

On the Inventory Routing Problem
with Stationary Stochastic Demand Rate

Enkele vraagstukken over het 'Inventory Routing'-probleem
met stationaire stochastische vraag

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Proefschrift ingediend tot het behalen van de graad van
Doctor in de Ingenieurswetenschappen: Bedrijfskundige Systeemtechnieken
en Operationeel Onderzoek

Vakgroep Technische Bedrijfsvoering
Voorzitter: prof. dr. E.-H. Aghezzaf
Faculteit Ingenieurswetenschappen en Architectuur
Academiejaar 2014 - 2015



ISBN 978-90-8578-764-8
NUR 950,804
Wettelijk depot: D/2015/10.500/8



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Thesis submitted to obtain the degree of
Doctor of Industrial Engineering
and Operations Research
Academic year 2014-2015

Acknowledgments

First and foremost, my sincere appreciation goes to my main supervisor, Prof. Dr. El-Houssaine Aghezzaf for all the guidance, the valuable suggestions and very helpful remarks during this PhD study. I will be forever thankful to him for his invaluable advices in inspiring ideas, his kind nature, his constant dedication in coaching and guiding me, his endless support and help throughout my PhD journey. I am truly delighted to have the chance to work with him.

To the members of the reading and examination committee, Prof. Dr. Hendrik Van Landeghem (Chairman), Prof. Dr. Dieter Fiems, Prof. Dr. Veronique Limère, Prof. Dr. ir. Stijn De Vuyst, Prof. Dr. Birger Raa (Secretary) and Prof. Nadine Meskens, thanks a lot for the thorough reading of my manuscripts and the constructive criticisms and suggestions which definitely helped me improve the quality of this dissertation.

I would like to especially thank Prof. Dr. Birger Raa and Prof. Dr. Veronique Limère, for their professional interactions and helping during my research activities. Many thanks go to Kurt De Cock and Veerle Van Beversluys, who always answer and fix all my questions regarding my research and life facilities. I also wish to thanks all staffs in the Department of Industrial Management, for the notable friendship and valuable help whenever needed. My dear Malaysian friends in Belgium and Malaysia, as well as my international friends in Ghent University, thank you for everything, for their friendship and help during my oversea life.

I am also grateful to the Malaysian Ministry of Education (MOE) and Universiti Utara Malaysia (UUM) for giving me the opportunity to pursue my dreams in becoming an academic and a researcher. I am also thankful to the Dean of School Technology Management and Logistics (STML) UUM, Assoc. Prof. Dr. Zulkifli Mohamed Udin, and all the faculty members, for all the supports and encouragements given from afar. Also thanks to Malaysian Embassy staff in Brussels, for their help during my study abroad.

A special thank goes to my lovely family. Moms and dads, brothers and sisters, no word can express how much I love you all and how grateful I am for your love, encouragement and support, without which none of this, I most certainly would not be where I am today. What a precious blessing it is to have the unconditional support of family. It has made this PhD journey so much easier and more pleasurable.

I also owe special thanks to my lovely wife, Nur Asyima Che Alias, word can't describe how lucky and blessed I am to have you throughout this PhD journey. Your endless love, patience, understanding, vigorous moral support and continuous encouragement are more than anyone can ever wish for. Thank you so much for the sacrifice you have made to our family and I pray that we will be together forever till 'jannah'.

And finally, to my two beautiful princesses Nur Aliyah Insyirah, 5 years old, Nur Amna Wafiyyah, 3 years old, and my newborn baby boy Aysar Wafiq, 3 months old, daddy appreciate your well-behaved when daddy was still struggling with the finalization of the PhD dissertation.

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January 2015

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List of Acronyms

C

CCLP	Capacitated Concentrator Location Problem
CIRP	Cyclic Inventory Routing Problem
CLMRP	Capacitated Warehouse Location Model with Risk Pooling
CVRP	Capacitated Vehicle Routing Problem

D

DDP	Delivery Dispatching Problem
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E

EOQ	Economic Order Quantity
-----	-------------------------

F

FP	Fixed Partition
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G

GA	Genetic Algorithm
GAP	Generalized Assignment Problem
GRASP	Greedy Randomized Adaptive Search Procedure

H

HAIR	Hybrid Approach to Inventory Routing
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I

IRP	Inventory Routing Problem
IRPDD	Inventory Routing Problem with Direct Deliver- ies
IRPSD	Inventory Routing Problem with Split Delivery

M

MDP	Markov Decision Process
MIP	Mixed Integer Programming
MIRP	Metered Inventory Routing Problem
ML	Maximum-level
MP-SIRP	Multi-period Stochastic Inventory Routing Prob- lem

O

OU	Order-up-to-level
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P

PDF	Probability Distribution Function
PHA	Progressive Hedging Algorithm

S

SCM	Supply Chain Management
SIRP	Stochastic Inventory Routing Problem
SP-SIRP	Single-Period Stochastic Inventory Routing Problem
STP	Scenario Tree Problem
SWMR	Single-warehouse, Multiple-retailer
SWMR-VMI	Single-warehouse, Multiple-retailer and Vendor Managed Inventory

T

TSP Travelling Salesman Problem

V

VMI Vendor Managed Inventory

VRP Vehicle Routing Problems

VRPTW Vehicle Routing Problem with Time Windows

Z

ZIO Zero Inventory Ordering

Nederlandse Samenvatting **–Summary in Dutch–**

Individuele bedrijven functioneren niet langer als volledig onafhankelijke entiteiten, ze maken nu veeleer deel uit van een bevoorradingsketen. Dat is een van de meest significante paradigmaverschuivingen van het huidige bedrijfsmanagement (Lambert and Cooper, 2000). Daarom wordt naar het beheer van de verschillende activiteiten binnen een bevoorradingsketen, zoals materiaal-, informatie en financiële stromen, verwezen als integraal ketenbeheer of supply chain management (SCM). SCM impliceert de coördinatie en integratie van die verschillende activiteiten binnen en tussen bedrijven, zodat een beter globaal resultaat voor de bevoorradingsketen bekomen kan worden. In dit proefschrift bespreken we de integratie van twee stappen in de bevoorradingsketen, namelijk het voorraadbeheer en de routeplanning. Het probleem van het gelijktijdig bepalen van de te leveren hoeveelheden en de routes voor de voertuigen staat bekend als het inventory routing probleem (IRP).

Het IRP is een van de belangrijke optimaliseringsproblemen bij het integraal keten- en logistiekbeheer. Het streeft naar een optimale integratie van het voorraadsbeheer en de routeplanning in een toeleveringsnetwerk. Over het algemeen doet IRP zich voor als een onderliggend optimaliseringsprobleem in situaties waar tegelijkertijd beslissingen over voorraadoptimalisering en distributie dienen genomen te worden. De belangrijkste doelstelling ervan is een optimaal distributiebeleid op te zetten, dat een set ruten voor de voertuigen, te leveren hoeveelheden en leveringstijden bepaalt, die de totale kosten voor de voorraadopslag en het vervoer tot een minimum beperken. Dat is een

typisch logistiek optimaliseringsprobleem dat optreedt in bevoorradingsketens die “Vendor Managed Inventory” (VMI) implementeren.

VMI is een overeenkomst tussen een leverancier en de kleinhandelaars die regelmatig door die leverancier worden bediend. De overeenkomst bepaalt dat de kleinhandelaars ermee instemmen dat de leverancier de tijdstippen en de omvang van de leveringen bepaalt. Volgens die overeenkomst verwerft de leverancier de volledige bevoegdheid om de voorraden van zijn klanten te onderhouden. Zo kan de leverancier proactief handelen en instaan voor het voorraadbeheer van zijn regelmatige klanten, in plaats van te wachten tot de kleinhandelaars hun bestellingen plaatsen. In de praktijk werd vastgesteld dat de implementatie van een strategie zoals VMI de algemene resultaten van het netwerk van de bevoorradingsketen verbetert; we verwijzen bijvoorbeeld naar Lee and Seungjin (2008), Andersson et al. (2010) and Coelho et al. (2014).

Dit proefschrift focust zich vooral op een single-warehouse-multiple-retailer systeem (SWMR), waarbij de leverancier een bepaald aantal kleinhandelaars bedient vanuit één enkel magazijn. In een eerste opzet veronderstellen we dat het om kleinhandelaars gaat met een deterministische, constante vraag en in een tweede opzet wordt aangenomen dat alle kleinhandelaars een stochastisch, stationair gebruiksniveau vertonen. De eerste doelstelling bestaat erin te bepalen wanneer en hoeveel eenheden moeten geleverd worden van de leverancier aan het magazijn en vanuit het magazijn aan de kleinhandelaars om de totale transport- en voorraadkosten tot een minimum te beperken, binnen een eindige tijdshorizon en zonder dat tekorten zich voordoen.

De rest van dit proefschrift is als volgt ingedeeld: de twee eerste hoofdstukken zijn een algemene inleiding tot IRP en omvatten ook een literatuuroverzicht van IRP-gerelateerde papers. In Hoofdstuk 2 stellen we een aantal benaderingswijzen voor het oplossen van de problemen bij de rittenplanning (VRP) die het mogelijk maken een efficiënte constructie- en verbeteringsheuristiek te ontwerpen.

Hoofdstuk 3 beschouwt een bevoorradingsketen met twee stappen, die bestaat uit een SWMR die moet voldoen aan een deterministisch vragenpatroon, en met een VMI-strategie. We stellen een optimaliseringsbenadering voor met twee fasen voor de coördinatie van de leveringen binnen dit VMI-systeem. In de eerste fase worden rechtstreekse leveringen gedaan van de leverancier naar alle kleinhandelaars om zo de algemene voorraadkosten te drukken. Vervolgens, in de tweede fase, worden de kleinhandelaars geclusterd door gebruik te maken van een constructieheuristiek om de transportkosten te optimaliseren en tegelijkertijd te voldoen aan een aantal bijkomende beperkingen. De betere resultaten van het systeem dat gebruikt maakt van gecoördineerde VMI- leveringen om de voorraden op peil te houden, ten opzichte van het systeem met enkel rechtstreekse transporten worden weergegeven en besproken bij de vergelijkende analyse.

In Hoofdstuk 4 wordt het stochastic-inventory-routing probleem met meerdere perioden (MP-SIRP) behandeld met als doelstelling de totale distributie- en voorraadkosten te drukken. Eerst wordt het probleem geformuleerd als een lineair gemengd geheeltallig programmeringsprobleem waarvoor we een deterministisch gelijkwaardig benaderingsmodel (MP-DAIRP _{α}) voorstellen. Dat laatste model kan dan worden opgesplitst in twee bekende deelproblemen: een voorraadallocatie- en een rittenplanningsdeelprobleem. Het stochastische aspect van de vraag wordt behandeld bij het voorraadallocatiedeelprobleem. Het rittenplanningsdeelprobleem wordt opgelost als een deterministisch gemengd geheeltallig probleem. Lagrangiaanse relaxatie wordt gebruikt om bijna-optimale, uitvoerbare oplossingen voor MP-DAIRP _{α} te bekomen. De resultaten van de voorgestelde Lagrangiaanse relaxatiebenadering op een aantal numerieke voorbeelden worden vermeld en grondig besproken.

Ten slotte worden in Hoofdstuk 5 een aantal slotopmerkingen uit recent onderzoek en een aantal richtlijnen voor toekomstige onderzoeksprojecten besproken.

English Summary

One of the most significant paradigm shifts of present business management is that individual businesses no longer participate as solely independent entities, but rather as supply chains (Lambert and Cooper, 2000). Therefore, the management of multiple relationships across the supply chain such as flow of materials, information, and finances is being referred to as supply chain management (SCM). SCM involves coordinating and integrating these multiple relationships within and among companies, so that it can improve the global performance of the supply chain. In this dissertation, we discuss the issue of integrating the two processes in the supply chain related, respectively, to inventory management and routing policies. The challenging problem of coordinating the inventory management and transportation planning decisions in the same time, is known as the inventory routing problem (IRP).

The IRP is one of the challenging optimization problems in logistics and supply chain management. It aims at optimally integrating inventory control and vehicle routing operations in a supply network. In general, IRP arises as an underlying optimization problem in situations involving simultaneous optimization of inventory and distribution decisions. Its main goal is to determine an optimal distribution policy, consisting of a set of vehicle routes, delivery quantities and delivery times that minimizes the total inventory holding and transportation costs. This is a typical logistical optimization problem that arises in supply chains implementing a vendor managed inventory (VMI) policy.

VMI is an agreement between a supplier and his regular retailers according to which retailers agree to the alternative that the supplier decides the timing and size of the deliveries. This agreement grants the supplier the full authority to manage inventories at his retailers'. This allows the supplier to act proactively and take responsibility for the inventory management of his regular retailers, instead of reacting to the orders placed by these retailers. In practice, implementing policies such as VMI has proven to considerably improve the overall performance of the supply network, see for example Lee and Seungjin (2008), Andersson et al. (2010) and Coelho et al. (2014).

This dissertation focuses mainly on the single-warehouse, multiple-retailer (SWMR) system, in which a supplier serves a set of retailers from a single warehouse. In the first situation, we assume that all retailers face a deterministic, constant demand rate and in the second condition, we assume that all retailers consume the product at a stochastic stationary rate. The primary objective is to decide when and how many units to be delivered from the supplier to the warehouse and from the warehouse to retailers so as to minimize total transportation and inventory holding costs over the finite horizon without any shortages.

The remainder of this dissertation is organized as follows. The first two chapters present a general introduction to the IRP, as well as a literature review of regular papers related to IRPs. We also highlight in Chapter 2 some routing solution approaches of the vehicle routing problem (VRP) that allows us to design an efficient improvement construction-heuristic.

Chapter 3 considers a two-stage supply chain, consisting of a SWMR facing deterministic demands, under a VMI policy. It presents a two-phase optimisation approach for coordinating the shipments in this VMI system. The first phase uses direct shipping from the supplier to all retailers to minimise the overall inventory costs. Then, in the second phase, the retailers are clustered using a construction heuristic in order to optimise the transportation costs while satisfying some additional restrictions. The improvement of the system's performance

through coordinated VMI replenishments against the system with direct shipping only is shown and discussed in the comparative analysis section.

Chapter 4 considers the multi-period stochastic inventory routing problem (MP-SIRP) with the objective of minimizing the total distribution and inventory costs. The problem is first formulated as a linear mixed-integer stochastic program for which we propose a deterministic equivalent approximation model (MP-DAIRP _{α}). This latter model can be decomposed into two well-known sub-problems: an inventory allocation sub-problem and a vehicle routing sub-problem. The stochastic aspect of the demand is accounted for in the inventory allocation sub-problem. The vehicle routing sub-problem is solved as a deterministic mixed-integer problem. Lagrangian relaxation is used to determine close to optimal feasible solutions for the MP-DAIRP _{α} . Results of the proposed Lagrangian relaxation approach on some numerical examples are reported and thoroughly discussed.

Finally, some concluding comments from recent research works and some directions for future research issues are discussed in this Chapter 5.

1

Introduction

1.1 General Introduction

The last three decades have witnessed an increased interest of researchers in the various research areas of supply chain management (SCM). In particular, the design and optimization problems have been arisen in the management of the supply chain operations. Supply chain operations involve the processes of sourcing the required materials, production of goods using the sourced materials, and transportation of these goods to retailers. Supply chain management can basically be defined as a set of approaches utilized to efficiently integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed in the right quantities, to the right locations, and at the right time, in order to minimize the system's wide costs while satisfying service level requirements (Simchi-levi et al. 2003). The key feature in supply chain management is the collaboration between different stages like purchasing, inventory control, production, sales and distribution, to achieve overall efficiency and continuous improvement.

The main concern of each company, which is usually involved in the supply chain operation, is to minimize its operational costs and maximize its profits. Inventory costs are among these major costs. Traditionally, a decentralized inventory system imposes itself for managing inventories across multiple stages in the supply chain. Each stage is responsible for managing its own inventory independently, and places its orders with the supplier based on its individual requirements without giving any consideration about others. Typically, retailers or distributors would focus on optimizing their own costs or profits, in spite of being a part of the supply chain, because the decisions concerning production and replenishment are made separately and independently by the members of that chain. This kind of inventory management has some disadvantages for the retailers and the other players in the supply chain. It may lead to demand uncertainty for the distributor as the time and the number of orders from retailers can vary. Hence, the distributor may face difficulty in managing their own inventory and scheduling the deliveries in an efficient way.

Kleywegt et al. (2002) have indicated that the lack of information with regard to retailer inventory level in conventional inventory management can affect the decision at the distributor level. They point out that without the visibility of inventory level information to the retailers', the supplier is unable to determine the priority of shipment between retailers. Thus, suppliers may possibly end up replenishing non-critical customers, causing a stock-out problem for other retailers who really require items are unable to fulfil their end customer demand. Thus, the decisions between the stages in the supply chain need to be integrated in a manner that is beneficial for the entire supply chain in both operational and economic terms, even though each member of the chain has their own different operational goals.

Vendor managed inventory (VMI) is one of the most widely discussed partnerships between the members of the supply chain for improving multi-firm supply chain efficiency. Also known as the continuous replenishment or supplier-managed inventory, it was popularized in the late 1980's by Wal-Mart and Procter & Gamble and resulted in significant benefits. After this successful application, many companies

have followed suit and implemented VMI in their supply chains. For example, this model has become very popular in the US retail sector, not only for consumable products, but also for electric appliances and high tech electronics. Needless to say, this model is also applied in the automotive industries in Europe and USA where component suppliers have served the assembly lines on VMI contract terms since the end of the last millennium. It is said that VMI represents a powerful tool to strike out costs from the supply chain if implemented properly (Van Weele, 2005).

VMI is an inventory management policy, in which the supplier assumes, in addition to its inbound inventory, the responsibility of maintaining inventory at the retailers and ensures that they do not run out of stock at any moment. The delivery times and quantities to be delivered to a retailer is no longer done after the retailer's orders. The supplier determines the quantity and when the delivery takes place. The replenishment is thus proactive as it is based on the available inventory information instead of being reactive in response to retailers' orders. This policy has many advantages for both the supplier and the retailers. The supplier has the possibility of combining multiple deliveries to optimize the truck loading and the routing cost. Moreover, as the deliveries are more uniform, the amount of inventory that must be held at the supplier can be drastically reduced. On the other hand, the retailers no longer need to dedicate resources to the management of their inventories. Also, the service level (i.e. product availability) increases, as the supplier can track inventory levels at the retailers to determine the precise replenishment urgency.

One reason why VMI has gained increased popularity nowadays is due to the availability of numerous technologies that enable to monitor retailer inventories online and cost effective. Accessibility to inventory data becomes much easier. On the contrary, implementing VMI does not always lead to improved results. Failure can, for example, happen due to the unavailability of the necessary information or due to the inability of the supplier to make the right decisions. The large amount of data makes it extremely hard to optimize this problem. It

involves managing inventory in supply chains and optimizing distribution, which are two particularly challenging problems.

The main focus of VMI is on the coordination issue of inventory replenishment and transportation. The transportation costs are reduced by shipping a large load to several retailers, in a coordinated manner, instead of delivering small loads to each retailer respectively. When implementing VMI, the crucial decision problem that frequently has to be addressed is how to determine optimal policies for the distribution of products from the supplier to each of the retailers, which is labelled as the inventory routing problem (IRP).

The IRP is one of the challenging optimization problems in the design and management of supply and distribution networks. It also provides a very good starting point for investigating the integration of different components in logistics and supply chain, for instance inventory management and transportation, which are traditionally dealt with separately. Such integration is expected to lead to a cost reduction in logistics and supply chain management.

Campbell et al. (1998) present a specific description of the IRP, which is concerned with the repeated distribution of a single product, from a single facility, to a set of n retailers over a given planning horizon of length T , possibly infinite. The retailers consume the product at a given rate u_j and have the capability to maintain a local inventory of the product up to a maximum of C_j . The inventory at retailer j is I_j at time 0. A fleet of m homogeneous vehicles, with limited capacity Q , is available for the distribution of the product. The objective is to minimize the average distribution costs during the planning period without causing stock-outs at any of the retailers.

The IRP differs from traditional vehicle routing problems (VRP) because it is based on retailer usage rather than retailer orders. The IRP as defined above is deterministic and static due to our assumption that usage rates are known and constant. Obviously, in real-life, the problem has come to be stochastic and dynamic.

In the remainder of this chapter, we will first review the definition and typology of the structure variants of IRP problems such as time horizon, demand, structure, routing, inventory policy, inventory decisions, fleet composition, and fleet size, which will be focused in this research. Then, in the second part of this chapter, we will discuss the contributions of this PhD research.

1.2 Definition and Typology of the IRP

Combined inventory management and routing gives rise to a large variety of aspects and assumptions. The IRP formulation and models are very hard to classify. Therefore, we classify IRPs according to the dimensions of the IRP which have been described over the past 30 years. We then concentrate on the basic versions of the IRP, on which most of the research effort is focused. We have adapted the categories by Andersson et al. (2010) to classify the IRP according to eight criteria which will be presented in Table 1.1.

Table 1.1 Structural Variants of the IRP

Criteria	Possible options		
Time horizon	Instant	Finite	Infinite
Demand	Stochastic	Deterministic	
Structure	One-to-one	One-to-many	Many-to-many
Routing	Direct	Multiple	Continuous
Inventory policy	Maximum-level	Order-up-to-level	
Inventory decisions	Lost sales	Back-order	Nonnegative
Fleet composition	Homogenous	Heterogeneous	
Fleet size	Single	Multiple	Unconstrained

Source: Adapted from Andersson et al. (2010)

1.2.1 Time Horizon

Time refers to the horizon taken into account by the IRP model. The instant planning horizon determines the solution that balances the transportation and inventory costs at the beginning of every single time period based on the inventory level information. When more than one visit to a retailer may be required, we are talking about a finite problem. In finite planning horizon, the improvement solutions for the delivery schedule and the routes for delivery are determined for a specific period of time. On the other hand, the infinite time horizon solves a long-term problem by evaluating the performance of replenishment policies and the routing approach that minimizes the total cost average. When analysing a problem within infinite planning horizon, the decisions of distribution strategies are used rather than schedules.

1.2.2 Demand

The demand pattern is another dimension that differentiates the IRP categories. Most researchers assume the retailer demand is deterministic to simplify the problem. However, in reality, the retailer demand becomes stochastic. Thus, this classification refers to the time when information on demand becomes known. If it is fully available to the decision maker at the beginning of the planning horizon, the cases then are called deterministic. However, when the method proposed incorporates uncertainty with respect to the demand, then the cases will be called stochastic.

1.2.3 Structure

The number of suppliers and retailers may vary, and therefore Baita et al. (1998) classified the topology of the problem into three modes; one-to-one, one-to-many and many-to-many, to describe the problem. The many-to-one case is included in the one-to-many case; it is not well studied and can easily be transformed to a one-to-many topology. The one-to-many is the most common case of IRP, where a single facility serves a set of retailers using a fleet of vehicles. The central

facility is a warehouse where the vehicles start and end their routes and where the goods are stored before being delivered to the retailers.

1.2.4 Routing

It is essential to differentiate the afore-mentioned three cases when classifying the routing component in the IRP. In the direct case, a vehicle picks up goods at the central warehouse and then distributes all goods to a single retailer before returning to the warehouse. When a vehicle can visit more than one retailer on a trip, we denote the case as multiple visits. In both instances, the trip starts and ends at the central warehouse and the underlying problem is a VRP. The trip can be seen as continuous routing, when there is no central warehouse, with no start or end.

1.2.5 Inventory Policy

Inventory policies describe pre-established rules to replenish retailers. The two most common used in the IRP are the order-up-to-level (OU) policy and the maximum-level (ML) policy. Under an OU policy, each retailer defines a minimum and a maximum inventory level and can be visited several times during the planning horizon. The supplier monitors the inventory of each retailer and guarantees that no stock-out will occur. Whenever a retailer is visited, the quantity delivered is that to fill its inventory capacity. Instead of imposing that every time a retailer is visited, the quantity delivered is such that the maximum level of inventory is reached, the only constraint on the shipping quantity is that it must not be greater than the maximum inventory level. Hence, under an ML inventory policy, the replenishment level is flexible, but bounded by the capacity available at each retailer.

1.2.6 Inventory Decisions

There are many inventory decisions that have to be made in the IRP. It determines how inventory management is modelled. In this classification, we will focus on the decisions concerning the retailers. In deterministic context, the inventory can be nonnegative, where the lowest inventory level is fixed either to zero or a level based on the safety

stock. If the inventory is allowed to become negative, then back-order occurs and the corresponding demand will be served at a later stage. If there are no back orders, then the extra demand is considered as lost sales. In both cases there may exist a penalty for the stock-out.

1.2.7 Vehicle Fleet

The last two criteria refer to fleet composition and size. The fleet used to distribute or collect goods can be classified according to composition and size. The fleet can be homogeneous, if all vehicles have the same characteristics such as speed, fixed cost, variable cost, equipment, and size. If the fleet is heterogeneous, some, or all, of the characteristics of the vehicles may differ.

The size of the fleet is an important aspect of the problem. We will apply the term single if the fleet consists of one vehicle. If the fleet consists of a number of vehicles, we will use multiple to describe this situation. This is the case where the distributor owns the fleet and cannot purchase extra vehicle capacity. If the distributor has the possibility to purchase extra vehicle capacity then we will use the term unconstrained.

1.3 PhD Contribution

We consider a two-stage supply system with uncertain demand, operating under VMI (see Figure 1). The two-echelon inventory system consists of two echelons. The first echelon is the warehouse and second one includes several retailers. The warehouse provides inventory to the retailers according to the retailers' demands. The inventory control policies for the warehouse and retailers are totally different. For example, the replenishment interval for the warehouse and retailers are not the same, the lead time for the order of the warehouse would be larger than the one for the retailers, and the demand during the lead time for the warehouse and retailers are also different.

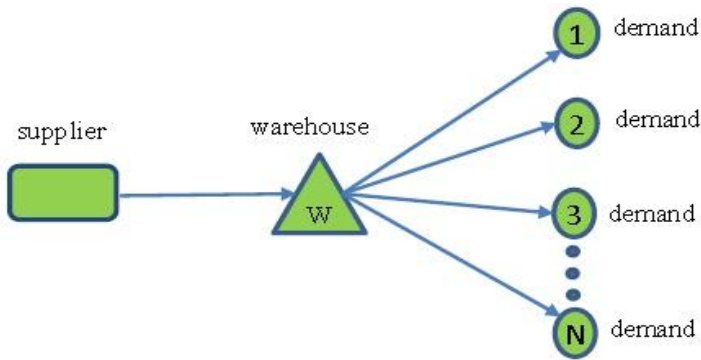


Figure 1. A two-echelon inventory system with SWMR

In fact, since the demand rates at the retailers are uncertain, these demand rates vary. Thus, the total supply to the warehouse may not equal the total demand, resulting in an unbalanced situation between the warehouse and the retailers. All back orders may not be fulfilled at a constant rate. Therefore, a new replenishment policy and inventory control strategy should be implemented to resolve these unbalances in replenishment intervals and reorder points.

In this dissertation, we concentrate on the case of the single-warehouse with multiple-retailers (SWMR). In this case, a supplier serves a set of retailers from a single warehouse. We first start with the case where we assume that all retailers face a constant deterministic demand rate, and then we consider a second case where we assume that all retailers consume the product at a stochastic stationary demand rate. Deliveries to these retailers are made from the warehouse with a fleet of vehicles having a limited capacity. The warehouse in turn places orders to an outside supplier to fill the demand of the retailers. Whenever the warehouse places an order, a fixed cost is incurred. Similarly, for each delivery to a retailer, a facility-dependent setup cost is charged. In addition, there is a facility-dependent holding cost for inventory at each facility in the system. The objective is to decide when and how many units to be delivered from the supplier to the warehouse and then from the warehouse to retailers so as to minimize total transportation and holding costs over a finite horizon without any shortages.

One main contribution is to propose a two-phase heuristic solution approach to minimize the overall inventory and transportation costs of the SWMR system under a VMI policy. Roundy (1985) and Chu and Leon (2008), amongst others studied the SWMR case before, however, they assumed that only direct shipping is used to replenish the retailers, i.e., each vehicle visits a single retailer and returns to the warehouse. Even under this assumption, it is shown that the problem cannot be solved in polynomial time. Therefore, in the first phase of the SWMR system under a VMI policy, retailers are partitioned into subsets in order to minimize the overall inventory costs of the system. Then, in the second phase, a VRP procedure is used to solve the routing in each of the retailer subsets with the objective of minimizing the travelled distance and hence the transportation costs. As such, we drop the assumption of direct shipments from warehouse to retailers, but also include the option of combining multiple outbound shipments in so-called *milk-runs*. To evaluate the impact of VMI and milk-runs on the SWMR system, a comparative analysis of the SWMR system before and after the adoption of VMI and milk-runs is carried out. In particular, inventory management practices of the different scenarios are examined and their related costs are compared. The obtained results and the analysis of the solution strategy for deterministic demand have been published in Rahim et al. (2014a).

Another important contribution of this dissertation is the investigation of the multi-period stochastic inventory routing problem (MP-SIRP) where the retailers consume the product at a stochastic stationary rate. It models the IRP problem as a stochastic multi-period problem. More precisely, we consider a distribution system in which a fleet of homogeneous vehicles is used to distribute some products from a single warehouse to a set of retailers consuming it at stationary demand rates, during a finite horizon H of consecutive periods (days). The objectives are to determine optimal quantities to be delivered to the retailers, the delivery times, and the vehicle delivery routes in order to minimize the total distribution and inventory costs. The resulting distribution plan must prevent stock-outs from occurring at all retailers during the planning horizon and assuring some predetermined service level. Based on the formulation of the cyclic IRP model (see,

e.g., Aghezzaf et al. 2006) and Multi-period IRP (see, e.g., Zhong and Aghezzaf 2012), we formulate a stochastic linear mixed-integer model for this MP-SIRP. A deterministic equivalent approximation reformulation (MP-DAIRP $_{\alpha}$) of the problem is proposed. This latter proposed model also determines the optimal vehicle fleet size in each period. A Lagrangian relaxation method to solve the proposed MP-DAIRP $_{\alpha}$ is developed and thoroughly discussed. Some numerical experiments are established to evaluate the effectiveness of the proposed solution approach. The detailed results for modelling and solving MP-SIRP case have been published in Rahim et al. (2014b).

1.4 Outline of Thesis

The remainder of this dissertation is organized as follows. Chapter 2 presents a literature review of regular papers regarding inventory routing problems and their related variants. As mentioned above, Anderson et al. (2010) discusses and classifies the IRPs and their variants according to the length of the planning horizon. Without loss of generality, the classification of reviewed papers is also based on the same criteria, by separating the existing work into finite and infinite planning horizon problems. Since the second phase of our proposed approach requires the transportation costs to be minimized, we highlight some routing solution approaches of the VRP that allow us to design an efficient improvement construction-heuristic.

Chapter 3 analyses the effectiveness of an inventory management policy before and after the implementation of VMI and milk-runs in a single-warehouse, multiple-retailer and vendor managed inventory (SWMR-VMI) system. It begins with partitioning retailers into subsets in order to minimize the overall inventory costs of the system. Then, a VRP procedure is used to solve the routing in each of the retailer subsets with the objective of minimizing the travelled distance and hence the transportation costs. Finally, a comparative analysis of the SWMR system before and after the adoption of VMI and milk-runs is carried out.

Chapter 4 investigates the multi-period stochastic inventory routing problem (MP-SIRP) with the objective to minimize the total distribution and inventory costs. Firstly, we formulate a stochastic linear mixed-integer model for this MP-SIRP. Next, a deterministic equivalent approximation reformulation (MP-DAIRP $_{\alpha}$) of the problem is proposed. Then, a Lagrangian relaxation method is proposed to solve the MP-DAIRP $_{\alpha}$. As a result of this method, some numerical experiments are demonstrated to evaluate the effectiveness of the proposed solution approach.

Finally, some concluding remarks from recent research works are conveyed in Chapter 5. Apart from that, some directions for future research issues are discussed in this chapter.

2

General Literature Review

2.1 Introduction

Manufacturers these days are interested in developing competitive strategies for coordinating their inventory management and vehicle routing in supply chain management (SCM). These two issues have traditionally been dealt with separately, but their integration can have a dramatic impact on overall system performance (Campbell and Savelsbergh 2004). Consequently, the coordination of inventory replenishment and transportation has been studied extensively and many approaches have been developed to solve these two activities simultaneously, which is commonly referred to as inventory routing problems (IRP). The IRP is an important optimization model that captures the essential characteristics of vendor managed inventory (VMI) agreements such as inventory control and transportation scheduling.

Additionally, the IRP applications arise in a variety of industries. For instance, Campbell and Savelsbergh (2004) were inspired by the

international industrial gas company, Praxair, where one of their activities is to separate air into gases such as oxygen, hydrogen, nitrogen and argon. Then, these gases are transported in their liquid form via trucks from the plants to the customers. The IRP is also implemented in petrochemical industry, the automotive industry, suppliers of supermarkets and department store chains, clothing industry and home products. Another application of IRP is in the marine industry where ships are used instead of trucks, and several products are often being shipped in separate compartments. However, the inventory activities are considered as both the sources and the destinations among other factors (Moin and Salhi, 2007). The number of these industries seems to be increasing, along with the need for approaches to the IRP that handle the additional constraints and influential complexities present in practical versions of the problem.

The IRP is a very challenging problem that arises in various distribution systems. It involves managing simultaneously inventory control and vehicle routing in organizations where one or several warehouses are responsible for the replenishment of a set of geographically dispersed retailers. These retailers face a demand for products spread over time, and are entitled to keep local inventory. Deliveries are made using a fleet of capacitated vehicles.

In the IRP, there are no retailer orders, and the routing decisions are dictated by the inventory behaviour of the retailers, which in itself is driven by their (daily) demand patterns. Given the retailers' inventory data and information regarding the retailers' demand, the supplier must subsequently make several decisions over a given planning horizon:

- Which retailer to visit in each period of the planning horizon?
- What are the quantities to deliver to each retailer?
- How to combine these deliveries into routes?

The goal of the IRP is to minimize the distribution costs in the system, over the planning horizon without causing stock-outs at any of the retailers. Therefore, in this literature review, we highlight some

existing studies of the IRP, which allows us to understand that this designation encompasses a wide range of situations that calls for various solution methods.

Moreover, a large variety of IRP research has been proposed and discussed in the literature over the last decades. Baita et al. (1998) classified the IRP by defining it as a class of problems having the following aspects in common: routing (necessity to organize a movement of goods between different sites), inventory (relevance of the volume and value of the goods moved), and dynamic behaviour (repeated decisions have to be made). Within this class of problems, a classification framework was proposed that took into account all the characteristics of the different approaches encountered in the literature.

Various models of the IRP exist depending mainly on the nature of demand by the retailers (whether it should be treated as deterministic or stochastic), and on the length of the planning horizon either (finite or infinite). Therefore, in the remaining of this section, our intention is to classify the IRP models according to two key characteristics, as featured below:

- Length of the planning horizon, which may be either finite or infinite.
- Demand pattern, which can be either deterministic or stochastic.

Actually, Baita et al. (1998) defined the IRP is deterministic due to the fact that the retailers' consumption rates are assumed to be known and constant. However, when looking at the coordination of inventory controlling and transportation scheduling from a practical point of view, stochastic models might better describe many real life cases. Particularly, when discussing the regular IRP models, Andersson et al. (2010) classified the literature according to the length of the planning horizon, where the reviewed IRPs were separated into finite and infinite planning horizon problems, and also included both stochastic and deterministic cases inside this arrangement.

The purpose of this chapter is to develop a comprehensive review of the related inventory routing problem (IRP) literature. Figure 2.1 shows the classification of main literature on the IRP, which is divided into stochastic IRP and deterministic IRP. Consequently, we have organized our literature review into three parts. In the first part, we separate the classical configuration of IRP into finite and infinite planning horizon problems, with deterministic IRP, while in the second part, we focus on stochastic IRP. In the last part, we discuss an overview of the vehicle routing problem (VRP) because of our main proposed approach in the second phase of the problem that requires transportation costs to be minimized. For the conclusion in this chapter, we identify research gaps between the previous and current issues discussed in the IRP literatures.

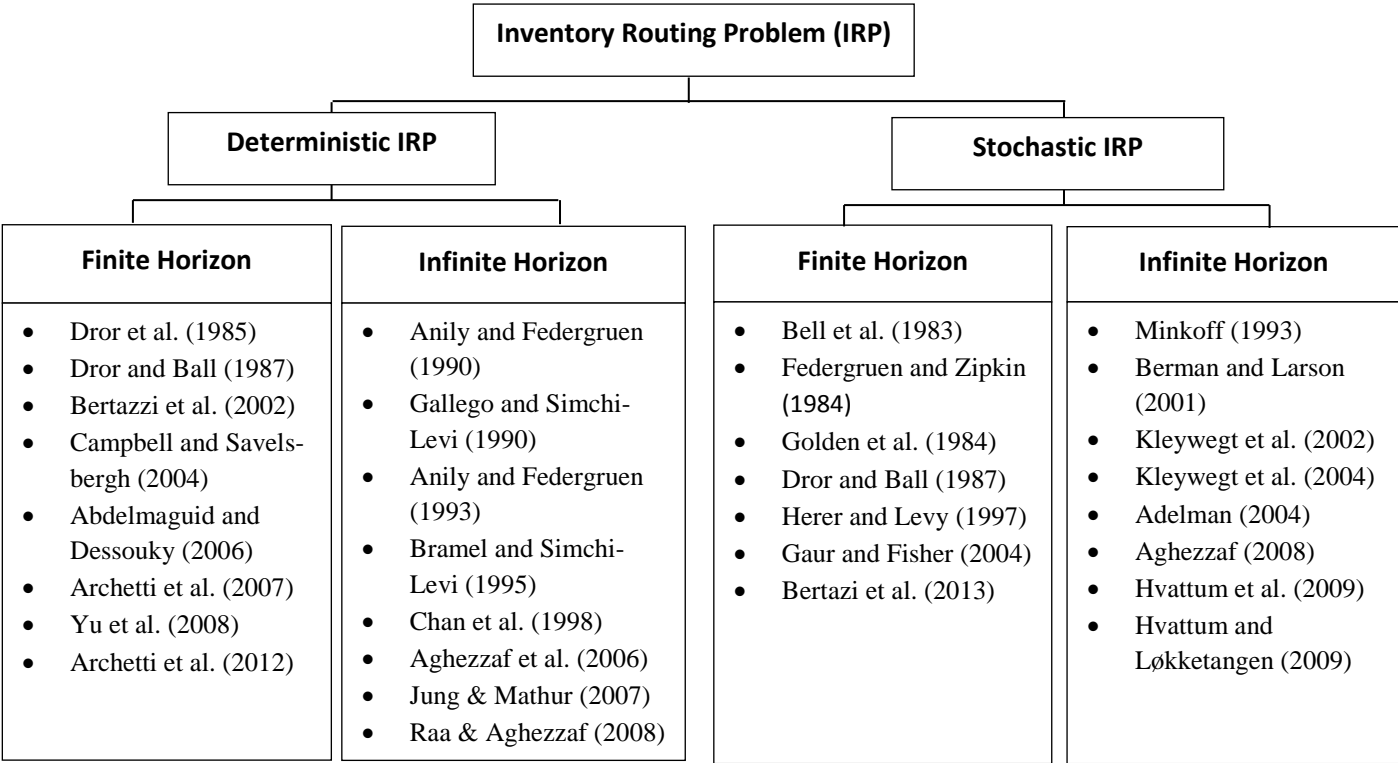


Figure 2.1 Classification of main literature on the Inventory Routing Problem (IRP)

2.2 Deterministic Inventory Routing Problem

We start by introducing the basic problem of the IRP. The problem is defined on a graph $G = (V, A)$ where $V = \{0, \dots, n\}$ is the vertex set and A is the arc set. Vertex 0 represents the supplier and the vertices of $V' = V \setminus \{0\}$ represent retailers. Both the supplier and retailers incur unit inventory holding costs h_j per period ($j \in V$), and each retailer has an inventory holding capacity C_j . The length of the planning horizon is p and, at each time period $t \in T = \{1, \dots, p\}$, the quantity of product made available at the supplier is r^t . We assume the supplier has sufficient inventory to meet all the demand during the planning horizon and those inventories are not allowed to be negative. The variables I_{0t} and I_{jt} are defined as the inventory levels at the end of period t , respectively at the supplier and at retailer j . At the beginning of the planning horizon, the decision maker knows the current inventory level of the supplier and of all retailers (I_{0t} and I_{jt} for $j \in V'$), and has full knowledge of the demand d_{jt} of each retailer j for each time in period t .

There is a set $K = \{1, \dots, K\}$ of vehicles available with capacity Q_k . Each vehicle is able to make one route per time period to deliver products from the supplier to a subset of retailers. A routing cost c_{ij} is associated with arc $(i, j) \in A$. The objective of the problem is to minimize the total inventory-distribution cost while meeting the demand of each retailer. The replenishment plan is subject to the following constraints:

- The inventory level at each retailer can never exceed its maximum capacity.
- Inventory levels at both the warehouse and the retailers are not allowed to be negative.
- The supplier's vehicles can perform at most one route per time period, each starting and ending at the warehouse.
- The vehicles' capacity cannot be exceeded.
- The final inventory level at each retailer at the end of the horizon shall cover its initial inventory.

The solution to the problem should determine which retailer is to be served in each time period using which of the supplier's vehicles, and how much to deliver to every visited retailer as well as which routes to be used.

2.2.1 Finite Horizon, Deterministic IRP

In the early 1980s some studies have started to incorporate inventory concerns within the existing vehicle routing literature. These were mostly variations of VRP models and heuristics developed to accommodate inventory costs. Most of the papers considered consumption rate at the retailers as known and deterministic. In a general setting, the IRP has a finite time horizon and a one-warehouse multiple-retailers inventory system dealing with a single product. The warehouse has enough goods to supply the retailers whose demands are known to the supplier at the beginning of the planning period. A homogeneous fleet of vehicles is available for the distribution of the problem and neither the warehouse nor the retailer faces any ordering or inventory costs. The objective is to minimize the distribution costs during the planning period without causing stock-outs at any of the retailers.

Dror et al. (1985) is among the earliest paper to address the IRP, and propose a short term solution approach to take into account what happens after the single day planning period. They described this problem over a short planning period, e.g. one week, and proposed a mixed integer programming model to display effects of present decisions on later periods. The solution is based on the assignment of retailers to their so called optimal replenishment period t , and then calculating the expected increase in cost if the retailer is visited in another period. The authors divided the problem based on two major assumptions: (1) once a retailer is visited, the amount of product delivered fills the retailer's capacity (order-up-to level policy), and (2) retailers are only visited once during the planning period (e.g. One week). Then, they created two subsets out of the retailers set, one containing retailers that must be visited where t falls within the short-term

planning period, and the other containing retailers that could be visited where t falls outside the short-term planning period.

They solved the problem in two phases as follows. (1) For the retailers that must be visited, they calculated the costs of visiting the retailers earlier than the latest period possible. (2) For retailers that could be visited, they computed the future cost difference between visiting and not visiting this retailer. Based on these costs, retailers are assigned to periods, and VRPs are solved for each period, followed by a node interchange improvement. In the solution approaches, the authors then proposed two algorithmic solutions. The first one assigns retailers to periods in a first step, and then solves a VRP for each period. Whereas, the other view is to assign retailers not only to periods, but also to vehicles, so that the second part needs only solve one travelling salesman problem (TSP) for each period and each vehicle. In the implementation of both solutions, an integer program is solved by assigning retailers to vehicles, minimizing transportation and inventory costs. However, in their model, the inventory holding costs are not included in the objective function. Only retailers who will reach their safety stock level during a particular time interval are serviced and the model only considered a fixed number of identical trucks. Their second part of algorithm works with the output of the first part, which was obtained heuristically and there is no guarantee of its quality.

Dror and Ball (1987) presented a procedure for reducing the annual optimization problem and selecting the set of retailers replenished in a short operational time period, with the objective to minimize annual costs subject to no retailer shortages. The relationship between the annual distribution cost, the fixed delivery cost, and the amount delivered to the retailers are examined and the retailers to be visited on a given day are selected according to these costs. They developed the conditions, which enable a transition from a stochastic problem to a deterministic problem in which the retailer's daily demand is treated as known. They clarified the problem by fixing the amount delivered to each retailer in order to fill up its inventory capacity using order up-to-level (OU) policy, and in this sense, the amount delivered only depends on the period of the delivery since their approach is determin-

istic. They have simplified real stochastic demands to a deterministic approach using three different variables: (1) to penalize early deliveries to retailers with sufficient inventory, (2) to motivate deliveries to retailers that are not restrictive (not required by the constraints), and (3) to identify retailers that must be served within the planning horizon. Finally, their solution involves assigning retailers to one period of the planning horizon through a generalized assignment algorithm, solving the VRP for each period of the planning horizon and then trying to improve the solution by promoting interchanges not only within routes but also within periods.

Bertazzi et al. (2002) addresses a multi-period model with a deterministic case in which a set of products is shipped from a common supplier to several retailers. For each product, a starting level of the inventory is given both for the supplier and for each retailer and the level of the inventory at the end of the time horizon can be different from the starting one. Therefore, each retailer should determine a lower and an upper level of the inventory of each product and can be visited several times during the time horizon. Every time a retailer is visited the quantity delivered is such that the maximum level of inventory is reached. This inventory policy is called the OU policy, decreasing the flexibility of the decision maker, but simplifying the set of possible decisions of the problem. The authors solved the problem heuristically in two steps. The first step creates a feasible solution, and the second one improves it as long as a given minimum improvement is made to the total cost function. This is achieved by removing all possible retailer pairs and computing a series of shortest paths to determine the periods in which the retailers should be reinserted. They considered both inventory and transportation costs and it is relevant to note that the supplier also incurs inventory costs in their model, which was generally not considered in other papers.

Campbell and Savelsbergh (2004) studied a multi-period IRP motivated by the application in the industrial gas industry, PRAXAIR, which is a large industrial gas company with about 60 production facilities and more than 10,000 retailers across North America. The authors propose a two-phase solution approach. In the first phase, they

determine which retailers receive a delivery on each day of the planning period and decide on the size of the deliveries. In the second phase, they then determine the actual delivery routes and schedules for each of the days. Their two-phase solution approach was based on the idea of a rolling horizon, which is typically used to solve the planning model and implement only the first period's decisions, update the model to reflect new information to resolve the model, and again implement the imminent decisions. In their rolling horizon framework, the authors planned on using only the first j days of the k day schedule being constructed. The value of j should be relatively small so as to be able to take advantage of new and updated information on inventory levels and usage rates. Then, they decomposed the solution process into two phases. A delivery schedule is created first, followed by the construction of a set of delivery routes. The first phase considered a coarse approximation of the problem, with daily decisions, over a k -day planning horizon. Based on the results of the previous phase, this second phase considered a model with decision accuracy in terms of minutes rather than days for the first j days, in which they solved a sequence of vehicle-routing problems with time windows (VRPTW).

Abdelmaguid and Dessouky (2006) propose a genetic algorithm (GA) approach for solving the integrated inventory distribution problem with multiple planning periods, in which backorders are permitted. Backorder decisions are generally established in two cases. In the first case is when there is insufficient vehicle capacity to deliver to a retailer, while the second case is when there is a transportation cost saving that is higher than the incurred backorder cost by a retailer. The authors designed a suitable genetic representation that focuses on the delivery schedule represented in the form of a 2-dimensional matrix and leave the vehicle routing part to be solved using any efficient polynomial time heuristic such as the savings algorithm. The problem is decomposed into one routing problem for each time period and one inventory problem. The link between the routing and inventory problem is the delivery quantities. These quantities are also the information in the chromosome. In the GA construction phase, they used a randomized version of a previously developed construction heuristic to generate the initial random population. In the GA improvement phase,

the authors then developed a suitable design which involves two random neighbourhood search mechanisms, the crossover and mutation operations. The crossover operator exchanges the delivery schedules for a set of retailers between two chromosomes. This will preserve the retailer inventory feasibility, but may introduce infeasibility in the vehicle capacity constraints. Once the infeasibilities are taken care of, a mutation operator that transfers part of the delivery quantities to different periods is applied.

Archetti et al. (2007) proposed an exact algorithm to the deterministic IRP over a given time horizon. Each retailer defines a maximum inventory level. The supplier monitors the inventory of each retailer and determines its replenishment policy, guaranteeing that no stock-out occurs at the retailer. These authors considered the case with only one vehicle, no backlogging and using the OU inventory policy, which is the quantity delivered by the supplier to the visited retailers is such that it reached the maximum inventory level. They developed a branch-and-cut algorithm and derived several valid inequalities to strengthen the linear relaxation of the model. The authors then compared the optimal solution of this problem, with the optimal solution of two problems obtained by relaxing in different ways the deterministic OU policy. The first problem is obtained by relaxing the constraint that every time a retailer is visited, the quantity delivered is such that the maximum level of the inventory is reached. The second problem is obtained by completely relaxing the OU policy, for instance by allowing the shipping quantity to be of any positive value. Computational results are presented on a set of randomly generated problem instances to evaluate the performance of the algorithm. They showed that by relaxing the constraints on the shipment quantities, substantial savings can be achieved.

Yu et al. (2008) consider a multi-period deterministic inventory routing problem with split delivery (IRPSD) where the retailers' demands in each period over a given planning horizon are assumed to be constant and must be satisfied without backorder. The delivery to each retailer in each period can be split and performed by multiple vehicles. In order to solve large scale instances, the authors proposed an ap-

proximate model for the multi-period IRP. The solution of such an approximate model only defines the quantity delivered to each retailer, the quantity transported through each directed arc and the number of times that each directed arc is visited by vehicles (i.e. a directed arc means the connection between two retailers or a retailer and the warehouse). A more robust Lagrangian relaxation method is developed to solve the relaxed problem, which is decomposed into an inventory problem, that can be solved by a linear programming algorithm and a routing problem, which is then transformed into a minimum cost flow algorithm. Moreover, in order to evaluate the quality of the final solution, a simple local search is used to further improve the quality of the routes, leading to a near-optimal solution of the IRPSD.

For recent study on the multi-period deterministic IRP, Archetti et al. (2012) consider the IRP in discrete time, where a supplier has to serve a set of retailers over a time horizon. A capacity constraint for the inventory is given for each retailer and the service cannot cause any stock-out situation. Two different replenishment policies are considered, the order-up-to level (OU) and the maximum level (ML) policies. To solve the IRP, the authors developed a powerful hybrid heuristic, which operates with a combination of a tabu search embedded within four neighbourhood searches and two mixed integer programming (MIP) models. They called the heuristic HAIR (hybrid approach to inventory routing). Their results show that the HAIR performs very well on small size instances, with the average error is 0.08% for the OU policy and 0.05% for the ML policy. On instances with up to 200 retailers and a horizon of $H = 6$ time units, the HAIR substantially improves the solutions obtained by a known heuristic for the case of the OU policy.

2.2.2 Infinite Horizon, Deterministic IRP

The following is a review of infinite horizon with deterministic IRP approaches. All the papers described in this section consider the same type of systems: a warehouse replenishes geographically dispersed retailers by a fleet of capacitated vehicles combining deliveries into efficient routes. These retailers face a constant, deterministic demand

rate. The appropriate objective is to determine long-term integrated replenishment strategies (i.e., inventory rules and routing patterns) that minimize the total transportation and inventory costs of the SWMR system. It is a strategy consisting of the construction of delivery routes, and the computation of the optimal replenishment frequency for each route.

Anily and Federgruen (1990) are among the fundamental contributions in the study of deterministic IRP with an infinite horizon. By using the same setting as above, they restricted that inventories must be kept at the retailers but not at the warehouse and an upper and a lower bound to the long run on average transportation and retailer inventory holding costs are determined. The authors considered a specific class of replenishment strategies which is fixed partition (FP) policies. The authors analysed an FP strategy for the IRP with constant deterministic demand rates and an unlimited number of vehicles. The class of replenishment strategies can be described as follows: the retailers are partitioned into regions and their demands are allowed to be split between several regions. The FP policy is a set of replenishment strategies where each time one of the retailers in a given region receives a delivery, this delivery is made by a vehicle that visits all other retailers in the region as well using an efficient route. This allowed the authors to transform this problem into a general partitioning problem, and to obtain several interesting results. Lower bounds on the optimal cost within the class are derived using dynamic programs to solve the partitioning problem and a heuristic based on a geographical division of the retailers is proposed.

Gallego and Simchi-Levi (1990) analyse the effectiveness of direct shipping strategies in the SWMR system. They assumed that the retailers face a constant demand rate and the inventory is only charged at the retailers but not at the warehouse. Their problem is to find an optimal inventory-routing strategy that consists of determining optimal re-order quantities and vehicle routes. The authors started with computing a lower bound of the system-wide cost over all inventory-routing strategies. An upper bound is identified for the case in which direct shipments are carried out by fully loaded trucks. The results

showed that the effectiveness of direct shipping is at least 94% whenever the economic lot sizes exceed 71% of the vehicle capacity. In other words, direct shipping may turn out to be an effective alternative to more complex strategies when the economic lot sizes for all retailers are close to the vehicle capacity. Moreover, it is stated that the error of direct shipping increases as the lot sizes decrease.

Anily and Federgruen (1993) consider an extension to their previous work by Anily and Federgruen (1990) to the case where the central warehouse is explicitly considered as a stock-keeping location. They considered the same problem as Anily and Federgruen (1990) but proposed a power-of-two policies, where the replenishment intervals are power-of-two multiples of the base planning period. The problems are compounded by determining a replenishment strategy for the warehouse, optimally coordinated with each of the retailers and synchronized with the transportation schedules. It is observed that the gap between the cost of the proposed strategy and a lower bound for the minimum cost is bounded by 6% for large numbers of retailers, which is usually better than the gap seen in the system without central inventories. In addition, they briefly explained how their results can be extended to cases where back-logging is allowed if all retailers face identical demand rates.

Bramel and Simchi-Levi (1995) present a general framework for solving several different routing problems. They applied the algorithm to two classical problems: the capacitated vehicle routing problem (CVRP) and the IRP. They then introduced a new heuristic for general routing problems, which is based on formulating the routing problem as a location problem commonly called the capacitated concentrator location problem (CCLP). This location problem is subsequently solved and the solution is transformed back into a solution to the routing problem. For solving the IRP that involves a more complex cost structure, they implemented fixed partition (FP) policies, in which the set of retailers is partitioned into disjoint subsets and each subset is served separately. As long as a retailer in a subset is visited, all other retailers in the set are visited as well. With that, the optimal deliveries occur at regular fixed intervals, and the optimal cycle time is deter-

mined by using the traditional economic order quantity (EOQ) formula. Then, a location based heuristic method is used to solve the problem, in which some seed sets of the retailers are elected in the first phase, then a CCLP is solved in the second phase, and in the final phase, the solution to the CCLP is transformed to a feasible solution to the IRP.

Chan et al. (1998) studied the zero inventory ordering (ZIO) policies and also the fixed partition (FP) policies in the SWMR system. The main objective of their study is to characterize the asymptotic effectiveness of the class of ZIO policies and the class of FP policies. Under the ZIO policy, a retailer is replenished if and only if their inventory level reaches zero. ZIO policies may fail to be optimal, with the presence of constraints on the vehicle capacity or the frequency with which retailers can be served. Under the FP strategy, a set of retailers is partitioned into a number of regions so that each region is served separately and independently from all other regions. Besides, a heuristic algorithm is developed for partitioning the retailers into regions. The FP policies are constructed by using a two-step procedure: (1) partition a given area where the retailers are distributed into sub-regions, (2) partition the retailers in each such sub-region into sets of retailers by solving a bin-packing problem with each set being served in an efficient way. Then, they computed a lower bound, built a fixed partition solution and gave a probabilistic analysis of the optimal gap for this solution. Computational results show that the algorithm is very effective on a set of randomly generated problems and it is seen that the gap between the heuristic solution (the upper bound) and the lower bound is less than 19%.

Aghezzaf et al. (2006) propose a model that extends the concept of vehicle tours to vehicle multi-tours, which is allowing vehicles to perform more than one route per period in order to minimize the number of vehicles used. The authors discussed a special case of the long-term IRP in which a single warehouse, supplying a single product, serves a set of retailers implementing economic order quantity (EOQ)-like policies to manage their inventories. They presented the cyclic inventory routing problem (CIRP), which is a vehicle can repeatedly travel

along the same routes to replenish retailers. The objective of their model is to minimize total fleet operation, inventory holding and distribution costs. To present their model, the authors defined some concepts with regards to 'cycle times', such as the minimal cycle time, the maximal cycle time and the EOQ cycle time (i.e. the theoretical optimal cycle time). They pointed out that for a tour to be feasible, it is necessary that the minimal cycle time does not exceed the maximal cycle time. Moreover, the EOQ cycle time might be not actually feasible and it could turn out to be greater than the maximal cycle time or smaller than the minimal cycle time. Thus, in this case, the actual optimal cycle time has to be chosen as close as possible to the EOQ cycle time, so it will be exactly the maximal (or minimal) cycle time. When the EOQ cycle time falls between the minimal and maximal cycle times, it is chosen to be the actual cycle time of the multi-tour. They then proposed a nonlinear mixed-integer programming model for the considered CIRP. Four different sets of decision variables are contained in the model: (1) to determine whether a vehicle is used or not, (2) to determine the movements of the vehicles, (3) to correspond to the amount delivered to each retailer, (4) to determine the cycle time. For the solution of the problem, they implemented a column generation for creating new multi-tours (columns), in which subproblems are solved using a savings-based approximation method. They tested this method to compare solutions obtained with the usual model, in which only routes or trips made of one tour is permitted. As a result, an average saving on the total cost rate ranging between 12% and 16% is achieved when using multi-tours (columns) model.

Jung and Mathur (2007) propose a power-of-two partitioning policies that partition retailers into a number of clusters, each of which is served separately and independently from all other clusters. The problem is to make joint inventory and routing decisions so as to minimize long-run average cost. The proposed model allows (1) vehicles with limited capacity and (2) both the warehouse and retailers to maintain inventories. The authors developed an efficient heuristic procedure that finds a re-order interval for the warehouse and each retailer, as well as the vehicle routes to deliver the associated quantities so as to minimize the long-run average inventory and routing cost. They then

decomposed the problem into the following three problems: (1) *Clustering*: partition the retailers into subgroups to be replenished by a single vehicle. (2) *Sequencing*: For each subgroup in (1), specify a nestedness sequence which implies that the reorder interval of a higher indexed retailer is not lower than the reorder interval of a lower indexed retailer. (3) *Inventory policy*: given the clustering and the nestedness sequence, determine the reorder intervals for each retailer and the warehouse. Finally, they compared the performance of the proposed algorithm to some of the possible alternative heuristics. In general, the computational results showed that their algorithm outperformed other methods and resulted in a cost savings of 2% - 45%.

Later, Raa and Aghezzaf (2008) extend the concept of multi-tour presented in Aghezzaf et al. (2006), and adopted the distribution pattern to represent tours by assigning possibly different frequencies to different tours. As a result, a cyclic-planning approach is adopted, in which a vehicle can repeatedly travel along the same routes to replenish retailers. The authors pointed out that a short tour to retailers with high demand rates can be performed more often than a longer tour to retailers with lower demand rates. Four nested tasks have to be tackled in this problem: (1) partitioning retailers over vehicles by a column generation approach, (2) partitioning retailers of a vehicle over different tours by a greedy heuristic, (3) determining tour frequencies by an iterative procedure, (4) scheduling the deliveries by a greedy algorithm so as to determine the optimal cycle time and the resulting total cost rate. To evaluate the performance of their proposed solution approach, a large number of tests were run on problem instances with varying characteristics.

Table 2.1 Classification of main papers on single item deterministic IRP with SWMR

Authors	Year	Time horizon		Routing			Inventory policy			Fleet composition		Fleet size		
		Finite	Infinite	Direct	Multiple	Continuous	Lost sales	Backlogging	Non-negative	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Dror et al.	1985	✓			✓		✓			✓			✓	
Dror and Ball	1987	✓			✓		✓			✓			✓	
Anily and Federgruen	1990		✓		✓		✓			✓				✓
Gallego and Simchi-Levi	1990		✓	✓			✓			✓				✓
Anily and Federgruen	1993		✓		✓		✓			✓				✓
Bramel and Simchi-Levi	1995		✓		✓		✓			✓				✓
Chan et al.	1998		✓		✓		✓			✓			✓	
Bertazzi et al.	2002	✓			✓				✓	✓		✓	✓	
Campbell and Savelsbergh	2004	✓			✓		✓			✓		✓	✓	
Abdelmaguid and Dessouky	2006	✓			✓			✓			✓		✓	
Aghezzaf et al.	2006		✓		✓		✓			✓			✓	
Archetti et al.	2007	✓			✓		✓		✓	✓		✓		
Jung and Mathur	2007		✓		✓		✓			✓				✓
Yu et al.	2008	✓			✓		✓			✓			✓	
Raa and Aghezzaf	2008		✓		✓		✓			✓			✓	
Archetti et al.	2012	✓			✓				✓	✓		✓		

Source: Adapted from Coelho et al. (2014)

2.3 Stochastic Inventory Routing Problem

Many studies consider the IRP with dynamic deterministic demand, which leads to more tractable yet less realistic models compared to those with stochastic demand. On the other hand, stochastic inventory routing problem (SIRP) models are intractable in that only very small instances can be solved optimally (Hvattum and Løkketangen 2009). The IRP defined above is deterministic and static because consumption rates are fixed and known beforehand. In real life the supplier does not always know in advance exactly how much each retailer will consume (stochastic demand), nor is this consumption static (dynamic demand). The basic idea behind the SIRP is the same as in the deterministic IRP, except that the level of realism and the difficulty of solving the problem are increased, given that some data are known only in a probabilistic sense and realizations of such data are revealed gradually to the decision maker. The unknown data can be the demand, the travelling time, the travelling cost, etc. It is easy to observe that many characteristics of the problem are stochastic in real life. These include demand, travelling times, vehicle loading and unloading times, even the availability of the road network.

In the SIRP, instead of knowing the consumption rate for each retailer, the supplier knows (or estimates) a probability distribution for r consumption. In this sense, the problem is no longer deterministic and future demands are uncertain. In the classical version of the SIRP, retailer demands are mutually independent. The stochasticity added to the problem creating a probability that shortages might occur. In order to discourage shortages, a penalty is imposed whenever a retailer runs out of stock, and this penalty is usually modelled as being proportional to the amount of unsatisfied demand. Unsatisfied demand is typically considered as lost demand, that is, there is no backlogging. Since decisions are made based on partially available information, decisions can lead to expensive course-correcting measures.

The knowledge of the supplier with respect to the dynamic problem can differ according to the problem at hand. The data can be completely unknown and periodically revealed, but usually the supplier

knows the information in some statistical way, such as a probability distribution estimated from historical data. The objective of the SIRP remains the same as in deterministic case, in which to minimize the overall inventory and transportation costs in the SWMR system, while avoiding retailers' stock-outs. The supplier must determine a distribution policy that accommodates the stochastic and unknown future parameters with a finite or infinite planning horizon.

2.3.1 Finite Horizon, Stochastic IRP

Bell et al. (1983) are among the pioneer researchers to discuss the stochastic IRP, to use forecasts to make the stochastic demands seem deterministic to the model. They proposed a linear programming model to solve a deterministic simplification of the problem. They used heuristics to generate forecasts of the unknown demand. Possible delivery routes are created heuristically, and continuous variables represent the amount to be delivered to the retailers. They solved a mixed integer programs with up to 800,000 variables and 200,000 constraints to near optimality. Due to the immense size of the model, Lagrangian relaxation is applied, after which the problem is decomposed into sub-problems (one for each vehicle) and an upper bound is obtained for the problem. A heuristic based on the Lagrangian relaxation approach is used to find a feasible solution (a lower bound) to the problem. It was observed that the gap between the upper and the lower bounds is at most 2% in the computational experiments performed. However, the heuristics used to generate forecasts was very simple, based on a simple exponential smoothing model tested with only 10 different values for the smoothing parameter.

Federgruen and Zipkin (1984) adopted the first approach to accommodate inventory and shortage costs in a random demand environment. The authors extended some of the available methods for the deterministic vehicle routing problem (VRP) to the case where the demand of the different retailers is considered as a random variable. They analysed a problem where the inventory levels are known to the supplier, who has to decide how much of a scarce resource to deliver to each retailer and how to route the fleet of vehicles. After the deliv-

eries are made, the demand is realized and the inventory, transportation and stock-out costs are computed. The authors proposed a nonlinear mixed-integer programming model. The solution approach is based on the observation that if the second set of variables is fixed, the problem decomposes into an inventory allocation problem and one travelling salesman problem (TSP) for each vehicle. The algorithm constructs an initial set of routes, with a feasible assignment of retailers to vehicles, and then calculates changes, using a modified interchange heuristic based on the r -opt methods of the VRP. In the second part, an exact algorithm based on generalized Benders' decomposition is proposed to solve the problem. They tested their heuristic algorithm with 50 retailers and 75 retailers problem and their results showed that a saving of 6% - 7% can be achieved in operating costs and consequently reducing the number of required vehicles by no less than 20%.

Golden et al. (1984) developed a heuristic for the optimization of an integrated delivery planning system for a large energy-product company that distributes liquid propane. They used a threshold to decide the retailers with the aim to minimize the daily operational costs, while attempting to ensure a sufficient level of products at each retailer location. Based on degrees of urgency, all retailers with inventory below a given threshold were considered as potential retailers to be visited. To be more specific, the approach undertaken is as follows: for each retailer, an 'emergency level' equal to the ratio of his current inventory level to his tank capacity is computed. All the retailers whose emergency levels are higher than a chosen critical level are designated as 'potential' retailers. Retailers are then ranked using the ratio of emergency to delivery cost, and a TSP is then iteratively built. The ranked retailers are added one at a time to the itinerary, until the total tour duration exceeds a pre-established maximum duration, T_{\max} . The tour is then split into routes. If no feasible solution is found, T_{\max} is decreased, and the procedure is repeated. Results from the simulated comparison of the proposed heuristic showed that the heuristics had a superior performance. The number of gallons/hour delivered was improved by 8.4%, with the number of stock-outs reduced by 50% and total costs were reduced by 23%.

Dror and Ball (1987) presented a procedure for reducing the long-term problem into short period problem that can be solved with the use of standard routing algorithms. The reduction procedure considers the definition of single-period costs that reflects long-term costs, the definition of safety stock level and a specification of the retailer subset to be considered during a single period. The authors constructed a replenishment routes, by taking into account the probability distribution function (PDF) of the retailers' demands. The authors used results based on one retailer, a deterministic demand system to compute 'incremental costs' incurred during the year-long planning whenever a retailer is replenished in the coming week before his inventory drops to zero. Using these incremental costs, as well as the costs charged for stockouts and the demand PDF of each retailer, the authors then computed the expected cost $E_i(t)$ for replenishing a specific retailer i on any day t . Under some assumptions, they showed the existence of t^* , the optimal replenishment day, that minimizes $E_i(t)$. In order to solve the problems, they developed a four-step heuristic: (1) retailers to be included in the coming week's schedule are selected based on their t^* . (2) a linear-programming-based on generalized assignment algorithm is solved to assign the retailers to delivery days. (3) a modified version of Clarke and Wright algorithm is used to build efficient routes for each day during the time period. (4) local improvements are made to obtain better solution.

Herer and Levy (1997) built and evaluated a heuristic for the metered inventory routing problem (MIRP). The MIRP involves a central warehouse, a fleet of trucks with a finite capacity, and a set of retailers, for each of whom there is an estimated consumption rate, and a known storage capacity. In the standard IRP, the retailer pays for the delivery in full when it is made (note that the timing of the delivery is determined by the supplier). In contrast, under the MIRP, the retailer pays for the inventory he uses, as he uses it. Thus, under the MIRP formulation, the supplier and not the retailer, pays for the inventory held at the retailer. The authors developed a heuristic method for determining the amount of capacity to be installed at each retailer. The problem is solved on a rolling horizon basis, taking into consideration holding, transportation, fixed ordering, and stock-out costs. They used

the concept of temporal distances, which was taken by the traditional Clarke and Wright algorithm and modified it by adding the temporal distance into the savings calculation. The temporal distance gives the solution method some flexibility by allowing retailers to be joined in the same route even when assigned to different periods. A simulation study is carried out to demonstrate the effectiveness of the procedure.

Gaur and Fisher (2004) analysed a periodic version of the problem where they assume a time-varying demand repeated over a week long period. They considered the fixed partitioning (FP) policy, in which the set of retailers is partitioned into disjoint subsets (called regions or clusters), with each region served separately and independently from all other regions. Because demand is time varying, they allowed two types of routes within each cluster: (1) shared routes that visit every store in the cluster with low-volume day, (2) direct shipments to individual stores in the cluster with high-volume day. The authors solved the routing part problem by a randomized sequential matching algorithm, with two main ideas: (1) repeated application of the generalized minimum weight matching algorithm, (2) randomized splitting of clusters, whereas the inventory management part is handled by stating a maximum time between deliveries. The computational results demonstrated that the potential cost savings on the inventory routing module and additional savings were obtained in truck assignment and workload balancing.

Recently, Bertazi et al. (2013) formulated the Stochastic IRP using dynamic programming, with a goal of minimizing the total inventory, distribution and shortage costs. An order-up-to level policy is applied to each retailer and an inventory cost is applied to any positive inventory level, while a penalty cost is charged and the excess demand is not backlogged whenever the inventory level is negative. They designed a hybrid rollout algorithm aimed at finding better quality solutions of the problem. Rollout algorithms are a class of heuristic algorithms that can be used to solve deterministic and stochastic dynamic programming problems. This algorithm applied a sampling approach to generate demand scenarios for the current period and considered the average demand for future ones. The authors then compared the hy-

brid rollout algorithm with the classical benchmark algorithm. The computational results showed that the rollout algorithm significantly dominates the benchmark algorithm, in particular when the variance increases. Moreover, the rollout algorithm tends to perform better than the benchmark algorithm in instances with a higher inventory cost.

2.3.2 Infinite Horizon, Stochastic IRP

Minkoff (1993) proposes a heuristic approach based on a Markov decision model to a problem somewhat similar to the IRP, called the delivery dispatching problem (DDP), which is too tough to solve because of its dynamic and stochastic nature. The problem is to determine a procedure for deriving itinerary assignments for each available vehicle that will minimize the long-run average costs of operating the system. He simplified the objectives function, making it a sum of smaller and simpler objective functions, one for each retailer and solved the problem heuristically. The heuristic is based partly on a decomposition of the problem by retailer, where retailers' sub-problems generate penalty functions that are applied to a master dispatching problem. He then described how to compute bounds on the algorithm's performance, and applied it to several examples with good results. The heuristic resulted in substantial reductions of computation time for determining dispatches, while maintaining good quality in dispatching performance. The author also provided a method for calculating bounds on the degree of sub-optimality of the heuristic for particular instances.

Berman and Larson (2001) use dynamic programming to solve the case where the demand probability distributions are known, adjusting the amount of goods delivered to each retailer so as to minimize the expected costs, comprising costs of earliness, lateness, product short-fall, and returning to the warehouse nonempty. The unique structures of their approach are as follows: (1) modelling the product usage and emptiness as stochastic processes which is called a Wiener process model; (2) providing incremental costs for early deliveries as well as late deliveries; (3) allowing the amount of product delivered to be determined by the driver which is not known until the driver is on scene

at the retailers' location, at which point the retailer is either restocked to capacity or left with some residual empty capacity, which is determined by stochastic dynamic programming. The authors have proposed four different versions of the dynamic program for solving the problems, utilizing two different state variables and allowing the possibility of 'dumping' excess product on the homeward leg of the tour. Certain properties of the dynamic programming model were derived that allow rapid and efficient computation of optimal delivery policies.

Kleywegt et al. (2002) formulated a Markov decision process for the stochastic inventory routing problem with direct deliveries (IRPDD), which consider only one delivery per trip, with the objective to maximize the expected total discounted value over an infinite horizon. They included constraints for the number of vehicles available, and allowed only direct deliveries. Immediate reward functions are composed of individual retailer immediate rewards (revenue minus the sum of travel, inventory and shortage costs), and an optimal expected value is the total discounted sum of all rewards. For their solution, they proposed an approximate dynamic programming approach, and studied the impact of the number of retailers, the number of vehicles, and the coefficient of variation of retailer demand. In general, computational experiments were conducted to demonstrate the practicability of using dynamic programming approximation methods for the IRPDD, and the optimal solutions were obtained on instances with up to 60 retailers and up to 16 vehicles. They concluded that taking available information about the future into account, through dynamic programming approximation methods, provides more benefits if the available information about the future is more accurate.

Kleywegt et al. (2004) extended the formulation and the approach from their previous research in Kleywegt et al. (2002), to handle multiple deliveries per trip using a Markov decision process. They used the same model but in this case they limited the routing to at most three retailers per route, instead of using one delivery per trip. The problem is formulated to maximize the expected discounted value over an infinite horizon as a discrete time Markov decision process. Retailer demands are stochastic and independent from each other, and

the supplier knows the joint probability distribution of their demands, which does not change over time. Although the demands are stochastic, the cost of each decision is known to the supplier. Thus, the authors took into account travelling costs, shortages that are proportional to the amount of unsatisfied and lost demand and holding costs. These models considered the revenue is proportional to the quantities delivered. They presented a solution approach that uses decomposition and optimization to approximate the value function. Specifically, the overall problem is decomposed into smaller sub-problems, and an optimization problem is defined to combine the solutions of the sub-problems in such a way that the value of a given state of the process is approximated by the optimal value of the optimization problem. Computational experiments demonstrated that their approach allows the construction of near optimal policies for small instances and these policies were better than their earlier policies that were proposed in their previous research. In this study, they obtained their optimal solutions on instances with up to 15 retailers and five vehicles.

Adelman (2004) considers a new approach to Stochastic IRP that approximates the future costs of current actions using optimal dual prices of a linear program. In the paper, the author did not restrict the number of retailers to be served in a route, except for the limits resulting from maximal route duration and vehicle capacity. He used the approximation, but takes a different approach. Instead of obtaining the approximate value of retailer i , V_i , through a heuristic sequential procedure, he obtained them as optimal dual prices from either of two linear programming relaxations of the underlying control problem. The control policy is called price directed because it uses these optimal prices to approximate future costs. His linear program takes into account inventory dynamics and economics simultaneously, rather than sequentially, allocating transportation costs and solving all local dynamic programs. He then implicitly optimizes overall itineraries that are generated by solving nonlinear discrete knapsack problems. During test instances, he compared the price directed policy against the other policies in the literature. The computational results showed that the price directed policy dominates all other policies in every instance. The median performance of the price-directed policy was 7%

from the lower bound, compared with 24.6% for Minkoff's procedure, 29.6% for the myopic policy, and 44.1% for direct shipment (which assigns exactly one retailer per route).

Aghezzaf (2008) studies the case where retailer demand rates and travel times are stochastic but have constant averages and bounded standard deviations. The author used robust optimization to determine the cyclic distribution plan through a non-linear mixed integer programming formulation which is feasible for all possible realizations of demand and travel times. To guarantee that a generated cyclic distribution plan is robust, it is enhanced with two simple and effective components. The first one is related to the use of 'fixed' safety stocks, which is reserved at the retailers. These safety stocks are used to hedge against any possible increase in demand rates and/or travel time. The second component is the 'mobile' safety stock, which is an extra amount of the product carried by the vehicle during each of its tours. This 'mobile' stock is used to guarantee that no additional or extra replenishments will take place to exclusively restore any particular safety stock. Monte Carlo simulation is used to improve the plan's critical parameters such as replenishment cycle times and safety stock levels. The results show that when the travel time variability increases the number and the magnitude of the stock-outs increases, nevertheless the plan remains practically robust even at a travel time variability level of 10% on average. Moreover, the magnitude of stock-outs increases from 1.2% of the realized demand to more than 20% when the travel time variability increases to 20% on average.

Hvattum et al. (2009) proposed a policy for the markov decision process (MDP) that maximizes the expected total discounted value over an infinite horizon, where the value is based on rewards and costs associated with the process. These rewards/costs are calculated based on four components, which include the travel costs, the holding costs at retailers' locations, penalties for stock-outs at the retailers, and possibly some revenue gathered for each delivery that is made. Even though their problem has an infinite planning horizon, most of the stochasticity can be captured in a finite scenario tree. In order to solve the problems, the authors created a scenario tree to capture the sto-

chastic IRP, formulated a scenario tree problem (STP) for a given state (i.e., a given set of inventory levels), and then estimated that the solution of the STP will correspond to decent decisions with respect to the underlying MDP. The main heuristic investigated in this paper is based on greedy randomized adaptive search procedure (GRASP), which successively increases the volume delivered to retailers. In GRASP, each iteration involves the construction of a solution from scratch, making greedy choices that are somewhat randomized in order to get different results from each iteration. The evaluations of the possible choices that can be made during the construction are updated as new solution evolves, and hence it is said to be adaptive.

Hvattum and Løkketangen (2009) describe the stochastic IRP by a discounted, infinite horizon markov decision problem. The authors investigated the progressive hedging algorithm (PHA) for solving the scenario tree based problems. This algorithm can be suitable for a large scale problem, by giving an effective decomposition, but there are no guarantees of convergence for non-convex problems. The PHA was examined as an alternative method for solving the sub-problems arising for each epoch under the scenario tree based regime. The problem is modelled as a discounted, infinite horizon MDP, and the optimal policy is thus the replenishment strategy that minimizes the total cost. In the MDP, the following sequence of events is assumed. First, the current inventory at each retailer can be observed. Second, vehicle routes are constructed and delivery is made. Third, the actual demand is observed and the inventory levels are updated, taking into account possible stock-outs. Holding costs are then calculated after demand is observed. The state of this MDP is uniquely identified through the current inventory levels of each retailer, and the transition probabilities are given once an action has been selected. In order to improve the solution processes, the standard algorithm is extended with locking mechanisms, dynamic multiple penalty parameters, and heuristic intermediate solutions. The results obtained are interesting. For some problems, the PHA produces good results alone, while on others the combination of the PHA and a previously developed GRASP in Hvattum et al. (2009) gives a much more robust result than any of the methods used separately.

Table 2.2 Classification of main papers on single item stochastic IRP with SWMR

Authors	Year	Time horizon		Routing			Inventory policy			Fleet composition		Fleet size		
		Finite	Infinite	Direct	Multiple	Continuous	Lost sales	Backlogging	Non-negative	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Bell et al.	1983	✓			✓		✓				✓		✓	
Federgruen and Zipkin	1984	✓			✓		✓				✓		✓	
Golden et al.	1984	✓			✓		✓			✓			✓	
Dror and Ball	1987	✓			✓		✓			✓			✓	
Minkoff	1993		✓		✓		✓			✓				✓
Herer and Levy	1997	✓			✓		✓				✓		✓	
Berman and Larson	2001		✓		✓		✓			✓		✓		
Kleywegt et al.	2002		✓	✓			✓			✓				✓
Gaur and Fisher	2004	✓			✓		✓			✓			✓	
Kleywegt et al.	2004		✓		✓		✓			✓			✓	
Adelman	2004		✓		✓		✓			✓			✓	
Aghezzaf	2008		✓		✓		✓			✓			✓	
Hvattum et al.	2009		✓		✓		✓			✓			✓	
Hvattum and Løkketangen	2009		✓		✓		✓			✓			✓	
Bertazi et al.	2013	✓			✓		✓			✓		✓		

Source: Adapted from Coelho et al. (2014)

2.4 Vehicle Routing Problem

The vehicle routing problem (VRP) creates a central role of distribution management. Most of companies and organisations have engaged in the delivery and pick-up of products every day. The problem is concerned with the construction of a plan that consists of trips, starting from a central warehouse, for vehicles servicing retailers with known demand. The fundamental objectives are to find the minimal number of vehicles, the minimal travel time or the minimal costs of the travelled routes. In other words, the VRP is used to design optimal routes for a fleet of vehicles to service a set of retailers, given a set of constraints. In practice, the objectives and constraints of the VRP are highly flexible because conditions differ from one setting to the next such as e.g. vehicle capacity or time interval in which each retailer has to be served, revealing the capacitated vehicle routing problem (CVRP) and the vehicle routing problem with time windows (VRPTW) respectively. The real-world problems mostly encompass the capacity and time constraints. In the last decades, many extensions of the basic VRP have been studied and most algorithmic research and software development in this area focus on a limited number of prototype problems.

Dantzig and Ramser (1959) are among the first to describe and define the VRP, as follows:

A number of identical vehicles with a given capacity are located at a central depot. They are available for servicing a set of customer orders, (all deliveries, or, alternatively, all pickups). Each customer order has a specific location and size. Travel costs between all locations are given. The goal is to design a least cost set of routes for the vehicles in such a way that all customers are visited once and vehicle capacities are adhered to.

The authors introduced the VRP to the research community and presented the first heuristic for the problem. Basically, this algorithm iteratively matches vertices, or vertices and partial routes, to form a set of vehicle routes. The VRP can be formally defined as follows. Let G

$= (V, A)$ be a graph where $V = \{v_0, v_i, \dots, v_n\}$ is a vertex set, and $A = \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$ is an arc set. Vertex v_0 represents a warehouse, while the remaining vertices correspond to retailers. Also, A is associated with a cost matrix (c_{ij}) and a travel time matrix (t_{ij}) . If these matrices are symmetrical, as is commonly the case, then it is standard to define the VRP on an undirected graph $G = (V, E)$, where $E = \{(v_i, v_j) : v_i, v_j \in V, i < j\}$ is an edge set. Each retailer has a non-negative demand q_i and a service time t_i . A fleet of m identical vehicles of capacity Q is based at the warehouse. The number of vehicles is either known in advance or treated as a decision variable.

The VRP consists of designing a set of at most m delivery or collection routes in such a manner that:

- Each route starts and ends at the warehouse.
- Each retailer is visited exactly once by exactly one vehicle.
- The total demand of each route does not exceed vehicle capacity, Q .
- The total duration of travel and service time on each route does not exceed a preset limit D .
- Total routing cost is minimized.

Nowadays, much progress has been made in this area and several variants of the basic routing problems have been put forward. Strong formulations have been proposed, together with polyhedral studies and exact decomposition algorithms. Numerous heuristics have also been developed for solving the VRP. For example, in the real world application that solves VRP, it is important to perform a fast selection of methods such as constructive heuristics, neighbourhood operators and escaping mechanisms, which produces the desired improvement on the objective function.

Therefore, in this study, we give an overview of the most important heuristics for the VRP literature, and highlight some of the solution approaches that inspired us when designing the heuristic described in the next chapter. Consequently, in the following section, we examine three different classes of classical heuristics which are constructive

methods, two phase algorithms and tour improvement heuristics. The constructive methods produce an approximately optimal tour for the distance matrix. The two phase algorithms, is to cluster first-route second procedures, whereby during the first phase, it groups or clusters demand nodes and then designs optimal routes in the second phase. The tour improving heuristics attempt to discover a better tour given an initial one.

2.4.1 Constructive Methods

Construction methods were among the first heuristics for the VRP and still form the principal of many software implementations for various routing applications. These construction algorithms start from an empty solution and iteratively build routes by inserting one or more retailers at each iteration, until all the retailers are routed. Construction algorithms are further subdivided into sequential and parallel, depending on the number of eligible routes for the insertion of a retailer. Sequential methods expand only one route at a time, whereas parallel methods consider more than one route simultaneously.

The most commonly used heuristic is the algorithm of Clarke and Wright (1964). This algorithm is based on a savings concept that is an estimate of the cost reduction obtained by serving two retailers sequentially in the same route, rather than in two separate ones. For example, the savings obtained by merging routes $(0, \dots, i, 0)$ and $(0, j, 0)$ are equal to $s_{ij} = c_{i0} + c_{0j} - c_{ij}$. The purpose of this algorithm is to cluster the retailers into routes. The savings algorithm finds pairs of retailers for which it is beneficial to combine them in a route and links as many of the pairs as possible. In the heuristic, a list of retailer pairs is generated and sorted in a descending order according to the savings. Then, from the top of the sorted list of retailer pairs, one pair of retailers is considered at a time. Each time when a pair of retailers' i - j is considered, it is evaluated if the two routes that visit i and j should be combined. Only if this can be done without deleting a previously established direct connection between two retailer pairs, and if the total demand on the resulting route does not exceed the vehicle capacity restriction, then the routes are combined. This algorithm is inherently

parallel since more than one route is active at any given time. However, it may easily be implemented in a sequential fashion.

Several enhancements have been proposed in order to improve the effectiveness of the savings method, (1) multiplying c_{ij} by a positive weight λ (Golden et al. 1977), (2) accelerating the savings computation (Paessens 1988), (3) making use of efficient data structures to speed up the computations (Nelson et al. 1985), (4) optimizing the route merges in a global fashion through the use of a matching algorithm (Altinkemer and Gavish 1991, Wark and Holt 1994). The first enhancement by Golden et al. (1977) is useful in avoiding circumferential routes that tend to occur in the original Clarke and Wright algorithm. The next two enhancements by Paessens (1988) and Nelson et al. (1985) are probably of little use these days given the state of the art in computer technology and improvement heuristics. The last enhancement presented by Altinkemer and Gavish (1991) and Wark and Holt (1994) is used to solve matching problems, which is highly time consuming and not worth the effort in comparison with other heuristics.

There exist other insertion heuristics for obtaining a feasible routing plan for the VRP. Two well-known methods are those proposed by Mole and Jameson (1976) and Christofides et al. (1979). Starting from an empty plan, the routes are established by iteratively inserting visits, which will incur the smallest additional costs. Mole and Jameson (1976) implemented a sequential version, in which only one route is constructed at a time. The selection of retailer is based on the extra distance resulted from the insertion of the retailer to the route, and the distance between the retailer and the warehouse. While Christofides et al. (1979) developed a more effective two-phase insertion heuristic using both sequential and parallel route constructions. In the first phase, a set of feasible routes are determined. In the second phase, a set of single-retailer routes are defined based on the routes obtained in the first phase. The remaining unrouted retailers are then inserted according to the difference between the best and the second-best insertion cost.

Another constructive method to obtain feasible routing plan is the iterative insertion (Golden 1991). The algorithm starts from an empty plan, where routes are developed by iteratively inserting visits that will incur the smallest additional cost. The insertion procedure takes a sub-tour of k nodes and attempts to determine which node should join the sub-tour next and then determines where in the sub-tour it should be inserted. The most known of these algorithms is the nearest insertion algorithm. The nearest insertion algorithm starts with a sub-graph consisting of only one node, i . Then, the algorithm defines node k such that c_{ik} is minimal and forms the sub-tour $i-k-i$. In the selection step, the algorithm will find again node k which is not already in the sub-tour. Thus, in the insertion step, this algorithm finds the arc (i, j) in the sub-tour which minimizes $c_{ik} + c_{kj} - c_{ij}$.

2.4.2 Two Phase Algorithms

In the two-phase heuristics, solving the VRP is decomposed into two parts, clustering retailers into subsets, each of which corresponds to a route, and routing the retailers in each subset. According to the order of solving these two parts, the heuristics can be divided into cluster-first-route-second methods and route-first-cluster-second methods.

In the cluster-first-route-second methods, retailers are first grouped into clusters and the routes are then determined by suitably sequencing the retailers within each cluster. Different clustering strategies are proposed in the literature. Gillett and Miller (1974) developed a sweep algorithm which divides retailers into clusters by rotating a ray centred at the warehouse. The approach explores a solution in two phases. It starts with assigning retailers to vehicles, and then decides on the sequence in which each vehicle visits the retailers assigned to it. This approach explores the retailers circularly, in increasing polar angle around the warehouse. In this order, each retailer is successively inserted at the end of the current route until the vehicle capacity constraint is reached. If this insertion is not feasible, then a new route is initiated. A vehicle route is optimized by solving the corresponding TSP.

Fisher and Jaikumar (1979) have probably developed the best-known cluster-first-route-second algorithm. In the first step, the algorithm chooses m retailers to be the seeds to the cluster zones, and a vehicle is allocated to each of these retailers. Then, it computes the insertion costs of adding each retailer j to each cluster k . Next, the clusters are created by solving a generalized assignment problem (GAP) based on the retailer weights. Once the clusters have been determined, the TSPs are solved optimally using a constraint relaxation based approach.

Some of the procedures for selecting the seeds are described in Bramel and Simchi-Levi (1995). This algorithm determines route seeds by solving a capacitated location problem, where m retailers are selected by minimizing the total distance between each retailer and its closest seed, and by imposing that the total demand associated with each seed be at most Q . Once seeds have been determined and the single-retailer routes are initialized, the remaining retailers are inserted in the current routes by minimizing insertion costs.

Foster and Ryan (1976) and Renaud et al. (1996) presented a so called petal algorithm, which first generates a large number of feasible routes and then selects the final subset by solving a set partitioning problem. The overall performance of petal algorithms is generally superior to that of the sweep algorithm.

In most of the route-first-cluster-second methods, a giant TSP tour over all the retailers is constructed and then decomposed into feasible vehicle routes. The basic route-first-cluster-second method was first put forward by Beasley (1983), where a central warehouse is surrounded by a number of retailers. The author first formed a 'giant tour' from the warehouse around all the retailers and back to the warehouse. This tour can be formed in a number of different ways. The key to the approach is that it is very easy to optimally partition such a tour into a set of feasible vehicle routes.

Another example of such algorithms can be found in Haimovich and Rinnooy Kan (1985) and Bertsimas and Simchi-Levi (1996), but they are not competitive with the cluster-first-route-second methods.

2.4.3 Tour Improving Heuristics

Given a solution, for example, generated by construction heuristics, we can apply some modifications to the solution to improve its quality. A large number of improvement heuristics have been proposed for this purpose, such as moving a retailer from one route to another, exchanging two retailers' positions in the solution in order to obtain neighbour solutions of possibly better cost. According to the number of routes modified at a time, the algorithms can be divided into intra-route operators, which work on a single route, and inter-route operators, which modify multiple routes at the same time.

One of the famous intra-route operators and the best-known improvement heuristics for VRP are the λ -opt heuristics. This λ -opt exchange is very simple and useful. It involves exhaustively considering exchanges of retailers in different routes. The 2-opt method was introduced by Lin (1965) and consists of eliminating two edges from the current solution and examining the ways to reconnect them. If a cost-saving combination is found, it is implemented. The procedure is repeated until no improvement is obtained. Lin (1965) also developed the 3-opt heuristic method which is quite similar to the 2-opt method, but it introduces more flexibility in modifying the existing solution. Firstly, it removes three edges from a tour and then reconnects the resulting three paths in order to form a tour. The procedure stops at a local minimum where no further improvement can be obtained.

Another intra-route operator is the Or-opt method proposed by Or (1976) which consists of displacing three, two or one edges to another cheaper location, until no improvement can be found. The interchanges allowed are thus restricted to 3-opt interchanges, where three arcs are removed and replaced by other three arcs. The solution is obtained when no insertion of arcs can decrease the cost of the tours any further.

Van Breedam (1994) classified the inter-route operators into four groups: i.e. string cross that exchanges two chains of nodes by crossing two edges, string exchange that exchanges two chains of nodes, string relocation that moves a chain of nodes to another route and string mix that consists of both string exchange and string relocation. In the literature, the string relocation with one single-vertex chain, which is also called insertion move, is very frequently used due to its simplicity, cheap computational cost and robustness. It can be viewed as a fundamental component undertaken by most operators. For example, swapping two nodes can be implemented by two insertion moves.

2.5 Conclusions

In the class of deterministic IRP, the majority of the models presented in the literature above optimize the inventory holding costs at the retailers but not at the warehouse. Thus, the model examined in this dissertation attempts to take into account all inventories and their related costs at the warehouse as well as at the retailers. Therefore, we extend the SWMR model proposed by Roundy (1985) and Chu and Leon (2008). In general, their models only considered the transportation costs as fixed costs. This means that there is no harmonization between inventory and transportation costs. Accordingly, we have to optimize the overall inventory and transportation costs at the warehouse as well as the retailers while satisfying some additional restrictions. In order to integrate these problems, some effective routing optimization procedures for VRP need to be used to design an efficient heuristic for the SWMR system. More details on these methods will be discussed in the next chapter.

The multi-period IRP case has been studied before in particular by Zhong and Aghezzaf (2012). They solved the multi-period IRP with deterministic demand rates. In this dissertation, we have extended their work to consider a multi-period stochastic inventory routing problem (MP-SIRP) where the retailers consume the product at a stochastic stationary rate. The IRP is stochastic when the demands of retailers are uncertain. Even though the stochastic nature of the IRP

input data, deterministic models have been extensively considered. The main reason is that, besides the complexity of the stochastic models solution approaches, it is often difficult to obtain the necessary information to derive probability distribution that represent the problem correctly. To solve the MP-SIRP, most research works have focused on heuristic solution approaches due to its NP-hard complexity. Furthermore, some methods need to be applied to decompose the integrated problem into sub-problems in order to determine upper and lower bounds. More details on the methods for the MP-SIRP will be discussed in Chapter 4.

3

Analysis of the Single-Warehouse, Multiple-Retailer System Operating Under VMI

3.1 Introduction

Vendor managed inventory (VMI) is an integrated inventory management policy in which the supplier assumes, in addition to its own inbound inventory, the responsibility of maintaining inventory at the retailers, and ensuring that they will not run out of stock at any moment. The replenishment of the retailers is thus no longer triggered by retailers placing orders, but instead it is the supplier who determines when each of the retailers is replenished, and what the replenishment quantities are. The supply is thus proactive, as it is based on the available inventory information, instead of being reactive, in response to retailers' orders. This proactive policy has many advantages for both the supplier and the retailers. On one hand, the supplier has the possibility to combine multiple deliveries to optimize truck loading and to minimize transportation costs. Moreover, since the supplier has direct information about retailers' demand, deliveries will become more uniform and predictable. As a consequence, the amount of inventory that must be held at the supplier can be drastically reduced. On the

other hand, retailers do not need to dedicate resources to the management of their inventories any longer. Furthermore, the service levels towards the retailers (i.e., product availability) can increase, as the supplier can track inventory levels and subsequently take into account the replenishment urgency.

VMI has gained popularity thanks to the availability of many technologies that enable to monitor retailer inventories in an online and cost-effective manner. Inventory data can be made accessible much easier. However, this does not guarantee that implementing VMI always leads to improved results. Failure can happen, for example due to the unavailability of sharing the right pieces of information, or due to the inability of the supplier to make the right decisions. These two problems have to be solved in an integrated manner when implementing VMI, which only adds to the complexity of the situation where managing inventory in a supply chain and optimizing transportation between stages are already particularly challenging problems.

In this chapter, we consider a two-stage supply system with deterministic demand, operating under VMI. In particular, we focus on the single-warehouse, multiple-retailer (SWMR) case in which a supplier serves a set of retailers from a single warehouse. We assume that all retailers face a deterministic, constant demand rate. Deliveries to these retailers are made from the warehouse with a fleet of vehicles having a limited capacity. The warehouse in turn is replenished from an outside source. Incoming shipments into the warehouse have to be coordinated with outgoing shipments to the retailers in order to minimize the total cost. This total cost consists of inventory holding costs at the central warehouse and all retailers, costs for incoming shipment into the warehouse, and outbound shipment costs for the retailer replenishments. The optimization problem of minimizing the total of inventory and transportation costs encountered in this VMI system is known in the literature as the inventory routing problem (IRP).

This SWMR case has been studied before in particular by Roundy (1985) and Chu and Leon (2008), amongst others. However, they assumed that only direct shipping is used to replenish the retailers, i.e.,

each vehicle visits a single retailer and returns to the warehouse. Even under this assumption, it is shown that the problem cannot be solved in polynomial time.

We propose a two-phase heuristic solution approach to minimize the overall inventory and transportation costs of the SWMR system under a VMI policy. In the first phase, retailers are partitioned into subsets in order to minimize the overall inventory costs of the system. Then, in the second phase, a vehicle routing problem (VRP) procedure is used to solve the routing in each of the retailer subsets with the objective of minimizing the travelled distance and hence the transportation costs. As such, we drop the assumption of direct shipments from warehouse to retailers, but also include the option of combining multiple outbound shipments in so-called *milk-runs* (see Figure 3.1). To evaluate the impact of VMI and milk-runs on the SWMR system, a comparative analysis of the SWMR system before and after the adoption of VMI and milk-runs is carried out. In particular, inventory management practices in the different scenarios are examined and their related costs are compared.

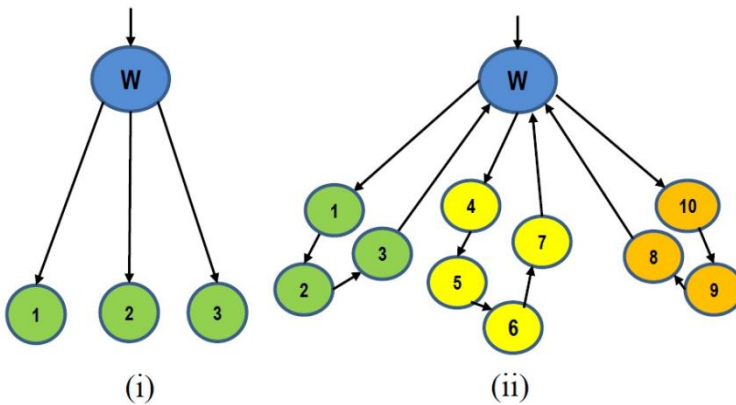


Figure 3.1. (i) A direct shipping tour (ii) A *milk-runs* tour

The remainder of this chapter is organized as follows. Section 3.2 reviews the model and solution approaches related to SWMR-VMI. Section 3.3 presents the formal description of the integrated SWMR-

VMI deterministic model. Section 3.4 reviews existing direct shipping solutions and Section 3.5 describes the proposed two-phase approach for milk-run solutions. A detailed analysis of an illustrative supply chain example is given in Section 3.6 whereas Section 3.7 provides conclusions and directions for future research.

3.2 Review of Models and Solution Approaches related to SWMR-VMI

An important stream of research related to the SWMR is the one that takes transportation costs explicitly into account. Federgruen and Zipkin (1984) were probably among the first to integrate the inventory allocation and routing problems. They have considered a plant with a limited amount of available inventory serving a set of retailers with random periodic demand rates. The objective of their model is to allocate available inventory in the warehouse to the retailers in a way that minimizes total transportation costs at the end of the period. They modelled the problem as a nonlinear mixed integer program, and proposed an approximation method for its solution. Federgruen et al. (1986) extended the work by Federgruen and Zipkin (1984) to the case in which the product considered is perishable. Chien et al. (1989) simulated a multiple period planning model based on a single period approach and formulated it as a mixed integer programming problem. Campbell et al. (1998) studied a two-phase heuristic based on a linear programming model. In the first phase, they calculated the exact visiting period and quantity to be delivered to each retailer. Then, in the second phase, retailers are sequenced into vehicle routes. Bertazzi et al. (2002) proposed a fast local search algorithm for the single vehicle case in which an Order-up-to level (OU) inventory policy is applied. Aghezzaf et al. (2006) formulated a model for the long-term IRP when demand rates are stable and economic order quantity-like policies are used to manage the inventory of the retailers. The authors then proposed a column generation based heuristic. Other examples of recent contributions in the SWMR system were carried out by Aghezzaf (2008), Solyali and Sural (2011), Solyali and Sural (2012), Aghezzaf et al. (2012) and Rahim and Aghezzaf (2012), Archetti et al. (2012), Coelho et al. (2012a, 2012b), Coelho and Laporte (2013a, 2013b).

In the context of replenishment strategies, Gallego and Simchi-Levi (1990, 1994) considered a direct shipping approach whereby every vehicle visits only a single retailer during every one of its trips. Retailers have deterministic demand rates and no shortages or backlogs are allowed. They assumed that a sufficient number of vehicles, each with a limited capacity, is available and that the storage capacities at the retailers are sufficiently large. Kim and Kim (2000) also examined a direct shipping method, but allowed for more than one trip per vehicle and time period. They formulated the problem as a mixed integer linear program and proposed a Lagrangian heuristic to solve it. More recent works in the direction of direct shipping strategies with deterministic demand can be found in Zhao et al. (2007), Bertazzi (2008), Li et al. (2008), Li et al. (2010). Barnes-Schuster and Bassok (1997) extended the direct shipping strategy to the case of independent stochastic stationary demand rates. Through simulation studies, they demonstrated that when the truck capacities are close to the mean of the demand, direct shipping strategy performs well. Kleywegt et al. (2002) developed an approach that is designed for a different setting in which vehicle routes are limited and only allowed for direct shipping. They introduced and modelled this as a Markov decision process and developed an approximate dynamic programming method in order to obtain good quality solutions with a reasonable computational effort. Direct shipping strategies are shown to be effective alternative to more complex strategies when the economic lot sizes for all the retailers are close to the capacities of the vehicles. However, it may not be the ideal policy when many retailers require significantly less than a vehicle load.

When a direct shipping strategy is proven ineffective, a milk-run approach should be considered, where each vehicle serves multiple retailers in one delivery (one route). For this reason, most studies concentrated on a special type of distribution policies called fixed partition (FP) policies, as they are easily implemented and effective in many situations. Anily and Federgruen (1990, 1993) are among the first to adopt the ideas of FP policy. They analysed the replenishment strategy where the set of retailers is partitioned into regions and each region is served independently. If a retailer in some region is visited,

all retailers in that region are visited. Viswanathan and Mathur (1997) extended the work of Anily and Federgruen (1990) for the multi-period, multi-product problem. They presented a new heuristic that generates a stationary nested joint replenishment policy for the problem with deterministic demands. Then, they adopted a power-of-two policy and the results showed that when the replenishment periods are power-of-two multiples of a common base planning period, better performance can be achieved. Chan et al. (1998) investigated the effectiveness of the class of FP policies and zero inventory ordering (ZIO) policies, and constructed an effective algorithm resulting in an FP policy that is asymptotically optimal. Jung and Mathur (2007) extended the replenishment strategy discussed in Anily and Federgruen (1993) by allowing a different reorder intervals for each retailer in a cluster. They developed a three-step heuristic and the solution is rounded to fit the power-of-two policy constraints. Interesting studies in this research stream are found in Chan et al. (2002), Anily and Bramel (2004), Gaur and Fisher (2004), Zhao et al. (2008), Raa and Aghezzaf (2008, 2009), Chu and Shen (2010) and Bertazzi et al. (2013).

An extension of this research line is concerned with models that involve location-inventory network design, integrating the location and inventory decisions. Barahona and Jensen (1998) studied a practical distribution network design problem for computer spare parts. Their model takes into account the inventory costs at the various warehouses. Erlebacher and Meller (2000) developed an analytical model to minimize the total fixed operating costs and inventory holding costs incurred by warehouses, together with the transportation costs. Shen et al. (2003) and Daskin et al. (2002) considered a case where retailers face uncertain demands following a poisson distribution. Shen et al. (2003) studied a facility location problem in which the facilities manage their inventory through an (r, Q) policy, while Daskin et al. (2002) presented an efficient solution based on Lagrangian relaxation approach. Meanwhile, Shu et al. (2005) solved the problem for general demand distributions. Shen and Qi (2007) defined a model for the stochastic supply chain design problem. Ozsen et al. (2008) introduced the capacitated warehouse location model with risk pooling

(CLMRP), which captures the interdependence between capacity issues and the inventory management at the warehouses. Chen et al. (2011) studied a reliable inventory-location model to optimize facility location decisions, allocation of retailers and management of inventory in case warehouses are at risk of disorder. More recent contributions in this research area are found in Tancrez et al. (2012), Berman et al. (2012), and Hamedani et al. (2013). In all these models, the inventory holding costs at the warehouse are ignored. The model examined here does not consider the design issue. However, it takes into account all the inventories at the warehouse as well as at the retailers.

In this chapter, we extend the SWMR model proposed by Roundy (1985) and Chu and Leon (2008) to allow for travel cost optimization. Roundy (1985) introduced two new types of policies, namely, integer-ratio policies and power-of-two policies. Power-of-two policies are a subset of the class of integer-ratio policies in which each facility orders at a power-of-two multiple of a base planning period. Roundy has shown that for the SWMR inventory model, the cost rate of the optimal power-of-two policy is within 6% of the cost percentage of any feasible policy. This result has made power-of-two policies very attractive. The complexity of each of the two policies developed by Roundy (1985) is $O(n \log(n))$, where n is the total number of retailers.

Chu and Leon (2008) considered the same problem as Roundy and proposed a solution method which only considers feasible power-of-two policies. Instead of successively checking whether the optimal reorder period of the warehouse falls within a certain interval Roundy (1985), Chu and Leon (2008) proposed a method that takes advantage of the property and that the total average cost of the system is convex.

The approaches proposed by Roundy (1985) and Chu and Leon (2008) regarded the transportation costs as fixed costs. This means that there is no coordination between retailers to minimize the transportation and fleet costs. Therefore, in order to integrate the inventory management and routing cost optimizations, we extended these approaches to include routing optimization. Some effective routing optimization procedures for VRP were used to design an efficient heuris-

tic for the SWMR. Laporte et al. (2000) classified the constructive techniques for solving the VRP into two main groups. The first group consists of methods that combine existing routes using a savings method, and the second group consists of techniques assigning vertices to vehicle routes using an insertion cost. In this chapter, we adopt the savings heuristic developed by Clarke and Wright (1964) and the improvement heuristic developed by Lin (1965) for the routing part of the problem.

3.3 The Integrated SWMR-VMI Deterministic Model

In this Section, the two-echelon single-warehouse multiple-retailer vendor managed inventory (SWMR-VMI) system is formally described in a mathematical model. This model enables to attain the optimal system order policy, i.e. minimizing the sum of all operational costs. For the model development, let R be the set of retailers, indexed by j , and $R^+ = R \cup \{0\}$ the set of facilities, where 0 indicates the warehouse. We also define the following parameters:

- t_{ij} : trip duration from facility $i \in R^+$ to facility $j \in R^+$;
- τ_{ij} : transportation cost from facility $i \in R^+$ to facility $j \in R^+$;
- φ_0 : fixed ordering cost incurred by the warehouse each time it places an order; the ordering cost is assumed independent of the order quantity;
- φ_j : fixed cost per delivery to retailer $j \in R$; the delivery cost is assumed independent of the replenishment quantity;
- h_0 : inventory holding cost rate per unit per period in warehouse 0;
- h_j : inventory holding cost rate per unit per period at retailer j ;
- d_j : constant demand rate per period faced by retailer j .

A solution to the problem is an *order policy*, which is described as the time between consecutive replenishments, or the replenishment interval, for all facilities in R^+ . All these replenishment intervals will be a power-of-two multiple of a base planning period, denoted as T_B .

Furthermore, for any retailer j , either of the two following cases must hold: (1) the retailer's replenishment interval, denoted T_j , is a power-of-two multiple of the warehouse's replenishment interval, denoted by T_0 , or (2) vice versa, that is T_0 is a power-of-two multiple of T_j .

Case 1: T_j is a power-of-two multiple of T_0

In the first situation, replenishments of retailer j (with a replenishment quantity of $d_j T_j$) can always be initiated at the moment when an inbound shipment in the warehouse arrives. As a result, the warehouse serves as a cross-dock and never holds any inventory destined for that retailer. The resulting inventory cost rate for retailer j in this first case is denoted IC_j^1 , and is given by:

$$IC_j^1 = \frac{\varphi_j}{T_j} + \frac{1}{2} h_j d_j T_j \quad (3.1)$$

Case 2: T_0 is a power-of-two multiple of T_j

In the second situation, a replenishment of retailer j (with a replenishment quantity of $d_j T_j$) can only be initiated at the moment an inbound shipment in the warehouse arrives every T_0/T_j times. The other times, replenishments are made from inventory in the warehouse. As a result, the warehouse does hold inventory for that retailer. The resulting inventory cost rate for retailer j in this second case, denoted IC_j^2 , is then given by:

$$IC_j^2 = \frac{\varphi_j}{T_j} + \frac{1}{2} h_j d_j T_j + \frac{1}{2} h_0 d_j (T_0 - T_j) \quad (3.2)$$

Thus, given all the replenishment intervals, the total inventory cost rate IC is:

$$IC_{MR} = \frac{\varphi_0}{T_0} + \sum_{j \in R} \left(\frac{\varphi_j}{T_j} + \frac{1}{2} h_j d_j T_j + \frac{1}{2} h_0 d_j [\max(T_0, T_j) - T_j] \right) \quad (3.3)$$

The second element in the total cost rate is the transportation cost rate. When milk-runs are used, decisions have to be made about clustering retailers and designing a trip per cluster, i.e., a VRP has to be solved. We assume that all the retailers in the same cluster have the same replenishment interval. The notation used for the milk-runs is the following: V is the set of available vehicles, indexed by v ; R^v is the cluster of retailers served by vehicle $v \in V$; Trip^v is the (shortest possible) milk-run trip that visits all retailers in R^v ; $\tau^v = \sum_{(i,j) \in \text{Trip}^v} \tau_{ij}$ is the transportation cost of making Trip^v ; and T^v is the replenishment interval of all retailers in R^v . The transportation cost rate when using milk-runs, TC_{MR} , is then given by:

$$TC_{MR} = \sum_{v \in V} \frac{\tau^v}{T^v} \quad (3.4)$$

The total cost rate TCR_{MR} , which is the sum of the inventory cost rate IC_{MR} and the transportation cost rate TC_{MR} , can then be written as follows:

$$TCR_{MR} = \frac{\varphi_0}{T_0} + \sum_{v \in V} \left(\frac{\tau^v + \sum_{j \in R^v} \varphi_j}{T^v} + \sum_{j \in R^v} \frac{h_j d_j}{2} T^v + \sum_{j \in R^v} \frac{h_0 d_j}{2} [\max(T_0, T^v) - T^v] \right) \quad (3.5)$$

For any Trip^v visiting cluster R^v with interval T^v to be feasible however, two conditions have to be met. First, the time between consecutive iterations, i.e., the interval T^v , must be longer than the duration of the trip, which results in a lower bound for the interval T^v , denoted T_{min}^v :

$$T^v \geq T_{min}^v = \sum_{(i,j) \in \text{Trip}^v} t_{ij} \quad (3.6)$$

Second, the total quantity delivered to all retailers in the trip cannot exceed the vehicle capacity k , which results in an upper bound for the interval T^v , denoted T_{max}^v :

$$T^v \leq T_{max}^v = \frac{k}{\sum_{j \in R^v} d_j} \quad (3.7)$$

Apart from the first term, TCR_{MR} (3.5) is separable per cluster/vehicle, and therefore, the intervals T^v can be optimized individually. The two possible cases identified above reappear here.

Case 1: $T^v \geq T_0$

In this case, the warehouse never holds inventory for retailers in R^v , and the last term of the cost rate function is zero. The interval value T^{v*} that minimizes the cost rate for vehicle v is as follows:

$$T^{v*} = \sqrt{\frac{2 \left(\tau^v + \sum_{j \in R^v} \varphi_j \right)}{\sum_{j \in R^v} h_j d_j}} \quad (\geq T_0) \quad (3.8)$$

Case 2: $T^v < T_0$

In this case, the warehouse does hold inventory for retailers in R^v , and the last term of the cost rate function is non-zero. The interval value T^{v*} that minimizes the cost rate for vehicle v is then:

$$T^{v*} = \sqrt{\frac{2 \left(\tau^v + \sum_{j \in R^v} \varphi_j \right)}{\sum_{j \in R^v} (h_j - h_0) d_j}} \quad (< T_0) \quad (3.9)$$

Since there is also a minimum and maximum value for the interval T^v , the optimal interval T_{opt}^v is as follows:

$$T_{opt}^v = \begin{cases} T^{v*}, & \text{if } T_{\min}^v \leq T^{v*} \leq T_{\max}^v \\ T_{\min}^v, & \text{if } T_{\min}^v > T^{v*} \\ T_{\max}^v, & \text{if } T^{v*} > T_{\max}^v \end{cases} \quad (3.10)$$

The problem to be solved is then to partition the set of retailers R into feasible clusters R^v , design a minimum cost trip per cluster, determine integer values for all T^v and T_0 , such that the total cost rate TCR_{MR} is minimized. To solve this SWMR-VMI problem efficiently, we propose an algorithm that combines a solution method for the direct shipping with an effective heuristic for the VRP as explained below.

3.4 Review of Classical Direct Shipping Solutions

To solve the SWMR-VMI problem, firstly, we describe the modelling algorithms developed in Roundy (1985) and the extension developed by Chu and Leon (2008).

3.4.1 Roundy's Algorithm

For the case of direct shipping, all retailers are in separate clusters, and the total cost rate TCR_{DS} is:

$$TCR_{DS} = \frac{\varphi_0}{T_0} + \sum_{j \in R} \left(\frac{\tau_{0j} + \tau_{j0} + \varphi_j}{T_j} + \frac{h_j d_j}{2} T_j + \frac{h_0 d_j}{2} [\max(T_0, T_j) - T_j] \right) \quad (3.11)$$

Apart from the first term, TCR_{DS} (3.11) is separable per retailer, and therefore, the intervals T_j can be optimized individually. Again, there are the same two possible cases.

Case 1: $T_j \geq T_0$

The interval value τ'_j that minimizes the cost rate for retailer j is as follows:

$$\tau'_j = \sqrt{\frac{2(\tau_{0j} + \varphi_j + \tau_{j0})}{h_j d_j}} \quad (3.12)$$

Case 2: $T_j < T_0$

The interval value τ_j that minimizes the cost rate for retailer j is then:

$$\tau_j = \sqrt{\frac{2(\tau_{0j} + \varphi_j + \tau_{j0})}{(h_j - h_0)d_j}} \quad (3.13)$$

It is easy to verify that $\tau'_j \leq \tau_j$.

Since TCR_{DS} is convex in T_0 , the optimal solution to the relaxed problem (without integer-ratio or power-of-two restrictions) given T_0 , is the following:

$$T_j = \begin{cases} T_0 & \text{if } \tau'_j \leq T_0 \leq \tau_j \\ \tau'_j & \text{if } T_0 < \tau'_j \\ \tau_j & \text{if } \tau_j < T_0 \end{cases} \quad (3.14)$$

Roundy's Algorithm starts by assuming that T_0 falls within the leftmost interval. After finding the optimal T_j based on (3.14) for all retailer j , optimal T_0 can be calculated by solving the relaxed problem (3.11). This procedure is repeated by successively assuming T_0 falls within each interval on the right until the calculated optimal T_0 falls within the same interval, in which the optimal solution is found. The algorithm introduced by Roundy (1985) assumes that no shortage or backlogging is permitted. Without loss of generality, replenishment is assumed to be instantaneous. Moreover, the base planning period T_B is assumed fixed, and only power-of-two policies are employed, i.e., the order intervals are all power-of-two multiples of T_B :

$$T_0 = 2^{k_0} T_B \quad k_0 \geq 0 \text{ and integer} \quad (3.15)$$

$$T_j = 2^{k_j} T_B \quad k_j \geq 0 \text{ and integer, } \forall j \quad (3.16)$$

3.4.2 Chu and Leon's Solution Algorithm

The algorithm proposed by Chu and Leon (2008), which extends the method developed in Roundy (1985), is the algorithm we will adopt in the direct shipping phase of our solution procedure. The method starts by allowing T_0 be a power-of-two of T_B . The proposed method then finds the corresponding optimal power-of-two multiples T_j for each retailer j , and calculates the corresponding total cost rate of the system. Then, T_0 is iteratively increased to the next power-of-two period until the total cost rate of the system increases. At this point, the optimal power-of-two policy is found (see Figure 3.2).

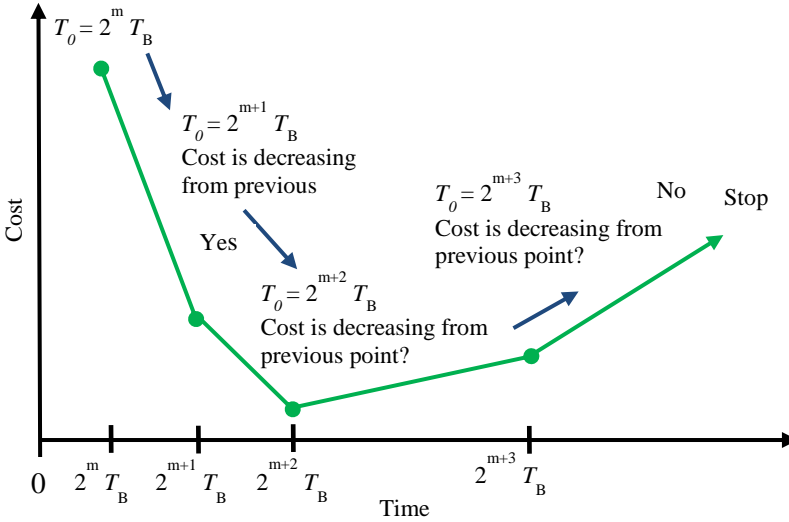


Figure 3.2. Steps of the algorithm of Chu and Leon (2008).

The optimal power-of-two solutions, denoted t'_j and t_j , are obtained by rounding the optimal solutions τ'_j and τ_j to the nearest power-of-two multiples of T_B .

Thus, this algorithm has proven that for a given T_0 , the optimal power-of-two policy is given by Chu and Leon (2008):

$$T_j = \begin{cases} T_0 & \text{if } t'_j \leq T_0 \leq t_j \\ t'_j & \text{if } T_0 < t'_j \\ t_j & \text{if } t_j < T_0 \end{cases} \quad (3.17)$$

Based on (3.17), and the fact that TCR_{DS} is convex in T_0 , Chu and Leon proposed an iterative heuristic that monitors the changes in total cost rate if interval T_i is used instead of interval T_j . The heuristic for the SWMR system is summarized as follows (see Chu and Leon, 2008):

- Calculate $\tau'_j = \sqrt{2(\tau_{0j} + \phi_j + \tau_{j0})h_j d_j}$ and $\tau_j = \sqrt{2(\tau_{0j} + \phi_j + \tau_{j0})(h_j - h_0)d_j}$, $\forall j \in R$ and round these to the nearest power-of-two to obtain t'_j and t_j . Find $t_{\min} = \min\{t'_j : j \in R\}$ and $t_{\max} = \max\{t_j : j \in R\}$. Let $i = 0$, $T_0 = t_{\min}$, $T^0 = \{\emptyset\}$, and $TCR_{DS}(T^0) = \infty$.
- Choose T_j according to condition (3.17). Set $i = i + 1$. Let $T^i = \{T_0, T_1, \dots, T_n\}$ and calculate $TCR_{DS}(T^i)$ using (3.11). If $TCR_{DS}(T^i) < TCR_{DS}(T^{i-1})$, go to Step 2. Otherwise, the best power-of-two policy has been found and is given by $T^* = T^{i-1}$. Stop.
- If $T_0 < t_{\max}$, set $T_0 = 2T_0$ and go back to Step 1. Otherwise, the optimal T_0 is in the range $[t_{\max}, \infty]$. For any $T_0 \geq t_{\max}$, the optimal T_j remain the same as the values last calculated (in Step 1). Therefore, given these optimal $T_j, \forall j, T_0$ can be found by first minimizing (3.11) with respect to T_0 and then rounding the solution such that $2^{k-1}\sqrt{2}T_B \leq T_0 = 2^k T_B \leq 2^k \sqrt{2}T_B$ with k integer. Stop.

3.5 Solution Approach for SWMR-VMI with Milk Runs

This section presents a solution approach for the problem presented in Section 3. Our method uses the work of Roundy (1985) and Chu and Leon (2008) for the case of direct shipping as a starting point, and

then builds upon it to be able to tackle the case of milk-runs. The solution framework is illustrated in Figure 3.5 and consists of the following steps.

We start by initializing the set of clusters, with each retailer in a separate cluster, i.e., the direct shipping case. We then use the algorithm of Chu and Leon (2008) presented above to find the replenishment interval for each retailer as well as the warehouse. These power-of-two order intervals are then used in the next phase, the vehicle routing problem phase.

For the vehicle routing problem phase, the retailers are clustered per replenishment interval. We then use the savings heuristic of Clarke and Wright (1964) for each of the clusters. This algorithm is based on a saving concept (see Figure 3.3). The main purpose of this algorithm is to optimize the transportation costs and to select retailers who can be replenished in a milk-run rather than with separate direct shipments. The solution must satisfy the restrictions that every retailer is visited exactly once, where the demanded quantities are delivered, and the total demand on every route must be within the vehicles capacity restriction. The transportation costs are specified as the cost of driving from the warehouse to any other point of the retailers.

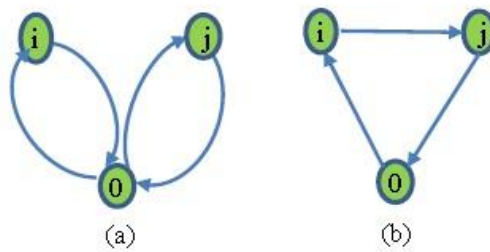


Figure 3.3. Illustration of the savings concept

For every cluster of retailers (who have the same replenishment interval after the previous step), we first perform the savings procedure as follows:

- Compute the savings, $S_{ij} = \tau_{i0} + \tau_{0j} - \tau_{ij}$, of combining every possible pair of retailers i and j in the cluster. Order the savings S_{ij} in a decreasing order.
- Find the first feasible link in the list which can be used to extend one of the two ends of the currently constructed route.
- If the route cannot be expanded further, terminate the route. Choose the first feasible link in the list to start a new route.
- Steps 2 and 3 are repeated until no further feasible links can be chosen.

In the final step, routing plans with lower costs can then be obtained using improvement heuristics that try to apply elementary modifications to the current solution. Thus, the best-known improvement heuristics for VRP, which is called a 2-opt improvement heuristic is applied to further reduce transportation costs. The 2-opt exchange is a very simple, yet very useful, improvement heuristic. It involves exhaustively considering exchanges of two retailers in different routes (see Figure 3.4). This consists of deleting and re-inserting sub-routes. The possible sub-routes are inserted into the existing solution, and the cheapest alternative is retained. If no cheaper alternative is found, the solution is restored and no improvement is realized. However, if a profitable reconnection is identified, it means that the solution can be improved. Thus, routing plans with lower costs can be obtained by using improvement heuristics that apply modifications to the current solution.

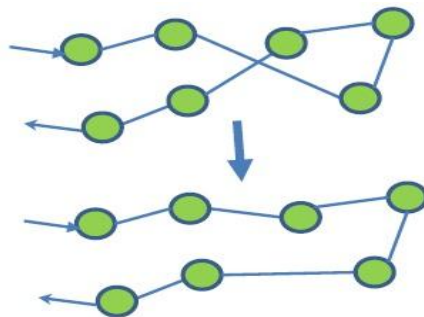


Figure 3.4. A basic arc interchange in the 2-opt procedure

During the vehicle routing problem phase, every route determines a new retailer cluster. If these retailer clusters are the same as before, then we stop the process. Otherwise, we return to the initial step to recalculate cycle times for each of the clusters from the central warehouse. Then, we can calculate the total cost for each of the clusters in the SWMR-VMI system.

To examine the impact of introducing milk-runs, we calculate the change in total inventory and transportation costs for each of the retailers and the warehouse.

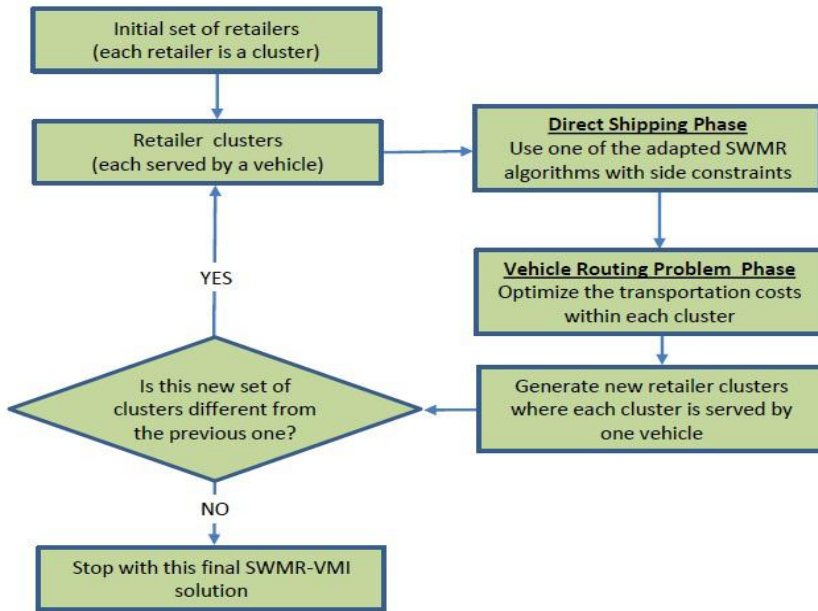


Figure 3.5. Solution framework for the SWMR-VMI system with milk-runs

3.6 A Detailed Analysis of an Illustrative Supply Chain Example

In this particular case, we consider 15 retailers as illustrated in Figure 3.6. These retailers are scattered around the warehouse and have demand rates that are assumed to be stable, adding up to 6.341 tons/hour

for all 15 retailers. We assume that a fleet of vehicles is available for product replenishment from the warehouse. The data of this case is obtained from Aghezzaf et al. (2006).

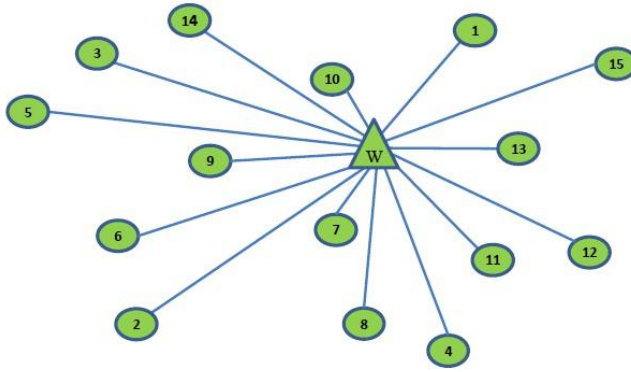


Figure 3.6. An example case with 15 retailers

Table 1 shows the distances (in kilometres) between the different retailers. Travel times can be obtained from Table 3.1 by considering an average speed of 50 km/hour for each vehicle. We assume that all vehicles in the fleet have a capacity of 60 tons and a transportation cost of €0.10 per kilometre. We also assume that the fixed ordering cost of the warehouse is €75 and all retailers have the same fixed cost per delivery of €50. Finally, we assume that there is a difference in inventory holding cost rates at the warehouse versus at the retailers, with $(h_j - h_0) > 0$.

For the illustrated 15-retailers example (see Figure 3.6), the following distribution pattern is considered where all retailers are served in a direct shipping tour, and the vehicle that leaves the warehouse, serves a retailer and then returns to the warehouse. If only one vehicle is used, the minimal cycle time of the vehicle is its total travel time, i.e. 195.6 hours, while the maximal cycle time is 72.9 hours. This tour solution is not feasible because the minimal cycle time exceeds the maximal cycle time. Therefore, when using the direct shipping tour for routing vehicles, adding more vehicles would be necessary for replenishing the 15 retailers or using a vehicle with a larger capacity.

Let us now consider what happens if a vehicle makes a TSP single-tour through 15 retailers. The best single-tour starts from the warehouse, goes to each retailer and then returns back to the warehouse at the end of the tour. Here we obtain an infeasible solution, since the minimal cycle time exceeds the maximal cycle time. The minimal cycle time of this tour is 66 hours, while the maximal cycle time is 9.46 hours. For the distribution of TSP single-tour solution to be feasible, some of the retailers must be clustered and served in one sub-tour of the vehicle.

To arrive at a feasible and a better solution, we use our solution method as presented above. In the first step, we start from the direct shipping solution and use the method of Chu and Leon (2008) to find the power-of-two order intervals for each retailer as well as the warehouse. We use $T_B = 1$ hour.

Initialization:

Step 0: $\tau'_j = \sqrt{2(\tau_{0j} + \varphi_j + \tau_{j0})/h_j d_j}$ and $\tau_j = \sqrt{2(\tau_{0j} + \varphi_j + \tau_{j0})/(h_j - h_0) d_j}$, $\forall j \in R$ and round these to the nearest power-of-two.

$\tau'_j = \{\tau'_1, \tau'_2, \tau'_3, \tau'_4, \tau'_5, \tau'_6, \tau'_7, \tau'_8, \tau'_9, \tau'_{10}, \tau'_{11}, \tau'_{12}, \tau'_{13}, \tau'_{14}, \tau'_{15}\} = \{63.09, 39.56, 73.53, 34.10, 90.75, 56.30, 47.69, 45.03, 70.84, 62.97, 33.68, 40.40, 72.15, 72.19, 85.94\}$ hours, and $t'_j = \{t'_1, t'_2, t'_3, t'_4, t'_5, t'_6, t'_7, t'_8, t'_9, t'_{10}, t'_{11}, t'_{12}, t'_{13}, t'_{14}, t'_{15}\} = \{64, 32, 64, 32, 64, 64, 32, 64, 64, 32, 64, 64, 32, 64, 64\}$ hours.

$\tau_j = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}\} = \{70.54, 43.33, 87.52, 38.13, 99.75, 65.01, 56.12, 52.00, 83.35, 77.12, 38.58, 45.17, 89.98, 88.41, 160.78\}$ hours, and $t_j = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\} = \{64, 32, 64, 32, 128, 64, 64, 64, 64, 64, 32, 32, 64, 64, 128\}$ hours.

We find $t_{\min} = \min\{t'_j; j \in R\} = 32$ hours and $t_{\max} = \max\{t_j; j \in R\} = 128$ hours. $i = 0$; $T_0 = t_{\min} = 32$ hours; and $TCR_{DS}(T^0) = \infty$.

Iteration 1:

Step 1: We set $i = 1$ and choose T_j according to condition (17): $T^1 = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}\} = \{32, 64, 32, 64, 32, 64, 64, 64, 32, 64, 64, 32, 32, 64, 64, 64\}$ and find $TCR_{DS}(T^1) = €70.00$ using (3.11). Since $TCR_{DS}(T^1) < TCR_{DS}(T^0)$, we go to Step 2.

Step 2: Set $T_0 = 64$ hours and return to Step 1.

Iteration 2:

Step 1: $i = 2$ and $T^2 = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}\} = \{64, 64, 32, 64, 32, 64, 64, 64, 64, 64, 32, 32, 64, 64, 64\}$. We find $TCR_{DS}(T^2) = €68.44$, which is less than $TCR_{DS}(T^1)$, so we go to Step 2.

Step 2: Set $T_0 = 128$ hours and return to Step 1.

Iteration 3:

Step 1: $i = 3$ and $T^3 = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}\} = \{128, 64, 32, 64, 32, 128, 64, 64, 64, 64, 32, 32, 64, 64, 128\}$. We find $TCR_{DS}(T^3) = €77.17$, which is more than $TCR_{DS}(T^2)$. Therefore, the optimal power-of-two policy is T^2 .

In the next step, we solve the VRP problem in order to reduce transportation costs. The problem is to define the allocation of retailers to routes, determine the sequence in which the retailers shall be visited on a route, and decide which vehicle shall cover which route.

To solve the constrained VRP sub-problems, firstly, we calculate the transportation costs between all pairs of points, as shown in Table 3.2, where 0 represents the warehouse. Because the costs are symmetric, only the upper half of the table is filled out. The transportation cost is given by $\tau_{ij} = \delta \cdot v \cdot t_{ij}$ euro per tour, where t_{ij} represents the travel time between the pairs of retailers at a speed of v km per hour, and δ is the travel cost per km.

Table 3.3. Quantities delivered to each of the retailers

Retailers	Demand (ton/hour)	Inventory holding cost (€)	Cycle time (hour)	Delivery (ton)
1	0.209	0.25	64	13.38
2	0.622	0.30	32	19.90
3	0.322	0.17	64	20.61
4	0.798	0.25	32	25.54
5	0.134	0.29	64	8.58
6	0.429	0.20	64	27.46
7	0.381	0.18	64	24.38
8	0.503	0.20	64	32.19
9	0.217	0.18	64	13.89
10	0.269	0.15	64	17.22
11	0.823	0.21	32	26.34
12	0.598	0.25	32	19.14
13	0.247	0.14	64	15.81
14	0.348	0.15	64	22.27
15	0.441	0.07	64	28.22

The replenishment quantity for each retailer is obtained by multiplying its cycle time with its demand rate. The resulting replenishment quantities are given in Table 3.3. The inventory holding cost rates are varying across at the retailers (see Table 3.3) and the inventory holding cost rate at the warehouse is 0.05.

The savings S_{ij} are calculated for each pair of retailers and presented in Table 3.4. Again, only the upper half of the table is completed, since the savings are symmetric due to symmetrical costs. We are now ready to continue with the vehicle routing problem phase of the algorithm presented in Section 3.5, to find feasible routes for the clustered retailers.

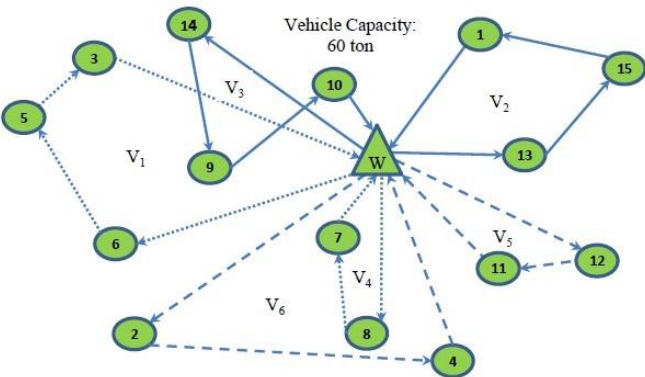


Figure 3.7. A VRP tour solution

Table 3.5. Distribution results for the different tours

Tours	Vehicle load (ton)	Capacity utilization (%)	Total cost (€)
$V_1 = (6, 5, 3)$	56.64	94.4	61.29
$V_2 = (13, 15, 1)$	57.41	95.7	58.55
$V_3 = (14, 9, 10)$	53.38	89.3	59.29
$V_4 = (8, 7)$	56.58	94.3	59.02
$V_5 = (12, 11)$	45.47	75.9	62.97
$V_6 = (2, 4)$	45.44	75.7	65.67

A VRP is solved for two sets of retailers: those with a cycle time of 64 hours {1, 3, 5, 6, 7, 8, 9, 10, 13, 14, 15}, and those with a cycle time of 32 hours {2, 4, 11, 12}. The result of the savings heuristic is shown in Figure 3.7. The retailers of the first set are assigned to four routes: route $V_1 = (6, 5, 3)$ with a total demand of 56.64 tons, route $V_2 = (13, 15, 1)$ with a total demand of 57.41 tons, route $V_3 = (14, 9, 10)$ with a total demand of 53.38 tons, route $V_4 = (8, 7)$ with a total demand of 56.58 tons. The retailers in the other set are assigned to two routes: route $V_5 = (12, 11)$ which delivers 45.47 tons, and route $V_6 = (2, 4)$ which delivers 45.44 tons.

In Table 3.5, the vehicle load and the total cost rate TCR_{MR} for each of the sub-tours are clearly shown. From the results above, it shows

that the truck loading is optimized efficiently with the average capacity utilization for all tours being 87.48%.

As can be seen in Figure 3.7, sub-tour $V_3 = (14, 9, 10)$ can be improved. This improvement is found in the 2-opt heuristic that we apply next (see Figure 3.8). The existing route $(0-14-9-10-0)$ is changed to a new route $(0-10-14-9-0)$. This decreases total transportation costs from €114 to €100.

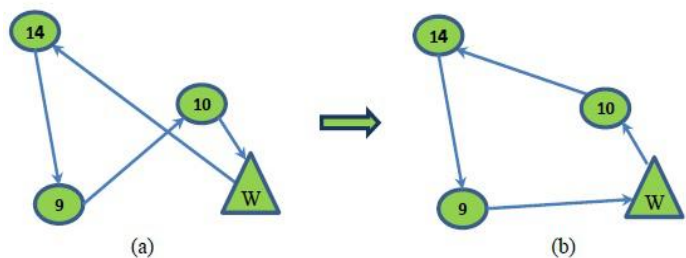


Figure 3.8. Solution of the sub-tour problem (improvement heuristic)

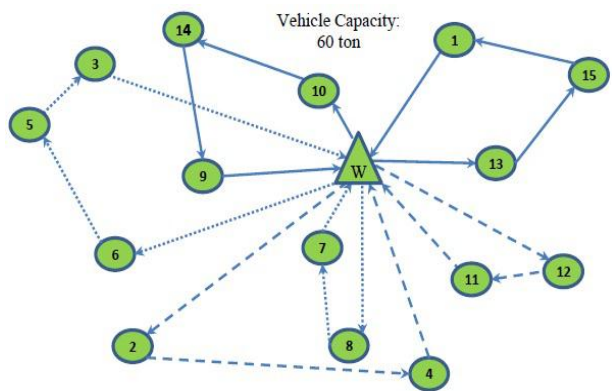


Figure 3.9. Solution of the VRP problem (savings + improvement heuristic)

Figure 3.9 shows the new solution for the SWMR-VMI after the savings and improvement heuristics. We now have six retailer clusters (one per route), which is different from the initial clustering (where we had one cluster for each retailer). Thus, the solution procedure starts a new iteration and will evaluate the reorder intervals for these new clusters.

Table 3.6. Inventory and transportation costs with direct shipping

Retailers	IC_{DS} (€/hour)	TC_{DS} (€/hour)
1	3.63	0.84
2	6.22	3.00
3	3.70	1.53
4	6.56	2.06
5	3.20	1.72
6	4.70	1.34
7	4.15	0.44
8	5.17	0.81
9	3.20	0.75
10	3.24	0.47
11	6.16	1.50
12	5.60	2.25
13	3.06	0.63
14	3.62	1.34
15	2.94	1.00
Total cost	65.15	19.68

Table 3.7. Inventory and transportation costs with milk-runs

Retailers Clusters	IC_{MR} (€/hour)	TC_{MR} (€/hour)
$V_1 = (6, 5, 3)$	9.26	2.03
$V_2 = (13, 15, 1)$	7.28	1.27
$V_3 = (10, 14, 9)$	7.73	1.56
$V_4 = (8, 7)$	8.15	0.88
$V_5 = (12, 11)$	10.59	2.38
$V_6 = (2, 4)$	11.61	4.06
Total cost	54.62	12.18

Table 3.6 shows the total of the inventory costs, IC_{DS} , and round trip transportation costs, TC_{DS} , for every retailer with direct shipping. Table 3.7 shows the total costs of the inventory, IC_{MR} , and transportation, TC_{MR} , for every retailers cluster after the adoption of VMI and milk-runs. As we know, before implementing VMI and milk-runs, each retailer is exclusively served by a vehicle in trip visiting that retailer only. Then, once VMI and milk-runs are implemented, some retailers are clustered and served in a sub-tour of the trip made by the vehicle.

Table 3.8. Summary of results for inventory and transportation costs

	Milk-runs (€/hour)	Direct shipping (€/hour)	Gap (%)
Inventory cost	54.62	65.15	16.16
Transportation cost	12.18	19.68	38.10
Total cost	66.80	84.83	21.25

Table 3.8 gives the comparisons between the results obtained by the inventory management policy before and then after the adoption of VMI and milk-runs. From the table above, we can see that the inventory cost is reduced by 16.16% and the transportation cost is decreased by 38.10% when implementing VMI and milk-runs in the system. Therefore, the total cost of the inventory and transportation costs in the system after implementing VMI and milk-runs is reduced by 21.25%.

In addition, Table 3.9 gives the summary of the results for the main characteristics of the distribution pattern. For example, the vehicle 1 with a 60 tons capacity makes the tour $V_1 = (6, 5, 3)$. The tour has $T_{min} = 26$ hours, $T_{max} = 67.8$ hours and $T^* = 64$ hours. The maximal cycle time is higher than the theoretical optimal cycle time. The actual cycle time is therefore, 64 hours, giving a total cost rate for this tour equal of €61.29/hour and the total demand is 56.64 tons.

From the table above, we also can evaluate the results of the vehicle storage capacity restrictions. In this case, capacities of 60 tons, 80 tons and 100 tons are used for delivering the product to each of the retailers clusters. The vehicle capacity factor is used to show that our solution approach not only helps to decide on the fleet size, but can also be used to select the most appropriate vehicle type for a particular problem in this instance. For the size of 15 retailers which are clustered by the same set partitions, the result shows that the average total cost rate is €366.79 when using a small vehicle of 60 tons, €315.08 when using a vehicle of 80 tons and €261.93 when using a larger vehicle of 100 tons. Therefore, we can see that the smaller the delivery quantities to each of the clusters, the lesser the number of retailers who are replenished per tour and more tours are made. Moreover, it also increases the number of vehicles and transportation costs. However, in this case a smaller vehicle capacity is utilized efficiently instead of a larger one.

Based on the assessment results summarized above, we will then investigate the possibility of generalizing the deterministic model to more complex stochastic inventory systems. One of the significant approaches is proposed by Chu and Shen (2010), to express the effect of demand variability and analyse its impact on the distribution strategy for the SWMR system, under a VMI policy. Therefore, in future research, we will be expanding the approach to the stochastic case where demand at each retailer has an average and standard deviation, and introducing safety stocks at the warehouse and each retailer site to ensure a certain target service level.

3.7 Conclusions

Managing inventory and routing in a supply chain is a very challenging optimization problem. In this chapter, we propose a global solution approach for a two-stage supply chain implementing vendor managed inventory (VMI). We focused on the problem, denoted by SWMR-VMI, where a single-warehouse delivers a single product to a set of independent retailers. These retailers draw the required material from the warehouse to satisfy their given individual demands. The

warehouse, in turn, places orders to an outside supplier to fill the accumulated demands of the retailers.

An approach is proposed to minimize the overall inventory and transportation costs of the SWMR-VMI system while satisfying the retailers demands. The approach integrates two effective algorithms, one for inventory management and the second for routing optimization. In particular, the algorithms proposed by Roundy (1985) and improved by Chu and Leon (2008) is used to solve the single-warehouse multiple-retailers direct shipping problem, and the heuristic of Clarke and Wright (1964) is used to solve the VRP sub-problem. The results of the proposed approach allowed us to investigate the effectiveness of an inventory management policy before and after implementation of VMI and milk-runs in a two-stage supply chain. We discovered that the transportation cost is relevant, the effect of VMI and milk-runs can result in a significant inventory and transportation cost savings.

Further research will focus on adapting this solution approach to enrich IRP problems, including larger sets of retailers, driving-time restrictions on the vehicles and their drivers, delivery time windows at the retailers, heterogeneous vehicle fleets, multiple warehouses, multiple products etc. Numerous experiments on large-scale problems are currently under investigation. Finally, the basic assumption that demand rates are constant is not always valid. So, it is worthwhile to investigate how the approach can be extended to explicitly take some demand variability into consideration. We will be extending this research to the stochastic case in the future research.

Acknowledgments: This chapter is derived in part from the article “Analyzing The Effectiveness of Vendor Managed Inventory in a Single-Warehouse, Multiple-Retailer System”, by M.K.I. Abdul Rahim, E.-H. Aghezzaf, V. Limère and B. Raa, published in *International Journal of Systems Science*, 13 October 2014, copyright Taylor & Francis, available online:

<http://www.tandfonline.com/doi/10.1080/00207721.2014.965771>

4

Modelling and Solving Multi-Period Stochastic Inventory Routing Problem

4.1 Introduction

The previous chapter of this dissertation focuses on the single-warehouse, multiple-retailer vendor managed inventory (SWMR-VMI), in which all retailers face a deterministic, constant demand rate. We consider a two-stage supply system where a supplier serves a set of retailers from a single warehouse using a fleet of vehicles having a limited capacity. The objective is to minimize the overall inventory and transportation costs of the SWMR system, under a VMI policy. We proposed a two-phase optimization approach for coordinating the shipments in this VMI system. The first phase uses direct shipping to minimize the overall inventory costs. Then, in the second phase, the retailers are clustered using a construction heuristic in order to optimize the transportation costs while satisfying some additional restrictions. The proposed solutions for the SWMR-VMI problem assumed that retailers demand rates were constant. However, in the realistic problems, demand rates are not usually constant and are stochas-

tic. Thus, the problem is better modelled as a multi-period replenishment problem with stochastic stationary demand rates.

In this chapter, we are concerned with a multi-period stochastic inventory routing problem (MP-SIRP) where the retailers consume the product at a stochastic stationary rate. More precisely, we consider a distribution system in which a fleet of homogeneous vehicles is used to distribute some product from a single warehouse to a set of retailers consuming it at stationary demand rates, during a finite horizon H of consecutive periods (days). The objective is to determine optimal quantities to be delivered to the retailers, the delivery times, and the vehicle delivery routes, so that the total distribution and inventory costs are minimized. The resulting distribution plan must prevent stock-outs from occurring at all retailers during the planning horizon and assuring some predetermined service level. Based on the formulation of the cyclic IRP model (see, e.g., Aghezzaf et al. 2006) and Multi-period IRP (see, e.g., Zhong and Aghezzaf 2012), we formulated a stochastic linear mixed-integer model for this MP-SIRP. A deterministic equivalent approximation reformulation (MP-DAIRP $_{\alpha}$) of the problem is proposed. This latter proposed model also determines the optimal vehicle fleet size, in each period. Moreover, initial inventories are typically set to predefined amounts in previous works (see for example Yu et al. 2008, Taarit et al. 2010 and references therein). A Lagrangian relaxation method to solve the proposed MP-DAIRP $_{\alpha}$ is developed and thoroughly discussed. Numerical experiments demonstrate the effectiveness of the proposed solution approach.

The remainder of this chapter is organized as follows. In Section 4.2, we review major papers on the modelling of MP-SIRP. In Section 4.3, a stochastic linear mixed-integer formulation for the multi-period inventory routing problem with stochastic stationary demand rates, MP-SIRP, is presented. Then a deterministic equivalent approximation model, MP-DAIRP $_{\alpha}$, is proposed. In Section 4.4, an illustrative example for the MP-SIRP is presented to illustrate the proposed model. In Section 4.5, a Lagrangian relaxation based decomposition approach is proposed to solve this deterministic equivalent approximation model MP-DAIRP $_{\alpha}$ and is thoroughly discussed. In Section 4.6,

some computational results are presented and compared with the results obtained by CPLEX. Finally, some concluding remarks are provided in Section 4.7.

4.2 Review of Major Papers on the Modelling MP-SIRP

Since Bell et al. (1983) first investigated the integrated inventory management and vehicle scheduling, varied versions of the IRP have been studied. A large variety of solution approaches have also been proposed to solving these problems. The IRP can be modelled and approached in many different ways depending on the characteristics of its parameters. Different models can be obtained for example, when retailers consume the product at a stable or at a variable rate; when retailer-demands are assumed deterministic or stochastic; when the planning horizon is finite or infinite, and so on. A classification of IRP models can be found in Andersson et al. (2010) and a recent thorough review of the literatures on IRPs during the last thirty years can be found in Coelho et al. (2013).

Federgruen and Zipkin (1984) address a single period IRP with stochastic demands and a fixed fleet vehicle size. Their work was extended by Federgruen et al. (1986) to consider multiple products. Aghezzaf (2008) considers the case of a cyclic IRP where retailer demand rates and travel times are stochastic but stationary and proposes a model that generates optimal robust distribution plans. All these contributions assume a stationary demand rate for the product(s). Dror and Ball (1987) decompose a multi-period IRP into series of single period problems. They studied the problem of constant demands and then proposed and compared two solution approaches for the resulting single period problem. Trudeau and Dror (1992) solved a similar problem with uncertain demands. Campbell et al. (2002) and Campbell and Savelsbergh (2004) also worked on multi-period IRPs where the decisions are executed over a finite horizon. For recent research devoted to the multi-period IRPs, we refer to, e.g., Lei et al. (2006), Archetti et al. (2007), Yu et al. (2008), Boudia et al. (2009) and Taarit et al. (2010). These papers consider periodic demands which are not necessarily constant over time.

Berman and Larson (2001) used stochastic dynamic programming to determine deliveries in an IRP with uncertain demands. Kleywegt et al. (2002, 2004) formulated a Markov decision process model of the stochastic inventory routing problem, and proposed the approximation dynamic programming methods to solve the problem. Hvattum and Løkketangen (2009) and Hvattum et al. (2009) solved the IRP with uncertain retailer-demands heuristically. They used scenario trees and a progressive hedging algorithm. Bertazzi et al. (2013) formulated the stochastic IRP as a dynamic program and have solved it by means of a hybrid rolling horizon algorithm. Solyali et al. (2012) modelled and solved the IRP problem when the probability distribution of the retailers is not fully specified as a robust mixed integer program.

Another important line of research consists of contemplating location decisions together with production, inventory and routing. Integrated supply chain models taking into account location and transportation aspects are investigated, among others, by Daskin et al. (2002), Ambrosino and Scutella (2005), Berger et al. (2007) and Shen and Qi (2007). A comprehensive review of these and other models can be found in Shen (2007). The multi-period version of the problem is discussed by Laporte and Dejax (1989) and Salhi and Nagy (1999). The stochastic location-transportation problem is studied among others by Laporte et al. (1989) and Albareda-Sambola et al. (2007). More recently, Klibiet al. (2010) investigated the stochastic multi-period location-transportation version of the problem, allowing multiple transportation options. Ma and Dai (2010) studied a stochastic dynamic location-routing-inventory problem in a two-echelon multiproduct distribution system. Likewise, integrating production, inventory and transportation has also been thoroughly studied. Some relevant contributions in this area are, among others, Fumero and Veccilly (1999), Park (2005), Lei et al. (2006), Boudia et al. (2007), Bard and Nananukul (2009), Chen (2010) and Safaeia et al. (2010).

4.3 Modelling and Reformulating the MP-SIRP

As mentioned above, the MP-SIRP, discussed in this chapter, consists of a single distribution centre using a fleet of homogeneous vehicles to

distribute a single product to a set of geographically dispersed retailers over a given planning horizon. It is assumed that retailer-demand rates are stochastic and stationary, and that travel-times are constant over time. The objective of this MP-SIRP is to determine optimal quantities to be delivered to the retailers, delivery time, and vehicle delivery routes, so that the total distribution and inventory costs is minimized, while preventing stock-outs from occurring at all retailers and assuring some predetermined service level during the entire planning horizon.

To formulate our model for the MP-SIRP, the following assumptions are made:

- The time required for loading and unloading a vehicle is neglected in the model.
- Inventory capacities at the warehouse and at the retailers are assumed to be large enough so that the corresponding capacity constraints are omitted in the model.
- Transportation costs are assumed to be proportional to travel times.
- Split deliveries are not allowed, each retailer is always completely replenished by a single vehicle.

The relevant variables, parameters as well as a linear mixed-integer formulation of the MP-SIRP are described in the following subsections:

Let $H = \{1, 2, \dots, T\}$ be the planning horizon set of consecutive periods indexed by t , and $H^+ = H \cup \{0\}$. Let τ_t be the size in time units of period t , for example 8 working hours. Let S be the set of retailers indexed by i and j ; and $S^+ = S \cup \{r\}$, which represents the warehouse. A fleet of vehicles V is used to serve these retailers. The other relevant parameters of the model are given below:

- φ_{jt} : the fixed handling cost (in euros) per delivery at location $j \in S^+$ (retailers and warehouse) in period $t \in H$;

- η_{jt} : the per unit per period holding cost of the product at location $j \in S^+$ (in euros per tons per period);
- ψ^v : the fixed operating cost of vehicle $v \in V$ (in euros per vehicle);
- δ_v : travel cost of vehicle $v \in V$ (in euros per km);
- κ^v : the capacity of vehicle $v \in V$ (in tons);
- v_v : average speed of vehicle $v \in V$ (in km per hour);
- θ_{ij} : duration of a trip from retailer $i \in S^+$ to retailer $j \in S^+$ (in hour);
- d_{jt} : the stochastic stationary demand rate at retailer j (in tons per hour) in period $t \in H$. We assume that d_{jt} is normally distributed with average $D_j = E(d_{jt})$ and standard deviation σ_j ;
- I_{j0} : the initial inventory levels (in tons) at each retailer $j \in S$.

The variables of the model are defined as follows:

- Q_{ijt}^v : the quantity (in tons) of product remaining in vehicle $v \in V$ when it travels directly to location $j \in S^+$ from location $i \in S^+$ in period $t \in H$. This quantity equals zero when the trip (i, j) is not on any tour of the route travelled by vehicle $v \in V$ in period t ;
- q_{jt} : the quantity (in tons) delivered to location $j \in S$ in period $t \in H$, and 0 otherwise;
- I_{jt} : the inventory level at location (retailers and warehouse) $j \in S^+$ by the end of period $t \in H$ (in tons);
- x_{ijt}^v : a binary variable set to 1 if location $j \in S^+$ is visited immediately after location $i \in S^+$ by vehicle $v \in V$ in period $t \in H$, and 0 otherwise;
- y_t^v : a binary variable set to 1 if vehicle $v \in V$ is being used in period t , and 0 otherwise.

Thus, if we let I_{jr} be the initial inventory level at the warehouse, the linear mixed-integer formulation for the multi-period IRP is given as follows:

MP-SIRP: Minimize

$$CV = \sum_{t \in H} \sum_{v \in V} \left[\psi^v y_t^v + \sum_{i \in S^+} \sum_{j \in S^+} (\delta_v v_v \theta_{ij} + \phi_{jt}) x_{ijt}^v \right] + \sum_{t \in H^+} \sum_{j \in S^+} \eta_{jt} I_{jt} \quad (4.1)$$

Subject to:

$$\sum_{v \in V} \sum_{i \in S^+} x_{ijt}^v \leq 1, \quad \forall j \in S, t \in H \quad (4.2)$$

$$\sum_{i \in S^+} x_{ijt}^v - \sum_{k \in S^+} x_{jkt}^v = 0, \quad \forall j \in S^+, t \in H, v \in V \quad (4.3)$$

$$\sum_{i \in S^+} \sum_{j \in S^+} \theta_{ij} x_{ijt}^v \leq \tau_t, \quad t \in H, v \in V \quad (4.4)$$

$$\sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v - \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v = q_{jt}, \quad \forall j \in S, t \in H \quad (4.5)$$

$$Q_{ijt}^v \leq k x_{ijt}^v, \quad \forall j \in S^+, t \in H, v \in V \quad (4.6)$$

$$I_{j,t-1} + q_{jt} - I_{jt} = d_{jt} \tau_t, \quad \forall j \in S, t \in H \quad (4.7)$$

$$I_{j0} \leq I_{jT}, \quad \forall j \in S \quad (4.8)$$

$$x_{rjt}^v \leq y_t^v, \quad \forall j \in S^+, t \in H, v \in V \quad (4.9)$$

$$x_{rjt}^v, y_t^v \in \{0,1\}, I_{j0}, I_{jT} \geq 0, Q_{ijt}^v \geq 0, q_{jt} \geq 0, \quad \forall j \in S^+, t \in H, v \in V$$

Constraints (4.2) ensures that each retailer is visited at most once in period t . Constraints (4.3) assures that if a vehicle arrives at a retailer, it must leave after it has served it to a next retailer or to the ware-

house. Constraints (4.4) ensures that vehicles complete their routes within one travel period, so the total travel time of a vehicle should not exceed the planned total working hours in each period. Constraints (4.5) determine the quantity delivered to a retailer, and these constraints also eliminate sub-tours. The vehicle capacity constraints are given by (4.6) and assures that the variables Q_{ijt}^v , cannot carry any cumulated flow unless x_{ijt}^v , equals 1. Constraints' (4.7) is the inventory balance equations at the retailers. Constraints (4.8) indicate that the final inventory level at retailer j at the end of period T is of the same magnitude as its initial inventory. Constraints (4.9) indicate that a vehicle cannot be used to serve any retailer unless it is selected. The objective function (4.1) includes four cost components, namely total fixed operating cost of using the vehicle(s), total transportation cost, total delivery handling cost and total inventory holding cost at the end of each period. Since demand rates are stochastic, the resulting optimization problem is stochastic and thus requiring a stochastic optimization treatment.

Now, observe that if the objective is to provide an optimal solution to the MP-SIRP that satisfies the stochastic demand in each period with some predetermined level of confidence $(1-\alpha)$, constraints (4.7) must be replaced by new restrictions that guarantee:

$$Probability \left\{ I_{j,t-1} + \sum_{s=t}^T q_{jt} \geq \sum_{s=t}^T d_{js} \tau_s \right\} = (1 - \alpha), \text{ for all } j \in S, t \in H$$

Notice that the restrictions $I_{j,t-1} + \sum_{s=t}^T q_{jt} \geq \sum_{s=t}^T d_{js} \tau_s$ for all $j \in S, t \in H$ are obtained as linear combinations of constraints (4.7). Using conditions (4.10), we obtain a deterministic equivalent approximation model for the multi-period stochastic inventory routing problem, denoted MP-DAIRP $_{\alpha}$ and given by:

MP-DAIRP_α: Minimize

$$CV_{\alpha} = \sum_{t \in H} \sum_{v \in V} \left[\psi^v y_t^v + \sum_{i \in S^+} \sum_{j \in S^+} (\delta_v v_v \theta_{ij} + \phi_{jt}) x_{ijt}^v \right] + \sum_{t \in H^+} \sum_{j \in S^+} \eta_{jt} I_{jt}$$

Subject to:

(4.2) - (4.6), (4.8) - (4.9) and

$$I_{j,t-1} + \sum_{s=t}^T q_{jt} \geq \sum_{s=t}^T E(d_{js}) \tau_t + z_{\alpha} \left(\sqrt{T-t+1} \right) \sigma_j, \forall j \in S, t \in H \quad (4.10)$$

$$x_{rjt}^v, y_t^v \in \{0,1\}, I_{j0}, I_{jt} \geq 0, Q_{ijt}^v \geq 0, q_{jt} \geq 0, \quad \forall j \in S, t \in H, v \in V$$

Constraints (4.10) prevent stock-outs from occurring at each retailer with a confidence level $(1-\alpha)$ during each period of the planning horizon. Thus, they guarantee a service level of $100(1-\alpha)\%$ defined by the standard normal value z_{α} .

For constraints (4.10), we assume that the per hour demand rates of the retailers are normally distributed, $d_{jt}^h \sim N(E(d_{jt}), \sigma_j^H)$, with the same average demand rates (in tons) per hour in period t . If we denote by $D_{jt}^P = \sum_{h=1}^{\tau_t} (d_{jt}^h)$ the demand rate in period t . Thus, we can calculate the average demand in period t , that is:

$$E(D_{jt}^P) = \sum_{h=1}^{\tau_t} E(d_{jt}^h) = \sum_{h=1}^{\tau_t} E(d_{jt}) = \tau_t \times E(d_{jt})$$

The variance of demand rate in period t , is given by:

$$\sigma_j^2 = \sum_{h=1}^{\tau_t} \text{var}(d_{jt}^h) = \sum_{h=1}^{\tau_t} (\sigma_j^H)^2 = \tau_t (\sigma_j^H)^2. \text{ If } \tau_t = \tau \text{ for all periods, then the standard deviation, that is } \sigma_j = \sqrt{\tau_t (\sigma_j^H)^2} = \sqrt{\tau} \sigma_j^H.$$

In the sequel, we focus on this equivalent deterministic approximation problem MP-DAIRP_α and discuss a decomposition approach for its solution.

4.4 An Illustrative Example for the MP-SIRP

Firstly, we start by introducing a small example case with 7 retailers for the single-period stochastic inventory routing problem (SP-SIRP) to illustrate the behaviour of our proposed model. We then *extend* from the *single-period* setting to a *multi-period*. Also, in the *multi-period* model, we present a small illustrative example for the multi-period stochastic inventory routing problem (MP-SIRP).

In this example case, we consider 7 retailers which are scattered around the warehouse (see Figure 4.1), and average demand rates of retailers d_{jt} are generated randomly and uniformly between 0.1 and 3 tons per hour with a standard deviation σ of 5% of the average over the planning horizon and the standard normal value z_α is set to 1.64. A fleet of homogeneous vehicles V with a capacity of vehicle κ^v is 30 tons, is available for the distribution of the product. The fixed operating cost of the vehicle ψ^v is €50 per vehicle. The vehicle's average speed v_v is 50 km per hour, and the travel cost δ_v is €1 per km. The inventory holding costs η_{jt} for each retailer is generated randomly and uniformly between 0.1 and 0.5 (in euro per ton per hour). We assume that the fixed delivery handling cost φ_{jt} is the same for all retailers and the size in time units τ_t of period t is set to 8 hours. The values of these parameters are then displayed in Table 4.1.

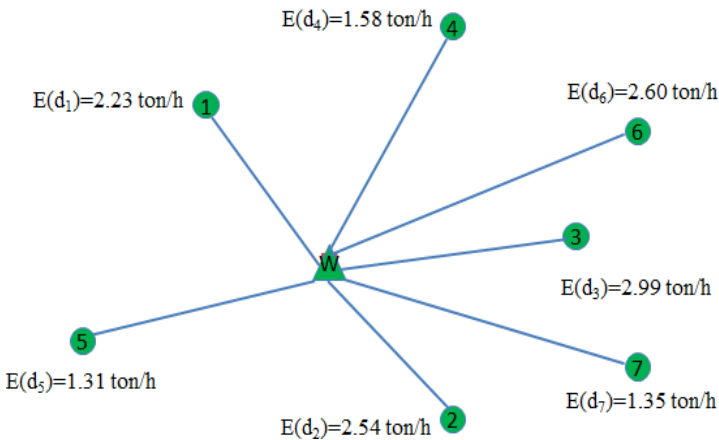


Figure 4.1. An example case with 7 retailers

Table 4.1. Parameters and delivery quantity to each of the retailers for the SP-SIRP

Retailers	Average demand (ton/hour)	Inventory holding cost (€)	Delivery cost (€)	Delivery quantity (ton)
1	2.23	0.112	25	19.44
2	2.54	0.131	25	30.00
3	2.99	0.117	25	30.00
4	1.58	0.130	25	19.12
5	1.31	0.109	25	10.56
6	2.60	0.144	25	30.00
7	1.35	0.115	25	10.88

The generated 7-clients instance of the SP-SIRP is solved by AMPL, with CPLEX 11.2. The solution is graphically displayed in Figure 4.2 and the quantity delivered to each of the retailers is presented in Table 4.1. In the solution, only one vehicle is used to replenish the product to each of the retailers. As illustrated in Figure 4.2, the retailers are assigned to five routes $\{(2), (3), (5,1), (6), (7,4)\}$. For example, route $\{(5, 1)\}$ delivers 10.56 tons and 19.44 tons respectively to retailer 5 and retailer 1, with a total demand of 30 tons. For this case, the solution gives the optimal objective value of €307.

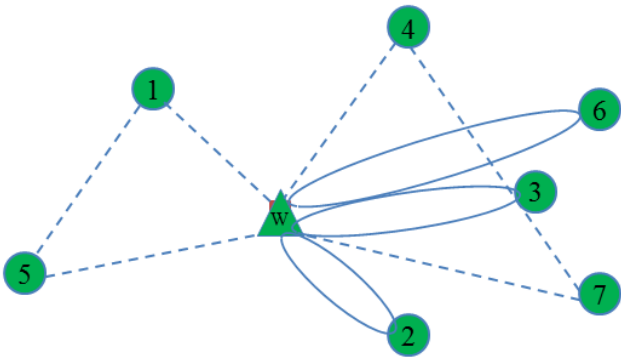


Figure 4.2. A VRP tour solution for the SP-SIRP

Now, in order to get a better understanding of the MP-SIRP model, we then construct a small example to illustrate the MP-SIRP model, based on the above 7 retailers instance (see Figure 4.1). We consider again another 7 retailers who are scattered around the warehouse, and the average demand rates of the retailers are generated randomly and uniformly between 0.1 and 3 tons per hour with a standard deviation of 5% on average over the planning horizon and the standard normal value z_α is set to 1.64. We also consider that the planning horizon set contains 3 consecutive periods and the size in time unit of each period is set to 8 hours. We follow the same notations and the same values for the vehicle's parameters, as well as the retailers' parameters such as the coordinate positions, and the delivery handling cost. The inventory holding costs for the retailers in each period is generated randomly and uniformly between 0.1 and 0.5 (in euro per ton per hour). The values of these parameters are then shown in Table 4.2.

Table 4.2. Parameters of the retailers for the MP-SIRP

Retailers	Average demand (ton/hour)			Inventory holding cost (€)			Delivery cost (€)
	t=1	t=2	t=3	t=1	t=2	t=3	
1	2.23	2.23	2.23	0.112	0.108	0.109	25
2	2.54	2.54	2.54	0.131	0.131	0.135	25
3	2.99	2.99	2.99	0.117	0.115	0.131	25
4	1.58	1.58	1.58	0.130	0.147	0.125	25
5	1.31	1.31	1.31	0.109	0.121	0.146	25
6	2.60	2.60	2.60	0.144	0.143	0.104	25
7	1.35	1.35	1.35	0.115	0.114	0.124	25

We solve the MP-SIRP problem by using AMPL, with CPLEX 11.2. Subsequently, the resulting optimal solution shows that a fleet of two vehicles is required to replenish these 7 retailers. The delivery routes in each period are shown below, with the optimal objective value of €745.

- Period: $t = 1$, Vehicle: $V = V_2 : \{(1), (2,7), (3), (4), (5), (6)\}$
- Period: $t = 2$, Vehicle: $V = V_1 : \{(1), (2), (4, 6, 3, 7)\}$
- Period: $t = 3$, Vehicle: $V = V_2 : \{(2), (5), (3, 6)\}$

The results' of the optimal solution of retailers for MP-SIRP is graphically presented in Figure 4.3. From the Figure 4.3(b), we can see that the vehicle V_1 , in period $t = 2$, construct three routes of solution in which the vehicle makes three separate tours: $\{(1), (2), (4, 6, 3, 7)\}$. For instance, within the time period of 8 hours in $t = 2$, the vehicle V_1 starts the direct shipping tour to retailer 1 and delivers 23.66 tons, and then returns back to the warehouse. Similar to retailer 2, vehicle V_1 also makes the direct shipping tour and delivers 30 tons, and then returns back to the warehouse. Next, the vehicle V_1 makes a multi-tour to deliver the products to retailer (4, 6, 3, and 7). Firstly, the vehicle arrives at retailer 4 and delivers 8.06 tons, and then the vehicle delivers 4.81 tons to retailer 6. The vehicle continues the tour to retailer 3 and delivers 11.90 tons, and finally goes to retailer 7 and delivers 5.23 tons. In the above case, it is clearly shown that the vehicle load is optimized efficiently with the vehicle capacity of 30 tons, and within the time period, only one vehicle is used for replenishing the retailers. Furthermore, for the full replenishment plan for all the retailers in the example is presented in Figure 4.4.

Apparently, from the results of both the examples above, both cases are solved to optimality. In terms of cost value, the result of the SP-SIRP is far better than that of the MP-SIRP. The objective value of the SP-SIRP is expressed in term of cost rate. From a long-term point of view, it represents the resulting average distribution and inventory costs when using a vehicle to replenish the retailers. The objective value of the MP-SIRP represents the total distribution and inventory costs for the use of a vehicle to carry out the replenishment, over the given planning horizon. As indicated previously, when demand rates are stochastic or volatile, it makes sense that the problem is modelled as a multi-period inventory routing problem with stochastic stationary rate.

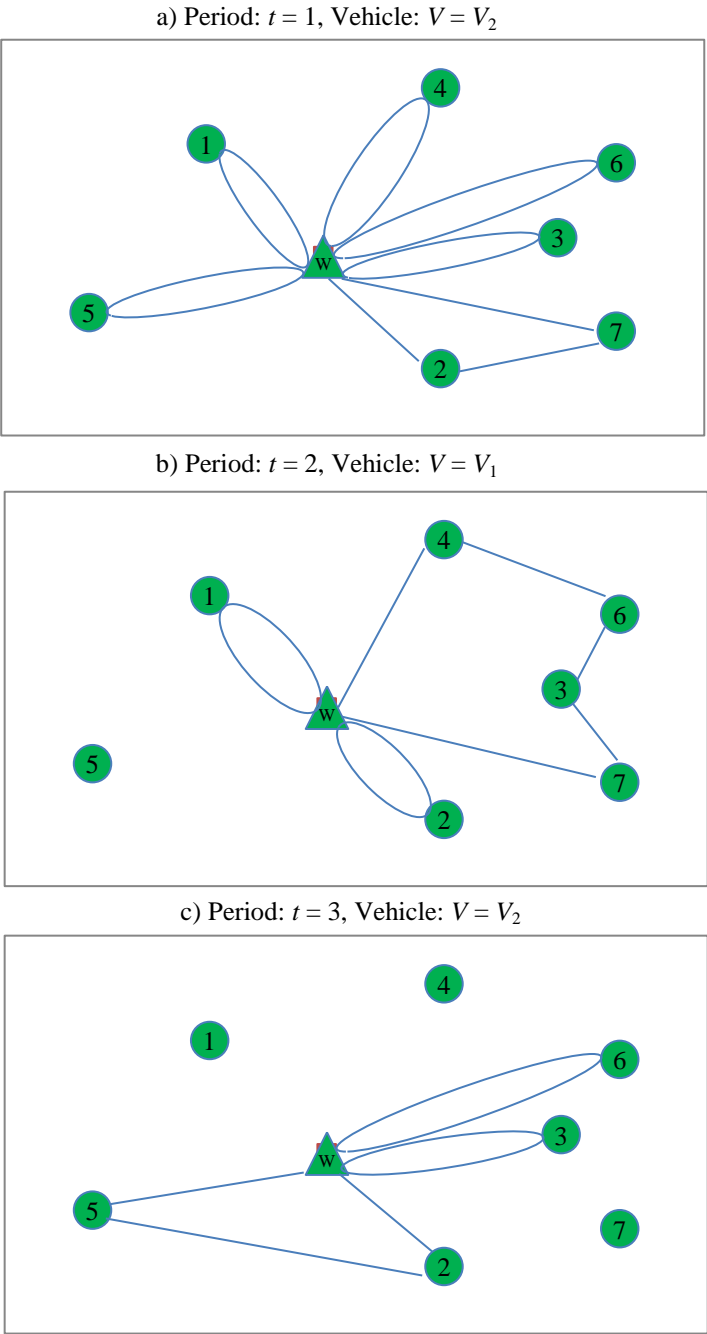


Figure 4.3. The optimal solution of the 7 retailers for MP-SIRP.

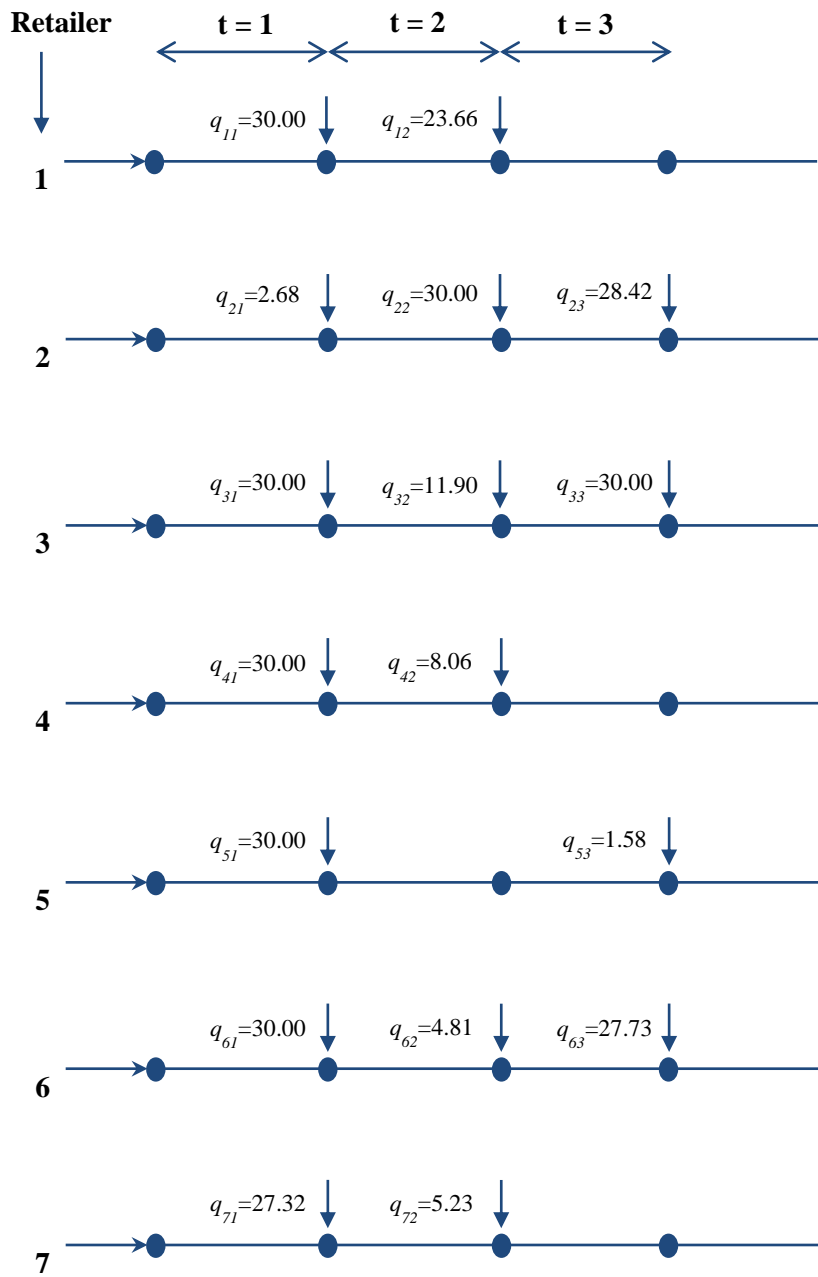


Figure 4.4. The replenishment plan for the 7 retailers of the MP-SIRP.

4.5 Lagrangian Relaxation for MP-DAIRP $_{\alpha}$

The MP-DAIRP $_{\alpha}$ is an NP-complete problem as it contains the vehicle routing problem (VRP) as a sub-problem in each period. As a consequence, large instances of the MP-DAIRP $_{\alpha}$ are hard to solve to optimality in a reasonable computational time. This section discusses a Lagrangian relaxation approach to decompose and possibly solve or generate some lower and upper bounds for the problem.

In the proposed Lagrangian relaxation procedure for MP-DAIRP $_{\alpha}$, constraints (4.5) are assumed to be the complicating restrictions. If they are relaxed, the resulting problem can be decomposed, along the same lines as in Taarit et al. (2010), into an inventory allocation sub-problem (denoted by IA-P) and a vehicle routing sub-problem (denoted by RT-P). These sub-problems involve fewer variables and constraints respectively and can be solved more efficiently by some standard optimization MIP-solver. Note, however, that the resulting inventory allocation problem in our case is inherently stochastic. It requires thus a special treatment and in particular inclusion of safety stocks to hedge against the variability of the demand. A sub-gradient algorithm (Fisher 1981, Shor 1985) will be used to update the Lagrangian multipliers and derive a lower bound on the optimal solution of the original problem. The Lagrangian relaxation implementation is carried out along the same lines as in the successful implementations used to solve complicated mixed-integer programs (see, e.g., Fisher 1985, Yu et al. 2008, Li et al. 2009).

4.5.1 Relaxation and Decomposition

Constraints (4.5) in the reformulation MP-DAIRP $_{\alpha}$ combine inventory allocation variables q and shipment flow variables Q . If these constraints are relaxed and incorporated in the objective function with unrestricted Lagrangian multipliers μ_{jt} for all $j \in S^+$ and $t \in H$, then the resulting relaxed problem, denoted by LR-MPIRP $_{\alpha}$, can be stated as follows:

$$\begin{aligned}
 CV_{LR}(\mu) = & \sum_{t \in H} \sum_{v \in V} \left[\psi^v y_t^v + \sum_{i \in S^+} \sum_{j \in S^+} (\delta_v v_v \theta_{ij} + \varphi_{jt}) x_{ijt}^v \right] \\
 & + \sum_{t \in H^+} \sum_{v \in V} \eta_{jt} I_{jt} + \sum_{t \in H} \sum_{v \in V} \mu_{jt} \left(q_{jt} - \sum_{v \in V} \sum_{i \in S^+} \mathcal{Q}_{ijt}^v - \sum_{v \in V} \sum_{k \in S^+} \mathcal{Q}_{jkt}^v \right) \quad (4.11)
 \end{aligned}$$

Subject to:

$$(4.2) - (4.4), (4.6) \text{ and } (4.8) - (4.10)$$

The two sub-problems, IA-P and RT-P, resulting from the relaxed problem LR-MPIRP_α are shown below. Observe that the stochastic aspect of the demand rates is taken into account in the inventory allocation sub-problem. This latter sub-problem is also strengthened with some additional safety stock related restrictions.

The inventory allocation sub-problem IA-P:

IA-P: Minimize

$$CV_{IA-P}(\mu) = \sum_{t \in H^+} \sum_{j \in S^+} \eta_{jt} I_{jt} + \sum_{t \in H} \sum_{j \in S} \mu_{jt} q_{jt} \quad (4.12)$$

Subject to:

$$(4.8) \text{ and } (4.10)$$

$$I_{j0} \geq 0, I_{jt} \geq 0, q_{jt} \geq 0, \quad \forall j \in S, \forall t \in H$$

Note that the sub-problem IA-P can be further decomposed into independent sub-problems associated with each retailer $j \in S$.

The routing sub-problem RT-P having as main objective to minimize transportation costs is given below:

RT-P: Minimize

$$CV_{RT-P}(\mu) = \sum_{t \in H} \sum_{v \in V} \left[\psi^v y_t^v + \sum_{i \in S^+} \sum_{j \in S^+} (\delta_v v_v \theta_{ij} + \phi_{jt}) x_{ijt}^v \right] \\ - \sum_{t \in H} \sum_{v \in V} \mu_{jt} \left(\sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v - \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v \right) \quad (4.13)$$

Subject to:

$$(4.2) - (4.4), (4.6) \text{ and } (4.9)$$

$$x_{ijt}^v, y_t^v \in \{0,1\}, Q_{ijt}^v \geq 0, \quad \forall j \in S^+, t \in H, t \in H, v \in V$$

In order to enhance the relaxed sub-problem RT-P, the following valid inequalities, extending those given in Taarit et al. (2010), are appended to the model:

$$E(d_{j1})\tau_1 + z_\alpha \sigma_j - I_{j0} \leq \sum_{v \in V} \sum_{i \in S^+} Q_{ij1}^v, \forall j \in S \quad (4.14)$$

These inequalities prevent stock-outs from occurring, with some level of confidence, at each retailer during the first period of the planning horizon guaranteeing the required service level. Note also that the sub-problem (RT-P) can further be decomposed into independent sub-problems each limited to only one period of the planning horizon H . The sub-problem RT-P is a mixed-integer program but with fewer variables and constraints than the original inventory routing problem. It can thus be solved relatively more efficiently, than the original problem, by effective standard Branch-and-Bound based solvers.

4.5.2 Lagrangian Procedure

Using the above decomposition of the problem MP-DAIRP_α , a lower bound on its optimal value can be generated for any given vector of Lagrangian multipliers μ . The best lower bound can be obtained from the optimal vector of Lagrangian multipliers, the solution of the following Lagrangian dual problem of LR-MPIRP_α , denoted by LD-MPIRP_α .

LD-MPIRP_α : Maximize $L(\mu_{jt})$

Where $L(\mu_{jt}) =$

$$\begin{aligned} \text{Minimize} \left(\sum_{v \in V} \psi^v y_t^v + \sum_{v \in V} \sum_{t \in H} \sum_{i \in S^+} \sum_{j \in S^+} (\delta_v^v \theta_{ij} + \phi_{jt}) x_{ijt}^v + \sum_{t \in H^+} \sum_{v \in V} \eta_{jt} I_{jt} \right) \\ + \sum_{t \in H} \sum_{v \in V} \mu_{jt} \left(q_{jt} - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v - \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v \right) \end{aligned} \quad (4.15)$$

is the solution value of the relaxed problem LR-MPIRP_α .

To solve the problem LD-MPIRP_α , the corresponding sub-problems IA-P and RT-P are first solved, and then a sub-gradient algorithm is used to improve the value of $L(\mu_{jt})$. In addition, a Lagrangian heuristic method is developed to provide a feasible solution of the MP-DAIRP_α . Thus, the sub-gradient optimization procedure generates lower bounds and upper bounds iteratively and updates the best lower bound and upper bound of the problem. The main procedure is summarized below.

Reconsider the formulation of MP-DAIRP_α , let $X = (x_{ijt}^v, y^v)$ be the binary variables of the model, $Z = (Q_{ijt}^v, q_{jt}, I_{jt})$ be the continuous variables and let $P = \{(X, Z) : (4.2) - (4.6), \dots, (4.8) - (4.10)\}$ be the set of feasible solutions satisfying the constraints (4.2) - (4.6) and (4.8) - (4.10). An instance of the MP-DAIRP_α can thus be stated as minimize

$(X, Z) \in P$ $CV_\alpha(X, Z)$, where CV_α is the cost function of the model MP-DAIRP $_\alpha$. In addition, if we let $g_{jt}(j \in S, t \in H)$ denote the corresponding sub-gradients in the optimization procedure, these can be determined as shown below:

$$g_{jt} = q_{jt} - \sum_{v \in V} \sum_{i \in S^+} Q_{ijt}^v + \sum_{v \in V} \sum_{k \in S^+} Q_{jkt}^v \quad (4.16)$$

The proposed algorithm can thus be summarized as follows:

Algorithm 1: (The Lagrangian procedure for MP-DAIRP $_\alpha$)

Step 0. (Inialization):

Let LB be the best lower bound, UB be the best upper bound, and (X^*, Z^*) be the best feasible solution found so far. Let ω be the sub-gradient agility and k be the iterations counter. Initialize $LB = 0, k = 1$, the values for the initial Lagrangian multipliers μ and the value for $\omega \in (0; 1)$.

Step 1. (Computing the first UB):

Generate a feasible solution $(X_0; Z_0)$ for the MP-DAIRP $_\alpha$ by making each retailer $j \in S$ be served separately, and then solving the corresponding inventory allocation problems to obtain the objective value $CV_\alpha(X_0; Z_0)$ of the MP-DAIRP $_\alpha$. Let the current best upper bound $UB := CV_\alpha(X_0; Z_0)$ and update $(X^*, Z^*) := (X_0; Z_0)$.

Step 2. (Computing the lower bound):

Solve the sub-problems IA-P(μ_k) and RT-P(μ_k) respectively. Let C_{IA-P}^k and C_{RT-P}^k denote the corresponding objective values, then the new lower bound value is given by $C_{LR}^k := C_{IA-P}^k + C_{RT-P}^k$. If this new lower bound is greater than LB , then set $LB := C_{LR}^k$; otherwise set $\omega := \omega/2$.

Step 3. (Computing the upper bound):

Based on the solutions of IA-P(μ_k) and RT-P(μ_k), the Lagrangian heuristic method (see the algorithm in Section 4.3) is called to derive a feasible solution ($X_k; Z_k$). If this solution improves the current best upper bound, then set $UB := CV_\alpha(X_k; Z_k)$, and update $(X^*; Z^*) := (X_k; Z_k)$.

Step 4. (Updating μ):

Set step size s_k by $s_k := \omega(UB - C_{LR}^k) / \|g_k\|^2$, where g_k are the current sub-gradients, determined by (4.16). Update the Lagrangian multipliers in iteration $k + 1$: $\mu_{k+1} := \mu_k + s_k g_k$.

Step 5. (Stopping rule):

If (1) k exceeds the maximal number of iterations, or (2) C_{LR}^k is not improved for a given number of iterations, then output the LB , UB and the current optimal solution $(X^*; Z^*)$, and stop; otherwise set $k := k + 1$, and then go to Step 2.

4.5.3 Lagrangian Heuristic Method

Solving the sub-problems of IA-P and RT-P generates a lower bound for the MP-DAIRP $_\alpha$. However due to the relaxation, solutions obtained by solving IA-P and RT-P at each iteration are usually not feasible for the original problem MP-DAIRP $_\alpha$. Therefore, referring to the IRP heuristic method developed in Aghezzaf et al. (2006) and the VRP heuristic method developed in Clarke and Wright (1964), we propose a saving-based heuristic approach that exploits the Lagrangian information to derive a feasible solution for the MP-DAIRP $_\alpha$ at each iteration.

The optimal solution of the sub-problem IA-P provides information on the quantities that should be delivered to each retailer during each period of the planning horizon. These quantities can be considered as

retailer orders and a vehicle routing problem is then solved for each period separately. The resulting solution is feasible if the corresponding constraints, such as the vehicle capacity constraints and the traveling time constraints, are not violated. Detailed steps of this algorithm are presented in the following paragraph.

For each period $t \in H$, we assume that q_j is the quantity to be delivered to the retailer $j \in S$. Let SR_t be the set of the be served retailers in period t , i.e., $SR_t = \{j : q_j > 0; \forall j \in S\} \forall t \in H$. Note that SR_t can possibly be an empty set for some $t \in H$. In this case no vehicle route needs to be generated in that period. The proposed saving-based heuristic method is outlined below:

Algorithm 2: (The Lagrangian heuristic algorithm for MP-DAIRP $_{\alpha}$)

Step 0. (Initialization):

Suppose one vehicle is available for serving the retailers at first. A temporary route is initiated with the basic tours, each serving one of the retailers by the vehicle (i.e., ignore the restriction for the total travelling time of the route made by the vehicle, but each separated vehicle tour in the route should satisfy the traveling time constraint). Thus, there are as many tours in the initial temporary route as there are retailers in the set SR_t .

Step 1. (Improvement Step):

The core of the saving-based heuristic is the process of combining two tours into one route to achieve some cost saving. This is implemented as follows:

- Suppose the current route L^* makes n tours. These n tours are put into a single list of tours C^1, \dots, C^n . We then calculate the cost values $CV^{C^1}, \dots, CV^{C^n}$ for each tour.
- For all $1 \leq i < j \leq n$, combine tours C^i and C^j into one tour, denoted by C^+ (by finding the TSP through all retailers covered by

both tours plus the warehouse. If this tour is infeasible (that is it doesn't satisfy the travel time constraint) then it is disregarded and a new route is generated that makes tours $C^1, \dots, C^{i-1}, C^{i+1}, \dots, C^{j+1}, C^{j+1}, \dots, C^n$ and C^+ . If the cost value CV^{C^+} is smaller than the sum of cost values of C^i and C^j , then we have achieved a saving $SV := CV^{C^i} + CV^{C^j} - CV^{C^+}$.

- By calculating all the combinations of two tours from the list, the best feasible combination is kept, i.e., the one which results in the largest saving. This best feasible combination is then added to the tour list and the two combined tours are removed from the list.

Step 2. (Stopping rule):

Repeat Step 1 until no further feasible combination resulting in a positive saving can be found. Calculate the total travelling time T_{min} for the current route L^* , if $T_{min} > \tau_t$, then calculate the vehicle number V , $N := \lceil T_{min}/\tau_t \rceil + 1$, and add the corresponding additional fixed operating costs of the vehicles to the route cost. Finally we get the output of the best feasible route and its cost value.

By calling Algorithm 2, we find a feasible solution for each period $t \in H$ ($SR_t \neq \emptyset$) separately. As a consequence, at each iteration of the Lagrangian relaxation approach, we generate a feasible solution that is an upper bound for the MPDAIRP $_{\alpha}$ using the above saving-based heuristic algorithm. The best upper bound is updated iteratively each time a better delivery schedule is obtained. To improve the upper bound of the MP-DAIRP $_{\alpha}$ further, we apply an adjustment procedure to the best feasible solution of the MP-DAIRP $_{\alpha}$ found by the Lagrangian relaxation approach. The main goal of this adjustment is to eliminate the unnecessary deliveries to the retailers, during the planning horizon, reducing by the way potential transportation costs.

The adjustment procedure tries thus to combine two or more deliveries to a retailer j ($j \in S$) into one delivery. If a better feasible solution of the MP-DAIRP $_{\alpha}$ is found, this adjustment is considered to be effective.

tive and is consequently kept. Details of the adjustment procedure are presented below:

Algorithm 3: (The adjustment procedure for MP-DAIRP _{α})

Step 0. (Initialization):

Let UB be the best upper bound and $(X^*; Z^*)$ be the best feasible solution found so far. Let $q_{jt}^* \in Z^*$ for all $(j \in S, t \in H)$ be the current best delivery schedule and $I_{j0}^* \in Z^*$ for all $(j \in S)$ be the current best initial inventory level. Let W be the set of the retailers that have more than one delivery during the planning horizon, i.e., $W = \{j: q_{jm}^* > 0 \text{ and } q_{jn}^* > 0, \text{ for some } m, n \in H, \text{ and } m \neq n\}$. Initialize the iteration number $k=0$.

Step 1. (Adjustment):

For all the remaining retailers in the set W , do the adjustment as follows:

(1-a): Select a retailer j from W , and delete it from the set W . For this retailer, starting from the final period during which the delivery takes place, do the delivery combination. If we let q_{jt}^k be the current delivery schedule, as a result of this combination we obtain $q_{jm}^k := (q_{jm}^* + \sum_{m < n \leq T} q_{jn}^*) \leq k$ and then $\sum_{m < n \leq T} q_{jn}^k = 0$, $q_{jn}^* > 0$, and $1 \leq m < n \leq T$, where k is the vehicle capacity. At the same time, adjust the corresponding inventory levels of retailer j . For this new delivery schedule, call the saving-based heuristic approach (Algorithm 2) to find a new feasible solution $(X_k; Z_k)$. If this new solution gives a better objective value such that $CV_\alpha(X_k; Z_k) < UB$, then update the current best upper bound by $UB := CV_\alpha(X_k; Z_k)$, and update the current best feasible solution by $(X^*; Z^*) := (X_k; Z_k)$

(1-b): For retailer j , according to the current best delivery schedule q_{jt}^* , if there still exist $q_{jm}^* > 0$ and $q_{jn}^* > 0$, for $1 \leq m < n \leq T$, then include

the delivered quantities at period m in the initial inventory. If we let I_{j0}^k be the current initial inventory level, as a result of this inclusion we obtain $I_{j0}^k := (I_{j0}^* + \sum_{1 < m \leq n} q_{jm}^*)$ and $q_{jn}^k := (q_{jn}^* + \sum_{1 < m \leq n} q_{jm}^*) \leq k$ and then set $\sum_{1 < m \leq n} q_{jm}^k = 0$, for $q_{jm}^* > 0$, $q_{jn}^* > 0$ and $1 \leq m < n \leq T$. At the same time, adjust the corresponding inventory levels of retailer j . Then call Algorithm 2 for the new delivery schedule to find a new feasible solution $(X_k; Z_k)$. If a better feasible solution is obtained, then update the current best upper bound by $UB := CV_\alpha(X_k; Z_k)$, and update the current best feasible solution by $(X^*; Z^*) := (X_k; Z_k)$.

Step 2. (Stopping rule):

If there are still retailers in set W , then set $k := k + 1$ and go to Step 1. Otherwise output UB and the current best solution $(X^*; Z^*)$, and stop.

If with this adjustment procedure, some unnecessary deliveries during the planning horizon are eliminated, then some additional savings in the total transportation cost can be realized. This in turn can improve the upper bound of the MP-DAIRP $_\alpha$.

4.6 Computational Results

In this section, we present some numerical experiments to evaluate the performance of the proposed Lagrangian relaxation approach for the MP-DAIRP $_\alpha$, using some randomly generated instances according to the generation scheme proposed by Yu et al. (2008). We consider different sets of problem instances with different retailer and planning horizon sizes. Each problem instances set is identified by the number of retailers N and time horizon T . The four tested problem instances sets that are considered in this analysis are: $(N=15; T=3)$, $(N=15; T=6)$, $(N=25; T=3)$ and $(N=25; T=6)$.

More specifically, for the set of the instances consisting of 15 retailers (denoted by Y15-x-T-y, where 'x' is the index of instances and 'y' is the index of time horizons), the retailers are scattered randomly and uniformly over a square of 30 by 30 km, and the distribution cen-

tre is always put in the centre of the square. Average demand rates of retailers are generated randomly and uniformly between 1 and 3 tons per hour with a standard deviation of 5% of the average over the planning horizon and the standard normal value z_α is set to 1.64. The inventory holding costs are generated randomly and uniformly between 0.1 and 0.15 (in euros per tons per period). Fixed delivery handling costs are the same for all the retailers, which are €25 per delivery. A fleet of homogeneous vehicles with a capacity of 60 tons is used to serve these retailers. The fixed operating cost of vehicle is €50 per vehicle. The vehicles can travel up to 50 km per hour, and the travel cost of a vehicle is €1 per km per hour. The time unit of one period τ_i is set to 8 hours in all instances.

For the set of the instances consisting of 25 retailers (denoted by A25-x-T-y), the retailers are scattered randomly over a square of 100 by 100 km (in clusters), and the warehouse is always placed in the centre of the square. Demand rates of the retailers are generated randomly and uniformly between 0.1 and 3 tons per hour with a standard deviation of 5% of the average over the planning horizon and the standard normal value z_α is set to 1.64. The inventory holding costs are also generated randomly and uniformly between 0.1 and 0.15 (in euros per tons per period) and fixed delivery handling costs are €10 per delivery. Again, a fleet of homogeneous vehicles with a capacity 60 tons is used to serve the retailers. The vehicles can also travel up to 50 km per hour, and the travel cost of the vehicle is €1 per km. The fixed operating cost of vehicle is €30 per vehicle. Also, the time unit of one period τ_i is still set to 8 hours in all instances. The proposed Lagrangian relaxation approach for MP-DAIRP $_\alpha$ is implemented in AMPL and all instances are tested on a PC with Intel (R) Core i7-3770 CPU @3.40GHz, 32.0GB RAM.

4.6.1 Effect of Changing the Vehicle Capacity

In order to evaluate the performance of our solution approach, the vehicle capacity is set between 60 tons and 80 tons. For different instances sets, Tables 4.3, 4.4 and 4.5 show the upper bounds found by

the MP-DAIRP $_{\alpha}$ with AMPL, and the gaps of the vehicle capacity restriction, i.e. *Gap1*, determined by:

$$Gap1 = \frac{UB_{80} - UB_{60}}{UB_{80}} \times 100\%$$

We observe that for all the problem sets, the average gap between the upper bound of 60 tons and the upper bound of 80 tons is around 0.22%, with the lowest gap being 0.16% and the highest gap being 0.29%, while the computational time is set to 1 and 2 hours, and the MIP gap tolerance for all instances is reached below 5%. Also, for all instances, the average optimal objective value for 60 tons is €1581.81, while the average optimal objective value for 80 tons is €1227.41.

On one hand, with the increase in vehicle capacity, the number of vehicles used decreases, which results in the decrease of transportation costs. On the other hand, when the vehicle capacity increases, the number of direct deliveries tends to increase, which makes it easier to find optimal routes in each period by the heuristic method.

Table 4.3. Results of the MP-DAIRP $_{\alpha}$ AMPL with instances (N=15; T=3)

Instances	MP-DAIRP $_{\alpha}$ UB-AMPL		
	60 tons (€)	80 tons (€)	<i>Gap1</i> (%)
Y15-0-T-3	967.00	726.50	0.25
Y15-1-T-3	882.00	721.00	0.18
Y15-2-T-3	851.59	690.50	0.19
Y15-3-T-3	900.58	681.50	0.24
Y15-4-T-3	878.63	717.00	0.18
Y15-5-T-3	953.88	695.50	0.27
Y15-6-T-3	828.80	615.50	0.26
Y15-7-T-3	654.00	527.50	0.19
Y15-8-T-3	780.29	655.50	0.16
Y15-9-T-3	886.35	674.50	0.24
Average	858.31	670.50	0.22

Table 4.4. Results of the MP-DAIRP $_{\alpha}$ AMPL with instances
(N=15; T=6)

Instances	MP-DAIRP $_{\alpha}$ UB-AMPL		
	60 tons (€)	80 tons (€)	Gap1 (%)
Y15-0-T-6	1647.00	1229.00	0.25
Y15-1-T-6	1556.00	1230.50	0.21
Y15-2-T-6	1413.50	1124.00	0.20
Y15-3-T-6	1538.00	1158.50	0.25
Y15-4-T-6	1472.29	1167.00	0.21
Y15-5-T-6	1635.50	1204.00	0.26
Y15-6-T-6	1375.17	1016.00	0.26
Y15-7-T-6	1363.00	966.50	0.29
Y15-8-T-6	1340.00	1124.00	0.16
Y15-9-T-6	1516.50	1141.00	0.25
Average	1485.70	1136.05	0.23

Table 4.5. Results of the MP-DAIRP $_{\alpha}$ AMPL with instances
(N=25; T=3) and (N=25; T=6)

Instances	MP-DAIRP $_{\alpha}$ UB-AMPL		
	60 tons (€)	80 tons (€)	Gap1 (%)
A25-0-T-3	1319.88	1044.50	0.21
A25-1-T-3	1662.37	1322.00	0.20
A25-2-T-3	1347.75	1029.00	0.24
A25-3-T-3	1211.64	976.00	0.19
A25-4-T-3	1678.87	1347.50	0.20
Average	1444.10	1143.80	0.21
A25-0-T-6	2379.97	1795.50	0.25
A25-1-T-6	3042.50	2345.50	0.23
A25-2-T-6	2299.48	1802.50	0.22
A25-3-T-6	2048.22	1581.50	0.23
A25-4-T-6	2925.58	2271.50	0.22
Average	2539.15	1959.30	0.23

4.6.2 Performance Comparison of the Lagrangian Relaxation and AMPL

To compare the solution values obtained from the Lagrangian relaxation approach, we have solved the problem MP-DAIRP_α directly for the generated instances using CPLEX 12.5. Due to the large number of variables and constraints in the problem MP-DAIRP_α, solving these instances with AMPL is quite time-consuming, in particular, when the size of the instances grows bigger. Therefore, for the problem sets of (N=15; T=3) and (N=15; T=6), we have decided to preset 1 hour as a limitation of the running time, and for the problem sets of (N=25; T=3) and (N=25; T=6), we preset 2 hours as a limitation of running time. The computational results of these two different problem instances are shown in the following tables.

For different instance sets, Tables 4.6, 4.7 and 4.8 show the lower bounds and upper bounds found by the Lagrangian relaxation approach, and the gaps between lower bounds and upper bounds, i.e. *Gap2*, determined by:

$$Gap2 = \frac{UB_{LR} - LB_{LR}}{UB_{LR}} \times 100\%$$

In addition, Tables 4.9, 4.10 and 4.11 present the comparisons between the results obtained from the Lagrangian relaxation approach and AMPL, and the values of *Gap3* illustrate the difference in the two upper bounds, where:

$$Gap3 = \frac{UB_{LR} - UB_{AMPL}}{UB_{LR}} \times 100\%$$

The results shown in the tables demonstrate the effectiveness of the proposed Lagrangian relaxation approach. Near optimal solutions are found for the problem sets (N=15, T=3) and (N=15, T=6) by the Lagrangian relaxation approach. For these two problem sets, the gaps between the solutions obtained from the Lagrangian relaxation ap-

proach and AMPL, in average, are 1.65% and 5.02%, respectively, and the worst cases are 4.07% and 6.66% respectively. The average gaps between the Lagrangian upper and lower bounds for these two problem sets are 5.62% and 18.79%, respectively. The computational time of the Lagrangian relaxation approach varies between 3 and 6 minutes for these instances.

Table 4.6. Results of the LR approach (N=15; T=3)

Instances (60 tons)	Lagrangian Relaxation (LR) Solutions			
	LB (€)	UB (€)	CPU time (s)	<i>Gap2</i> (%)
Y15-0-T-3	929.92	967.00	108.35	3.83
Y15-1-T-3	793.83	890.30	79.46	10.84
Y15-2-T-3	802.20	864.16	80.50	7.17
Y15-3-T-3	837.05	933.85	85.22	10.37
Y15-4-T-3	869.04	915.88	104.63	5.11
Y15-5-T-3	916.33	968.51	115.91	5.39
Y15-6-T-3	789.32	846.28	103.48	6.73
Y15-7-T-3	654.00	654.00	119.56	0.00
Y15-8-T-3	775.14	803.77	114.71	3.56
Y15-9-T-3	857.61	886.35	144.98	3.24
Average	822.44	873.01	105.68	5.62

Table 4.7. Results of the LR approach (N=15; T=6)

Instances (60 tons)	Lagrangian Relaxation (LR) Solutions			
	LB (€)	UB (€)	CPU time (s)	Gap2 (%)
Y15-0-T-6	1393.41	1711.81	241.60	18.60
Y15-1-T-6	1349.99	1627.22	209.35	17.04
Y15-2-T-6	1243.36	1503.67	163.72	17.31
Y15-3-T-6	1233.79	1624.01	242.00	24.03
Y15-4-T-6	1240.43	1549.87	341.37	19.97
Y15-5-T-6	1407.08	1709.92	254.16	17.71
Y15-6-T-6	1212.50	1433.47	240.82	15.42
Y15-7-T-6	1210.31	1460.24	235.66	17.12
Y15-8-T-6	1118.96	1425.84	195.94	21.52
Y15-9-T-6	1285.01	1589.61	332.32	19.16
Average	1269.48	1563.57	245.69	18.79

Table 4.8. Results of the LR approach (N=25; T=3) and (N=25; T=6)

Instances (60 tons)	Lagrangian Relaxation (LR) Solutions			
	LB (€)	UB (€)	CPU time (s)	Gap2 (%)
A25-0-T-3	1130.55	1390.15	119.84	18.67
A25-1-T-3	1509.15	1937.40	80.47	22.10
A25-2-T-3	1188.70	1487.57	123.60	20.09
A25-3-T-3	1000.10	1233.81	129.69	18.94
A25-4-T-3	1504.33	1794.87	129.98	16.19
Average	1266.57	1568.76	116.72	19.20
A25-0-T-6	2073.63	2542.53	339.93	18.44
A25-1-T-6	2696.40	3272.11	286.27	17.59
A25-2-T-6	1834.65	2433.63	265.14	24.61
A25-3-T-6	1610.83	2135.01	203.12	24.55
A25-4-T-6	2524.29	3169.49	194.67	20.36
Average	2147.96	2710.55	257.83	21.11

When the size of the instances increases, as for the problem sets (N=25, T=3) and (N=25, T=6), the averaged values of *Gap2* are 19.20% and 21.11%, respectively, whereas the average values of *Gap3* are 7.38% and 6.14%, respectively. Observe that the increase of the planning horizon from T=3 to T=6 creates more flexibility for the MP-DAIRP_α to achieve a trade-off between the distribution costs and the inventory costs. Consequently, the averaged values of *Gap3* rise to 5.02% and 6.14% for the problems sets (N=15, T=6) and (N=25, T=6), respectively, when they are compared with the corresponding values of problem sets with (N=15, T=3) and (N=25, T=3). Finally, observe that, in terms of computational time, quite good solutions can be obtained by the proposed Lagrangian relaxation approach within a few minutes, while the solver for the original problem takes many hours of the running time.

Table 4.9. Solution comparisons of the Lagrangian relaxation approach and AMPL (N=15; T=3)

Instances (60 tons)	LR Solutions UB (€)	AMPL Solutions UB (€)	<i>Gap3</i> (%)
Y15-0-T-3	967.00	967.00	0.00
Y15-1-T-3	890.30	882.00	0.93
Y15-2-T-3	864.16	851.59	1.45
Y15-3-T-3	933.85	900.58	3.56
Y15-4-T-3	915.88	878.63	4.07
Y15-5-T-3	968.51	953.88	1.51
Y15-6-T-3	846.28	828.80	2.07
Y15-7-T-3	654.00	654.00	0.00
Y15-8-T-3	803.77	780.29	2.92
Y15-9-T-3	886.35	886.35	0.00
Average	873.01	858.31	1.65

Table 4.10. Solution comparisons of the Lagrangian relaxation approach and AMPL (N=15; T=6)

Instances (60 tons)	LR Solutions UB (€)	AMPL Solutions UB (€)	Gap3 (%)
Y15-0-T-6	1711.81	1647.00	3.79
Y15-1-T-6	1627.22	1556.00	4.38
Y15-2-T-6	1503.67	1413.50	6.00
Y15-3-T-6	1624.01	1538.00	5.30
Y15-4-T-6	1549.87	1472.29	5.01
Y15-5-T-6	1709.92	1635.50	4.35
Y15-6-T-6	1433.47	1375.17	4.07
Y15-7-T-6	1460.24	1363.00	6.66
Y15-8-T-6	1425.84	1340.00	6.02
Y15-9-T-6	1589.61	1516.50	4.60
Average	1563.57	1485.70	5.02

Table 4.11. Solution comparisons of the Lagrangian relaxation approach and AMPL (N=25; T=3) and (N=25; T=6)

Instances (60 tons)	LR Solutions UB (€)	AMPL Solutions UB (€)	Gap3 (%)
A25-0-T-3	1390.15	1319.88	5.05
A25-1-T-3	1937.40	1662.37	14.20
A25-2-T-3	1487.57	1347.75	9.40
A25-3-T-3	1233.81	1211.64	1.80
A25-4-T-3	1794.87	1678.87	6.46
Average	1568.76	1444.10	7.38
A25-0-T-6	2542.53	2379.97	6.39
A25-1-T-6	3272.11	3042.50	7.02
A25-2-T-6	2433.63	2299.48	5.51
A25-3-T-6	2135.01	2048.22	4.07
A25-4-T-6	3169.49	2925.58	7.70
Average	2710.55	2539.15	6.14

4.7 Concluding Remarks

We investigated the multi-period inventory routing problem (MP-SIRP) in which a single warehouse is distributing a single product to a set of retailers consuming it at stochastic stationary demand rates, using a fleet of homogeneous vehicles over a given finite horizon. The objective is to determine the optimal quantities to be delivered to the retailers, the delivery time, and to design vehicle delivery routes, so that the total distribution and inventory costs are minimized while some service level is guaranteed at each retailer during each period of the planning horizon. The MP-SIRP is first formulated as a linear mixed-integer program and then a deterministic equivalent approximation model, MP-DAIRP $_{\alpha}$, is proposed. In this deterministic model, the stochastic demand constraints are replaced with deterministic ones guaranteeing some predetermined service level at each retailer. A Lagrangian relaxation approach is used to decompose the MP-DAIRP $_{\alpha}$ and to derive both the lower and upper bounds for it. The two sub-problems resulting from this decomposition are an inventory allocation and a vehicle routing optimization problem in each period. The stochastic aspect of the demand rates is taken care of within the inventory allocation sub-problem through provision of safety stocks at the retailers. Computational results on some medium size instances demonstrate the effectiveness of the proposed Lagrangian relaxation approach. Good quality solutions for the MP-DAIRP $_{\alpha}$ were found within a reasonable computational time. Numerical experiments on large scale problems and other deterministic approximation to the MP-SIRP are currently under investigation.

Acknowledgments: This chapter is derived in part from the article “Modeling and Solving the Multi-period Inventory Routing Problem with Stochastic Stationary Demand Rates” by M.K.I. Abdul Rahim, Y. Zhong, E.-H. Aghezzaf and T. Aouam, published in *International Journal of Production Research*, 52(14), 4351-4363, 10 February 2014, copyright Taylor & Francis, available online: <http://www.tandfonline.com/doi/10.1080/00207543.2014.883470>

5

Conclusions

5.1 Concluding Summary

In this dissertation, we study the integrated inventory and routing management with the main objectives: (1) to analyze the effectiveness of an inventory management policy before and after the implementation of vendor managed inventory (VMI) and milk-runs in a single-warehouse, multiple-retailer (SWMR) system, and (2) to investigate the multi-period stochastic inventory routing problem (MP-SIRP). More specifically, in the first situation, we assume that all the retailers face a deterministic and constant demand rate whereas in the second condition, we assume that all the retailers consume the product at a stochastic stationary rate. In both cases, the supplier serves a set of retailers from a single warehouse and deliveries to these retailers are made from the warehouse with a fleet of vehicles having a limited capacity. The warehouse in turn is replenished from an outside source. Incoming shipments into the warehouse have to be coordinated with

outgoing shipments to the retailers in order to minimize the total cost. This total cost consists of inventory holding costs at the central warehouse and all the retailers, costs for incoming shipment into the warehouse, and outbound shipment costs for the retailer replenishments. The objective in both the situation is to decide when and how many units to be delivered from the supplier to the warehouse and from the warehouse to retailers respectively in order to minimize total distribution and inventory holding costs over the finite horizon without any shortages.

For the class of infinite horizon with deterministic IRP, we discussed an optimization approach, for a two-stage supply chain implementing VMI with the objective to minimize the overall inventory and transportation costs of the SWMR system. We then propose a two-phase heuristic solution approach, one for inventory management and the second for routing optimization.

For the inventory management problem, retailers are partitioned into subsets in order to minimize the overall inventory costs of the system. In this phase, we start by initializing the set of clusters, with each retailer in a separate cluster, i.e., the direct shipping case. We then use the algorithms proposed by Roundy (1985) and improved by Chu and Leon (2008) to find the replenishment interval for each retailer as well as the warehouse. These power-of-two order intervals are then used in the next phase.

For the routing problem, a VRP procedure is used to solve the routing in each of the retailer subsets with the objective of minimizing the travelled distance and the transportation costs. As such, we drop the assumption of direct shipments from warehouse to retailers, but also include the option of combining multiple outbound shipments in the so-called *milk-runs*. In this phase, the retailers are clustered per replenishment interval. Then, we use the savings heuristic of Clarke and Wright (1964) for each of the clusters in order to optimize the transportation costs and to select retailers who can be replenished in a milk-run rather than with separate direct shipments.

The results of the proposed approach allowed us to investigate the effectiveness of an inventory management policy before and after the implementation of VMI and milk-runs in a two-stage supply chain. We discovered that the transportation cost is relevant as the effect of VMI and milk-runs can result in a significant inventory and transportation cost savings. Additionally, we evaluated the results of the vehicle storage capacity restrictions. The vehicle capacity factor is used to show that our solution approach not only helps to decide on the fleet size, but can also be used to select the most appropriate vehicle type for a particular problem in this instance.

For the class of finite horizon with stochastic IRP, we investigated the MP-SIRP where the retailers consume the product at a stochastic stationary rate. In particular, we considered a distribution system in which a fleet of homogeneous vehicles is used to distribute some products from a single warehouse to a set of retailers consuming it at stochastic stationary demand rates, during a finite horizon of consecutive periods. The objective is to determine optimal quantities to be delivered to the retailers, delivery time, and to design vehicle delivery routes, so that the total distribution and inventory costs are minimized while some service level is guaranteed at each retailer during each period of the planning horizon.

Based on the formulation of the cyclic IRP model (see, e.g., Aghezzaf et al. 2006) and Multi-period IRP (see, e.g., Zhong and Aghezzaf 2012), the MP-SIRP is first formulated as a linear mixed-integer program and then a deterministic equivalent approximation model, MP-DAIRP α , is proposed. In this deterministic model, the stochastic demand constraints are replaced with deterministic ones guaranteeing some predetermined service level at each retailer. This latter proposed model also determines the optimal vehicle fleet size in each period.

To solve the problem, a Lagrangian relaxation approach is used to decompose the MP-DAIRP α and to derive both lower and upper bounds for it. The two sub-problems resulting from this decomposition are an inventory allocation and a vehicle routing optimization

problem in each period. The stochastic aspect of the demand rates is taken care of within the inventory allocation sub-problem through the provision of safety stocks at the retailers.

Computational results demonstrate the effectiveness of the proposed Lagrangian relaxation approach. More specifically, we considered different sets of problem instances with different retailers and planning horizon sizes. Each problem instances set is identified by the number of retailers N and time horizon T . The four tested problem instances sets that are considered in this analyses are: ($N=15$; $T=3$), ($N=15$; $T=6$), ($N=25$; $T=3$) and ($N=25$; $T=6$). To compare the solution values obtained from the Lagrangian relaxation approach, we have solved the problem MP-DAIRP α directly for the generated instances using AMPL CPLEX 12.5. In terms of computational time, relatively better solutions could be obtained by the proposed Lagrangian relaxation approach within a few minutes, while the solver of the original problem takes many hours to complete a run.

5.2 Further Extensions

For the SWMR-VMI problem, further research will focus on adapting the model with more complex distribution patterns including larger sets of retailers, driving-time restrictions on the vehicles and their drivers, delivery time windows at the retailers, heterogeneous vehicle fleets, multiple warehouses, multiple products etc. For instance, in this dissertation, we assumed that a vehicle can make a tour at any time of day. However, in most real-case situations, the driving time of the vehicles is restricted. Deliveries can only occur during the day. Thus, replenishment cycle times are allowed during the day, on weekdays, so a vehicle can only drive 8 hours a day, and 5 days a week. Consequently, it is worthwhile to investigate how the SWMR-VMI approach can be extended to explicitly take into consideration some demands and travel times variability. These two parameters need to be taken into account during the development of the distribution plan, because it will become the case in more realistic problems and will increase the performance of the solution approach. We believe that this is an

interesting extension of the SWMR problem and it may be suitable for a distribution system with multiple clusters of retailers.

In the second part, the model and solution approaches presented in the multi-period stochastic inventory routing problem (MP-SIRP) consider that the retailer-demand is stochastic stationary rates. Beyond the possible algorithmic extensions to the multi-period stochastic inventory routing problem, future research could be extended to accommodate more complex distributions such as non-stationary stochastic demand, multiple products, heterogeneous vehicle fleet, retailer time windows for delivery, etc. For instance, the demand for products is not only stochastic stationary rates, but may also be non-stationary. This leads to an irregular pattern of the decision when to produce, and how much. This challenging problem needs to be solved efficiently. Also, one of the approaches to hedge against the variability of demand rates at each retailer is to introduce safety stock, which are called ‘fixed’ safety stock (kept at the retailers) and ‘mobile’ safety stock (carried by the vehicle during each of the tours). This approach needs to be very effective in resolving the issue of uncertainty. Moreover, with respect to the proposed Lagrangian relaxation approach, more intelligent heuristic/meta-heuristic or approximation methods are worth being investigated, so that the upper bound of the problem could be further improved, especially in large-size instances.

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M.K.I. Abdul Rahim, Y. Zhong, E.-H. Aghezzaf and T. Aouam (2014) Modeling and Solving the Multi-period Inventory Routing Problem with Stochastic Stationary Demand Rates, *International Journal of Production Research*, 52(14), 4351-4363.

M.K.I. Abdul Rahim, E.-H. Aghezzaf, V. Limère and B. Raa (2014) Analyzing The Effectiveness of Vendor Managed Inventory in a Single-Warehouse, Multiple-Retailer System, *International Journal of System Science*, DOI: 10.1080/00207721.2014.965771.

Papers at International Conferences - Published in Full in Proceedings (C1/P1)

M.K.I. Abdul Rahim and E.-H. Aghezzaf (2014) Effectiveness of Vendor Managed Inventory Approach in a Two-Stage Supply Chain when Demand Rates are Static, *Proceedings of 12th International Conference on Numerical Analysis and Applied Mathematics (ICNAAM 2014)*, September 22-28, Rhodes, Greece. (oral presentation)

M.K.I. Abdul Rahim and E.-H. Aghezzaf (2012) Implementing a Vendor Managed Inventory Policy in a Two-Stage Supply Chain with Stochastic Demands, *Proceedings of 14th IFAC Symposium on Information Control Problems in Manufacturing (INCOM 2012)*, May 23-25, Bucharest, Romania, pp. 602-607 ISBN: 978-3-902661-98-2, available online: 10.3182/20120523-3-RO-2023.00215 (oral presentation)

Meeting Abstracts - Presented at National and International Conferences (C3)

M.K.I. Abdul Rahim and E.-H. Aghezzaf (2012) Managing Inventory and Routing in a Two-Stage Supply Network with Stochastic Demands, Proceedings of 26th Annual Conference of the Belgian Operations Research Society (ORBEL 26), February 2-3, Brussels, Belgium, p. 44-45. (oral presentation)

M.K.I. Abdul Rahim and E.-H. Aghezzaf (2011) Location and Inventory Routing Problems with Stochastic Demand and Lead Times, In: Book of Abstracts of the 12th UGent - Firw Doctoraats Symposium, Gent, Belgium, December 7, 2011. (poster presentation)

M.K.I. Abdul Rahim and E.-H. Aghezzaf (2011) Location and Inventory Routing Problems with Stochastic Demand and Lead Times, Proceedings of 25th Annual Conference of the Belgian Operations Research Society (ORBEL 25), February 10-11, Ghent, Belgium, p. 49-51. (oral presentation)

