150 | 17 | 280 | 166 | 1 | 102 |
| 40 | 312 | 322 | 330 | 1600 | 339 | 116 | 244 | 234 | 105 | 172 |
| 41 | 1560 | 81 | 194 | 610 | 45 | 162 | 176 | 201 | 228 | 78 |
| 42 | 2040 | 152 | 120 | 2 | 399 | 296 | 6 | 170 | 68 | 210 |
| 43 | 1632 | 352 | 164 | 800 | 246 | 65 | 140 | 26 | 251 | 108 |
| 44 | 544 | 360 | 9 | 1040 | 120 | 352 | 160 | 136 | 147 | 39 |
| 45 | 2400 | 520 | 304 | 272 | 210 | 212 | 44 | 128 | 28 | 3 |
| 46 | 3264 | 11 | 480 | 272 | 19 | 8 | 244 | 57 | 116 | 120 |
| 47 | 2081 | 530 | 427 | 1760 | 300 | 136 | 400 | 212 | 162 | 153 |
| 48 | 960 | 100 | 160 | 850 | 366 | 176 | 364 | 324 | 72 | 83 |
| 49 | 2080 | 320 | 68 | 20 | 435 | 56 | 128 | 8 | 194 | 192 |
| 50 | 4160 | 560 | 185 | 580 | 63 | 256 | 91 | 262 | 10 | 21 |

## Appendix G. Isoperimetric distributions of leader classes of the third polytope shell

Table G.13 consists of 11 columns and represents the isoperimetric distributions of leader classes of polytope shell $P_7^3 \setminus P_7^2$. The first column is the index of the integer sequence of frequencies of the respective isoperimetric distributions. The other columns contain each the first 50 frequencies of the isoperimetric distribution corresponding to the leader classes with $s = 3$ and respective $Id$ from the measure polytope $P_7^3$. Observe that minimum frequencies $f_{min} = 1$ are present in the distributions. Listing the vertices that correspond to those frequency minima results in finding the leader classes that have *unique* ternary operations.

Table G.13: Truncated ($n \leq 50$) integer sequences of the frequencies of the isoperimetric distributions of leader classes of the polytope shell $P_7^3 \setminus P_7^2$ .

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl9$ | $cl14$ | $cl15$ | $cl22$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 12 | 1 | 2 | 1 | 3 | 1 | 1 | 1 | 2 | 1 |
| 4 | 60 | 10 | 1 | 1 | 3 | 1 | 2 | 2 | 1 | 1 |
| 5 | 160 | 10 | 8 | 10 | 1 | 1 | 1 | 1 | 2 | 2 |
| 6 | 60 | 1 | 2 | 1 | 6 | 1 | 2 | 2 | 2 | 1 |
| 7 | 1 | 1 | 16 | 10 | 3 | 8 | 1 | 6 | 2 | 2 |
| 8 | 252 | 10 | 8 | 40 | 18 | 9 | 7 | 2 | 6 | 2 |
| 9 | 160 | 40 | 3 | 10 | 18 | 1 | 13 | 2 | 4 | 6 |
| 10 | 312 | 40 | 8 | 10 | 6 | 9 | 9 | 8 | 12 | 8 |
| 11 | 12 | 1 | 26 | 10 | 6 | 1 | 8 | 2 | 6 | 3 |
| 12 | 252 | 10 | 48 | 40 | 6 | 9 | 2 | 13 | 12 | 1 |
| 13 | 544 | 10 | 28 | 1 | 19 | 24 | 15 | 1 | 12 | 8 |
| 14 | 60 | 10 | 16 | 1 | 39 | 8 | 26 | 8 | 4 | 13 |
| 15 | 312 | 80 | 2 | 80 | 3 | 9 | 9 | 14 | 2 | 13 |
| 16 | 960 | 40 | 24 | 40 | 18 | 32 | 30 | 15 | 12 | 2 |
| 17 | 544 | 80 | 48 | 40 | 42 | 9 | 34 | 26 | 16 | 7 |
| 18 | 160 | 10 | 26 | 1 | 18 | 32 | 26 | 13 | 28 | 1 |
| 19 | 1020 | 40 | 64 | 40 | 36 | 1 | 2 | 14 | 6 | 14 |
| 20 | 960 | 1 | 64 | 80 | 18 | 10 | 15 | 6 | 24 | 13 |
| 21 | 252 | 10 | 49 | 11 | 50 | 33 | 43 | 13 | 20 | 26 |
| 22 | 876 | 41 | 1 | 90 | 42 | 35 | 38 | 30 | 30 | 13 |
| 23 | 1020 | 90 | 16 | 1 | 60 | 57 | 35 | 38 | 29 | 21 |
| 24 | 1 | 90 | 74 | 10 | 44 | 32 | 1 | 1 | 24 | 30 |
| 25 | 312 | 40 | 74 | 80 | 42 | 33 | 34 | 27 | 2 | 26 |
| 26 | 1560 | 80 | 51 | 1 | 1 | 1 | 70 | 32 | 32 | 6 |
| 27 | 876 | 1 | 48 | 80 | 18 | 24 | 14 | 46 | 28 | 15 |
| 28 | 12 | 80 | 120 | 90 | 78 | 56 | 46 | 40 | 12 | 22 |
| 29 | 544 | 90 | 3 | 80 | 96 | 66 | 1 | 40 | 40 | 8 |
| 30 | 2400 | 40 | 72 | 50 | 44 | 1 | 61 | 2 | 56 | 25 |
| 31 | 1560 | 112 | 112 | 10 | 66 | 40 | 43 | 32 | 52 | 1 |
| 32 | 2080 | 112 | 49 | 112 | 99 | 25 | 78 | 32 | 65 | 45 |
| 33 | 960 | 90 | 128 | 90 | 84 | 25 | 15 | 14 | 30 | 31 |
| 34 | 60 | 90 | 8 | 40 | 60 | 64 | 66 | 57 | 16 | 56 |
| 35 | 2400 | 91 | 120 | 10 | 84 | 66 | 90 | 80 | 56 | 33 |
| 36 | 2040 | 10 | 176 | 90 | 42 | 65 | 70 | 60 | 62 | 30 |
| 37 | 1020 | 1 | 72 | 10 | 6 | 57 | 26 | 82 | 2 | 9 |
| 38 | 160 | 130 | 24 | 112 | 116 | 34 | 62 | 39 | 40 | 44 |
| 39 | 2080 | 240 | 76 | 120 | 168 | 9 | 9 | 10 | 64 | 1 |
| 40 | 3264 | 241 | 2 | 90 | 174 | 96 | 71 | 68 | 106 | 50 |
| 41 | 876 | 170 | 122 | 240 | 152 | 128 | 143 | 50 | 12 | 43 |
| 42 | 252 | 40 | 192 | 113 | 36 | 97 | 61 | 44 | 30 | 62 |
| 43 | 2040 | 112 | 72 | 40 | 3 | 136 | 164 | 84 | 90 | 14 |
| … | … | … | … | … | … | … | … | … | … | … |

| $n$ | $cl1$ | $cl2$ | $cl3$ | $cl4$ | $cl5$ | $cl6$ | $cl7$ | $cl8$ | $cl11$ | $cl12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 44 | 4160 | 122 | 267 | 81 | 99 | 40 | 103 | 132 | 17 | 28 |
| 45 | 1560 | 41 | 194 | 112 | 120 | 9 | 43 | 13 | 38 | 52 |
| 46 | 312 | 192 | 26 | 40 | 60 | 88 | 8 | 24 | 80 | 75 |
| 47 | 3264 | 320 | 112 | 240 | 145 | 83 | 90 | 92 | 64 | 2 |
| 48 | 4092 | 10 | 160 | 1 | 240 | 40 | 108 | 40 | 5 | 39 |
| 49 | 2400 | 330 | 74 | 40 | 19 | 152 | 66 | 60 | 104 | 53 |
| 50 | 544 | 112 | 224 | 170 | 225 | 216 | 146 | 100 | 32 | 48 |

## Appendix H. Classification of common physical quantities

Table H.14 contains 5 columns. The first column represents the name of a common physical quantity. The second column indicates to which shell that the physical quantity belongs. The third column gives the *Id* of the leader class within the respective polytope shell. The fourth column lists the leader class that contains the physical quantity. The fifth column identifies the physical quantity by its integer lattice point in $\mathbb{Z}^7$ .

Table H.14: Classification of common physical quantities.

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| plane angle | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| solid angle | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| linear strain | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| shear strain | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| bulk strain | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| relative elongation | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| refractive index | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| electric susceptibility | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| mass ratio | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| fine-structure constant ($\alpha_e$) | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| ($\alpha_w$) | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| ($\alpha_s$) | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| ($\alpha_G$) | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| redshift | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| Poisson's ratio | 0 | 1 | $[0^7]$ | (0,0,0,0,0,0,0) |
| length | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| height | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| breadth | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| thickness | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| distance | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| radius | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| diameter | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| path length | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| persistence length | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| length of arc | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| Planck length | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| ... | ... | ... | ... | ... |

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| wavelength | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| Compton wavelength | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| relaxation length | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| luminosity distance | 1 | 1 | $[10^6]$ | (1,0,0,0,0,0,0) |
| mass | 1 | 1 | $[10^6]$ | (0,1,0,0,0,0,0) |
| reduced mass | 1 | 1 | $[10^6]$ | (0,1,0,0,0,0,0) |
| Planck mass | 1 | 1 | $[10^6]$ | (0,1,0,0,0,0,0) |
| time | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| period | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| relaxation time | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| time constant | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| time interval | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| proper time | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| Planck time | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| half-life time | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| specific impulse | 1 | 1 | $[10^6]$ | (0,0,1,0,0,0,0) |
| electric current | 1 | 1 | $[10^6]$ | (0,0,0,1,0,0,0) |
| thermodynamic temperature | 1 | 1 | $[10^6]$ | (0,0,0,0,1,0,0) |
| Planck temperature | 1 | 1 | $[10^6]$ | (0,0,0,0,1,0,0) |
| thermal expansion coefficient | 1 | 1 | $[10^6]$ | (0,0,0,0,-1,0,0) |
| amount of substance | 1 | 1 | $[10^6]$ | (0,0,0,0,0,1,0) |
| luminous intensity | 1 | 1 | $[10^6]$ | (0,0,0,0,0,0,1) |
| luminous flux | 1 | 1 | $[10^6]$ | (0,0,0,0,0,0,1) |
| wave number | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| optical power | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| spatial frequency | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| absorption coefficient | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| laser gain | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| rotational constant | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| Rydberg constant | 1 | 1 | $[10^6]$ | (-1,0,0,0,0,0,0) |
| frequency | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| angular frequency | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| circular frequency | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| activity | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| specific material permeability | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| angular velocity | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| decay constant | 1 | 1 | $[10^6]$ | (0,0,-1,0,0,0,0) |
| Avogadro constant | 1 | 1 | $[10^6]$ | (0,0,0,0,0,-1,0) |
| velocity | 1 | 2 | $[1^2 0^5]$ | (1,0,-1,0,0,0,0) |
| group velocity | 1 | 2 | $[1^2 0^5]$ | (1,0,-1,0,0,0,0) |
| volumetric flux | 1 | 2 | $[1^2 0^5]$ | (1,0,-1,0,0,0,0) |
| speed | 1 | 2 | $[1^2 0^5]$ | (1,0,-1,0,0,0,0) |
| speed of light in vacuum | 1 | 2 | $[1^2 0^5]$ | (1,0,-1,0,0,0,0) |
| magnetic field strength | 1 | 2 | $[1^2 0^5]$ | (-1,0,0,1,0,0,0) |
| magnetisation | 1 | 2 | $[1^2 0^5]$ | (-1,0,0,1,0,0,0) |
| temperature gradient | 1 | 2 | $[1^2 0^5]$ | (-1,0,0,0,1,0,0) |
| electric charge | 1 | 2 | $[1^2 0^5]$ | (0,0,1,1,0,0,0) |
| ... | ... | ... | ... | ... |

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| electric flux | 1 | 2 | $[1^2 0^5]$ | (0,0,1,1,0,0,0) |
| catalytic activity | 1 | 2 | $[1^2 0^5]$ | (0,0,-1,0,0,1,0) |
| molar mass | 1 | 2 | $[1^2 0^5]$ | (0,1,0,0,0,-1,0) |
| second radiation constant | 1 | 2 | $[1^2 0^5]$ | (1,0,0,0,1,0,0) |
| luminous energy | 1 | 2 | $[1^2 0^5]$ | (0,0,1,0,0,0,1) |
| linear density | 1 | 2 | $[1^2 0^5]$ | (-1,1,0,0,0,0,0) |
| mass flow rate | 1 | 2 | $[1^2 0^5]$ | (0,1,-1,0,0,0,0) |
| electric dipole moment | 1 | 3 | $[1^3 0^4]$ | (1,0,1,1,0,0,0) |
| linear momentum | 1 | 3 | $[1^3 0^4]$ | (1,1,-1,0,0,0,0) |
| Faraday constant | 1 | 3 | $[1^3 0^4]$ | (0,0,1,1,0,-1,0) |
| dynamic viscosity | 1 | 3 | $[1^3 0^4]$ | (-1,1,-1,0,0,0,0) |
| fluidity | 1 | 3 | $[1^3 0^4]$ | (1,-1,1,0,0,0,0) |
| magnetogyric ratio | 1 | 3 | $[1^3 0^4]$ | (0,-1,1,1,0,0,0) |
| vacuum condensate of Higgs field ($\eta$) | 1 | 3 | $[1^3 0^4]$ | (0,1,-1,-1,0,0,0) |
| area | 2 | 1 | $[20^6]$ | (2,0,0,0,0,0,0) |
| elastic modulus | 2 | 1 | $[20^6]$ | (2,0,0,0,0,0,0) |
| Thomson cross section | 2 | 1 | $[20^6]$ | (2,0,0,0,0,0,0) |
| spacetime curvature | 2 | 1 | $[20^6]$ | (-2,0,0,0,0,0,0) |
| angular acceleration | 2 | 1 | $[20^6]$ | (0,0,-2,0,0,0,0) |
| acceleration | 2 | 1 | $[210^5]$ | (1,0,-2,0,0,0,0) |
| areal velocity | 2 | 2 | $[210^5]$ | (2,0,-1,0,0,0,0) |
| mass attenuation coefficient | 2 | 2 | $[210^5]$ | (2,-1,0,0,0,0,0) |
| radiant exposure | 2 | 2 | $[210^5]$ | (0,1,-2,0,0,0,0) |
| diffusion constant | 2 | 2 | $[210^5]$ | (2,0,-1,0,0,0,0) |
| thermal diffusivity | 2 | 2 | $[210^5]$ | (2,0,-1,0,0,0,0) |
| kinematic viscosity | 2 | 2 | $[210^5]$ | (2,0,-1,0,0,0,0) |
| quantum of circulation | 2 | 2 | $[210^5]$ | (2,0,-1,0,0,0,0) |
| electric current density | 2 | 2 | $[210^5]$ | (-2,0,0,1,0,0,0) |
| luminance | 2 | 2 | $[210^5]$ | (-2,0,0,0,0,0,1) |
| illuminance | 2 | 2 | $[210^5]$ | (-2,0,0,0,0,0,1) |
| luminous emittance | 2 | 2 | $[210^5]$ | (-2,0,0,0,0,0,1) |
| irradiance | 2 | 2 | $[210^5]$ | (-2,0,0,0,0,0,1) |
| magnetic dipole moment | 2 | 2 | $[210^5]$ | (2,0,0,1,0,0,0) |
| Bohr magneton | 2 | 2 | $[210^5]$ | (2,0,0,1,0,0,0) |
| surface density | 2 | 2 | $[210^5]$ | (-2,1,0,0,0,0,0) |
| surface tension | 2 | 2 | $[210^5]$ | (0,1,-2,0,0,0,0) |
| stiffness | 2 | 2 | $[210^5]$ | (0,1,-2,0,0,0,0) |
| compliance | 2 | 2 | $[210^5]$ | (0,-1,2,0,0,0,0) |
| moment of inertia | 2 | 2 | $[210^5]$ | (2,1,0,0,0,0,0) |
| accelerator luminosity | 2 | 2 | $[210^5]$ | (-2,0,-1,0,0,0,0) |
| force | 2 | 3 | $[21^2 0^4]$ | (1,1,-2,0,0,0,0) |
| energy density | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| radiant energy density | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| sound energy density | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| toughness | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| pressure | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| modulus of elasticity | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| ... | ... | ... | ... | ... |

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| Young's modulus | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| shear modulus | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| compression modulus | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| normal stress | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| shear stress | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| energy momentum tensor | 2 | 3 | $[21^2 0^4]$ | (-1,1,-2,0,0,0,0) |
| Planck constant | 2 | 3 | $[21^2 0^4]$ | (2,1,-1,0,0,0,0) |
| angular momentum | 2 | 3 | $[21^2 0^4]$ | (2,1,-1,0,0,0,0) |
| action | 2 | 3 | $[21^2 0^4]$ | (2,1,-1,0,0,0,0) |
| spin | 2 | 3 | $[21^2 0^4]$ | (2,1,-1,0,0,0,0) |
| acoustic impedance | 2 | 3 | $[21^2 0^4]$ | (-2,1,-1,0,0,0,0) |
| mass flux | 2 | 3 | $[21^2 0^4]$ | (-2,1,-1,0,0,0,0) |
| magnetic flux density | 2 | 3 | $[21^2 0^4]$ | (0,1,-2,-1,0,0,0) |
| magnetic induction | 2 | 3 | $[21^2 0^4]$ | (0,1,-2,-1,0,0,0) |
| surface charge density | 2 | 3 | $[21^2 0^4]$ | (-2,0,1,1,0,0,0) |
| dielectric polarisation | 2 | 3 | $[21^2 0^4]$ | (-2,0,1,1,0,0,0) |
| electrical displacement | 2 | 3 | $[21^2 0^4]$ | (-2,0,1,1,0,0,0) |
| electrical quadrupole moment | 2 | 3 | $[21^2 0^4]$ | (2,0,1,1,0,0,0) |
| luminous exposure | 2 | 3 | $[21^2 0^4]$ | (-2,0,1,0,0,0,1) |
| absorbed dose | 2 | 4 | $[2^2 0^4]$ | (2,0,-2,0,0,0,0) |
| dose equivalent | 2 | 4 | $[2^2 0^4]$ | (2,0,-2,0,0,0,0) |
| specific energy | 2 | 4 | $[2^2 0^4]$ | (2,0,-2,0,0,0,0) |
| gravitational potential | 2 | 4 | $[2^2 0^4]$ | (2,0,-2,0,0,0,0) |
| molar Planck constant | 2 | 5 | $[21^3 0^3]$ | (2,1,-1,0,0,-1,0) |
| magnetic vector potential | 2 | 5 | $[21^3 0^3]$ | (1,1,-2,-1,0,0,0) |
| thermal conductivity | 2 | 5 | $[21^3 0^3]$ | (1,1,-2,0,-1,0,0) |
| thermal resistivity | 2 | 5 | $[21^3 0^3]$ | (-1,-1,2,0,1,0,0) |
| torque | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| moment of a force | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| specific heat capacity | 2 | 6 | $[2^2 10^4]$ | (2,0,-2,0,-1,0,0) |
| energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| potential energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| kinetic energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| work | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| Lagrange function | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| Hamilton function | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| Hartree energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| ionization energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| electron affinity | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| electronegativity | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| dissociation energy | 2 | 6 | $[2^2 10^4]$ | (2,1,-2,0,0,0,0) |
| magnetic constant | 2 | 8 | $[2^2 1^2 0^3]$ | (1,1,-2,-2,0,0,0) |
| permeability | 2 | 8 | $[2^2 1^2 0^3]$ | (1,1,-2,-2,0,0,0) |
| magnetic flux | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,-1,0,0,0) |
| magnetic moment | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,-1,0,0,0) |
| entropy | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,0,-1,0,0) |
| specific heat | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,0,-1,0,0) |
| ... | ... | ... | ... | ... |

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| Boltzmann constant | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,0,-1,0,0) |
| Josephson constant | 2 | 8 | $[2^2 1^2 0^3]$ | (-2,-1,2,1,0,0,0) |
| magnetic flux quantum | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,-1,0,0,0) |
| chemical potential | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,0,0,-1,0) |
| molar energy | 2 | 8 | $[2^2 1^2 0^3]$ | (2,1,-2,0,0,-1,0) |
| molar heat capacity | 2 | 8 | $[2^2 1^3 0^2]$ | (2,1,-2,0,-1,-1,0) |
| molar gas constant | 2 | 11 | $[2^2 1^3 0^2]$ | (2,1,-2,0,-1,-1,0) |
| molar entropy | 2 | 11 | $[2^2 1^3 0^2]$ | (2,1,-2,0,-1,-1,0) |
| inductance | 2 | 12 | $[2^3 1 0^3]$ | (2,1,-2,-2,0,0,0) |
| self-inductance | 2 | 12 | $[2^3 1 0^3]$ | (2,1,-2,-2,0,0,0) |
| mutual inductance | 2 | 12 | $[2^3 1 0^3]$ | (2,1,-2,-2,0,0,0) |
| magnetisability | 2 | 12 | $[2^3 1 0^3]$ | (2,-1,2,2,0,0,0) |
| volume | 3 | 1 | $[3 0^6]$ | (3,0,0,0,0,0,0) |
| Loschmidt constant | 3 | 1 | $[3 0^6]$ | (-3,0,0,0,0,0,0) |
| number density | 3 | 1 | $[3 0^6]$ | (-3,0,0,0,0,0,0) |
| mass density | 3 | 2 | $[3 1 0^5]$ | (-3,1,0,0,0,0,0) |
| specific volume | 3 | 2 | $[3 1 0^5]$ | (3,-1,0,0,0,0,0) |
| amount of substance concentration | 3 | 2 | $[3 1 0^5]$ | (-3,0,0,0,0,1,0) |
| molar volume | 3 | 2 | $[3 1 0^5]$ | (3,0,0,0,0,-1,0) |
| heat flux density | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| Poynting vector | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| radiative flux | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| thermal emittance | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| sound intensity | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| radiance | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| irradiance | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| radiant exitance | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| radiant emittance | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| radiosity | 3 | 2 | $[3 1 0^5]$ | (0,1,-3,0,0,0,0) |
| volume rate of flow | 3 | 2 | $[3 1 0^5]$ | (3,0,-1,0,0,0,0) |
| jerk | 3 | 2 | $[3 1 0^5]$ | (1,0,-3,0,0,0,0) |
| electric field gradient | 3 | 3 | $[3 1^2 0^4]$ | (0,1,-3,-1,0,0,0) |
| electric charge density | 3 | 3 | $[3 1^2 0^4]$ | (-3,0,1,1,0,0,0) |
| heat transfer coefficient | 3 | 3 | $[3 1^2 0^4]$ | (0,1,-3,0,-1,0,0) |
| thermal insulance | 3 | 3 | $[3 1^2 0^4]$ | (0,-1,3,0,1,0,0) |
| spectral exitance | 3 | 3 | $[3 1^2 0^4]$ | (-1,1,-3,0,0,0,0) |
| spectral radiance | 3 | 3 | $[3 1^2 0^4]$ | (-1,1,-3,0,0,0,0) |
| spectral irradiance | 3 | 3 | $[3 1^2 0^4]$ | (-1,1,-3,0,0,0,0) |
| spectral power | 3 | 3 | $[3 1^2 0^4]$ | (1,1,-3,0,0,0,0) |
| spectral intensity | 3 | 3 | $[3 1^2 0^4]$ | (1,1,-3,0,0,0,0) |
| luminous energy density | 3 | 3 | $[3 1^2 0^4]$ | (-3,0,1,0,0,0,1) |
| catalytic activity concentration | 3 | 3 | $[3 1^2 0^4]$ | (-3,0,-1,0,0,1,0) |
| reaction rate | 3 | 3 | $[3 1^2 0^4]$ | (-3,0,-1,0,0,1,0) |
| absorbed dose rate | 3 | 4 | $[3 2 0^5]$ | (2,0,-3,0,0,0,0) |
| thermal conductivity | 3 | 5 | $[3 1^3 0^3]$ | (1,1,-3,0,-1,0,0) |
| first hyper-susceptibility | 3 | 5 | $[3 1^3 0^3]$ | (-1,-1,3,1,0,0,0) |
| electric field | 3 | 5 | $[3 1^3 0^3]$ | (1,1,-3,-1,0,0,0) |
| ... | ... | ... | ... | ... |

| physical quantity | $s$ | Id | leader class | vertex |
|---|---|---|---|---|
| radiant intensity | 3 | 6 | $[3210^4]$ | (2,1,-3,0,0,0,0) |
| radiant flux | 3 | 6 | $[3210^4]$ | (2,1,-3,0,0,0,0) |
| Newton constant of gravitation | 3 | 6 | $[3210^4]$ | (3,-1,-2,0,0,0,0) |
| power | 3 | 6 | $[3210^4]$ | (2,1,-3,0,0,0,0) |
| sound energy flux | 3 | 6 | $[3210^4]$ | (2,1,-3,0,0,0,0) |
| bolometric luminosity | 3 | 6 | $[3210^4]$ | (2,1,-3,0,0,0,0) |
| responsivity | 3 | 6 | $[321^20^3]$ | (-2,-1,3,1,0,0,0) |
| electric potential difference | 3 | 9 | $[321^20^3]$ | (2,1,-3,-1,0,0,0) |
| electric potential | 3 | 9 | $[321^20^3]$ | (2,1,-3,-1,0,0,0) |
| thermal conductance | 3 | 9 | $[321^20^3]$ | (2,1,-3,0,-1,0,0) |
| thermal resistance | 3 | 9 | $[321^20^3]$ | (-2,-1,3,0,1,0,0) |
| electromotive force | 3 | 9 | $[321^20^3]$ | (2,1,-3,-1,0,0,0) |
| luminous efficacy | 3 | 9 | $[321^20^3]$ | (-2,1,3,0,0,0,1) |
| electrical resistance | 3 | 14 | $[32^210^3]$ | (2,1,-3,-2,0,0,0) |
| reactance | 3 | 14 | $[32^210^3]$ | (2,1,-3,-2,0,0,0) |
| impedance | 3 | 14 | $[32^210^3]$ | (2,1,-3,-2,0,0,0) |
| conductance | 3 | 14 | $[32^210^3]$ | (-2,-1,3,2,0,0,0) |
| admittance | 3 | 14 | $[32^210^3]$ | (-2,-1,3,2,0,0,0) |
| susceptance | 3 | 14 | $[32^210^3]$ | (-2,-1,3,2,0,0,0) |
| characteristic impedance of vacuum | 3 | 14 | $[32^210^3]$ | (2,1,-3,-2,0,0,0) |
| von Klitzing constant | 3 | 14 | $[32^210^3]$ | (2,1,-3,-2,0,0,0) |
| specific resistance | 3 | 15 | $[3^21^20^3]$ | (3,1,-3,-1,0,0,0) |
| electrical resistivity | 3 | 22 | $[3^2210^3]$ | (3,1,-3,-2,0,0,0) |
| electrical conductivity | 3 | 22 | $[3^2210^3]$ | (-3,-1,3,2,0,0,0) |
| second moment of area | 4 | 1 | $[40^6]$ | (4,0,0,0,0,0,0) |
| jounce | 4 | 2 | $[410^5]$ | (1,0,-4,0,0,0,0) |
| electric polarisability | 4 | | $[4210^4]$ | (0,-1,4,2,0,0,0) |
| Stefan-Boltzmann constant | 4 | | $[4310^4]$ | (0,1,-3,0,-4,0,0) |
| first radiation constant | 4 | | $[4310^4]$ | (4,1,-3,0,0,0,0) |
| electrical mobility | 4 | | $[431^20^3]$ | (3,1,-4,-1,0,0,0) |
| electric capacitance | 4 | | $[42^210^3]$ | (-2,-1,4,2,0,0,0) |
| electric constant | 4 | | $[43210^3]$ | (-3,-1,4,2,0,0,0) |
| permittivity | 4 | | $[43210^3]$ | (-3,-1,4,2,0,0,0) |
| second hyper-susceptibility | 6 | | $[62^30^3]$ | (-2,-2,6,2,0,0,0) |
| first hyper-polarisability | 7 | | $[73210^3]$ | (-1,-2,7,3,0,0,0) |
| second hyper-polarisability | 10 | | $[(10)4320^3]$ | (-2,-3,10,4,0,0,0) |

## Appendix I. Gödel walk in 7-dimensional integer lattice

Table I.15 contains in the first column the row identifier. In the second column we list the vertices in the order of appearance in the Gödel walk. The third column gives the value of the Gödel number up to the number 100. The fourth column shows the dimension $d$ of $\mathbb{Z}^d \times \{0\}^{7-d}$ in which the lattice point is embedded. The fifth column indicates to which measure polytope $P_7^s$ the lattice point belongs. The sixth column shows the leader class containing the lattice point.

Table I.15: Gödel walk in $\mathbb{Z}^7$.

| Id | vertex | Gödel number | dimension | $\|\check{x}\|_\infty = s$ | leader class |
|---|---|---|---|---|---|
| 1 | (0,0,0,0,0,0,0) | 1 | 0 | 0 | $[0^7]$ |
| 2 | (1,0,0,0,0,0,0) | 2 | 1 | 1 | $[10^6]$ |
| 3 | (0,1,0,0,0,0,0) | 3 | 2 | 1 | $[10^6]$ |
| 4 | (2,0,0,0,0,0,0) | 4 | 1 | 2 | $[20^6]$ |
| 5 | (0,0,1,0,0,0,0) | 5 | 3 | 1 | $[10^6]$ |
| 6 | (1,1,0,0,0,0,0) | 6 | 2 | 1 | $[1^20^5]$ |
| 7 | (0,0,0,1,0,0,0) | 7 | 4 | 1 | $[10^6]$ |
| 8 | (3,0,0,0,0,0,0) | 8 | 1 | 3 | $[30^6]$ |
| 9 | (0,2,0,0,0,0,0) | 9 | 2 | 2 | $[20^6]$ |
| 10 | (1,0,1,0,0,0,0) | 10 | 3 | 1 | $[1^20^5]$ |
| 11 | (0,0,0,0,1,0,0) | 11 | 5 | 1 | $[10^6]$ |
| 12 | (2,1,0,0,0,0,0) | 12 | 2 | 2 | $[210^5]$ |
| 13 | (0,0,0,0,0,1,0) | 13 | 6 | 1 | $[10^6]$ |
| 14 | (1,0,0,1,0,0,0) | 14 | 4 | 1 | $[1^20^5]$ |
| 15 | (0,1,1,0,0,0,0) | 15 | 3 | 1 | $[1^20^5]$ |
| 16 | (4,0,0,0,0,0,0) | 16 | 1 | 4 | $[40^6]$ |
| 17 | (0,0,0,0,0,0,1) | 17 | 7 | 1 | $[10^6]$ |
| 18 | (1,2,0,0,0,0,0) | 18 | 2 | 2 | $[210^5]$ |
| 19 | (2,0,1,0,0,0,0) | 20 | 3 | 2 | $[210^5]$ |
| 20 | (0,1,0,1,0,0,0) | 21 | 4 | 1 | $[1^20^5]$ |
| 21 | (1,0,0,0,1,0,0) | 22 | 5 | 1 | $[1^20^5]$ |
| 22 | (3,1,0,0,0,0,0) | 24 | 2 | 3 | $[310^5]$ |
| 23 | (0,0,2,0,0,0,0) | 25 | 3 | 2 | $[20^6]$ |
| 24 | (1,0,0,0,0,1,0) | 26 | 6 | 1 | $[1^20^5]$ |
| 25 | (0,3,0,0,0,0,0) | 27 | 2 | 3 | $[30^6]$ |
| 26 | (2,0,0,1,0,0,0) | 28 | 4 | 2 | $[210^5]$ |
| 27 | (1,1,1,0,0,0,0) | 30 | 3 | 1 | $[1^30^4]$ |
| 28 | (5,0,0,0,0,0,0) | 32 | 1 | 5 | $[50^6]$ |
| 29 | (0,1,0,0,1,0,0) | 33 | 5 | 1 | $[1^20^5]$ |
| 30 | (1,0,0,0,0,0,1) | 34 | 7 | 1 | $[1^20^5]$ |
| 31 | (0,0,1,1,0,0,0) | 35 | 4 | 1 | $[1^20^5]$ |
| 32 | (2,2,0,0,0,0,0) | 36 | 2 | 2 | $[2^20^5]$ |
| 33 | (0,1,0,0,0,1,0) | 39 | 6 | 1 | $[1^20^5]$ |
| 34 | (3,0,1,0,0,0,0) | 40 | 3 | 3 | $[310^5]$ |
| 35 | (1,1,0,1,0,0,0) | 42 | 4 | 1 | $[1^30^4]$ |
| 36 | (2,0,0,0,1,0,0) | 44 | 5 | 2 | $[210^5]$ |
| 37 | (0,2,1,0,0,0,0) | 45 | 3 | 2 | $[210^5]$ |
| 38 | (4,1,0,0,0,0,0) | 48 | 2 | 4 | $[410^5]$ |
| 39 | (0,0,0,2,0,0,0) | 49 | 4 | 2 | $[20^6]$ |
| 40 | (1,0,2,0,0,0,0) | 50 | 3 | 2 | $[210^5]$ |
| 41 | (0,1,0,0,0,0,1) | 51 | 7 | 1 | $[1^20^5]$ |
| 42 | (2,0,0,0,0,1,0) | 52 | 6 | 2 | $[210^5]$ |
| 43 | (1,3,0,0,0,0,0) | 54 | 2 | 3 | $[310^5]$ |
| 44 | (0,0,1,0,1,0,0) | 55 | 5 | 1 | $[1^20^5]$ |
| 45 | (3,0,0,1,0,0,0) | 56 | 4 | 3 | $[310^5]$ |
| ... | ... | ... | ... | ... | ... |

| Id | vertex | Gödel number | dimension | $\|\breve{x}\|_\infty = s$ | leader class |
|---|---|---|---|---|---|
| 46 | (2,1,1,0,0,0,0) | 60 | 3 | 2 | $[21^2 0^4]$ |
| 47 | (0,2,0,1,0,0,0) | 63 | 4 | 2 | $[210^5]$ |
| 48 | (6,0,0,0,0,0,0) | 64 | 1 | 6 | $[60^6]$ |
| 49 | (0,0,1,0,0,1,0) | 65 | 6 | 1 | $[1^2 0^5]$ |
| 50 | (1,1,0,0,1,0,0) | 66 | 5 | 1 | $[1^3 0^4]$ |
| 51 | (2,0,0,0,0,0,1) | 68 | 7 | 2 | $[210^5]$ |
| 52 | (1,0,1,1,0,0,0) | 70 | 4 | 1 | $[1^3 0^4]$ |
| 53 | (3,2,0,0,0,0,0) | 72 | 2 | 3 | $[320^5]$ |
| 54 | (0,1,2,0,0,0,0) | 75 | 3 | 2 | $[210^5]$ |
| 55 | (0,0,0,1,1,0,0) | 77 | 5 | 1 | $[1^2 0^5]$ |
| 56 | (1,1,0,0,0,1,0) | 78 | 6 | 1 | $[1^3 0^4]$ |
| 57 | (4,0,1,0,0,0,0) | 80 | 3 | 4 | $[410^5]$ |
| 58 | (0,4,0,0,0,0,0) | 81 | 2 | 4 | $[40^6]$ |
| 59 | (2,1,0,1,0,0,0) | 84 | 4 | 2 | $[21^2 0^4]$ |
| 60 | (0,0,1,0,0,0,1) | 85 | 7 | 1 | $[1^2 0^5]$ |
| 61 | (3,0,0,0,1,0,0) | 88 | 5 | 3 | $[310^5]$ |
| 62 | (1,2,1,0,0,0,0) | 90 | 3 | 2 | $[21^2 0^4]$ |
| 63 | (0,0,0,1,0,1,0) | 91 | 6 | 1 | $[1^2 0^5]$ |
| 64 | (5,1,0,0,0,0,0) | 96 | 2 | 5 | $[510^5]$ |
| 65 | (1,0,0,2,0,0,0) | 98 | 4 | 2 | $[210^5]$ |
| 66 | (0,2,0,0,1,0,0) | 99 | 5 | 2 | $[210^5]$ |
| 67 | (2,0,2,0,0,0,0) | 100 | 3 | 2 | $[2^2 0^5]$ |

## Appendix J. Cardinality of sets of pairwise orthogonal vertices resulting in the representative vertex of the leader class

Table J.16 contains 11 columns. The first column is the row identifier. The second column represents the infinity norm $s$ of the polytope shell. The third column lists the leader classes $[\breve{z}]$. The fourth column gives the total degree "deg" of the leader classes. The fifth column contains $N(\breve{z}) \equiv m \pmod 2$ where $N(\breve{z})$ is the square of the radius of the 7-sphere associated to the representative vertex $\breve{z}$ of the leader class $[\breve{z}]$. From column 6 to column 11 the cardinality $n_d$ is given as function of the dimension $d \in \mathbb{N}$ of $\mathbb{Z}^d \times \{0\}^{7-d}$ where $2 \leq d \leq 7$.

Table J.16: Cardinality of the sets of pairwise orthogonal vertices resulting in the representative vertex of leader class $[\breve{z}]$.

| Id | $s$ | leader class | deg | $N(\breve{z}) \equiv m \pmod 2$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $n_6$ | $n_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $[10^6]$ | 1 | $N \equiv 1 \pmod 2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | $[1^2 0^5]$ | 2 | $N \equiv 0 \pmod 2$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | $[1^3 0^4]$ | 3 | $N \equiv 1 \pmod 2$ | 2 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | $[1^4 0^3]$ | 4 | $N \equiv 0 \pmod 2$ | 3 | 7 | 7 | 7 | 7 | 7 |
| 5 | 1 | $[1^5 0^2]$ | 5 | $N \equiv 1 \pmod 2$ | 3 | 7 | 15 | 15 | 15 | 15 |
| 6 | 1 | $[1^6 0]$ | 6 | $N \equiv 0 \pmod 2$ | 3 | 7 | 15 | 15 | 31 | 31 |
| 7 | 1 | $[1^7]$ | 7 | $N \equiv 1 \pmod 2$ | 3 | 7 | 15 | 15 | 31 | 63 |
| 8 | 2 | $[20^6]$ | 2 | $N \equiv 0 \pmod 2$ | 1 | 2 | 3 | 4 | 5 | 6 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

| Id | $s$ | leader class | deg | $N(\breve{z}) \equiv m \pmod 2$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $n_6$ | $n_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 2 | $[210^5]$ | 3 | $N \equiv 1 \pmod 2$ | 1 | 3 | 5 | 7 | 9 | 11 |
| 10 | 2 | $[21^20^4]$ | 4 | $N \equiv 0 \pmod 2$ | 1 | 3 | 7 | 11 | 15 | 19 |
| 11 | 2 | $[2^20^5]$ | 4 | $N \equiv 0 \pmod 2$ | 1 | 5 | 11 | 19 | 29 | 41 |
| 12 | 2 | $[21^30^3]$ | 5 | $N \equiv 1 \pmod 2$ | 3 | 7 | 7 | 15 | 23 | 31 |
| 13 | 2 | $[2^210^4]$ | 5 | $N \equiv 1 \pmod 2$ | 4 | 4 | 12 | 24 | 40 | 60 |
| 14 | 2 | $[2^21^20^3]$ | 6 | $N \equiv 0 \pmod 2$ | 3 | 9 | 11 | 27 | 51 | 83 |
| 15 | 2 | $[2^30^4]$ | 6 | $N \equiv 0 \pmod 2$ | 1 | 3 | 15 | 39 | 79 | 139 |
| 16 | 2 | $[2^21^30^2]$ | 7 | $N \equiv 1 \pmod 2$ | 3 | 9 | 11 | 27 | 59 | 107 |
| 17 | 2 | $[2^310^3]$ | 7 | $N \equiv 1 \pmod 2$ | 3 | 7 | 13 | 39 | 89 | 171 |
| 18 | 3 | $[30^6]$ | 3 | $N \equiv 1 \pmod 2$ | 0 | 4 | 12 | 24 | 40 | 60 |
| 19 | 3 | $[310^5]$ | 4 | $N \equiv 0 \pmod 2$ | 3 | 3 | 11 | 27 | 51 | 83 |
| 20 | 3 | $[31^20^4]$ | 4 | $N \equiv 1 \pmod 2$ | 7 | 11 | 11 | 27 | 59 | 107 |
| 21 | 3 | $[320^5]$ | 5 | $N \equiv 1 \pmod 2$ | 1 | 3 | 13 | 39 | 89 | 171 |
| 22 | 3 | $[31^30^3]$ | 6 | $N \equiv 0 \pmod 2$ | 7 | 11 | 31 | 31 | 63 | 127 |
| 22 | 3 | $[3^20^5]$ | 6 | $N \equiv 0 \pmod 2$ | u | u | u | u | u | u |
| 23 | 3 | $[3210^4]$ | 6 | $N \equiv 0 \pmod 2$ | 3 | 7 | 15 | 39 | 95 | 199 |
| 24 | 3 | $[321^20^3]$ | 7 | $N \equiv 1 \pmod 2$ | 4 | 20 | 60 | 260 | 620 | 1460 |
| 25 | 3 | $[32^210^3]$ | 8 | $N \equiv 0 \pmod 2$ | 3 | 15 | 25 | 61 | 145 | 341 |
| 26 | 3 | $[3^21^20^3]$ | 8 | $N \equiv 0 \pmod 2$ | 3 | 23 | 47 | 63 | 207 | 479 |
| 27 | 3 | $[3^2210^3]$ | 9 | $N \equiv 1 \pmod 2$ | 3 | 7 | 23 | 79 | 263 | 671 |
| 28 | 4 | $[40^6]$ | 4 | $N \equiv 0 \pmod 2$ | 1 | 2 | 11 | 44 | 125 | 286 |
| 29 | 4 | $[410^5]$ | 5 | $N \equiv 1 \pmod 2$ | 1 | 7 | 17 | 47 | 129 | 311 |
| 30 | 4 | $[4210^4]$ | 7 | $N \equiv 1 \pmod 2$ | 7 | 7 | 31 | 79 | 191 | 471 |
| 31 | 4 | $[4310^4]$ | 8 | $N \equiv 1 \pmod 2$ | 5 | 11 | 27 | 99 | 339 | 923 |
| 32 | 4 | $[42^210^3]$ | 9 | $N \equiv 1 \pmod 2$ | 7 | 23 | 30 | 120 | 330 | 796 |
| 33 | 4 | $[431^20^3]$ | 9 | $N \equiv 1 \pmod 2$ | 5 | 23 | 39 | 111 | 327 | 975 |
| 34 | 4 | $[43210^3]$ | 10 | $N \equiv 1 \pmod 2$ | 5 | 23 | 47 | 135 | 415 | 1287 |

## Appendix K. Pairwise orthogonal vertices resulting in the vertex representing energy

Table K.17 contains 5 columns. The first column is the row identifier. The second column represents the perimeter of the parallelogram that is also a rectangle. The third column contains vertex $\breve{x}$. The fourth column contains the vertex $\breve{y}$. The fifth column contains the squared area $A_p^2$ of the parallelograms. We see that the squared areas are all *even*. We observe that the vertices with $Id = 1$ , $Id = 35$ and $Id = 36$ were already found through the two-factoring of the leader class $[2^210^4]$ . We find in total 60 non-degenerated rectangles for the vertex $\breve{z} = (2, 1, -2, 0, 0, 0, 0)$ with the 4 vertices of each rectangle incident on the same hypersphere $(\breve{x} - \frac{\breve{z}}{2})^2 = (\frac{\breve{z}}{2})^2$ .

Table K.17: Orthogonal pairwise vertices resulting in the physical quantity *energy*.

| $Id$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $A_p^2$ |
|---|---|---|---|---|
| 1 | 7,657 | (0,1,0,0,0,0,0) | (2,0,-2,0,0,0,0) | 8 |
| 2 | 8,120 | (0,0,-1,-1,0,0,0) | (2,1,-1,1,0,0,0) | 14 |
| 3 | 8,120 | (0,0,-1,0,-1,0,0) | (2,1,-1,0,1,0,0) | 14 |
| ... | ... | ... | ... | ... |

| $Id$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $A_p^2$ |
|---|---|---|---|---|
| 4 | 8,120 | (0,0,-1,0,0,-1,0) | (2,1,-1,0,0,1,0) | 14 |
| 5 | 8,120 | (0,0,-1,0,0,0,-1) | (2,1,-1,0,0,0,1) | 14 |
| 6 | 8,120 | (0,0,-1,0,0,0,1) | (2,1,-1,0,0,0,-1) | 14 |
| 7 | 8,120 | (0,0,-1,0,0,1,0) | (2,1,-1,0,0,-1,0) | 14 |
| 8 | 8,120 | (0,0,-1,0,1,0,0) | (2,1,-1,0,-1,0,0) | 14 |
| 9 | 8,120 | (0,0,-1,1,0,0,0) | (2,1,-1,-1,0,0,0) | 14 |
| 10 | 8,120 | (1,0,0,-1,0,0,0) | (1,1,-2,1,0,0,0) | 14 |
| 11 | 8,120 | (1,0,0,0,-1,0,0) | (1,1,-2,0,1,0,0) | 14 |
| 12 | 8,120 | (1,0,0,0,0,-1,0) | (1,1,-2,0,0,1,0) | 14 |
| 13 | 8,120 | (1,0,0,0,0,0,-1) | (1,1,-2,0,0,0,1) | 14 |
| 14 | 8,120 | (1,0,0,0,0,0,1) | (1,1,-2,0,0,0,-1) | 14 |
| 15 | 8,120 | (1,0,0,0,0,1,0) | (1,1,-2,0,0,-1,0) | 14 |
| 16 | 8,120 | (1,0,0,0,1,0,0) | (1,1,-2,0,-1,0,0) | 14 |
| 17 | 8,120 | (1,0,0,1,0,0,0) | (1,1,-2,-1,0,0,0) | 14 |
| 18 | 8,363 | (0,1,-1,-1,0,0,0) | (2,0,-1,1,0,0,0) | 18 |
| 19 | 8,363 | (0,1,-1,0,-1,0,0) | (2,0,-1,0,1,0,0) | 18 |
| 20 | 8,363 | (0,1,-1,0,0,-1,0) | (2,0,-1,0,0,1,0) | 18 |
| 21 | 8,363 | (0,1,-1,0,0,0,-1) | (2,0,-1,0,0,0,1) | 18 |
| 22 | 8,363 | (0,1,-1,0,0,0,1) | (2,0,-1,0,0,0,-1) | 18 |
| 23 | 8,363 | (0,1,-1,0,0,1,0) | (2,0,-1,0,0,-1,0) | 18 |
| 24 | 8,363 | (0,1,-1,0,1,0,0) | (2,0,-1,0,-1,0,0) | 18 |
| 25 | 8,363 | (0,1,-1,1,0,0,0) | (2,0,-1,-1,0,0,0) | 18 |
| 26 | 8,363 | (1,-1,-1,0,0,0,0) | (1,2,-1,0,0,0,0) | 18 |
| 27 | 8,363 | (1,0,-2,-1,0,0,0) | (1,1,0,1,0,0,0) | 18 |
| 28 | 8,363 | (1,0,-2,0,-1,0,0) | (1,1,0,0,1,0,0) | 18 |
| 29 | 8,363 | (1,0,-2,0,0,-1,0) | (1,1,0,0,0,1,0) | 18 |
| 30 | 8,363 | (1,0,-2,0,0,0,-1) | (1,1,0,0,0,0,1) | 18 |
| 31 | 8,363 | (1,0,-2,0,0,0,1) | (1,1,0,0,0,0,-1) | 18 |
| 32 | 8,363 | (1,0,-2,0,0,1,0) | (1,1,0,0,0,-1,0) | 18 |
| 33 | 8,363 | (1,0,-2,0,1,0,0) | (1,1,0,0,-1,0,0) | 18 |
| 34 | 8,363 | (1,0,-2,1,0,0,0) | (1,1,0,-1,0,0,0) | 18 |
| 35 | 8,472 | (0,0,-2,0,0,0,0) | (2,1,0,0,0,0,0) | 20 |
| 36 | 8,472 | (0,1,-2,0,0,0,0) | (2,0,0,0,0,0,0) | 20 |
| 37 | 8,472 | (1,0,-1,-1,-1,0,0) | (1,1,-1,1,1,0,0) | 20 |
| 38 | 8,472 | (1,0,-1,-1,0,-1,0) | (1,1,-1,1,0,1,0) | 20 |
| 39 | 8,472 | (1,0,-1,-1,0,0,-1) | (1,1,-1,1,0,0,1) | 20 |
| 40 | 8,472 | (1,0,-1,-1,0,0,1) | (1,1,-1,1,0,0,-1) | 20 |
| 41 | 8,472 | (1,0,-1,-1,0,1,0) | (1,1,-1,1,0,-1,0) | 20 |
| 42 | 8,472 | (1,0,-1,-1,1,0,0) | (1,1,-1,1,-1,0,0) | 20 |
| 43 | 8,472 | (1,0,-1,0,-1,-1,0) | (1,1,-1,0,1,1,0) | 20 |
| 44 | 8,472 | (1,0,-1,0,-1,0,-1) | (1,1,-1,0,1,0,1) | 20 |
| 45 | 8,472 | (1,0,-1,0,-1,0,1) | (1,1,-1,0,1,0,-1) | 20 |
| 46 | 8,472 | (1,0,-1,0,-1,1,0) | (1,1,-1,0,1,-1,0) | 20 |
| 47 | 8,472 | (1,0,-1,0,0,-1,-1) | (1,1,-1,0,0,1,1) | 20 |
| 48 | 8,472 | (1,0,-1,0,0,-1,1) | (1,1,-1,0,0,1,-1) | 20 |
| 49 | 8,472 | (1,0,-1,0,0,1,-1) | (1,1,-1,0,0,-1,1) | 20 |
| 50 | 8,472 | (1,0,-1,0,0,1,1) | (1,1,-1,0,0,-1,-1) | 20 |
| ... | ... | ... | ... | ... |

| $Id$ | $p_p$ | $\breve{x}$ | $\breve{y}$ | $A_p^2$ |
|---|---|---|---|---|
| 51 | 8,472 | (1,0,-1,0,1,-1,0) | (1,1,-1,0,-1,1,0) | 20 |
| 52 | 8,472 | (1,0,-1,0,1,0,-1) | (1,1,-1,0,-1,0,1) | 20 |
| 53 | 8,472 | (1,0,-1,0,1,0,1) | (1,1,-1,0,-1,0,-1) | 20 |
| 54 | 8,472 | (1,0,-1,0,1,1,0) | (1,1,-1,0,-1,-1,0) | 20 |
| 55 | 8,472 | (1,0,-1,1,-1,0,0) | (1,1,-1,-1,1,0,0) | 20 |
| 56 | 8,472 | (1,0,-1,1,0,-1,0) | (1,1,-1,-1,0,1,0) | 20 |
| 57 | 8,472 | (1,0,-1,1,0,0,-1) | (1,1,-1,-1,0,0,1) | 20 |
| 58 | 8,472 | (1,0,-1,1,0,0,1) | (1,1,-1,-1,0,0,-1) | 20 |
| 59 | 8,472 | (1,0,-1,1,0,1,0) | (1,1,-1,-1,0,-1,0) | 20 |
| 60 | 8,472 | (1,0,-1,1,1,0,0) | (1,1,-1,-1,-1,0,0) | 20 |

Table K.18: Partitions of leader classes with Gödel number $\leq 1500$

| leader class | deg | psc $(\breve{z})$ | $N(\breve{z})$ | vertices | Gödel number | $F2$ | $F3$ | $F4$ | $F5$ |
|---|---|---|---|---|---|---|---|---|---|
| $[0^7]$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $[10^6]$ | 1 | 1 | 1 | 14 | 2 | 0 | 0 | 0 | 0 |
| $[20^6]$ | 2 | 0 | 4 | 14 | 4 | 0 | 0 | 0 | 0 |
| $[1^2 0^5]$ | 2 | 0 | 2 | 84 | 6 | 1 | 0 | 0 | 0 |
| $[30^6]$ | 3 | 1 | 9 | 14 | 8 | 1 | 0 | 0 | 0 |
| $[210^5]$ | 3 | 1 | 5 | 168 | 12 | 2 | 0 | 0 | 0 |
| $[310^5]$ | 4 | 0 | 10 | 168 | 24 | 3 | 1 | 0 | 0 |
| $[1^3 0^4]$ | 3 | 1 | 3 | 280 | 30 | 3 | 1 | 0 | 0 |
| $[2^2 0^5]$ | 4 | 0 | 8 | 84 | 36 | 3 | 1 | 0 | 0 |
| $[21^2 0^4]$ | 4 | 0 | 6 | 840 | 60 | 5 | 3 | 0 | 0 |
| $[320^5]$ | 5 | 1 | 13 | 168 | 72 | 5 | 3 | 0 | 0 |
| $[31^2 0^4]$ | 5 | 1 | 11 | 840 | 120 | 7 | 7 | 1 | 0 |
| $[2^2 10^4]$ | 5 | 1 | 9 | 840 | 180 | 8 | 8 | 1 | 0 |
| $[1^4 0^3]$ | 4 | 0 | 4 | 560 | 210 | 7 | 6 | 1 | 0 |
| $[3^2 0^5]$ | 6 | 0 | 18 | 84 | 216 | 7 | 8 | 1 | 0 |
| $[3210^4]$ | 6 | 0 | 14 | 1680 | 360 | 11 | 17 | 5 | 0 |
| $[21^3 0^3]$ | 5 | 1 | 7 | 2240 | 420 | 11 | 15 | 4 | 0 |
| $[31^3 0^3]$ | 6 | 0 | 12 | 2240 | 840 | 15 | 29 | 13 | 1 |
| $[2^3 0^4]$ | 6 | 0 | 12 | 280 | 900 | 12 | 20 | 7 | 0 |
| $[3^2 10^4]$ | 7 | 1 | 19 | 840 | 1080 | 15 | 33 | 17 | 1 |
| $[2^2 1^2 0^3]$ | 6 | 0 | 10 | 3360 | 1260 | 17 | 35 | 16 | 1 |

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