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# From the Moufang world to the structurable world and back again

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# Contents

# Introduction

The main theme of this thesis is to study some algebraic structures related to exceptional linear algebraic groups of relative rank 1 or 2. More precisely, the goal of this thesis is to connect the Moufang world and the structurable world, which seem to have been two isolated islands up to now.

## Some history

Since the beginning of the 20th century mathematicians try to get hold on linear algebraic groups. These are certain structures that arise in different areas of mathematics and gave rise to the development of a lot of interesting theories. The theory of linear algebraic groups is a very active area of research, that contains a lot of open questions. Amongst others it is not clear how certain forms of exceptional linear algebraic groups can be described explicitly. From classification results we know they exist, but for certain forms of exceptional linear algebraic groups no explicit description is known.

In order to get a deeper understanding of the isotropic linear algebraic groups, Jacques Tits introduced the notion of a building, which is a geometrical structure built up from the parabolics of the linear algebraic group. On the other hand, these geometries can be defined axiomatically, giving rise to a more general notion, where more examples occur.

If a spherical building arises from a linear algebraic group, then the linear algebraic group has a natural action on its building. Therefore studying buildings could be useful to get a better understanding of those groups.

In 1974, Jacques Tits published his lecture notes "Buildings of Spherical Type and Finite BN-Pairs" [Tit74], in which he classified all spherical buildings of rank at least 3. The following quote is taken from *loc. cit.*:

> *The origin of the notions of buildings and BN-pairs lies in an attempt to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups.*

There is no hope to classify spherical buildings of rank two. For example, the class of buildings of rank 2 contains the class of all projective planes, and these are impossible to classify. This is why it is necessary to impose an extra condition: the Moufang condition.

It is only 26 years later that Jacques Tits and Richard Weiss in [TW02] finished the classification of spherical buildings of rank 2 satisfying the Moufang property, also called *Moufang polygons*; see Chapter 2. There is no doubt that the hardest part in the whole classification is precisely where the quadrangles coming from exceptional linear algebraic groups of relative rank 2 turn up. All the Moufang quadrangles can in turn be described using an algebraic structure, that is well understood when the Moufang polygon is coming from an (infinite dimensional) classical linear algebraic group of rank 2. However the algebraic structure that determines Moufang quadrangles of type $E_6$, $E_7$ and $E_8$, has a very artificial coordinate-based definition, that came up during the classification, containing (in the case of $E_8$) a 12-dimensional quadratic space, a 32-dimensional vector space, and several maps between those spaces.

In fact, there had been several attempts to try to understand the structure of these exceptional quadrangles. For example, Tom De Medts introduced the notion of *quadrangular systems* [DM05], a uniform algebraic structure describing all Moufang quadrangles. From a different point of view, Richard Weiss defined *quadrangular algebras* [Wei06b], which are a set of algebraic data coordinatizing the exceptional Moufang quadrangles. If we exclude the characteristic 2 case, then a quadrangular algebra is either obtained from a pseudo-quadratic space, or it is the structure that coordinatizes Moufang quadrangles of type $E_6$, $E_7$ and $E_8$.

Spherical Moufang buildings of rank one are called *Moufang sets*, as they can be described without using any geometry only using a set and a collection of groups that acts on this set; see Section 2.4. One is far from giving a classification of Moufang sets, it is even not clear if it will be ever possible to give such a classification. In view of this, it is of interest to find descriptions of Moufang sets. In particular (and this is one of the main motivations for studying Moufang sets) it is interesting to try to describe Moufang sets obtained from exceptional linear algebraic groups of relative rank one, and we hope that they will give new insights in the study of those linear algebraic groups. As an example, we mention that the Kneser–Tits problem for groups of type $E_{8,2}^{66}$ has recently been solved using the theory of Moufang quadrangles [PTW12], and for many of the exceptional groups of relative rank one, the Kneser–Tits problem is still open.

In this thesis we take a new point of view towards these not-so-well-understood-Moufang-objects. We apply the theory of structurable algebras to give new interpretations of these Moufang structures. Structurable algebras (see Chapter 3) are certain non-associative algebras with involution, this class of algebras contains, amongst others, Jordan algebras (with trivial involution) and associative algebras with involution.

In order to provide explicit constructions of exceptional Lie algebras, several authors (I.L. Kantor, B. Allison, J. Tits, J. Faulkner, W. Hein and many others) have introduced various algebraic structures. In each case, the starting point of those constructions is a non-associative binary or ternary algebra, or a pair of them. One of the earliest is the Tits–Kantor–Koecher[1] construction of Lie algebras from Jordan algebras (see [Jac68, Section VIII.5]); its main purpose was to provide models of Lie algebras of type $E_7$.

Structurable algebras were introduced by Bruce Allison, he gives a very successful generalization of the Tits–Kantor–Koecher construction starting from structurable algebras. They provide a construction of all isotropic simple Lie algebras in characteristic 0. Structurable algebras can only be defined over fields of characteristic different from 2 and 3. If $\mathrm{char}(k) > 3$, all simple Lie algebras satisfying an additional (more technical) condition can be obtained. Central simple structurable algebras have been classified (over fields of characteristic bigger than five) and consist of six different classes.

## New progress

In Chapters 4 and 5, we relate quadrangular algebras with structurable algebras in various ways. Since quadrangular algebras of type $E_8$ are the most interesting and least understood case, we will focus on this case in this introduction. A quadrangular algebra of type $E_8$ is totally determined by a so-called quadratic form of type $E_8$. It is shown in Lemma 5.15 that such an anisotropic quadratic form can be written as

$$q \sim N \otimes (\langle\!\langle s_2, s_3 \rangle\!\rangle \perp -\langle\!\langle s_4 s_6, s_5 s_6 \rangle\!\rangle)$$

where $N$ is the norm of a separable quadratic field extension $E/k$ and $s_2, \ldots, s_6 \in k$ such that $s_2 s_3 s_4 s_5 s_6 = -1$. This quadratic form is thus entirely determined by the field $E$ and the two quaternion algebras[2] $Q_1 = (-s_2, -s_3)_k$ and $Q_2 = (-s_4 s_6, -s_5 s_6)_k$.

Our starting point was the strong belief that since these algebraic data determine the quadratic form of type $E_8$ completely, there had to be some nice algebraic structure, built from such a quadratic field extension and two quaternion algebras, relating these data directly to the quadrangular algebra itself.

[1] This construction is also referred to by various permutations of subsets of these three names.

[2] However, this field and these quaternion algebras are *not* uniquely determined by the quadratic form.

Eventually, we found two different ways to achieve this. One could say that the second one is more successful than the first, but nonetheless both of them are interesting.

**In Chapter 4** we identify the quadrangular algebra with the biquaternion algebra $Q_1 \otimes_k Q_2$ 'twisted' by the field $E$. More precisely, we prove that each quadrangular algebra in characteristic different from 2 and 3 carries the structure of a Freudenthal triple system in a very natural way.

Freudenthal triple systems are vector spaces equipped with a trilinear symmetric map $V \times V \times V \to V$ and bilinear skew-symmetric map $V \times V \to k$, satisfying certain conditions. Initially, they were introduced as 56-dimensional structures that describe the minimal representation of a Lie algebra of type $E_7$, but the notion turned out to be interesting in a larger generality.

For each simple Freudenthal triple system, there is a corresponding isotopy class of structurable algebras. We determine a 'nice' representative of this class. In the $E_8$-case, the structurable algebra that does the trick is the one that is constructed from $Q_1 \otimes_k Q_2$ using the Cayley-Dickson process for structurable algebras w.r.t. the quadratic extension field $E$.

As subcases, we as well investigate what happens in the case of a pseudo-quadratic quadrangular algebra and in the case of quadrangular algebras of type $E_6$ and $E_7$.

**In Chapter 5** we define two octonion algebras $C_1$ and $C_2$ by applying the classical Cayley-Dickson process to $Q_1$ and $Q_2$, respectively, in both cases using a parameter arising from the quadratic field extension $E/k$.

We then reconstruct the quadrangular algebra of type $E_8$ with all its maps from the bioctonion algebra $C_1 \otimes_k C_2$ in characteristic different from two. (When $\mathrm{char}(k)$ is not 3 either, then this bioctonion algebra is also a structurable algebra.) We were inspired by the construction of $J$-ternary algebras out of structurable algebras given in [ABG02]. A $J$-ternary algebra is a module for a Jordan algebra $J$ equipped with a triple product.

As a byproduct we obtain a uniform description in characteristic different from 2 of all algebraic structures defining Moufang quadrangles. This was already accomplished in general characteristic in [DM05] by the use of quadrangular systems, but these are complicated algebraic structures consisting of various vector spaces equipped with several maps, satisfying no less than 24 axioms. Our description is more elegant, more algebraic, and gives more insight into the structure.

**In Chapter 6** we discuss a different kind of problem: the construction of Moufang sets. As we mentioned before, these are Moufang spherical buildings of rank 1.

In [DMW06], Tom De Medts and Richard Weiss gave a construction of a Moufang set starting from a Jordan division algebra, giving rise to Moufang sets with abelian root groups. In [DMVM10], Tom De Medts and Hendrik Van Maldeghem gave a construction of a Moufang set starting from an octonion division algebra with standard involution, giving rise to Moufang sets with non-abelian root groups; these are precisely the Moufang sets corresponding to algebraic groups of type $F_4$.

Both Jordan division algebras and octonion algebras with standard involution are examples of structurable division algebras. The idea was that it should be possible to generalize these constructions simultaneously to an arbitrary structurable division algebra. We have succeeded in doing so, and we explain and prove this construction in Chapter 6. With our construction we obtain as good as all examples of Moufang sets over fields of characteristic different from 2 and 3 that already have been explicitly described; but we conjecture that our construction gives rise to Moufang sets coming from linear algebraic groups of rank one that have not been described explicitly before.

**In Appendix A** we give a technical construction that gives an alternative approach to the construction of the structurable algebras in Chapter 4.

**In Appendix B** we give some more details on the computer programs written in the computer algebra package Sage, that are used to prove two results in this thesis (one of these calculations is only essential in characteristic three). We do not include the actual program listings; these can be found on my web page `cage.ugent.be/~lboelaer` (together with some more technical details on how they are organized).

The material covered in Chapter 4 and Appendix A has been published in [BDM13a]; the material covered in Chapter 5 will appear in [BDM13b]. Whereas most of the material in Chapters 4, 5 and 6 is new, most of the material in Chapters 1, 2 and 3 is taken from the existing literature:

**In Chapter 1** we give definitions to certain algebraic structures, which are more or less well known. **In Chapter 2** we welcome the reader in the world of the Moufang polygons and Moufang sets. **In Chapter 3** we continue our journey to the world of the structurable algebras with some special attention to Freudenthal triple systems and $J$-ternary algebras.

**Thanks to** (in alphabetical order) Bruce Allison, Tom De Medts, John Faulkner, Skip Garibaldi, Hendrik Van Maldeghem and Richard Weiss for their indispensable mathematical help. Many thanks also to the algebra girls and boys for the motivating algebra-tea (and cookie) breaks, and to Ellen Waelbroeck for designing the drawing on the front page.

# Chapter 1

# General preliminaries

*We always assume that $k$ is an arbitrary field of characteristic different from 2.*

With a $k$-algebra $A$, we mean that $A$ is a $k$-vector space equipped with a $k$-bilinear map $A \times A \to A$, called the multiplication and usually denoted by juxtaposition. We will almost always assume that an algebra is unital, i.e. there exists an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

We stress that we do in general not assume that the multiplication is commutative or associative. A good reference for general facts on non-associative algebras is [Sch85].

In the preliminary parts of this thesis we introduce several kinds of algebras: composition algebras in Section 1.3; Jordan algebras in Section 1.4; Lie algebras in Section 1.5; structurable algebras in Chapter 3. We point out that the quadrangular algebras defined in Section 2.2 are no algebras in the above sense as they do not admit a multiplication.

When $A$ is an algebraic structure with a zero (a group, a field, an algebra), will often use the notation $A^*$ to denote the set of non-zero elements of $A$.

In Section 1.1 we give some definitions concerning forms of degree $2, 3, 4$ and their respective linearizations. In Section 1.2 we recall the definitions of hermitian and skew-hermitian spaces and show that these are equivalent notions.

## 1.1 Quadratic forms and forms of higher degree

Let $V$ be a $k$-vector space. A *quadratic form* on $V$ is a map $q : V \to k$ that satisfies

- $q(tv) = t^2 q(v)$ for all $t \in k,\ v \in V$,
- the linearization $f : V \times V \to k : (v,w) \mapsto q(v+w) - q(v) - q(w)$ is $k$-bilinear.

A quadratic form is *anisotropic* if $q(v) \neq 0$ for all $v \in V \setminus \{0\}$. A quadratic form is *non-degenerate* if

$$\{v \in V \mid f(v,V) = 0\} = \{0\}.$$

For definitions of concepts such as Witt-index, isometry (denoted by $\cong$), similarity (denoted by $\sim$) etc. we refer to [Lam05]. We will use notations as introduced in [Lam05]; in particular, $\mathbb{H}$ will denote the hyperbolic plane, i.e. a quadratic form isometric to $\langle 1, -1 \rangle$. We will later need the following definitions concerning Pfister forms; see also [Lam05, Definition X.5.11].

**Definition 1.1.** An $n$-fold *Pfister form* is a quadratic form of dimension $2^n$ isometric to $\langle\langle a_1, \dots, a_n \rangle\rangle := \otimes_{i=1}^n \langle 1, a_i \rangle$ for some $a_1, \dots, a_n \in k$.

Two $n$-fold Pfister forms $q_1$, $q_2$ are *$r$-linked* if there is an $r$-fold Pfister form $h$ such that $q_1 \cong h \otimes q_3$ and $q_2 \cong h \otimes q_4$ for some Pfister forms $q_3, q_4$.

The *linkage number* of $q_1$ and $q_2$ is the number $r \in \mathbb{N}$ such that $q_1$ and $q_2$ are $r$-linked but not $(r+1)$-linked.

Below we define forms of degree 3 or 4, they can be seen as a generalization of the definition of a quadratic form. We also define the linearizations of such forms.

**Definition 1.2.** Let $V$ be a $k$-vector space and let $m = 2, 3, 4$. If $V$ is finite dimensional, a *form of degree* $m$ is a map $Q : V \to k$ such that, relative to some choice of basis for $V$, $Q$ is induced by a homogeneous polynomial (possible zero) of degree $m$ in $\dim_k V$ variables over $k$, and such that $Q$ extends uniquely to $V_K = V \otimes_k K$ for every field extension $K/k$. (This extension of $Q$ will also be denoted by $Q$.)

If $V$ is infinite dimensional, $Q : V \to k$ is a form of degree $m$ if $Q$ is a form of degree $m$ on every finite dimensional subspace of $V$.

Let $Q$ be a form of degree $m$, we say that $Q$ has a *basepoint* $c \in V$ if $Q(c) = 1$.

We define the linearizations of $Q : V \to k$, a form of degree $m$. Let $x, y, z, u$ be arbitrary elements in $V$.

$\boldsymbol{m = 2}$: Since $\mathrm{char}(k) \neq 2$ we have $|k| > 2$, hence $Q$ automatically extends uniquely to $V_K$ for every field extension $K/k$.
The definition of a form of degree 2 is equivalent with the definition of a quadratic form and $f(x,y) = q(x+y) - q(x) - q(y)$ is its (full) linearization[1], it is symmetric and $k$-bilinear. Note that $f(x,x) = 2q(x)$ and that $q$ can be reconstructed when $f$ is given.

[1] If $\lambda$ is transcendental over $k$, $q(x + \lambda y) = q(x) + \lambda f(x,y) + \lambda^2 q(y)$.

$\boldsymbol{m = 3}$: In the case that $|k| > 3$, $Q$ automatically extends uniquely to $V_K$ for every field extension $K/k$.
Let $\lambda$ be transcendental over $k$, we define the *first linearization* $Q(x; y)$ as follows

$$Q(x + \lambda y) = Q(x) + \lambda Q(x; y) + \lambda^2(\dots) + \lambda^3 Q(y),$$

it is quadratic in $x$ and $k$-linear in $y$. The *full linearization* is defined as

$$Q(x, y, z) = Q(x + z; y) - Q(x; y) - Q(z; y),$$

it is $k$-trilinear and symmetric in its 3 variables. We have $Q(x, x, x) = 3!\, Q(x)$.

$\boldsymbol{m = 4}$: In the case that $|k| > 4$, $Q$ automatically extends uniquely to $V_K$ for every field extension $K/k$.
Let $\lambda$ be transcendental over $k$, we define the *first linearization* $Q(x; y)$ as follows

$$Q(x + \lambda y) = Q(x) + \lambda Q(x; y) + \lambda^2(\dots) + \lambda^3(\dots) + \lambda^4 Q(y).$$

$Q(x; y)$ is cubic in $x$ and $k$-linear in $y$. The *second linearization* $Q(x; y, z)$ is defined by

$$Q(x + \lambda y; z) = Q(x; z) + \lambda Q(x; y, z) + \lambda^2(\dots) + \lambda^3 Q(y; z).$$

$Q(x; y, z)$ is quadratic in $x$ and $k$-bilinear and symmetric in $y$ and $z$. The *full linearization* is defined by

$$Q(x, y, z, u) = Q(x + u; y, z) - Q(x; y, z) - Q(u; y, z).$$

It is $k$-linear in its 4 variables and is symmetric.
We have that $Q(x, x, x, x) = 4!\, Q(x)$.

If we assume that the characteristic of $k$ is different from 2 and 3, we have automatically that $|k| > 4$ and the second condition of the definition of a form of degree 3 or 4 can be omitted. Furthermore, if $\mathrm{char}(k) \neq 2, 3$ a form of degree 2, 3 or 4 can be reconstructed from its full linearization.

We also point out that it is clear how to generalize the definition of a form of degree $m$ to *maps of degree $m$* taking values in a vector space $W$ instead, and we will in fact encounter this more general notion when we deal with Freudenthal triple systems later.

**Remark 1.3.** There exist various approaches to the study of these forms of higher degree. The most general approach, which makes sense for arbitrary modules over rings, is the one by Norbert Roby [Rob63]. Another approach, which is closer to ours, was taken by Robert W. Fitzgerald and Susanne Pumplün; see, for example, [FP09].

## 1.2 Hermitian and skew-hermitian spaces

**Definition 1.4.** Let $L$ be a unital associative $k$-algebra with an involution $\sigma$, i.e. $\sigma : L \to L$ is $k$-linear and $(\alpha\beta)^\sigma = \beta^\sigma\alpha^\sigma$ for all $\alpha, \beta \in L$.

(i) Let $X$ be a *right* $L$-module, then $h\colon X \times X \to L$ is a *skew-hermitian form* if
- $h$ is bi-additive and $h(x, y\alpha) = h(x, y)\alpha$,
- $h(x, y)^\sigma = -h(y, x)$,

for all $x, y \in X$ and all $\alpha \in L$. We call $X$ a *skew-hermitian space*[2] . A skew-hermitian form is *non-degenerate* if

$$\{x \in X \mid h(x, X) = 0\} = \{0\}.$$

(ii) Let $W$ be a *left* $L$-module, then $h : W \times W \to L$ is a *hermitian form* if
- h is bi-additive and $h(\alpha v, w) = \alpha h(v, w)$,
- $h(v, w)^\sigma = h(w, v)$,

for all $v, w \in W$ and all $\alpha \in L$. We call $W$ a *hermitian space.* A hermitian form is *non-degenerate* if

$$\{x \in W \mid h(x, W) = 0\} = \{0\}.$$

If $L$ is an associative division $k$-algebra with non-trivial involution, then skew-hermitian forms and hermitian forms are equivalent concepts. The following construction is based on [TW02, 16.18].

**Lemma 1.5.** *Let $L$ be an associative division $k$-algebra with involution $\sigma \neq \mathrm{id}$. Let $X$ be a skew-hermitian space over $L$ with skew-hermitian form $h$. Let $0 \neq s \in L$ such that $s^\sigma = -s$.*

*Define a new involution $\tau$ of $L$ by $v^\tau := sv^\sigma s^{-1}$ and a new left scalar multiplication $v \circ x := xv^\tau$, for all $v \in L, x \in X$. Then $X$ is a left $L$-module w.r.t. $\circ$ and $sh$ is a hermitian form w.r.t. $\tau$ and $\circ$.*

*Proof.* Since $\sigma \neq \mathrm{id}$, there exists an $s \in L$ such that $s^\sigma = -s$. We verify that $\circ$ makes $X$ into a left $L$ module. Indeed, for all $v, w \in L,\ x \in X$, we have

$$v \circ (w \circ x) = (xw^\tau)v^\tau = x(w^\tau v^\tau) = x(vw)^\tau = (vw) \circ x.$$

Finally, it is easily checked that $sh$ is a hermitian form:

$$sh(v \circ x, y) = sh(xv^\tau, y) = s(sv^\sigma s^{-1})^\sigma h(x, y) = vsh(x, y),$$
$$(sh(x, y))^\tau = -sh(x, y)^\sigma ss^{-1} = sh(y, x),$$

for all $x, y \in X$ and all $v \in L$. □

[2] Some authors have the convention that a *space* is only used for non-degenerate forms, but we will not adopt this convention, thereby following [TW02].

## 1.3 Composition algebras

A *composition algebra* is a unital $k$-algebra $C$ equipped with a non-degenerate quadratic form $q\colon C \to k$ that is multiplicative, i.e. $q(xy) = q(x)q(y)$ for all $x, y \in C$. This quadratic form $q$ is called the *norm form*, its associated bilinear form will be denoted by $f$. With the norm form we associate an involution on $C$ by defining

$$\sigma\colon C \to C\colon x \mapsto \overline{x} := f(x,1)1 - x.$$

By a classical result (see for example [SV00, Theorem 1.6.2]) each composition algebra has dimension 1, 2, 4 or 8. We describe the structure of each type of composition algebra and give the norm form and associated involution.

(i) If $\dim_k C = 1$, then $C = k$, $q(x) = x^2$ and the involution is trivial.

(ii) If $\dim_k C = 2$, then $C/k$ is a *quadratic étale extension* of $k$: there exists $a \in k$ such that $C = k[\mathbf{i}]/(\mathbf{i}^2 - a)$ and the norm form is $\langle\langle -a \rangle\rangle$. Either $C/k$ is a separable quadratic field extension and $\sigma$ is the non-trivial element of $\mathrm{Gal}(C/k)$, or $C \cong k \oplus k$ and $\sigma$ interchanges the two components.

(iii) If $\dim_k C = 4$, then $C/k$ is a *quaternion algebra* over $k$: there exist $a, b \in k$ such that $C = k \oplus k\mathbf{i} \oplus k\mathbf{j} \oplus k(\mathbf{ij})$ with multiplication defined by
$$\mathbf{i}^2 = a, \mathbf{j}^2 = b, \mathbf{ij} = -\mathbf{ji}.$$
This quaternion algebra is denoted by $(a,b)_k$. The norm form is equal to $\langle\langle -a, -b \rangle\rangle$, the involution fixes $k$ and maps $\mathbf{i} \mapsto -\mathbf{i}$, $\mathbf{j} \mapsto -\mathbf{j}$.

(iv) If $\dim_k C = 8$, then $C/k$ is an *octonion algebra* over $k$: there exist $a, b, c \in k$ such that $C = Q \oplus Q\mathbf{k}$ where $Q = (a,b)_k$ and multiplication is given by
$$(x_1 + x_2\mathbf{k})(y_1 + y_2\mathbf{k}) = (x_1y_1 + c\overline{y_2}x_2) + (y_2x_1 + x_2\overline{y_1})\mathbf{k} \quad \text{for all } x_i, y_i \in Q.$$
The norm form is equal to $\langle\langle -a, -b, -c \rangle\rangle$ and the involution is given by $\overline{x_1 + x_2\mathbf{k}} = \overline{x_1} - x_2\mathbf{k}$. for all $x_1, x_2 \in Q$.

The construction of an octonion algebra from a pair of quaternion algebras in (iv) is an example of the *Cayley-Dickson doubling process* for composition algebras; see, for example, [SV00, Sec. 1.5].

**Remark 1.6.** In each case, the norm form of a composition algebra is a Pfister form (see Definition 1.1). The norm form is anisotropic when $C$ is a division algebra, and it is hyperbolic otherwise (i.e. when $C$ is a split algebra).

The similarity class of the norm form is completely determined by the algebra structure of the composition algebra. It is a well known but somewhat deeper fact (see e.g. [SV00]) that the converse also holds, i.e. the composition algebra is determined up to isomorphism by the similarity class of the norm.

Quaternion algebras are not commutative, but associative. Octonion algebras are not commutative nor associative, but they are alternative:

**Definition 1.7.** A $k$-algebra $A$ is called *alternative* if $x(xy) = (xx)y$ and $(yx)x = y(xx)$ for all $x, y \in A$. It is known that each subalgebra of $A$ generated by two elements is associative; see, for example, [Sch85, Chapter III].

In the lemma below we summarize some useful identities that hold in each composition algebra.

**Lemma 1.8** ([SV00, Lemma 1.3.2, 1.3.3 and 1.4.1])**.** *Let $C$ be an arbitrary composition algebra with norm $q$, with associated bilinear form $f$, and involution denoted by $x \mapsto \overline{x}$. Then for all $x, y, z \in C$ we have*

(i) $x^2 - f(x, e)x + q(x)e = 0$;

(ii) $f(xy, z) = f(y, \overline{x}z), \quad f(xy, z) = f(z, y\overline{x}), \quad f(xy, z) = f(y\overline{z}, \overline{x})$;

(iii) *$C$ is alternative, and therefore each subalgebra generated by two elements is associative;*

(iv) $x(\overline{x}y) = q(x)y, \quad (x\overline{y})y = q(y)x$;

(v) $(zx)(yz) = z((xy)z), \quad z(x(zy)) = (z(xz))y, \quad x(z(yz)) = ((xz)y)z$.

The identities in (v) are called the *Moufang identities*.

The Bruckfeld–Klein theorem states that each simple alternative algebra, is either associative or an octonion algebra.

## 1.4 Jordan algebras

We only consider Jordan algebras over fields of characteristic different from two. In characteristic equal to two, Jordan algebras come in a different flavor; the definition does not involve a multiplication, only a quadratic operator. Two good references to study the theory of Jordan algebras are [McC04] and [Jac68]. Our construction of exceptional quadrangular algebras in Chapter 5 uses the Peirce decomposition of a Jordan algebra. Below we summarize the main properties of Peirce subspaces.

### 1.4.1 Definition and basic properties

**Definition 1.9.** Let $k$ be a field of characteristic different from two. A *Jordan $k$-algebra $J$* is a unital commutative $k$-algebra such that for all $x, y \in J$ we have $(x^2y)x = x^2(yx)$.

We define the *U-operator* and its linearization for $x, y, z \in J$ by

$$U_x y := 2x(xy) - x^2 y, \quad U_{x,z} y := (U_{x+z} - U_x - U_z)y.$$

An element $x \in J$ is *invertible* if and only if there exists a $y \in J$ such that $xy = 1$ and $x^2 y = x$; this condition is equivalent with $U_x y = x, U_x y^2 = 1$. The element $y$ is the *inverse* of $x$, and we denote it by $x^{-1} := y$.

A *Jordan division algebra* is a Jordan algebra $J$ for which each element in $J \setminus \{0\}$ is invertible.

An element $u \in J$ is invertible if and only if $U_u$ is an invertible operator. If this is the case, then $u^{-1} = U_u^{-1} u$, and for all $x \in J$ we have $u^{-1}x = U_u^{-1}(ux)$ and $u(u^{-1}x) = u^{-1}(ux)$; see, for example, [McC04, Chapter II.6].

We describe the main examples of Jordan algebras and we refer to Chapter II.3 and II.4 of [McC04] for more details.

**Definition 1.10.** (i) **Special Jordan algebras**
Let $A$ be a unital $k$-associative algebra. The algebra $A^+$ with product $x \cdot y = \frac{1}{2}(xy + yx)$ for all $x, y \in A$, is a Jordan $k$-algebra. For all $x, y \in A^+$, we have $U_x y = xyx$.
A Jordan algebra is *special* if it can be embedded in an algebra $A^+$ for some unital associative algebra $A$; otherwise it is called *exceptional*.

(ii) **Jordan algebras of hermitian type**
Let $A$ be a unital associative $k$-algebra with involution $^-$. The Jordan algebra of hermitian type $\mathcal{H}(A, ^-) = \{x \in A \mid \overline{x} = x\}$ is a subalgebra of $A^+$.

(iii) **Jordan algebras of quadratic form type**
Let $q : J \to k$ be a quadratic form with basepoint $c$. We denote by $f$ the linearization of $q$. A Jordan algebra $J$ of quadratic form type has multiplication given by

$$x \cdot y := \tfrac{1}{2}(f(x, c)y + f(y, c)x - f(x, y)c)$$

for all $x, y \in J$.
We have $U_x y = f(x, \overline{y})x - q(x)\overline{y}$ for all $x \in J$, with $\overline{x} = f(x, c)c - x$. For all $x \in J$ the identity $x^2 - f(x, c)x + q(x)c = 0$ holds, which means that $J$ is of degree 2.

(iv) **Cubic Jordan algebras**
Let $N : J \to k$ be a cubic form (i.e. a form of degree 3; see Definition 1.2) with basepoint $c$ such that its trace form $T(x, y) = N(c; x)N(c; y) - N(c, x, y)$ is non-degenerate. Define the map $\sharp : J \to J$ such that $T(x^\sharp, y) = N(x; y)$ for all $x, y \in J$ and define its linearization as $x \times y = (x + y)^\sharp - x^\sharp - y^\sharp$. If we have $(x^\sharp)^\sharp = N(x)c$, then we say that $N$ is a *non-degenerate admissible form*.

Let $N : J \to k$ be a non-degenerate admissible form. A cubic Jordan algebra on $J$ has multiplication given by

$$x \cdot y = \tfrac{1}{2}(x \times y + T(x,c)y + T(y,c)x - N(x,y,c)c)$$

for all $x, y \in J$.
We have $U_x y = T(x,y)x - x^\sharp \times y$ for all $x, y \in J$ and all elements $x \in J$ satisfy $x^3 - T(x,c)x^2 + T(x^\sharp, c)x - N(x)c = 0$; therefore $J$ has degree 3. Conversely, each separable Jordan algebra of degree 3 can be given the structure of a cubic Jordan algebra where the admissible form is the generic norm of the Jordan algebra; see [KMRT98, 38.4].

All exceptional Jordan algebras are 27 dimensional cubic Jordan algebras, called Albert algebras.

The previous list of examples describes each possible type of Jordan algebras. Indeed, by [MZ88, Theorem 15.5], each central simple Jordan algebra is either of type (ii) (where $A$ is either central simple over a field or $A$ is central simple over a quadratic étale algebra where $^-$ is an involution of the second kind), or of type (iii) (for a non-degenerate quadratic form), or it is an Albert algebra. See also [KMRT98, 37.2] for a good summary.

### 1.4.2 Peirce decomposition

**Definition 1.11** ([McC04, II.8.1 and II.8.2 on p. 235])**.** Let $J$ be a Jordan $k$-algebra.

(i) An element $e \in J$ is an *idempotent* if $e^2 = e$. An idempotent is *proper* if it is different from 0 and 1. If $e$ is an idempotent, then $1 - e$ is also an idempotent. Two idempotents $e, e'$ are *supplementary* if $e + e' = 1$. Observe that two supplementary idempotents are always orthogonal, i.e. $ee' = 0$.

(ii) Let $e$ be a proper idempotent in $J$. The *Peirce decomposition* with respect to $e$ is defined as follows. For each $i \in k$, let

$$J_i = \{x \in J \mid ex = ix\};$$

then we have

$$J = J_0 \oplus J_{1/2} \oplus J_1;$$

in particular, $J_\ell = 0$ if $\ell \notin \{0, \frac{1}{2}, 1\}$. For a nondegenerate Jordan algebra we have $J_0 \neq 0$ (see [McC04, II.10.1.2]). Let $i \in \{0,1\}$ and $j = 1 - i$; then

$$J_i^2 \subseteq J_i, \quad J_i J_{1/2} \subseteq J_{1/2}, \quad J_{1/2}^2 \subseteq J_0 + J_1, \quad J_i J_j = 0.$$

For all $\ell, m \in \{0, \frac{1}{2}, 1\}$, we have

$$U_{J_m} J_\ell \subseteq J_{2m-\ell}.$$

To construct quadrangular algebras in Chapter 5 we will use the following two types of Jordan algebras that contain supplementary idempotents.

**Definition 1.12** ([McC04, II.3.4 on p. 180])**.** Consider a quadratic form $q\colon V \to k$ over $k$. Starting from the vector space $V$ we construct a Jordan algebra by adjoining two supplementary idempotents to $V$.

As a vector space, we define $J$ by adjoining two copies of $k$ to $V$:

$$J := ke_0 \oplus V \oplus ke_1.$$

We define the following multiplication:

$$\begin{aligned}(t_1e_i)(t_2e_j) &= \delta_{ij}t_1t_2e_i,\\ (te_i)v &= \tfrac{1}{2}tv,\\ vw &= \tfrac{1}{2}f(v,w)(e_0+e_1),\end{aligned} \tag{1.1}$$

for all $i, j \in \{0, 1\}, v, w \in V, t, t_1, t_2 \in k$. This defines a Jordan algebra[3] on $ke_0 \oplus V \oplus ke_1$, called the *reduced spin factor of the quadratic form $q$*. We say that $J$ is of *reduced spin type*.

The unit of this Jordan algebra is $e_0 + e_1$, and for all $v, w \in V$ we have

$$U_ve_0 = q(v)e_1, \quad U_ve_1 = q(v)e_0, \quad U_vw = f(v,w)v - q(v)w.$$

It is clear that $e_0$ and $e_1$ are supplementary idempotents and that we have the following Peirce subspaces with respect to $e_1$:

$$J_0 = ke_0, \quad J_{1/2} = V, \quad J_1 = ke_1.$$

**Definition 1.13** ([McC04, Example II.3.2.4])**.** Let $L$ be a skew field with involution $\sigma$, define $L_\sigma := \mathrm{Fix}_\sigma(L)$ and $k := Z(L)$. The matrix algebra $M_2(L)$ is associative with involution $\sigma T$, where $T$ denotes the transpose of the matrix. Now let $J$ be the Jordan $k$-algebra

$$J = \mathcal{H}(M_2(L), \sigma T) = \left\{ \begin{bmatrix} \alpha_1 & \ell^\sigma \\ \ell & \alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2 \in L_\sigma, \ell \in L \right\};$$

see Definition 1.10.(ii).

We define the supplementary idempotents $e_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], e_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in J$. With respect to $e_1$, we have

$$J_0 = L_\sigma e_0, \quad J_1 = L_\sigma e_1 \quad \text{and} \quad J_{1/2} = \left\{ \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \mid \ell \in L \right\}.$$

[3]Notice that this is the same Jordan algebra as the Jordan algebra of the quadratic form $Q : ke_0 \oplus V \oplus ke_1 \to k : t_0e_0 + v + t_1e_1 \mapsto t_0t_1 - q(v)$.

We have

$$\begin{aligned}
(\alpha_1 e_i)(\alpha_2 e_j) &= \delta_{ij}\tfrac{1}{2}(\alpha_1\alpha_2+\alpha_2\alpha_1)e_i,\\
(\alpha e_0)v &= \tfrac{1}{2}\begin{bmatrix}0 & \alpha\ell^\sigma\\ \ell\alpha & 0\end{bmatrix},\\
(\alpha e_1)v &= \tfrac{1}{2}\begin{bmatrix}0 & \ell^\sigma\alpha\\ \alpha\ell & 0\end{bmatrix},\\
v_1v_2 &= \tfrac{1}{2}(\ell_1^\sigma\ell_2+\ell_2^\sigma\ell_1)e_0+\tfrac{1}{2}(\ell_1\ell_2^\sigma+\ell_2\ell_1^\sigma)e_1,
\end{aligned}$$

for all $i,j \in \{0,1\}, v = \begin{bmatrix}0 & \ell^\sigma\\ \ell & 0\end{bmatrix}, v_1 = \begin{bmatrix}0 & \ell_1^\sigma\\ \ell_1 & 0\end{bmatrix}, v_2 = \begin{bmatrix}0 & \ell_2^\sigma\\ \ell_2 & 0\end{bmatrix} \in J_{1/2}$, $\alpha,\alpha_1,\alpha_2 \in L_\sigma$. For the $U$-operators we find

$$U_v(\alpha e_0) = (\ell\alpha\ell^\sigma)e_1, \quad U_v(\alpha e_1) = (\ell^\sigma\alpha\ell)e_0, \quad U_{v_1}v_2 = \begin{bmatrix}0 & \ell_1^\sigma\ell_2\ell_1^\sigma\\ \ell_1\ell_2^\sigma\ell_1 & 0\end{bmatrix}.$$

**Remark 1.14.** If we consider the above definition in the case that $(L,\sigma)$ is a quadratic pair (see Definition 2.11 below), then $k = L_\sigma$. Now there exists a non-degenerate anisotropic quadratic form $q : L \to k : \ell \mapsto \ell\ell^\sigma = \ell^\sigma\ell$, and the Peirce subspaces $J_0, J_1$ of $\mathcal{H}(M_2(L),\sigma T)$ are one-dimensional.

Define a quadratic form $Q$ on $J_{1/2} \subseteq \mathcal{H}(M_2(L),\sigma T)$ given by $Q(\begin{bmatrix}0 & \ell^\sigma\\ \ell & 0\end{bmatrix}) = q(\ell)$. By comparing the multiplication in Definition 1.13 above and the one in Definition 1.12 we conclude that $\mathcal{H}(M_2(L),\sigma T)$ is the reduced spin factor of the quadratic space $(J_{1/2}, k, Q)$.

In the following proposition we use Osborn's Capacity Two theorem to show that the two families of Jordan algebras we discussed above can be characterized in a unified way. The proof of this Proposition uses some results and concepts of Jordan theory that we will not use in the remaining part of this thesis.

**Proposition 1.15.** *Let $J$ be a non-degenerate Jordan $k$-algebra with supplementary proper idempotents $e_0$ and $e_1$. Let $J_0, J_{1/2}, J_1$ be the Peirce subspaces of $J$ with respect to $e_1$. We assume that each element in $J_{1/2} \setminus \{0\}$ is invertible and that there exists $u \in J_{1/2}$ such that $u^2 = 1$.*

- *If $\dim(J_0) = 1$, then $J$ is the reduced spin factor of some non-degenerate anisotropic quadratic space with basepoint $u$.*[4]
- *If $\dim(J_0) > 1$, then $J$ is isomorphic to $\mathcal{H}(M_2(L),\sigma T)$ for some skew field $L$ with involution $\sigma$, such that $(L,\sigma)$ is not a quadratic pair.*

*Proof.* We will show that the assumptions imply that $J$ is a simple nondegenerate Jordan algebra of capacity 2; an algebra has capacity 2 if the unit

[4]Notice that $\mathcal{H}(M_2(L),\sigma T)$ for $(L,\sigma)$ a quadratic pair is included in this case.

is the sum of two supplementary idempotents $e_0, e_1$ such that the Peirce subspaces $J_0, J_1$ are division algebras.

Since $u^2 = 1$ it follows from [McC04, II.6.1.10] that $U_u$ is a Jordan isomorphism[5] of $J$ such that $(U_u)^2$ is the identity map. Since $U_u(J_1) \subseteq J_0$, $U_u$ is an isomorphism between $J_0$ and $J_1$. Therefore it is enough to show that $J_0$ is a division algebra. It follows from [McC04, II.6.1.2] that it is sufficient to show that for each element $t \in J_0 \setminus \{0\}$ the operator $U_t$ is surjective on $J_0$.

Let $t, s \in J_0 \setminus \{0\}$; using [McC04, II.8.4.1] one can verify that $U_t s = U_u U_{2ut} s$. We have $2ut \in J_{1/2} \setminus \{0\}$: if $2ut = 0$ it would follow that $U_u(t) = -u^2 t = -t$ which implies that $t \in J_0 \cap J_1 = \{0\}$. It follows that $U_{2ut}$ is invertible. Now let $r \in J_0$; since $U_{2ut}^{-1} U_u r \in J_0$ we have

$$r = U_t(U_{2ut}^{-1} U_u r)$$

and hence $U_t$ is surjective on $J_0$.

This proves that $J$ has capacity 2.

We now show that $J$ is simple. From [McC04, II.20.2.4] a nondegenerate algebra with capacity is simple if and only if its capacity is connected (i.e. if and only if $e_0, e_1$ are connected [McC04, II.10.1.3]). In fact, $e_0, e_1$ are even strongly connected since $u \in J_{1/2}$ is an involution, i.e. $u^2 = 1$.

We proved all the conditions necessary to apply Osborn's Capacity Two theorem; see [McC04, II.22.2.1 on p. 351]. This theorem states that a simple nondegenerate Jordan algebra of capacity 2 belongs to exactly one of the following three disjoint classes from which we can exclude the first:

(i) Full type $M_2(L)^+$, for a noncommutative skew field $L$. In this case, let $e_0$ be an arbitrary proper idempotent of $M = M_2(L)$, and let $e_1 = 1 - e_0$. Then

$$J_0 = e_0 M e_0, \quad J_1 = e_1 M e_1, \quad J_{1/2} = e_0 M e_1 \oplus e_1 M e_0,$$

and in particular each non-trivial element of $e_0 M e_1 \subset J_{1/2}$ is not invertible (since its square is zero).

(ii) Hermitian type $\mathcal{H}(M_2(L), \sigma T)$ with $L$ a skew field with involution $\sigma$ such that $(L, \sigma)$ is not a quadratic pair. In this case $\dim(J_0) > 1$.

(iii) Reduced spin factor of a non-degenerate quadratic space $(k, V, q)$. Since the unit of $J$ is $e_0 + e_1$ and $1 = u^2 = q(u)(e_0 + e_1) = q(u)1$, $u$ is a basepoint of $q$.
Suppose there exists a $v \in J_{1/2} \setminus \{0\}$ such that $q(v) = 0$; then it would follow that $vw = 0$ for all $w \in J_{1/2}$, which implies that $v$ is not invertible. Therefore $q$ is anisotropic. In this case $\dim(J_0) = 1$. □

[5] This means that $U_u(xy) = U_u(x)U_u(y)$ for all $x, y \in J$.

## 1.5 Lie algebras and linear algebraic groups

Linear algebraic groups are affine varieties equipped with a compatible group structure. They can be seen (in a non-intrinsic way) as subgroups of $\mathrm{GL}_n(k)$, i.e. the invertible $n \times n$-matrices over a field $k$.

Semisimple connected linear algebraic groups can be divided into different classes, according to their Tits index, which is the Dynkin diagram endowed with some "circled vertices" representing the distinguished orbits. These Tits indices already contain a lot of information about the specific linear algebraic group; for instance, the number of vertices of the diagram is the absolute rank, and the number of distinguished orbits is the relative rank. In Table 1.1 below we give a few examples of Tits indices.

Linear algebraic groups with Dynkin diagrams of type $A_n$, $B_n$, $C_n$, $D_n$ are *classical*, linear algebraic groups with Dynkin diagrams of type $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ are *exceptional*. $D_4$ is sometimes also considered as exceptional.

We do not go into more details concerning linear algebraic groups. Linear algebraic groups are the main motivation for the mathematics done in this thesis, but we only mention a few times the type or Tits index of a linear algebraic group and do not rely on the theory of the linear algebraic groups. We refer to [Tit66] and [Sel76a] for details on the classification.

As mentioned in the introduction linear algebraic groups give rise to spherical buildings (see [Tit74]) and a linear algebraic group has a natural action on its building. If the linear algebraic group has relative rank $r$, its corresponding building has rank $r$. In this thesis we are interested in spherical Moufang buildings of rank 1 and 2 (i.e. Moufang sets and Moufang polygons) coming from exceptional linear algebraic groups. Therefore we list all the exceptional Tits indices of relative rank 1 and 2 (see Table 1.1). For further reference we will adopt the notation ${}^gX^t_{n,r}$ for Tits indices introduced in [Tit66], where $X \in \{A, B, C, D, G, F, E\}$ is the type, $n$ is the absolute rank, $r$ the relative rank and $g, t \in \mathbb{N}$ are some other invariants of the linear algebraic group defined in [Tit66] (note that $g$ is omitted from the notation if $g = 1$).

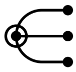
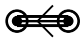
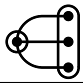
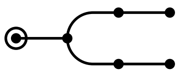
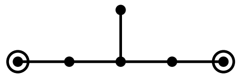
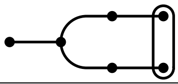
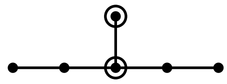
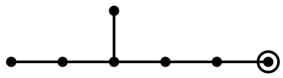
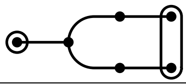
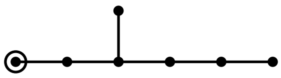
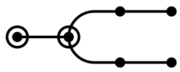
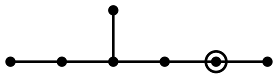
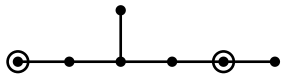
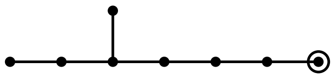
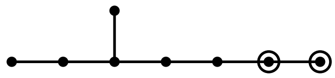
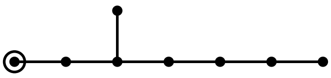
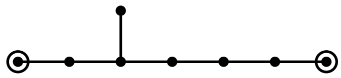
| | | | |
|---|---|---|---|
| ${}^{3,6}D_{4,1}^{9}$ |  | $G_{2,2}^{0}$ |  |
| $F_{4,1}^{21}$ |  | ${}^{3,6}D_{4,2}^{2}$ |  |
| ${}^{2}E_{6,1}^{35}$ |  | $E_{6,2}^{28}$ |  |
| ${}^{2}E_{6,1}^{29}$ |  | $E_{6,2}^{16}$ |  |
| $E_{7,1}^{78}$ |  | ${}^{2}E_{6,2}^{16'}$ |  |
| $E_{7,1}^{66}$ |  | ${}^{2}E_{6,2}^{16''}$ |  |
| $E_{7,1}^{48}$ |  | $E_{7,2}^{31}$ |  |
| $E_{8,1}^{133}$ |  | $E_{8,2}^{78}$ |  |
| $E_{8,1}^{91}$ |  | $E_{8,2}^{66}$ |  |

Table 1.1: Exceptional Tits indices of relative rank 1 and 2

We will only make explicit use of the following very basic concepts in the theory of Lie algebras. We refer to [Jac62], for example, for more advanced concepts concerning Lie algebras.

**Definition 1.16.** (i) A $k$-algebra $\mathcal{L}$ is a *Lie algebra* if the $k$-bilinear 'multiplication' (called the *Lie bracket*) $\mathcal{L} \times \mathcal{L} \to \mathcal{L} : (x, y) \mapsto [x, y]$ satisfies the following conditions for all $x, y, z \in \mathcal{L}$:
- $[x, x] = 0$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

The first identity implies that $[x, y] = -[y, x]$, the second identity is called the *Jacobi identity*.

(ii) Let $\mathcal{L}$ and $\mathcal{L}'$ be $k$-Lie algebras. A $k$-linear map $\psi : \mathcal{L} \to \mathcal{L}'$ is a *morphism* if $\psi[x, y] = [\psi x, \psi y]$ for all $x, y \in \mathcal{L}$.

(iii) Let $\mathcal{L}$ and $\mathcal{L}'$ be $k$-Lie algebras. A $k$-linear map $\psi : \mathcal{L} \to \mathcal{L}'$ is a *derivation* if $\psi[x,y] = [\psi x, y] + [x, \psi y]$ for all $x, y \in \mathcal{L}$. For each $x \in \mathcal{L}$, the map $\mathrm{ad}(x) : \mathcal{L} \to \mathcal{L} : y \mapsto [x,y]$ is an example of a derivation of $\mathcal{L}$, a so-called inner derivation.

(iv) A ($\mathbb{Z}$-)*graded* Lie-algebra is a Lie algebra $\mathcal{L}$ together with a decomposition into subvector spaces $\{\mathcal{L}_i \mid i \in \mathbb{Z}\}$ such that

$$\mathcal{L} = \bigoplus_{i\in\mathbb{Z}} \mathcal{L}_i \quad \text{and} \quad [\mathcal{L}_i, \mathcal{L}_j] = \mathcal{L}_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

We call the grading a $(2n+1)$-*grading* if $\mathcal{L}_i = 0$ for all $i \in \mathbb{Z}$ with $|i| > n$.

(v) Let $\mathcal{L}$ and $\mathcal{L}'$ be two graded Lie algebras. A morphism $\psi : \mathcal{L} \to \mathcal{L}'$ is a *graded morphism* if $\psi(\mathcal{L}_i) \subseteq \mathcal{L}'_i$ for all $i \in \mathbb{Z}$.

It is an important fact that any linear algebraic group gives rise to a Lie algebra; see, for example, [Mil12a, Chapter XI]. In general, this relation is not one-to-one, and not every Lie algebra can be obtained from a linear algebraic group.

However when $\mathrm{char}(k) = 0$, any semisimple Lie algebra (i.e. a Lie algebra that is the direct sum of simple[6] Lie algebras) is obtained from a semisimple linear algebraic group (see [Mil12b, Theorem II.3.20]). From this point of view we can say that a semisimple Lie algebra is of a certain type and has a certain Tits index ${}^{g}X^{t}_{n,r}$. The relative rank of a semisimple Lie algebra is then also the dimension of a maximal split toral subalgebra (see [Sel76b]).

The theory of Lie algebras in characteristic $p > 0$ gives rise to many complications, compared to the theory in characteristic zero; see [Sel67]. However if a Lie algebra in characteristic $p > 0$ arises from a linear algebraic group of type $X_n$, we will also say that this Lie algebra is of type $X_n$.

[6] A Lie algebra is *simple* if it does not contain any proper non-trivial two sided ideals.

# Chapter 2

# The Moufang world

In this chapter we give an overview of the theory of Moufang polygons and Moufang sets. As we mentioned in the introduction, these are spherical Moufang buildings of rank two and one, respectively.

In Section 2.1 we explain the concept of a Moufang polygon, which is graph-theoretic. By Theorem 2.3, however, this notion reduces quickly to some algebraic data.

In Section 2.2 we introduce the quadrangular algebras. These algebraic structures characterize the Moufang quadrangles of type $E_6$, $E_7$ and $E_8$. The aim of Chapters 4 and 5 is to get a better understanding of the structure of the exceptional quadrangular algebras.

In Section 2.3, we give an overview of all the classes of Moufang quadrangles that are defined over fields of characteristic different from 2.

In Section 2.4 we give some equivalent definitions of Moufang sets and we describe many of the known classes of Moufang sets in characteristic different from 2 and 3.

## 2.1 Definition and basic properties of Moufang polygons

We only give a brief summary of the theory of Moufang polygons, and we refer to [TW02] or to the survey article [DMVM03] for more details.

A *generalized $n$-gon* $\Gamma$ is a connected bipartite graph with diameter $n$ and girth $2n$, where $n \geq 2$. If we do not want to specify the value of $n$, then we call this a *generalized polygon.* We call a generalized polygon *thick* if every vertex has at least 3 neighbors. A *root* in $\Gamma$ is a (non-stammering) path of length $n$ in $\Gamma$. Observe that the two extremal vertices of such a path are always *opposite*, i.e. their distance is equal to the diameter $n$ of $\Gamma$. An *apartment* in $\Gamma$ is a circuit of length $2n$.

Let $\Gamma$ be a thick generalized $n$-gon with $n \geq 3$, and let $\alpha = (x_0, \dots, x_n)$ be a root of $\Gamma$. The *root group* of $\alpha$ is the group of all automorphisms of $\Gamma$ fixing all neighbors of $x_1, \dots, x_{n-1}$; it is denoted by $U_\alpha$.

A root group acts freely on the set of vertices incident with $x_0$ but different from $x_1$. If $U_\alpha$ acts transitively on this set (and hence regularly), then we say that $\alpha$ is a *Moufang root*. It turns out that $\alpha$ is a Moufang root if and only if $U_\alpha$ acts regularly on the set of apartments through $\alpha$.

**Definition 2.1.** A *Moufang polygon* is a thick generalized $n$-gon for which every root is a Moufang root.

The group generated by all the root groups is called the *little projective group* of $\Gamma$.

Moufang polygons have been classified by Jacques Tits and Richard Weiss in [TW02]. Loosely speaking, the result is the following.

**Theorem 2.2** ([TW02])**.** *Let $\Gamma$ be a Moufang $n$-gon. Then $n \in \{3, 4, 6, 8\}$. Moreover, every Moufang polygon arises from an absolutely simple linear algebraic group of relative rank 2, or from a corresponding classical group or group of mixed type, or (when $n = 8$) from a Ree group of type ${}^2F_4$.*

With a classical group, we mean a form of a special linear group, of a unitary group, of a symplectic group or of an orthogonal group, where in each case we explicitly allow the possibility that the underlying vector space is infinite-dimensional (in which case the group is not a linear algebraic group). Mixed Moufang polygons are more exotic; they are related to groups which are in some sense linear algebraic groups defined over a pair of fields $k$ and $K$ in characteristic $p$ where $K^p \subset k \subset K$ and $p$ is equal to 2 or 3. As we will always work over fields of characteristic different from 2 and 3, we will not be bothered by the Moufang polygons of mixed type.

In order to describe a Moufang polygon in terms of algebraic data, we will use a sequence of root groups. Let $(x_1, \dots, x_n, \dots, x_{2n-1}, x_0, x_1)$ be an apartment of $\Gamma$, and define the sequence

$$U_1 = U_{(x_1,\dots,x_{n+1})},\ U_2 = U_{(x_2,\dots,x_{n+2})},\ \dots\ , U_n = U_{(x_n,\dots,x_0)}.$$

**Theorem 2.3.** *Let $\Gamma$ be a Moufang $n$-gon. The root groups $U_1, \dots, U_n$ are (up to order and isomorphism) independent of the choice and of the labeling of the apartment.*

*Moreover, $\Gamma$ is completely determined by the root groups $U_1, \dots, U_n$ together with their commutator relations, which describe the commutators of elements of two different root groups.*

*Proof.* See [TW02, Chapter 7]. □

More precisely, it is always the case that the commutator of an element of $U_i$ and an element of $U_j$ (with $i < j$) belongs to the group $\langle U_{i+1}, \ldots, U_{j-1} \rangle$. In particular, it follows that for all $1 \le i \le n-1$,

$$[U_i, U_{i+1}] = 1.$$

For each type of Moufang polygons, we will describe an *algebraic structure* which will allow us to parametrize the root groups and describe the commutator relations. Below we describe the Moufang triangles, hexagons and octagons. In each case, we will mention, in view of Theorem 2.2, from which type of algebraic, classical or mixed group the Moufang polygon arises; for the notation of the Tits index of a linear algebraic group, we refer to Table 1.1 on page 13.

In Section 2.3, we will describe the structure of the root groups of Moufang quadrangles. In order to give this description we will need to introduce quadrangular algebras, which we do in Section 2.2 below.

**Moufang triangles** Every Moufang triangle can be described in terms of an *alternative division algebra*, i.e. an alternative algebra (see p. 6) such that for each $a \in A \setminus \{0\}$, there is a (unique) element $a^{-1} \in A \setminus \{0\}$ satisfying the condition

$$a^{-1}(ab) = b = (ba)a^{-1} \quad \text{for all } b \in A.$$

If $A$ is such an alternative division algebra, then we define $U_1 \cong U_2 \cong U_3 \cong (A, +)$; we denote the explicit isomorphisms from $A$ to $U_i$ by $x_i$, and we call this the *parametrization* of the groups $U_i$ by $(A, +)$. The commutator relations are then given by

$$[x_1(a), x_3(b)] = x_2(ab)$$

for all $a, b \in A$. Every Moufang triangle can be described in this fashion.

If $A$ is a finite-dimensional division algebra of degree $d$, then this Moufang triangle arises from a linear algebraic group of absolute type $A_{3d-1}$. If $A$ is infinite-dimensional, then the associated group is no longer an algebraic group, but it can still be viewed as a classical group, namely $PSL_3(A)$. The case where $A$ is an octonion division algebra is exceptional; it arises from a linear algebraic group of type $E_{6,2}^{28}$.

**Moufang hexagons** Every Moufang hexagon can be described in terms of an anisotropic cubic norm structure; this is essentially the same as a *(quadratic) Jordan division algebra of degree* 3 (see Definition 1.10).

We will not give a precise definition of these algebraic structures since we will not need them explicitly, but we refer to [DMVM03, KMRT98, TW02] instead. We will only mention that if $J$ is such an anisotropic cubic norm structure over a field $k$, then either $J/k$ is a purely inseparable cubic extension, or $\dim_k J \in \{1, 3, 9, 27\}$.

If $J$ is such an anisotropic cubic norm structure, then we define $U_1 \cong U_3 \cong U_5 \cong (J, +)$ and $U_2 \cong U_4 \cong U_6 \cong (k, +)$. The commutator relations can be expressed in terms of the norm, trace and Freudenthal cross product, but their explicit form is not important for us; we refer to [DMVM03, TW02] for more details.

The case where $J = k$ gives rise to the so-called *split Cayley hexagon*, which arises from a split linear algebraic group of type $G_2$. If $J$ is a cubic separable extension field of $k$, then the corresponding Moufang hexagon is the so-called *twisted triality hexagon*, which arises from a quasi-split linear algebraic group of type ${}^3D_{4,2}^2$ or ${}^6D_{4,2}^2$. If $J$ is a purely inseparable cubic extension field of $k$, then the corresponding Moufang hexagon arises from a group of mixed type $G_2$. The next case is where $J$ is a 9-dimensional $k$-algebra. There are two cases to distinguish; either $J$ is a central simple cubic cyclic division algebra, or it is a twisted form of such an algebra, arising from an involution of the second kind on such an algebra. The resulting Moufang hexagon arises from a linear algebraic group of type $E_{6,2}^{16}$ and ${}^2E_{6,2}^{16''}$, respectively. Finally, if $J$ is 27-dimensional over $k$, then it is an Albert division algebra; the resulting Moufang hexagon arises from a linear algebraic group of type $E_{8,2}^{78}$.

**Moufang octagons** The Moufang octagons have a fairly simple structure from an algebraic point of view. Every Moufang octagon can be described from a commutative field $k$ with $\mathrm{char}(k) = 2$ equipped with a *Tits endomorphism* $\sigma$, i.e. an endomorphism such that $(x^\sigma)^\sigma = x^2$ for all $x \in k$. The root groups $U_1, U_3, U_5, U_7$ are parametrized by $(k, +)$, and the root groups $U_2, U_4, U_6, U_8$ are parametrized by the non-abelian group $T$ with underlying set $k \times k$, and with group operation

$$(a, b) \cdot (c, d) := (a + c, b + d + a^\sigma c) \quad \text{for all } a, b, c, d \in k.$$

The corresponding groups are Ree groups of type ${}^2F_4$.

## 2.2 Quadrangular algebras

A quadrangular algebra is an algebraic structure that was constructed to describe the exceptional Moufang quadrangles. For more information on quadrangular algebras, including the case where the characteristic is 2, we refer to [Wei06b].

### 2.2.1 Definition and basic properties

We emphasize that we will only be dealing with quadrangular algebras over fields of characteristic different from 2, in which case the structure of a quadrangular algebra simplifies significantly; see Remark 2.5 below.

**Definition 2.4.** A *quadrangular algebra*, in characteristic different from 2, is a 7-tuple $(k, L, q, 1, X, \cdot, h)$, where

(i) $k$ is a commutative field with $\operatorname{char}(k) \neq 2$,
(ii) $L$ is a $k$-vector space,
(iii) $q$ is an anisotropic quadratic form from $L$ to $k$,
(iv) $1 \in L$ is a *basepoint* for $q$, i.e. an element such that $q(1) = 1$,
(v) $X$ is a non-trivial $k$-vector space,
(vi) $(x, v) \mapsto x \cdot v$ is a map from $X \times L$ to $X$,
(vii) $h$ is a map from $X \times X$ to $L$,

satisfying the following axioms, where

$$\begin{aligned}
&f\colon L \times L \to k\colon (x,y) \mapsto f(x,y) := q(x+y) - q(x) - q(y)\,;\\
&\sigma\colon L \to L\colon v \mapsto f(1,v)1 - v\,;\\
&v^{-1} := v^\sigma / q(v)\,.
\end{aligned}$$

(A1) The map $\cdot$ is $k$-bilinear.
(A2) $x \cdot 1 = x$ for all $x \in X$.
(A3) $(x \cdot v) \cdot v^{-1} = x$ for all $x \in X$ and all $v \in L \setminus \{0\}$.

(B1) $h$ is $k$-bilinear.
(B2) $h(x, y \cdot v) = h(y, x \cdot v) + f(h(x,y),1)v$ for all $x, y \in X$ and all $v \in L$.
(B3) $f(h(x \cdot v, y), 1) = f(h(x,y), v)$ for all $x, y \in X$ and all $v \in L$.

(C) $\theta(x, v) := \frac{1}{2} h(x, x \cdot v)$.

(D1) Let $\pi(x) := \theta(x, 1)$ for all $x \in X$, hence

$$\pi(x) = \tfrac{1}{2} h(x, x).$$

Then $x \cdot \theta(x, v) = (x \cdot \pi(x)) \cdot v$ for all $x \in X$ and all $v \in L$.
(D2) For all $x \in X \setminus \{0\}$ we have $\pi(x) \neq 0$.

Moreover, we define a map $g\colon X \times X \to k$ by

$$g(x,y) := \tfrac{1}{2} f(h(x,y), 1)$$

for all $x, y \in X$.

**Remark 2.5.** When one compares our definition of quadrangular algebras with the general definition in [Wei06b, Definition 1.17] there are two differences which are due to the fact that the definition simplifies when the characteristic is different from 2.

(i) Our axiom (C) is a mere definition and not really an axiom. In [Wei06b], this axiom (C) consists of 4 much more involved axioms, which are only necessary in the characteristic 2 case (see [Wei06b, Remark 4.8]). By defining $\theta(x, v) = \frac{1}{2}h(x, x \cdot v)$ we actually assume that the quadrangular algebra is standard. Every quadrangular algebra is equivalent to a standard quadrangular algebra (see [Wei06b, Proposition 4.2, 3.14 and 4.5].)

(ii) In [Wei06b], axiom (D2) has the seemingly weaker condition that $\pi(x) \equiv 0 \pmod{k}$ if and only if $x = 0$ (where $k$ has been identified with its image under the map $t \mapsto t1$ from $k$ to $L$). We show that this is equivalent to our axiom (D2).
Indeed, assume that this weaker version of (D2) holds. Applying (B2) with $x = y$ and $v = 1$, we get $f(h(x, x), 1) = 0$. If we suppose $\pi(x) = \frac{1}{2}h(x, x) \in k1$, we have $f(h(x, x), 1) = 2h(x, x) = 0$ and it follows that $\pi(x) = 0$, so $x = 0$.

A quadrangular algebra is a module for the Clifford algebra with basepoint of $q$:

**Definition 2.6.** (i) Let $(k, V, q)$ be a quadratic space with basepoint $\epsilon$. Then the *Clifford algebra of* $q$ *with basepoint* $\epsilon$ is defined as

$$C(q, \epsilon) := T(V)/\langle u \otimes u^\sigma - q(u)1,\ \epsilon - 1\rangle,$$

where $T(V)$ is the tensor algebra of $V$, and where $\sigma$ is defined as in Definition 2.4. It is not too hard to show that $C(q, \epsilon) \cong C_0(q)$, the even Clifford algebra of $q$. The notion of a Clifford algebra with basepoint was introduced by Jacobson and McCrimmon; see [TW02, Chapter 12] for more details.

(ii) Since $q$ is anisotropic, the axioms (A1)–(A3) of an arbitrary quadrangular algebra say precisely that $X$ is a $C(q, 1)$-module, such that the action of $C(q, 1)$ on $X$ is an extension of the action of $L$ on $X$ (see [Wei06b, Proposition 2.22]).

(iii) A $C(q, 1)$-module $X$ is *irreducible* if we have $x \cdot C(q, 1) = X$ for all $x \in X \setminus \{0\}$ .

We will use the following formulas in the sequel.

**Theorem 2.7.** *Let* $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ *be a quadrangular algebra, with* $\mathrm{char}(k) \neq 2$. *For all* $a, b \in X$ *and all* $u, v \in L$, *we have:*

(i) $h(a, b) = -h(b, a)^\sigma$,

(ii) $f(h(a, b \cdot v), 1) = f(h(a, b), v^\sigma)$,

(iii) $(a \cdot u) \cdot v = -(a \cdot v^\sigma) \cdot u^\sigma + af(u, v^\sigma)$,

(iv) $h(a \cdot \pi(a), b) + \theta(a, h(a, b)) = 0$,

(v) $\theta(a \cdot v, w) = \theta(a, w^\sigma)^\sigma q(v) - f(w, v^\sigma)\theta(a, v)^\sigma + f(\theta(a, v), w^\sigma)v^\sigma$.

(vi) $g(x \cdot v, y \cdot v) = g(x, y)q(v)$.

*Proof.* Identities (i)–(iii) are precisely [Wei06b, (3.6), (3.7) and (3.8)]. Identity (iv) is identity (e) in the proof of [TW02, (13.67)]; the proof holds without any change in the pseudo-quadratic case as well. Identity (v) is precisely axiom (C4) in [Wei06b, Definition 1.17], taking into account that the map $\phi$ occurring in this axiom is trivial by [Wei06b, Proposition 4.5]. Identity (vi) is [Wei06b, Proposition 4.18]. □

In the following subsections we will describe two examples of quadrangular algebras. The following theorem tells us that these are all examples in characteristic different from 2.

**Theorem 2.8.** *A quadrangular algebra in characteristic not* 2 *is either obtained from an anisotropic pseudo-quadratic space over a quadratic pair (see Section* 2.2.2*), or it is of type* $E_6$*,* $E_7$ *or* $E_8$ *(see Section* 2.2.3*).*

*Proof.* Since the characteristic of $k$ is not 2, it follows from [Wei06b, 2.3 and 2.4] that the quadrangular algebra is regular, i.e. $f$ is non-degenerate. (From [Wei06b, 3.14], it follows that it is also proper, i.e. $\sigma \neq \text{id}$). Now it follows from [Wei06b, 3.2] that if the quadrangular algebra is not special (i.e. not arising from a pseudo-quadratic space), then it is of type $E_6$, $E_7$ or $E_8$. □

### 2.2.2 Pseudo-quadratic spaces

Below we give the definition of a pseudo-quadratic space in characteristic different from 2. Just as for quadrangular algebras, this definition simplifies significantly compared to the general definition given in [TW02, 11.16] for arbitrary characteristic; see Remark 2.10 below.

The simplification could be compared to the fact that when one defines a quadratic form in characteristic different from 2, it is actually enough to give the bilinear form, since $q(x) = \frac{1}{2}f(x, x)$, whereas in general characteristic, a quadratic form is not uniquely determined by its bilinear form.

**Definition 2.9** ([Wei06b, Definition 1.16])**.** A *pseudo-quadratic space* over a field of characteristic not 2 is a quintuple $(L, \sigma, X, h, \pi)$ where

(i) $L$ is a skew field of characteristic different from 2;

(ii) $\sigma$ is an involution of $L$, and we let

$$L_\sigma := \{\ell \in L \mid \ell^\sigma = \ell\} = \{\ell + \ell^\sigma \mid \ell \in L\};$$

(iii) $X$ is a right vector space over $L$;
(iv) $h\colon X \times X \to L$ is a skew-hermitian form (see Definition 1.4.(i));
(v) $\pi : X \to L : x \mapsto \pi(x) = \frac{1}{2}h(x,x)$.

The pseudo-quadratic space is *anisotropic* if $\pi(x) \neq 0$ for all $x \in X \setminus \{0\}$.

In the following remark we explain why the definition of a pseudo-quadratic space in [TW02, 11.16] reduces to the one given above if the characteristic is different from 2.

**Remark 2.10.** Definition [TW02, 11.16] makes use of an *involutory set* in $L$ (see [TW02, 11.1]). In the case that the characteristic is different from 2, however, any involutory set in $L$ is equal to $L_\sigma$.

It follows from [TW02, 11.28] that in characteristic different from 2, a pseudo-quadratic space as defined in [TW02, 11.16] is always isomorphic to a pseudo-quadratic space for which $\pi(x) = \frac{1}{2}h(x,x)$. Therefore we can assume this in our definition.

In the original definition of an anisotropic pseudo-quadratic space, one should have that $\pi(x) \in L_\sigma$ if and only if $x = 0$. But since $\pi(x)^\sigma = \frac{1}{2}h(x,x)^\sigma = -\frac{1}{2}h(x,x) = -\pi(x)$, we have $\pi(x) \in L_\sigma$ if and only if $\pi(x) = 0$.

Not every pseudo-quadratic space is a quadrangular algebra; to be a quadrangular algebra the skew field has to satisfy some additional properties.

**Definition 2.11** ([Wei06b, Definition 1.12])**.** Let $L$ be a skew field with involution $\sigma$. We call $(L,\sigma)$ a *quadratic pair*[1], if $k := L_\sigma$ is a field and if either

(i) $L/k$ is a separable quadratic field extension and $\sigma$ is the generator of the Galois group; or
(ii) $L$ is a quaternion algebra over $k$ and $\sigma$ is the standard involution.

Define $q(u) = uu^\sigma$; then $(k, L, q, 1)$ is a pointed anisotropic non-degenerate quadratic space.

A result of Dieudonné (see for example [Wei06b, Theorem 1.15]) says that if $\sigma$ is not trivial, then either $L$ is generated by $L_\sigma$ as a ring, or $(L,\sigma)$ is a quadratic pair (and in this case $L_\sigma$ is a field). From this point of view, quadratic pairs are an exceptional class of skew fields with involution.

[1]This notion, taken from [Wei06b, Definition 1.12], is quite different from the notion of a quadratic pair as defined in the Book of Involutions [KMRT98], and has nothing to do with the notion of a quadratic pair in (finite) group theory either.

**Lemma 2.12** ([Wei06b, Proposition 1.18])**.** *Let* $(L, \sigma)$ *be a quadratic pair and* $(L, \sigma, X, h, \pi)$ *an anisotropic pseudo-quadratic space such that* $X \neq 0$. *Then* $(k, L, q, 1, X, \textit{scalar multiplication}, h)$ *is a quadrangular algebra.*

### 2.2.3 Quadrangular algebras of type $E_6, E_7$ and $E_8$

We give a concise overview of the structure of quadrangular algebras of type $E_6$, $E_7$ and $E_8$. More information can be found in [TW02, Chapter 12 and 13]. Some care is needed, since the map $g$ in [Wei06b] is equal to $-g$ in [TW02]. We use the notation from [Wei06b].

**Definition 2.13.** A *quadratic space* $(k, L, q)$ *with basepoint* is of *type* $E_6$, $E_7$ *or* $E_8$ if it is anisotropic and there exists a separable quadratic field extension $E/k$, with norm denoted by $N$, such that:

$E_6$ : there exist $s_2, s_3 \in k^*$ such that

$$(k, L, q) \cong (k, E^3, N \otimes \langle 1, s_2, s_3 \rangle);$$

$E_7$ : there exist $s_2, s_3, s_4 \in k^*$ such that $s_2 s_3 s_4 \notin N(E)$ and

$$(k, L, q) \cong (k, E^4, N \otimes \langle 1, s_2, s_3, s_4 \rangle);$$

$E_8$ : there exist $s_2, s_3, s_4, s_5, s_6 \in k^*$ such that $-s_2 s_3 s_4 s_5 s_6 \in N(E)$ and

$$(k, L, q) \cong (k, E^6, N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle).$$

We always assume that $s_2 s_3 s_4 s_5 s_6 = -1$, which can be achieved by rescaling the quadratic form if necessary.

*A quadratic space of type* $E_6, E_7$ *or* $E_8$ is a quadratic space that is similar to a quadratic space with basepoint of type $E_6$, $E_7$ or $E_8$.

**Lemma 2.14.** *If* $s_2 s_3 s_4 s_5 s_6 = -1$, *then* $\langle 1, s_2, s_3, s_4, s_5, s_6 \rangle \perp \mathbb{H}$ *is similar to* $\langle\langle s_2, s_3 \rangle\rangle \perp -\langle\langle s_4 s_6, s_5 s_6 \rangle\rangle$.

*Proof.* Since $t\mathbb{H} \cong \mathbb{H}$ for $t \in k^*$, it follows that

$$\begin{aligned} s_2 s_3(\langle 1, s_2, s_3, s_4, s_5, s_6 \rangle \perp \mathbb{H}) &\cong s_2 s_3 \langle s_3, s_2, 1, s_5, s_4, s_6 \rangle \perp \mathbb{H} \\ &\cong \langle s_2, s_3, s_2 s_3, s_2 s_3 s_5, s_2 s_3 s_4, s_2 s_3 s_6 \rangle \perp \mathbb{H} \\ &\cong \langle s_2, s_3, s_2 s_3, -s_4 s_6, -s_5 s_6, -s_4 s_5 \rangle \perp \langle 1, -1 \rangle \\ &\cong \langle\langle s_2, s_3 \rangle\rangle \perp -\langle\langle s_4 s_6, s_5 s_6 \rangle\rangle. \end{aligned}$$

□

Let $(k, L, q, 1, X, \cdot, h)$ be a quadrangular algebra of type $E_6$, $E_7$ or $E_8$; then $(k, L, q)$ is a quadratic space of type $E_6$, $E_7$ or $E_8$, respectively, with basepoint denoted by 1. This quadratic space determines the quadrangular algebra entirely up to isomorphism (see [Wei06b, Theorem 6.24]).

As we are working in characteristic not 2, we can always assume that $E = k(\gamma)$ for some $\gamma \in E$ with $\gamma^2 \in k$.

The vector space $X$ has $k$-dimension 8, 16 or 32, respectively. First we explain how this space is constructed in the $E_8$-case, which is the most interesting case; afterwards we explain the $E_6$- and $E_7$-case, which we will formally describe as vector subspaces of the $E_8$-case.

These constructions are very technical and ad hoc. This is precisely what we try to overcome in Chapters 4 and 5.

**The quadrangular algebra of type $E_8$** In order to construct a quadrangular algebra of type $E_8$, we start with a quadratic space of type $E_8$ with basepoint.

We fix the anisotropic quadratic space

$$(k, L, q) = (k, E^6, N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle)$$

with $s_2 s_3 s_4 s_5 s_6 = -1$, as in Definition 2.13. If we consider $L$ as a vector space over $E$, then it has dimension 6, so we can fix a basis $\{v_1, \ldots, v_6\}$ of $L$ over $E$ such that for all $e_1, \ldots, e_6 \in E$,

$$q(\sum_{i=1}^{6} e_i v_i) = N(e_1) + \sum_{i=2}^{6} N(e_i) s_i;$$

notice that $v_1$ is a basepoint of $q$.

It follows that $\{v_1, \gamma v_1, \ldots, v_6, \gamma v_6\}$ is a basis of $L$ over $k$. We define the 6-dimensional $k$-vector space $L_k = k v_1 \oplus \cdots \oplus k v_6$ and denote by $q|_k : L_k \to k$ the restriction of $q$ to $L_k$.

Next we consider the Clifford algebra with basepoint $C(q|_k, v_1)$ (see Definition 2.6). In $C(q|_k, v_1)$, we have for all $x, y \in C(q|_k, v_1)$ that (see [TW02, (12.48)])

$$x\, y + \overline{y}\, \overline{x} = f(x, \overline{y})$$

where $\overline{x} = f(v_1, x) v_1 - x$ is as in Definition 2.4. Let $i \neq j \in \{2, \ldots, 6\}$; since $f(v_i, v_j) = 0$ and $f(v_1, v_i) = 0$ we have

$$\begin{aligned} &\overline{v_1} = v_1,\ \overline{v_i} = -v_i, \\ &v_1^2 = 1,\ v_i^2 = -q(v_i) = -s_i,\ v_i v_1 = v_1 v_i,\ v_i v_j = -v_j v_i. \end{aligned} \tag{2.1}$$

**Notation 2.15.** We define $S$ as the set of all sequences of the interval $[2, 6]$ in ascending order, including the empty sequence, i.e.

$$S := \{\emptyset\} \cup \{j_1 \ldots j_p \mid 1 \leq p \leq 5 \text{ and } j_1 < \cdots < j_p \in [2, 6]\}.$$

We define the *length* of elements in $S$ as $\text{len}(\emptyset) = 0$ and $\text{len}(j_1 \dots j_p) = p$. For each $I \in S$ we denote its complement in $[2, 6]$ with $I' \in S$. Now define

$$v_I := \begin{cases} v_1 & \text{if } I = \emptyset, \\ v_{j_1} v_{j_2} \dots v_{j_p} \in C(q|_k, v_1) & \text{if } I = j_1 j_2 \dots j_p, \end{cases}$$

and similarly

$$s_I := \begin{cases} 1 \in k & \text{if } I = \emptyset, \\ s_{j_1} s_{j_2} \dots s_{j_p} \in k & \text{if } I = j_1 j_2 \dots j_p. \end{cases}$$

It follows from (2.1) that for all $I \in S$, we have $v_2 v_3 v_4 v_5 v_6 v_I = \pm s_I v_{I'}$. We define the unique map $\text{sgn} : S \to \{-1, +1\}$ by

$$v_2 v_3 v_4 v_5 v_6 v_I = \text{sgn}(I) s_I v_{I'}.$$

By [TW02, 12.49], the set $\{v_I \mid I \in S\}$ is a basis of $C(q|_k, v_1)$, and in particular $\dim_k C(q|_k, v_1) = 32$.

Now we define $\widetilde{X} := E \otimes_k C(q|_k, v_1)$; this is a vector space of dimension 64 over $k$. We do not consider the $E$-vector space structure that $\widetilde{X}$ has, but we do denote $ex := e \otimes x$ for all $e \in E$ and $x \in C(q|_k, v_1)$. Consider the $k$-subspace

$$M = \text{Span}_k\{ev_I - \text{sgn}(I) s_I e^\sigma v_{I'} \mid e \in E \text{ and } I \in S \text{ such that } \text{len}(I) \leq 2\};$$

this subspace has $k$-dimension $2 \cdot 16 = 32$.

We can now finally define the $k$-vector space $X := \widetilde{X}/M$; this space is 32-dimensional. In the vector space $X$ all elements of the form $ev_I$ with $e \in E$ and $I \in S$ such that $\text{len}(I) \geq 3$ can be substituted by elements in $\{ev_I \mid e \in E, \text{len}(I) \leq 2\}$. Therefore an arbitrary element $x \in X$ can be written as

$$x = e_1 v_1 + \sum_{i \in [2,6]} e_i v_i + \sum_{i<j \in [2,6]} e_{ij} v_i v_j \quad \text{with } e_I \in E. \tag{2.2}$$

The maps $\cdot : X \times L \to X$ and $h : X \times X \to L$ are defined explicitly using these coordinates in [TW02, 13.6 and 13.19]. In the sequence we will not need the exact definitions of these maps, but we will make use of identities that these maps satisfy, given in Definition 2.4 and 2.7.

The structure $(k, L, q, 1, X, \cdot, h)$ is a quadrangular algebra of type $E_8$. We implemented these algebraic structures in the computer algebra package Sage; this implementation is very useful to verify if certain identities concerning quadrangular algebras of type $E_8$ hold. We refer to Appendix B for more information concerning this implementation.

**The quadrangular algebra of type $E_6$ and $E_7$** We will always consider the quadrangular algebras of type $E_6$ and $E_7$ as subspaces of quadrangular algebras of type $E_8$. Of course, not every quadratic form of type $E_6$ or $E_7$ over a given field extends to a quadratic form of type $E_8$ over the same field $k$, but it nevertheless makes sense to describe the structure of quadrangular algebras of type $E_6$ and $E_7$ as substructures of a quadrangular algebra of type $E_8$, which can then be interpreted independently of the surrounding $E_8$-structure.

It is shown in [TW02, (12.37)] that if

$$(k, E^6, N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle)$$

is a quadratic space of type $E_8$, then $(k, E^4, N \otimes \langle 1, s_2, s_3, s_4 \rangle)$ is a quadratic space of type $E_7$ and $(k, E^3, N \otimes \langle 1, s_2, s_3 \rangle)$ is a quadratic space of type $E_6$. Below we explain what the vector space $X$ and the maps $\cdot$ and $h$ are in the $E_6$- and $E_7$-case.

**$\mathbf{E_7}$**: Assume that $(k, L, q) = (k, E^4, N \otimes \langle 1, s_2, s_3, s_4 \rangle)$ for certain constants $s_2, s_3, s_4 \in k$ such that $s_2 s_3 s_4 \notin N(E)$. We let $L$ have an $E$-basis $\{v_1, v_2, v_3, v_4\}$ such that for all $e_1, \dots, e_4 \in E$

$$q(\sum_{i=1}^{4} e_i v_i) = N(e_1) + \sum_{i=2}^{4} N(e_i) s_i.$$

In this case $X$ is defined as $X := E \otimes_k C(q|_k, v_1)$, which has dimension 16. Each element of $X$ can be written as

$$\sum_{i=1}^{4} e_i v_i + e_5 v_2 v_3 + e_6 v_2 v_4 + e_7 v_3 v_4 + e_8 v_2 v_3 v_4 \text{ for all } e_i \in E.$$

We will embed this $X$ in the $X$ of the $E_8$ case as follows:

$$\begin{aligned} E_7 \quad & E_8 & \\ e v_i \mapsto\ & e v_i & \text{for all } i \in \{1, \dots, 4\}, \\ e v_i v_j \mapsto\ & e v_i v_j & \text{for all } i < j \in \{2, 3, 4\}, \\ e v_2 v_3 v_4 \mapsto\ & s_2 s_3 s_4 e^{\sigma} v_5 v_6. & \end{aligned}$$

With this identification the maps $\cdot$ and $h$ on the $X$ in the $E_7$-case are the restrictions of those maps in the $E_8$-case (see [TW02, Remark 13.20]).

**$\mathbf{E_6}$**: Assume that $(k, L, q) = (k, E^3, N \otimes \langle 1, s_2, s_3 \rangle)$ for constants $s_2, s_3 \in k$. We let $L$ have an $E$-basis $\{v_1, v_2, v_3\}$ such that for all $e_1, e_2, e_3 \in E$

$$q(e_1 v_1 + e_2 v_2 + e_3 v_3) = N(e_1) + N(e_2) s_2 + N(e_3) s_3.$$

In this case $X$ is defined as $X := E \otimes_k C(q|_k, v_1)$, which has dimension 8. Each element of $X$ can be written as

$$e_1v_1 + e_2v_2 + e_3v_3 + e_4v_2v_3 \text{ for all } e_i \in E.$$

We will embed this $X$ in the $X$ of the $E_8$ case as follows:

$$\begin{aligned} E_6 \quad & E_8 \\ ev_i \mapsto & \ ev_i \qquad \text{for all } i \in \{1,2,3\}, \\ ev_2v_3 \mapsto & \ ev_2v_3. \end{aligned}$$

With this identification the maps $\cdot$ and $h$ on the $X$ in the $E_6$-case are the restrictions of those maps in the $E_8$-case (see [TW02, Remark 13.20]).

The next Theorem shows that if $(k, L, q, 1, X, \cdot, h)$ is a quadrangular algebra of type $E_6$, $E_7$ or $E_8$, then $X$ is an irreducible $C(q,1)$-module (see Definition 2.6).

**Theorem 2.16** ([Wei06b, 2.26]). *Let $(k, V, q)$ be a quadratic space of type $E_6$, $E_7$ or $E_8$ with basepoint 1 and let $X$ be a right $C(q,1)$-module. Then $X$ is an irreducible right $C(q,1)$-module if and only if* $\dim_k(X) = 8$, 16 *or* 32, *respectively.*

## 2.3 Description of Moufang quadrangles

The classification of Moufang quadrangles as given by Tits and Weiss in [TW02], distinguishes six different (non-disjoint) classes:

(1) Moufang quadrangles of quadratic form type;
(2) Moufang quadrangles of involutory type;
(3) Moufang quadrangles of pseudo-quadratic form type;
(4) Moufang quadrangles of type $E_6$, $E_7$ and $E_8$;
(5) Moufang quadrangles of type $F_4$;
(6) Moufang quadrangles of indifferent type.

Moufang quadrangles of type (5) and (6) only exist over fields of characteristic 2 and arise from groups of mixed type; we exclude these classes from further discussion. The other types of Moufang quadrangles arise from classical or algebraic groups. We mention below from which types of groups they arise and use the notation for the Tits indices given in Table 1.1 on page 13.

**Remark 2.17.** It is possible to define a single algebraic structure to describe all possible Moufang quadrangles; this gives rise to the so-called *quadrangular systems* which have been introduced by Tom De Medts in [DM05].

These structures, however, have some disadvantages from an algebraic point of view; most notably, the definition does not mention an underlying field of definition (although it is possible to construct such a field from the data), and the axiom system looks very wild and complicated, with no less than 20 defining identities.

In Section 5.5, we give a more elegant algebraic structure that describes all possible Moufang quadrangles in characteristic different from 2 in a unified way. This algebraic structure is a module for a certain type of Jordan algebras.

Below we give the case by case description of Moufang quadrangles as is given in [TW02].

**Moufang quadrangles of quadratic form type** Moufang quadrangles of quadratic form type are determined by an anisotropic quadratic form $q\colon V \to k$, where $V$ is an arbitrary (possibly infinite-dimensional) vector space over some commutative field $k$. The bilinear map associated by $q$ is denoted by $f$.

The root groups $U_1$ and $U_3$ are parametrized by $(k, +)$ and the root groups $U_2$ and $U_4$ are parametrized by $(V, +)$. We consider the isomorphisms $x_1\colon k \to U_1$, $x_3\colon k \to U_3$, $x_2\colon V \to U_2$ and $x_4\colon V \to U_4$. The commutator relations are given by

$$\begin{aligned} [x_2(v), x_4(w)^{-1}] &= x_3(f(v, w)) \\ [x_1(t), x_4(v)^{-1}] &= x_2(tv)x_3(tq(v)) \\ [U_1, U_3] &= 1 \end{aligned} \tag{2.3}$$

for all $v, w \in V$ and $t \in k$.

If $d = \dim_k V$ is finite, then these Moufang quadrangles arise from linear algebraic groups; they are of absolute type $B_{\ell+2}$ if $d = 2\ell + 1$ is odd, and of type $D_{\ell+2}$ if $d = 2\ell$ is even.

**Moufang quadrangles of involutory type** Moufang quadrangles of involutory type are determined by a skew field $L$ equipped[2] with an involution $\sigma$. The root groups $U_2$ and $U_4$ are parametrized by $(L, +)$ and the root groups $U_1$ and $U_3$ are parametrized by $(L_\sigma, +)$. We consider the isomorphisms $x_1\colon L_\sigma \to U_1$, $x_3\colon L_\sigma \to U_3$, $x_2\colon L \to U_2$ and $x_4\colon L \to U_4$. The commutator relations are given by

$$\begin{aligned} [x_2(a), x_4(b)^{-1}] &= x_3(a^\sigma b + b^\sigma a) \\ [x_1(t), x_4(a)^{-1}] &= x_2(ta)x_3(a^\sigma ta) \\ [U_1, U_3] &= 1 \end{aligned} \tag{2.4}$$

[2] If $\mathrm{char}(L) = 2$, some more data are required, but this is not relevant for our purposes.

for all $a, b \in L$ and $t \in L_\sigma$.

If $L$ is finite-dimensional over its center, of degree $d$, then these Moufang quadrangles arise from algebraic groups; they are outer forms of $A_{4d-1}$ if the involution is of the second kind, and they are (inner or outer forms) of absolute type $D_{2d}$ if the involution is of the first kind.

**Moufang quadrangles of pseudo-quadratic form type** Moufang quadrangles of pseudo-quadratic form type are determined[3] by an anisotropic pseudo-quadratic space, $(L, \sigma, X, h, \pi)$ (see Section 2.2.2) of possibly infinite degree.

Let

$$T = \{(v, a) \in X \times L \mid \pi(v) - a \in L_\sigma\} \tag{2.5}$$

be a group with addition defined for all $v, w \in X$ and $a, b \in L$ by

$$(v, a) + (w, b) = (v + w, a + b + h(w, v)).$$

This group is in general non-abelian and has identity $(0, 0)$ and inverse $-(v, a) = (-v, -a^\sigma)$ (see Lemma [TW02, 11.24]).

The root groups $U_2$ and $U_4$ are parametrized by $(L, +)$; the root groups $U_1$ and $U_3$ are parametrized by $T$. We consider the isomorphisms $x_1 : T \to U_1$, $x_3 : T \to U_3$, $x_2 : L \to U_2$ and $x_4 : L \to U_4$. The commutator relations are given by

$$\begin{aligned} [x_1(v, a), x_3(w, b)^{-1}] &= x_2(h(v, w)) \\ [x_2(a), x_4(b)^{-1}] &= x_3(0, a^\sigma b + b^\sigma a) \\ [x_1(v, a), x_4(b)^{-1}] &= x_2(ab)x_3(vb, b^\sigma ab) \end{aligned} \tag{2.6}$$

for all $a, b \in L$ and $v, w \in X$.

If $L$ is finite-dimensional over its center, of degree $d$, and $X$ is finite-dimensional over $L$, then these Moufang quadrangles arise from algebraic groups. If the involution is of the second kind, they are outer forms of absolute type $A_\ell$. If the involution is of the first kind, they are of absolute type $C_\ell$ or $D_\ell$.

On several occasions the following equivalent description of these Moufang quadrangles will be more useful.

We define the group $S = X \times L_\sigma$ with addition for all $v, w \in X$ and $t, s \in L_\sigma$ by

$$(v, t) + (w, s) = (v + w, t + s + \tfrac{1}{2}(h(w, v) - h(v, w))). \tag{2.7}$$

[3] Again, the situation is slightly more complicated in characteristic 2, but we omit the details. Notice that when the characteristic is not 2, the pseudo-quadratic form $\pi$ is in fact uniquely determined by the skew-hermitian form $h$.

The map

$$\varphi : T \to S : (v, a) \mapsto (v, -\pi(v) + a) \tag{2.8}$$

is a group isomorphism. Now we let $U_1$ and $U_3$ be parametrized by $S$ and $U_2$ and $U_4$ be parametrized by $L$; we consider the isomorphisms $x'_1 = x_1 \circ \varphi^{-1} \colon S \to U_1$, $x'_3 = x_3 \circ \varphi^{-1} \colon S \to U_3$, $x'_2 = x_2 \colon L \to U_2$ and $x'_4 = x_4 \colon L \to U_4$. Using (2.6), we obtain the following commutator relations (we denote $x'_i$ by $x_i$)

$$\begin{aligned} [x_1(v,t), x_3(w,s)^{-1}] &= x_2(h(v,w)) \\ [x_2(a), x_4(b)^{-1}] &= x_3(0, a^\sigma b + b^\sigma a) \\ [x_1(v,t), x_4(b)^{-1}] &= x_2(\pi(v)b + tb) x_3(vb, b^\sigma tb) \end{aligned} \tag{2.9}$$

for all $t, s \in L_\sigma$, $a, b \in L$ and $v, w \in X$.

**Moufang quadrangles of type $E_6$, $E_7$ and $E_8$** Moufang quadrangles of type $E_6$, $E_7$ and $E_8$ are parametrized by a quadrangular algebra $\Omega = (k, L, q, 1, X, \cdot, h)$ of type $E_6$, $E_7$ or $E_8$, respectively.

Let $S = X \times k$ be a group with addition defined by

$$(v, t) + (w, s) = (v + w, t + s + g(w, v)), \tag{2.10}$$

for all $v, w \in X$ and $t, s \in k$; this group is in general non-abelian, with identity $(0, 0)$ and inverse $-(v, t) = (-v, -t)$.

Now the root groups $U_1$ and $U_3$ are parametrized by $S$ and the root groups $U_2$ and $U_4$ are parametrized by $L$. We consider the isomorphisms $x_1 \colon S \to U_1$, $x_3 \colon S \to U_3$, $x_2 \colon L \to U_2$ and $x_4 \colon L \to U_4$. The commutator relations are given by

$$\begin{aligned} [x_1(v,t), x_3(w,s)^{-1}] &= x_2(h(v,w)) \\ [x_2(a), x_4(b)^{-1}] &= x_3(0, f(a,b)) \\ [x_1(v,t), x_4(b)^{-1}] &= x_2(\theta(v,b) + tb) x_3(vb, tq(v)) \end{aligned} \tag{2.11}$$

for all $t, s \in k$, $a, b \in L$ and $v, w \in X$.

These Moufang quadrangles arise from linear algebraic groups of type ${}^2E_{6,2}^{16'}$, $E_{7,2}^{31}$ and $E_{8,2}^{66}$.

**Remark 2.18.** Applying the construction of the previous paragraph to an *arbitrary* quadrangular algebra always gives rise to a Moufang quadrangle. Indeed, assume that $\Omega$ is a quadrangular algebra $(k, L, q, 1, X, \cdot, h)$ arising from a pseudo-quadratic space. Then $L_\sigma = k$, $f(a, b) = a^\sigma b + b^\sigma a$,

$$\theta(v, b) = \tfrac{1}{2}h(v, vb) = \tfrac{1}{2}h(v, v)b = \pi(v)b,$$

$$g(w,v) = \tfrac{1}{2}f(h(w,v),1) = \tfrac{1}{2}(h(w,v) + h(w,v)^\sigma) = \tfrac{1}{2}(h(w,v) - h(v,w)),$$

for all $a, b \in L$ and $v, w \in X$. Therefore the addition of the group $S$ given in (2.7) and the group $S$ given in (2.10) coincide; also the commutator relations given by (2.9) and (2.11) coincide. See also [Wei06b, Theorem 11.11].

## 2.4 Moufang sets

Moufang sets are precisely the Moufang spherical buildings of rank one. They can be defined from a purely group theoretic view point: A Moufang set is essentially a doubly transitive permutation group $G$ such that the point stabilizer contains a normal subgroup which is regular on the remaining vertices. These regular normal subgroups are called the *root groups*, and they are assumed to be conjugate and to generate $G$. A good reference to learn about Moufang sets is [DMS09].

### 2.4.1 Definitions and basic properties

There are some equivalent definitions of the notion of a Moufang set. First we give the standard definition:

**Definition 2.19.** Let $X$ be a set (with $|X| \geq 3$) and $\{U_x \mid x \in X\}$ be a collection of groups. The set $(X, \{U_x\}_{x\in X})$ is a *Moufang set* if the following two properties are satisfied:

($\mathbf{M}_1$) For each $x \in X$, $U_x$ is a subgroup of $\mathrm{Sym}(X)$, $U_x$ fixes $x$ and $U_x$ acts regularly on $X \setminus \{x\}$. We will always denote the action of $U_x$ on $X$ from the right.

($\mathbf{M}_2$) We demand that for all $g \in U_x$

$$U_y^g := g^{-1}U_y g = U_{yg}.$$

The *little projective group* is defined by $G^+ := \langle U_x \mid x \in X\rangle$. The groups $U_x$ are called the *root groups* of the Moufang set.

Note that ($\mathbf{M}_1$) implies that $G^+$ acts doubly transitively on $X$.

Moufang sets are essentially equivalent to saturated split BN-pairs and to abstract rank one groups, a notion that was introduced by Franz Timmesfeld. In Chapter 6 we will use the viewpoint of an abstract rank one group:

**Definition 2.20.** An *abstract rank one group* is a group $G$ together with a pair $(A, B)$ of subgroups such that $A, B \leq G$ are distinct nilpotent subgroups, $G = \langle A, B\rangle$, for each $a \in A^*$ there exists an element $b(a) \in B^*$ such that $A^{b(a)} = B^a$, and for each $b \in B^*$ there exists an element $a(b) \in A^*$ such that $B^{a(b)} = A^b$.

We then say that $G$ is an abstract rank one group with *unipotent subgroups* $A$ and $B$.

The following lemma shows that the two previous definitions are essentially equivalent.

**Lemma 2.21.** (i) *Let* $(X, \{U_x\}_{x\in X})$ *be a Moufang set with nilpotent root groups, and choose two arbitrary elements* $0, \infty \in X$. *Then* $G^+ = \langle U_0, U_\infty\rangle$ *is an abstract rank one group.*

(ii) *Let* $G = \langle A, B\rangle$ *be an abstract rank one group. Define*

$$\begin{aligned} Y &:= \{A^g \mid g \in G\} = \{A^b \mid b \in B\} \cup \{B\} \\ &= \{B^g \mid g \in G\} = \{B^a \mid a \in A\} \cup \{A\}. \end{aligned}$$

*For all* $x \in Y$, *define* $U_x := x \leq G$. *Let* $y \in Y$ *and* $h \in U_x \leq G$, *define* $(y)h := y^h$. *Therefore*

$$(A^g)h = A^{gh}.$$

*Then* $(Y, \{U_x\}_{x\in Y})$ *is a Moufang set with*[4] $G^+ \cong G/Z(G)$.

*Proof.* (i) is proven in [DMS09, Prop. 2.2.2]. (ii) is proven by combining [DMS09, Prop. 2.2.3] and [DMS09, Prop. 2.1.3], although it is not hard to verify this directly without doing the detour over BN-pairs. □

We remark that it is an open problem whether there exist Moufang sets with root groups that are not nilpotent. All known examples of Moufang sets have root groups of nilpotency class at most three, i.e. $[[[U_x, U_x], U_x], U_x] = 1$ for all $x \in X$.

In the following construction we show how one can describe a Moufang set using only one group $U$ and a permutation of $U^*$. The advantage of this description is that the only required data are a group and a permutation. This should be compared with Definition 2.19, where one needs a set $X$ and a collection of subgroups of $\mathrm{Sym}(X)$, and with Definition 2.20, where one needs to describe the ambient group $G$ (and two of its subgroups).

**Construction 2.22.** Let $\mathbb{M} = (X, (U_x)_{x\in X})$ be an arbitrary Moufang set, and let $0 \neq \infty \in X$ be two arbitrary elements.

Define the set $U := X \setminus \{\infty\}$. For each $a \in U$ define the unique element $\alpha_a \in U_\infty$ that maps 0 to $a$. It follows from axiom $(\mathbf{M}_1)$ that $U_\infty = \{\alpha_a \mid a \in U\}$.

We give $U$ the structure of a group by defining $a + b := a\alpha_b = 0\alpha_a\alpha_b$ for all $a, b \in U$. It is clear that this is a (not necessarily commutative) group, with neutral element $\alpha_0$, and with $\alpha_a^{-1} = \alpha_{-a}$. It follows that $U \cong U_\infty$.

[4]We embed $U_x$ in $\mathrm{Sym}(Y)$ via the conjugation action.

**Lemma 2.23.** *For each* $a \in U \setminus \{0\}$ *there exists a unique permutation* $\mu_a \in U_0^* \alpha_a U_0^* \subseteq G^+$ *interchanging* $0$ *and* $\infty$.

*Proof.* Let $\gamma_1, \gamma_2 \in U_0^*$, and let $\mu_a = \gamma_1 \alpha_a \gamma_2$. We require $0\mu_a = \infty$ and $\infty\mu_a = 0$. The condition that $0\gamma_1\alpha_a\gamma_2 = \infty$ is equivalent with $a\gamma_2 = \infty$ and the condition $\infty\gamma_1\alpha_a\gamma_2 = 0$ is equivalent with $\infty\gamma_1 = -a$.

By ($\mathbf{M}_1$), $\gamma_1$ and $\gamma_2$ exist and are uniquely determined. Therefore $\mu_a$ exists and is uniquely determined. □

Fix an element $e \in U \setminus \{0\}$ and define $\tau := \mu_e$, this is a permutation of $X \setminus \{0, \infty\} = U^*$ that interchanges 0 and $\infty$.

Since $\tau$ and $\alpha_a$ are both in $G^+$ for all $a \in U$, we have $U_\infty^\tau = U_{\infty\tau} = U_0$ and $U_0^{\alpha_a} = U_{0\alpha_a} = U_a$.

It follows that in order to describe the Moufang set $\mathbb{M} = (X, (U_x)_{x\in X})$, it is sufficient to know of the group $U$ with the permutation $\tau$. Therefore we denote the Moufang set $\mathbb{M} = (X, (U_x)_{x\in X})$ by $\mathbb{M}(U, \tau)$.

**Remark 2.24.** A slight disadvantage of this description is that the permutation $\tau$ is not uniquely determined by the Moufang set. However, it is shown in [DMS09, Lemma 4.1.2] that $\mathbb{M}(U, \tau) = \mathbb{M}(U, \mu_a)$ for all $a \in U^*$, and in fact, the data $\big(U, (\mu_a)_{a\in U^*}\big)$ is uniquely determined by the Moufang set.

**Remark 2.25.** Starting from an arbitrary group $U$ together with a permutation $\tau$ of $U^*$, one can apply the previous construction the other way around. However, the result is not always a Moufang set. We refer to [DMS09, section 3] for the details of this construction, and for the condition needed on $U$ and $\tau$ in order to get a Moufang set. If it does, we denote this Moufang set by $\mathbb{M}(U, \tau)$.

The following lemma, which we will need in the following section and in Chapter 6, is given in [DMS09, Prop. 4.1.1].

**Lemma 2.26.** *For all* $a \in U^*$, *we have*

$$\mu_a = \alpha^\tau_{(-a)\tau^{-1}} \alpha_a \alpha^\tau_{-(a\tau^{-1})}.$$

*Proof.* It follows from $U_0 = U_\infty^\tau$ that $U_0 = \{\alpha_a^\tau \mid a \in U\}$. Using the proof of Lemma 2.23, we know that $\mu_a = \gamma_1\alpha_a\gamma_2$ with $\gamma_1 = \alpha_x^\tau$ and $\gamma_2 = \alpha_y^\tau$ for some $x, y \in U^*$ such that $\infty\gamma_1 = -a$ and $a\gamma_2 = \infty$.

We find that $x\tau = -a$ and $a\tau^{-1}\alpha_y = 0$. We conclude that $x = (-a)\tau^{-1}$ and $y = -(a\tau^{-1})$. □

The following concept plays an important role in the theory of Moufang sets.

**Definition 2.27.** Let $\mathbb{M}(U, \tau)$ be a Moufang set, let $a \in U^*$. Define the Hua-map $h_a = \tau\mu_a \in \mathrm{Sym}(X)$. It is clear that the map $h_a$ fixes 0 and $\infty$; it can be shown that $(b + c)h_a = bh_a + ch_a$ for all $b, c \in U$; see the second paragraph of the proof of [DMS09, Theorem 3.5].

Define the Hua group

$$H = \langle h_a \mid a \in U^* \rangle = \langle \mu_a\mu_b \mid a, b \in U^* \rangle \leq \mathrm{Aut}(U).$$

A Moufang set is called *proper* if $H \neq 1$. The non-proper Moufang sets are precisely the ones for which the little projective group is sharply 2-transitive.

Now we introduce what it means for Moufang sets to be isomorphic.

**Definition 2.28.** Let $\mathbb{M}_1 = (X_1, \{U_x\}_{x\in X_1})$ and $\mathbb{M}_2 = (X_2, \{U_y\}_{y\in X_2})$ be two Moufang sets. We say that $\mathbb{M}_1$ and $\mathbb{M}_2$ are *isomorphic* if there exists a bijection $\varphi : X_1 \to X_2$ such that the induced map $\mathrm{Sym}(X_1) \to \mathrm{Sym}(X_2) : g \mapsto \varphi^{-1}g\varphi$ maps each root group $U_x$ isomorphically to the corresponding root group $U_{x\varphi}$.

We call $\varphi$ an *isomorphism* from $\mathbb{M}_1$ to $\mathbb{M}_2$.

Next we translate the definition of isomorphic Moufang sets into a more useful criteria.

**Lemma 2.29.** (i) *Let $\mathbb{M}_1 = \mathbb{M}(U_1, \tau_1)$ and $\mathbb{M}_2 = \mathbb{M}(U_2, \tau_2)$ be two Moufang sets. Then $\mathbb{M}_1$ and $\mathbb{M}_2$ are isomorphic if and only if there exists a group isomorphism $\varphi : U_1 \to U_2$ such that $\tau_2 h = \varphi^{-1}\tau_1\varphi$ for some element $h \in H_2$, the Hua group of $\mathbb{M}_2$.*

(ii) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two Moufang sets with corresponding abstract rank one groups $G_1 = \langle A_1, B_1\rangle$ and $G_2 = \langle A_2, B_2\rangle$. Then $\mathbb{M}_1$ and $\mathbb{M}_2$ are isomorphic if and only if there exists a group isomorphism $\varphi : G_1 \to G_2$ such that $A_1.\varphi = A_2$ and $B_1.\varphi = B_2$.*

*Proof.* (i) First assume that $\mathbb{M}_1 = (X_1, \{U_x\}_{x\in X_1}) = \mathbb{M}(U_1, \tau_1)$ and $\mathbb{M}_2 = (X_2, \{U_y\}_{y\in X_2}) = \mathbb{M}(U_2, \tau_2)$ are isomorphic with isomorphism $\varphi : X_1 \to X_2$. We follow Construction 2.22 and choose two elements $0_1, \infty_1 \in X_1$ and fix $0_2 = 0_1.\varphi, \infty_2 = \infty_1.\varphi \in X_2$. Since the little projective group of $\mathbb{M}_2$ acts 2-transitively on $X_2$, this is equivalent with choosing $0_2, \infty_2 \in X_2$ arbitrary.
It is clear that for all $a \in U_1^*$, $\alpha_a^\varphi = \varphi^{-1}\alpha_a\varphi = \alpha_{a.\varphi} \in U_{\infty_2}$; therefore $\mu_a^\varphi = \mu_{a.\varphi}$. Now the restriction $\varphi : U_1 = X_1\backslash\{\infty_1\} \to U_2 = X_2\backslash\{\infty_2\}$ is a group isomorphism, since for all $a, b \in U_1$

$$(a + b).\varphi = (a\alpha_b).\varphi = a.\varphi\alpha_b^\varphi = a.\varphi\alpha_{b.\varphi} = a.\varphi + b.\varphi.$$

Now suppose that $\tau_1 = \mu_{e_1}$ and $\tau_2 = \mu_{e_2}$ for $e_1 \in U_1^*$ and $e_2 \in U_2^*$, then

$$\tau_1^\varphi = \mu_{e_1.\varphi} = \tau_2 \mu_{e_2}^{-1} \mu_{e_1.\varphi} = \tau_2 \mu_{-e_2} \mu_{e_1.\varphi}.$$

This proves the claim since by definition, $\mu_{-e_2}\mu_{e_1.\varphi} \in H_2$.
Conversely, assume that there exists a group isomorphism $\varphi : U_1 \to U_2$ and that $\tau_1^\varphi = \tau_2 h$ for some $h \in H_2$. Notice that for $i = 1, 2$ it follows from Construction 2.22 that

$$U_{\infty_i} = \{\alpha_a \mid a \in U_i\},\ U_{0_i} = U_{\infty_i}^{\tau_i},\ U_a = U_{0_i}^{\alpha_a}, U_{0_i}^h = U_{0_i}$$

for all $a \in U_i$.
First extend $\varphi : X_1 \to X_2$ to a bijection of sets by defining $\infty_1.\varphi = \infty_2$. Now

$$b\alpha_a^\varphi = (b.\varphi^{-1} + a).\varphi = b + a.\varphi = b\alpha_{a.\varphi}$$

for all $a \in U_1$ and $b \in U_2$, since $\varphi$ is a group morphism. It follows that $\alpha_a^\varphi = \alpha_{a.\varphi}$, so in particular $U_{\infty_1}^\varphi = U_{\infty_2}$, and therefore

$$U_{0_1}^\varphi = U_{\infty_1}^{\tau_1\varphi} = U_{\infty_1}^{\varphi\tau_1^\varphi} = U_{\infty_2}^{\tau_2 h} = U_{0_2}$$

and

$$U_a^\varphi = U_{0_1}^{\alpha_a\varphi} = U_{0_1}^{\varphi\alpha_a^\varphi} = U_{0_2}^{\alpha_{a.\varphi}} = U_{a.\varphi}.$$

It is now clear that $\varphi$ induces group isomorphisms between corresponding root groups.

(ii) This is clear from Lemma 2.21. □

### 2.4.2 Examples of Moufang sets

Moufang sets are far from being classified. Each linear algebraic group of relative rank one gives rise to a Moufang set. However, only for a few Moufang sets obtained from exceptional linear algebraic groups, an explicit construction was known. Below we give the construction of Moufang sets of type $E_{7,1}^{78}, F_{4,1}^{21}$ and $E_{7,1}^{66}$. Recently, a construction of Moufang sets of type ${}^2E_{6,1}^{29}$ has been given in [Cal13].

Below we give a description of most of the known Moufang sets in characteristic[5] different from 2 and 3. If the characteristic is equal to 2 or 3, certain more complicated structures of mixed type arise. In view of Theorem 2.2, it is a cautious conjecture to assume that in characteristic different from 2 and 3 all Moufang sets are obtained from linear algebraic groups or from infinite dimensional classical groups.

[5] There is no well-defined notion (yet) of the characteristic of a Moufang set, but what we mean is that the root groups have no elements of order 2 or 3. In all known examples, however, there is a certain underlying field, and this condition is then equivalent to the fact that this field has characteristic unequal to 2 or 3.

#### 2.4.2.1 Moufang sets from Jordan algebras

Let $J$ be a Jordan division algebra (see Definition 1.9). Define $U$ to be the additive group of $J$, considered as vector space. Define for each $x \in J \setminus \{0\}$ the permutation $x.\tau = -x^{-1}$. Then $\mathbb{M}(U, \tau)$ is a Moufang set.

In this case the Hua-map $h_a$ coincides with $U_a$, the $U$-operator in the Jordan algebra. Therefore the Hua subgroup is equal to $\langle U_a \mid a \in J \setminus \{0\} \rangle$, which is known as the inner structure group of $J$ and is known to play an important role in the theory of Jordan algebras.

The group $U$ is abelian. All *known* examples of proper Moufang sets with abelian rout groups can be described in this fashion. Below we give a few examples.

**Example 2.30.** (i) Let $J = A^+$ for some associative division $k$-algebra $A$. The inverse in $J$ is the same as the inverse in $A$. Therefore $\mathbb{M}((A, +), x \mapsto -x^{-1})$ is a Moufang set. If $A = k$, the little projective group of the Moufang set is $PSL_2(k)$.

(ii) Let $J$ be of quadratic form type with quadratic form $q : J \to k$ with basepoint. It follows from [McC04, Theorem II.6.1.6] that $x^{-1} = \overline{x}/q(x)$ for all $x \in J$, therefore $\mathbb{M}(J, x \mapsto -\overline{x}/q(x))$ is a Moufang set.

If the Jordan algebra is exceptional, the corresponding Moufang set arises from a linear algebraic group of type $E_{7,1}^{78}$. In all the other cases the Moufang set arises from a classical group.

#### 2.4.2.2 Moufang sets of skew-hermitian type

Let $(L, \sigma, X, h, \pi)$ be an anisotropic pseudo-quadratic space. Define the group $T \subset X \times L$ as in (2.5) on page 29 and the involution $(v, a).\tau = (-va^{-1}, -a^{-1})$ for all $(v, a) \in T^*$. Then $\mathbb{M}(T, \tau)$ is a Moufang set of skew-hermitian type.

We know that the group $T$ is isomorphic to the group $S = X \times L_\sigma$ (see (2.7)) by the isomorphism $\varphi$ defined in (2.8). We determine a permutation $\tau'$ of $S^*$ such that $\mathbb{M}(T, \tau)$ is isomorphic to $\mathbb{M}(S, \tau')$. It follows from Lemma 2.29 that $\tau' = \varphi^{-1}\tau\varphi$ will give rise to isomorphic Moufang sets:

$$\begin{aligned}(v,t).\tau' &= \big(-v(\pi(v)+t)^{-1},\ -(\pi(v)+t)^{-1}\big).\varphi \\ &= \big(-v(\pi(v)+t)^{-1},\ -\pi(v(\pi(v)+t)^{-1}) - (\pi(v)+t)^{-1}\big) \\ &= \big(-v(\pi(v)+t)^{-1},\ -(-\pi(v)+t)^{-1}t(\pi(v)+t)^{-1}\big). \end{aligned} \tag{2.12}$$

#### 2.4.2.3 Moufang sets of type $F_4$

Let $C$ be a division composition $k$-algebra with norm $N(a) = a\overline{a}$ and trace $T(a) = a + \overline{a}$, define

$$U := \{(a, b) \in C \times C \mid N(a) + T(b) = 0\}$$

with addition

$$(a, b) + (c, d) := (a + c, b + d - \overline{c}a),$$

and the permutation $\tau$ on $U^*$ by

$$(a, b).\tau := (-ab^{-1}, b^{-1}).$$

It is shown in [DMVM10] that $\mathbb{M}(U, \tau)$ is a Moufang set. When $C$ is an octonion algebra, this Moufang set is of type $F_{4,1}^{21}$; in all other cases it comes from a classical linear algebraic group. In [Cal13], it is shown that the Hua maps are given by the formula

$$(x, y).h_{(a,b)} = \big((ab^{-1})((\overline{b}a^{-1})(x\overline{b})), (ba^{-1})((ay)\overline{b})\big)$$

for all $(a, b), (x, y) \in U$ such that $a \neq 0$. If $a = 0$, we have $(x, y).h_{(0,b)} = (xb, -byb)$.

Let $t \in k$; then $(2t1, -2t^2 1) \in U$, and

$$(x, y).h_{(2t1,-2t^2 1)} = (-2t^2 x, 4t^4 y), \tag{2.13}$$

for all $(x, y) \in U$.

In Chapter 6, we will make use of the following isomorphic description of this Moufang set. Let $U' = \{(a, b) \in C \times C \mid N(a) + T(b) = 0\}$ with addition

$$(a, b) + (c, d) := (a + c, b + d - a\overline{c})$$

and permutation

$$(a, b).\tau' := (2b^{-1}a, 4b^{-1}).$$

It is easy to verify that $\mathbb{M}(U, \tau)$ is isomorphic to $\mathbb{M}(U', \tau')$ with isomorphism $\varphi : U \to U' : (a, b) \to (\overline{a}, \overline{b})$, making use of (2.13) for $t = 1$. The underlying reason for this isomorphism is the fact that a composition algebra is isomorphic with its opposite algebra.

#### 2.4.2.4 Moufang sets obtained from Moufang polygons

To each Moufang polygon, one can associate two different Moufang sets (for Moufang triangles, there is only one). The Moufang sets are described by considering the local structure of the Moufang polygon. Below, we describe to what types of Moufang sets the Moufang polygons give rise.

Let $\Gamma$ be a Moufang $n$-gon, fix two opposite vertices $a_0, a_n \in \Gamma$. Since the girth of $\Gamma$ is $2n$, each neighbor of $a_0$ uniquely determines a root with extremal vertices $a_0$ and $a_n$. Let $X$ be the set of neighbors of $a_0$, thus $X$ is in one-to-one correspondence with the set of roots with extremal vertices $a_0$ and $a_n$.

Let $x \in X$ and let $\alpha$ be the corresponding root; we will denote the root group $U_\alpha$ by $\mathcal{U}_x$. Now $(X, (\mathcal{U}_x)_{x\in X})$ is a Moufang set. Indeed:

($\mathbf{M}_1$): Let $x \in X$. It follows from the definition of a Moufang polygon that the root group $\mathcal{U}_x$ stabilizes $x$ and acts regularly on $X \setminus \{x\}$.

($\mathbf{M}_2$): For $x, y \in X$, let $g \in \mathcal{U}_x$. Since $g \in \mathrm{Aut}(\Gamma)$ that fixes $a_0$ and $a_n$, it is clear that if $\alpha$ is a root with extremal vertices $a_0$ and $a_n$, $\alpha.g$ is also a root with extremal vertices $a_0$ and $a_n$. It follows that $\mathcal{U}_y^g = \mathcal{U}_{y.g}$.

Actually, this is a special case of the fact that the residue at a vertex of a Moufang spherical building of rank $n$ is a Moufang spherical building of rank $n-1$, see [AB08, Prop 7.32]. In the above procedure we described a rank one residue of the Moufang polygon.

We will describe the resulting Moufang set as $\mathbb{M}(\mathcal{U}, \tau)$. We start by fixing some notations.

By Theorem 2.3 all apartments are equivalent, so we fix an apartment

$$\Sigma = (b_1, b_2, \ldots, b_n, \ldots, b_{2n-1}, b_0, b_1)$$

and define the sequence of root groups as in Theorem 2.3:

$$U_1 = U_{(b_1,\ldots,b_{n+1})}, U_2 = U_{(b_2,\ldots,b_{n+2})}, \ldots, U_n = U_{(b_n,\ldots,b_0)}. \tag{2.14}$$

We will also make use of the following root groups:

$$U_0 = U_{(b_0,\ldots,b_n)}, U_{n+1} = U_{(b_{n+1},\ldots,b_1)}. \tag{2.15}$$

By the classification of Moufang polygons, we know that these root groups are parametrized by an algebraic structure. If $n = 3$, $U_1 \cong U_2 \cong U_3 \cong (A, +)$ for $A$ an alternative division ring. If $n > 3$, $U_1 \cong U_3 \cong \ldots \cong U_{n-1} \cong (A, +)$ and $U_2 \cong U_4 \cong \ldots \cong U_n \cong (B, +)$ for some algebraic structures $A$ and $B$ which we do not specify for the moment.

We fix two opposite vertices $a_0$ and $a_n$ in the apartment $\Sigma$. If $n > 3$, the isomorphism class of the Moufang set we will obtain depends on this choice.

If necessary we change the labeling of the apartment such that either $a_0 = b_1$ and $a_n = b_{n+1}$, or $a_0 = b_0$ and $a_n = b_n$. In the first case the root groups of the resulting Moufang set will be conjugate to $U_1 \cong (A, +)$; in the second case the root groups of the resulting Moufang set will be conjugate to $U_n \cong (B, +)$.

We follow Construction 2.22 to describe the Moufang set as $\mathbb{M}(\mathcal{U},\tau)$.

As before, let $X$ denote the set of neighbors of $a_0$. We fix two vertices $0,\infty \in X \cap \Sigma$ such that in the first case $0 := b_0, \infty := b_2$ and in the second case $0 := b_1, \infty := b_{2n-2}$. In the terminology of (2.14) and (2.15), it follows that $\mathcal{U}_0 := \mathcal{U}_{b_0} = U_{n+1}$, $\mathcal{U}_\infty := \mathcal{U}_{b_2} = U_1$ and $\mathcal{U}_0 = \mathcal{U}_{b_1} = U_0$, $\mathcal{U}_\infty := \mathcal{U}_{b_{2n-2}} = U_n$, respectively[6].

We define $\mathcal{U} = X \setminus \{\infty\}$; this is as a group isomorphic to $\mathcal{U}_\infty$ with isomorphism

$$\mathcal{U} \to \mathcal{U}_\infty : u \mapsto \alpha_u.$$

We have $\mathcal{U} \cong \mathcal{U}_\infty = U_1 \cong (A,+)$ or $\mathcal{U} \cong \mathcal{U}_\infty = U_n \cong (B,+)$ respectively, so we will identify $\mathcal{U}$ with $A$ or $B$, respectively. As in [TW02, p. 163], we parametrize the root groups $U_1, \ldots, U_n$ by fixing isomorphisms $x_i$ from $A$ or $B$ to $U_i$.

Let $i = 1$ or $i = n$ respectively. For each $u \in \mathcal{U}$, we have $\alpha_u \in \mathcal{U}_\infty = U_i$. It follows that there exists a unique $a \in A$ or $a \in B$ such that $\alpha_u = x_i(a)$. We define $\gamma : \mathcal{U} \to A$ (or $B$) such that $\alpha_u = x_i(\gamma(u))$. We show that $\gamma$ is a group isomorphism:

$$x_i(\gamma(u+v)) = \alpha_{u+v} = \alpha_u\alpha_v = x_i(\gamma(u))x_i(\gamma(v)) = x_i(\gamma(u)+\gamma(v)),$$

for all $u, v \in \mathcal{U}$. From now on we will simply say that $\mathcal{U} = A$ or $B$ respectively, implicitly making use of the isomorphism $\gamma$.

To determine the permutation $\tau$ of $\mathcal{U}$, we can make use of calculations done in [TW02, Chapter 32]. Although the result is written in the language of Moufang quadrangles, we can easily deduce the permutation $\tau$. In [TW02, Chapter 32] the $\mu$-maps are calculated; these maps are defined in Theorem [TW02, 6.1]. When we translate this theorem to our terminology for $i = a_0$, we obtain the following:

**Lemma 2.31** ([TW02, 6.1])**.** *There exist unique functions $\kappa, \lambda : \mathcal{U}_\infty^* \to \mathcal{U}_0^*$ such that $0^{g\lambda(g)} = \infty$ and $\infty^{\kappa(g)g} = 0$ for all $g \in \mathcal{U}_\infty^*$. The product*

$$\mu(g) = \kappa(g).g.\lambda(g)$$

*fixes $a_0$ and $a_n$ and reflects the apartment. This implies that* 0 *and* $\infty$ *are interchanged by $\mu(g)$.*

Let $a \in \mathcal{U}^*$, and consider the corresponding map $\mu_a \in U_0^*\alpha_a U_0^*$ defined in Lemma 2.23. It follows from the uniqueness of this $\mu$-map that

$$\mu_a = \mu(\alpha_a) = \mu(x_i(a)).$$

[6]Observe that we distinguish between $\mathcal{U}_0$ defined in the beginning of this section and $U_0$ defined in (2.15).

Let $i = 1$ or $i = n$. In each case, we define the permutation $\tau = \mu_{e_i} = \mu(x_i(e_i))$ where we choose $e_1 \in A^* \cong U_1^*$ and $e_n \in B^* \cong U_n^*$ as in [TW02, Fig. 5 on p. 354].

Note that we have $\mathcal{U}_0 = U_{n+1} \cong \mathcal{U}_\infty = U_1 \cong A$ or $\mathcal{U}_0 = U_0 \cong \mathcal{U}_\infty = U_n \cong B$, respectively. We have to choose a certain parametrization of $\mathcal{U}_0$. As in [TW02, p. 354], we define

$$\begin{aligned} x_{n+1}(a) &:= x_1(a)^\tau = \alpha_a^\tau && \text{for all types of } \Gamma; \\ x_0(b) &:= x_n(\overline{b})^\tau = \alpha_{\overline{b}}^\tau && \text{if } \Gamma \text{ is of quadratic form type or of type } E_6, E_7, E_8, \\ x_0(b) &:= x_n(b)^\tau = \alpha_b^\tau && \text{for the remaining types of } \Gamma. \end{aligned}$$

It follows from Lemma 2.26 and the above parametrizations that

$$\mu_a = \mu(x_1(a)) = x_{n+1}((-a)\tau^{-1})x_1(a)x_{n+1}(-(a\tau^{-1})), \tag{2.16}$$

$$\mu_b = \mu(x_n(b)) = x_0((-b)\tau^{-1})x_n(b)x_0(-(b\tau^{-1})), \tag{2.17}$$

$$\text{or } \mu_b = \mu(x_n(b)) = x_0(\overline{(-b)\tau^{-1}})x_n(b)x_0(\overline{-(b\tau^{-1})}). \tag{2.18}$$

In [TW02, Chapter 32], thee $\mu$-maps have been calculated, and we can therefore describe the $\tau$ of the Moufang set. As usual, we only describe the cases that appear in characteristic not 2 and 3.

(1) In the case of Moufang triangles, the root groups are parametrized by an alternative division algebra $A$. Then for all $t \in A \setminus \{0\}$, we have

$$\mu(x_1(t)) = x_4(t^{-1})\, x_1(t)\, x_4(t^{-1}).$$

Therefore by (2.16), $\mathcal{U} = (A, +)$ and $t.\tau = -t^{-1}$. If $A$ is associative, this is given in Example 2.30.(i). If $A$ is an octonion algebra, its norm $N$ makes $A$ into a quadratic space with basepoint. Since $x^{-1} = \overline{x}/N(x)$, this Moufang set is described in Example 2.30.(ii).

(2) In the case of Moufang hexagons, the root group $U_1$ is parametrized by an anisotropic cubic norm structure $J$. For each $a \in J \setminus \{0\}$, we define $a^{-1} := a^\sharp/N(a)$. We have

$$\mu(x_1(a)) = x_7(a^{-1})\, x_1(a)\, x_7(a^{-1}).$$

Therefore by (2.16), $\mathcal{U} = (J, +)$ and $a.\tau = -a^{-1}$.
In the case that $J$ is a cubic Jordan division algebra, it is shown in [McC04, Theorem II.6.1.6] that $a^{-1}$ is the Jordan inverse of $a$; therefore we obtain a Moufang set as in Section 2.4.2.1.
The root group $U_1$ is parametrized by the field $k$; we have

$$\mu(x_6(t)) = x_0(t^{-1})x_6(t)x_0(t^{-1}),$$

for all $t \in k \setminus \{0\}$. Therefore by (2.17), $\mathcal{U} = (k, +)$ and $t.\tau = -t^{-1}$, this Moufang set is described in Example 2.30.(i).

(3) In the case of Moufang quadrangles of quadratic form type, the root group $U_4$ is parametrized by a vector space $V$ equipped with an anisotropic quadratic form $q$. We have

$$\mu(x_4(v)) = x_0(v/q(v))\, x_4(v)\, x_0(v/q(v)),$$

for all $v \in V \setminus \{0\}$. Therefore by (2.18), $\mathcal{U} = V$ and $v.\tau = -\overline{v}/q(v)$, this Moufang set is described in Example 2.30.(ii).
The root group $U_1$ is parametrized by the field $k$; we have

$$\mu(x_1(t)) = x_5(t^{-1})x_1(t)x_5(t^{-1}),$$

for all $t \in k \setminus \{0\}$. The obtained Moufang set is described in Example 2.30.(i).

(4) In the case of Moufang quadrangles of involutory type, we let $L$ be a skew field. The root group $U_4$ is parametrized by the additive group of $L$; we have

$$\mu(x_4(u)) = x_0(u^{-1})x_4(u)x_0(u^{-1}),$$

for all $u \in L \setminus \{0\}$. Therefore by (2.17), $\mathcal{U} = (L, +)$ and $u.\tau = -u^{-1}$, this Moufang set is described in Example 2.30.(i). The root group $U_1$ is parametrized by the additive group of $L_\sigma$; we have

$$\mu(x_1(a)) = x_5(a^{-1})x_1(a)x_5(a^{-1}),$$

for all $a \in L_\sigma \setminus \{0\}$. Therefore by (2.16), $\mathcal{U} = (L_\sigma, +)$ and $a.\tau = -a^{-1}$.

(5) In the case of Moufang quadrangles of pseudo-quadratic form type, we let $(L, \sigma, X, h, \pi)$ be an anisotropic pseudo-quadratic space. The root group $U_4$ is parametrized by the additive group of $L$; we have

$$\mu(x_4(u)) = x_0(u^{-1})x_4(u)x_0(u^{-1}),$$

for all $u \in L \setminus \{0\}$. Therefore this Moufang set is described in Example 2.30.(i).
The root group $U_1$ is parametrized by $T$ defined in (2.5); we have

$$\mu(x_1(v, a)) = x_5(va^{-\sigma}, a^{-\sigma})x_1(v, a)x_5(-va^{-1}, a^{-\sigma}),$$

for all $(v, a) \in T \backslash \{(0, 0)\}$. Therefore by (2.16), $(v, a).\tau^{-1} = (va^{-1}, -a^{-1})$. It follows that $(v, a).\tau = (-va^{-1}, -a^{-1})$, thus this Moufang set coincides with the one of skew-hermitian form type described in Section 2.4.2.2.

(6) In the case of Moufang quadrangles of type $E_6, E_7$ or $E_8$, we let $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra of type $E_6, E_7$ or $E_8$ with characteristic different from 2.
The root group $U_1$ is parametrized by $X \times k$ with addition given by (2.10) on page 30. For all $(x, t) \in X \times k \setminus \{(0, 0)\}$, we have

$$\mu(x_1(x,t)) =$$
$$x_5\Big(\frac{x\cdot\pi(x)+tx}{q(\pi(x))+t^2}, \frac{t}{q(\pi(x))+t^2}\Big)x_1(x,t)x_5\Big(\frac{x\cdot\pi(x)-tx}{q(\pi(x))+t^2}, \frac{t}{q(\pi(x))+t^2}\Big). \tag{2.19}$$

Therefore by (2.16),

$$(x,t).\tau^{-1} = \Big(\frac{-x\cdot\pi(x)+tx}{q(\pi(x))+t^2}, \frac{-t}{q(\pi(x))+t^2}\Big).$$

One can verify using some identities in [TW02, (13.56) and (b) on p. 118] that

$$(x,t).\tau = \Big(\frac{x\cdot\pi(x)-tx}{q(\pi(x))+t^2}, \frac{-t}{q(\pi(x))+t^2}\Big). \tag{2.20}$$

When the Moufang quadrangle is of type $E_8$, this Moufang set is obtained from a linear algebraic group of type $E_{7,1}^{66}$. When the Moufang quadrangle is of type $E_6$ and $E_7$, it is shown in [DM06] that this Moufang set is isomorphic to a Moufang set of skew-hermitian type.
The root group $U_4$ is parametrized by $L$; we have

$$\mu(x_4(v)) = x_0(v/q(v))\,x_4(v)\,x_0(v/q(v)),$$

for all $v \in L \setminus \{0\}$. Just as in (3) we find that this is a Moufang set of quadratic form type.

**Remark 2.32.** In the case when we consider a quadrangular algebra $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ that is a pseudo-quadratic space (see Remark 2.18), the permutation $\tau$ in (2.12) coincides with the one in (2.20). Indeed, this is clear when we rewrite (2.12) using the facts that $L_\sigma = k$ and $q(\pi(v)) = \pi(v)^\sigma\pi(v) = -\pi(v)\pi(v) \in k$. We then find for all $(v,t) \in X \times k \setminus \{(0,0)\}$ that

$$\begin{aligned}(v,t).\tau' &= \big(-v(\pi(v)+t)^{-1},\ -(-\pi(v)+t)^{-1}t(\pi(v)+t)^{-1}\big)\\ &= \big(-v(-\pi(v)+t)(-\pi(v)+t)^{-1}(\pi(v)+t)^{-1},\\ &\qquad\qquad -t(-\pi(v)+t)^{-1}(\pi(v)+t)^{-1}\big)\\ &= \Big(\frac{v\pi(v)-tv}{q(\pi(v))+t^2}, \frac{-t}{q(\pi(v))+t^2}\Big).\end{aligned}$$

# Chapter 3

# The structurable world

Structurable algebras have been introduced by Bruce Allison in 1978. The main motivation was to construct exceptional Lie algebras, by generalizing the Tits–Kantor–Koecher construction, which constructs Lie algebras from Jordan algebras.

Unfortunately, there is no good single source to get familiar with the theory of structurable algebras, but instead, this material is spread out over more than 10 articles written by Bruce Allison, John Faulkner and some coauthors. In this chapter we tried to write an introduction to the theory of structurable algebras, in such a way that this should give the reader enough feeling with structurable algebras to find his way through this thesis. Of course, we gave more attention to topics we will need, so this should not be seen as a complete introduction to structurable algebras. We do not include any proofs, but we tried to give precise references to where the proofs can be found.

In Sections 3.1 and 3.2 we discuss some basic notions and notations. In Section 3.3 we discuss the classification of structurable algebras and give the main examples of structurable algebras. The structurable algebras discussed in Section 3.3.4 will play a leading role in Chapter 4; the ones in Section 3.3.5 play a leading role in Chapter 5.

In Section 3.4 we discuss the Tits–Kantor–Koecher construction of Lie algebras. In Section 3.5 we discuss the notion of an isotopy between structurable algebras, this turns out to be the right notion of 'equivalent' structurable algebras. In Section 3.6 we go a bit further back in history and discuss some aspects of Freudenthal triple systems, and their relation with structurable algebras. In Section 3.7 we give the definition of $J$-ternary algebras. They are an early attempt of Bruce Allison to define a model to describe all isotropic Lie algebras.

## 3.1 Definitions and basic properties of structurable algebras

Structurable algebras are algebras equipped with an involution and triple product. This triple product is in some sense of more importance than the multiplication of the algebra. Structurable algebras can only be defined over fields of characteristic different from 2 and 3.

**Definition 3.1.** A *structurable algebra* over a field $k$ of characteristic not 2 or 3 is a finite-dimensional, unital $k$-algebra with involution[1] $(\mathcal{A}, ^-)$ such that

$$[V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}} \tag{3.1}$$

for $x, y, z, w \in \mathcal{A}$ where $V_{x,y}z := \{x, y, z\} := (x\overline{y})z + (z\overline{y})x - (z\overline{x})y$.

For all $x, y, z \in \mathcal{A}$, we write $U_{x,y}z := V_{x,z}y$ and $U_x y := U_{x,x}y$. We will refer to the maps $V_{x,y} \in \mathrm{End}_k(\mathcal{A})$ as *$V$-operators*, and to the maps $U_{x,y} \in \mathrm{End}_k(\mathcal{A})$ and $U_x \in \mathrm{End}_k(\mathcal{A})$ as *$U$-operators*.

Structurable algebras are not necessarily associative nor commutative; they are generalizations of both associative algebras with involution and Jordan algebras. In particular structurable algebras with identity involution are exactly the Jordan algebras, see Section 3.3.2.

In [All79] and [All78], a structurable algebra is defined as an algebra with involution which satisfies

$$[T_z, V_{x,y}] = V_{T_z x,y} - V_{x,T_{\overline{z}}y} \tag{3.2}$$

for all $x, y, z \in \mathcal{A}$ with $T_x := V_{x,1}$. The equivalence of (3.1) and (3.2) follows from [All79, Corollary 5.(v)].

As usual, the commutator and the associator are defined as

$$[x, y] = xy - yx, \quad [x, y, z] = (xy)z - x(yz),$$

for all $x, y, z \in \mathcal{A}$. For each $x \in \mathcal{A}$, we define the operator $L_x : \mathcal{A} \to \mathcal{A}$ by

$$L_x(y) := xy.$$

Let $(\mathcal{A}, ^-)$ be a structurable algebra; then $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$ for

$$\mathcal{H} = \{h \in \mathcal{A} \mid \overline{h} = h\} \quad \text{and} \quad \mathcal{S} = \{s \in \mathcal{A} \mid \overline{s} = -s\}.$$

The dimension of $\mathcal{S}$ is called the *skew-dimension* of $\mathcal{A}$; we sometimes call elements of $\mathcal{S}$ *skew-hermitian elements* or briefly *skew-elements*. Skew-elements tend to behave 'nicer' than arbitrary elements of the structurable algebra.

[1] An involution is a $k$-linear map of order 2 such that $\overline{xy} = \overline{y}\,\overline{x}$.

A structurable algebra $(\mathcal{A}, ^{-})$ is skew-alternative, i.e.

$$[s,x,y] = -[x,s,y] = [x,y,s], \quad \forall s \in \mathcal{S},\ \forall x,y \in \mathcal{A}, \tag{3.3}$$

see [All78, Prop. 1]. This implies that for all $s,t \in \mathcal{S}$ and $x \in \mathcal{A}$

$$[s,s,x] = [x,s,s] = [s,x,s] = 0, \tag{3.4}$$

$$s[t,s,x] = -[s,ts,x], \quad [x,s,t]s = -[x,st,s]. \tag{3.5}$$

The identities (3.5) are weak versions of two of the Moufang identities (see Lemma 1.8(v)). The following map is of crucial importance in the study of structurable algebras:

$$\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{S} : (x,y) \mapsto x\overline{y} - y\overline{x}.$$

We list some identities. For all $x,y \in \mathcal{A}$ and $s \in \mathcal{S}$, we have

$$U_{x,y} - U_{y,x} = L_{\psi(x,y)}, \tag{3.6}$$

$$V_{x,sy} - V_{y,sx} = -L_{\psi(x,y)}L_s, \tag{3.7}$$

$$s\psi(x,y)s = -\psi(sx,sy), \tag{3.8}$$

$$L_sU_{x,y}L_s = -U_{sx,sy}. \tag{3.9}$$

Identity (3.6) follows from the definition of the $U$-operator; (3.7) is proven in [All79, Lemma 2]; (3.8) is [AH81, Lemma 11.2], writing this out yields $s(x\overline{y})s - s(y\overline{x})s = (sx)(\overline{y}s) - (sy)(\overline{x}s)$, which is a weak form of the first classical Moufang identity; (3.9) is [AH81, Prop. 11.3].

A structurable algebra $(\mathcal{A}, ^{-})$ is *simple* if its only ideals[2] are $\{0\}$ and $\mathcal{A}$; $(\mathcal{A}, ^{-})$ is *central* if its center

$$\begin{aligned} Z(\mathcal{A}, ^{-}) &= Z(\mathcal{A}) \cap \mathcal{H} \\ &= \{c \in \mathcal{A} \mid [c,\mathcal{A}] = [c,\mathcal{A},\mathcal{A}] = [\mathcal{A},c,\mathcal{A}] = [\mathcal{A},\mathcal{A},c] = 0\} \cap \mathcal{H} \end{aligned}$$

is equal to $k1$.

We quote some interesting results that give a structure theory for structurable algebras. The radical of $\mathcal{A}$ is the largest solvable[3] ideal of $\mathcal{A}$. A structurable algebra is semisimple if its radical is zero.

If $\mathrm{char}(k) \neq 2,3,5$, a semisimple structurable algebra is the direct sum of simple structurable algebras (see [Smi92, Section 2] for the general case; [Sch85, Section 2] for the characteristic zero case). If $\mathrm{char}(k) = 0$, [Sch85, Theorem 10] states that if $\mathcal{A}$ is a structurable algebra with radical $R$, there exists a semisimple structurable subalgebra $\mathcal{B} \leq \mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus R$.

[2] An ideal of $\mathcal{A}$ is a two-sided ideal stabilized by $^{-}$.

[3] An ideal $I$ is solvable if there exists a $k \in \mathrm{N}$ such that $I^{2^k} = 1$.

## 3.2 Conjugate invertibility in structurable algebras

For structurable algebras there is a notion of invertibility that generalizes the invertibility in Jordan algebras.

**Definition 3.2.** Let $(\mathcal{A}, ^-)$ be a structurable algebra. An element $u \in \mathcal{A}$ is said to be *conjugate invertible* if there exists an element $\hat{u} \in \mathcal{A}$ such that

$$V_{u,\hat{u}} = \mathrm{id}, \text{ or equivalently } V_{\hat{u},u} = \mathrm{id}. \tag{3.10}$$

If $u$ is conjugate invertible, then the element $\hat{u}$ is uniquely determined, and is called the *conjugate inverse* of $u$. For long expressions we will denote

$$u^{\wedge} := \hat{u} = \widehat{u}.$$

If $u$ is conjugate invertible, the operator $U_u$ is invertible and we have

$$\hat{u} = U_u^{-1} u. \tag{3.11}$$

All the above facts are proven in [AH81, Sec. 6].

If each element in $\mathcal{A} \setminus \{0\}$ is conjugate invertible, $\mathcal{A}$ is called a *conjugate division algebra* or a *structurable division algebra*.

If $\mathcal{A}$ is an associative algebra with involution and $u \in \mathcal{A}$ is invertible, then $\hat{u} = \overline{u}^{-1}$; this motivates the term "conjugate inverse".

For skew-elements, a more elegant criterion for conjugate invertibility exists: an element $s \in \mathcal{S}$ is conjugate invertible if and only if $L_s$ is invertible. If $s \in \mathcal{S}$ is conjugate invertible, we have

$$\hat{s} = -L_s^{-1} 1 \in \mathcal{S}, \tag{3.12}$$

$$L_{\hat{s}} L_s = L_s L_{\hat{s}} = -\mathrm{id}. \tag{3.13}$$

All those facts are proven in [AH81, Prop 11.1]. Note that $s\hat{s} = \hat{s}s = -1$.

The following formula (see [All86a, Prop 2.6]) allows to determine the conjugate inverse of any invertible element, if one can determine the conjugate inverse of any invertible skew-element. Let $u \in \mathcal{A}$ and $s \in \mathcal{S}$ be both conjugate invertible; then $\psi(u, U_u(su))$ is conjugate invertible and

$$\widehat{u} = 2\big(\psi(u, U_u(su))\big)^{\wedge} U_u(su). \tag{3.14}$$

An interesting article that provides a criterion for invertibility of elements is [AF92]. The authors define the *conjugate norm* of a structurable algebra as the exact denominator of the conjugate inversion map. If $\mathcal{A}$ is either a Jordan algebra or an alternative algebra with involution, the conjugate norm is given by the generic norm of $\mathcal{A}$. An element $x \in \mathcal{A}$ is conjugate invertible if and only if $N(x) \neq 0$, where $N$ denotes the conjugate norm of $\mathcal{A}$. The *conjugate degree* of a structurable algebra is defined as the degree of the conjugate norm.

In the following section we will give some criteria for invertibility of elements in some specific examples of structurable algebras.

## 3.3 Examples of structurable algebras

Central simple structurable algebras over fields of characteristic different from 2, 3 and 5 are classified; they consist of six (non-disjoint) classes:

(1) central simple associative algebras with involution,
(2) central simple Jordan algebras,
(3) structurable algebras constructed from a non-degenerate hermitian form over a central simple associative algebra with involution,
(4) simple structurable algebras of skew-dimension 1,
(5) forms of the tensor product of two composition algebras,
(6) an exceptional 35-dimensional case, which can be constructed from an octonion algebra.

When the characteristic is zero the classification was carried out by Allison in [All78], but in this paper class (6) was overlooked. The classification was completed and generalized to fields of characteristic different from $2, 3, 5$ by Smirnov in [Smi92].

Below we describe classes (1)-(5) and give some specific properties of each class. Since we do not need it, we do not describe class (6) but refer to [Smi90] and [AF93].

Although we defined structurable algebras to be finite dimensional algebras, the examples (1), (2), (3) can of course also be defined in the infinite dimensional case.

### 3.3.1 Associative algebras with involution

Let $(\mathcal{A}, \overline{\phantom{x}})$ be an associative algebra with involution. It is proven in [All78, Example 8.i] or in [Sch85, p. 411] that $\mathcal{A}$ is a structurable algebra. An element of $\mathcal{A}$ is conjugate invertible if and only if it is invertible in the usual associative sense[4] (see [All86b, p. 142]). If $u \in \mathcal{A}$ is invertible, it is clear that $V_{u,\overline{u}^{-1}} = \mathrm{id}$, therefore $\hat{u} = \overline{u}^{-1}$.

[4] An element $u$ in an associative algebra is invertible if there exists an $u^{-1} \in \mathcal{A}$ such that $uu^{-1} = u^{-1}u = 1$.

### 3.3.2 Jordan algebras

Let $\mathcal{A}$ be a Jordan algebra and let $^{-} = \mathrm{id}$. In this case the $V$-operator of structurable algebras is equal to the $V$-operator of Jordan algebras defined as $V_{x,y}z := U_{x,z}y = (L_{xy} + [L_x, L_y])z$ (see Definition 1.9); the defining identity of structurable algebras is a known identity in the theory of Jordan algebras, see for example formula (FFV)' of [McC04, p. 202].

In particular, each structurable algebras with trivial involution (equivalently with skew-dimension zero) is a Jordan algebra; this is verified in [All78, p.135, Remark (ii)]).

Let $u \in \mathcal{A}$ be invertible; then $V_{u,u^{-1}} = L_{uu^{-1}} + [L_u, L_{u^{-1}}] = \mathrm{id}$. On the other hand, if $u$ is conjugate invertible, then $U_u$ is invertible and hence $u$ is invertible. Therefore an element is conjugate invertible if and only if it is invertible in the usual Jordan sense and the conjugate inverse is equal to the Jordan inverse.

### 3.3.3 Hermitian structurable algebras

This class of structurable algebras is constructed from a hermitian space.

**Definition 3.3.** Let $(E, ^{-})$ be a unital associative algebra with involution $^{-}$ over a field $k$. Let $W$ be a unital left $E$-module, and let $h : W \times W \to E$ be a hermitian form (see Definition 1.4.(ii)).

It is proven in [All78, Sec. 8.(iii)] that $E \oplus W$ is a structurable algebra with the following involution and multiplication:

$$\overline{e + w} = \overline{e} + w,$$
$$(e_1 + w_1)(e_2 + w_2) = (e_1e_2 + h(w_2, w_1)) + (e_2w_1 + \overline{e_1}w_2),$$

for all $e, e_1, e_2 \in E$ and all $w, w_1, w_2 \in W$.

It is clear that $\mathcal{S} = \{e \in E \mid \overline{e} = -e\}$. After some calculations we find that

$$\psi(e_1 + w_1, e_2 + w_2) = (e_1\overline{e_2} - e_2\overline{e_1}) - (h(w_1, w_2) - \overline{h(w_1, w_2)}), \tag{3.15}$$

$$\begin{aligned} V_{e_1+w_1,e_2+w_2}(e_3 + w_3) = \\ &(e_1\overline{e_2} + h(w_1, w_2))e_3 + (e_1\overline{e_2} + h(w_1, w_2))w_3 \\ &+ (e_3\overline{e_2} + h(w_3, w_2))e_1 + (e_3\overline{e_2} + h(w_3, w_2))w_1 \\ &+ (-e_3\overline{e_1} + h(w_3, w_1))e_2 + (e_3\overline{e_1} - h(w_3, w_1))w_2, \end{aligned} \tag{3.16}$$

for all $e_1, e_2, e_3 \in E$, $w_1, w_2, w_3 \in W$.

[All86b, Prop. 4.1] states that the structurable algebra $E \oplus W$ is central simple if and only if $E$ is central simple, $h$ is non-degenerate and $\dim(E)\dim(W) \neq 1$. Furthermore it states that $e + w \in E \oplus W$ is conjugate invertible if and only if $e\overline{e} - h(w,w)$ is invertible in $E$; if $e + w$ is conjugate invertible the conjugate inverse is given by

$$\widehat{e+w} = (e\overline{e} - h(w,w))^{-1}(e - w). \tag{3.17}$$

Therefore $\mathcal{A} = E \oplus W$ is a conjugate division algebra if and only if $E$ is a division algebra and if for all $0 \neq w \in W$ we have $h(w,w) \neq 0, 1$. If $s \in \mathcal{S}$ is conjugate invertible, then $\hat{s} = -s^{-1}$.

Notice that the decomposition of $\mathcal{A}$ in the disjoint sum of $E$ and $W$ is not uniquely defined. We want to be able to define a structurable algebra on the hermitian space $W$, and not only on the bigger space $E \oplus W$.

**Remark 3.4.** (i) On the structurable algebra $\mathcal{A} = E \oplus W$ we can define various hermitian spaces. Let $\mathcal{A}$ be an $E$-module by defining

$$e_1 \cdot (e_2 + w) = e_1e_2 + e_1w$$

for all $e_1, e_2 \in E$, $w \in W$. Then for all choices of $t_1, t_2 \in k$, the map

$$H : \mathcal{A} \times \mathcal{A} \to E : (e_1 + w_1, e_2 + w_2) \mapsto t_1e_1\overline{e_2} + t_2h(w_1, w_2)$$

is a hermitian form on $\mathcal{A}$.

(ii) Assume now that $E$ is a skew field with involution and that $X$ is a left $E$-module equipped with a hermitian form $H : X \times X \to E$ such that $H(x,x) \neq 0$ for all $0 \neq x \in X$. We will define a structurable algebra on $X$. Fix $0 \neq \xi \in X$. We embed $E$ into $X$ by identifying it with $E\xi$, and we define the orthogonal complement in $X$ of $E\xi$ by

$$(E\xi)^\perp = \{x \in X \mid h(\xi, x) = 0\};$$

it follows that $X = (E\xi) \oplus (E\xi)^\perp$. We make $E\xi$ into a skew field by defining

$$(a\xi)(b\xi) = (ab)\xi, \quad \overline{a\xi} = \overline{a}\xi$$

for all $a, b \in E$. We define an action of $E\xi$ on $(E\xi)^\perp$ by $(a\xi)x = ax$ for all $a \in E$ and $x \in (E\xi)^\perp$. We now restrict our hermitian form $H$ to $(E\xi)^\perp$ by defining

$$h : (E\xi)^\perp \times (E\xi)^\perp \to E\xi : (x,y) \mapsto h(x,y)\xi.$$

Using the hermitian form $h$, the hermitian space $X = (E\xi) \oplus (E\xi)^\perp$ becomes a structurable algebra with the involution and multiplication given by

$$\overline{e\xi + x} = \overline{e}\xi + x,$$

$$(e_1\xi + x_1)(e_2\xi + x_2) = (e_1e_2 + h(x_2, x_1))\xi + (e_2x_1 + \overline{e_1}x_2).$$

This construction depends on the choice of $\xi$. We believe that different choices for $\xi$ give rise to isotopic structurable algebras, but we were not (yet) able to prove this.

(iii) The constructions given in (i) and in (ii) are inverses of each other, in the following sense. When we start with $\mathcal{A}$ equipped with the hermitian form $H$ with $t_1 = t_2 = 1$ from (i), and apply (ii) with $\xi = 1 \in E$, we obtain exactly the structurable algebra $\mathcal{A}$. Conversely, when we start with a hermitian space $X$ equipped with a hermitian form $H$ as in (ii), we see that $H(e_1\xi + x_1, e_2\xi + x_2)\xi = e_1H(\xi, \xi)\overline{e_2}\xi + h(x_1, x_2)$ is a hermitian form given in (i) if $h(\xi, \xi) \in k$.

### 3.3.4 Structurable algebras of skew-dimension one

Structurable algebras of skew-dimension one are, among the structurable algebras, closest to Jordan algebras. In [All90] and [AF84], this type of structurable algebras has been studied. We give some useful results obtained in those articles. Structurable algebras of skew-dimension one have an interesting connection with Freudenthal triple systems; see Section 3.6.

In this section $\mathcal{A}$ is a structurable algebra of skew-dimension one. We fix a non-zero element $s_0 \in \mathcal{S}$, so $\mathcal{S} = ks_0$. By [AF84, Lemma 2.1], $s_0^2 = \mu 1$ for some $\mu \in k^*$, therefore $s_0(s_0x) = (xs_0)s_0 = \mu x$; it is also shown that a simple structurable algebra of skew-dimension one is always central.

If $\mathcal{A}$ is simple, then the bilinear map $\psi$ is non-degenerate (see [AF84, Lemma 2.2]). Define the quartic form $\nu : \mathcal{A} \to k$ given by

$$\nu(x) = \frac{1}{6\mu}\psi(x, U_x(s_0x))s_0, \tag{3.18}$$

by identifying $k1$ with $k$. It is easy to see that $\nu(1) = 1$ and that $\nu$ is independent of the choice of $s_0$. This map plays the role of the norm of $\mathcal{A}$.

By [AF84, Prop. 2.11], an element $x \in \mathcal{A}$ is conjugate invertible if and only if $\nu(x) \neq 0$. When this is the case, we have

$$\hat{x} = -\frac{1}{3\mu\nu(x)}s_0U_x(s_0x). \tag{3.19}$$

We will now discuss an important class of structurable algebras of skew-dimension one, which we will call *structurable matrix algebras*. The multiplication of these algebras is similar to the multiplication of split octonions when represented by Zorn matrices.

**Definition 3.5.** Let $J$ be a Jordan algebra over a field $k$, let $T : J \times J \to k$ be a symmetric bilinear form, let $\times : J \times J \to J$ be a symmetric bilinear map, and let $N : J \to k$ be a cubic form such that one of the following holds:

(i) $J$ is a cubic Jordan algebra with a non-degenerate admissible form $N$ with basepoint 1 and trace form $T$ and $\times$ as in Definition 1.10(iv).
(ii) $J$ is a Jordan algebra of a non-degenerate quadratic form $q$ with basepoint 1, and $T$ is the linearization of $q$. In this case, $N$ and $\times$ are the zero maps.
(iii) $J = 0$, $N = 0$, $T = 0$ and $\times = 0$. In this case, $J$ is not unital.

We define the *structurable matrix algebras* as follows. Fix a constant $\eta \in k$, and define

$$\mathcal{A} = \left\{ \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \;\middle|\; k_1, k_2 \in k,\, j_1, j_2 \in J \right\}.$$

Define the involution and multiplication as follows:

$$\overline{\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}} = \begin{pmatrix} k_2 & j_1 \\ j_2 & k_1 \end{pmatrix},$$

$$\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \begin{pmatrix} k_1' & j_1' \\ j_2' & k_2' \end{pmatrix} = \begin{pmatrix} k_1 k_1' + \eta T(j_1, j_2') & k_1 j_1' + k_2' j_1 + \eta (j_2 \times j_2') \\ k_1' j_2 + k_2 j_2' + j_1 \times j_1' & k_2 k_2' + \eta T(j_2, j_1') \end{pmatrix},$$

for all $k_1, k_2, k_1', k_2' \in k$, $j_1, j_2, j_1', j_2' \in J$. We denote this structurable matrix algebra by $M(J, \eta)$. It is proven in [All78, Sec 8.v] and [AF84, Sec 4] that $M(J, \eta)$ is a simple structurable algebra.

The following proposition explains the importance of structurable matrix algebras.

**Proposition 3.6** ([AF84, Prop. 4.5])**.** *Let $\mathcal{A}$ be a structurable algebra of skew-dimension one with $s_0^2 = \mu 1$. Then $\mathcal{A}$ is isomorphic to a structurable matrix algebra $M(J, \eta)$ if and only if $\mu$ is a square in $k$.*

It follows that all structurable algebras of skew-dimension one are forms of structurable matrix algebras, which can always be split by a field extension of degree at most 2.

The algebras in classes (1) and (3) on page 47 have skew-dimension 1 if and only if the associative algebra with involution has skew-dimension 1.

It follows by combining Proposition 4.4 and Theorem 4.11 of [All90] that $\mathcal{A}$ is a form of a structurable matrix algebra with $N = 0$ if and only if $\mathcal{A}$ is isomorphic to a structurable algebra of a non-degenerate hermitian form over a 2-dimensional composition algebra; in this case the conjugate degree of the algebra is equal to 2.

In all the other cases the conjugate degree of $\mathcal{A}$ is 4 and the quartic form $\nu$ defined in (3.18) is the conjugate norm of $\mathcal{A}$.

It is an open problem to determine explicitly all structurable algebras of skew-dimension one. Examples of structurable algebras of skew-dimension one that are not isomorphic to structurable matrix algebras can be obtained by applying a Cayley–Dickson process to a certain class of Jordan algebras.

In [AF84, p. 200] it is shown how this procedure can be seen as a generalization of the classical Cayley–Dickson process to construct composition algebras.

**The Cayley–Dickson process for structurable algebras** We will now briefly explain the Cayley–Dickson process, and refer to [AF84] for more details. In order to obtain a structurable algebra, one needs to start from a Jordan algebra equipped with a Jordan norm of degree 4.

**Definition 3.7.** Let $J$ be a Jordan algebra over $k$. A form $Q\colon J \to k$ of degree 4 is a *Jordan norm of degree 4* if:

(i) $1 \in J$ is a basepoint of $Q$, i.e. $Q(1) = 1$;
(ii) $Q(U_j j') = Q(j)^2 Q(j')$ for all $j, j' \in J \otimes_k K$ for all field extensions $K/k$;
(iii) The trace form

$$T\colon J \times J \to k\colon (j, j') \mapsto Q(1; j)Q(1; j') - Q(1; j, j')$$

is a $k$-bilinear non-degenerate form.

The main examples of Jordan algebras with a Jordan norm of degree 4 are separable Jordan algebras of degree 4 with their generic norm and separable Jordan algebras of degree 2 with the square of their generic norm. In [All90, Prop. 5.1] Jordan norms of degree 4 are classified.

If $J$ is a separable Jordan algebra of degree 4, it is shown in [AF84, Theorem 5.4] that one can give the space $J_0 = \{x \in J \mid T(x, 1) = 0\}$ the structure of a separable Jordan algebra of degree 3 and thus of a cubic Jordan algebra.

**Definition 3.8.** Let $J$ be a Jordan algebra with $Q$ a Jordan norm of degree 4 with trace $T$. Consider the $k$-linear bijection $\theta$ on $J$ given by

$$b^\theta = -b + \tfrac{1}{2}T(b, 1)1,$$

for all $b \in J$; observe that $\theta^2 = 1$.

Let $\mu \in k^*$, and define the algebra $\mathrm{CD}(J, Q, \mu) := J \oplus s_0 J$, with multiplication and involution given by

$$\begin{aligned}(j_1 + s_0 j_1')(j_2 + s_0 j_2') &= j_1 j_2 + \mu\big(j_1' j_2'^\theta\big)^\theta + s_0\big(j_1^\theta j_2' + (j_1'^\theta j_2^\theta)^\theta\big),\\ \overline{j + s_0 j'} &= j - s_0 j'^\theta,\end{aligned}$$

for all $j_1, j_1', j_2, j_2', j, j' \in J$. By [AF84, Theorem 6.6], this is a simple structurable algebra with skew-dimension one and $\mathcal{S} = k s_0$.

Since $(ts_0)^2 = t^2\mu$, it follows from Proposition 3.6 that the structurable algebra $CD(J, Q, \mu)$ is isomorphic to a structurable matrix algebra if and only if $\mu$ is a square in $k$.

This procedure can be used to construct central simple structurable division algebras of skew-dimension one.

**Lemma 3.9** ([AF84, Theorem 7.1]). *Let $J$ be a Jordan division $k$-algebra with Jordan norm of degree 4. Define the field $E = k(\xi)$ with $\xi$ transcendental over $k$, let $J' = J \otimes_k E$ and let $Q'$ be the extension of $Q$ to $J'$. Then* $\mathrm{CD}(J', Q', \xi)$ *is a central simple structurable division algebra over $k$.*

### 3.3.5 Forms of the tensor product of two composition algebras

For the first part of this section we allow the characteristic of $k$ to be equal to 3, but we still demand that it is different from 2.

Let $C_1$ and $C_2$ be two composition algebras over $k$ (possibly of different dimension) with involution $\sigma_1$ and $\sigma_2$, respectively. We denote $m_1 := \dim_k(C_1)$ and $m_2 := \dim_k(C_2)$. If $\mathcal{A} = C_1 \otimes_k C_2$, we say that $\mathcal{A}$ is an *$(m_1, m_2)$-product algebra.* We let $\mathcal{A}$ be equipped with the involution

$$\bar{\ } = \sigma := \sigma_1 \otimes \sigma_2.$$

If $\mathrm{char}(k) \neq 2, 3$, it is proven in [All78, Sec 8.(iv)] that $(\mathcal{A}, \bar{\ })$ is a structurable algebra.

**Definition 3.10.** A $k$-algebra $\mathcal{A}$ is a form of the tensor product of two composition algebras, if there exists a field extension $E/k$ and two composition algebras $C_1, C_2$ over $E$ such that $A \otimes_k E \cong C_1 \otimes_E C_2$.

The article [All88] gives a lot of information on this type of algebras. This article assumes that the base field has characteristic zero, but several results remain valid in general characteristic different from 2.

**Lemma 3.11** ([All88, Th. 2.1] and [AF92, Prop. 7.9]). *If $\mathcal{A}$ is a form of the tensor product of two composition algebras, then either $\mathcal{A}$ is isomorphic to the tensor product of two composition algebras defined over $k$,* or *in the case that* $\dim_E(C_1) = \dim_E(C_2) > 1$ *we can choose $E$ such that it has degree 2 over $k$.*

If $\mathcal{A}$ is a form of the tensor product of two composition algebras of the same dimension $m$ such that $\mathcal{A}$ is itself not isomorphic to the tensor product of two composition algebras, then $\mathcal{A}$ can be constructed from one composition algebra $C$ of dimension $m$ over a quadratic field extension $E/k$

using the corestriction functor; see [All88, p. 671] for the precise construction and the proof that this algebra is structurable in characteristic different from 2 and 3. In this case, $\mathcal{A}$ is called a *twisted $(m,m)$-product algebra.*

By [All88, Prop. 2.2], the $(m_1,m_2)$-product algebras and the twisted $(m,m)$-product algebras are central simple, except for $(m_1,m_2)=(2,2)$ and $(m,m)=(2,2)$. Each (twisted) $(m_1,m_2)$-product algebra for $m_1,m_2\leq 4$ is associative and each $(8,1)$- or $(8,2)$-product algebra is alternative (see p. 6).

In this thesis we will only make use of $(m_1,m_2)$-product algebras and not of twisted $(m,m)$-product algebras. Therefore we only give the following definitions and properties in the case where $\mathcal{A}=C_1\otimes_k C_2$; these definitions and properties can be analogously defined for twisted $(m,m)$-product algebras (see [All88, Sec. 3]). We will need these properties in fields of characteristic different from 2, but sometimes equal to 3. In this case $\mathcal{A}$ is strictly speaking not a structurable algebra.

Let $C_1$ and $C_2$ be two composition algebras with involution $\sigma_1$ and $\sigma_2$ and norm form $q_1$ and $q_2$, respectively. Let $S_i$ be the set of skew-elements in $C_i$, i.e.

$$S_i=\{x\in C_i\mid x^{\sigma_i}=-x\}.$$

Let $C_1\otimes_k C_2$ be a $(m_1,m_2)$-product algebra equipped with the involution $^- = \sigma := \sigma_1\otimes\sigma_2$. It follows that the set of skew-elements in $C_1\otimes_k C_2$ is equal to

$$\mathcal{S}=\{x\in C_1\otimes_k C_2\mid \overline{x}:=x^\sigma=-x\}=(S_1\otimes 1)\oplus(1\otimes S_2);$$

observe that $\dim_k\mathcal{S}=\dim_k C_1+\dim_k C_2-2$.

**Definition 3.12.** (i) We will associate a quadratic form $q_A$ to $C_1\otimes_k C_2$, called the *Albert form*, by setting

$$q_A\colon\mathcal{S}\to k\colon (s_1\otimes 1)+(1\otimes s_2)\mapsto q_1(s_1)-q_2(s_2)$$

for all $s_1\in S_1$ and $s_2\in S_2$. We have $q_A\perp\mathbb{H}=q_1\perp(-1)q_2$ and when we denote $q_i'=q_i|_{S_i}$ for the pure part of the Pfister form $q_i$, we have $q_A=q_1'\perp\langle -1\rangle q_2'$.

(ii) For each $s=s_1\otimes 1+1\otimes s_2\in S$, we define

$$(s_1\otimes 1+1\otimes s_2)^\natural=s_1\otimes 1-1\otimes s_2.$$

(iii) Let $s\in\mathcal{S}$ such that $q_A(s)\neq 0$. Then we say that $s$ is *invertible* and define the *inverse* of $s$ by

$$s^{-1}:=-\frac{1}{q_A(s)}s^\natural.$$

The Albert form was introduced by Abraham A. Albert in the case where $C_1$ and $C_2$ are both quaternion algebras.

Tensor products of two composition algebras are far from being associative or alternative, but (as usual) the skew-elements behave more nicely than arbitrary elements:

**Lemma 3.13.** *For all $x \in C_1 \otimes_k C_2$, $s_1, s_2, s \in \mathcal{S}$, we have*

(i) $s_1(s_2 s_1) = (s_1 s_2)s_1$;
(ii) $(s_1 s_2 s_1)x = s_1(s_2(s_1 x))$;
(iii) *If $s$ is invertible then $s(s^{-1}x) = s^{-1}(sx) = x$.*

*Proof.* These identities can be easily verified using Lemma 1.8. □

In the case that both $C_1$ and $C_2$ are quaternion algebras, $C_1 \otimes_k C_2$ is associative; Albert proved that $C_1 \otimes_k C_2$ is a division algebra if and only if its Albert form is anisotropic (see [Lam05, Theorem III.4.8].) It is not obvious to generalize this result to arbitrary composition algebras. The proof of the following theorem makes use of the Lie algebra associated to the structurable algebra constructed in section 3.4.

**Theorem 3.14** ([All86b, Theorem 5.1] ). *Let* $\operatorname{char}(k) = 0$ *and $\mathcal{A}$ a (twisted) $(8, m_2)$-product algebra. Then $\mathcal{A}$ is a conjugate division algebra if and only if the Albert form on $\mathcal{S}$ is anisotropic.*

It seems plausible that this result can be generalized to arbitrary fields of characteristic different from 2 and 3. However the method followed in the proof of the above theorem can not be generalized.

Note that it follows from Lemma 3.13.(iii) and (3.12) that $s \in \mathcal{S}$ is invertible (equivalent with $q_A(s) \neq 0$) if and only if $s$ is conjugate invertible, in which case $\hat{s} = -s^{-1}$. Therefore it is clear that if $C_1 \otimes_k C_2$ is a conjugate division algebra, $q_A$ has to be anisotropic and $C_1$ and $C_2$ are (conjugate) division algebras.

The following theorem is useful in the case that $\mathcal{A}$ is an $(8,1)$- or $(8,2)$-algebra, since then $\mathcal{A}$ is alternative.

**Theorem 3.15** ([All86a, Cor. 3.7]). *Let $\mathcal{A}$ be an alternative algebra with involution over a field of characteristic different from 2 and 3. Then $\mathcal{A}$ is a conjugate division algebra if and only if $\mathcal{A}$ is a division algebra, i.e. for all $0 \neq x \in \mathcal{A}$ there exists an $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = 1$.*

If $C_1 \otimes_k C_2$ is a conjugate division algebra, then by Definition 3.12(iii) we do have an nice expression to determine the conjugate inverse of skew-elements. Formula (3.14) can be used to give an expression for the conjugate inverse of an arbitrary element of $C_1 \otimes_k C_2$.

**Lemma 3.16.** *Let* $\mathrm{char}(k) \neq 2, 3$. *Let* $\mathcal{A} = C_1 \otimes_k C_2$ *with* $C_1$ *an octonion* $k$*-algebra. Suppose that* $q_A$ *is anisotropic.*

- *If* $\dim_k(C_2) = 2, 4$, *then* $\mathcal{A}$ *is a conjugate division algebra.*
- *If* $\dim_k(C_2) = 8$ *and if for all* $0 \neq x \in \mathcal{A}$ *there exists an* $s \in \mathcal{S}$ *such that* $\psi(x, U_x(sx)) \neq 0$ *then* $\mathcal{A}$ *is a conjugate division algebra.*

*Proof.* Since $q_1 \perp -q_2 = q_A \perp \mathbb{H}$, it follows from the fact that $q_A$ is anisotropic that $q_1$ and $q_2$ are both anisotropic, therefore $C_1$ and $C_2$ are division algebras. Furthermore all skew-elements different from 0 are conjugate invertible.

- If $\dim_k(C_2) = 2$, $C_1 \otimes_k C_2$ is alternative. By [All86a, Theorem 3.4], an element $x \in \mathcal{A}$ is conjugate invertible if and only if $x\overline{x}$ is invertible. Therefore we only have to check invertibility for elements in $\mathcal{H}$. Let $t1 \otimes 1 + s \otimes e \in \mathcal{H}$; then

  $$(t1 \otimes 1 + s \otimes e)(t1 \otimes 1 - s \otimes e) = (t^2 - N(s)N(e))1.$$

  Since $q_a$ is anisotropic, we have $N(s^{-1}) \neq N(e)$, thus $t^2 \neq N(e)N(s)$; hence we constructed an inverse for each element in $\mathcal{H}$.
- If $\dim_k(C_2) = 4$, [AF92, Theorem 8.7] implies that $x \in \mathcal{A}$ is conjugate invertible if there exists an $s_0 = 1 \otimes s_2 \in \mathcal{S} \setminus \{0\}$ such that $q_A(\psi(x, xs_0)) \neq 0$. Since $q_A$ is anisotropic, $x \in \mathcal{A}$ is conjugate invertible if there exists an $s_0 = 1 \otimes s_2 \in \mathcal{S} \setminus \{0\}$ such that $\psi(x, xs_0) \neq 0$.
  Below we indicate how one can show that for all $x \in \mathcal{A} \setminus \{0\}$ there exists an $s_0 = 1 \otimes s_2 \in \mathcal{S} \setminus \{0\}$ such that $\psi(x, xs_0) \neq 0$. It follows that $\mathcal{A}$ is a conjugate division algebra.
  We take $x \in \mathcal{A}$ arbitrary and assume that $\psi(x, xs_0) = 0$ for all $s_0 = 1 \otimes s_2 \in \mathcal{S}$. Since $s_0 \in 1 \otimes C_2$ and $C_2$ is associative, $s_0$ associates with all elements in $C_1 \otimes_k C_2$. Therefore

  $$\psi(x, xs_0) = x(\overline{xs_0}) - (xs_0)\overline{x} = -2x(s_0\overline{x}).$$

  Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ be a basis of $C_2$; then each element in $C_1 \otimes_k C_2$ can be written as $x_1 \otimes 1 + x_2 \otimes \mathbf{i} + x_3 \otimes \mathbf{j} + x_4 \otimes \mathbf{ij}$ for some $x_1, \ldots, x_4 \in C_1$. By expanding the equality $x(s_0\overline{x}) = 0$ for $s_0 = 1 \otimes \mathbf{i}$ and for $s_0 = 1 \otimes \mathbf{j}$, one can show that $x = 0$ using the fact that $q_A$ is anisotropic.
- If $\dim_k(C_2) = 8$, [AF92, Theorem 9.6] implies that $x \in \mathcal{A}$ is conjugate invertible if there exists a conjugate invertible element $s \in \mathcal{S}$ such that $q_A(\psi(x, U_x(sx))) \neq 0$. ☐

**Remark 3.17.** (i) In [AF92], it is shown that the conjugate degree (i.e. the degree of the conjugate norm) of a (twisted) $(8, m)$-product algebra is given by 2, 4, 4 or 8 for $m = 1, 2, 4, 8$ respectively.

(ii) In an entirely different context the authors of [HT98] study Albert forms of the (twisted) tensor product of two octonion algebras. In [HT98, Theorem 2.1] it is mentioned that every anisotropic 14-dimensional quadratic form with trivial discriminant and Clifford invariant, is the Albert form of a twisted tensor product of two octonion algebras. Furthermore it is shown that there exist quadratic forms that are the Albert form of a twisted tensor product of two octonion algebras, but that are not similar to the Albert form of a non-twisted tensor product of two octonion algebras. These are certain quadratic forms over fields that do not satisfy condition D(14).
It follows from Example (5) on page 64 below that there exists a twisted tensor product of two octonion algebras that is not isotopic to the tensor product of two composition algebras.

## 3.4 Construction of Lie algebras from structurable algebras

In order to describe the construction of Lie algebras from a structurable algebra, we need to introduce some more concepts; we give a brief summary of [All79]. The ring $\mathrm{End}(\mathcal{A})$ denotes the $k$-linear maps from $\mathcal{A}$ to $\mathcal{A}$. For each $A \in \mathrm{End}(\mathcal{A})$, we define new $k$-linear maps

$$A^\epsilon = A - L_{A(1)+\overline{A(1)}},$$
$$A^\delta = A + R_{\overline{A(1)}}.$$

One can verify that

$$V^\epsilon_{x,y} = -V_{y,x}, \tag{3.20}$$
$$V^\delta_{x,y}(s) = -\psi(x, sy), \tag{3.21}$$

for all $x, y \in \mathcal{A}$ and $s \in \mathcal{S}$. Define the Lie subalgebra $\mathrm{Strl}(\mathcal{A},^-)$ of $\mathrm{End}(\mathcal{A})$ as[5]

$$\mathrm{Strl}(\mathcal{A},^-) = \{A \in \mathrm{End}(\mathcal{A}) \mid [A, V_{x,y}] = V_{Ax,y} + V_{x,A^\epsilon y}\}. \tag{3.22}$$

It follows from the definition of structurable algebras that $V_{x,y} \in \mathrm{Strl}(\mathcal{A},^-)$, so we can define the Lie subalgebra

$$\mathrm{Instrl}(\mathcal{A},^-) = \mathrm{Span}\{V_{x,y} \mid x, y \in \mathcal{A}\}.$$

The Lie subalgebra $\mathrm{Instrl}(\mathcal{A},^-)$ is even an ideal of $\mathrm{Strl}(\mathcal{A},^-)$. Notice that for all $s, t \in \mathcal{S}$, we have $L_sL_t \in \mathrm{Instrl}(\mathcal{A})$, since it follows from (3.3) that

$$L_sL_t = \tfrac{1}{2}(V_{st,1} - V_{s,t}) = \tfrac{1}{2}(V_{1,ts} - V_{s,t}). \tag{3.23}$$

[5] This definition follows from [All78, Cor. 5].

It follows from skew-alternativity and the definition of $\delta$ and $\epsilon$ that

$$(L_r L_t)^\epsilon = -L_t L_r, \tag{3.24}$$

$$(L_r L_t)^\delta(s) = s(tr) + r(ts) \tag{3.25}$$

for all $r, t, s \in \mathcal{S}$. For all $A \in \mathrm{Strl}(\mathcal{A}, ^-)$ we have a version of triality for endomorphisms:

$$A(sx) = A^\delta(s)x + sA^\epsilon(x) \quad \text{for all } x \in \mathcal{A}, s \in \mathcal{S}. \tag{3.26}$$

By [All79, Lemma 1], we have for all $A \in \mathrm{Strl}(\mathcal{A}, ^-)$ and $x, y \in \mathcal{A}$ that

$$A^\delta \psi(x, y) = \psi(Ax, y) + \psi(x, Ay). \tag{3.27}$$

For all $A \in \mathrm{Strl}(\mathcal{A}, ^-)$, the map $A \mapsto A^\epsilon$ is a Lie algebra automorphism of $\mathrm{Strl}(\mathcal{A}, ^-)$ of order 2; the map $A \mapsto A^\delta|_\mathcal{S}$ is a Lie algebra homomorphism of $\mathrm{Strl}(\mathcal{A}, ^-)$ into $\mathrm{End}_k(\mathcal{S})$.

It follows that $\mathcal{A} \oplus \mathcal{S}$ is a $\mathrm{Strl}(\mathcal{A}, ^-)$-module under the action $A(x, s) = (Ax, A^\delta s)$ for all $A \in \mathrm{Strl}(\mathcal{A}, ^-)$ and $(x, s) \in \mathcal{A} \oplus \mathcal{S}$.

**Definition 3.18.** Consider two copies $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ with corresponding isomorphisms $\mathcal{A} \to \mathcal{A}_+ : x \mapsto x_+$ and $\mathcal{A} \to \mathcal{A}_- : x \mapsto x_-$, and let $\mathcal{S}_+ \subset A_+$ and $\mathcal{S}_- \subset A_-$ be the corresponding subspaces of skew-elements. Define as a vector space

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+.$$

As in [All79, par. 3], we define a Lie algebra on $K(\mathcal{A})$ as the unique extension of the Lie algebra on $\mathrm{Instrl}(\mathcal{A})$:

- $[\mathrm{Instrl}, K(\mathcal{A})]$

$$[V_{a,b}, V_{a',b'}] = V_{\{a,b,a'\},b'} - V_{a',\{b,a,b'\}} \in \mathrm{Instrl}(\mathcal{A}),$$

$$[V_{a,b}, x_+] := (V_{a,b}x)_+ \in \mathcal{A}_+, \qquad [V_{a,b}, y_-] := (V^\epsilon_{a,b}y)_- \in \mathcal{A}_- = (-V_{b,a}y)_- \in \mathcal{A}_-,$$

$$[V_{a,b}, s_+] := (V^\delta_{a,b}s)_+ \in \mathcal{S}_+ = -\psi(a, sb)_+ \in \mathcal{S}_+, \qquad [V_{a,b}, t_-] := (V^{\epsilon\delta}_{a,b}t)_- \in \mathcal{S}_- = \psi(b, ta)_- \in \mathcal{S}_-,$$

- $[\mathcal{S}_\pm, \mathcal{A}_\pm]$

$$[s_+, x_+] := 0, \qquad [t_-, y_-] := 0,$$

$$[s_+, y_-] := (sy)_+ \in \mathcal{A}_+, \qquad [t_-, x_+] := (tx)_- \in \mathcal{A}_-,$$

- $[\mathcal{A}_\pm, \mathcal{A}_\pm]$

$$[x_+, y_-] := V_{x,y} \in \mathrm{Instrl}(\mathcal{A}),$$

$$[x_+, x'_+] := \psi(x, x')_+ \in \mathcal{S}_+, \qquad [y_-, y'_-] := \psi(y, y')_- \in \mathcal{S}_-$$

- $[\mathcal{S}_\pm, \mathcal{S}_\pm]$

$$[s_+, s'_+] := 0, \qquad [t_-, t'_-] := 0,$$
$$[s_+, t_-] := L_s L_t \in \mathrm{Instrl}(\mathcal{A}).$$

for all $x, x', y, y' \in \mathcal{A}$, $s, s', t, t' \in \mathcal{S}$, $V_{a,b}, V_{a',b'} \in \mathrm{Instrl}(\mathcal{A})$.

From the definition of the Lie bracket we clearly see that the Lie algebra $K(\mathcal{A})$ has a 5-grading given by $K(\mathcal{A})_j = 0$ for all $|j| > 2$ and

$$K(\mathcal{A})_{-2} = \mathcal{S}_-, \quad K(\mathcal{A})_{-1} = \mathcal{A}_-, \quad K(\mathcal{A})_0 = \mathrm{Instrl}(\mathcal{A}),$$
$$K(\mathcal{A})_1 = \mathcal{A}_+, \quad K(\mathcal{A})_2 = \mathcal{S}_+.$$

In the case where $\mathcal{A}$ is a Jordan algebra, we have $\mathcal{S} = 0$, and thus the Lie algebra $K(\mathcal{A})$ has a 3-grading; in this case $K(\mathcal{A})$ is exactly the Tits–Kantor–Koecher construction of a Lie algebra from a Jordan algebra (see for example [Jac68, Section VIII.5]).

It is proven in [All79, par 5] that the structurable algebra $\mathcal{A}$ is simple if and only if $K(\mathcal{A})$ is a simple Lie algebra; $\mathcal{A}$ is central if and only if $K(\mathcal{A})$ is central. The following strong result motivates the construction of structurable algebras.

**Theorem 3.19** ([All79, Theorem 4 and 10])**.** *Let $\mathcal{L}$ be a simple Lie algebra over a field $k$ of characteristic different from 2, 3 and 5. We define the following condition on $\mathcal{L}$:*

($\star$) *$\mathcal{L}$ contains an $sl_2$-triple*[6] *$\{e, f, h\}$, such that $\mathcal{L}$ is the direct sum of an arbitrary number of irreducible $sl_2$-modules over this triple of highest weight* 0, 2, *or* 4.

*Then $\mathcal{L} \cong K(\mathcal{A})$ for some simple structurable algebra $\mathcal{A}$ if and only if $\mathcal{L}$ satisfies ($\star$).*

*In the case that* $\mathrm{char}(k) = 0$ *the condition ($\star$) on $\mathcal{L}$ is fulfilled if and only if $\mathcal{L}$ is isotropic, i.e. contains a non-trivial split toral subalgebra (see [Sel76b]).*

The above theorem can also be formulated for non-simple Lie algebras, using a Lie algebra that contains $K(\mathcal{A})$.

If we start from a simple structurable algebra $\mathcal{A}$, the Lie algebra $K(\mathcal{A})$ is simple. If $\mathrm{char}(k) = 0$, it follows that $K(\mathcal{A})$ can be obtained from a semisimple linear algebraic group.

[6] A triple $\{e, f, h\}$ is called an $sl_2$-triple if $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$; such a triple spans an $sl_2$ Lie subalgebra.

**Theorem 3.20** ([All86b, Theorem 3.1])**.** *Let* $\text{char}(k) = 0$*, and let* $\mathcal{A}$ *be a central simple structurable algebra. Then* $K(\mathcal{A}, ^-)$ *has relative rank 1 if and only if* $\mathcal{A}$ *is a conjugate division algebra.*

*Moreover, each central simple Lie algebra of relative rank 1 can be obtained in this way.*

Below we give an overview of which types of simple Lie algebras are obtained starting from central simple structurable algebras. In the case that $\text{char}(k) = 0$, we mention, in view of the previous theorem, the Tits index (see Table 1.1 on page 13) when the structurable algebra is division. If $\text{char}(k) \neq 0$, it can be shown that these Lie algebras are also obtained from linear algebraic groups, by passing to the algebraic closure of $k$. We do not go into detail about the classical types that are obtained.

(1) If $\mathcal{A}$ is an associative algebra, then the corresponding Lie algebra is classical.
(2) Let $\mathcal{A}$ be a central simple Jordan algebra. In view of the remark after Definition 1.10, we have 3 classes of Jordan algebras. Only if the Jordan algebra is an exceptional Jordan algebra, $K(\mathcal{A})$ is not classical; it is then of type $E_7$. If $J$ is an exceptional Jordan division algebra, it can be verified that the corresponding Lie algebra has index $E_{7,1}^{78}$.
(3) If $\mathcal{A}$ is a structurable algebra arising from a hermitian form, then the corresponding algebra is classical.
(4) Let $\mathcal{A}$ be of skew-dimension 1, thus $\mathcal{A}$ is a form of the algebra $M(J, \eta)$ for several possibilities of the Jordan algebra $J$ given in Definition 3.5. If $N = 0$, $\mathcal{A}$ is isomorphic to a structurable algebra of hermitian form type; $K(\mathcal{A})$ is classical. So assume that $N \neq 0$, and $J$ is a Jordan algebra of a non-degenerate cubic norm as in Definition 1.10.(iv). It follows from [Jac68, Theorem V.4,V.8 and V.9] that $J$ has one of the following dimensions:

| $\dim_k(J)$ | type of $K(\mathcal{A})$ |
|---|---|
| 1 | $G_2$ |
| 3 | $D_4$ |
| 6 | $F_4$ |
| 9 | $E_6$ |
| 15 | $E_7$ |
| 27 | $E_8$ |

The algebra of type $G_2$ is split (with Tits index $G_{2,2,}^0$) if and only if $\mathcal{A}$ is a structurable matrix algebra.
In Lemma 3.9 there is given a construction of structurable division algebras of skew-dimension one; one has to apply the Cayley-Dickson process to a central simple Jordan division algebra of degree 4. Let $\text{char}(k) = 0$; in the case that the Jordan division algebra has dimension 10, 16 or

28, the index of $K(\mathcal{A})$ is ${}^2E_{6,1}^{35}$, $E_{7,1}^{66}$ or $E_{8,1}^{133}$, respectively (see [AF84, Example 7.2]), where $\mathcal{A}$ is constructed from $J$ as in Lemma 3.9.

(5) When $C_1, C_2$ are composition algebras, the Lie algebra $K(C_1 \otimes_k C_2)$ coincides with the Lie algebra constructed from $C_1$ and $C_2$ using Tits' second Lie algebra construction, which is visualized in Freudenthal's magic square (see [All88, p. 672]).
In the case where $\mathcal{A}$ is a non-associative (twisted) product algebra, we have

| (twisted) $(i,j)$-product algebra | type of $K(\mathcal{A})$ |
|---|---|
| $(8,1)$ | $F_4$ |
| $(8,2)$ | $E_6$ |
| $(8,4)$ | $E_7$ |
| $(8,8)$ | $E_8$ |

If $\mathrm{char}(k) = 0$, it is shown in [All88, Theorem 6.22] that if the algebras in the above list are conjugate division algebras, $K(\mathcal{A})$ has index $F_{4,1}^{21}$, ${}^2E_{6,1}^{29}$, $E_{7,1}^{48}$ or $E_{8,1}^{91}$, respectively.

(6) The exceptional 35-dimensional structurable algebras give rise to split Lie algebras of type $E_7$; see [AF93].

## 3.5 Isotopies of structurable algebras

Although isomorphisms of structurable algebras are well defined, it turns out that it is better to allow the unit element 1 to be mapped to a different element. This idea is encapsulated in the notion of an isotopy.

**Definition 3.21** ([AH81, par. 8])**.** Two structurable algebras $(\mathcal{A}, ^-)$ and $(\mathcal{A}', ^-)$ over a field $k$ are *isotopic* if there exists a $k$-vector space isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ such that there exists a $\chi \in \mathrm{End}_k(\mathcal{A}, \mathcal{A}')$ such that

$$\psi(V_{x,y}z) = V_{\psi(x),\chi(y)}\psi(z) \quad \forall x, y, z \in \mathcal{A}.$$

The map $\psi$ is then called an *isotopy* between $(\mathcal{A}, ^-)$ and $(\mathcal{A}', ^-)$. The map $\chi$ is entirely determined by the map $\psi$; we call $\chi$ the inverse dual of $\psi$ and denote $\chi := \hat{\psi}$.

Then $\chi$ is again an isotopy, with inverse dual $\psi$. Therefore $\hat{\hat{\psi}} = \psi$. Isotopy defines an equivalence relation on structurable algebras.

If $\psi$ maps the identity of $\mathcal{A}$ to the identity of $\mathcal{A}'$, then $\psi$ is an isomorphism of structurable algebras.

[AF84, Lemma 1.20] states that if $\mathcal{A}$ and $\mathcal{A}'$ are isotopic, then $\mathcal{A}$ is (central) simple if and only if $\mathcal{A}'$ is (central) simple.

The following theorem indicates why isotopies are the right definition of "equivalent" structurable algebras.

**Theorem 3.22** ([AH81, Prop. 12.3])**.** *Two structurable algebras $\mathcal{A}$ and $\mathcal{A}'$ are isotopic if and only if $K(\mathcal{A})$ and $K(\mathcal{A}')$ are graded-isomorphic graded Lie algebras.*

From [AH81, Prop. 11.3] it follows that,

$$L_s\{x, y, z\} = \{L_s x, L_{\hat{s}} y, L_s z\} \tag{3.28}$$

for all $x, y, z \in \mathcal{A}$ and $s \in \mathcal{S}$ conjugate invertible. Therefore $L_s$ is an isotopy with $\widehat{L_s} = L_{\hat{s}}$. Let $u \in \mathcal{A}$ be conjugate invertible and let $\alpha$ be an isotopy; then by [AH81, Prop. 8.2],

$$\widehat{\alpha u} = \hat{\alpha}\hat{u}, \tag{3.29}$$

and in particular if $s \in \mathcal{S}$ is conjugate invertible, then

$$\widehat{su} = \hat{s}\hat{u}. \tag{3.30}$$

If $(\mathcal{A}', ^-)$ is isotopic to $(\mathcal{A}, ^-)$, there exists a conjugate invertible $u \in \mathcal{A}$ such that $(\mathcal{A}', ^-)$ is isomorphic to a certain isotope of $(\mathcal{A}, ^-)$, denoted by $(\mathcal{A}, ^-)^{\langle u \rangle}$, which we describe below.

**Construction 3.23.** Let $u \in \mathcal{A}$ be a conjugate invertible element. We give the definition of the $u$-conjugate isotope $\mathcal{A}^{\langle u \rangle}$ of $\mathcal{A}$ following the approach in [All86a, p 364]; see [AH81, Par. 7] for the original definition.

The algebra $\mathcal{A}^{\langle u \rangle}$ will be a structurable algebra with underlying vector space $\mathcal{A}$. Its involution is defined by

$$\tau^{\langle u \rangle} x = \bar{x}^{\langle u \rangle} = 2x - \{x, u, \hat{u}\} = x - \psi(x, \hat{u})u. \tag{3.31}$$

We have ${\tau^{\langle u \rangle}}^2 = \mathrm{id}$, and we define $\mathcal{S}^{\langle u \rangle}$ and $\mathcal{H}^{\langle u \rangle}$ as the $(-1)$- and 1-eigenspace, respectively, for $\tau^{\langle u \rangle}$. Then

$$\mathcal{A} = \mathcal{S}^{\langle u \rangle} \oplus \mathcal{H}^{\langle u \rangle};$$

moreover, one can show that $\mathcal{S}^{\langle u \rangle} = \mathcal{S}u$.

Next, we define the operator $P_u$ given by

$$P_u x = \tfrac{1}{3} U_u (2\tau^{\langle u \rangle} + \mathrm{id}) x = \frac{1}{3} U_u (5x - 2V_{x,u}\hat{u}). \tag{3.32}$$

This operator is invertible and has the following nice properties:

$$P_u P_{\hat{u}} = P_{\hat{u}} P_u = \mathrm{id}, \tag{3.33}$$

$$P_u \hat{u} = u, \tag{3.34}$$

$$P_u(su) = -\frac{1}{3} U_u(su) \quad \text{for all } s \in \mathcal{S}, \tag{3.35}$$

$$P_u\{x,y,z\} = \{P_u x, P_{\hat{u}} y, P_u z\} \quad \text{for all } x,y,z \in \mathcal{A}. \tag{3.36}$$

This last identity says that $P_u$ is an isotopy on $\mathcal{A}$ with $\widehat{P_u} = P_{\hat{u}}$.

Finally, if $x, y \in \mathcal{A}$ we can write $x = su + a$ where $s \in \mathcal{S}$ and $a \in \mathcal{H}^{\langle u \rangle}$, and we define

$$x_{\langle u \rangle} y = (su+a)_{\langle u \rangle} y = sP_u y + V_{a,u} y. \tag{3.37}$$

This defines a product on the vector space $\mathcal{A}$. Then $1^{\langle u \rangle} = \hat{u}$ is a unit for the product and $\tau^{\langle u \rangle}$ is an involution for this product. We denote the algebra with this product and involution $\tau^{\langle u \rangle}$ by $\mathcal{A}^{\langle u \rangle}$; this algebra is again structurable.

The $V$-operator of the algebra $\mathcal{A}^{\langle u \rangle}$ is given by

$$V^{\langle u \rangle}_{x,y} z = \{x,y,z\}^{\langle u \rangle} = \{x, P_u y, z\}, \tag{3.38}$$

for all $x,y,z \in \mathcal{A}$. We denote by $L^{\langle u \rangle}_x$ the left multiplication with $x$ in $\mathcal{A}^{\langle u \rangle}$ and by $\psi^{\langle u \rangle}(x,y) := L^{\langle u \rangle}_x \overline{y}^{\langle u \rangle} - L^{\langle u \rangle}_y \overline{x}^{\langle u \rangle}$. When we summarize Construction 3.23, we have

$$1^{\langle u \rangle} = \hat{u}, \tag{3.39}$$

$$\mathcal{S}^{\langle u \rangle} = \mathcal{S} u, \tag{3.40}$$

$$\psi^{\langle u \rangle}(x,y) = \psi(x,y)u, \tag{3.41}$$

$$L^{\langle u \rangle}_{su} = L_s P_u, \tag{3.42}$$

for all $x, y \in \mathcal{A}$, $s \in \mathcal{S}$.

Let $x$ be conjugate invertible in $\mathcal{A}$. It follows from the identity $V^{\langle u \rangle}_{x, P_{\hat{u}} \widehat{x}} = V_{x,\widehat{x}} = \mathrm{id}$ that $x$ is also conjugate invertible in $\mathcal{A}^{\langle u \rangle}$, with

$$\widehat{x}^{\langle u \rangle} = P_{\hat{u}} \widehat{x} \tag{3.43}$$

where $\widehat{\ }^{\langle u \rangle}$ denotes the conjugate inverse in $\mathcal{A}^{\langle u \rangle}$.

We give an overview of some interesting facts about isotopies for the different classes of central simple algebras.

(1) Let $\mathcal{A}$ be an associative algebra with involution. It is proven in [All86a] that two associative algebras are isotopic if and only if they are isomorphic. If $u \in \mathcal{A}$ is invertible, $P_u = L_{u\overline{u}}$.
(2) Let $\mathcal{A}$ be a Jordan algebra. It follows from (3.31) and (3.37) that two Jordan algebras are isotopic as Jordan algebras (see [Jac68, Section 12]) if and only if they are isotopic as structurable algebras. If $u \in \mathcal{A}$ is invertible, $P_u = U_u$.

(4) Let $\mathcal{A}$ be of skew-dimension 1. In [All90], various interesting properties of the isotope $(\mathcal{A}, ^{-})^{\langle u\rangle}$ for $u \in k1 \oplus ks_0$ are shown.
(5) If $\mathrm{char}(k) = 0$, [All88, Theorem 5.4] states that two structurable algebras that are forms of an $(m_1, m_2)$-product algebra are isotopic if and only if their respective Albert forms are similar.
In [All86a] it is proven that two alternative algebras are isotopic if and only if they are isomorphic, in this case $P_u = L_{u\overline{u}}$ for invertible $u \in \mathcal{A}$.
(6) Two exceptional 35-dimensional structurable algebras are always isotopic; see [AF93].

## 3.6 Freudenthal triple systems and structurable algebras

Freudenthal triple systems were studied by Freudenthal, Meyberg and Brown. The motivation was to introduce an axiomatic approach to the study of the algebraic structure of the 56-dimensional representation of Lie algebras of type $E_7$.

The following definitions are taken from [Fer72], where more details can be found.

**Definition 3.24.** A *Freudenthal triple system* $(V, b, t)$ is a vector space $V$ over a field $k$ of characteristic not 2 or 3, endowed with a trilinear symmetric product

$$t\colon V \times V \times V \to V\colon (x, y, z) \mapsto t(x, y, z) =: xyz$$

and a skew-symmetric bilinear form

$$b\colon V \times V \to k\colon (x, y) \mapsto b(x, y) =: \langle x, y\rangle$$

such that

(i) the map $(x, y, z, w) \mapsto \langle x, yzw\rangle$ is a nonzero symmetric 4-linear form;
(ii) $(xxx)xy = \langle y, x\rangle xxx + \langle y, xxx\rangle x \quad \forall x, y \in V$.

When it is clear which triple product and skew-symmetric form are considered, we do not explicitly mention $b$ and $t$, but we use juxtaposition and $\langle ., .\rangle$ instead.

**Definition 3.25.** Two Freudenthal triple systems $(V, b, t)$, $(V', b', t')$ over a field $k$ are *similar* if there exists a $k$-vector space isomorphism $\psi : V \to V'$ and $\lambda \in k^*$ such that

$$t'(\psi(x), \psi(y), \psi(z)) = \lambda\psi(t(x, y, z)).$$

In [Fer72, Lemma 6.6] it is proven that this condition is equivalent with

$$\begin{cases} b'(\psi(x), \psi(y)) = \lambda b(x, y) \text{ and} \\ b'(\psi(x), t'(\psi(x), \psi(x), \psi(x))) = \lambda^2 b(x, t(x, x, x)). \end{cases}$$

The map $\psi$ is then called a *similarity* with *multiplier* $\lambda$. We say that two Freudenthal triple systems are *isometric* if they are similar with $\lambda = 1$; in this case $\psi$ is called an *isometry*.

**Definition 3.26.** Let $V$ be a Freudenthal triple system.

(i) An element $u \in V \setminus \{0\}$ is called *strictly regular* if $uVV \subseteq ku$.
(ii) A pair of strictly regular elements $u_1, u_2$ is called *supplementary* if $\langle u_1, u_2 \rangle = 1$.
(iii) $V$ is called *reduced* if it contains a strictly regular element.
(iv) $V$ is called *simple* if it does not contain a proper ideal, i.e. a subspace $I \neq 0, V$ such that $IVV \subseteq I$. By [Fer72, Theorem 2.1], a Freudenthal triple system is simple if and only if the skew-symmetric bilinear form is non-degenerate.

The main example of Freudenthal triple systems are given in the following theorem.

**Theorem 3.27** ([AF84, Proposition 2.8])**.** *Let $(\mathcal{A}, ^-)$ be a simple structurable algebra of skew-dimension one and let $0 \neq s_0 \in \mathcal{S}$. The following bilinear form and triple product give $\mathcal{A}$ the structure of a simple Freudenthal triple system:*

$$\begin{aligned} \langle x, y \rangle 1 &= \psi(x, y)s_0, \\ yzw &= 2\{y, s_0 z, w\} - \langle z, w \rangle y - \langle z, y \rangle w - \langle y, w \rangle z, \end{aligned}$$

*for all $x, y, z, w \in \mathcal{A}$.*

**Remark 3.28.** (i) Let $s_0, s_0' \in \mathcal{S} \setminus \{0\}$, then $s_0' = \lambda s_0$ for some $\lambda \in k$. Let $(\mathcal{A}, t, b)$ and $(\mathcal{A}, t', b')$ be the Freudenthal triple systems constructed in Theorem 3.27 by starting from $s_0$ and $s_0'$, respectively. It is clear that the identity is a similarity between $(\mathcal{A}, t, b)$ and $(\mathcal{A}, t', b')$ with multiplier $\lambda$.

(ii) Let $\nu$ be the norm on $\mathcal{A}$ defined in (3.18). Then it is clear that

$$\nu(x) = \frac{1}{12\mu} \langle x, xxx \rangle$$

for all $x \in \mathcal{A}$.

(iii) Applying the above formulas on a structurable matrix algebra with $s_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ gives the Freudenthal triple system with bilinear product

$$\left\langle \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}, \begin{pmatrix} k_1' & j_1' \\ j_2' & k_2' \end{pmatrix} \right\rangle = k_1 k_2' - k_2 k_1' + \eta T(j_1, j_2') - \eta T(j_2, j_1'),$$

and with conjugate norm

$$\nu \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} = 4k_1\eta N(j_1) + 4k_2\eta^2 N(j_2) - 4\eta^2 T(j_1^\sharp, j_2^\sharp) + \big(\eta T(j_1, j_2) - k_1k_2\big)^2.$$

**Definition 3.29.** Let $\mathcal{A}$ be a simple structurable algebra of skew-dimension one with $s_0 \in \mathcal{S} \setminus \{0\}$. We call the Freudenthal triple system obtained in Theorem 3.27 *the Freudenthal triple system associated to* $\mathcal{A}$.

If the element $s_0 \in \mathcal{S} \setminus \{0\}$ is not specified, the Freudenthal triple system associated to $\mathcal{A}$ is only determined up to similarity.

Moreover one can show that each simple Freudenthal triple system can be obtained from a structurable algebra of skew-dimension one. On a structurable algebra we have the notion of isotopy; on a Freudenthal triple system we have the notion of similarity. The notions of isotopy and similarity coincide:

**Theorem 3.30** ([Gar01])**.** (i) *Let $(V, t, b)$ be a simple Freudenthal triple system. There exists a simple structurable algebra $\mathcal{A}$ of skew-dimension one, such that $(V, t, b)$ is isometric to the Freudenthal triple system associated to $\mathcal{A}$.*

(ii) *Let $\mathcal{A}$ and $\mathcal{A}'$ be structurable algebras of skew-dimension one. Consider the associated Freudenthal triple systems. Then $\mathcal{A}$ and $\mathcal{A}'$ are similar as Freudenthal triple systems if and only if they are isotopic as structurable algebras.*

*Proof.* See [Gar01, Lemma 4.15] and [Gar01, Proposition 4.11], respectively. All Freudenthal triple systems that are considered in [Gar01] are 56-dimensional, but the proofs of these theorems remain valid in arbitrary dimension. □

In [Fer72, Theorem 5.1], it is shown that a Freudenthal triple system is associated to an isotope of a structurable matrix algebra if and only if the Freudenthal triple system is reduced. Given a reduced simple Freudenthal triple system $(V, t, b)$, the proof of [Fer72, Theorem 5.1] gives an explicit construction of a structurable matrix algebra that has $(V, t, b)$ as associated Freudenthal triple system. We apply this explicit construction in Appendix A.

We mention a few results for later use.

**Lemma 3.31.** *If the map $x \mapsto \langle x, xxx\rangle$ is anisotropic, then the Freudenthal triple system is not reduced and simple.*

*Proof.* Suppose $u \in V$ is strictly regular, then $uuu = su$ for some $s \in k$, so

$$\langle u, uuu\rangle = s\langle u, u\rangle = 0.$$

This implies that $u = 0$, so the Freudenthal triple system is not reduced. It is clearly simple, since $\langle x, xxx \rangle \neq 0$ for $x \neq 0$. □

**Lemma 3.32** ([Fer72, Corollary 3.4]). *A simple Freudenthal triple system $V$ is reduced if and only there exists $x \in V$ such that $\langle x, xxx \rangle = 12t^2$ for $t \in k^*$. If this is the case, then*

$$u_1 = \frac{1}{2}x + \frac{1}{12t}xxx, \qquad u_2 = -\frac{1}{2t}x + \frac{1}{12t^2}xxx$$

*is a pair of supplementary strictly regular elements.*

## 3.7 $J$-ternary algebras

$J$-ternary algebras have been introduced by Allison in [All76]; they are ternary algebras that are a module for a Jordan algebra. Just like structurable algebras, they can be used to construct exceptional Lie algebras, using a generalization of the Tits–Kantor–Koecher construction.

**Definition 3.33.** Let $\mathrm{char}(k) \neq 2, 3$, let $J$ be a Jordan $k$-algebra. Let $X$ be a $k$-vector space equipped with a $k$-bilinear map $\bullet : J \times X \to X$ such that

$$(j_1 j_2) \bullet x = \tfrac{1}{2}(j_1 \bullet (j_2 \bullet x) + j_2 \bullet (j_1 \bullet x)) \quad \text{and} \quad 1 \bullet x = x, \tag{3.44}$$

for all $j_1, j_2 \in J, x \in X$.

Let $(\ ,\ ) : X \times X \to J$ be a skew-symmetric $k$-bilinear map, and $(\ ,\ ,\ ) : X \times X \times X \to X$ a $k$-trilinear product. Then $X$ is a *$J$-ternary algebra* if the following axioms hold for all $j \in J$, $x, y, z, v, w \in X$:

(JT1) $j\,(x, y) = \frac{1}{2}\,(j \bullet x, y) + \frac{1}{2}\,(x, j \bullet y)$

(JT2) $j \bullet (x, y, z) = (j \bullet x, y, z) - (x, j \bullet y, z) + (x, y, j \bullet z)$

(JT3) $(x, y, z) = (z, y, x) - (x, z) \bullet y$

(JT4) $(x, y, z) = (y, x, z) + (x, y) \bullet z$

(JT5) $((x, y, z)\,, w) + (z, (x, y, w)) = (x, (z, w) \bullet y)$

(JT6) $(x, y, (z, w, v)) = ((x, y, z)\,, w, v) + (z, (y, x, w)\,, v) + (z, w, (x, y, v))$.

As we show below every structurable algebra with at least one invertible skew-element can be given the structure of a $J$-ternary algebra. From this point of view $J$-ternary algebras can be seen as an early version of structurable algebras. Below, we elaborate the proof of [ABG02, Remark 6.7]. We thank Bruce Allison for providing us a sketch of this proof.

**Theorem 3.34.** *Let $(\mathcal{A}, ^-)$ be a structurable algebra such that there exists an $s \in \mathcal{S}$ that is conjugate invertible. Then $J := L_{\mathcal{S}} L_s \subset \mathrm{End}_k(\mathcal{A})^+$ is a Jordan algebra. Define*

$$L_t L_s \bullet x := L_t L_s x = t(sx), \quad (x, y) = L_{\psi(x,y)} L_s \quad \text{and} \quad (x, y, z) = -V_{x,sy} z$$

*for all $t \in \mathcal{S}$, $x, y, z \in \mathcal{A}$. Then $\mathcal{A}$ is a $J$-ternary algebra.*

*Proof.* Fix an invertible $s \in \mathcal{S}$ and let $r, t \in \mathcal{S}$, $x, y, z, v, w \in \mathcal{A}$ be arbitrary. The algebra $\mathrm{End}_k(\mathcal{A})^+$ is a Jordan algebra; we show that $J = L_{\mathcal{S}} L_s = \{L_t L_s \mid t \in \mathcal{S}\}$ is a Jordan subalgebra. It follows from the first identity of (3.5) that $L_r L_t L_r = L_{(rt)r} = L_{r(tr)}$; by linearizing this expression and since $L_s$ is invertible it follows that

$$\begin{aligned} \tfrac{1}{2}((L_{t_1} L_s)(L_{t_2} L_s) + (L_{t_2} L_s)(L_{t_1} L_s)) &= L_{\frac{1}{2}((t_1 s)t_2 + (t_2 s)t_1)} L_s \\ &= L_{\frac{1}{2}(t_1(st_2) + t_2(st_1))} L_s. \end{aligned} \tag{3.45}$$

Since $\mathrm{id} = L_{-\hat{s}} L_s$, it follows that $J = L_{\mathcal{S}} L_s$ is a Jordan subalgebra of $\mathrm{End}_k(\mathcal{A})^+$. It is immediately clear that $\bullet$ satisfies (3.44). Let $j = L_t L_s \in J$, we verify the remaining axioms:

(JT1) We expand (3.27) for $A = L_t L_s \in \mathrm{Instrl}(\mathcal{A}, ^-)$ using (3.25) and (3.45):

$$\begin{aligned} & t(s\psi(x,y)) + \psi(x,y)(st) = \psi(t(sx), y) + \psi(x, t(sy)) \\ \iff \quad & L_{t(s\psi(x,y)) + \psi(x,y)(st)} L_s = L_{\psi(t(sx),y)} L_s + L_{\psi(x,t(sy))} L_s \\ \iff \quad & L_t L_s L_{\psi(x,y)} L_s + L_{\psi(x,y)} L_s L_t L_s = L_{\psi(t(sx),y)} L_s + L_{\psi(x,t(sy))} L_s. \end{aligned}$$

(JT2) We expand (3.22) for $A = L_s L_t \in \mathrm{Strl}(\mathcal{A}, ^-)$ using (3.24):

$$L_t L_s V_{x,sy} z - V_{x,sy} L_t L_s z = V_{L_t L_s x, sy} z + V_{x, -L_s L_t sy} z.$$

(JT3) It is clear from the definition of the $V$-operator that $V_{x,sy} z - V_{z,sy} x = L_{\psi(x,z)} L_s y$.

(JT4) This follows immediately from (3.7).

(JT5) We expand (3.27) for $A = V_{x,sy}$ using (3.21):

$$\begin{aligned} & V^{\delta}_{x,sy} \psi(z, w) = \psi(V_{x,sy} z, w) + \psi(z, V_{x,sy} w) \\ \iff \quad & -\psi(x, \psi(z, w)(sy)) = \psi(V_{x,sy} z, w) + \psi(z, V_{x,sy} w). \end{aligned}$$

(JT6) We expand (3.22) for $A = V_{x,sy}$ using (3.20) and (3.9):

$$\begin{aligned} & V_{x,sy}(V_{z,sw} v) - V_{z,sw}(V_{x,sy} v) = V_{V_{x,sy} z, sw} v + V_{z, -V_{sy,x} sw} v \\ \iff \quad & V_{x,sy}(V_{z,sw} v) - V_{z,sw}(V_{x,sy} v) = V_{V_{x,sy} z, sw} v + V_{z, sV_{y,sx} w} v. \end{aligned}$$

□

# Chapter 4

# Structurable algebras on quadrangular algebras

The purpose of this and of the next chapter is to get a better understanding of the structure of exceptional quadrangular algebras. In this chapter, we essentially capture the rank-one structure of quadrangular algebras.

In Section 4.1 we will show that there lives a Freudenthal triple system on each quadrangular algebra in a natural way. By Theorem 3.30, we know there exists a family of isotopic structurable algebras of skew-dimension one associated to this Freudenthal triple system. We give some interesting properties of such structurable algebras in Section 4.2; these properties give a new interpretation of some rank-one aspects of the quadrangular algebra (see Theorem 4.7).

More specifically, in Section 4.3 we give a structurable algebra of hermitian type that has an isometric Freudenthal triple system as a quadrangular algebra of pseudo-quadratic type. In Section 4.4 we show the following result:

**Theorem 4.1.** *Let* $\Omega$ *be a quadrangular algebra of type* $E_6$, $E_7$ *or* $E_8$ *over* $k$, *with* $\operatorname{char}(k) \neq 2, 3$. *Let* $A$ *be one of the following associative division algebras:*

- (i) *$A$ is a quaternion algebra $Q$ if $\Omega$ is of type $E_6$;*
- (ii) *$A$ is a tensor product $Q \otimes_k L$ with $Q$ a quaternion algebra and $L/k$ a quadratic field extension, if $\Omega$ is of type $E_7$;*
- (iii) *$A$ is a biquaternion algebra $Q_1 \otimes_k Q_2$ if $\Omega$ is of type $E_8$.*

*We define the structurable algebra* $\mathcal{A} = \mathrm{CD}\big(A^+, \mathrm{Nrd}, c\big)$ *of skew-dimension one (see p. 52) for some* $c \in k$. *Then the Freudenthal triple system associated to* $\mathcal{A}$ *(see Definition 3.29) is similar to the Freudenthal triple system given in Theorem 4.2 starting from* $\Omega$.

We will give more precise statements below; in particular, we will explicitly construct the algebra $A$ and the constant $c$ in each case. The proof of this theorem gives rise to lengthy computations for which we use the computer algebra package Sage. In [Wei06a] it is shown that quadrangular algebras of type $E_6$ and $E_7$ carry the structure of pseudo-quadratic spaces; in Section 4.4.2 we point out that this is compatible from the viewpoint of Freudenthal triple systems and structurable algebras.

Originally, we came up with a family of structurable algebras with Freudenthal triple system isometric to the one in Theorem 4.2 by going through the proofs of Theorem 3.30.(i) and [Fer72, Theorem 5.1] in a very explicit way. This technical approach is given in Appendix A.

## 4.1 Quadrangular algebras are Freudenthal triple systems

We show that each quadrangular algebra $(k, L, q, 1, X, \cdot, h, \theta)$ (see Definition 2.4) defined over a field of characteristic not 2 or 3 can be given the structure of a Freudenthal triple system. Notice that the definition of this Freudenthal triple system only uses the maps $g$, $x \mapsto x \cdot \pi(x)$ and $q \circ \pi$, which already appear in the description of the Moufang set obtained from a quadrangular algebra on page 42. In this sense, the Freudenthal triple system only captures rank-one information of the Moufang quadrangle and not the structure imposed by the commutator relations.

By the definition $\pi(x) = \frac{1}{2}h(x, x)$ and the maps $h : X \times X \to L$ and $\cdot : X \times L \to X$ are $k$-bilinear. Since $q$ is a quadratic form, the map $X \to k : x \mapsto q(\pi(x))$ is a form of degree 4 (see Section 1.1). Strictly speaking, the map $X \to X : x \mapsto x \cdot \pi(x)$ is not a form of degree 3 since its image is not the base field. However $x \mapsto \xi_i(x \cdot \pi(x))$ is a form of degree 3 for each $1 \leq i \leq \dim_k X$, where $\xi_i : X \to k$ denotes the projection on the $i$-th coordinate. In this way we can as well speak of the linearization of the map $x \mapsto x \cdot \pi(x)$.

**Theorem 4.2.** *Let $(k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra over a field $k$ with $\mathrm{char}(k) \neq 2, 3$. Then $X$ is a Freudenthal triple system with triple product*

$$t(x, y, z) := \tfrac{1}{2}(x(h(y, z) + h(z, y)) + y(h(x, z) + h(z, x)) + z(h(x, y) + h(y, x)))$$

*and skew symmetric bilinear form $\langle x, y\rangle := g(x, y)$, for all $x, y, z \in X$. This Freudenthal triple system is simple and not reduced.*

*Furthermore we have that the map $t$ is the linearization of $x \mapsto x \cdot \pi(x)$ and $(x, y, z, w) \mapsto \langle x, t(y, z, w)\rangle$ is the linearization of $x \mapsto -\frac{1}{2}q(\pi(x))$.*

*In particular, $t(x, x, x) = 6x \cdot \pi(x)$ and $\langle x, t(x, x, x)\rangle = -12q(\pi(x))$.*

*Proof.* It is clear that the triple product is symmetric and trilinear. It follows from Theorem 2.7.(i) and the fact that $f(x^\sigma, 1) = f(x, 1)$ that $g$ is skew symmetric and bilinear, and that $\langle x, t(y, z, w)\rangle$ is linear in its four variables. Since $\pi(x) = \frac{1}{2}h(x, x)$ we have that $t(x, x, x) = 6x \cdot \pi(x)$, so $t(x, y, z)$ is the linearization of $x \cdot \pi(x)$. To prove the first axiom of Definition 3.24 we expand $\langle x, t(y, z, w)\rangle$, and we find

$$\begin{aligned}
&\langle x, t(y, z, w)\rangle \\
&\quad = \tfrac{1}{2} f(h(x, t(y, z, w)), 1) \\
&\quad = \tfrac{1}{4}\big(f(h(x, w(h(y, z) + h(z, y))), 1) + f(h(x, y(h(w, z) + h(z, w))), 1) \\
&\qquad\qquad + f(h(x, z(h(w, y) + h(y, w))), 1)\big) \\
&\quad = \tfrac{1}{4}\big(f(h(x, w), \overline{h(y, z) + h(z, y)}) + f(h(x, y), \overline{h(w, z) + h(z, w)}) \\
&\qquad\qquad + f(h(x, z), \overline{h(w, y) + h(y, w)})\big) \\
&\quad = -\tfrac{1}{4}\big(f(h(x, w), h(y, z) + h(z, y)) + f(h(x, y), h(w, z) + h(z, w)) \\
&\qquad\qquad + f(h(x, z), h(w, y) + h(y, w))\big).
\end{aligned}$$

Therefore $\langle x, t(y, z, w)\rangle$ is indeed symmetric and linear in its four variables. When we put $x = y = z = w$, this expression equals $-12q(\pi(x))$. Thus it is the linearization of $-\frac{1}{2}q(\pi(x))$. This map is non-zero since both $q$ and $\pi$ are anisotropic.

In order to establish the second axiom of Definition 3.24, we show that

$$t(x \cdot \pi(x), x, y) = \tfrac{1}{2}\big(f(h(y, x), 1)x \cdot \pi(x) + f(h(y, x \cdot \pi(x)), 1)x\big). \tag{4.1}$$

We expand the left side of this identity, and we get

$$\begin{aligned}
t(x\cdot\pi(x), x, y) = \tfrac{1}{2}\Big(&x\cdot\pi(x)\big(h(x, y)+h(y, x)\big)+x\big(h(x\cdot\pi(x), y)+h(y, x\cdot\pi(x))\big)\Big) \\
&+ y\big(h(x, x \cdot \pi(x)) + h(x \cdot \pi(x), x)\big).
\end{aligned}$$

It follows from Theorem 2.7(iv) that the third term is zero. To reduce the two other terms we use

$$h(y, z) + h(z, y) = h(y, z) - \overline{h(y, z)} = 2h(y, z) - f(h(y, z), 1)\, 1,$$

and we get

$$\begin{aligned}
t(x \cdot \pi(x), x, y) &= \tfrac{1}{2}\Big(2x \cdot \pi(x)h(x, y) - x \cdot \pi(x)f(h(x, y), 1) \\
&\qquad\qquad + 2xh(x \cdot \pi(x), y) - xf(h(x \cdot \pi(x), y), 1)\Big) \\
&= x\big(\theta(x, h(x, y)) + h(x \cdot \pi(x), y)\big) \\
&\qquad\qquad + \tfrac{1}{2}\big(x \cdot \pi(x)f(h(y, x), 1) + xf(h(y, x \cdot \pi(x)), 1)\big),
\end{aligned}$$

where we have used (D1). It follows from Theorem 2.7(iv) that the first term is zero, establishing (4.1).

As $\langle x, t(x,x,x)\rangle = -12q(\pi(x))$ is anisotropic it follows from Lemma 3.31 that the Freudenthal triple system we obtained is simple and not reduced. □

Next we show that the triple product $t$ behaves well with respect to the $C(q,1)$-module structure on $X$ defined in Definition 2.6.

**Lemma 4.3.** *For $x, y, z \in X$ and $v \in L \setminus \{0\}$ we have that*

$$t(x,y,z)\cdot v = \frac{t(x\cdot v, y\cdot v, z\cdot v)}{q(v)}.$$

*Proof.* It is enough to show that this identity holds for $x = y = z$, since the general result then follows by linearizing. Thus we have to show that

$$(x\cdot \pi(x))\cdot v = \frac{(x\cdot v)\cdot \pi(x\cdot v)}{q(v)}.$$

This follows from [Wei06a, Theorem 3.18], since $(x\cdot\pi(x))v = x\theta(x,v)$; the map $\phi$ occurring in that formula is identically zero for fields of characteristic not 2. (In *loc. cit.*, only quadrangular algebras of type $E_6, E_7$ and $E_8$ are considered, but this proof is also valid for pseudo-quadratic spaces.) □

Two quadrangular algebras are isotopic if and only if they describe the same Moufang quadrangle. For a precise definition and some properties, we refer to [Wei06b, Chapter 8]. In the following lemma we observe that when we construct two Freudenthal triple systems starting from two isotopic quadrangular algebras, we end up with similar Freudenthal triple systems.

**Lemma 4.4.** *Let $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ and $\Omega' = (k, L', q', 1', X', \cdot', h', \theta')$ be two isotopic quadrangular algebras. Let $(X, t, b)$ and $(X', t', b')$ be the respective Freudenthal triple systems constructed as in Theorem 4.2. Then $(X, t, b)$ and $(X', t', b')$ are similar.*

*Proof.* If $\Omega$ and $\Omega'$ are isotopic, then $\Omega'$ is isomorphic to the isotope $\Omega_u$ for some $u \in L$; we denote the corresponding isomorphisms from $L_u$ to $L'$ and from $X_u$ to $X'$ by $\alpha$ and $\psi$, respectively. We use the following formulas from [Wei06b, Proposition 8.1]:

$$\begin{aligned} 1' &= \alpha(u), \\ \theta'(\psi(x), \alpha(v)) &= q(u)^{-1}\theta(x,v), \\ \psi(x)\cdot' \alpha(v) &= (xv)u^{-1}, \end{aligned}$$

for all $x \in X$ and all $v \in L$. It follows that

$$\begin{aligned}\psi(x) \cdot' \pi'(\psi(x)) &= \psi(x) \cdot' \theta'(\psi(x), 1') = q(u)^{-1}(x\theta(x,u))u^{-1} \\ &= q(u)^{-1}\big((x \cdot \pi(x))u\big)u^{-1} = q(u)^{-1}x \cdot \pi(x).\end{aligned}$$

By linearizing this expression we obtain that the Freudenthal triple systems are similar with similarity $\psi$ and multiplier $q(u)^{-1}$. □

**Remark 4.5.** We do not know whether the converse also holds, in other words, whether the fact that the Freudenthal triple systems are similar implies that the quadrangular algebras are isotopic.

## 4.2 Structurable algebras on general quadrangular algebras

From Theorem 3.30 we know that there exist structurable algebras of skew-dimension one whose associated Freudenthal triple system is similar to the Freudenthal triple system as in Theorem 4.2. In the following lemma we give some interesting properties of such structurable algebras.

**Lemma 4.6.** *Let $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra with* $\mathrm{char}(k) \neq 2, 3$ *equipped with the Freudenthal triple system $(X, t, \langle .,. \rangle)$ as in Theorem 4.2.*

*Let $\mathcal{A}$ be a simple structurable algebra of skew-dimension one such that its associated Freudenthal triple system is similar to $(X, t, \langle .,. \rangle)$. We choose $s_0 \in \mathcal{S}$ in such a way that the multiplier of the similarity is equal to one. Let $\chi : \mathcal{A} \to X$ denote the isometry. Then $\mathcal{A}$ satisfies the following properties:*

(i) *Each simple structurable algebra whose associated Freudenthal triple system is similar to $(X, t, \langle .,. \rangle)$ is isotopic to $\mathcal{A}$.*

(ii) *Let $\nu$ be the norm on $\mathcal{A}$ defined in* (3.18)*, then*

$$\nu(a) = -\frac{1}{\mu} q(\pi(\chi(a)))$$

*for all $a \in \mathcal{A}$, where $s_0^2 = \mu 1$.*

(iii) *We have that*

$$\chi\big(U_a(s_0 a)\big) = 3\chi(a) \cdot \pi(\chi(a)) \quad \text{and } \psi(a,b)s_0 = g(\chi(a), \chi(b))1$$

*for all $a, b \in \mathcal{A}$.*

(iv) *For all $0 \neq u \in \mathcal{A}$ the conjugate inverse is given by*

$$\hat{u} = s_0 \frac{1}{q(\pi(\chi(u)))} \chi^{-1}(\chi(u) \cdot \pi(\chi(u))).$$

(v) *The $V$-operator of $\mathcal{A}$ is compatible with the $C(q,1)$-module structure of $X$. Indeed, if for all $a \in \mathcal{A}$ and $v \in L$ we denote $a \odot v := \chi^{-1}(\chi(a) \cdot v) \in \mathcal{A}$, then*

$$(V_{a,s_0 b}\, c) \odot v = \frac{1}{q(v)} V_{a\odot v,\, s_0(b\odot v)}\, (c \odot v).$$

(vi) *$\mathcal{A}$ is a central simple division algebra.*

*Proof.* By Remark 3.28.(i) we can choose $s_0 \in \mathcal{S}$ such that the similarity has multiplier one.

(i) This follows from Theorem 3.30.ii.

(ii) Remark 3.28.(ii) states that $\nu(a) = \frac{1}{12\mu}\langle a, aaa\rangle$ for all $a \in \mathcal{A}$.
In Theorem 4.2 it is shown that $\langle x, t(x,x,x)\rangle = -12q(\pi(x))$ for all $x \in X$. It follows from Definition 3.25 that

$$\nu(a) = \frac{1}{12\mu}\langle \chi(a), t(\chi(a),\chi(a),\chi(a)\rangle = -\frac{1}{\mu}q(\pi(\chi(a))).$$

(iii) By Theorem 3.27 $U_a(s_0 a) = \frac{1}{2}aaa$ and $\langle a,b\rangle 1 = \psi(a,b)s_0$. Since $\chi$ is an isometry of Freudenthal triple systems we have that

$$\chi(U_a(s_0 a)) = \frac{1}{2}t(\chi(a),\chi(a),\chi(a)) = 3\chi(a)\cdot \pi(\chi(a)),$$

$$\psi(a,b)s_0 = \langle \chi(a),\chi(b)\rangle 1 = g(\chi(a),\chi(b))1.$$

(iv) The formula follows immediately from (ii), (iii) above and (3.19).

(v) Using Theorem 4.3 and 2.7.(vi) we have

$$\begin{aligned}
V_{a\odot v,\, s_0(b\odot v)}\,(c\odot v) &= \frac{1}{2}\Big(t(a\odot v, b\odot v, c\odot v) + \langle b\odot v, c\odot v\rangle(a\odot v) \\
&\qquad + \langle b\odot v, a\odot v\rangle(c\odot v) + \langle a\odot v, c\odot v\rangle(b\odot v)\Big) \\
&= \frac{q(v)}{2}\chi^{-1}\Big(t(\chi(a),\chi(b),\chi(c))\cdot v + \langle\chi(b),\chi(c)\rangle(\chi(a)\cdot v) \\
&\qquad + \langle\chi(b),\chi(a)\rangle(\chi(c)\cdot v) + \langle\chi(a),\chi(c)\rangle(\chi(b)\cdot v)\Big) \\
&= \frac{q(v)}{2}\Big(t(a,b,c)\odot v + \langle b,c\rangle(a\odot v) + \langle b,a\rangle(c\odot v) + \langle a,c\rangle(b\odot v)\Big) \\
&= q(v)(V_{a,\,s_0 b}\, c)\odot v.
\end{aligned}$$

(vi) It follows from Definition 2.4 that the map $q\circ\pi$ is anisotropic, we conclude from (ii) that $\nu$ is anisotropic. This implies that $\mathcal{A}$ is a conjugate division algebra (see Section 3.3.4). $\mathcal{A}$ is central since each simple algebra of skew-dimension one is central. □

In the following theorem we give the quadrangular algebra $X$ itself the structure of a structurable algebra. Doing so, we obtain some nice identifications of expressions defined in quadrangular algebras and in structurable algebras.

**Theorem 4.7.** *Let $\mathcal{A}$ be an algebra as in Theorem 4.6 above with $\chi : \mathcal{A} \to X$ the isometry. We give $X$ itself the structure of a structurable algebra by defining the following multiplication and involution*

$$x \star y := \chi(\chi^{-1}(x)\chi^{-1}(y)), \quad \overline{x} := \chi(\overline{\chi^{-1}(x)}),$$

*for all $x, y \in X$. Define $1 = \chi^{-1}(1) \in X$, let $0 \neq s_0 \in \mathcal{S} \subseteq X$ and $\mu 1 = s_0^2$.*

*Hence $X$ is a central simple structurable division algebra of skew-dimension one, let $\nu$ denote the norm of this structurable algebra. We have the following nice expressions for $q \circ \pi$, $x \cdot \pi(x)$ and $g$:*

$$\nu(x) = -\frac{1}{\mu} q(\pi(x)), \quad U_x(s_0 \star x) = 3x \cdot \pi(x), \quad \psi(x, y)s_0 = g(x, y)1,$$

*for all $x, y \in X$. We have that the $V$-operator is compatible with the $C(q, 1)$-module structure of $X$:*

$$(V_{x, s_0 \star y}\, z) \cdot v = \frac{1}{q(v)} V_{x \cdot v,\, s_0 \star (y \cdot v)}\, (z \cdot v),$$

*for all $x, y, z \in X$, $v \in L$. Consider the Moufang set on $X \times k$ and $\tau$ given by 2.20, let $0 \neq x \in X$ then*

$$(x, 0).\tau = (\frac{1}{\mu} s_0 \star \hat{x}, 0).$$

*Proof.* All the claims follow immediately from the previous lemma if we let the isometry $\chi$ be the identity map $X \to X$. For the last claim, use the fact that $s_0 \star (s_0 \star y) = \mu y$ for all $y \in X$. □

In Section 2.4.2.4 we showed that each Moufang quadrangle gives rise to two Moufang sets. When the coordinatizing structure of the root groups is one of the following: a Jordan division algebra, a skew field or an anisotropic quadratic form space; the permutation $\tau$ is given by considering the inverse. In the previous theorem we showed that in the case of a quadrangular algebra $(x, 0).\tau = (\frac{1}{\mu} s_0 \star \hat{x}, 0)$; therefore also in this case, $\tau$ acts as the inverse on $X$.

## 4.3 Structurable algebras on pseudo-quadratic spaces

We take a closer look at Lemma 4.6 in the case that the quadrangular algebra is obtained from a pseudo-quadratic space $(L, \sigma, X, h, \pi)$ as in Lemma 2.12. In particular we will give an example of a structurable algebra $\mathcal{A}$ of hermitian type and skew-dimension one such that the Freudenthal triple system obtained in Theorem 4.2 is associated to $\mathcal{A}$ in the sense of Definition 3.29.

Structurable algebras of hermitian type are constructed from hermitian forms. Since $h$ is a skew-hermitian form, we need to apply Lemma 1.5 to have a hermitian form at our disposal. Then we apply Remark 3.4.(ii) to make $X$ into a structurable algebra, if we would just apply the procedure in Definition 3.3 we would obtain a structurable algebra on $L \oplus X$.

**Theorem 4.8.** *Let $(L, \sigma)$ be a quadratic pair and $(L, \sigma, X, h, \pi)$ be an anisotropic pseudo-quadratic space with $X \neq 0$. Choose an element $0 \neq s \in L$ such that $s^\sigma = -s$ and an element $0 \neq \xi \in X$. Since $X = (\xi L) \oplus (\xi L)^\perp$, each element in $X$ can be written in a unique way as $\xi v + x$ for $v \in L$ and $x \in (\xi L)^\perp$.*

*The following involution and multiplication define a structurable algebra on $X$:*

$$\begin{aligned} \overline{\xi v + x} &= \xi(s v^\sigma s^{-1}) + x, \\ (\xi v + x) \cdot (\xi u + y) &= \xi\big(uv + sh(x, y)\big) + \big(xu + y(s v^\sigma s^{-1})\big), \end{aligned} \tag{4.2}$$

*for all $u, v \in L$ and all $x, y \in (\xi L)^\perp$. This structurable algebra is simple and has skew-dimension one.*

*When we fix $s = h(\xi, \xi)$, the Freudenthal triple system associated to the structurable algebra $X$ is similar to the Freudenthal triple system obtained by Theorem 4.2. For other choices of $\xi$ we obtain isotopic structurable algebras.*

*Proof.* Let $0 \neq s \in L$ such that $s^\sigma = -s$, as in Lemma 1.5 we define an involution for all $v \in L$ by $v^\tau = s v^\sigma s^{-1}$, a left scalar multiplication $v \circ x := x v^\tau$ and we consider the hermitian form $sh$.

Now we apply Remark 3.4.(ii). Let $0 \neq \xi \in X$ on the hermitian space $X$ with hermitian form $sh$. Then

$$(\xi L)^\perp = (L \circ \xi)^\perp = \{x \in X \mid sh(x, \xi) = 0\} = \{x \in X \mid h(x, \xi) = 0\}.$$

We obtain the following involution and multiplication:

$$\begin{aligned} \overline{\xi v + x} &= \overline{v^\tau \circ \xi + x} = v \circ \xi + x = \xi v^\tau + x = \xi(s v^\sigma s^{-1}) + x, \\ (\xi v + x)(\xi u + y) &= (v^\tau \circ \xi + x)(u^\tau \circ \xi + y) \\ &= (v^\tau u^\tau + sh(y, x)) \circ \xi + (u^\tau \circ x + v \circ y) \\ &= \xi\big(uv + sh(x, y)\big) + \big(xu + y(s v^\sigma s^{-1})\big), \end{aligned}$$

for all $u, v \in L$ and all $x, y \in (\xi L)^\perp$. We denote this structurable algebra by $\mathcal{A}_h$. Since $sh$ is non-degenerate and $L$ is central simple, it follows from Section 3.3.3 that $\mathcal{A}_h$ is simple.

The skew-elements of $^-$ are given by $\mathcal{S} = \{v \in L \mid sv^\sigma s^{-1} = -v\}$, since $L$ is either a quaternion algebra or a quadratic field extension of $k$ we have

$$\begin{aligned} sv^\sigma s^{-1} = -v &\iff sv^\sigma = -vs \\ &\iff vs = (vs)^\sigma \\ &\iff vs \in \mathrm{Fix}_L(\sigma) = k. \end{aligned}$$

It follows that $\dim_k \mathcal{S} = 1$; since $s \in \mathcal{S}$, we conclude that $\mathcal{S} = ks$.

Now take $s = h(\xi, \xi) \in L$; we have $\overline{s} = -s$. We will determine the trilinear map of the Freudenthal triple system associated to $\mathcal{A}_h$ (see Definition 3.29) for $s_0 = -\frac{1}{2}s^{-1}$. Let $y = \xi v + x \in (\xi L) \oplus (\xi L)^\perp$ be arbitrary. Then

$$\begin{aligned} 2V_{y,\,s_0 y}\, y &= 2(2(y \cdot \overline{s_0 \cdot y})y - (y \cdot \overline{y}) \cdot (s_0 \cdot y)) && (4.3)\\ &= 3\big(\xi v(-v^\sigma s^{-1} v + h(x,x)) + x(-v^\sigma s^{-1} v + h(x,x))\big) && (4.4)\\ &= 3\big((\xi v + x)((s^{-1}v)^\sigma h(\xi,\xi)(s^{-1}v) + h(x,x))\big) && (4.5)\\ &= 6(\xi v + x)\pi(\xi s^{-1} v + x), && (4.6) \end{aligned}$$

where equation (4.3) follows from the definition of the $V$-operator of a structurable algebra; (4.4) follows after a straightforward calculation using the multiplication on $\mathcal{A}_h$ defined by (4.2).

Let $\psi$ be the vector space automorphism

$$\psi \colon \mathcal{A}_h \to \mathcal{A}_h \colon \xi v + x \mapsto \xi s v + x;$$

then by applying (4.6) with $y$ replaced by $\psi(y) = \xi s^{-1} v + x$, we get

$$\begin{aligned} 2V_{\psi(y),\,s_0\psi(y)}\, \psi(y) &= 12\big((\xi s v + x)\pi(\xi v + x)\big) \\ &= 6\psi\big((\xi v + x)\pi(\xi v + x)\big) \\ &= \psi\big(6y\pi(y)\big). \end{aligned}$$

Therefore $\psi$ is a isometry from the Freudenthal triple system given by Theorem 3.27 and the Freudenthal triple system given in Theorem 4.2. It follows from Theorem 3.30 that the algebras obtained from different elements $\xi \in X$ are isotopic. □

## 4.4 Structurable algebras on quadrangular algebras of type $E_6$, $E_7$ and $E_8$

Now we take a closer look at Lemma 4.6 in the case that the quadrangular algebra is of type $E_6$, $E_7$ or $E_8$.

### 4.4.1 The quadrangular algebra of type $E_8$

Let $\Omega = (k, L, q, 1, X, \cdot, h)$ be a quadrangular algebra of type $E_8$ over $k$, we described its structure in Section 2.2.3. We start by associating a biquaternion algebra to $X_k$.

By definition $q = N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle$ with $N$ the norm of a quadratic separable field extension $E = k[\gamma]/(\gamma^2 - c)$. We denote the elements of the set $S$ (see Notation 2.15) of length less or equal to 2 and different from $\varphi$ by

$$\mathcal{I} = \{2, 3, 4, 5, 6, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56\}$$

Remember that each element of $X$ is of the form given in (2.2). We start by investigating the set

$$X_k := \Big\{ t_1 v_1 + \sum_{i \in \mathcal{I}} t_i v_i \mid t_1, \dots, t_{56} \in k \Big\} \leqslant X.$$

This is a 16-dimensional vector space over $k$. Define as in Section 2.2.3 $L_k = kv_1 \oplus \cdots \oplus kv_6 \leqslant L$ and denote the restriction of the quadratic form $q$ to $L_k$ by $q|_k \colon L_k \to k$. By construction $X_k$ is isomorphic as a vector space to $C(q|_k, 1)/M_k$, where $M_k$ is the submodule

$$M_k = \mathrm{Span}_k\{v_I - \mathrm{sgn}(I) s_I v'_I \mid I \in \emptyset \cup \mathcal{I}\} = (v_2 v_3 v_4 v_5 v_6 - 1) C(q|_k, 1).$$

of $C(q|_k, 1)$. Since $v_i v_j = -v_j v_i \in C(q|_k, 1)$ for $i \neq j \in \{2, \dots, 5\}$, the element $v_2 v_3 v_4 v_5 v_6$ is in the center of $C(q|_k, 1)$; therefore $M_k$ is a two-sided ideal of $C(q|_k, 1)$.

**Theorem 4.9.** *We consider $X_k = C(q|_k, 1)/M_k$ as an associative algebra, endowed with the multiplication induced by the Clifford algebra with base-point. Then $X_k$ is isomorphic to a biquaternion algebra, as an algebra.*

*In particular, $X_k \cong Q_1 \otimes_k Q_2$ for the quaternion $k$-algebras $Q_1 := (-s_2, -s_3)_k$ and $Q_2 := (-s_{46}, -s_{56})_k$ and $Q_1 \otimes_k Q_2$ is a division algebra.*

*Proof.* The multiplication on $X_k = C(q|_k, 1)/M_k$ is induced by the multiplication in the Clifford algebra $C(q|_k, 1)$ by reducing the result modulo $M_k$. We define the two quaternion algebras over $k$ with the following generators:

$$\begin{aligned} Q_1 := (-s_2, -s_3)_k &= \langle \ell, m \mid \ell^2 = -s_2, m^2 = -s_3, \ell m = -m\ell \rangle, \\ Q_2 := (-s_{46}, -s_{56})_k &= \langle n, r \mid n^2 = -s_{46}, r^2 = -s_{56}, nr = -nr \rangle. \end{aligned}$$

In order to construct an isomorphism $\psi \colon Q_1 \otimes_k Q_2 \to X_k$, we have to describe two isomorphisms $\psi_i \colon Q_i \to X_k$ for $i \in \{1, 2\}$, such that the images $\psi_1(Q_1)$ and $\psi_2(Q_2)$ commute elementwise, and together generate $X_k$. We can achieve this by the choice

$$\psi_1(\ell) = v_2, \quad \psi_1(m) = v_3, \quad \psi_1(\ell m) = v_{23},$$

$$\psi_2(n) = v_{46}, \quad \psi_2(r) = v_{56}, \quad \psi_2(nr) = s_6 v_{45}.$$

Observe that the subspaces $\langle 1, v_2, v_3, v_{23}\rangle$ and $\langle 1, v_{46}, v_{56}, v_{45}\rangle$ do indeed commute elementwise, and together they generate $X_k$.

It follows from Lemma 2.14 that the Albert form $q_A$ of $Q_1 \otimes_k Q_2$ is similar to $\langle 1, s_2, s_3, s_4, s_5, s_6\rangle$; since $q$ is of type $E_8$ and therefore anisotropic, $q_A$ is anisotropic and $Q_1 \otimes_k Q_2$ is division. □

We can summarize the isomorphism $\psi : Q_1 \otimes_k Q_2 \to X_k$ in the following table:

| $\otimes$ | $1$ | $n$ | $r$ | $nr$ |
|---|---|---|---|---|
| $1$ | $v_1$ | $v_{46}$ | $v_{56}$ | $s_6 v_{45}$ |
| $\ell$ | $v_2$ | $-s_{246} v_{35}$ | $s_{256} v_{34}$ | $s_{2456} v_{36}$ |
| $m$ | $v_3$ | $s_{346} v_{25}$ | $-s_{356} v_{24}$ | $-s_{3456} v_{26}$ |
| $\ell m$ | $v_{23}$ | $-s_{2346} v_5$ | $s_{2356} v_4$ | $-v_6$ |

(4.7)

**Remark 4.10.** The construction of the biquaternion algebra depends on the similarity class of $q$ and on the norm splitting for $q$, and in fact, this biquaternion algebra is *not* an invariant of the quadrangular algebra.

The algebra $Q_1 \otimes_k Q_2$ has dimension 16, whereas the structurable algebra we want to construct should have dimension 32. We will apply the Cayley-Dickson process on the Jordan algebra $(Q_1 \otimes_k Q_2)^+$, see Definition 3.8.

We start with the central simple biquaternion algebra equipped with the reduced norm Nrd.

**Lemma 4.11.** *Let $Q_1 \otimes_k Q_2$ be a biquaternion algebra over a field $k$. Then the reduced norm* Nrd *is a Jordan norm of degree 4 of the Jordan algebra $(Q_1 \otimes_k Q_2)^+$.*

*Proof.* The central simple algebra $Q_1 \otimes_k Q_2$ has degree 4, so its reduced norm is indeed a form of degree 4 with basepoint 1; and the trace form is bilinear nondegenerate. Since the Jordan algebra arises from a biquaternion algebra, we have $U_j j' = jj'j$, and it follows that $\mathrm{Nrd}(U_j j') = \mathrm{Nrd}(j)^2 \,\mathrm{Nrd}(j')$. □

In order to apply the Cayley–Dickson construction to $(Q_1 \otimes Q_2)^+$, we have to determine the trace map $T$ associated to Nrd explicitly. Since $T$ is bilinear, it suffices to compute its value for elements of the form $a \otimes b$ in $Q_1 \otimes Q_2$. It turns out that

$$T(a \otimes b, a' \otimes b') = \mathrm{Trd}(a, a')\,\mathrm{Trd}(b, b') \quad \forall a, a' \in Q_1, b, b' \in Q_2,$$

where Trd is the reduced trace. For $a = a_1 + a_2\ell + a_3m + a_4\ell m \in Q_1$, $b = b_1 + b_2n + b_3r + b_4nr \in Q_2$, we have that $T(a \otimes b, 1 \otimes 1) = 4a_1b_1$, as in Definition 3.8 we define

$$(a \otimes b)^\theta = -a \otimes b + 2a_1b_1(1 \otimes 1). \tag{4.8}$$

The following theorem gives us the $E_8$-case of Theorem 4.1.

**Theorem 4.12.** *Let $Q_1 = (-s_2, -s_3)_k$ and $Q_2 = (-s_{46}, -s_{56})_k$. The Freudenthal triple system associated to $\mathrm{CD}\big((Q_1 \otimes Q_2)^+, \mathrm{Nrd}, \gamma^2\big)$ is similar to the Freudenthal triple system $(X, t, \langle ., . \rangle)$ defined in Theorem 4.2.*

*Proof.* Since $(Q_1 \otimes_k Q_2)^+ \cong X_k$ as vector spaces, the structurable algebra $\mathrm{CD}\big((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2\big)$ is as a vector space isomorphic to $X_k \oplus s_0X_k$.

We use the isomorphism $\psi : Q_1 \otimes_k Q_2 \to X_k$ defined in (4.7) to define a $k$-vector space isomorphism $\chi$

$$\begin{aligned} \mathrm{CD}\big((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2\big) &\to X \\ (t_1\ 1 \otimes 1 + x_1) + s_0(t_2\ 1 \otimes 1 + x_2) &\mapsto (t_1 - \gamma t_2)v_1 + \psi(x_1) + \gamma\psi(x_2) \end{aligned}$$

for all $t_1, t_2 \in k$ and for all $x_1, x_2 \in Q_1 \otimes_k Q_2$ such that $T(x_1, 1 \otimes 1) = T(x_2, 1 \otimes 1) = 0$.

For the Freudenthal triple system in Theorem 4.2 we have that $t(x, x, x) = 6x \cdot \pi(x)$ for $x \in X$; for the Freudenthal triple system associated to $\mathrm{CD}\big((Q_1 \otimes Q_2)^+, \mathrm{Nrd}, \gamma^2\big)$ we have $t(x, x, x) = U_x(s_0x)$. We implemented the necessary algebraic structures into the program [Sea11] to show that for all $x \in \mathrm{CD}\big((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2\big)$

$$3\chi(x)\pi(\chi(x)) = \chi\big(U_x(s_0x)\big), \tag{4.9}$$

this implies that the Freudenthal triple systems are similar. Verifying the above identity is done by calculating the left and right-hand side and verifying that they are equal, for more details see Appendix B. □

**Remark 4.13.** (i) The structurable algebra described above, consisting of a twisted Jordan algebra of a biquaternion algebra, is defined up to isotopy by the quadrangular algebra; in particular its isotopy class is determined by the quadratic form of type $E_8$.

(ii) The map $-\frac{1}{\gamma^2}q \circ \pi$ can be seen as the norm on the quadrangular algebra $X$. Using [AF84, Proposition 6.7] and Theorem 4.7, we find an elegant expression for $q \circ \pi$ on $X_k$:

$$q(\pi(x)) = -\gamma^2\nu(z) = N(\gamma)\,\mathrm{Nrd}(z)$$

for all $x \in X_k$, where $x = \psi(z)$ for $z \in Q_1 \otimes_k Q_2$.

(iii) In Chapter 6, we will see more generally how we can construct a Moufang set from any structurable division algebra, and in particular, we will recover the non-abelian Moufang set which is the residue of a Moufang quadrangle of type $E_8$; see Lemma 6.31.

### 4.4.2 The pseudo-quadratic spaces on $E_6$ and $E_7$

Now assume that $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ is a quadrangular algebra of type $E_6$ or $E_7$. On page 26 we explained that quadrangular algebras of type $E_6$ and $E_7$ are subspaces of the quadrangular algebra of type $E_8$.

It follows from Theorem 4.9 that in the $E_6$-case $X_k$ is isomorphic to $Q_1$, and in the $E_7$-case $X_k$ is isomorphic to $Q_1 \otimes_k k(r)$. Therefore it follows from Theorem 4.12 that in the $E_6$-case and $E_7$-case the Freudenthal triple systems associated to $\mathrm{CD}(Q_1^+, \mathrm{Nrd}, \gamma^2)$, $\mathrm{CD}\big((Q_1 \otimes_k k(r))^+, \mathrm{Nrd}, \gamma^2\big)$ respectively, are similar to the Freudenthal triple system defined in Theorem 4.2; this proves the $E_6$- and $E_7$-case of Theorem 4.1.

Whereas for quadrangular algebras of type $E_8$ the Moufang set defined by (2.20) is of type $E_7$, in the case of $E_6$ and $E_7$ we have that the Moufang set defined by (2.20) is classical. We show that this fact is also visible at the level of the structurable algebras.

Indeed, in [Wei06a] it is shown that $X$ can be made into a 4-dimensional vector space over $E$ or over the quaternion algebra $D = (E/k, s_2s_3s_4)$, respectively; moreover, there is, in both cases, an anisotropic pseudo-quadratic form on this vector space $X$, denoted by $\hat{Q}$, with the property that

$$x \cdot \pi(x) = x * \hat{Q}(x) \tag{4.10}$$

for all $x \in X$. (We have used the symbol $*$ to denote the scalar multiplication of $X$ over $E$ or $D$, respectively.) We refer to [Wei06a, Definition 3.6, and Theorems 5.3 and 5.4] for more details.

Equation (4.10) exactly implies that the Freudenthal triple system corresponding to the pseudo-quadratic form space $X$ is similar to the Freudenthal triple system of $X$ as a quadrangular algebra of type $E_6$ or $E_7$. From Lemma 4.6 it follows that structurable algebras associated to those Freudenthal triple systems are isotopic.

It is also interesting to note that it is shown in [Wei06a, Theorem 5.12] that

$$q(\pi(x)) = N(\hat{Q}(x)) \tag{4.11}$$

for all $x \in X$; both sides of this expression are, up to a constant, equal to the conjugate norm of a structurable algebra defined in Theorem 4.7 related to the respective Freudenthal triple systems. Identity (4.11) could as well be proven by combining these two remarks.

We conclude that our result in Theorem 4.7 is a kind of generalization of the construction of the pseudo-quadratic spaces on the quadrangular algebra of type $E_6$ and $E_7$ in [Wei06a].

# Chapter 5

# A coordinate-free construction of quadrangular algebras

In this chapter we give a coordinate-free construction of quadrangular algebras in characteristic different from two. The most interesting case is of course the one of the quadrangular algebras of type $E_6$, $E_7$ and $E_8$; we can summarize their construction as follows:

**Construction 5.1.** Let $\mathrm{char}(k) \neq 2$. We start with a quadratic space $(k, V, q)$ of type $E_6$, $E_7$ or $E_8$ with basepoint (see also Definition 2.13).

By Theorem 5.14 below, there exist an octonion division algebra $C_1$ and a division composition algebra $C_2$ of dimension 2, 4 or 8, respectively such that $C_1$ and $C_2$ contain an isomorphic quadratic field extension, but no isomorphic quaternion algebra, and such that $q$ is similar to the anisotropic part of the Albert form, $q_A$, of $C_1 \otimes_k C_2$. It follows that there exist $\mathbf{i}_1 \in C_1$ and $\mathbf{i}_2 \in C_2$ such that $\mathbf{i}_1^2 = \mathbf{i}_2^2 = a \in k \setminus \{k^2\}$.

We define a subspace $V$ of the skew-elements of $C_1 \otimes_k C_2$ of dimension 6, 8 or 12, respectively, as[1]

$$V := \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^{\perp}.$$

We choose an arbitrary $u \in V \setminus \{0\}$ and define the quadratic form

$$Q := \frac{1}{q_A(u)} q_A|_V;$$

this form has basepoint $u$ and is similar to the quadratic form of type $E_6, E_7$ or $E_8$ we started with.

[1] The orthogonal complement is taken w.r.t. the bilinear form associated to the Albert form; this quadratic form is defined on the skew-elements of $C_1 \otimes_k C_2$.

We then define the subspace $X_0$ of $C_1 \otimes_k C_2$ of dimension 8, 16 or 32 as

$$X_0 := \Big\langle \big(a\, x \otimes y + \mathbf{i}_1 x \otimes \mathbf{i}_2 y\big) \mid x \in C_1, y \in C_2 \Big\rangle.$$

Next, we define a suitable element $r \in \mathcal{S}$ as in Definition 5.19(iii) below, and we define the bilinear map $X_0 \times L_0 \to X_0$ as

$$x \cdot v = v(r(u(rx))),$$

and the bilinear map $h\colon X_0 \times X_0 \to V$ as

$$h(x, y) = (u(rx))\overline{y} - y((\overline{x}r)u).$$

In Theorem 5.23 we prove that the 7-tuple $(k, V, Q, u, X_0, \cdot, h)$ is a quadrangular algebra of type $E_6$, $E_7$ or $E_8$, respectively. It follows that this is the structure described in [TW02, Chapter 13] giving rise to the Moufang quadrangles of type $E_6$, $E_7$ and $E_8$, and hence to the corresponding rank two forms of exceptional linear algebraic groups of type $E_6$, $E_7$ and $E_8$.

For the idea behind Theorem 5.14 we are indebted to Skip Garibaldi. We have inspired our construction on several properties of $J$-ternary algebras (see Definition 3.33) and a construction of $J$-ternary algebras from the tensor product of composition algebras given in [ABG02]. It is not clear at all how to generalize the theory of $J$-ternary algebras to fields of characteristic 2 and 3.

In the theory of quadrangular algebras fields of characteristic 3 play no special role (but those of characteristic 2 behave differently). Therefore we want our construction of quadrangular algebras to work in characteristic 3 in the same way as in characteristic not 2 and 3. We extended the methods used in [ABG02] to work also over fields of characteristic equal to 3. The side effect is that this gives rise to lenghty computations; we need the computer algebra package Sage to verify these.

In Section 5.1 we define special $J$-modules over fields of characteristic different from two, where $J$ is a Jordan algebra. In Section 5.2 we show that special $J$-modules satisfying some conditions give rise to quadrangular algebras (see Theorem 5.6). We also show that a $J$-ternary algebra is a special $J$-module that satisfies these conditions if the characteristic is different from 3.

In Section 5.3 we make the method in Theorem 5.6 more explicit by giving the construction of the quadrangular algebras of pseudo-quadratic form type. In Section 5.4 we elaborate Construction 5.1.

In Section 5.5 we give a uniform description of all of the 4 classes of Moufang quadrangles, over fields of characteristic different from 2, starting from a special $J$-module by extending the construction given in Theorem 5.6.

## 5.1 Special $J$-modules

In this section we let $k$ be a field of characteristic different from 2. In the next lemma we introduce a module for Jordan algebras.

**Lemma 5.2.** *Let $J$ be a Jordan $k$-algebra and let $X$ be a $k$-vector space. Let $\bullet : J \times X \to X$ be a $k$-bilinear map such that $1 \bullet x = x$ for all $x \in X$. Then the following identities are equivalent*

$$\begin{aligned} U_j j' \bullet x &= j \bullet (j' \bullet (j \bullet x)), \\ (jj') \bullet x &= \tfrac{1}{2}(j \bullet (j' \bullet x) + j' \bullet (j \bullet x)), \end{aligned} \tag{5.1}$$

*for all $j, j' \in J$, $x \in X$.*

*Proof.* We first assume that the first identity holds. Since $U_j 1 = j^2$, we have $j^2 \bullet x = j \bullet (j \bullet x)$. Linearizing this expression gives us the second identity.

Now we assume that the second identity holds, we have

$$\begin{aligned} U_j j' \bullet x &= (2j(jj') - j^2 j') \bullet x \\ &= \tfrac{1}{2}(2j \bullet ((jj') \bullet x) + 2(jj') \bullet (j \bullet x) \\ &\qquad - j^2 \bullet (j' \bullet x) - j' \bullet (j^2 \bullet x)) \\ &= \tfrac{1}{2}(j^2 \bullet (j' \bullet x) + j' \bullet (j^2 \bullet x) + 2j \bullet (j' \bullet (j \bullet x)) \\ &\qquad - j^2 \bullet (j' \bullet x) - j' \bullet (j^2 \bullet x)) \\ &= j \bullet (j' \bullet (j \bullet x)). \end{aligned}$$ □

**Definition 5.3** ([ABG02, 3.12])**.** Let $J$ be a Jordan $k$-algebra and let $X$ be a $k$-vector space, let $\bullet : J \times X \to X$ be a $k$-bilinear map such that $1 \bullet x = x$ for all $x \in X$. We call $X$ a *special $J$-module* if the equivalent identities (5.1) are satisfied.

As in the definition of a $J$-ternary algebra, we equip a special $J$-module with a skew-symmetric form that satisfies (JT1) of Definition 3.33.

**Lemma 5.4.** *Let $X$ be a special $J$-module and let $(\cdot,\cdot) : X \times X \to J$ be a $k$-bilinear skew-symmetric form. Then the following identities are equivalent*

$$\begin{aligned} U_j (x, y) &= (j \bullet x, j \bullet y), \\ j (x, y) &= \tfrac{1}{2}((j \bullet x, y) + (x, j \bullet y)), \end{aligned} \tag{5.2}$$

*for all $j \in J$, $x, y \in X$.*

*Proof.* The second identity follows immediately from the first identity since for all $j, j' \in J$ we have $U_{j+1} j' - U_j j' - U_1 j' = 2jj'$.

Now we assume that the second identity holds. Then we have for all $j \in J$, $x, y \in X$ that

$$\begin{aligned} U_j\,(x,y) &= 2j(j\,(x,y)) - j^2\,(x,y) \\ &= \tfrac{1}{2}\big((j \bullet (j \bullet x), y) + (x, j \bullet (j \bullet y)) + 2\,(j \bullet x, j \bullet y)\big) \\ &\qquad - \tfrac{1}{2}\big((\!(j^2 \bullet x, y)\!) + (\!(x, j^2 \bullet y)\!)\big) \end{aligned}$$

The first identity follows since $j \bullet (j \bullet x) = j^2 \bullet x$. □

In the following lemma we consider the Peirce decomposition of special $J$-modules; we took this material from [ABG02, 6.61], there it is elaborated (without proof) for $J$-ternary algebras in characteristic 0.

**Lemma 5.5.** *Let $J$ be a Jordan $k$-algebra with supplementary proper idempotents $e_0$ and $e_1$. Let $J_0, J_{1/2}, J_1$ be the Peirce subspaces of $J$ with respect to $e_1$ (see Definition 1.11). Let $X$ be a special $J$-module and define*

$$X_0 := \{x \in X \mid e_0 \bullet x = x\} = \{x \in X \mid e_1 \bullet x = 0\},$$
$$X_1 := \{x \in X \mid e_0 \bullet x = 0\} = \{x \in X \mid e_1 \bullet x = x\}.$$

*Then*

(i) *We have $e_0 \bullet X = X_0$, $e_1 \bullet X = X_1$ and $X = X_0 \oplus X_1$.*
(ii) *For $i \in \{0,1\}$ and $j = 1 - i$ we have*

$$J_i \bullet X_i \subseteq X_i, \quad J_i \bullet X_j = 0, \quad J_{1/2} \bullet X_i \subseteq X_j. \tag{5.3}$$

(iii) *Let $(\cdot,\cdot) : X \times X \to J$ be a skew-symmetric bilinear form satisfying* (5.2)*, then*

$$(X_i, X_i) \subseteq J_i, \quad (X_i, X_j) \subseteq J_{1/2} \tag{5.4}$$

*for $i \in \{0,1\}$ and $j = 1 - i$.*
(iv) *Assume there exists an element $u \in J_{1/2}$ such that $u^2 = 1$. The map*

$$X_0 \to X_1 : x \mapsto u \bullet x$$

*is a vector space isomorphism, called the* connecting morphism.

*Proof.* (i) Let $i \in \{0,1\}$, we have $e_i \bullet (e_i \bullet x) = (e_i e_i) \bullet x = e_i \bullet x$ for all $x \in X$, therefore $e_i \bullet X = X_i$. Since $(e_0 + e_1) \bullet x = x$, we have $X = X_0 \oplus X_1$.
(ii) This follows by evaluating

$$e_1 \bullet (j \bullet x) = 2(e_1 j) \bullet x - j \bullet (e_1 \bullet x)$$

for all the combinations of $j \in J_0$, $J_1$ or $J_{1/2}$ and $x \in X_0$ or $X_1$; making use of the fact that $X = X_0 \oplus X_1$.
(iii) This follows from evaluating $e_1\,(x,y) = \frac{1}{2}((e_1 \bullet x, y) + (x, e_1 \bullet y))$ for $x, y \in X_0$ or $X_1$.
(iv) It follows from (ii) that $u \bullet x \in X_1$ if and only if $x \in X_0$. Since $u \bullet (u \bullet x) = (uu) \bullet x = x$ the connecting morphism is an isomorphism. □

## 5.2 Construction of general quadrangular algebras

In the following theorem we consider a special $J$-module in the case where $J$ is a reduced spin factor (see Definition 1.12). These structures give rise to quadrangular algebras.

**Theorem 5.6.** *Let* $\operatorname{char}(k) \neq 2$.

*Let $J$ be the reduced spin factor of the non-degenerate anisotropic quadratic space $(k, V, q)$ with basepoint $u$: $J = ke_0 \oplus V \oplus ke_1$.*

*Let $X$ be a non-trivial special $J$-module equipped with a bilinear skew-symmetric form $\boldsymbol{(}\cdot,\cdot\boldsymbol{)} : X \times X \to J$ satisfying* (5.2).

*Define $X_0 = \{x \in X \mid e_0 \bullet x = x\}$ as in Lemma 5.5, assume that the following holds:*

$$\forall x \in X_0, v \in V : \qquad \boldsymbol{(}v \bullet x, x\boldsymbol{)} \bullet x = v \bullet (u \bullet (\boldsymbol{(}u \bullet x, x\boldsymbol{)} \bullet x)), \tag{5.5}$$
$$\forall x \in X_0 \setminus \{0\} : \qquad \boldsymbol{(}u \bullet x, x\boldsymbol{)} \neq 0. \tag{5.6}$$

*We define*

$$\begin{aligned} \cdot : X_0 \times V \to X_0 : x \cdot v = v \bullet (u \bullet x) \\ h : X_0 \times X_0 \to V : (x, y) \mapsto \boldsymbol{(}u \bullet x, y\boldsymbol{)} . \end{aligned}$$

*Then $(k, V, q, u, X_0, \cdot, h)$ is a quadrangular algebra.*

*Proof.* Notice that $e_0, e_1 \in J$ are supplementary proper idempotents and that $u \in J_{1/2}$ such that $u^2 = q(u)1 = 1$. Thus we can apply Lemma 5.5 with $J_0 = ke_0$, $J_{1/2} = V$, $J_1 = ke_1$. It follows from (5.3) and (5.4) that the maps $\cdot$ and $h$ are well defined.

To start we show that $U_u(v) = v^\sigma$ for all $v \in J_{1/2}$ with $\sigma$ as in Definition 2.4:

$$\begin{aligned} U_u(v) &= 2u(uv) - v \\ &= u(f(u, v)1) - v \\ &= f(u, v)u - v = v^\sigma . \end{aligned}$$

We verify that all the axioms of a quadrangular algebra given in Definition 2.4 hold.

(A1) This follows from bi-linearity of $\bullet$.
(A2) Let $x \in X_0$, then $x \cdot u = u \bullet (u \bullet x) = u^2 \bullet x = 1 \bullet x = x$.
(A3) Let $x \in X_0$ and $v \in J_{1/2}$, then

$$(x \cdot v) \cdot v^\sigma = U_u(v) \bullet (u \bullet (v \bullet (u \bullet x)))$$

$$\begin{aligned}&= u \bullet (v \bullet (v \bullet (u \bullet x))) \\ &= \tfrac{1}{2} f(v,v)\ u \bullet (1 \bullet (u \bullet x)) \\ &= q(v)x.\end{aligned}$$

(B1) This follows by from bilinearity of $(.,.)$.

(B2) Let $x, y \in X_0$ and $v \in J_{1/2}$, then by applying consecutively (1.1); (5.2); (A2) and (5.1) we find

$$\begin{aligned}& h(x, y \cdot v) = h(y, x \cdot v) + f(h(x,y),u)v \\ \Longleftrightarrow\quad & (u \bullet x, v \bullet (u \bullet y)) = (u \bullet y, v \bullet (u \bullet x)) + f((u \bullet x, y), u)v \\ \Longleftrightarrow\quad & (u \bullet x, v \bullet (u \bullet y)) \\ & \quad + (v \bullet (u \bullet x), u \bullet y) = 2((u \bullet x, y)\, u)v \\ \Longleftrightarrow\quad & 2v\,(u \bullet x, u \bullet y) = ((u \bullet x, u \bullet y) + (x,y))v \\ \Longleftrightarrow\quad & vU_u\,(x,y) = (x,y)\, v\end{aligned}$$

From (5.4) we know that $(x,y) = te_0$ for some $t \in k$. It follows from Definition 1.12 that

$$vU_u(te_0) = v(te_1) = \tfrac{1}{2} tv = (te_0)v.$$

(B3) Let $x, y \in X_0$ and $v \in J_{1/2}$, then by applying consecutively (1.1); (5.2) we find

$$\begin{aligned}& f(h(x \cdot v, y), u) = f(h(x,y), v) \\ \Longleftrightarrow\quad & f((u \bullet (v \bullet (u \bullet x)), y), u) = f((u \bullet x, y), v) \\ \Longleftrightarrow\quad & (u \bullet (v \bullet (u \bullet x)), y)\, u = (u \bullet x, y)\, v \\ \Longleftrightarrow\quad & (u \bullet (v \bullet (u \bullet x)), u \bullet y) \\ & \quad + (v \bullet (u \bullet x), y) = (u \bullet x, v \bullet y) + (v \bullet (u \bullet x), y) \\ \Longleftrightarrow\quad & (u \bullet (v \bullet (u \bullet x)), u \bullet y) = (u \bullet x, v \bullet y)\end{aligned}$$

Since $U_u = \sigma$ on $J_{1/2}$ is an involution, by (5.2) this is equivalent to

$$\begin{aligned}\Longleftrightarrow\quad & (v \bullet (u \bullet x), y) = (x, u \bullet (v \bullet y)) \\ \Longleftrightarrow\quad & (v \bullet (u \bullet x), v \bullet (v \bullet y)) = q(v)\,(u \bullet (u \bullet x), u \bullet (v \bullet y))\end{aligned}$$

the last equivalence follows from (1.1). From (5.4) we know that $(u \bullet x, v \bullet y) = te_1$ for some $t \in k$. The last equation reduces to

$$U_v(te_1) = q(v)U_u(te_1)$$

This holds since $U_v(e_1) = q(v)e_0$ and $U_u(e_1) = e_0$ by Definition 1.12.

(C) $\theta(x,v) := \frac{1}{2}\,(u \bullet x, v \bullet (u \bullet x))$.

(D1) Let $x \in X_0$ and $v \in J_{1/2}$. Since $f$ is non-degenerate, we have $V = ku \oplus u^\perp$. Now

$$\begin{aligned} & x \cdot h(x, x \cdot v) = (x \cdot h(x,x)) \cdot v \\ \iff \quad & \left(u \bullet x, v \bullet (u \bullet x)\right) \bullet (u \bullet x) = v \bullet (u \bullet (\left(u \bullet x, x\right) \bullet (u \bullet x))) \end{aligned}$$

Since this expression is linear in $v$ and trivial for $v \in ku$, we can assume $v \in u^\perp$ and thus $f(u,v) = uv = 0$ and hence $u \bullet (v \bullet x) = -v \bullet (u \bullet x)$. In this case we continue as follows:

$$\begin{aligned} \iff \quad & -\left(u \bullet x, u \bullet (v \bullet x)\right) \bullet (u \bullet x) = v \bullet (U_u \left(u \bullet x, x\right) \bullet x) \\ \iff \quad & U_u \left(x, v \bullet x\right) \bullet (u \bullet x) = -v \bullet (\left(x, u \bullet x\right) \bullet x) \\ \iff \quad & u \bullet (\left(x, v \bullet x\right) \bullet x) = -v \bullet (\left(x, u \bullet x\right) \bullet x) \\ \iff \quad & \left(x, v \bullet x\right) \bullet x = -u \bullet (v \bullet (\left(x, u \bullet x\right) \bullet x)) \\ \iff \quad & \left(x, v \bullet x\right) \bullet x = v \bullet (u \bullet (\left(x, u \bullet x\right) \bullet x)). \end{aligned}$$

This is exactly (5.5).

(D2) This is assumption (5.6). □

For the remaining part of this section we assume that $\mathrm{char}(k) \neq 2, 3$. In the following theorem we prove that an arbitrary 'anisotropic' non-trivial $J$-ternary algebra(see Definition 3.33), where $J$ is a reduced spin factor, satisfies the assumptions of Theorem 5.6. It follows that we can construct quadrangular algebras from $J$-ternary algebras.

**Theorem 5.7.** *Let* $\mathrm{char}(k) \neq 2, 3$. *Let $J$ be the reduced spin factor of the non-degenerate anisotropic quadratic space $(k, V, q)$ with basepoint $u$. Let $X$ be a non-trivial $J$-ternary algebra such that $\left(u \bullet x, x\right) \neq 0$ for all $x \in X_0 \backslash \{0\}$, here $X_0$ is defined as in Lemma 5.5.*

*Then $X_0$ satisfies* (5.5). *Therefore $(k, V, q, u, X_0, \cdot, h)$ is a quadrangular algebra, with $\cdot$ and $h$ as in Theorem 5.6.*

*Proof.* Let $i \in \{0, 1\}$, we will first show that for all $x \in X_i$ and $v \in J_{1/2}$

$$v \bullet \left(x, x, x\right) = 3 \left(v \bullet x, x\right) \bullet x = 3 \left(v \bullet x, x, x\right) = \frac{3}{2} \left(x, x, v \bullet x\right). \tag{5.7}$$

From (JT2) we find that $e_1 \bullet \left(x, v \bullet x, x\right) = t \left(x, v \bullet x, x\right)$ with $t = -1$ if $i = 0$ and $t = 2$ if $i = -1$, therefore

$$\left(x, v \bullet x, x\right) = 0.$$

Using (JT3) and (JT4) respectively we get

$$\left(x, v \bullet x\right) \bullet x = \left(v \bullet x, x, x\right) - \left(x, x, v \bullet x\right) \text{ and } \left(x, v \bullet x\right) \bullet x = -\left(v \bullet x, x, x\right).$$

Combining these equations, we obtain

$$\left(x, x, v \bullet x\right) = 2\left(v \bullet x, x, x\right) = 2\left(v \bullet x, x\right) \bullet x.$$

From (JT2) we have

$$v \bullet \left(x, x, x\right) = \left(v \bullet x, x, x\right) + \left(x, x, v \bullet x\right).$$

Combining the two last formulas proves (5.7). Since $\mathrm{char}(k) \neq 3$, it follows from (5.7) that (5.5) is equivalent with

$$v \bullet \left(x, x, x\right) = v \bullet (u \bullet (u \bullet \left(x, x, x\right))).$$

Since this last equation holds we have proved that (5.5) holds. It follows from Definition 3.33 that $X$ is a special $J$-module that satisfies (5.2). Since we assume that (5.6), we can apply Theorem 5.6. □

**Remark 5.8.** (i) We give an alternative characterization of an 'anisotropic' $J$-ternary algebra. We show that for all $x \in X_0$, $\left(u \bullet x, x\right) = 0 \iff \left(x, x, x\right) = 0$.
First remark that if $v \bullet x = 0$ we have $v = 0$ or $x = 0$: it follows from $v \bullet x = 0$ that $v \bullet (v \bullet x) = q(v)x = 0$, since $q$ is anisotropic we have $v = 0$ or $x = 0$. Now we have

$$\begin{aligned} & \left(u \bullet x, x\right) = 0 \\ \iff \quad & \left(u \bullet x, x\right) \bullet x = 0 \\ \iff \quad & u \bullet \left(x, x, x\right) = 0 \quad \text{by (5.7)} \\ \iff \quad & \left(x, x, x\right) = 0. \end{aligned}$$

(ii) In the previous theorem we have to demand that $\left(x, x, x\right) \neq 0$ for all $x \in X_0 \setminus \{0\}$, because there exist $J$-ternary algebras which fulfill all the requirements but where $\left(x, x, x\right) = 0$ for some $x \neq 0 \in X_0$. For example, consider [ABG02, Example 6.81] with the zero skew-hermitian form. Examples like this we clearly want to avoid.

## 5.3 Construction of pseudo-quadratic spaces

Let $k$ be a field of characteristic not 2.

We rely on the example [ABG02, 6.81] to obtain a quadrangular algebra of pseudo-quadratic form type using Theorem 5.6. In combination with Section 5.4 this will show that all quadrangular algebras of characteristic not 2 can be obtained using the construction in Theorem 5.6.

Let $(k, L, q, u, X, \cdot, h)$ be a quadrangular algebra that is obtained from a pseudo-quadratic space $(L, \sigma, X, h, \pi)$ (see Theorem 2.12). It follows that $q : L \to k : \ell \mapsto \ell\ell^\sigma$ is a quadratic form.

**Definition 5.9.** (i) Define $J = \mathcal{H}(M_2(L), \sigma T)$ (see Definition 1.13 and Remark 1.14); this Jordan algebra is a reduced spin factor of the quadratic form

$$Q : J_{1/2} \to k : \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \mapsto q(\ell).$$

As before we define $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in J : u$ is a basepoint of $Q$.

(ii) Define $\widetilde{X} = X^2$, the $1 \times 2$ row vectors over X.

(iii) We define the action of $J$ on $\widetilde{X}$ as[2] $j \bullet x := xj \in \widetilde{X}$ for $j \in J, x \in \widetilde{X}$.

(iv) Define $\psi : \widetilde{X} \times \widetilde{X} \to M_2(L) : \psi([x_1, x_2], [y_1, y_2]) := \big[h(x_i, y_j)\big]$, now we define the skew-symmetric bilinear map $\widetilde{X} \times \widetilde{X} \to J$

$$\begin{aligned} \boldsymbol{(}[x_1, x_2], [y_1, y_2]\boldsymbol{)} &:= \psi([x_1, x_2], [y_1, y_2]) - \psi([y_1, y_2], [x_1, x_2]) \\ &= \begin{bmatrix} h(x_1, y_1) - h(y_1, x_1) & h(x_1, y_2) - h(y_1, x_2) \\ -h(y_2, x_1) + h(x_2, y_1) & h(x_2, y_2) - h(y_2, x_2) \end{bmatrix}. \end{aligned}$$

With respect to $e_1$ we have

$$\widetilde{X}_0 = \{[x, 0] \mid x \in X\}, \ \widetilde{X}_1 = \{[0, x] \mid x \in X\}.$$

**Lemma 5.10.** *The space $\widetilde{X}$ is a non-trivial special $J$-module with skew-symmetric bilinear form $\boldsymbol{(}\cdot, \cdot\boldsymbol{)}$ that satisfies* (5.2), (5.5) *and* (5.6) *hold as well.*

*Under the identifications*

$$J_{1/2} \cong L : \ell \leftrightarrow \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \text{ and } \widetilde{X}_0 \cong X : [x, 0] \leftrightarrow x,$$

*the quadrangular algebra defined in Theorem 5.6 is exactly the quadrangular algebra we started with.*

*Proof.* Verifying that $\widetilde{X}$ is a special $J$-module and satisfies (5.2) requires some straightforward calculations. We will verify (5.5) and (5.6).

Define $v = \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \in J_{1/2}, \widetilde{x} = [x, 0] \in \widetilde{X}_0$. Notice that $u \bullet \widetilde{x} = [0, x] \in X_1$, and

$$\boldsymbol{(}u \bullet \widetilde{x}, \widetilde{x}\boldsymbol{)} = \begin{bmatrix} 0 & -h(x, x) \\ h(x, x) & 0 \end{bmatrix}.$$

Hence $\boldsymbol{(}u \bullet \widetilde{x}, \widetilde{x}\boldsymbol{)}$ is equal to 0 if and only if $h(x, x) = 0$. It follows from the anisotropity of the pseudo-quadratic space that $\pi(x) = \frac{1}{2}h(x, x)$ is anisotropic (see Remark 2.5.(ii)), so (5.6) holds.

[2] On the right hand side the usual matrix multiplication is used.

Condition (5.5) holds since

$$(v \bullet \widetilde{x}, \widetilde{x}) \bullet \widetilde{x} = [0, -xh(x,x)l^\sigma] = v \bullet [-xh(x,x), 0] = v \bullet (u \bullet ((u \bullet \widetilde{x}, \widetilde{x}) \bullet \widetilde{x})).$$

From Theorem 5.6 we conclude that $(k, J_{1/2}, Q, u, \widetilde{X}_0, \cdot, \widetilde{h})$ is a quadrangular algebra with

$$[x,0] \cdot \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} = [0, x \cdot \ell],$$

$$\widetilde{h}([x,0],[y,0]) = (u \bullet [x,0],[y,0]) = \begin{bmatrix} 0 & -h(y,x) \\ h(x,y) & 0 \end{bmatrix}. \qquad \square$$

## 5.4 Construction of quadrangular algebras of type $E_6, E_7, E_8$

Let $k$ be a field of characteristic not 2.

In this section we work with quadrangular algebras of type $E_6, E_7, E_8$. In Section 2.2.3 we described the structure of the vector spaces $X, L$ and the maps $\cdot$ and $h$ defined in [TW02]. In this section we give a new coordinate-free construction of the $X$, $L$, $\cdot$ and $h$ starting from the tensor product of composition algebras.

### 5.4.1 A characterization of quadratic forms of type $E_6, E_7, E_8$

Let $C_1$ and $C_2$ be division composition algebras, in Definition 3.12 we introduced the Albert form on $\mathcal{S} \subset C_1 \otimes C_2$. The case where the Albert form has Witt index one will give rise a new characterization of quadratic forms of type $E_6$, $E_7$ and $E_8$. In the following lemma we give three equivalent characterizations of the fact that the Witt index of $q_A$ is equal to one.

**Lemma 5.11.** *Let $C_1$ be an octonion division algebra with norm $q_1$ and let $C_2$ be a separable quadratic field extension, quaternion division algebra or an octonion division algebra, with norm $q_2$. The following are equivalent:*

(i) *$C_1$ and $C_2$ contain isomorphic separable quadratic field extensions, but $C_1$ and $C_2$ do not contain isomorphic quaternion algebras.*

(ii) *The linkage number[3] of $q_1$ and $q_2$ is one, i.e. $q_1$ and $q_2$ are 1-linked but not 2-linked.*

(iii) *The Witt index of the Albert form $q_A$ of $C_1 \otimes_k C_2$ is equal to one.*

*Proof.* Since the Witt index of $q_A$ is one less than the Witt index of $q_1 \perp -q_2$, the equivalence of (ii) and (iii) is given by a result of Elman–Lam (see for example [Lam05, Theorem X.5.13]).

[3] see Definition 1.1.

The following observations follow from [SV00, Prop. 1.5.1]. Let $C$ be a composition algebra over $k$ with norm $q$.

If $\dim(C) = 4$ or 8: $C$ contains a separable extension field isomorphic to $k(\mathbf{i})/(\mathbf{i}^2 - a)$ with $a \in k$ if and only if there exists a Pfister form $\varphi$, of dimension 2 or 4 respectively, such that $q \cong \langle\langle -a \rangle\rangle \otimes \varphi$.

If $\dim(C) = 8$: $C$ contains a quaternion algebra isomorphic to $(a, b)_k$ with $a, b \in k$ if and only if $q \cong \langle\langle -a, -b, -c \rangle\rangle$ for some $c \in k$.

From this it follows immediately that (i) and (ii) are equivalent. □

**Definition 5.12.** We define the *linkage number* of $C_1$ and $C_2$ as the linkage number of their norm forms $q_1$ and $q_2$. Hence $C_1$ and $C_2$ have linkage number one if one of the three equivalent conditions of Lemma 5.11 is satisfied.

**Remark 5.13.** Suppose that $C_1$ and $C_2$ have linkage number one. Notice that it is possible that $C_1$ and $C_2$ contain more than one isomorphic separable quadratic field extension up to isomorphism.

The following theorem gives a new way to describe quadratic forms of type $E_6, E_7$ and $E_8$ (see Definition 2.13). This illuminating observation was made by Skip Garibaldi.

**Theorem 5.14.** *Let $q$ be an anisotropic form over $k$ of dimension $6, 8$ or $12$. Then $q$ is of type $E_6$, $E_7$ or $E_8$ respectively if and only if there exist an octonion division algebra $C_1$ and a division composition algebra $C_2$, of dimension $2, 4$ or $8$ respectively, that have linkage number one such that $q$ is similar to the anisotropic part of the Albert form of $C_1 \otimes_k C_2$.*

The 'only if'-direction of this theorem is proved in the following, more technical, lemma.

**Lemma 5.15.** *We consider a quadratic form $q$ of type $E_6$, $E_7$ or $E_8$. Let $N$ denote the norm of a separable quadratic field extension $E = k(x)/(x^2 - a)$ for $a \notin k^2$;*

(i) *If $q = N \otimes \langle 1, s_2, s_3 \rangle$ of type $E_6$, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = E.$$

(ii) *If $q = N \otimes \langle 1, s_2, s_3, s_4 \rangle$ of type $E_7$, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = (a, s_2 s_3 s_4)_k.$$

(iii) *If $q = N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle$ of type $E_8$, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = (a, -s_4 s_6, -s_5 s_6)_k.$$

*Then $q$ is similar to the anisotropic part of the Albert form of $C_1 \otimes_k C_2$ and $C_1$ and $C_2$ are division algebras that have linkage number 1.*

*Proof.* Denote the norm form of $C_1$ by $q_1$, the norm form of $C_2$ by $q_2$ and the Albert form of $C_1 \otimes_k C_2$ by $q_A$. In the case that $q$ is of type $E_8$ we have $q_1 = N \otimes \langle\langle s_2, s_3\rangle\rangle$ and $q_2 = N \otimes \langle\langle s_4s_6, s_5s_6\rangle\rangle$; the following identity follows by multiplying the quadratic forms in Lemma 2.14 by $N$.

$$q \perp 2\mathbb{H} \sim q_A \perp \mathbb{H} \sim q_1 \perp -q_2, \tag{5.8}$$

The above formula holds as well in the cases $E_6$ and $E_7$, the verification can be done in a similar way as in the $E_8$-case.

Note that $q$ is anisotropic. Therefore $q_1 \perp -q_2$ has Witt index 2; it follows that $q_1$ and $q_2$ are anisotropic and both $C_1$ and $C_2$ are division algebras. It follows from (5.8) that $q_A$ has Witt index 1, and now Lemma 5.11 implies that $C_1$ and $C_2$ have linkage number 1. □

*Proof of Theorem 5.14.* The 'only if'-direction is proven in the Lemma above. The 'if'-direction follows in a similar way. We elaborate the case where $C_1$ and $C_2$ are octonion division algebras.

Since $C_1$ and $C_2$ contain an isomorphic field extension, by [SV00, Prop. 1.5.1] we can assume that $C_1 = (a, b_1, c_1)_k$ and $C_2 = (a, b_2, c_2)_k$ for some $a, b_1, b_2, c_1, c_2 \in k$. We denote the Albert form of $C_1 \otimes_k C_2$ by $q_A$.

Define $N := \langle\langle -a\rangle\rangle$, this is anisotropic since $C_1$ is division. By going through the computation in Lemma 2.14 from bottom to top with

$$s_2 := -b_1, s_3 := -c_1, s_4 := \frac{1}{b_1c_1c_2}, s_5 := \frac{1}{b_1c_1b_2}, s_6 := -b_1b_2c_1c_2$$

we find that $q_A$ is similar to $N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6\rangle \perp \mathbb{H}$. Since the Witt index of $q_A$ is one, $N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6\rangle$ is the anisotropic part of $q_A$; since $s_2s_3s_4s_5s_6 = -1$ it is of type $E_8$. □

### 5.4.2 The construction

In order to construct quadrangular algebras of type $E_6, E_7$ and $E_8$ we follow Example 6.82 in [ABG02] closely. In *loc. cit.* a $J$-ternary algebra is constructed out of the structurable algebra $C_1 \otimes_k C_2$ in characteristic zero as in Theorem 3.34, but this restriction is not necessary. Below, we elaborate this Example 6.82 in [ABG02] in full detail in arbitrary characteristic.

First we give a intuitive motivation of the approach we will be following in our construction.

**Remark 5.16.** Let $C_1$ be an octonion division algebra and $C_2$ a separable quadratic field extension, quaternion division algebra or octonion division algebra and assume that $C_1$ and $C_2$ have linkage number one.

The dimension of $C_1 \otimes_k C_2$ is 16, 32 or 64, respectively. The space of skew-elements is $\mathcal{S} = S_1 \otimes 1 + 1 \otimes S_2$ and has dimension 8, 10 or 14, respectively. Let $(k, L, q, 1, \widetilde{X}, \cdot, h)$ be a quadrangular algebra of type $E_6$, $E_7$ or $E_8$, respectively. We summarize some dimensions:

| | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|
| $\dim_k \mathcal{S}$ | 8 | 10 | 14 |
| $\dim_k L$ | 6 | 8 | 12 |
| $\dim_k(C_1 \otimes_k C_2)$ | 16 | 32 | 64 |
| $\dim_k \widetilde{X}$ | 8 | 16 | 32 |

We see that in all three cases $\dim_k \mathcal{S} = \dim_k L + 2$ and $\dim_k(C_1 \otimes C_2) = 2 \dim_k \widetilde{X}$.

In Theorem 5.6 we considered some objects the dimensions of which behave similarly: Let $J$ be a Jordan algebra of reduced spin type with basepoint and let $X$ be a special $J$-module. Then $\dim_k(J) = \dim J_{1/2} + 2$ and $\dim_k X = 2 \dim_k X_0$.

From Lemma 5.15, the Albert form from $\mathcal{S}$ to $k$ can be written as the sum of a hyperbolic plane and a quadratic form of type $E_6, E_7$ or $E_8$, respectively. Note that a hyperbolic plane is two-dimensional.

In the following pages, we will give $\mathcal{S}$ the structure of a reduced spin factor of a quadratic form of type $E_6$, $E_7$ or $E_8$, respectively, and identify $J_{1/2}$ with $L$. Then we will give $C_1 \otimes C_2$ the structure of a special $J$-module equipped with a bilinear skew-hermitian form, and identify $(C_1 \otimes C_2)_0$ with $\widetilde{X}$.

We start by fixing some notation.

**Notation 5.17.** (i) We fix a basis for the composition algebras $C_1$ and $C_2$ that have linkage number 1. We let $C_1$ be the octonion division algebra

$$C_1 = \langle 1, \mathbf{i}_1, \mathbf{j}_1, \mathbf{i}_1\mathbf{j}_1, \mathbf{k}_1, \mathbf{i}_1\mathbf{k}_1, \mathbf{j}_1\mathbf{k}_1, (\mathbf{i}_1\mathbf{j}_1)\mathbf{k}_1 \rangle.$$

If $C_2$ is a separable quadratic field extension, we define $C_2 = \langle 1, \mathbf{i}_2 \rangle$. In the case $C_2$ is a quaternion division algebra we define $C_2 = \langle 1, \mathbf{i}_2, \mathbf{j}_2, \mathbf{i}_2\mathbf{j}_2 \rangle$. In the case $C_2$ is an octonion division algebra we define

$$C_2 = \langle 1, \mathbf{i}_2, \mathbf{j}_2, \mathbf{i}_2\mathbf{j}_2, \mathbf{k}_2, \mathbf{i}_2\mathbf{k}_2, \mathbf{j}_2\mathbf{k}_2, (\mathbf{i}_2\mathbf{j}_2)\mathbf{k}_2 \rangle.$$

Since $C_1$ and $C_2$ have linkage number 1, we can choose these bases in such a way that

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 =: a \in K.$$

(ii) From now on we denote $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}$ for the set of skew-elements of $C_1$, $C_2$ and $C_1 \otimes_k C_2$, respectively.
(iii) We denote the Albert form of $C_1 \otimes_k C_2$ by $q_A : \mathcal{S} \to k$ and its associated bilinear form by $f_A$.
(iv) Let $V := \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^\perp$ denote the orthogonal complement of the subspace $\langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle$ of $\mathcal{S}$ with respect to the non-degenerate bilinear form $f_A$.

We want to make $\mathcal{S}$ into a Jordan algebra of reduced spin type. In particular it should contain supplementary proper idempotents $e_0$ and $e_1$ and an element $u \in J_{1/2}$ such that $u^2$ is the identity. It will become clear that the elements constructed in the following lemma will be the ones we need.

**Lemma 5.18.** *Let $u \in V \setminus \{0\}$ be arbitrary. Then up to order and up to scalars, there exists a unique pair $(e_0, e_1)$ of elements in $\mathcal{S}$ such that*

$$q_A(e_0) = q_A(e_1) = 0,$$
$$f_A(e_0, V) = f_A(e_1, V) = 0, \quad f_A(e_0, e_1) = -q_A(u) \neq 0.$$

*Explicitly, there exists an element $\lambda \in k$ such that $(\lambda e_0, \lambda^{-1} e_1)$ is equal to*

$$\left( \mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2, \frac{q_A(u)}{4a} (\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2) \right).$$

*Proof.* Since $q_A$ has Witt index one, $q_A$ is anisotropic on $V = \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^\perp$. Hence $q_A(u) \neq 0$.

We demand that $e_0, e_1$ are isotropic elements in $V^\perp = \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle$. This implies that they are of the form $\lambda(\mathbf{i}_1 \otimes 1 \pm 1 \otimes \mathbf{i}_2)$. Since $f_A(e_0, e_1)$ should be different from 0 we can take without loss of generality $e_0 = \lambda_0(\mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2)$ and $e_1 = \lambda_1(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2)$ for some $\lambda_0, \lambda_1 \in k \setminus \{0\}$. Now we determine the scalars $\lambda_0, \lambda_1$, such that $f_A(e_0, e_1) = -q_A(u)$. We have

$$f_A(\lambda_0(\mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2), \lambda_1(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2)) = \lambda_0 \lambda_1 (-4a).$$

So we find that $\lambda_1 = \frac{q_A(u)}{4a\lambda_0}$. □

Since $\dim \mathcal{S} = \dim V + 2$, we want to make $V$ into a quadratic space. If we want that $u.u = 1$ in the Jordan algebra of reduced spin type we will define, the element $u$ should be the basepoint of the quadratic form that determines the reduced spin factor. In the following definition we define a Jordan algebra on $\mathcal{S}$; in Lemma 5.20 we will show that this Jordan algebra has a natural interpretation in the endomorphism ring of $C_1 \otimes_k C_2$.

**Definition 5.19.** Let $u \in V \setminus \{0\}$ and

$$e_0 = \mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2, \quad e_1 = \frac{q_A(u)}{4a}(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2).$$

(i) We define a quadratic form on the vector space $V$,

$$Q := \frac{1}{q_A(u)} q_A|_V.$$

We denote the corresponding bilinear form by $F$.
It follows from Theorem 5.14 that $(k, V, Q)$ is a quadratic space of type $E_6, E_7$ or $E_8$, respectively, with basepoint $u$.

(ii) We have $\mathcal{S} = ke_0 \oplus V \oplus ke_1$, we define the Jordan multiplication as in Definition 1.12:

$$\begin{aligned}(t_1e_i).(t_2e_j) &= \delta_{ij}t_1t_2e_i,\\ (te_i).v &= \tfrac{1}{2}tv,\\ v.w &= \tfrac{1}{2}F(v,w)(e_0+e_1),\end{aligned}$$

for all $i, j \in \{0, 1\}, v, w \in V, t, t_1, t_2 \in k$. We denote this Jordan algebra by $J$, this is the reduced spin factor of $(k, V, Q)$.

(iii) As $q_A(e_0 + e_1) = f_A(e_0, e_1) = -q_A(u) \neq 0$, $e_0 + e_1$ is invertible and we define

$$r := (e_0 + e_1)^{-1} = -\frac{1}{q_A(e_0 + e_1)}(e_0 + e_1)^\natural \in \mathcal{S},$$

where the inverse and $\natural$ is as in Definition 3.12. Notice that $e_0 + e_1$ is the identity in the Jordan algebra $J$ on $\mathcal{S}$, the definition of $r$ has nothing to do with the inverse in $J$.

(iv) Let $s \in \mathcal{S}$, define $L_s \in \mathrm{End}_k(C_1 \otimes_k C_2)$ as $L_s x := sx$ for all $x \in C_1 \otimes_k C_2$.

Consider the Jordan algebra of the associative algebra $\mathrm{End}_k(C_1 \otimes_k C_2)$, denoted by $\mathrm{End}_k(C_1 \otimes_k C_2)^+$. We show that the algebra of reduced spin type we defined above, is isomorphic to a Jordan subalgebra of $\mathrm{End}_k(C_1 \otimes_k C_2)^+$. The following lemma is given in [ABG02, Example 6.82] without proof; for completeness we include it here. In the case that $\mathrm{char}(k) \neq 2, 3$ part of the lemma follows as well from the proof of Theorem 3.34.

**Lemma 5.20.** *Let $s_1, s_2 \in \mathcal{S}$, we have*

$$\frac{1}{2}(L_{s_1}L_rL_{s_2}L_r + L_{s_2}L_rL_{s_1}L_r) = L_{s_1.s_2}L_r,$$

*where $s_1.s_2$ denotes the multiplication in the algebra $J$ defined in Definition 5.19.*(ii).

*Therefore $L_{\mathcal{S}}L_r$ is a Jordan subalgebra of* $\mathrm{End}_k(C_1 \otimes_k C_2)^+$ *isomorphic to $J$.*

*Proof.* We will make use of [All86b, Proposition 3.3 (3.8)]. In [All86b] only characteristic 0 is considered; however this proposition can be generalized to characteristic different from 2 without any adjustments. The proof of this proposition uses basic identities of octonions (see Lemma 1.8) and the identity $s_1(s_2(s_1x)) = (s_1s_2s_1)x$ for $x \in C_1 \otimes_k C_2$ (see Lemma 3.13).

Linearizing [All86b, Prop 3.3 (3.8)] gives

$$\begin{aligned} &L_{s_1}L_{(e_0+e_1)^\natural}L_{s_2} + L_{s_2}L_{(e_0+e_1)^\natural}L_{s_1} \\ &\qquad = -f_A(s_1, e_0+e_1)L_{s_2} - f_A(s_2, e_0+e_1)L_{s_1} + f_A(s_1, s_2)L_{e_0+e_1}. \end{aligned}$$

Since $r = (e_0+e_1)^{-1} = -\frac{1}{q_A(e_0+e_1)}(e_0+e_1)^\natural = \frac{1}{q_A(u)}(e_0+e_1)^\natural$, we find that

$$\begin{aligned} &\frac{1}{2}(L_{s_1}L_rL_{s_2}L_r + L_{s_2}L_rL_{s_1}L_r) \\ &= \frac{1}{2q_A(u)}(-f_A(s_1, e_0+e_1)L_{s_2}L_r - f_A(s_2, e_0+e_1)L_{s_1}L_r + f_A(s_1, s_2)L_{e_0+e_1}L_r). \end{aligned}$$

It follows from $u \in V = \langle e_0, e_1\rangle^\perp$, $q_A(e_0) = q_A(e_1) = 0, f_A(e_0, e_1) = -q_A(u)$, that for $i, j \in \{0, 1\}$ and for all $v, w \in V$

$$\begin{aligned} \tfrac{1}{2}(L_{e_i}L_rL_{e_j}L_r + L_{e_j}L_rL_{e_i}L_r) &= \delta_{ij}L_{e_i}L_r, \\ \tfrac{1}{2}(L_{e_i}L_rL_vL_r + L_vL_rL_{e_i}L_r) &= \tfrac{1}{2}L_vL_r, \\ \tfrac{1}{2}(L_vL_rL_wL_r + L_wL_rL_vL_r) &= \frac{f_A(v, w)}{2q_A(u)}L_{e_0+e_1}L_r. \end{aligned}$$

This is exactly the multiplication of $J$. □

In order to define an action of $J$ on $C_1 \otimes_k C_2$, we use the isomorphism of the previous Lemma.

**Definition 5.21.** We define the bilinear action

$$\bullet : \mathcal{S} \times (C_1 \otimes_k C_2) \to C_1 \otimes_k C_2 : (s, x) \mapsto L_sL_rx = s(rx).$$

We define the *skew-symmetric bilinear map*

$$(.,.) : (C_1 \otimes_k C_2) \times (C_1 \otimes_k C_2) \to \mathcal{S} : (x, y) \mapsto \psi(x, y) = x\overline{y} - y\overline{x}.$$

**Remark 5.22.** (i) After some computation we find

$$\begin{aligned} e_0 \bullet (x_1 \otimes x_2) &= \tfrac{1}{2}\left(x_1 \otimes x_2 + \frac{1}{a}\mathbf{i}_1x_1 \otimes \mathbf{i}_2x_2\right), \\ e_1 \bullet (x_1 \otimes x_2) &= \tfrac{1}{2}\left(x_1 \otimes x_2 - \frac{1}{a}\mathbf{i}_1x_1 \otimes \mathbf{i}_2x_2\right), \end{aligned}$$

for all $x_1 \in C_1, x_2 \in C_2$. Note that this is independent of the choice of the basepoint $u$.

(ii) For all $x_1, y_1 \in C_1, x_2, y_2 \in C_2$ we have,

$$\boldsymbol{(}x_1 \otimes x_2, y_1 \otimes y_2\boldsymbol{)} = f_2(x_2, y_2)\psi(x_1, y_1) \otimes 1 + 1 \otimes f_1(x_1, y_1)\psi(x_2, y_2).$$

In the following theorem we show that when applying Theorem 5.6 with $X = C_1 \otimes_k C_2$ and $J$, $\bullet$ and $\boldsymbol{(}.,.\boldsymbol{)}$ as above, we obtain indeed a quadrangular algebra of type $E_6, E_7$ or $E_8$. In the proof we make a distinction between the cases $\mathrm{char}(k) \neq 2$ and $\mathrm{char}(k) \neq 2, 3$.

When $\mathrm{char}(k) \neq 2, 3$, $C_1 \otimes_k C_2$ is a structurable algebra, thus we can apply known results from the theory of structurable algebras.

If $\mathrm{char}(k) = 3$ we can not make use of the theory of structurable algebras. Therefore we prove this in a direct way only making use of identities in octonions. Regrettably, this gives rise to lengthy computations and for one particular identity we had to rely on the computer algebra software [Sea11]. This proof does not use the fact that the characteristic is equal to 3, but only that it is different from 2.

**Theorem 5.23.** *Let* $\mathrm{char}(k) \neq 2$. *Let* $e_0, e_1, u \in \mathcal{S}$ *be as in Lemma 5.18, let the quadratic form* $Q$ *of type* $E_6, E_7, E_8$ *and the reduced spin factor* $J$ *be as in Definition 5.19. Let* $X := C_1 \otimes_k C_2$, *let* $\bullet$ *and* $\boldsymbol{(}.,.\boldsymbol{)}$ *be defined as above.*

*Then* $X$ *is a special* $J$*-module and* $\boldsymbol{(}.,.\boldsymbol{)}$ *satisfies* (5.2). *Conditions* (5.5) *and* (5.6) *of Theorem 5.6 are satisfied. As in Theorem 5.6 we define*

$$\cdot : X_0 \times V \to X_0 : x \cdot v = v \bullet (u \bullet x)$$
$$h : X_0 \times X_0 \to V : (x, y) \mapsto \boldsymbol{(}u \bullet x, y\boldsymbol{)}\,.$$

*Then* $(k, V, Q, u, X_0, \cdot, h)$ *is a quadrangular algebra of type* $E_6, E_7, E_8$.

*Proof.* We have from Lemma 3.13 that $(e_0 + e_1) \bullet x = x$ for all $x \in X$. The fact that $X$ is a special $J$-module now follows from Lemma 5.20. It follows from Theorem 2.8 that if $(k, V, Q, u, X_0, \cdot, h)$ is a quadrangular algebra, it has to be of type $E_6, E_7, E_8$ due to the dimension of $V$.

### $\mathrm{char}(k) \neq 2, 3$

Since $C_1 \otimes_k C_2$ is a structurable algebra, it is a $J$-ternary algebra by Theorem 3.34. Notice that $J$, $\bullet$ and the skew-symmetric bilinear map defined above are identical to the ones used in Theorem 3.34. By applying Theorem 5.7 we find that (5.2) and (5.5) are satisfied.

For the proof of (5.6) we refer to the general characteristic case below.

### $\mathbf{char}(\boldsymbol{k}) \neq \mathbf{2}$

We first verify that the second identity of (5.2) holds, this takes a rather lengthy but straightforward computation:

Since the condition is linear in $x$ and $y$, one can choose $x = x_1 \otimes x_2$ and $y = y_1 \otimes y_2$ for $x_1, y_1 \in C_1, x_2, y_2 \in C_2$. Let $s = s_1 \otimes 1 + 1 \otimes s_2 \in \mathcal{S}$ and denote $r = r_1 \otimes 1 + 1 \otimes r_2$, instead of using its definition with coordinates. Using Remark 5.22.(ii) it is not hard to show that the following identities hold for $i \in \{1, 2\}$

- $f_i(s_i, \psi(x_i, y_i)) = -2f_i(s_i x_i, y_i)$,
- $\psi(s_i x_i, y_i) + \psi(s_i y_i, x_i) = 2s_i f_i(x_i, y_i)$,
- $f_i(s_i(r_i x_i), y_i) + f_i(s_i(r_i y_i), x_i) = -f_i(s_i, r_i) f_i(x_i, y_i)$.

Using these identities, (5.2) can be simplified to

$$\begin{aligned}\psi(s_i(r_i x_i), y_i) &- \psi(s_i(r_i y_i), x_i) \\ &= -f_i(r_i, \psi(x_i, y_i))s_i + f_i(s_i, \psi(x_i, y_i))r_i - f_i(s_i, r_i)\psi(x_i, y_i),\end{aligned}$$

and this identity can be checked using Lemma 1.8, especially the Moufang identities (v).

We were not able to verify (5.5) by hand. The problem is that (5.5) has degree 3 in $x$, so we can not assume that $x$ is of the form $e_0 \bullet (x_1 \otimes x_2)$. We did a computation based on a coordinatization of $X$, we used the software [Sea11] to do the symbolic computations:

Now $x$ is an arbitrary element in $X_0 = e_0 \bullet X$, therefore $x$ is a sum of elements of the form $a\ x_1 \otimes x_2 + \mathbf{i}_1 x_1 \otimes \mathbf{i}_2 x_2$ (see Remark 5.22). We implemented octonions and the tensor product of two octonions in Sage in a symbolic way (see Appendix B), and we verified that (5.5) holds.

The only fact that remains to be verified is (5.6). In fact, this is exactly axiom (D2) and in the proof of Theorem 5.6 the condition (5.6) is not used to prove any of the other axioms. Since we already know that the axioms A-B-C-D1 are true, we will use these to prove $(u \bullet x, x) \neq 0$ for all $x \in X_0 \backslash \{0\}$.

First we show that

$$\text{there exists an } x \in X_0 \text{ such that } (u \bullet x, x) \neq 0. \tag{5.9}$$

Let $x = e_0 \bullet (x_1 \otimes x_2) \in X_0, u = s_1 \otimes 1 + 1 \otimes s_2 \in V$, with some calculation using Lemma 1.8, Remark 5.22 and the coordinate expression for $r$ we find that

$$(u \bullet x, x) = \frac{1}{4a}(q_2(x_2)\psi(s_1 x_1, \mathbf{i}_1 x_1) \otimes 1 + 1 \otimes q_1(x_1)\psi(s_2 x_2, \mathbf{i}_2 x_2)).$$

Since $C_1$ and $C_2$ are division algebras, it is enough to show that for all $y \in C_1 \setminus \{0\}$ we have $\psi(s_1 y, \mathbf{i}_1 y) \neq 0$. We assume that $y \neq 0$ and

$\psi(s_1y, \mathbf{i}_1y) = 0$ and deduce a contradiction.

$$\begin{aligned}
& \psi(s_1y, \mathbf{i}_1y) = 0 & \\
\Rightarrow\quad & (s_1y)(\overline{y}\mathbf{i}_1) - (\mathbf{i}_1y)(\overline{y}s_1) = 0 & \\
\Rightarrow\quad & 2(s_1y)(\overline{y}\mathbf{i}_1) = f_1(\mathbf{i}_1y, s_1y)1 & \text{since } y\overline{z} + z\overline{y} = f_1(y,z)1 \\
\Rightarrow\quad & s_1y = \tfrac{1}{2}f_1(\mathbf{i}_1y, s_1y)(\overline{y}\mathbf{i}_1)^{-1} & \text{since } \overline{y}\mathbf{i}_1 \neq 0 \\
\Rightarrow\quad & s_1y = -\frac{f_1(\mathbf{i}_1y, s_1y)}{2q_1(\overline{y}\mathbf{i}_1)}\mathbf{i}_1y & \text{since } y^{-1} = \frac{1}{q_1(y)}\overline{y} \\
\Rightarrow\quad & s_1 = -\frac{f_1(\mathbf{i}_1y, s_1y)}{2q_1(\overline{y}\mathbf{i}_1)}\mathbf{i}_1. &
\end{aligned}$$

This is a contradiction since $s_1 \perp \mathbf{i}_1$.

The rest of the proof is inspired by the proof given in [TW02, Theorem 13.47].

We fix an arbitrary $x \neq 0 \in X_0$, notice that we no longer assume that $x$ has the form $e_0 \bullet (x_1 \otimes x_2)$. We suppose that $(u \bullet x, x) = 0$ and aim to get a contradiction. It follows [4] from (5.5) that $(v \bullet x, x) = 0$ for all $v \in V$.

We first show that there exists an element $y \in X_0$ such that $(u \bullet x, y) \neq 0$. Suppose that $(u \bullet x, X_0) = 0$. It follows from (B2) that for all $y \in X_0, v \in V$

$$\begin{aligned}
(u \bullet x, v \bullet (u \bullet y)) &= (u \bullet y, v \bullet (u \bullet x)) \\
&= -(v \bullet (u \bullet x), u \bullet y) \\
&= -U_u\,(u \bullet (v \bullet (u \bullet x)), y)
\end{aligned}$$

Therefore $(u \bullet (x \cdot v), X_0) = 0$ and by repeating this procedure we obtain

$$(u \bullet (x \cdot C(Q,u)), X_0) = 0.$$

From Definition 2.6.(ii) and Theorem 2.16 it follows that $X_0$ is an irreducible $C(Q,u)$-module, therefore we obtain $(u \bullet X_0, X_0) = 0$. This contradicts (5.9).

From now on we assume that $y \in X_0$ is such that $(u \bullet x, y) \neq 0$. Next we show that

$$(x \cdot (u \bullet x, y)) \cdot v = x \cdot (u \bullet x, y \cdot v)\,. \tag{5.10}$$

Since this identity is trivial for $u = v$, we assume $v \perp u$. Then (5.10) is equivalent to

$$\Longleftrightarrow \qquad v \bullet u \bullet (u \bullet x, y) \bullet u \bullet x = (u \bullet x, v \bullet u \bullet y) \bullet u \bullet x$$

[4] Since $q_A$ is anisotropic on $V$ : $v \bullet x = 0 \iff v \bullet v \bullet x = q_A(v)x = 0 \iff v = 0$ or $x = 0$.

$$\begin{aligned}
&\iff & v \bullet U_u (u \bullet x, y) \bullet x &= - (u \bullet x, u \bullet v \bullet y) \bullet u \bullet x \\
&\iff & v \bullet (x, u \bullet y) \bullet x &= -U_u (x, v \bullet y) \bullet u \bullet x \\
&\iff & v \bullet (x, u \bullet y) \bullet x &= -u \bullet (x, v \bullet y) \bullet x \\
&\iff & v \bullet u \bullet (u \bullet y, x) \bullet x &= (v \bullet y, x) \bullet x. \qquad (5.11)
\end{aligned}$$

We consider (5.5) for $y + tx$ for a parameter $t \in k$, we compare the terms that have degree one in $t$ using the assumption that $(v \bullet x, x) = 0$ for all $v \in V$, and we get

$$\begin{aligned}
& & (v \bullet x, y) \bullet x + (v \bullet y, x) \bullet x &= v \bullet u \bullet ((u \bullet x, y) + (u \bullet y, x)) \bullet x \\
&\iff & 2 (v \bullet y, x) \bullet x + 2(v (x, y)) \bullet x &= 2v \bullet u \bullet ((u \bullet y, x) + (u (x, y))) \bullet x \\
&\iff & (v \bullet y, x) \bullet x &= v \bullet u \bullet (u \bullet y, x) \bullet x,
\end{aligned}$$

since $v (x, y) \in J_1$ and $x \in X_0$. This proves (5.11).

From (5.10) we have $(x \cdot (u \bullet x, y)) \cdot V \subseteq x \cdot V$; since $(u \bullet x, y) \neq 0$, the dimension of those two vector spaces is equal and we find that

$$(x \cdot (u \bullet x, y)) \cdot V = x \cdot (u \bullet x, y \cdot V) = x \cdot V. \qquad (5.12)$$

For arbitrary $w \in V$ it follows from (5.10) that

$$(x \cdot (u \bullet x, y \cdot w)) \cdot v = x \cdot (u \bullet x, (y \cdot w) \cdot v) \in x \cdot V.$$

From (5.12) we find that $(x \cdot V) \cdot V = x \cdot V$ and hence

$$x \cdot C(q, u) = x \cdot V \neq X_0$$

contradicting the irreducibility of $X_0$. This finishes the proof of Theorem 5.23. □

The previous theorem completes the proof of Construction 5.1, therefore we have, in characteristic not 2, a new coordinate-free definition of the various maps introduced in [TW02, Chapter 13].

**Remark 5.24.** The map $g : X_0 \times X_0 \to k : (x, y) \mapsto \frac{1}{2} f(h(y, x), 1)$ takes an elegant expression. Indeed,

$$\begin{aligned}
g(x, y)e_0 &= \tfrac{1}{2} f((u \bullet y, x), u)e_0 \\
&= ((u \bullet y, x)\, u)e_0 \\
&= \tfrac{1}{2}((y, x) + (u \bullet y, u \bullet x))e_0.
\end{aligned}$$

Since $(y, x) \in ke_0$ and $(u \bullet y, u \bullet x) \in ke_1$, we conclude that $g(x, y)e_0 = \frac{1}{2} (y, x)$. When we identify $k$ and $ke_0$, we have

$$g(x, y) = \tfrac{1}{2} (y, x).$$

**Remark 5.25.** Let $(k, L, q, 1, X, \cdot, h)$ be a quadrangular algebra of type $E_8$. In the previous chapter in Theorem 4.12 we identified $X$ with $CD((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2)$. In the Theorem above we identified $X$ with $(C_1 \otimes C_2)_0$.

$CD((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2)$ is describing the rank one structure (i.e. the Moufang set) related to the Moufang quadrangle, whereas $(C_1 \otimes C_2)_0$ describes the entire Moufang quadrangle. However we are not yet aware of a direct way of relating these two structurable algebras.

**Remark 5.26.** The reader might wonder what will happen if we apply our construction in the case that both $C_1$ and $C_2$ are composition algebras of dimension 2 or 4 with linkage number 1. In the three different cases that arise in this way, we get the following dimensions for the different relevant vector spaces.

| | $E \otimes E$ | $E \otimes Q$ | $Q_1 \otimes Q_2$ |
|---|---|---|---|
| $\dim_k \mathcal{S}$ | 2 | 4 | 6 |
| $\dim_k L$ | 0 | 2 | 4 |
| $\dim_k (C_1 \otimes_k C_2)$ | 4 | 8 | 16 |
| $\dim_k X_0$ | 2 | 4 | 8 |

In the first case, the vector space $L$ is trivial, so our construction no longer applies (we cannot find an element $u \in V \setminus \{0\}$ needed in Definition 5.19).

In the two other cases we can apply Theorem 5.23 to obtain a quadrangular algebra. It follows from the dimensions of $L$ that the obtained quadrangular algebras are of pseudo-quadratic form type.

## 5.5 Construction of arbitrary Moufang quadrangles from special Jordan modules

We will show that each type of Moufang quadrangle in characteristic not 2 can be described in a unified way from a special $J$-module. We generalize the procedure that we used in Theorem 5.6 to obtain quadrangular algebras. In order to obtain all Moufang quadrangles we allow the Jordan algebra in Theorem 5.6 to have $\dim(J_0) > 1$ and we allow the special $J$-module to be the trivial module.

In Section 2.3 we have written the case by case description of Moufang quadrangles from [TW02].

**Construction 5.27.** Let $J$ be a non-degenerate Jordan algebra that contains supplementary proper idempotents $e_0$ and $e_1$. Let $J_0$, $J_{1/2}$, $J_1$ be the Peirce subspaces of $J$ with respect to $e_1$. We assume that each element in $J_{1/2} \setminus \{0\}$ is invertible and that there exists $u \in J_{1/2}$ such that $u^2 = 1$.

Let $X$ be a special $J$-module equipped with a skew-symmetric bilinear form $(\cdot,\cdot) : X \times X \to J$.

- Define the abelian group $V := (J_{1/2}, +)$.
- Define the (not necessary abelian) group $W := X_0 \times J_0$ with addition

$$[a_1, t_1] \boxplus [a_2, t_2] = [a_1 + a_2, t_1 + t_2 + \tfrac{1}{2}\,(a_2, a_1)].$$

Notice that the inverse is $\boxminus[a, t] = [-a, -t]$.

Let $U_1$ and $U_3$ be two groups isomorphic to $W$, and let $U_2$ and $U_4$ be two groups isomorphic to $V$. Denote the corresponding isomorphisms by

$$\begin{aligned}
x_1 &: W \to U_1 : [a,t] \mapsto x_1(a,t) \;;\\
x_2 &: V \to U_2 : v \mapsto x_2(v) \;;\\
x_3 &: W \to U_3 : [a,t] \mapsto x_3(a,t) \;;\\
x_4 &: V \to U_4 : v \mapsto x_4(v) \;;
\end{aligned}$$

we say that $U_1$ and $U_3$ are *parametrized* by $W$ and that $U_2$ and $U_4$ are *parametrized* by $V$.

Now, we implicitly define the group $U_+ = \langle U_1, U_2, U_3, U_4\rangle$ by the following commutator relations:

$$\begin{aligned}
[x_1(a_1,t_1), x_3(a_2,t_2)^{-1}] &= x_2\big((u \bullet a_1, a_2)\big) \;,\\
[x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, 2(v_1 v_2)e_0\big) \;,\\
[x_1(a,t), x_4(v)^{-1}] &= x_2\big(\tfrac{1}{2}\,(u \bullet a, v \bullet (u \bullet a)) + 2(U_u t)v\big)\\
&\qquad x_3\big(v \bullet (u \bullet a), U_v U_u t\big) \;,\\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1,2,3\} \;,
\end{aligned}$$

for all $[a,t], [a_1,t_1], [a_2,t_2] \in W$ and all $v, v_1, v_2 \in V$.

It follows from Lemma 1.15 that $J$ is either of reduced spin type or of type $\mathcal{H}(M_2(L), \sigma T)$. For each of these two cases, we will distinguish between the zero special $J$-module and a non-zero $J$-module. Case by case, we will show that in this way the root groups $U_1, U_2, U_3, U_4$ and commutator relations given above coincide with the description given in Section 2.3 of the Moufang quadrangles in characteristic not 2.

**Remark 5.28.** In Remark 2.17 we discussed quadrangular systems. These are structures that as well describe in a unified way all Moufang quadrangles (including characteristic 2.) We believe it should be possible to start with Construction 5.27, impose a few more axioms that look like the ones in Theorem 5.6 and prove all the axioms defining a quadrangular system. However the verifications of the axioms that use the map $\kappa$, this is a kind of "multiplicative inverse" in the group $W$, get very complicated.

**Moufang quadrangles of quadratic form type** Let $J$ be a reduced spin factor of an anisotropic, non-degenerate quadratic space $(k, V, q)$ with basepoint $u$. Let $X$ be the zero module over $J$.

Remember that $J_0 = ke_0, J_{1/2} = V, J_1 = ke_1$.

- Define the abelian group $V = (J_{1/2}, +)$.
- Define the group $W = X_0 \times J_0 = \{[0, te_0] | t \in k\} \cong k$ with addition $[0, t_1e_0] \boxplus [0, t_2e_0] = [0, (t_1 + t_2)e_0]$. Therefore $W$ is isomorphic to the additive group of $k$ with corresponding isomorphism $W \cong k : [0, te_0] \leftrightarrow t$. So we will write $x_1(t) := x_1(0, te_0)$ and $x_3(t) := x_3(0, te_0)$.

Let $U_1$ and $U_3$ be parametrized by $W$ and $U_2$ and $U_4$ be parametrized by $V$. Let $t, t_1, t_2 \in k$, $v, v_1, v_2 \in V$; using the formulas for the multiplication and the $U$-operator in a Jordan algebra of reduced spin type (see Definition 1.12) we find for the commutator relations

$$\begin{aligned}
[x_1(t_1), x_3(t_2)^{-1}] &= [x_1(0, t_1e_0), x_3(0, t_2e_0)^{-1}] = x_2(0) = 1 \ , \\
[x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, f(v_1, v_2)e_0\big) = x_3\big(f(v_1, v_2)\big) \ , \\
[x_1(t), x_4(v)^{-1}] &= [x_1(0, te_0), x_4(v)^{-1}] = x_2\big(2(U_ute_0)v\big)x_3\big(0, U_vU_ute_0\big) \\
&= x_2\big(2(te_1)v\big)x_3\big(0, U_vte_1\big) = x_2\big(tv\big)x_3\big(0, q(v)te_0\big) \\
&= x_2\big(tv\big)x_3\big(q(v)t\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ .
\end{aligned}$$

We obtain exactly the same description as in (2.3).

**Moufang quadrangles of involutory type** Let $L$ be a skew field with involution $\sigma$. Let $J = \mathcal{H}(M_2(L), \sigma T)$ (see Definition 1.13) and let $X$ be the zero module. Remember that with idempotents $e_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], e_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in J$, we have

$$J_0 = L_\sigma e_0, \quad J_1 = L_\sigma e_1 \quad \text{and} \quad J_{1/2} = \left\{ \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \mid \ell \in L \right\} .$$

- Define the abelian group $V = (J_{1/2}, +) \cong (L, +)$ with isomorphism
$$\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \leftrightarrow \ell.$$
We will write $x_2(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_2(\ell)$ and $x_4(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_4(\ell)$.
- Define the group $W = X_0 \times J_0 = \{[0, \alpha e_0] | \alpha \in L_\sigma\} \cong L_\sigma$ with addition $[0, \alpha_1 e_0] \boxplus [0, \alpha_2 e_0] = [0, (\alpha_1 + \alpha_2)e_0]$. Therefore $W$ is isomorphic to the additive group of $L_\sigma$, we use the isomorphism $W \cong L_\sigma : [0, \alpha e_0] \leftrightarrow \alpha$. We will write $x_1(0, \alpha e_0) = x_1(\alpha)$ and $x_3(0, \alpha e_0) = x_3(\alpha)$.

Let $U_1$ and $U_3$ be parametrized by $W$ and $U_2$ and $U_4$ be parametrized by $V$. Let $\alpha, \alpha_1, \alpha_2 \in L_\sigma$ and consider the following elements of $V$

$$v = \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 & \ell_1^\sigma \\ \ell_1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & \ell_2^\sigma \\ \ell_2 & 0 \end{bmatrix} \in J_{1/2},$$

using the formulas for the multiplication and the $U$-operator in $\mathcal{H}(M_2(L), \sigma T)$, we find for the commutator relations:

$$\begin{aligned}
[x_1(\alpha_1), x_3(\alpha_2)^{-1}] &= [x_1(0, \alpha_1 e_0), x_3(0, \alpha_2 e_0)^{-1}] = x_2(0) = 1 \ , \\
[x_2(\ell_1), x_4(\ell_2)^{-1}] &= [x_2(v_1), x_4(v_2)^{-1}] \\
&= x_3\big(0, ((\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1)e_0 + (\ell_1 \ell_2^\sigma + \ell_2 \ell_1^\sigma)e_1)e_0\big) \\
&= x_3\big(0, (\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1)e_0\big) = x_3\big(\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1\big) \ , \\
[x_1(\alpha), x_4(\ell)^{-1}] &= [x_1(0, \ell e_0), x_4(v)^{-1}] = x_2\big(2(U_u \alpha e_0)v\big) x_3\big(0, U_v U_u \alpha e_0\big) \\
&= x_2\big(2(\alpha e_1)v\big) x_3\big(0, U_v \alpha e_1\big) \\
&= x_2\left(\begin{bmatrix} 0 & \ell^\sigma \alpha \\ \alpha \ell & 0 \end{bmatrix}\right) x_3\big(0, \ell^\sigma \alpha \ell e_0\big) = x_2\big(\alpha \ell\big) x_3\big(\ell^\sigma \alpha \ell\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ .
\end{aligned}$$

This is exactly (2.4).

**Moufang quadrangles of pseudo-quadratic type** These are obtained in similar fashion as the quadrangular algebras in Section 5.3, but here we start from an arbitrary skew field with involution instead of starting from a quadratic pair. We repeat part of the setup from Section 5.3.

Let $L$ be a skew field with involution $\sigma$. Let $(L, \sigma, X, h, \pi)$ be a pseudo-quadratic space, so $\pi(a) = \frac{1}{2} h(a, a)$ for all $a \in X$.

Let $J = \mathcal{H}(M_2(L), \sigma T)$ and let $\widetilde{X} = X^2$, the $1 \times 2$ row vectors over X. For the action of $J$ on $\widetilde{X}$, for $j \in J, a \in \widetilde{X}$ we have $j \bullet a = aj \in \widetilde{X}$.

As before we define $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in J$. We have $\widetilde{X}_0 = \{[a, 0] \mid a \in X\}, \widetilde{X}_1 = \{[0, a] \mid a \in X\}$.

We define the skew-symmetric bilinear map $\widetilde{X} \times \widetilde{X} \to J$ as

$$\big([a_1, a_2], [b_1, b_2]\big) = \begin{bmatrix} h(a_1, b_1) - h(b_1, a_1) & h(a_1, b_2) - h(b_1, a_2) \\ -h(b_2, a_1) + h(a_2, b_1) & h(a_2, b_2) - h(b_2, a_2) \end{bmatrix}.$$

- Define the abelian group $V = (J_{1/2}, +) \cong (L, +)$ with isomorphism

$$\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \leftrightarrow \ell.$$

  We will write $x_2(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_2(\ell)$ and $x_4(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_4(\ell)$.

- Define the group $W = \widetilde{X}_0 \times J_0 \cong X \times L_\sigma$, when we identify $J_0 \cong L_\sigma : \alpha e_0 \leftrightarrow \alpha$ and is $\widetilde{X}_0 \cong X : [a, 0] \leftrightarrow a$,[5] we get the addition

$$[a_1, \alpha_1] \boxplus [a_2, \alpha_2] = [a_1 + a_2, \alpha_1 + \alpha_2 + \tfrac{1}{2}(h(a_2, a_1) - h(a_1, a_2))].$$

We will write $x_1(a, \alpha) := x_1([a, 0], \alpha e_0)$ and $x_3(a, \alpha) := x_3([a, 0], \alpha e_0)$.

For the commutator relations we obtain

$$\begin{aligned}
[x_1(a_1, \alpha_1), x_3(a_2, \alpha_2)^{-1}] &= x_2\big(h(a_1, a_2)\big) \ , \\
[x_2(\ell_1), x_4(\ell_2)^{-1}] &= x_3\big(0, \ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1\big) \ , \\
[x_1(a, \alpha), x_4(\ell)^{-1}] &= x_2\big(\theta(a, \ell) + \alpha\ell\big) x_3\big(a\ell, \ell^\sigma \alpha \ell\big) \\
&= x_2\big(\pi(a)\ell + \alpha\ell\big) x_3\big(a\ell, \ell^\sigma \alpha \ell\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ ,
\end{aligned}$$

for all $[a, t], [a_1, t_1], [a_2, t_2] \in W$ and all $\ell, \ell_1, \ell_2 \in L$.

This is exactly (2.9).

**Moufang quadrangles of type $E_6, E_7, E_8$** This case was actually already handled in Theorem 5.23, since from quadrangular algebras one can define the root groups and commutator relations of the corresponding Moufang quadrangles. Now we quickly verify that we get indeed the right commutator relations using Construction 5.27.

Let $J$ be a reduced spin factor of an anisotropic, non-degenerate quadratic space $(k, V, q)$ with basepoint $u$, let $X = C_1 \otimes_k C_2$ and let the skew-symmetric form $\boldsymbol{(}., .\boldsymbol{)}$ be as in Section 5.4.

Quadrangular algebras of type $E_6, E_7$ and $E_8$ are entirely determined by the similarity class of their quadratic space. It follows that the quadrangular algebras we constructed in Theorem 5.23 are identical to the ones in [TW02, Chapter 13]. Therefore we have that the following maps coincide with the maps defined in [TW02, Chapter 13]:

$$a \cdot v = v \bullet (u \bullet a), \quad h(a, b) = \boldsymbol{(}u \bullet a, b\boldsymbol{)}\,, \quad g(a, b)e_0 = \tfrac{1}{2}\,\boldsymbol{(}b, a\boldsymbol{)}\,.$$

Now define

- the abelian group $V = (J_{1/2}, +)$;
- the group $W = X_0 \times J_0 \cong X_0 \times k$ with addition $[a_1, t_1 e_0] \boxplus [a_2, t_2 e_0] = [a_1 + a_2, t_1 e_0 + t_2 e_0 + \frac{1}{2}\,\boldsymbol{(}a_2, a_1\boldsymbol{)}]$. When we identify $J_0 \cong k : te_0 \leftrightarrow t$, we get

$$[a_1, t_1] \boxplus [a_2, t_2] = [a_1 + a_2, t_1 + t_2 + g(a_1, a_2)].$$

We will write $x_1(a, t) := x_1(a, te_0)$ and $x_3(a, t) := x_3(a, te_0)$.

[5] We have to denote elements of $\widetilde{X}$ and of $W$ both by $[., .]$, from now on we only will make use of elements in $\widetilde{X}_0$ and not those contained in $X$ in general. We use the notation $[., .]$ exclusively for elements of $W$ from now on.

Let $U_1$ and $U_3$ be parametrized by $W$ and $U_2$ and $U_4$ be parametrized by $V$. Let $t, t_1, t_2 \in k$, $v, v_1, v_2 \in V$; we find the following commutator relations:

$$\begin{aligned}
[x_1(a_1,t_1), x_3(a_2,t_2)^{-1}] &= x_2\big(\boldsymbol{(}u \bullet a_1, a_2\boldsymbol{)}\big) = x_2\big(h(a_1,a_2)\big) \ , \\
[x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, f(v_1,v_2)e_0\big) = x_3\big(0, f(v_1,v_2)\big) \ , \\
[x_1(a,t), x_4(v)^{-1}] &= [x_1(a,te_0), x_4(v)^{-1}] \\
&= x_2\big(\tfrac{1}{2}\,\boldsymbol{(}u \bullet a, a \cdot v\boldsymbol{)} + 2(te_1)v\big) x_3\big(a \cdot v, U_v te_1\big) \\
&= x_2\big(\theta(a,v) + tv\big) x_3\big(a \cdot v, q(v)t\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1,2,3\} \ .
\end{aligned}$$

We obtain exactly the same description as in (2.11).

# Chapter 6

# A construction of Moufang sets from structurable division algebras

In this chapter we will show how every structurable division algebra $\mathcal{A}$ gives rise to a Moufang set. It was known since some time that every Jordan division algebra gives rise to a Moufang set with abelian root groups [DMW06]. We were inspired by this result and by the known examples of Moufang sets with non-abelian root groups, in particular those of type $F_4$ (see Section 2.4.2.3) and those arising as residues of exceptional Moufang quadrangles (see Section 2.4.2.4). It turns out that each of these cases arises from a specific type of structurable algebra. Indeed, every Jordan division algebra is also a structurable algebra (with trivial involution); the Moufang sets of type $F_4$ arise from an octonion division algebra with standard involution; and the residues of exceptional Moufang quadrangles arise from structurable algebras as described in Chapter 4.

The Moufang sets we obtain have root groups of nilpotency class 2 (i.e. $[[U_x, U_x], U_x] = 1$). Notice that the theory of structurable algebras forces us to omit fields of characteristic 2 and 3. Furthermore we have to assume that the characteristic of the field is different from 5.

The structure of the Lie algebra corresponding to $\mathcal{A}$ convinced us that the root group $U$ of the corresponding Moufang set $\mathbb{M}(U, \tau)$ had to have underlying set $\mathcal{A} \times \mathcal{S}$. On the other hand, the permutation $\tau$ should act as some kind of inverse on $\mathcal{A} \times \mathcal{S}$ such that $(x, 0).\tau = (-\hat{x}, 0)$. If $\mathcal{A}$ is a Jordan algebra, then $\mathcal{S} = 0$ and we obtain indeed the Moufang set described in Section 2.4.2.1.

For a long time we were trying to find out how to define in a good way the inverse of an element in $\mathcal{A} \times \mathcal{S}$, until we found out that in [AF99], Bruce

Allison and John Faulkner already introduced such a notion, namely one-invertibility. Even more, they prove a property of one-invertible elements which turns out to be essential for our construction of a Moufang set.

In Section 6.1 we define one-invertibility of elements in $\mathcal{A}\times\mathcal{S}$. In Section 6.2 we show that if $\mathcal{A}$ is a structurable division algebra each element in $(x, s) \in \mathcal{A}\times\mathcal{S}\setminus\{(0, 0)\}$ is one-invertible. We are indebted to John Faulkner for providing the main idea of how to prove this.

In Section 6.3 we give the description of the exact construction of the Moufang sets and determine the group $U$ and permutation $\tau$; see Theorem 6.25 for the main result. We also give an elegant expression for the Hua-maps. In Section 6.4 we show that we obtain all the examples of Moufang sets described in Section 2.4.2 using our new construction; but we conjecture that our construction gives rise to Moufang sets coming from linear algebraic groups of rank one that have not been described explicitly before, see Conjecture 6.28.

*In this chapter we always assume that $k$ is a field of characteristic different from* 2, 3 *and* 5.

## 6.1 One-invertibility in $\mathcal{A}\times\mathcal{S}$

In [AF99] the notion of $n$-invertibility for Kantor pairs is introduced. Kantor pairs are generalizations of Jordan pairs; an example of a Kantor pair is a pair of structurable algebras.

Since we will only apply the results of [AF99] in the context of a pair of structurable algebras, we only explain the necessary terminology and results of [AF99] in this context. This makes the exposition less technical, and in Remark 6.3 we explain why our point of view is the same as in [AF99].

Let $\mathcal{A}$ be an arbitrary structurable $k$-algebra. In Section 3.4 we described the 5-graded Lie algebra $K(\mathcal{A})$ constructed from $\mathcal{A}$. In [AF99] an isomorphic Lie algebra is used, doing this will make the formulas for one-invertibility more elegant.

**Definition 6.1.** Let $\mathcal{A}$ be a structurable algebra.

(i) Consider two copies $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ with corresponding isomorphisms $\mathcal{A}\to\mathcal{A}_+ : x \mapsto x_+$ and $\mathcal{A}\to\mathcal{A}_- : x \mapsto x_-$, and let $\mathcal{S}_+ \subset A_+$ and $\mathcal{S}_- \subset A_-$ be the corresponding subspaces of skew-elements. We define the Lie algebra

$$K'(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+,$$

with Lie bracket given by

- $[\mathrm{Instrl}, K'(\mathcal{A})]$

$$
\begin{aligned}
&[V_{a,b}, V_{a',b'}] = V_{\{a,b,a'\},b'} - V_{a',\{b,a,b'\}} \in \mathrm{Instrl}(\mathcal{A}),\\
&[V_{a,b}, x_+] := (V_{a,b}x)_+ \in \mathcal{A}_+, \qquad [V_{a,b}, y_-] := (V^{\epsilon}_{a,b}y)_- \in \mathcal{A}_- = (-V_{b,a}y)_- \in \mathcal{A}_-,\\
&[V_{a,b}, s_+] := (V^{\delta}_{a,b}s)_+ \in \mathcal{S}_+ = -\psi(a, sb)_+ \in \mathcal{S}_+, \qquad [V_{a,b}, t_-] := (V^{\epsilon\delta}_{a,b}t)_- \in \mathcal{S}_- = \psi(b, ta)_- \in \mathcal{S}_-,
\end{aligned}
$$

- $[\mathcal{S}_\pm, \mathcal{A}_\pm]$

$$
\begin{aligned}
&[s_+, x_+] := 0, \qquad [t_-, y_-] := 0,\\
&[s_+, y_-] := (sy)_+ \in \mathcal{A}_+, \qquad [t_-, x_+] := (tx)_- \in \mathcal{A}_-,
\end{aligned}
$$

- $[\mathcal{A}_\pm, \mathcal{A}_\pm]$

$$
\begin{aligned}
&[x_+, y_-] := -2V_{x,y} \in \mathrm{Instrl}(\mathcal{A}),\\
&[x_+, x'_+] := -2\psi(x, x')_+ \in \mathcal{S}_+, \qquad [y_-, y'_-] := -2\psi(y, y')_- \in \mathcal{S}_-
\end{aligned}
$$

- $[\mathcal{S}_\pm, \mathcal{S}_\pm]$

$$
\begin{aligned}
&[s_+, s'_+] := 0, \qquad [t_-, t'_-] := 0,\\
&[s_+, t_-] := L_sL_t \in \mathrm{Instrl}(\mathcal{A}).
\end{aligned}
$$

for all $x, x', y, y' \in \mathcal{A}$, $s, s', t, t' \in \mathcal{S}$, $V_{a,b}, V_{a',b'} \in \mathrm{Instrl}(\mathcal{A})$. The Lie algebra $K'(\mathcal{A})$ has a 5-grading with $K'(\mathcal{A})_0 = \mathrm{Instrl}(\mathcal{A})$, $K'(\mathcal{A})_{\pm1} = \mathcal{A}_\pm$ and $K'(\mathcal{A})_{\pm2} = \mathcal{S}_\pm$.

(ii) We define the *grading derivation* $\zeta \in \mathrm{End}_k(K'(\mathcal{A}))$ as the $k$-linear map $\zeta(x_i) = ix_i$ for all $x_i \in K'(\mathcal{A})_i$ with $i \in [-2, 2]$. It is clear that $\zeta$ is indeed a derivation of Lie algebra $K'(\mathcal{A})$.

(iii) In the sequel it is convenient to consider $\zeta$ as an element of the Lie algebra, we define

$$\mathcal{G} := \mathcal{S}_- \oplus \mathcal{A}_- \oplus (\mathrm{Instrl}(\mathcal{A}) + k\zeta) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$$

with the same Lie bracket as $K'(\mathcal{A})$ and with $[\zeta, x_i] = \zeta(x_i) = ix_i$ for all $x_i \in K'(A)_i$ with $i \in [-2, 2]$. It follows that also $\mathcal{G}$ has a 5-grading with

$$
\begin{aligned}
\mathcal{G}_0 &= K'(\mathcal{A})_0 + k\zeta = \mathrm{Instrl}(\mathcal{A}) + k\zeta,\\
\mathcal{G}_{\pm1} &= K'(\mathcal{A})_{\pm1} = \mathcal{A}_\pm,\\
\mathcal{G}_{\pm2} &= K'(\mathcal{A})_{\pm2} = \mathcal{S}_\pm.
\end{aligned}
$$

**Remark 6.2.** Notice that the only difference between $K(\mathcal{A})$ and $K'(\mathcal{A})$ is in the bracket of $[\mathcal{A}_\pm, \mathcal{A}_\pm]$. It is straightforward to verify that the following map is a Lie algebra isomorphism:

$$\begin{aligned} K(\mathcal{A}) &\to K'(\mathcal{A}) \\ \mathrm{Instrl}(\mathcal{A}) \to \mathrm{Instrl}(\mathcal{A}) &: V_{a,b} \mapsto V_{a,b}, \\ \mathcal{A}_1 \to \mathcal{A}_1 &: \quad x \mapsto x, \\ \mathcal{A}_{-1} \to \mathcal{A}_{-1} &: \quad y \mapsto -\frac{1}{2}y, \\ \mathcal{A}_2 \to \mathcal{A}_2 &: \quad s \mapsto -2s, \\ \mathcal{A}_{-2} \to \mathcal{A}_{-2} &: \quad t \mapsto -\frac{1}{2}t, \end{aligned}$$

It is clear that $K'(\mathcal{A})$ is an ideal of $\mathcal{G}$ and that $[\mathcal{G}, \mathcal{G}] = K'(\mathcal{A})$.

The Lie algebra $\mathcal{G}$ we defined here, is the same algebra as the one in [AF99] in the case the considered Kantor pair is a pair of structurable algebras. In the following remark we give an overview of the construction in [AF99], and indicate why the two Lie algebras are identical.

**Remark 6.3.** [This remark uses definitions not mentioned in this thesis.] A Kantor pair is a pair of vector spaces $(K_+, K_-)$ with a triple product $\{\cdot,\cdot,\cdot\} : K_\sigma \times K_{-\sigma} \times K_\sigma \to K_\sigma$ for all $\sigma \in \{-1, 1\}$ that satisfies the two conditions (KP 1) and (KP 2) in [AF99].

Let $\mathcal{A}$ be a structurable algebra and let $\mathcal{A}_+$, $\mathcal{A}_-$ be two isomorphic copies of $\mathcal{A}$, then the pair $(\mathcal{A}_+, \mathcal{A}_-)$ with triple product $\{x, y, z\} := 2V_{x,y}z$ for $x, z \in \mathcal{A}_\sigma$ and $y \in \mathcal{A}_{-\sigma}$ is a Kantor pair.

In [AF99, Theorem 7] it is shown that this gives rise to a sign-graded Lie triple system $\mathcal{L}(\mathcal{A}) = \mathcal{A}_+ \oplus \mathcal{A}_-$. Notice that for this Lie triple system, we define $[x, y, z] = -\{x, y, z\} = -2V_{x,y}z$ for $x, z \in \mathcal{A}_\sigma$ and $y \in \mathcal{A}_{-\sigma}$.

In [AF99, p. 532] the Lie algebra $\mathcal{G}(\mathcal{L}(\mathcal{A}))$ is defined, which is called the standard graded embedding of $\mathcal{L}(\mathcal{A})$. If we identify $L_s \in \mathcal{G}(\mathcal{L}(\mathcal{A}))_{\pm 2}$ with $s \in \mathcal{G}_{\pm 2}$ and represent the elements of $\mathcal{G}(\mathcal{L}(\mathcal{A}))_0$ with their action on $\mathcal{G}(\mathcal{L}(\mathcal{A}))_1$, the algebras $\mathcal{G}(\mathcal{L}(\mathcal{A}))$ and $\mathcal{G}$ are identical. This can be verified making use of $K_{a,b} = 2L_{\psi(a,b)}$ and [AF99, identity (KP 2)]. This Lie algebra $\mathcal{G}(\mathcal{L}(\mathcal{A}))$ is described more explicitly in [AF99, p. 535].

We will define some subgroups of $\mathrm{End}_k(\mathcal{G})$. In [AF99] the action of $\mathrm{End}_k(\mathcal{G})$ on $\mathcal{G}$ is denoted on the left, whereas we need an action on the right in order to be compatible with the conventions in the theory of Moufang sets. This is why some formulas differ slightly from [AF99].

**Definition 6.4.** Let $\sigma \in \{-1,+1\}$, $x \in \mathcal{G}_\sigma$, $s \in \mathcal{G}_{2\sigma}$; we define

$$e_\sigma(x,s) = \exp(\mathrm{ad}(x+s)) = \sum_{i=0}^{4} \frac{1}{i!}(\mathrm{ad}(x+s))^i \in \mathrm{End}_k(\mathcal{G}).$$

Define the set

$$U_\sigma = \{e_\sigma(x,s) \mid x \in \mathcal{G}_\sigma, s \in \mathcal{G}_{2\sigma}\}.$$

The following lemma shows that $U_\sigma$ is in fact a subgroup of $\mathrm{Aut}(\mathcal{G})$. We will explicitly need the assumption that $\mathrm{char}(k) \neq 5$ in part (i). This condition was omitted in the proof in [AF99]; we thank Ottmar Loos for bringing this to our attention. We include a correct proof of (i) below.

**Lemma 6.5** ([AF99, Theorem 8])**.** *Let $\sigma \in \{-1,+1\}$, we have the following properties for all $x, y \in \mathcal{G}_\sigma$ and $s, t \in \mathcal{G}_{2\sigma}$:*

(i) *$e_\sigma(x,s)$ is an automorphism of the Lie algebra $\mathcal{G}$, i.e. $[a,b].e_\sigma(x,s) = [a.e_\sigma(x,s), b.e_\sigma(x,s)]$ for all $a, b \in \mathcal{G}$.*
(ii) *$e_\sigma(x,s)e_\sigma(y,t) = e_\sigma(x+y, s+t+\psi(x,y))$,*
(iii) *$e_\sigma(x,s)^{-1} = e_\sigma(-x,-s)$.*
(iv) *The map $e_\sigma : \mathcal{G}_\sigma \times \mathcal{G}_{2\sigma} \to U_\sigma : (x,s) \mapsto e_\sigma(x,s)$ is a bijection.*

*Idea of proof.* (i) Let $x \in \mathcal{G}_\sigma$ and $s \in \mathcal{G}_{2\sigma}$, and let

$$D_0 = \mathrm{id}, \quad D_i = \frac{1}{i!}\,\mathrm{ad}(x+s)^i \text{ for all } 0 < i \le 6, \text{ and } D_i = 0 \text{ for all } i > 6.$$

(Notice that also for $i = 5$ or $6$, we have $D_i = 0$, but we will nevertheless need the explicit formula later.)
Thus $e_\sigma(x,s) = \sum_{i=0}^{\infty} D_i$. To show that $e_\sigma(x,s)$ is an automorphism, it suffices by degree considerations to show that

$$[u,v].D_n = \sum_{i+j=n} [u.D_i, v.D_j] \tag{6.1}$$

for all $u, v \in \mathcal{G}$ and for all $n \ge 0$.
If $n > 4$, $D_n = 0$. If $n > 6$, $\sum_{i+j=n}[u.D_i, v.D_j] = 0$ by degree considerations. Indeed, notice that for $i + j = n$, either $D_i$ or $D_j$ is 0 unless $n = 7$ and $\{i,j\} = \{3,4\}$ or $n = 8$ and $i = j = 4$; but also in these cases,

$$[u.D_3, v.D_4] + [u.D_4, v.D_3] \in [\mathcal{G}_\sigma + \mathcal{G}_{2\sigma}, \mathcal{G}_{2\sigma}] + [\mathcal{G}_{2\sigma}, \mathcal{G}_\sigma + \mathcal{G}_{2\sigma}] = 0;$$
$$[u.D_4, v.D_4] \in [\mathcal{G}_{2\sigma}, \mathcal{G}_{2\sigma}] = 0.$$

It remains to show (6.1) for all $1 \le n \le 6$. Notice that (6.1) is trivial for $n = 0$, and follows from the fact that $D_1$ is a derivation for $n = 1$.

We proceed by induction on $n < 6$, and notice that the condition on $\mathrm{char}(k)$ ensures that we can divide by $n+1$:

$$\begin{aligned}
&[u,v].D_{n+1} \\
&= ([u,v].D_n).\frac{\mathrm{ad}(x+s)}{n+1} = \sum_{i+j=n} [u.D_i, v.D_j].\frac{\mathrm{ad}(x+s)}{n+1} \\
&= \frac{1}{n+1} \sum_{i+j=n} \big((i+1)[u.D_{i+1}, v.D_j] + (j+1)[u.D_i, v.D_{j+1}]\big) \\
&= \frac{1}{n+1}\Big((n+1)[u.D_{n+1}, v.D_0] + (n+1)[u.D_0, v.D_{n+1}] \\
&\qquad + \sum_{i+j=n, j\geq 1} (i+j+1)[u.D_{i+1}, v.D_j]\Big) \\
&= \sum_{i+j=n+1} [u.D_i, v.D_j].
\end{aligned}$$

(ii) The Campbell-Baker-Hausdorff theorem for an arbitrary Lie algebra over a field of characteristic 0 states that

$$\exp(\mathrm{ad}\, a) \circ \exp(\mathrm{ad}\, b) = \exp(\mathrm{ad}(a + b + \frac{1}{2}[a,b] - \frac{1}{12}[[a,b],a] + \dots)),$$

for all $a, b$ in this Lie algebra. In our case $a = x+s$ and $b = y+t$, $\frac{1}{2}[a,b] = -\psi(x,y)$ and all terms beyond $[a,b]$ are zero by degree considerations. The $-\psi(x,y)$ becomes $\psi(x,y)$ since we work with the action on the right.
Some work is needed to prove that Campbell-Baker-Hausdorff theorem can be reformulated to hold over fields of characteristic different from 2 and 3.

(iii) This is an easy consequence of (ii).

(iv) The injectivity follows from the fact that $\zeta.e_\sigma(x,s) = \zeta - \sigma x - 2\sigma s$. □

**Definition 6.6.** (i) The *elementary group* of the structurable algebra $\mathcal{A}$ is defined as

$$G := \langle U_+, U_- \rangle \leq \mathrm{Aut}(\mathcal{G}).$$

(ii) We define the subset[1] $H_-$ of $G$ as the set of homomorphisms that reverse the gradation of $\mathcal{G}$, i.e.

$$\begin{aligned}
H_- &:= \{h \in G \mid \zeta.h = -\zeta\} \\
&= \{h \in G \mid \mathcal{G}_i.h = \mathcal{G}_{-i} \text{ for all } i \in \{-2,-1,0,1,2\}\}.
\end{aligned}$$

(iii) Define $\varphi \in \mathrm{End}_k(\mathcal{G})$, which reverses the gradation of $\mathcal{G}$, as

$$\mathcal{G}_0 \to \mathcal{G}_0 : \quad V_{a,b} \mapsto -V_{b,a},$$

[1] In [AF99] $H_+$ is defined as the automorphisms in $G$ that preserve the gradation of $\mathcal{G}$.

$$\begin{aligned} & \zeta \mapsto -\zeta, \\ \mathcal{A}_1 \to \mathcal{A}_{-1}: \quad & x \mapsto x, \\ \mathcal{A}_{-1} \to \mathcal{A}_1: \quad & x \mapsto x, \\ \mathcal{A}_2 \to \mathcal{A}_{-2}: \quad & s \mapsto s, \\ \mathcal{A}_{-2} \to \mathcal{A}_2: \quad & s \mapsto s. \end{aligned}$$

In Theorem 6.20 we will show that if all elements in $\mathcal{A} \times \mathcal{S}$ are one-invertible, the group $G = \langle U_+, U_- \rangle$ is an abstract rank one group.

**Lemma 6.7.** *Let $\sigma \in \{-1, +1\}$ and $(x, s) \in \mathcal{G}_\sigma \times \mathcal{G}_{2\sigma}$.*

(i) *For all $h \in H_-$, we have $e_\sigma(x, s)^h = e_{-\sigma}(x.h, s.h)$ and ${U_\sigma}^h = U_{-\sigma}$.*
(ii) *We have $\varphi \in \mathrm{Aut}(\mathcal{G})$, $e_\sigma(x, s)^\varphi = e_{-\sigma}(x, s)$ and ${H_-}^\varphi = H_-$.*

*Proof.* (i) Let $\sigma \in \{-1, +1\}$. For all $h \in H_-$, for all $(x, s) \in \mathcal{G}_\sigma \times \mathcal{G}_{2\sigma}$ we find

$$a.\,\mathrm{ad}(x + s)^h = a.(h^{-1}\,\mathrm{ad}(x + s)h) = a.\,\mathrm{ad}(x.h + s.h),$$

for all $a \in \mathcal{G}$, since $h$ is an automorphism of $\mathcal{G}$. As $(x.h, s.h) \in \mathcal{G}_{-\sigma} \times \mathcal{G}_{-2\sigma}$ we have that $e_\sigma(x, s)^h = e_{-\sigma}(x.h, s.h)$. Since $h$ is an isomorphism, we conclude that ${U_\sigma}^h = U_{-\sigma}$.

(ii) Using the definition of the Lie bracket of $\mathcal{G}$ one can easily verify that $\varphi \in \mathrm{Aut}_k(\mathcal{G})$. In order to verify that $[s, t].\varphi = [s.\varphi, t.\varphi]$ for $s \in \mathcal{G}_2$ and $t \in \mathcal{G}_{-2}$, one should make use of identity (3.23).
Since $\mathrm{ad}(x + s)^\varphi = \mathrm{ad}(x.\varphi + s.\varphi)$, we have $e_\sigma(x, s)^\varphi = e_{-\sigma}(x, s)$.
Now $\mathcal{G}_i.\varphi^{-1}h\varphi = \mathcal{G}_{-i}.h\varphi = \mathcal{G}_i.\varphi = \mathcal{G}_{-i}$; since it is clear that $G^\varphi = G$ we have that $H_-^\varphi = H_-$. □

It is not clear that the homomorphism $\varphi$ is actually contained in $H_-$, this will be shown in Lemma 6.23.

From now on we will use the following convention, which will allow us to consider one-invertibility of an element in $\mathcal{A} \times \mathcal{S}$.

**Notation 6.8.** Let $(x, s) \in \mathcal{A} \times \mathcal{S}$. As $\mathcal{A}_+$ and $\mathcal{A}_-$ are copies of $\mathcal{A}$ and $\mathcal{S}_-$ and $\mathcal{S}_+$ copies of $\mathcal{S}$, we can write $e_\sigma(x, s)$ without causing any confusion:

If we write $e_+(x, s)$ we consider $(x, s)$ as an element of $\mathcal{A}_+ \times \mathcal{S}_+$, whereas if we write $e_-(x, s)$ we consider $(x, s)$ as an element of $\mathcal{A}_- \times \mathcal{S}_-$.

Now we have enough background information to give the definition of one-invertibility.

**Definition 6.9.** (i) Let $(x, s) \in \mathcal{A} \times \mathcal{S}$. We say that $(x, s)$ is *one-invertible* if there exist $(y, t), (z, r) \in \mathcal{A} \times \mathcal{S}$ such that

$$e_-(z, r)e_+(x, s)e_-(y, t) \in H_-.$$

Using Lemma 6.7.(ii), we see that this condition is equivalent with

$$e_+(z,r)e_-(x,s)e_+(y,t) \in H_-.$$

(ii) If $(x, s)$ is one-invertible, we say that $(x, s)$ has *left*[2] *inverse* $(y, t)$ and *right inverse* $(z, r)$. [AF99, Lemma 11] states that the left and right inverses are unique.

(iii) Let $(x, s) \in \mathcal{A} \times \mathcal{S}$ be one-invertible with right inverse $(z, r)$, define the linear map $P_{(x,s)} : \mathcal{A} \to \mathcal{A}$ given by

$$P_{(x,s)}a = U_x(a + \frac{2}{3}\psi(z,a)x) + s(a + 2\psi(z,a)x) \quad \text{for all} a \in \mathcal{A}.$$

In [AF99, Section 5] $n$-invertibility for an $n$-tuple in

$$(\mathcal{G}_\sigma \times \mathcal{G}_{2\sigma}) \times (\mathcal{G}_{-\sigma} \times \mathcal{G}_{-2\sigma}) \times \cdots \times (\mathcal{G}_{(-1)^{n-1}\sigma} \times \mathcal{G}_{(-1)^{n-1}2\sigma})$$

is defined in a similar way.

The following theorem gives us a very useful characterization of one-invertibility.

**Theorem 6.10** ([AF99, Theorem 13]). (i) *An element $(x, s) \in \mathcal{A} \times \mathcal{S}$ is one-invertible if and only if there exists $(u, t) \in \mathcal{A} \times \mathcal{S}$ such that*

$$\begin{aligned} V_{x,u} &= \mathrm{id} + L_s L_t, \\ su &= -\frac{1}{3}U_x(tx), \\ \psi(x, s(tx)) &= 0. \end{aligned} \tag{6.2}$$

*This system of equations has either no solutions or exactly one solution.*

(ii) *Let $(x, s) \in \mathcal{A}\times\mathcal{S}$ be one-invertible with $(u, t)$ the solution of the system of equations (6.2). Then the left inverse of $(x, s)$ is $(u - tx, t)$ and its right inverse is $(u + tx, t)$.*

(iii) *Let $(x, s) \in \mathcal{A}\times\mathcal{S}$ be one-invertible with $(u, t)$ the solution of the system of equations (6.2). For $\sigma \in \{-1, 1\}$ denote*

$$h_\sigma := e_{-\sigma}(u + tx, t)e_\sigma(x, s)e_{-\sigma}(u - tx, t) \in H_-.$$

*Then $h_\sigma|_{\mathcal{G}_\sigma} = P_{(u-tx,t)} = P_{(u+tx,t)}$ and $h_\sigma|_{\mathcal{G}_{-\sigma}} = P_{(x,s)}$.*

*Proof.* We transfer [AF99, Theorem 13] to our setup using Remark 6.3.

[2]Note that the action of $G$ is on the right.

It is shown in the proof in [AF99] that $(x, s) \in \mathcal{G}_\sigma \times \mathcal{G}_{2\sigma}$ is one-invertible if and only if there exist $(u, t) \in \mathcal{G}_{-\sigma} \times \mathcal{G}_{-2\sigma}$ such that

$$\begin{aligned} 2[s,t] + [x,u] + 2\sigma\zeta &= 0, \\ [s,u] - \frac{1}{6}[x,[x,[x,t]]] &= 0, \\ -\frac{1}{3}[x,[x,[s,t]]] &= 0, \end{aligned} \tag{6.3}$$

and that in this case the left inverse of $(x, s)$ is $(u - tx, t)$ and the right inverse is $(u + tx, t)$.

The last two equations of (6.3) immediately give the last two equations of (6.2).

The first equation needs a bit more explanation. It states $2[s, t]+[x, u]+ 2\sigma\zeta = 0$, by identifying $\mathcal{G}(\mathcal{L}(\mathcal{A}))$ and $\mathcal{G}$ we made the convention in Remark 6.3, that we would represent elements of $\mathcal{G}(\mathcal{L}(\mathcal{A}))_0$ with their action on $\mathcal{A}_+$. Let $w \in A_+$, if $\sigma = +1$ we get

$$\operatorname{ad}(2[s,t] + [x,u] + 2\sigma\zeta)(w) = 2L_sL_tw - 2V_{x,u}w + 2w = 0,$$

and if $\sigma = -1$, we get

$$\operatorname{ad}(2[s,t] + [x,u] + 2\sigma\zeta)(w) = -2L_tL_sw + 2V_{u,x}w - 2w = 0.$$

Thus we have two conditions $V_{x,u} = \mathrm{id} + L_sL_t$ and $V_{u,x} = \mathrm{id} + L_tL_s$. These two identities are equivalent when we consider $x, u \in \mathcal{A}$ and $s, t \in \mathcal{S}$, since $V^\epsilon_{u,x} = \mathrm{id}^\epsilon + (L_tL_s)^\epsilon$ gives $-V_{x,u} = -\mathrm{id} - L_sL_t$ with $\epsilon$ as defined on page 57.

In [AF99] those equations are not necessarily equivalent since one considers Kantor pairs. In [AF99] the identity that should be satisfied is given by $V_{x,u} = 2(\mathrm{id} + L_sL_t)$; notice that the difference with the first equation of (6.2) is caused by the fact that the $V$-operator in [AF99] is the double of the $V$-operator of the structurable algebra

Since the left and right inverses of an one-invertible element are uniquely determined, the system of equations (6.2) has either no solutions or it has a unique solution. □

The following lemma shows that one-invertibility in $\mathcal{A} \times \mathcal{S}$ is a generalization of conjugate invertibility in $\mathcal{A}$.

**Lemma 6.11.** (i) *Let $x \in \mathcal{A}$. Then $(x, 0) \in \mathcal{A} \times \mathcal{S}$ is one-invertible if and only if $x$ is conjugate invertible in $\mathcal{A}$. The left and right inverse of $(x, 0)$ is given by $(\hat{x}, 0)$.*

(ii) *Let $s \in \mathcal{S}$. Then $(0, s) \in \mathcal{A} \times \mathcal{S}$ is one-invertible if and only if $s$ is conjugate invertible in $\mathcal{A}$. The left and right inverse of $(0, s)$ is given by $(0, \widehat{s})$.*

*Proof.* (i) If we want to determine the one-invertibility of $(x, 0)$, the system of equations (6.2) reduces to

$$\begin{aligned} V_{x,u} &= \mathrm{id}, \\ 0 &= -\frac{1}{3}U_x tx, \\ 0 &= 0. \end{aligned}$$

From the first equation it follows that this system of equations can only have a solution if $x$ is conjugate invertible. In this case, by (3.10), $u = \hat{x}$ and $t = 0$ is the solution.

(ii) If we want to determine the one-invertibility of $(0, s)$, the system of equations (6.2) reduces to

$$\begin{aligned} 0 &= \mathrm{id} + L_s L_t, \\ su &= 0, \\ 0 &= 0. \end{aligned}$$

From the first equation it follows that this system of equations can only have a solution if $s$ is conjugate invertible. In this case, by (3.13), $u = 0$ and $t = \hat{s}$ is the solution. □

The map $P_{(x,s)}$ defined in Definition 6.9 is a generalization of the map $P_x$ on the structurable algebra $\mathcal{A}$ defined in (3.32); indeed using (3.6) we obtain

$$\begin{aligned} P_{(x,0)}a = U_x(a + \frac{2}{3}\psi(\hat{x}, a)x) &= U_x(a + \frac{2}{3}(V_{\hat{x},x}a - V_{a,x}\hat{x})) \\ &= \frac{1}{3}U_x(5a - 2V_{a,x}\hat{x}) = P_x a. \end{aligned}$$

## 6.2 Structurable division algebras are one-invertible

It follows from Lemma 6.11 that if $\mathcal{A}$ is a structurable algebra such that each element in $\mathcal{A} \times \mathcal{S} \setminus (0, 0)$ is one-invertible, then $\mathcal{A}$ is a conjugate division algebra. In Theorem 6.18 we show that the converse is true as well. If $\mathcal{A}$ is a conjugate division algebra, we show that each element in $\mathcal{A} \times \mathcal{S} \setminus (0, 0)$ is one-invertible, and we determine the left and right inverse.

We start by showing that $(1, s) \in \mathcal{A} \times \mathcal{S}$ is always one-invertible if $\mathcal{A}$ is conjugate division.

**Lemma 6.12** (J. Faulkner)**.** *Let $\mathcal{A}$ be a structurable division algebra, let $s \neq 0 \in \mathcal{S}$. We have that $s + \hat{s} \neq 0$ and*

$$u := -\hat{s}\widehat{(s+\hat{s})} = \widehat{1-s^2}, \quad t := \widehat{s+\hat{s}}$$

*is the solution of*

$$\begin{aligned} V_{1,u} &= \mathrm{id} + L_s L_t, \\ su &= -\frac{1}{3} U_1 t, \\ \psi(1, st) &= 0. \end{aligned} \tag{6.4}$$

*By Theorem 6.10, $(1, s)$ is one-invertible.*

*Proof.* It follows from (3.30) and (3.13) that $-\hat{s}\widehat{(s+\hat{s})} = \widehat{1-s^2}$. It follows from (3.13) that $s + \hat{s} = 0$ if and only if $s^2 = 1$.

We suppose that $s^2 = 1$ and deduce a contradiction. Define $a = 1 + s$, it follows from (3.3) that $L_s L_s = L_{s^2} = \mathrm{id}$, therefore $(xa)\overline{a} = (x\overline{a})a = 0$ for all $x \in \mathcal{A}$. Since $a \neq 0$, it is conjugate invertible, we find that

$$a = V_{a,\hat{a}} a = 2(a\overline{\hat{a}})a - (a\overline{a})\hat{a} = 2(a\overline{\hat{a}})a$$

and

$$a = V_{\hat{a},a} a = (\hat{a}\overline{a})a + (a\overline{a})\hat{a} - (a\overline{\hat{a}})a = -(a\overline{\hat{a}})a,$$

a contradiction. It follows that $s + \widehat{s} \neq 0$. Next we will prove the following identity for $t = \widehat{s+\hat{s}}$

$$L_{st} = L_s L_t = L_t L_s. \tag{6.5}$$

By (3.13) and the fact that $L_s^2 = L_{s^2}$ we have $[L_s, L_{s+\hat{s}}] = 0$, therefore $[L_s, -L_{s+\hat{s}}^{-1}] = [L_s, L_t] = 0$. Hence $[s,t] = [L_s, L_t](1) = 0$ and from (3.3) it follows that for all $y \in \mathcal{A}$

$$\begin{aligned} 2(L_{st} - L_s L_t)y = 2[s,t,y] &= [s,t,y] - [t,s,y] \\ &= (L_{[s,t]} - [L_s, L_t])y = 0, \end{aligned}$$

and (6.5) follows. Now we can verify that $u = -\hat{s}\widehat{(s+\hat{s})}$ and $t = \widehat{s+\hat{s}}$ are solutions of (6.4).

(1) Since the conjugate inverse of a skew-element is again a skew-element and $\hat{s}$ and $t$ commute, we have $\overline{u} = u$; hence $V_{1,u} = L_u$. From (6.5) it follows that $\mathrm{id} + L_s L_t = \mathrm{id} + L_t L_s = L_{1+st}$, the first equation of (6.4) is satisfied since
$$1 + st = -(s+\hat{s})t + st = -\hat{s}t = u.$$

(2) We have $\frac{1}{3} U_1 t = -t = L_s L_{\hat{s}} t = -su$.

(3) We have $\psi(1, st) = \overline{st} - st = ts - st = 0$ by (6.5). □

From now on let $\mathcal{A}$ be a conjugate division algebra and let $(x, s) \in \mathcal{A} \times \mathcal{S}$ with $x \neq 0$ and $s \neq 0$. Our aim is to determine the solution of (6.2). It follows from (3.39) that $x = \widehat{\hat{x}}$ is the unity in the structurable algebra $A^{\langle \hat{x} \rangle}$, an isotope of $\mathcal{A}$ described in Construction 3.23. From the previous lemma we know that $(x, s)$ is one-invertible in the algebra $A^{\langle \hat{x} \rangle}$. We will show that this implies that $(x, s)$ is one-invertible in $\mathcal{A}$. We are indebted to John Faulkner for bringing this method to our attention.

**Definition 6.13.** Let $(x, s) \in \mathcal{A} \times \mathcal{S}$, define

$$\alpha_x : \mathcal{A} \to \mathcal{A}^{\langle \hat{x} \rangle} : y \mapsto y,$$

note that $\alpha_x(1)$ is not the unit in $\mathcal{A}^{\langle \hat{x} \rangle}$, but that $x$ is the unit. The map $\alpha_x$ is an isotopy (see Definition 3.21) with $\hat{\alpha}_x = P_x$. Indeed using (3.38) we get

$$\alpha_x\{x, y, z\} = \{\alpha_x x, \hat{\alpha}_x y, \alpha_x z\}^{\langle \hat{x} \rangle} \iff \{x, y, z\} = \{x, P_{\hat{x}} P_x y, z\}.$$

which holds by (3.33).

We need to determine a map from $\mathcal{S}$ onto $\mathcal{S}^{\langle \hat{x} \rangle} = \mathcal{S}\hat{x}$ that is compatible with $\alpha_x$ and $\hat{\alpha}_x$.

**Definition 6.14.** Let $x \in \mathcal{A}$, define the $k$-linear map

$$q_x : \mathcal{S} \to \mathcal{S} : s \mapsto \frac{1}{6}\psi(x, U_x(sx))$$

Define the $k$-linear maps

$$\beta_x : \mathcal{S} \mapsto \mathcal{S}^{\langle \hat{x} \rangle} : s \mapsto s\hat{x},$$
$$\hat{\beta}_x : \mathcal{S} \mapsto \mathcal{S}^{\langle \hat{x} \rangle} : s \mapsto q_x(s)\hat{x}.$$

**Lemma 6.15.** *Let $\mathcal{A}$ be a structurable division algebra. Then for all $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$, we have*

(i) $q_x(s)\hat{x} = P_x(sx) = -\frac{1}{3}U_x sx = \frac{1}{2}\psi(P_x s, P_x 1)\hat{x}$.
(ii) $(q_x)^{-1} = q_{\hat{x}}$, *hence* $\hat{\beta}$ *is a bijection.*
(iii) $\widehat{q_x(s)} = q_{\hat{x}}(\hat{s})$.
(iv) *We have for all $y \in \mathcal{A}$ and $s \in \mathcal{S}$ that*

$$\alpha_x(L_s y) = L^{\langle \hat{x} \rangle}_{\beta_x(s)} \hat{\alpha}_x y \quad \text{and} \quad \hat{\alpha}_x(L_s y) = L^{\langle \hat{x} \rangle}_{\hat{\beta}_x(s)} \alpha_x y.$$

*Proof.* (i) (3.35) states that indeed $P_x(sx) = -\frac{1}{3}U_x(sx)$. It follows from (3.14) that $U_x sx = -\frac{1}{2}\psi(x, U_x sx)\hat{x}$.

Combining (3.36) and (3.6) gives for all $u, y, z \in \mathcal{A}$

$$P_u L_{\psi(y,z)} = L_{\psi(P_u y, P_u z)} P_{\hat{u}}, \tag{6.6}$$

by (3.34) we have

$$\tfrac{1}{2}\psi(P_x s, P_x 1)\hat{x} = \tfrac{1}{2}(P_x(\psi(s,1)x)) = P_x(sx).$$

(ii) We show that $q_{\hat{x}}(q_x(s))x = sx$, which implies that $q_{\hat{x}} \circ q_x = \mathrm{id}$. Similarly one shows that $q_x \circ q_{\hat{x}} = \mathrm{id}$.
Using (i) we find

$$q_{\hat{x}}(q_x(s))x = P_{\hat{x}}(q_x(s)\hat{x}) = P_{\hat{x}}(P_x(sx)) = sx.$$

(iii) By (3.30) and (3.29)

$$\widehat{(q_x(s))}x = (\widehat{q_x(s)\hat{x}}) = \widehat{P_x(sx)} = P_{\hat{x}}(\hat{s}\hat{x}) = q_{\hat{x}}(\hat{s})x,$$

it follows that $\widehat{q_x(s)} = q_{\hat{x}}(\hat{s})$.

(iv) The first equality follows by (3.42):

$$L^{\langle \hat{x} \rangle}_{\beta_x(s)} \hat{\alpha}_x y = L^{\langle \hat{x} \rangle}_{s\hat{x}} P_x y = L_s P_{\hat{x}} P_x y = L_s y.$$

The second equality follows by (i), (3.42) and (6.6):

$$\begin{aligned} L^{\langle \hat{x} \rangle}_{\hat{\beta}_x(s)} \alpha_x y &= L^{\langle \hat{x} \rangle}_{\frac{1}{2}\psi(P_x s, P_x 1)\hat{x}} y = L_{\frac{1}{2}\psi(P_x s, P_x 1)} P_{\hat{x}} y \\ &= P_x(L_{\frac{1}{2}\psi(s,1)} y) = P_x L_s y. \end{aligned}$$

□

**Lemma 6.16.** *Let $(x, s) \in \mathcal{A} \times \mathcal{S} \setminus \{0, 0\}$ and $(u, t) \in \mathcal{A} \times \mathcal{S} \setminus \{0, 0\}$. Then $(u, t)$ is the solution of the equations (6.2) w.r.t. $(x, s)$ in $\mathcal{A}$ if and only if $(\hat{\alpha}_x u, \hat{\beta}_x t) \in \mathcal{A}^{\langle \hat{x} \rangle} \times \mathcal{S}^{\langle \hat{x} \rangle}$ is the solution of the equations (6.2) w.r.t. $(\alpha_x x, \beta_x s)$ in $\mathcal{A}^{\langle \hat{x} \rangle}$.*

*Proof.* Let $(x, s) \in \mathcal{A} \times \mathcal{S} \setminus \{0, 0\}$ and denote $\alpha := \alpha_x, \hat{\alpha} := \hat{\alpha}_x, \beta := \beta_x, \hat{\beta} := \hat{\beta}_x$. Suppose that the following equations hold in $\mathcal{A}$

$$\begin{aligned} V_{x,u} &= \mathrm{id} + L_s L_t, \\ su &= -\frac{1}{3} U_x t x, \\ \psi(x, s(tx)) &= 0. \end{aligned}$$

Applying the isotopy $\alpha$ we obtain using Lemma 6.15 that

$$V^{\langle \hat{x} \rangle}_{\alpha x, \hat{\alpha} u} \alpha = \alpha + L^{\langle \hat{x} \rangle}_{\beta s} L^{\langle \hat{x} \rangle}_{\hat{\beta} t} \alpha,$$

$$\begin{aligned} L^{\langle\hat{x}\rangle}_{\beta s}\hat{\alpha}u &= -\frac{1}{3}U^{\langle\hat{x}\rangle}_{\alpha x}L^{\langle\hat{x}\rangle}_{\hat{\beta}t}\alpha x, \\ \beta(\psi(x, s(tx))) &= 0. \end{aligned}$$

We determine what $\beta\psi(x, s(tx))$ is in terms of the multiplication of $\mathcal{A}^{\langle\hat{x}\rangle}$. By (3.41), we have

$$\beta\psi(x, s(tx)) = \psi(x, s(tx))\hat{x} = \psi^{\langle\hat{x}\rangle}(\alpha x, \alpha s(tx)) = \psi^{\langle\hat{x}\rangle}(\alpha x, L^{\langle\hat{x}\rangle}_{\beta s}L^{\langle\hat{x}\rangle}_{\hat{\beta}t}\alpha x).$$

Since $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ are bijections we conclude that $(u, t)$ is the solution of the equations for $(x, s)$ in $\mathcal{A}$ if and only if $(\hat{\alpha}u, \hat{\beta}t)$ is the solution of the equations for $(\alpha x, \beta s)$ in $\mathcal{A}^{\langle\hat{x}\rangle}$. □

**Remark 6.17.** Define the Lie algebra $\mathcal{G}^{\langle\hat{x}\rangle}$ as the Lie algebra obtained by applying Definition 6.1 to $\mathcal{A}^{\langle\hat{x}\rangle}$. Define the graded bijection $\gamma_x : \mathcal{G} \to \mathcal{G}^{\langle\hat{x}\rangle}$ given by $\gamma_x : \mathcal{G}_i \to \mathcal{G}^{\langle\hat{x}\rangle}_i$ for $i \in [-2, 2]$ such that

$$\begin{aligned} \gamma_x|_{\mathcal{G}_1} &= \alpha_x, & \gamma_x|_{\mathcal{G}_{-1}} &= \widehat{\alpha}_x, \\ \gamma_x|_{\mathcal{G}_2} &= \beta_x, & \gamma_x|_{\mathcal{G}_{-2}} &= \widehat{\beta}_x, \\ \gamma_x(V_{a,b}) &= V^{\langle\hat{x}\rangle}_{a,P_x b}, & \gamma_x(\zeta) &= \zeta. \\ &= V_{a,b} \end{aligned}$$

Using (3.41), (3.42), (6.6) and Lemma 6.15.(iii), one can verify that $\gamma_x$ is a Lie algebra isomorphism. One could use this fact to prove Lemma 6.16, but to prove this lemma we needed to verify less identities than one needs to verify to show that $\gamma_x$ is an isomorphism.

In [AH81, Section 12] it is shown that each isomorphism of $K(A)$ and $K(A')$ can be obtained in a similar way from an isotopy between $\mathcal{A}$ and $\mathcal{A}'$.

We can now prove the main theorem of this section.

**Theorem 6.18.** *Let $\mathcal{A}$ be a structurable division algebra, and let $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$. Then $(x, s)$ is one-invertible. The solution of the system of equations (6.2) is*

$$u = \big(x - s(q_{\hat{x}}(s)x)\big)^{\wedge} = -\hat{s}\big((q_{\hat{x}}(s) + \hat{s})^{\wedge}\hat{x}\big) \quad \textit{and} \quad t = \big(s + q_x(\hat{s})\big)^{\wedge},$$

*where the expressions of which the conjugate inverse is taken are different from zero.*

*Proof.* Let $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$ and denote $\alpha := \alpha_x, \hat{\alpha} := \hat{\alpha}_x, \beta := \beta_x, \hat{\beta} := \hat{\beta}_x$. We have $(\alpha x, \beta s) = (x, s\hat{x})$. Now $x = \hat{\hat{x}} = 1^{\langle\hat{x}\rangle}$, by (3.39). We can apply Lemma 6.12 to find the solution of the equations (6.2) for $(x, s\hat{x})$ in $\mathcal{A}^{\langle\hat{x}\rangle}$. We use Lemma 6.16 to translate this solution back to $\mathcal{A}$.

In Lemma 6.12 we found that for $(1, s)$ we have that $u = -L_{\hat{s}} t = \widehat{1 - s^2}$ and $t = \widehat{s + \hat{s}}$. It also follows from this lemma that the expressions of which the conjugate inverses are considered, are never zero.

In $\mathcal{A}^{\langle \hat{x} \rangle}$ we have that $\hat{\beta}(t)$ is equal to, by (3.43), Lemma 6.15.(i) and (3.30)

$$\begin{aligned} \hat{\beta}(t) = \big(\beta(s) + (\beta(s))^{\wedge\langle \hat{x} \rangle}\big)^{\wedge\langle \hat{x} \rangle} &= P_x\big(\big(s\hat{x} + P_x(\widehat{s\hat{x}})\big)^{\wedge}\big) \\ &= P_x\big(\big(s\hat{x} + P_x(\hat{s}x)\big)^{\wedge}\big) \\ &= P_x\big(\big(s\hat{x} + q_x(\hat{s})\hat{x}\big)^{\wedge}\big) \\ &= P_x\big(\big(s + q_x(\hat{s})\big)^{\wedge} x\big) \\ &= q_x\big(\big(s + q_x(\hat{s})\big)^{\wedge}\big)\hat{x}. \end{aligned}$$

It follows that $t = \big(s + q_x(\hat{s})\big)^{\wedge}$. In order to determine $\hat{\alpha_x}(u)$ in $\mathcal{A}^{\langle \hat{x} \rangle}$ we simplify,

$$L_{\beta s}^{\langle \hat{x} \rangle 2} \alpha x = L_{s\hat{x}}^{\langle \hat{x} \rangle 2} x = L_{s\hat{x}}^{\langle \hat{x} \rangle} s P_{\hat{x}} x = L_{s\hat{x}}^{\langle \hat{x} \rangle} s\hat{x} = s P_{\hat{x}}(s\hat{x}) = s(q_{\hat{x}}(s)x).$$

Now we have that

$$\hat{\alpha}(u) = \big(\alpha x - L_{\beta s}^{\langle \hat{x} \rangle 2} \alpha x\big)^{\wedge\langle \hat{x} \rangle} = P_x\big(\big(x - L_{s\hat{x}}^{\langle \hat{x} \rangle 2} x\big)^{\wedge}\big) = P_x\big(\big(x - s(q_{\hat{x}}(s)x)\big)^{\wedge}\big),$$

therefore

$$u = \big(x - s(q_{\hat{x}}(s)x)\big)^{\wedge} = \big(-s(\hat{s}x) - s(q_{\hat{x}}(s)x)\big)^{\wedge} = -\hat{s}\big(\big(q_{\hat{x}}(s) + \hat{s}\big)^{\wedge}\hat{x}\big). \qquad \square$$

Combining Theorem 6.10, Lemma 6.11 and the previous theorem, we obtain an expression for the left and right inverses of elements in $\mathcal{A} \times \mathcal{S}$:

**Corollary 6.19.** *Let $\mathcal{A}$ be a structurable division algebra. Then all elements in $\mathcal{A} \times \mathcal{S} \setminus \{(0, 0)\}$ are one-invertible. Let $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$,*

- *The left and right inverse of $(x, 0)$ is $(\hat{x}, 0)$.*
- *The left and right inverse of $(0, s)$ is $(0, \hat{s})$.*
- *The right inverse of $(x, s)$ is*

$$\big(-\hat{s}\big((q_{\hat{x}}(s) + \hat{s})^{\wedge}\hat{x}\big) + \big(s + q_x(\hat{s})\big)^{\wedge} x,\ \big(s + q_x(\hat{s})\big)^{\wedge}\big),$$

*the left inverse of $(x, s)$ is*

$$\big(-\hat{s}\big((q_{\hat{x}}(s) + \hat{s})^{\wedge}\hat{x}\big) - \big(s + q_x(\hat{s})\big)^{\wedge} x,\ \big(s + q_x(\hat{s})\big)^{\wedge}\big).$$

## 6.3 The construction of Moufang sets

In Section 2.4 we introduced several concepts from the theory of Moufang sets. We start by proving that structurable division algebras give rise to abstract rank one groups; we use the terminology introduced in Section 6.1.

**Theorem 6.20.** *Let $\mathcal{A}$ be a structurable division algebra over a field $k$ of characteristic different from 2, 3 and 5. Then the elementary group $G$ of $\mathcal{A}$ is an abstract rank one group with unipotent subgroups $U_+$ and $U_-$.*

*Proof.* By definition $G = \langle U_+, U_- \rangle$; $U_+$ and $U_-$ are nilpotent subgroups (of nilpotency class 2), since by Lemma 6.5 we have that

$$[[e_\sigma(x,s), e_\sigma(y,t)], e_\sigma(z,r)] = [e_\sigma(0, 2\psi(x,y)), e_\sigma(z,r)] = 0$$

for all $x, y, z \in \mathcal{A}$, $s, t, r \in \mathcal{S}$.

Let $(x,s) \in \mathcal{A} \times \mathcal{S} \setminus \{(0,0)\}$, let $\sigma = \pm 1$, denote $a = e_\sigma(x,s) \in U_\sigma \setminus \{0\}$. By Corollary 6.19 $(x,s)$ is one-invertible; thus by Definition 6.9 there exist unique elements $y, z \in \mathcal{A}, t \in \mathcal{S}$ such that

$$h := e_{-\sigma}(z,t) e_\sigma(x,s) e_{-\sigma}(y,t) \in H_-. \tag{6.7}$$

Define $b(a) := e_{-\sigma}(-y,-t) = h^{-1} e_{-\sigma}(z,t) a$. Then using Lemma 6.7.(i) we obtain

$$U_\sigma^{b(a)} = U_{-\sigma}^{e_{-\sigma}(z,t)a} = (U_{-\sigma})^a.$$

This proves the condition in Definition 2.20. □

Applying Lemma 2.21 to the abstract rank one group of the previous theorem, gives us a Moufang set $\mathbb{M}$. We will use Construction 2.22 to bring this Moufang set in the form $\mathbb{M}(\mathcal{U}, \tau)$ with $\tau$ a permutation of $\mathcal{U}^*$. The groups $U_+$ and $U_-$ are root groups of the Moufang set we constructed. We define an addition on $\mathcal{A} \times \mathcal{S}$, such that $\mathcal{A} \times \mathcal{S}$ is a group isomorphic to $U_+$ and $U_-$.

**Definition 6.21.** Let $\mathcal{U} := \mathcal{A} \times \mathcal{S}$ be the (non-abelian) group with addition

$$(x,s) + (y,t) = (x + y, s + t + \psi(x,y)).$$

In particular, $\mathcal{U} \cong U_+ \cong U_-$. We will also write 0 for $(0,0) \in \mathcal{U}$, and we will use the notation $\mathcal{U}^*$ for $\mathcal{U} \setminus \{0\}$.

For each element $u = (x,s) \in \mathcal{U}$, we set

$$e_+(u) = e_+(x,s) \quad \text{and} \quad e_-(u) = e_-(x,s).$$

**Construction 6.22.** (i) Let $G = \langle U_+, U_- \rangle$ be the abstract rank one group from Theorem 6.20. The set $Y$ of the Moufang set $\mathbb{M}$ obtained from Lemma 2.21 is given by

$$Y = \{(U_-)^{e_+(u)} \mid u \in \mathcal{U}\} \cup \{U_+\}.$$

We identify $Y$ with $X = \mathcal{U} \cup \{\infty\}$ by identifying

$$(U_-)^{e_+(u)} \longleftrightarrow u$$

$$U_+ \longleftrightarrow \infty.$$

The action of elements in $G = \langle U_+, U_- \rangle$ on $Y$ is given by conjugation, and this induces an equivariant action of $G$ on $X = \mathcal{U} \cup \{\infty\}$. We denote

$$U_\infty := \infty = U_+ \quad \text{and} \quad U_0 := 0 = U_-.$$

(ii) Let $a \in \mathcal{U}$, then the unique element in $U_\infty$ that maps $0 \mapsto a$ is given by $\alpha_a = e_+(a)$, indeed $e_+(a) \in U_\infty$ and

$$(0)e_+(a) = (U_-)^{e_+(0)e_+(a)} = (U_-)^{e_+(a)} = a.$$

It follows that for all $a, b \in \mathcal{U}$ we have $a + b = a\alpha_b$ and $\mathcal{U} \cong U_\infty = \{\alpha_a \mid a \in \mathcal{U}\}$.

(iii) For each $u = (x, s) \in \mathcal{U}^*$, we define $\mu_u$ to be the unique element in the double coset $U_0\alpha_u U_0$ interchanging the elements 0 and $\infty$ of $X$ (see Lemma 2.23). By (6.7) and Lemma 6.7.(i), we have

$$\mu_u = \mu_{(x,s)} = e_-(z, t)\, e_+(x, s)\, e_-(y, t) \in H_-, \tag{6.8}$$

where $(y, t) = (u - tx, t)$ and $(z, t) = (u + tx, t)$ are the left and right inverse of $(x, s)$, respectively.

(iv) Let $e = (1, 0) \in \mathcal{U}^*$, we define $\tau = \mu_e$. We have that $U_0 = U_\infty^\tau$.

As in (ii) define $\alpha_u = e_+(u)$ for each $u \in \mathcal{U}$, then

$$U_\infty = \{\alpha_u \mid u \in \mathcal{U}\} \quad \text{and} \quad U_0 = \{\alpha_u^\tau \mid u \in \mathcal{U}\}.$$

We still need to describe explicitly the action of $\tau$ on $\mathcal{U}^*$. First we determine the explicit action of $\tau$ on the Lie algebra $\mathcal{G}$.

**Lemma 6.23.** *The automorphism $\tau = \mu_{(1,0)} \in H_-$ is equal to the "gradation flipping" automorphism $\varphi$ defined in Definition 6.6.*

*This implies that $\tau$ is an involution and that for each $u \in \mathcal{U}$, we have $e_-(u) = e_+(u)^\tau$.*

*Proof.* We first observe that the left and right inverse of $e = (1, 0)$ are both equal to $e = (1, 0)$ again, and hence

$$\tau = \mu_{(1,0)} = e_-(1, 0)\, e_+(1, 0)\, e_-(1, 0) \in H_-.$$

Since $\tau \in H_-$, we have that $\zeta.\tau = -\zeta$. We now verify that $\tau$ maps every element $x \in \mathcal{G}_\pm$ to the corresponding element $x \in \mathcal{G}_\mp$.

Indeed, by Theorem 6.10.(iii), we know that $\tau|_{\mathcal{G}_+} = P_{(1,0)}$ and $\tau|_{\mathcal{G}_-} = P_{(1,0)}$, which we can compute explicitly. We get for all $a \in \mathcal{G}_\pm$

$$P_{(1,0)}a = U_1(a + \tfrac{2}{3}\psi(1, a)1) = U_1(a + \tfrac{2}{3}(\overline{a} - a)1)$$

$$= \tfrac{1}{3}U_1(a+2\overline{a}) = \tfrac{1}{3}(2(\overline{a}+2a)-(a+2\overline{a})) = a \in \mathcal{G}_{\mp}.$$

Since $\tau$ is an automorphism of $\mathcal{G}$, we find for all $s \in \mathcal{G}_{\pm 2}$ that

$$s.\tau = \tfrac{1}{2}\psi(s,1).\tau = -\tfrac{1}{4}[s,1].\tau = -\tfrac{1}{4}[s.\tau, 1.\tau] = \tfrac{1}{2}\psi(s,1) = s \in \mathcal{G}_{\mp 2},$$

and for all $a \in \mathcal{G}_+$, $b \in \mathcal{G}_-$ that

$$V_{a,b}.\tau = -\tfrac{1}{2}[a,b].\tau = -\tfrac{1}{2}[a.\tau, b.\tau] = \tfrac{1}{2}[b,a] = -V_{b,a}.$$

We conclude that $\tau = \varphi$, which is clearly an involution. Since $\tau \in H_-$, it follows from Lemma 6.7.(i) that

$$e_+(x,s)^\tau = e_-(x.\tau, s.\tau) = e_-(x,s)$$

for all $(x,s) \in \mathcal{U}$. □

We can now determine the action of $\tau$ on $\mathcal{U}^*$ using Lemma 2.26.

**Theorem 6.24.** *The map $\tau = \mu_{(1,0)}$ maps each element $(x,s) \in \mathcal{U}^*$ to $(-y,-t)$, where $(y,t)$ is the left inverse of $(x,s)$.*

*Proof.* Let $u = (x,s) \in \mathcal{U}^*$ be arbitrary. By (6.8) and Lemma 6.23, we have

$$\mu_u = \mu_{(x,s)} = \alpha^\tau_{(z,t)}\,\alpha_{(x,s)}\,\alpha^\tau_{(y,t)}.$$

On the other hand, it follows from Lemma 2.26 that

$$\mu_u = \alpha^\tau_{(-u)\tau^{-1}}\,\alpha_u\,\alpha^\tau_{-(u\tau^{-1})}.$$

By the uniqueness of the $\mu$-maps (see Lemma 2.23), the last terms are equal

$$(x,s).\tau^{-1} = (-y,-t).$$

The lemma follows since $\tau$ is an involution. □

Now we can prove the main theorem of this section:

**Theorem 6.25.** *Let $\mathcal{A}$ be a structurable division algebra over a field of characteristic different from 2, 3 and 5. Define the group $\mathcal{U} := A \times S$ with addition*

$$(x,s)+(y,t) = (x+y, s+t+\psi(x,y)).$$

*Define $q_x : \mathcal{S} \to \mathcal{S} : s \mapsto \frac{1}{6}\psi(x, U_x(sx))$ and define the permutation $\tau$ of $\mathcal{U}^*$ for all $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$*

$$\begin{aligned}(x,0) &\mapsto (-\hat{x}, 0),\\ (0,s) &\mapsto (0, -\hat{s}),\\ (x,s) &\mapsto \big(\,\hat{s}\big((q_{\hat{x}}(s)+\hat{s})^\wedge \hat{x}\big) + \big(s+q_x(\hat{s})\big)^\wedge x,\ -\big(s+q_x(\hat{s})\big)^\wedge\big).\end{aligned}$$

*Then $\mathbb{M}(\mathcal{U},\tau)$ is a Moufang set.*

*Proof.* Using Corollary 6.19, we find the explicit form of the left inverses of $(x, s)$, those describe $\tau$ using Theorem 6.24.

We started from the Moufang set $\mathbb{M}$ obtained from the abstract rank one group in Theorem 6.20 and described $\mathcal{U}$ and $\tau$ such that $\mathbb{M}(\mathcal{U}, \tau)$ is equal to $\mathbb{M}$. Therefore $\mathcal{U}$ and $\tau$ give rise to a Moufang set. □

**Remark 6.26.** Let $\mathcal{A}$ and $\mathcal{A}'$ be isotopic structurable algebras, by Theorem 3.22 the Lie algebras $K(\mathcal{A})$ and $K(\mathcal{A}')$ are graded isomorphic. It follows from Lemma 2.29 and Theorem 6.20 that the Moufang sets constructed by Theorem 6.25 from $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic.

It is not clear to us whether the fact that the Moufang sets obtained from $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic implies that $\mathcal{A}$ and $\mathcal{A}'$ are isotopic.

We will determine the Hua maps, defined in Definition 2.27, of the Moufang set obtained in the previous theorem. We get a (surprisingly) elegant expression.

**Theorem 6.27.** *Let $\mathbb{M}(\mathcal{U}, \tau)$ be as in Theorem 6.25. Then for all $(x, s) \in \mathcal{U}^*$ and $(a, r) \in \mathcal{U}$*

$$\begin{aligned}(a, r).h_{(x,s)} &= (P_{(x,s)}a,\ q_x(r) + \psi(x, s(rx)) - s(rs)) \\ &= (P_{(x,s)}a,\ \tfrac{1}{2}\psi(P_{(x,s)}r, P_{(x,s)}1)).\end{aligned}$$

*Proof.* Let $(x, s) \in \mathcal{U}^*$ and $(a, r) \in \mathcal{U}$, let $(z, t)$ denote the right inverse of $(x, s)$ and let $(y, t)$ denote the left inverse of $(x, s)$.

By definition $h_{(x,s)} = \tau\mu_{(x,s)} \in \operatorname{Aut}(\mathcal{G})$ and $h_{(x,s)}$ preservers the gradation of $\mathcal{G}$. Hence we have

$$e_+(a, r)^{h_{(x,s)}} = e_+(a.h_{(x,s)}, r.h_{(x,s)}).$$

By Lemma 6.23 we find

$$e_+(a, r)^{h_{(x,s)}} = e_-(a, r)^{\mu_{(x,s)}} = e_+(a.\mu_{(x,s)}, r.\mu_{(x,s)}).$$

Combining the last two equalities, we find that

$$(a.r).h_{(x,s)} = (a.\mu_{(x,s)}, r.\mu_{(x,s)}), \tag{6.9}$$

in the right hand side expression we have to consider $a \in \mathcal{G}_-$, $r \in \mathcal{G}_{-2}$ and $x \in \mathcal{G}_+$, $s \in \mathcal{G}_{+2}$.

By Construction 6.22.(iii) have $\mu_{(x,s)} = e_-(z, t)e_+(x, s)e_-(y, t)$. By Theorem 6.10.(iii), we have that $\mu_{(x,s)}|_{\mathcal{G}_-} = P_{(x,s)}$. Hence in (6.9) we find $a.\mu_{(x,s)} = P_{(x,s)}a$. Since $\mu_{(x,s)}$ is a Lie algebra morphism we have that

$$r.\mu_{(x,s)} = \tfrac{1}{2}[r, 1].\mu_{(x,s)} = \tfrac{1}{2}[r.\mu_{(x,s)}, 1.\mu_{(x,s)}] = \tfrac{1}{2}\psi(P_{(x,s)}r, P_{(x,s)}1).$$

Using the theory developed in [AF99], we can obtain an equivalent formula.

In the proof of [AF99, Theorem 12] it shown that for $h = \mu_{(x,s)} = e_-(z,t)e_+(x,s)e_-(y,t)$ there holds that $h|_{\mathcal{G}_{-2}} = \epsilon_2 e_+(x,s)|_{\mathcal{G}_{-2}}$, where $\epsilon_2$ denotes the projection $\mathcal{G} \to \mathcal{G}_{-2}$. In the proof of [AF99, Theorem 13] it is shown that[3]

$$\epsilon_2 e_+(x,s)|_{\mathcal{G}_{-2}} = \tfrac{1}{24}(\mathrm{ad}(x))^4 + \tfrac{1}{2}(\mathrm{ad}(x))^2\,\mathrm{ad}(s) + \tfrac{1}{2}(\mathrm{ad}(s))^2.$$

Using the definition of the Lie bracket of $\mathcal{G}$ we find for (6.9)

$$r.\mu_{(x,s)} = \tfrac{1}{6}\psi(x, U_x(rx)) + \psi(x, s(rx)) - s(rs). \qquad \square$$

## 6.4 Examples

We believe that the construction of Moufang sets in Theorem 6.25 will give rise to descriptions of Moufang sets from linear algebraic groups that have not been described explicitly before. We were not able to prove this yet, but we formulate the following conjecture. For the notations of the Tits index see Table 1.1 on page 13.

**Conjecture 6.28.** *Let $\mathcal{A}$ be a central simple structurable division algebra over a field of characteristic different from 2,3 and 5, such that $K(\mathcal{A})$ is a Lie algebra of type ${}^gX^t_{n,1}$. Then the Moufang set constructed in Theorem 6.25 starting from $\mathcal{A}$ is isomorphic to the Moufang set obtained from a linear algebraic group of type ${}^gX^t_{n,1}$.*

In this section we show that we can obtain all Moufang sets described in Section 2.4.2 using Theorem 6.25. These examples confirm the above conjecture, see especially Lemma 6.30 and 6.31 for two examples of exceptional Moufang sets.

If this conjecture is true, Theorem 3.19 ought to imply that we can give an explicit construction of the Moufang set obtained from any linear algebraic group of relative rank one in characteristic different from 2 and 3. We hope that this might give a clue to prove that any Moufang set in characteristic different from 2 and 3 is obtained from a linear algebraic group or from a classical group. This fact is true for Moufang buildings of rank greater than one, see for example Theorem 2.2.

Below, we apply Theorem 6.25 to several central simple structurable division algebras. We show that in the case that the structurable algebra is a Jordan algebra, associative, of hermitian type or equal to an octonion algebra, the described Moufang set is isomorphic to a Moufang set described

[3]This can be easily verified out of the definition of $e_+(x,s)$.

in Section 2.4.2. We show also that certain structurable algebras of skew-dimension one give rise to Moufang sets described in Section 2.4.2.

First we show that in the case that $s^2 \in k1$ and $q_x(s) \in ks$ the formulas for the calculation of the $\tau$-map in Theorem 6.25 simplify significantly.

**Lemma 6.29.** *Let $\mathcal{A}$ be a structurable division algebra such that $n(s) := s\overline{s} = -s^2$ is a quadratic form on $\mathcal{S}$ and such that $q_x = \nu(x)\mathrm{id}$ for all $x \in \mathcal{A}$ for some form $\nu$ of degree 4 on $\mathcal{A}$.*

*Let $0 \neq x \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$, then the solution of the system of equations (6.2) given in Theorem 6.18 simplifies to*

$$u = \frac{1}{n(s) + \nu(x)}\nu(x)\hat{x}, \quad t = \frac{1}{n(s) + \nu(x)}s,$$

*Hence*

$$(x, s).\tau = \big(\frac{-1}{n(s) + \nu(x)}(-\nu(x)\hat{x} + sx),\ -\frac{1}{n(s) + \nu(x)}s\big),$$

*for all $(x, s) \in \mathcal{A} \times \mathcal{S} \setminus \{(0, 0)\}$.*

*Proof.* First we notice that in the above formula $(x, 0).\tau = (-\hat{x}, 0)$ and $(0, s).\tau = (0, s^{-1} = -\hat{s})$ which is equivalent with the $\tau$ in Theorem 6.25. From now on assume that $0 \neq x$ and $0 \neq s$. We first calculate $t$:

$$\begin{aligned} t &= (s + q_x(\hat{s}))^\wedge = (s + \nu(x)\hat{s})^\wedge \\ &= \big((n(s) + \nu(x))\hat{s}\big)^\wedge = (n(s) + \nu(x))^{-1}s \end{aligned}$$

To calculate $u$, note that $n(\hat{s}) = -(\hat{s})^2 = -(s^2)^\wedge = n(s)^{-1}$; by Lemma 6.15.(iii) we have $\nu(\hat{x}) = \nu(x)^{-1}$ since

$$\nu(\hat{x})s = q_{\hat{x}}(s) = \widehat{q_x(\hat{s})} = \widehat{\nu(x)\hat{s}} = \nu(x)^{-1}s.$$

We find

$$\begin{aligned} u &= -\hat{s}\big((q_{\hat{x}}(s) + \hat{s})^\wedge \hat{x}\big) \\ &= -\hat{s}\big(((\nu(x)^{-1} + n(s)^{-1})s)^\wedge \hat{x}\big) \\ &= (\nu(x)^{-1} + n(s)^{-1})^{-1} n(\hat{s})\hat{x} \\ &= \big(\frac{\nu(x) + n(s)}{\nu(x)n(s)}\big)^{-1} n(s)^{-1}\hat{x} \\ &= (\nu(x) + n(s))^{-1}\nu(x)\hat{x}. \end{aligned}$$

The left inverse of $(x, s)$ is $(u - tx, t)$, by Theorem 6.24 $\tau$ maps $(x, s)$ on $(-y, -t) = (-u + tx, -t)$. □

**Jordan algebras** Let the structurable division algebra $\mathcal{A} = J$ be a Jordan algebra, by Example 3.3.2 we know that this is a Jordan division algebra with $\hat{x} = x^{-1}$, where $x^{-1}$ is the Jordan inverse in $J$.

In this case $\mathcal{S} = 0$, and by Theorem 6.25 $\mathcal{U}$ is the additive group of $J$ and $x.\tau = -\hat{x} = -x^{-1}$ for $x \in \mathcal{U}^*$. Thus our new construction of Moufang sets generalizes the one given in Example 2.4.2.1.

If $\mathcal{A}$ is an exceptional Jordan algebra, the Lie algebra $K(\mathcal{A})$ is of type $E_{7,1}^{78}$ (see (2) on page 60). On the other hand we already know that the Moufang set obtained from an exceptional Jordan algebra is also of type $E_{7,1}^{78}$ (see Section 2.4.2.1). This confirms Conjecture 6.28.

**Octonion algebras** Let $\mathcal{A}$ be an octonion division algebra (or a lower dimensional division composition algebra), in this case we can apply Lemma 6.29. It is indeed clear that $n(s) = -s^2 \in k$ is a quadratic form, and when evaluating $q_x$

$$\begin{aligned} q_x(s)\hat{x} = -\tfrac{1}{3}U_x(sx) &= -\tfrac{1}{3}(2x\overline{(sx)})x - (x\,\overline{x})(sx) \\ &= N(x)sx = N(x)^2(sx)(\overline{x}x)^{-1} = N(x)^2 s\overline{x}^{-1} = N(x)^2 s\hat{x}, \end{aligned}$$

we find that $q_x = N(x)^2\mathrm{id}$ and $\nu(x) = N(x)^2$. By applying Lemma 6.29 using that $\hat{x} = \overline{x}^{-1}$ we find

$$\begin{aligned} u &= \frac{1}{N(x)^2 - s^2}N(x)^2\hat{x}, & t &= \frac{1}{N(x)^2 - s^2}s, \\ &= \frac{1}{N(x)^2 - s^2}N(x)(x\overline{x})\overline{x}^{-1}, \\ &= \frac{1}{N(x)^2 - s^2}N(x)x. \end{aligned}$$

Therefore

$$(x,s).\tau = \big(\frac{1}{N(x)^2 - s^2}(-N(x)x + sx),\ -\frac{1}{N(x)^2 - s^2}s\big).$$

**Lemma 6.30.** *Let $\mathcal{A}$ be an octonion division algebra (or a lower dimensional division composition algebra). Then the Moufang set $\mathbb{M}(U,\tau)$ for $U = \mathcal{A}\times\mathcal{S}$ with addition*

$$(x,s) + (y,t) = (x + y, s + t + \psi(x,y))$$

*and*

$$(x,s).\tau = \left(\frac{1}{N(x)^2 - s^2}(-N(x)x + sx), -\frac{1}{N(x)^2 - s^2}s\right)$$

*is isomorphic to the Moufang set described in Section 2.4.2.3. If $\mathcal{A}$ is an octonion algebra, this Moufang set is of type $F_{4,1}^{21}$.*

*Proof.* We use the second description of the Moufang set given in Section 2.4.2.3, thus $U' = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid N(a) + T(b) = 0\}$ with addition $(a,b) + (c,d) = (a+c, b+d-a\bar{c})$ and $(a,b).\tau' = (2b^{-1}a, 4b^{-1})$.

We have that $U = \mathcal{A} \times \mathcal{S}$ and consider the following bijection:

$$\begin{aligned} \varphi : U' &\to U \\ (a,b) &\mapsto \Big[\frac{a}{2}, -\frac{b}{2} - N\Big(\frac{a}{2}\Big)\Big] \\ (2a, 2(-b - N(a))) &\leftarrow [a,b] \end{aligned}$$

Indeed, if $N(a) + T(b) = 0$ then $T\big(-\frac{1}{2}(b + \frac{N(a)}{2})\big) = -\frac{1}{2}(T(b) + N(a)) = 0$; Conversely, if $T(b) = 0$, then $T(2(-b - N(a))) = -4N(a) = -N(2a)$.

We verify that $\varphi$ is a group morphism, indeed for all $(a,b), (c,d) \in U$,

$$\begin{aligned} ([a,b].\varphi^{-1} + [c,d].\varphi^{-1}).\varphi &= ((2a, 2(-b - N(a))) + (2c, 2(-d - N(c)))).\varphi \\ &= ((2a + 2c, -2b - 2d - 2(N(a) + N(c)) - 4a\bar{c})).\varphi \\ &= [a + c, b + d + N(a) + N(c) + 2a\bar{c} - N(a + c)] \\ &= [a + c, b + d \\ &\quad + N(a) + N(c) + 2a\bar{c} - N(a) - N(c) - a\bar{c} - c\bar{a}] \\ &= [a + c, b + d + \psi(a, c)] = [a, b] + [c, d]. \end{aligned}$$

We verify that $\tau = \tau'^{\varphi}$, for all $[a,b] \in U \setminus \{0\}$ we have

$$\begin{aligned} [a,b].\varphi^{-1}\tau'\varphi &= (2a, 2(-b - N(a))).\tau'\varphi \\ &= (2(-b - N(a))^{-1}a, 2(-b - N(a))^{-1}).\varphi \\ &= \big[-(b + N(a))^{-1}a, (b + N(a))^{-1} - N((b + N(a))^{-1}a)\big] \\ &= \left[\frac{1}{N(a)^2 - b^2}(-N(a) + b)a, \frac{1}{N(a)^2 - b^2}(-b + N(a) - N(a))\right] \\ &= \left[\frac{1}{N(a)^2 - b^2}(-N(a) + b)a, -\frac{1}{N(a)^2 - b^2}b\right] = [a,b].\tau. \end{aligned}$$

It follows from Lemma 2.29 that these two Moufang sets are isomorphic. □

If $\mathcal{A}$ is an octonion division algebra, the Lie algebra $K(\mathcal{A})$ is of type $F_{4,1}^{21}$ (see (5) on page 60). This confirms Conjecture 6.28.

**Structurable algebras of skew-dimension 1** Let $\mathcal{A}$ be a structurable division algebra of skew-dimension 1, let $0 \neq s_0 \in \mathcal{S}$ with $s_0^2 = \mu$. Then $n(\alpha s_0) = \alpha^2\mu$ for $\alpha \in k$, hence $n$ is a quadratic form on $\mathcal{S}$. We have

$$q_x(\alpha s_0) = \frac{1}{6}\psi(x, U_x(\alpha s_0 x))$$

$$= \alpha \frac{1}{6\mu} \psi(x, U_x(s_0 x)) s_0 s_0 = \nu(x) \alpha s_0,$$

here $\nu(x)$ denotes the conjugate norm defined in (3.18) on page 50. It follows that we can apply Lemma 6.29. Since $\mathcal{S} = ks_0$,

$$\begin{aligned}
&(x, \alpha s_0).\tau \\
&\quad = \big(\frac{1}{\nu(x) - \alpha^2\mu}(-\nu(x)\frac{-1}{3\mu\nu(x)} s_0 U_x(s_0 x) + \alpha s_0 x),\ -\frac{1}{\nu(x) - \alpha^2\mu}\alpha s_0\big) \\
&\quad = \big(\frac{1}{\nu(x) - \alpha^2\mu}(\frac{1}{3\mu} s_0 U_x(s_0 x) + \alpha s_0 x),\ -\frac{1}{\nu(x) - \alpha^2\mu}\alpha s_0\big)
\end{aligned}$$

It follows from Definition 6.9.(iii) and Theorem 6.27 that

$$(a, r).h_{(0, -s_0/\mu)} = (-s_0 a/\mu, -s_0(r s_0)/\mu^2) = (-s_0 a/\mu, -r/\mu).$$

We evaluate $\tau' = \tau h_{(0, s_0)}$

$$(x, \alpha s_0).\tau' = \big(-\frac{1}{\nu(x)\mu - \alpha^2\mu^2}(\frac{1}{3}U_x(s_0 x) + \alpha\mu x),\ \frac{1}{\nu(x)\mu - \alpha^2\mu^2}\alpha s_0\big). \tag{6.10}$$

It follows from Lemma 2.29 that $\mathbb{M}(U, \tau)$ and $\mathbb{M}(U, \tau')$ are isomorphic

From now on we focus on structurable algebras described in Theorem 4.7.

**Lemma 6.31.** *Let $(k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra in characteristic not 2 and 3 and let $\mathcal{A} = X$ be the structurable algebra of skew-dimension 1 described in Theorem 4.7.*

*Then the Moufang set $\mathbb{M}(\mathcal{A} \times ks_0, \tau')$ described above is isomorphic to the Moufang set obtained from Moufang quadrangles of type $E_6$, $E_7$ and $E_8$ described on page 42. It follows that in the $E_8$-case this Moufang set is of type $E_{7,1}^{66}$.*

*Proof.* In this case (6.10) becomes

$$(x, \alpha s_0).\tau' = \big(\frac{1}{q(\pi(x)) + \alpha^2\mu^2}(x \cdot \pi(x) + \alpha\mu x),\ -\frac{1}{q(\pi(x)) + \alpha^2\mu^2}\alpha s_0\big)$$

We use Lemma 2.29 to define an isomorphism of the two Moufang sets. We define a the bijection $\varphi : X \times k \to X \times ks_0 : (x, t) \mapsto (x, -\frac{t}{\mu}s_0)$. We verify this is a group morphism:

$$((x_1, t_1) + (x_2, t_2)).\varphi = (x_1 + x_2, t_1 + t_2 + g(x_2, x_1)).\varphi$$

$$\begin{aligned}
&= (x_1 + x_2, -\frac{1}{\mu}(t_1 + t_2 + \psi(x_2, x_1)s_0)s_0) \\
&= (x_1 + x_2, -\frac{1}{\mu}(t_1 + t_2)s_0 + \psi(x_1, x_2)) \\
&= (x_1, t_1).\varphi + (x_2, t_2).\varphi.
\end{aligned}$$

We determine $\tau'^{\varphi}$, let $(x, t) \in X \times k \setminus \{(0, 0\}$

$$\begin{aligned}
(x, t).\tau'^{\varphi} &= (x, -\frac{t}{\mu}s_0).\tau'\varphi^{-1} \\
&= \big(\frac{1}{q(\pi(x)) + t^2}(x \cdot \pi(x) - tx),\ \frac{1}{q(\pi(x)) + t^2}\frac{t}{\mu}s_0\big).\varphi^{-1} \\
&= \big(\frac{1}{q(\pi(x)) + t^2}(x \cdot \pi(x) - tx),\ -\frac{1}{q(\pi(x)) + t^2}t\big),
\end{aligned}$$

this is indeed identical to the $\tau$ in (2.20). □

We have that as a structurable algebra $X$ is isotopic to $CD((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2)$ by Theorem 4.12 and Lemma 4.6. Since $(Q_1 \otimes_k Q_2)^+$ is a 16-dimensional division Jordan algebra of degree 4, it follows from (4) on page 60 that the Tits–Kantor–Koecher process on $X$ yields a Lie algebra of type $E_{7,1}^{66}$. This fact confirms Conjecture 6.28.

**Structurable algebras of hermitian type** Let $\mathcal{A} = E \oplus W$ be a structurable division algebra of hermitian type, see Section 3.3.3.. This implies that $E$ be an associative division algebra with involution $^-$ and that $W$ is a left $E$-module equipped with a hermitian form $h$, such that $h(x, x) \neq 0, 1$ for all $0 \neq x \in W$.

For the sake of readability, we introduce the notation

$$\delta_{e+w} = e\bar{e} - h(w, w) \in \mathcal{H}(E),$$

for all $e \in E$ and $w \in W$. Hence by (3.17) $\widehat{e + w} = \delta_{e+w}(e - w)$ and

$$\delta_{\widehat{x}} = \delta_x^{-1}$$

for all $0 \neq x = e + w \in \mathcal{A}$. Using formulas (3.15) and (3.16), it is a straight forward calculation to verify that

$$q_{(e+w)}(s) = \delta_{e+w} s \delta_{e+w} \in \mathcal{S},$$

for all $e \in E$, $w \in W$ and $s \in \mathcal{S}(E)$, where $q_x$ is defined in Definition 6.14.

Now we apply Theorem 6.25. It follows that $\mathcal{U} = \mathcal{A} \times \mathcal{S}(E)$ with addition

$$(e_1 + w_1, s_1) + (e_2 + w_2, s_2)$$

$$= \big(e_1 + e_2 + w_1 + w_2, s_1 + s_2 + e_1\overline{e_2} - e_2\overline{e_1} + h(w_2, w_1) - h(w_1, w_2)\big). \tag{6.11}$$

For all $0 \neq x \in \mathcal{A}$, $0 \neq s \in \mathcal{S}$ we have that $(x, s).\tau = (-u + tx, -t)$ where $(u, t)$ is as in Theorem 6.18. We simplify the expressions for $u$ and $t$ using the associativity of $E$ and the identity $(a + b)^{-1}a(a - b)^{-1} = (a - ba^{-1}b)^{-1}$:

$$\begin{aligned} t &= (s + q_{e+w}(\hat{s}))^\wedge \\ &= \overline{(s + \delta_{e+w}\overline{s}^{-1}\delta_{e+w})}^{-1} \\ &= (-s + \delta_{e+w}s^{-1}\delta_{e+w})^{-1} \\ &= (\delta_{e+w} + s)^{-1}s(\delta_{e+w} - s)^{-1}. \end{aligned}$$

Evaluating $tx$ gives

$$t(e + w) = (\delta_{e+w} + s)^{-1}s(\delta_{e+w} - s)^{-1}e - (\delta_{e+w} + s)^{-1}s(\delta_{e+w} - s)^{-1}w.$$

The expression for $u$ simplifies to

$$\begin{aligned} u &= -\hat{s}\big((\hat{s} + q_{\widehat{e+w}}(s))^\wedge(\widehat{e+w})\big) \\ &= s^{-1}\big((s^{-1} - \delta_{e+w}^{-1}s\delta_{e+w}^{-1})^{-1}\delta_{e+w}^{-1}(e - w)\big) \\ &= s^{-1}\big((s^{-1} - \delta_{e+w}^{-1}s\delta_{e+w}^{-1})^{-1}\delta_{e+w}^{-1}e + (s^{-1} - \delta_{e+w}^{-1}s\delta_{e+w}^{-1})^{-1}\delta_{e+w}^{-1}w\big) \\ &= s^{-1}(s^{-1} - \delta_{e+w}^{-1}s\delta_{e+w}^{-1})^{-1}\delta_{e+w}^{-1}e - s^{-1}(s^{-1} - \delta_{e+w}^{-1}s\delta_{e+w}^{-1})^{-1}\delta_{e+w}^{-1}w \\ &= (\delta_{e+w} + s)^{-1}\delta_{e+w}(\delta_{e+w} - s)^{-1}e - (\delta_{e+w} + s)^{-1}\delta_{e+w}(\delta_{e+w} - s)^{-1}w. \end{aligned}$$

Therefore

$$\begin{aligned} (e + w, s).\tau = \big(&-(s + \delta_{e+w})^{-1}e + (s + \delta_{e+w})^{-1}w, \\ &(s + \delta_{e+w})^{-1}s(s - \delta_{e+w})^{-1}\big) \end{aligned} \tag{6.12}$$

Notice that $(e + w, 0).\tau = (-\delta_{e+w}^{-1}(e + w), 0) = (-\widehat{e + w}, 0)$ and $(0, s).\tau = (0, s^{-1})$, which is equivalent with the $\tau$ in Theorem 6.25.

Therefore formula (6.12) is valid for all $(e + w, s) \in \mathcal{A} \times \mathcal{S} \setminus \{(0, 0)\}$.

**Lemma 6.32.** *The Moufang set we obtained above is isomorphic to a Moufang set of skew-hermitian type defined in Section 2.4.2.2.*

*Proof.* We start with an anisotropic pseudo-quadratic space $(L, \sigma, X, h, \pi)$ with corresponding Moufang set $\mathbb{M}(U', \tau')$ as in Section 2.4.2.2. We make $X$ into a structurable algebra of hermitian type by applying Lemma 1.5 and Remark 3.4.(ii) successively. This obtained hermitian structurable algebra will have Moufang set $\mathbb{M}(U = \mathcal{A} \times \mathcal{S}, \tau)$ with $\tau$ as in (6.12). We show that those two Moufang sets are isomorphic. It follows from Remark 3.4.(iii) that any Moufang set from a hermitian structurable algebra is isomorphic to a Moufang set of skew-hermitian type.

Notice that since for $0 \neq x \in X$, $h(x,x)^\sigma = -h(x,x) \neq 0$, the involution $\sigma \neq \mathrm{id}$. Fix for the rest of this proof

$$0 \neq \xi \in X \text{ and } s_0 := -2h(\xi,\xi)^{-1} \in \mathcal{S}(L).$$

Define for all $e \in L$ the involution $e^\varsigma := s_0 e^\sigma s_0^{-1}$. Notice that $e^\sigma = e$ if and only if $(s_0 e)^\varsigma = -s_0 e$, which implies that $L_\sigma \to \mathcal{S}_\varsigma(L) : e \mapsto s_0 e$ is a bijection.

We let $X$ be a left $L$-module for the scalar multiplication $e \circ x = xe^\varsigma$, now $s_0 h : X \times X \to L$ is a hermitian form with respect to the involution $\varsigma$.

We have that $X = E \oplus W$ for $W := (L \circ \xi)^\perp$ and $E := L \circ \xi$ which is an associative algebra $(e_1 \circ \xi)(e_2 \circ \xi) = e_1 e_2 \circ \xi$ with involution $(e \circ \xi)^\varsigma = e^\varsigma \circ \xi$.

We let $W$ be a left $E$-module for $(e \circ \xi) \circ w = e \circ w$ and define the hermitian form

$$H(w_1, w_2) = \tfrac{1}{2} s_0 h(w_1, w_2) \circ \xi,$$

for all $w_1, w_2 \in W$. In this way $X$ becomes a structurable algebra of hermitian type.

Notice that for $x_1 = e_1 \circ \xi + w_1$ and $x_2 = e_2 \circ \xi + w_2$, we have

$$\begin{aligned} h(x_1, x_2) \circ \xi &= \big(h(e_1 \circ \xi, e_2 \circ \xi) + h(w_1, w_2)\big) \circ \xi \\ &= \big(e_1^{\varsigma\sigma} h(\xi,\xi) e_2^\varsigma + 2 s_0^{-1} H(w_1, w_2)\big) \circ \xi \\ &= \big(s_0^{-1} e_1 s_0 (-2 s_0^{-1}) e_2^\varsigma + 2 s_0^{-1} H(w_1, w_2)\big) \circ \xi \\ &= -2 s_0^{-1} \big(e_1 e_2^\varsigma - H(w_1, w_2)\big) \circ \xi. \end{aligned} \tag{6.13}$$

When we work with the structurable algebra obtained from the hermitian form $H$, we obtain by (6.13) that

$$\delta_{e \circ \xi + w} \circ \xi = e e^\varsigma \circ \xi - H(w,w) = -s_0 \pi(e \circ \xi + w) \circ \xi.$$

Since we started from an anisotropic pseudo-hermitian space, the pseudo-quadratic form $\pi$ is anisotropic. Therefore $\delta_{e \circ \xi + w} \neq 0$ for $e \circ \xi + w \neq 0$, thus the obtained structurable algebra is division.

Now we define the bijection

$$\begin{aligned} \varphi : X \times L_\sigma &\to E \oplus W \times \mathcal{S}(E) \\ (x, a) &\mapsto (x = e \circ \xi + w,\ s_0 a \circ \xi). \end{aligned}$$

It follows from (6.13) after an easy verification that $\varphi((x_1, a_1) + (x_2, a_2)) = \varphi(x_1, a_1) + \varphi(x_2, a_2)$ for all $x_1, x_2 \in X$, $a_1, a_2 \in L_\sigma$, where the addition in the left side is (2.7) and the addition in the right side is (6.11).

We will show that $\varphi^{-1}\tau'\varphi = \tau h_{(0,s_0)}$, then it follows from Lemma 2.29 that the two Moufang sets are isomorphic.

It follows from Theorem 6.27 that

$$(e\circ\xi+w, a\circ\xi).\tau h_{(0,s_0)} = \big(-s_0(a\circ\xi+\delta_{e\circ\xi+w})^{-1}e\circ\xi - s_0(a\circ\xi+\delta_{e\circ\xi+w})^{-1}w, \\ - s_0(a\circ\xi+\delta_{e\circ\xi+w})^{-1}(a\circ\xi)(a\circ\xi-\delta_{e\circ\xi+w})^{-1}s_0\big) \quad (6.14)$$

We find that

$$\begin{aligned}(e\circ\xi+w, a\circ\xi).\varphi^{-1}\tau\varphi &= \big(-(e\circ\xi+w)(-s_0^{-1}\delta_{e\circ\xi+w}\circ\xi+s_0^{-1}a\circ\xi)^{-1}, \\ &\quad - s_0(s_0^{-1}\delta_{e\circ\xi+w}\circ\xi+s_0^{-1}a\circ\xi)^{-1}(s_0^{-1}a\circ\xi)(-s_0^{-1}\delta_{e\circ\xi+w}\circ\xi+s_0^{-1}a\circ\xi)^{-1}\big) \\ &= \big(-s_0(a+\delta_{e\circ\xi+w})^{-1}e\circ\xi - s_0(a+\delta_{e\circ\xi+w})^{-1}w, \\ &\quad - s_0(a\circ\xi+\delta_{e\circ\xi+w})^{-1}(a\circ\xi)(a\circ\xi-\delta_{e\circ\xi+w})^{-1}s_0\big)\end{aligned}$$

which is exactly (6.14). □

**Associative algebras with involution** Let $\mathcal{A}$ be an associative algebra with involution. $\mathcal{A}$ can be seen as structurable algebra of hermitian type with the zero hermitian space. It follows from Lemma 6.32 that the Moufang set constructed in Theorem 6.25 is isomorphic to Moufang set of skew-hermitian type.

In order to obtain an associative algebra in the proof of Lemma 6.32, one has to start with the 1-dimensional skew-hermitian space $h : E \times E \to E : (e, e') \mapsto et\overline{e'}$ for some element $t \in \mathcal{S}(E)$.

# Appendix A

# Explicit construction of a structurable algebra associated to a quadrangular algebra

Let $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra with $\mathrm{char}(k) \neq 2, 3$. Let $(X, t, \langle ., . \rangle)$ be the Freudenthal triple system constructed in Theorem 4.2 from $\Omega$. In Sections 4.3 and 4.4 we defined structurable algebras whose associated Freudenthal triple system is isometric to $(X, t, \langle ., . \rangle)$.

In this appendix we take another point of view towards these structurable algebras. We start with the Freudenthal triple system $(X, t, \langle ., . \rangle)$ and construct in a very explicit way a family of isotopic structurable algebras whose associated Freudenthal triple system is isometric to $(X, t, \langle ., . \rangle)$.

We achieve this by going through the constructions in the proof of [Fer72, Theorem 5.1] and [Gar01, Lemma 4.15] in a very explicit way; we proceed in 3 steps:

*Step* 1. We tensor the simple Freudenthal triple system $X$ with a quadratic field extension $\Delta$ such that it becomes reduced.

*Step* 2. We apply the proof of [Fer72, Theorem 5.1] to construct a structurable matrix algebra that is isometric to $X \otimes_k \Delta$.

*Step* 3. We use the methods from [Gar01, Lemma 4.15] to apply Galois descent and find a structurable algebra that is isometric to $X$.

## A.1 The construction

**Construction A.1.** Let $\Omega = (k, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra with $\mathrm{char}(k) \neq 2, 3$, and consider $X$ as a simple non-reduced Freudenthal triple system as in Theorem 4.2.

*Step* 1: *Extending scalars to make $X$ reduced.*

To reduce $X$ we use Lemma 3.32. For all $x \in X$, we have $\langle x, t(x,x,x)\rangle = 12(-q(\pi(x)))$. Since $X$ is not reduced, $-q(\pi(x))$ is never a square in $k$.

We fix an arbitrary $a \in X^*$ and define $\delta := \sqrt{-q(\pi(a))}$ in the algebraic closure of $k$, so that $\Delta = k(\delta)$ is a quadratic field extension of $k$; let $\iota$ be the non-trivial element of $\mathrm{Gal}(\Delta/k)$.

We now linearly extend the trilinear product and the bilinear form on $X$ to $X \otimes_k \Delta$. This makes $X \otimes_k \Delta$ into a Freudenthal triple system. By our choice of $\Delta$, the Freudenthal triple system $X \otimes_k \Delta$ is reduced. By Lemma 3.32 and Theorem 4.2,

$$u_1' = \tfrac{1}{2}\Big(a + \frac{a\pi(a)}{\delta}\Big), \quad u_2' = \tfrac{1}{2\delta}\Big(-a + \frac{a\pi(a)}{\delta}\Big)$$

form a supplementary pair of strictly regular elements.

*Step* 2: *Construction of a structurable matrix algebra isometric to $X \otimes_k \Delta$.*

We point out that if we say that a structurable matrix algebra $M(J,\eta)$ is isometric to $X \otimes_K \Delta$, we mean that the Freudenthal triple system $M(J,\eta)$, defined by the formulas for $\langle .,. \rangle$ and $\nu$ in Remark 3.28.(iii), is isometric to the Freudenthal triple system $X \otimes_K \Delta$.

In order to construct a structurable matrix algebra that is isometric to $X \otimes_K \Delta$, we have to construct a Jordan algebra over $\Delta$. We proceed as in [Fer72], but we slightly modify the construction which is presented there. We only give the necessary ingredients, referring the reader to *loc. cit.* for more details.

For $\epsilon \in \{1, -1\}$, we let

$$M_\epsilon := \{x \in X \otimes_K \Delta \mid t(u_1', u_2', x) = \epsilon x\}.$$

As in *loc. cit.*, we will define a Jordan algebra on the vector space $M_1$. This Jordan algebra will be constructed either from a quadratic form or from an admissible cubic form.

If the expression $g(u_1', y\pi(y))$ is identically zero for $y \in M_1$, then there is a quadratic form $Q$ on $M_1$ making $M_1$ into a Jordan algebra; in this case we define $N = 0$ and $\lambda = 1$.

On the other hand, if there exists an $e \in M_1$ such that $g(u_1', e\pi(e)) \neq 0$, then

$$N(x) := \frac{g(u_1', x\pi(x))}{g(u_1', e\pi(e))}$$

is an admissible cubic form on $M_1$ with basepoint $e$, making $M_1$ into a Jordan algebra, and we let

$$\lambda := \tfrac{1}{2} g(u_1', e\pi(e)) \in \Delta.$$

It is shown in *loc. cit.* that in both cases, $X \otimes_K \Delta$ is isometric to the structurable matrix algebra $M(M_1, \lambda)$ (as defined in Example 3.3.4). However, we prefer to slightly modify the construction so that $X \otimes_K \Delta$ becomes isometric to $M(M_1, 1)$. One obvious way to do this is to redefine the pair of strictly regular elements as $u_1 = \lambda^{-1} u_1'$ and $u_2 = \lambda u_2'$, so that

$$u_1 = \frac{1}{2\lambda}\Big(a + \frac{a\pi(a)}{\delta}\Big), \quad u_2 = \frac{\lambda}{2\delta}\Big(-a + \frac{a\pi(a)}{\delta}\Big);$$

then $X \otimes_K \Delta$ will indeed be isometric to $M(M_1, 1)$. Note that the spaces $M_\epsilon$ are unchanged by replacing $u_1'$ and $u_2'$ by $u_1$ and $u_2$, respectively.

In *loc. cit.* it is shown that $X \otimes_K \Delta = \Delta u_1 \oplus \Delta u_2 \oplus M_1 \oplus M_{-1}$, and that there exists an isomorphism $t\colon M_1 \to M_{-1}$. This allows us to explicitly write down the isometry $\psi$ between $X \otimes_K \Delta$ and $M(M_1, 1)$:

$$\begin{aligned} \psi : \Delta u_1 \oplus \Delta u_2 \oplus M_1 \oplus M_{-1} &\to \begin{pmatrix} \Delta & M_1 \\ M_1 & \Delta \end{pmatrix} : \\ d_1 u_1 + d_2 u_2 + j_1 + t(j_2) &\mapsto \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix}, \end{aligned}$$

for all $d_1, d_2 \in \Delta$ and all $j_1, j_2 \in M_1$. So we obtain a structurable algebra $M(M_1, 1)$ that is defined over $\Delta$ and isometric to $X \otimes_K \Delta$.

*Step* 3: *Galois descent.*

Our next step is to apply Galois descent to obtain a structurable algebra over $k$ isometric to $X$. We follow the ideas of [Gar01, Lemma 4.15], but we use a more explicit approach in order to obtain exact formulas.

Let $\widetilde{\eta}$ be the extension of $\iota$ to $X \otimes_K \Delta$ given by

$$\widetilde{\eta}(x \otimes d) := x \otimes \iota(d).$$

Since the fixed point set of $\widetilde{\eta}$ in $X \otimes_K \Delta$ is $X$, we determine how this map acts on $M(M_1, 1)$. As $\widetilde{\eta}(t(x, y, z)) = t(\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\eta}(z))$, the map $\widetilde{\eta}$ is an isometry of the Freudenthal triple system. We have $\widetilde{\eta}(u_1) = \frac{-\delta}{N(\lambda)} u_2$, and it follows from $\widetilde{\eta}(u_1 u_2 x) = -u_1 u_2 \widetilde{\eta}(x)$ that $x \in M_{\pm 1}$ if and only if $\widetilde{\eta}(x) \in M_{\mp 1}$.

The explicit formula for $\widetilde{\eta}$ is given by

$$\begin{aligned}\widetilde{\eta}(d_1u_1 + d_2u_2 + j_1 + t(j_2)) \\ &= \iota(d_1)\frac{-\delta}{N(\lambda)}u_2 + \iota(d_2)\frac{N(\lambda)}{\delta}u_1 + \widetilde{\eta}(j_1) + \widetilde{\eta}(t(j_2)) \\ &= \iota(d_2)\frac{N(\lambda)}{\delta}u_1 + \iota(d_1)\frac{-\delta}{N(\lambda)}u_2 + \widetilde{\eta}(t(j_2)) + t(t^{-1}(\widetilde{\eta}(j_1))),\end{aligned}$$

for all $d_1, d_2 \in \Delta$ and all $j_1, j_2 \in M_1$. Since $\widetilde{\eta}(t(j_2)), t^{-1}(\widetilde{\eta}(j_1)) \in M_1$, we can translate this into matrix notation using $\psi$, and we get

$$\widetilde{\eta} : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{N(\lambda)}{\delta}\iota(d_2) & \widetilde{\eta}(t(j_2)) \\ t^{-1}(\widetilde{\eta}(j_1)) & \frac{-\delta}{N(\lambda)}\iota(d_1) \end{pmatrix}.$$

We denote the Freudenthal triple system on $\mathcal{A} := M(M_1, 1)$ from Example 3.3.4 by $(\mathcal{A}, b, t)$; it follows that $\widetilde{\eta}$ is an isometry of $(\mathcal{A}, b, t)$. It is important to note, however, that $\widetilde{\eta}$ is in general *not* an algebra automorphism of $\mathcal{A}$, and the fixed points of $\widetilde{\eta}$ in $\mathcal{A}$ do *not* form a structurable algebra.

Following [Gar01], consider the structurable algebra $\mathcal{A}' := M(M_1, \frac{\delta}{N(\lambda)})$; denote the corresponding Freudenthal triple system by $(\mathcal{A}', b', t')$. We now modify this Freudenthal triple system once more. Let

$$s_0' = \frac{N(\lambda)}{\delta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and consider the Freudenthal triple system associated to $\mathcal{A}'$ with respect to $s_0'$ as in Definition 3.29. Then we obtain the Freudenthal triple system $(\mathcal{A}', b'', t'')$, where

$$b'' = \frac{N(\lambda)}{\delta}b' \quad \text{and} \quad t'' = \frac{N(\lambda)}{\delta}t'.$$

The map

$$\widetilde{f} : \mathcal{A} \to \mathcal{A}' : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\delta}{N(\lambda)}d_1 & j_1 \\ j_2 & d_2 \end{pmatrix}$$

is an isometry from $(\mathcal{A}, b, t)$ to $(\mathcal{A}', b'', t'')$. Now consider the map $\widetilde{\pi} := \widetilde{f}\widetilde{\eta}\widetilde{f}^{-1} : \mathcal{A}' \to \mathcal{A}'$, which is explicitly given by

$$\widetilde{\pi} : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \iota(d_2) & \widetilde{\eta}(t(j_2)) \\ t^{-1}(\widetilde{\eta}(j_1)) & \iota(d_1) \end{pmatrix}.$$

It is now obvious that $\widetilde{\pi}$ is an isometry of $(\mathcal{A}', b'', t'')$. Using some properties of norm similarities of Jordan algebras, one can show that $\widetilde{\pi}$ is, in fact, an algebra automorphism of $\mathcal{A}'$.

It follows that $\mathcal{A}'^{\widetilde{\pi}}$, the fixed points of $\mathcal{A}'$ under $\widetilde{\pi}$, is a structurable algebra. Considered as Freudenthal triple systems, $\mathcal{A}'^{\widetilde{\pi}}$ and $\mathcal{A}^{\widetilde{\eta}}$ are isometric. Since $\mathcal{A}^{\widetilde{\eta}}$ is in turn isometric to $X$, the map

$$\tau := \widetilde{f} \circ \psi : X \to \mathcal{A}'^{\widetilde{\pi}}$$

is an isometry.

We now use this isometry to make $X$ into a structurable algebra isomorphic to $\mathcal{A}'^{\widetilde{\pi}}$, by defining the following multiplication and involution:

$$x \star y := \tau^{-1}(\tau(x)\tau(y)) \quad \text{and} \quad \overline{x} := \tau^{-1}(\overline{\tau(x)})$$

for all $x, y \in X$, where the multiplication and involution in the right hand sides are as in Definition 3.5 applied on $\mathcal{A}'$.

We will denote this structurable algebra by

$$X = X(\Omega, a, \lambda)\,,$$

where $\Omega$ is the quadrangular algebra we started from, and where $a \in X^*$ and $\lambda \in \Delta$ are as in Step 1 and Step 2, respectively.

`[End of Construction A.1]`

We can now explicitly write down the structurable algebra $X$ in terms of the original quadrangular algebra.

**Theorem A.2.** *Let $X = X(\Omega, a, \lambda)$ be as above. Let*

$$\begin{aligned}
\mathbf{1} &:= \frac{1}{2\delta}\Big(\lambda^\sigma\Big(a + \frac{a\pi(a)}{\delta}\Big) + \lambda\Big(-a + \frac{a\pi(a)}{\delta}\Big)\Big),\\
s_0 &:= \frac{N(\lambda)}{2\delta^2}\Big(\lambda^\sigma\Big(a + \frac{a\pi(a)}{\delta}\Big) - \lambda\Big(-a + \frac{a\pi(a)}{\delta}\Big)\Big),\\
\mu &:= \frac{N(\lambda)^2}{\delta^2}.
\end{aligned}$$

*Then $X$ is a structurable algebra with zero element $0 \in X$ and identity element $\mathbf{1} \in X$; $X$ has skew-dimension one, and the subspace $\mathcal{S}$ of skew-symmetric elements is generated by $s_0$, with $s_0^2 = \mu$. The subspace $\mathcal{H}$ of symmetric elements is*

$$\mathcal{H} = \{x \in X \mid \overline{x} = x\} = \{\ell\mathbf{1} + j + \widetilde{\eta}(j) \mid \ell \in k, j \in M_1\}.$$

*Moreover, for all $x, y, z \in X$, we have*

$$\begin{aligned}
V_{x,\, s_0 \star y}\, z &= \tfrac{1}{2}\Big(xh(y,z) + yh(x,z) + zh(y,x)\Big),\\
(x \star \overline{y} - y \star \overline{x}) \star s_0 &= g(x,y)\mathbf{1},
\end{aligned}$$

$$\nu(x) = -\frac{\delta^2}{N(\lambda)^2}\, q(\pi(x)),$$

*where $\nu$ is the norm of $X$ (see* (3.18)*). If we make other choices for $a' \in X^*$ and $\lambda' \in \Delta$, then the structurable algebras $X(\Omega, a, \lambda)$ and $X(\Omega, a', \lambda')$ are isotopic.*

*Proof.* By definition, the isometry $\tau$ is an isomorphism from the structurable algebra $X$ to the structurable algebra $\mathcal{A}'^{\widetilde{\pi}}$, which is known to be of skew-dimension one. In particular, the zero element and the identity element of $X$ are equal to $0 = \tau^{-1}(0)$ and $\mathbf{1} := \tau^{-1}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, respectively. Moreover, $X$ has skew-dimension one, and the set $\mathcal{S}$ of skew-symmetric elements is generated by $s_0 := \tau^{-1}(s_0')$. We now perform some explicit calculations.

Notice that all elements in $\mathcal{A}'^{\widetilde{\pi}}$ are of the form

$$\begin{pmatrix} d & j \\ t^{-1}(\widetilde{\eta}(j)) & \iota(d) \end{pmatrix}$$

for some $d \in \Delta$ and $j \in M_1$. We compute $\tau^{-1} := \psi^{-1} \circ \widetilde{f}^{-1} \colon \mathcal{A}'^{\widetilde{\pi}} \to X$:

$$\begin{pmatrix} d & j \\ t^{-1}(\widetilde{\eta}(j)) & \iota(d) \end{pmatrix} \xmapsto{\widetilde{f}^{-1}} \begin{pmatrix} \frac{N(\lambda)}{\delta} d & j \\ t^{-1}(\widetilde{\eta}(j)) & \iota(d) \end{pmatrix} \in \mathcal{A}^{\widetilde{\eta}}$$
$$\xmapsto{\psi^{-1}} \frac{N(\lambda)}{\delta} du_1 + \iota(d)u_2 + j + \widetilde{\eta}(j) \in X.$$

Now we can determine

$$\begin{aligned}
\mathbf{1} = \tau^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{N(\lambda)}{\delta} u_1 + u_2 \\
&= \frac{1}{2\delta}\Big(\lambda^\sigma\Big(x + \frac{x\pi(x)}{\delta}\Big) + \lambda\Big(-x + \frac{x\pi(x)}{\delta}\Big)\Big), \\
s_0 = \tau^{-1}\begin{pmatrix} \frac{N(\lambda)}{\delta} & 0 \\ 0 & -\frac{N(\lambda)}{\delta} \end{pmatrix} &= \frac{N(\lambda)^2}{\delta^2} u_1 - \frac{N(\lambda)}{\delta} u_2 \\
&= \frac{N(\lambda)}{2\delta^2}\Big(\lambda^\sigma\Big(x + \frac{x\pi(x)}{\delta}\Big) - \lambda\Big(-x + \frac{x\pi(x)}{\delta}\Big)\Big), \\
s_0 \star s_0 = \tau^{-1}(s_0' s_0') = \tau^{-1}\left(\frac{N(\lambda)^2}{\delta^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= \frac{N(\lambda)^2}{\delta^2}\mathbf{1}.
\end{aligned}$$

We determine how the involution of the structurable algebra $\mathcal{A}'$ acts on $X$. We have $\overline{x} = \tau^{-1}(\overline{\tau(x)})$, therefore

$$\overline{\frac{N(\lambda)}{\delta} du_1 + \iota(d)u_2 + j + \widetilde{\eta}(j)} = \tau^{-1}\left(\overline{\begin{pmatrix} d & j \\ t^{-1}(\widetilde{\eta}(j)) & \iota(d) \end{pmatrix}}\right)$$

$$= \tau^{-1}\left(\begin{pmatrix} \iota(d) & j \\ t^{-1}(\widetilde{\eta}(j)) & d \end{pmatrix}\right) = \frac{N(\lambda)}{\delta}\iota(d)u_1 + du_2 + j + \widetilde{\eta}(j).$$

Since $\mathbf{1} = \frac{N(\lambda)}{\delta}u_1 + u_2$, it follows that each element in the $k$-vector subspace $\{\ell\mathbf{1} + j + \widetilde{\eta}(j) \mid \ell \in k, j \in M_1\}$ is fixed by the involution. Since

$$\{\ell\mathbf{1} + j + \widetilde{\eta}(j) \mid \ell \in k, j \in M_1\} \oplus ks_0 = X,$$

we conclude that $\mathcal{H} = \{\ell\mathbf{1} + j + \widetilde{\eta}(j) \mid \ell \in k, j \in M_1\}$.

By the definition of $\star$ and $x \mapsto \overline{x}$, we have

$$V_{x,\, s_0\star y}\, z = \tau^{-1}\big(V_{\tau(x),\, s_0'\tau(y)}\, \tau(z)\big).$$

It follows from Theorem 3.27 that

$$\begin{aligned} V_{x,\, s_0\star y}\, z = \tau^{-1}\Big(&\tfrac{1}{2}(\tau(x)\tau(y)\tau(z)) + \langle\tau(y),\tau(z)\rangle\tau(x) \\ &+ \langle\tau(y),\tau(x)\rangle\tau(z) + \langle\tau(x),\tau(z)\rangle\tau(y)\Big). \end{aligned}$$

Since $\tau$ is an isometry we have

$$\begin{aligned} &V_{x,\, s_0\star y}\, z \\ &= \tfrac{1}{2}(xyz + \langle y,z\rangle x + \langle y,x\rangle z + \langle x,z\rangle y) \\ &= \tfrac{1}{2}\Big(\tfrac{1}{2}\big(x(h(y,z)+h(z,y)) + y(h(x,z)+h(z,x)) + z(h(x,y)+h(y,x))\big) \\ &\qquad + g(y,z)x + g(y,x)z + g(x,z)y\Big) \\ &= \tfrac{1}{2}(xh(y,z) + yh(x,z) + zh(y,x)), \end{aligned}$$

where the last step follows from

$$\begin{aligned} h(y,z) + h(z,y) &= h(y,z) - \overline{h(y,z)} \\ &= 2h(y,z) - f(h(y,z),1)1 = 2\big(h(y,z) - g(y,z)1\big). \end{aligned}$$

Again it follows from Theorem 3.27 and Theorem 4.2 that

$$\begin{aligned} (x\star\overline{y} - y\star\overline{x})\star s_0 &= \tau^{-1}\Big(\big(\tau(x)\overline{\tau(y)} - \tau(y)\overline{\tau(x)}\big)s_0'\Big) \\ &= \tau^{-1}\big(\langle\tau(x),\tau(y)\rangle 1\big) \\ &= \langle x,y\rangle\mathbf{1} = g(x,y)\mathbf{1}; \end{aligned}$$

$$\nu(x) = \frac{1}{12\mu}\langle x, t(x,x,x)\rangle = -\frac{\delta^2}{N(\lambda)^2}q(\pi(x)).$$

Finally, if we make other choices for $a' \in X^*$ and $\lambda' \in \Delta$, then the structurable algebras $X(\Omega, a, \lambda)$ and $X(\Omega, a', \lambda')$ are, by construction, isometric as Freudenthal triple systems to the Freudenthal triple system $X$. It follows from Lemma 3.30.(ii) that they are isotopic. □

In the case of quadrangular algebras of type $E_6$, $E_7$ and $E_8$, we can actually take $\lambda = 1$, in which case the formulas of Theorem A.2 look nicer; see Lemma A.3 below. In the following section we show that the structurable algebras described in Theorem 4.1 are isomorphic to a structurable algebra described in the theorem above.

We do not know whether we can always take $\lambda = 1$ in the pseudo-quadratic case. We have that the structurable algebra described in Theorem 4.8 is isotopic to the class of structurable algebras we described in the above theorem, but we do not know if it belongs to this class.

## A.2 Structurable algebras on quadrangular algebras of type $E_6$, $E_7$ and $E_8$

In this section we consider quadrangular algebras of type $E_6, E_7$ and $E_8$. We show that the structurable algebra obtained in Theorem A.2 is isomorphic to $CD((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2)$ for a "nice" choice of $a$ and $e$. The following rather technical lemma will assure that we can choose the structurable algebra $X$ obtained in Theorem A.2 in a nice way.

**Lemma A.3.** *Let $\Omega$ be a quadrangular algebra of type $E_8$. When applying Theorem* A.2 *with $a = -v_1/\gamma$, we can always choose $e$ in such a way that $\lambda = 1$. For those choices we have that $\mathbf{1} = v_1, s_0 = -\gamma v_1$ and $\mu = \gamma^2$.*

*Proof.* Recall that $\gamma \in E \setminus k$ and $\gamma^2 \in k$. For $a = -v_1/\gamma$ we have that $q(\pi(a)) = -\frac{1}{\gamma^2}$ and $a\pi(a) = \frac{v_1}{\gamma^2}$. So $\delta = \frac{1}{\gamma}$, and hence $\Delta = k(\delta) = E$. We point out that in $X \otimes_K \Delta$, the element $1 \otimes \delta$ is not equal to $\frac{1}{\gamma} \otimes 1$. For instance, we have

$$\frac{a\pi(a)}{\delta} = \frac{v_1}{\gamma^2} \otimes \gamma \neq \frac{v_1}{\gamma} \otimes 1 = -a.$$

If we assume that $\lambda = 1$ using the formulas in Theorem A.2, we obtain $\mathbf{1} = v_1, s_0 = -\gamma v_1, \mu = \gamma^2$. In order to prove that we can always find an $e$ such that $\lambda = 1$, we will do some explicit calculations.

We have

$$u_1' = \tfrac{1}{2}\Big(-\frac{v_1}{\gamma} \otimes 1 + v_1 \otimes \frac{1}{\gamma}\Big), \quad u_2' = \frac{1}{2}\Big(\frac{v_1}{\gamma} \otimes \gamma + v_1 \otimes 1\Big).$$

We determine explicitly the subspaces $M_1$ and $M_{-1}$. One can calculate that for all $x = \sum_{I \in \mathcal{I}} t_I v_I \in X$,

$$u_1' u_2'(x \otimes 1) = x/\gamma \otimes \gamma, \quad u_1' u_2'(x \otimes \gamma) = \gamma x \otimes 1, \tag{A.1}$$

and for all $x = t_1 v_1 \in X$,

$$u_1' u_2'(x \otimes 1) = 2(x/\gamma \otimes \gamma), \quad u_1' u_2'(x \otimes \gamma) = 2(\gamma x \otimes 1). \tag{A.2}$$

We find that

$$\begin{aligned} M_1 &= \Big\{ \gamma x \otimes 1 + x \otimes \gamma \;\Big|\; x = \sum_{I \in \mathcal{I}} t_I v_I \text{ with } t_I \in E \Big\}, \\ M_{-1} &= \Big\{ \gamma x \otimes 1 - x \otimes \gamma \;\Big|\; x = \sum_{I \in \mathcal{I}} t_I v_I \text{ with } t_I \in E \Big\}. \end{aligned} \tag{A.3}$$

Following an idea of Richard Weiss, we introduce the following notation: let $i, j, k, l, m$ denote five different indices in $\{2, \ldots, 6\}$, then $\beta_{ijkl} = \pm 1$ is defined by $v_{ij} v_{kl} = \beta_{ijkl} s_i s_j s_k s_l v_m$.

Next we need an expression for $g(u_1', e\pi(e))$ for an arbitrary $e \in M_1$. (Recall that $g$ is now a map from $(X \otimes_K \Delta) \times (X \otimes_K \Delta)$ to $\Delta$.)

So let $x = \sum_{2 \le i \le 6} t_i v_i + \sum_{2 \le i < j \le 6} t_{ij} v_{ij} \in X$ for $t_i, t_{ij} \in E$, and consider the expression

$$\rho(x) := \sum_{ij/kl/m} \beta_{ijkl} t_m t_{ij} t_{kl} \in E = \Delta,$$

where the summation runs over all partitions of $\{2, \ldots, 6\}$ into two sets of two elements and one set of one element.

Since in the $E_6$-case no such partition with non-zero coefficients exists, $\rho(x)$ is identically 0 in this case.

Take $e = \gamma x \otimes 1 + x \otimes \gamma \in M_1$; we wrote a computer program in Sage [Sea11] that shows that (see Appendix B)

$$g(u_1', e\pi(e)) = 16\gamma^4 \rho(x) \in \Delta. \tag{A.4}$$

In the $E_6$-case, this expression is identically 0, so $\lambda = 1$ by definition. In the $E_7$- and the $E_8$-case, we look for an $e \in M_1$ such that $g(u_1', e\pi(e)) = 2$, so $8\gamma^4 \rho(x)$ should be equal to 1. This is indeed the case for

$$x = \tfrac{1}{2\gamma^2}(v_2 + \gamma v_{34} + \gamma v_{56}),$$

since $\beta_{3456} = 1$. □

In Theorem 4.9 we showed that $X_k$ is isomorphic to a biquaternion algebra. We now show that with the above choices of $a$ and $e$ the structurable algebra $X$ we obtained in Theorem A.2 is isomorphic to $CD((Q_1 \otimes_k Q_2)^+, \mathrm{Nrd}, \gamma^2)$.

**Theorem A.4.** *Let $\Omega$ be a quadrangular algebra of type $E_8$. We choose $a = -v_1/\gamma$, and we let $e$ be as in Lemma* A.3*, such that $\lambda = 1$.*

*The structurable algebra $X$ defined in Theorem A.2 is isomorphic to $\mathcal{A} = CD\big(\big((-s_2, -s_3)_k \otimes_k (-s_{46}, -s_{56})_k\big)^+, \mathrm{Nrd}, \gamma^2\big)$.*

*Proof.* In the proof of Theorem 4.12 we defined a vector space isomorphism $\chi : \mathcal{A} \to X$ such that $3\chi(x)\pi(\chi(x)) = \chi(U_x(s_0 x))$ for all $x \in \mathcal{A}$. It follows from Theorem A.2 that in the structurable algebra $X$, we have $U_x(s_0 \star x) = 3x\pi(x)$ for all $x \in X$. It follows that $\chi$ is an similarity of the associated Freudenthal triple systems of $\mathcal{A}$ and $X$. It is easy to see that $\chi$ is an isotopy from $\mathcal{A}$ to $X$.

It follows from Lemma A.3 that the identity of $X$ is $v_1$. Since $\chi(1 \otimes 1) = v_1$, it follows from Definition 3.21 that $\chi$ is an isomorphism of structurable algebras. ☐

It is as well possible to prove the above theorem without making use of Theorem 4.12. This can be achieved by determining the multiplication of $X$ explicitly.

**Lemma A.5.** *Let $\circ$ denote the multiplication of the Jordan algebra*

$$(X_k)^+ \cong \big((-s_2, -s_3)_k \otimes_k (-s_{46}, -s_{56})_k\big)^+.$$

*We choose $a = -v_1/\gamma$, and we let $e$ be as in Lemma* A.3*, such that $\lambda = 1$. Then the multiplication of the structurable algebra $X$ obtained in Theorem* A.2*, which we will denote by $\star$, is given by*

$$\begin{aligned}
Av_1 \star Bv_1 &= AB\, v_1 \\
Av_1 \star Bv_I &= A^\sigma B\, v_I \\
Av_I \star Bv_1 &= AB\, v_I \\
Av_I \star Bv_I &= A^\sigma B\, (v_I \circ v_I) = (-s_I) A^\sigma B\, v_1 \\
Av_I \star Bv_J &= A^\sigma B^\sigma\, (v_I \circ v_J)
\end{aligned}$$

*for all $A, B \in E$ and all $I \neq J \in \mathcal{I}$. The involution of $X$ is given by*

$$\overline{Av_1} = A^\sigma v_1, \quad \overline{Av_I} = Av_I$$

*for all $A \in E$ and all $I \in \mathcal{I}$.*

*Proof.* By Lemma A.3, we have $\mathbf{1} = v_1$, $s_0 = -\gamma v_1$ and $\mu = \gamma^2$. We know that $s_0 \star s_0 = \mu \mathbf{1}$, so $\gamma v_1 \star \gamma v_1 = \gamma^2 v_1$. Since $v_1$ is the identity of $\star$, we have $Av_1 \star Bv_1 = ABv_1$ for all $A, B \in E$.

Since $X$ has skew-dimension one, it follows from the definition of $s_0$ that

$$\mathcal{S} = \{x \in X \mid \overline{x} = -x\} = ks_0 = k\gamma v_1.$$

From Theorem A.2 we have

$$\mathcal{H} = \{x \in X \mid \overline{x} = x\} = \{\ell\mathbf{1} + j + \widetilde{\eta}(j) \mid \ell \in k, j \in M_1\}.$$

It follows from equation (A.3) that every $j \in M_1$ is of the form $\gamma x \otimes 1 + x \otimes \gamma$ for some $x \in \bigoplus_{I \in \mathcal{I}} Ev_I$. Then $\widetilde{\eta}(j) = \gamma x \otimes 1 - x \otimes \gamma \in M_{-1}$. It follows that

$$\{j + \widetilde{\eta}(j) \mid j \in M_1\} = \{2\gamma x \otimes 1 \mid x \in \bigoplus_{I \in \mathcal{I}} Ev_I\} = \bigoplus_{I \in \mathcal{I}} Ev_I.$$

Therefore

$$\mathcal{H} = kv_1 \oplus \bigoplus_{I \in \mathcal{I}} Ev_I.$$

It follows that we have for $A \in E$ that

$$\overline{Av_1} = A^\sigma v_1, \quad \overline{Av_I} = Av_I \text{ for all } I \in \mathcal{I}.$$

For all $x \in \mathcal{H}$ and $y \in X$, we have

$$V_{x,\mathbf{1}}\, y = (x \star \overline{\mathbf{1}}) \star y + (y \star \overline{\mathbf{1}}) \star x - (y \star \overline{x}) \star \mathbf{1} = x \star y.$$

It now follows from Theorem A.2 that

$$x \star y = V_{x,\, s_0 \star \frac{1}{\mu} s_0}\, y = -\frac{1}{2\gamma^2}\Big(xh(\gamma v_1, y) + (\gamma v_1)h(x,y) + yh(x, \gamma v_1)\Big). \quad \text{(A.5)}$$

Now we can compute $x \star y$ for all different values that can occur, using the formulas from [TW02, (13.6) and (13.19)]. Let $i, j, k, l$ be distinct indices in $\{2, 3, 4, 5, 6\}$; then one can verify the following multiplication table:

| $x$ | $y$ | $x \star y$ |
|---|---|---|
| $Av_i$ | $Bv_1$ | $ABv_i$ |
| $Av_i$ | $Bv_i$ | $-s_i A^\sigma B v_1$ |
| $Av_i$ | $Bv_k$ | $0$ |
| $Av_i$ | $Bv_{ik}$ | $0$ |
| $Av_i$ | $Bv_{kl}$ | $2A^\sigma B^\sigma v_i v_{kl}$ |
| $Av_{ij}$ | $Bv_1$ | $ABv_{ij}$ |
| $Av_{ij}$ | $Bv_{ij}$ | $-s_i s_j A^\sigma B v_1$ |
| $Av_{ij}$ | $Bv_{ik}$ | $0$ |
| $Av_{ij}$ | $Bv_{kl}$ | $2A^\sigma B^\sigma v_{ij} v_{kl}$ |

Observe that this multiplication coincides with the Jordan multiplication in $(X_k)^+$ if $A, B$ are restricted to $k$.

Note that the formula (A.5) is not valid for $x = \gamma v_1 \in \mathcal{S}$; this case is obtained by

$$Av_1 \star Bv_I = \overline{\overline{Bv_I} \star \overline{Av_1}} = A^\sigma B v_I. \qquad \square$$

Let $\chi : \mathcal{A} \to X$ be the vector space morphism defined in the proof of Theorem 4.12. It is straightforward to verify that $\chi(xy) = \chi(x) \star \chi(y)$ and $\chi(\overline{x}) = \overline{\chi(x)}$ for all $x, y \in \mathcal{A}$. Therefore $\chi$ is an isomorphism between $\mathcal{A}$ and $X$.

# Appendix B

# Implementations in Sage

We relied on symbolic calculations in the computer algebra package Sage [Sea11] at two points of this thesis:

- A computation that shows that the Freudenthal triple systems in Theorem 4.12 are similar.
- A computation that verifies identity (5.5) in order to prove Theorem 5.23. Note that this computation is only essential when the characteristic of the field is equal to 3.

Sage is also used once in Appendix A in the proof of the technical Lemma A.3.

In particular Sage played a big role in the coming into being of the material in Chapter 4 and Appendix A. It was very useful to be able to verify if a certain complicated identity in a quadrangular algebra of type $E_8$ could be true before trying to prove it.

Since the program listings itself are long and not very interesting to read as literature, we do not display it here. The interested reader can download it at `cage.ugent.be/~lboelaer`. We hope that one day it will be of use to someone who wants to verify identities in a quadrangular algebra of type $E_8$, in octonions or in the tensor product of two quaternions or octonions.

Sage is an open-source mathematics system available for download on `www.sagemath.org`, it is an alternative to Magma, Maple, Matlab etc. Sage is much better equipped to perform computations in algebraic structures than any of those programs. Another advantage is that Sage is based on the Python programming language (see `www.python.org`), which makes it convenient to write own programs in Sage. The codes I wrote should be readable for people who know some basics of Python.

All the objects we have to implement are vector spaces over an arbitrary field of characteristic different from 2 (and 3), these vector spaces are equipped with various multiplication and various mappings.

While vector spaces are implemented in Sage, we are using our own (very basic) implementation. The alternative was to adapt the source code of vector spaces to our purposes, which would have been very advanced programming.

However it is impossible to implement the concept of ‘a general field’ in Sage. In the following remark we explain how we implemented the following:

*Let $k$ be a field of characteristic different from 2 (and 3) with $t_1, \ldots, t_n \in k$.*

**Remark.** We define the ring of multinomial polynomials $R = \mathbb{Q}[t_1, \ldots, t_n]$ and then take the fraction field $k$ of $R$. These concepts are very well implemented in Sage. Since we only do basic computations, we will only make use of the addition and multiplication in $k$. Therefore it does not matter that the field we are using has actually a very specific form instead of being a random field.

Since $t_1, \ldots, t_n$ are transcendental elements of $k/\mathbb{Q}$, we can consider $t_1, \ldots, t_n$ as arbitrary elements in the field $k$.

As in our computations we will never divide with integers other than 2 (and 3), it does not matter that the field $k$ that we use for our calculations has actually characteristic 0.

The problem with this approach is that doing extended computations in $\mathbb{Q}(t_1, \ldots, t_n)$ can give rise to long computation times when $n$ is big. However all our verifications can be done in a very reasonable time (less than 10 minutes on a ‘good’ computer).

For more experimental long calculations one can assign random (not too large) values in $\mathbb{Q}$ to $t_1, \ldots, t_n$, this reduces the computation time significantly. Actually once an identity is valid for a few very random values for $t_1, \ldots, t_n$, it has to be true in general. However this is not a mathematically sound argument.

Below we give an overview of all the different files that can be downloaded at `cage.ugent.be/~lboelaer`. We indicate which algebraic structures are exactly implemented and how we verified that the codes are free of errors, or at least do not contain any essential mathematical mistakes. For more technical explanations on how the codes are build up and should be used, we refer to the comments inside the codes itself.

**Implementation of the quadrangular algebra of type $E_8$** In Section 2.2.3 we introduced the structure of the quadrangular algebra $(k, L, q, 1, X, \cdot, h)$ of type $E_8$, this implementation is given in `implementation_quad_alg_E8.sage`.

The base field $k$ should be the fraction field of $\mathbb{Q}[s_2,\dots,s_5,t_1,\dots,t_n]$ where $t_1,\dots,t_n$ are not specified and after defining $s_6 = -(s_2\dots s_5)^{-1}$ we assume that $s_2,\dots,s_6$ are as in Definition 2.13.

We implemented the field $E$ (see Definition 2.13) as a 2-dimensional vector space over $k$ equipped with the multiplication. The generator of this field is denoted by $\gamma$ and is chosen in such a way that $\gamma^2 = c \in k$. $E$ is equipped with the standard involution and the norm.

We implemented the quadratic form space $L$ as a 6-dimensional vector space over $E$, equipped with the quadratic form of type $E_8$ and its linearization. Actually $L$ is only a vector space over $k$, but it is convenient to consider it as a vector space over $E$. As well we implemented the standard involution on $L$. In the code $L$ is denoted by `L0`, as in [TW02].

The vector space $X$ is implemented as a 16-dimensional vector space over $E$. In the code $X$ is denoted by `X0` as in [TW02].

Finally, we implemented the crucial maps $\cdot : X \times L \to X$ (called `qemap` in the code[1]) and $h : X \times X \to L$. For the exact definitions we refer to [TW02, 13.6 and 13.19]. Both maps are $k$-bilinear, therefore it is sufficient to define $(ev_I)\cdot(e'v_i)$ and $h(ev_I, e'v_{I'})$ for all $e,e' \in E$, $i \in \{1,\dots,6\}$ and $I, I' \in 1 \cup \mathcal{I}$. One has to distinguish between a lot of different cases for the relative positions of the elements $i, I, I'$, the details can be found in the code.

The definitions of $\cdot$ and $h$ make use of the multiplication on $X_k = C(q|_k, 1)/M_k$ induced by the Clifford algebra $C(q,1)$ (see Theorem 4.9). To implement this we make use of the following functions, following an idea of Richard Weiss. Let $\{i,j,k,l,m\} = \{2,\dots,6\}$ and define

$\boldsymbol{\alpha_{ij} = \pm 1}$ such that $v_iv_j = -\alpha_{ij}v_{ij}$ or equivalently $v_{ij}v_j = \alpha_{ij}s_jv_i$. It follows from (2.1) that $\alpha_{ij} = 1$ if $i > j$ and $\alpha_{ij} = -1$ if $i < j$.

$\boldsymbol{\delta_{lmk} = \pm 1}$ such that $v_{lm}v_k = \delta_{lmk}s_{lmk}v_{ij}$. In the case that $l < m < k$ it is clear from (2.1) that

$$v_{23456}v_lv_mv_k = -(-1)^{l+m+k}s_{lmk}v_{ij}.$$

It follows that $\delta_{lmk} = -(-1)^{l+m+k}$ if $k < l,m$ or $k > l,m$ and that $\delta_{lmk} = (-1)^{l+m+k}$ in all other cases.

$\boldsymbol{\beta_{ijlm} = \pm 1}$ such that $v_{ij}v_{lm} = \beta_{ijlm}s_{ijlm}v_k$. It follows from the previous definition that $\beta_{ijlm} = \delta_{ijk}$.

As well, we implemented the maps $g : X \times X \to k$ and $\pi : X \to L$.

There is a risk of making small mistakes with big consequences in implementing the maps $\cdot$ and $h$. We verified that $\cdot$ satisfies the characterizing properties [TW02, 13.7] and that the map $h$ satisfies [TW02, 13.15], these

[1] This name comes from the fact that the map $\cdot$ is called a $(q,\epsilon)$-map in [TW02].

properties even define $h$ uniquely. Therefore we can be sure that the code is correct. The interested reader can run these tests by executing `tests_of_implementation_quad_alg_E8.sage`. Note that it is necessary to execute `implementation_quad_alg_E8.sage` first.

**Implementation of a biquaternion algebra and the Cayley-Dickson process** A quaternion algebra is already implemented in Sage. The tensor product of two quaternion algebras is not implemented, therefore we made this implementation ourselves.
In `implementation_QxQ_and_CayleyDickson.sage` the following objects are implemented:

A biquaternion algebra $Q_1 \otimes_k Q_2$ as a 16-dimensional vector space over $k$. We implemented the usual associative multiplication (denoted by *) and also the Jordan multiplication. Furthermore we implemented the standard involution of the biquaternion algebra, the Albert quadratic form on the skew-elements and the bijection $\theta$ of order 2 defined in (4.8).

Then we implemented the Cayley-Dickson structurable algebra as in Definition 3.8 starting from the associative biquaternion algebra equipped with Jordan multiplication and $\theta$. Elements of this structurable algebra consist of two elements of the biquaternion algebra. We also implemented the $V$- and the $U$-operators.

To make sure there are no mistakes in the implementation we verified some characterizing properties of the implemented structures. We verify that the multiplication on the biquaternion algebra is indeed associative, that the standard involution is indeed an involution and that Jordan$(x, y) = (x * y + y * x)/2$.

We verified if the multiplication and involution of the Cayley-Dickson structurable algebra do indeed satisfy (3.2), this calculation requires a couple of minutes. The interested reader can run these tests by executing `tests_of_implementation_QxQ_and_CayleyDickson.sage`. Note that it is necessary to execute `implementation_QxQ_and_CayleyDickson.sage` first.

**The proof of Theorem 4.12 and Lemma A.3**
First run `implementation_QxQ_and_CayleyDickson.sage` and `implementation_quad_alg_E8.sage`.

In `proof_theorem_4_12.sage` we defined the quadrangular algebra of type $E_8$, the quaternion algebras $Q_1 = (-s_2, -s_3)_k$, $Q2 = (-s_4s_6, -s_5s_6)$ and $Q_1 \otimes_k Q_2$. We implemented the isomorphism $\psi$ from Lemma 4.9 using table (4.7). Then we defined the map $\chi : CD \to X$ defined in Theorem 4.12 and we verified (4.9).

In `proof_lemma_A_4.sage` we implemented the tensor product $X \otimes_k E$.

We implemented the addition in $X \otimes_k E$ and multiplication with elements in $k$. Furthermore we defined the triple product of the Freudenthal triple system on $X$ (see Theorem 4.2). We extended this triple product to the Freudenthal triple system $X \otimes_k E$ as in Step 1 of Construction A.1; we also extended the skew-symmetric map $g$ to the Freudenthal triple system $X \otimes_k E$. We defined $u_1'$ and $u_2'$ as in Lemma A.3. We verify that (A.1) and (A.2) hold. We implemented the partition over which the summation runs in the definition of $\rho(x)$. Then we verified formula (A.4).

**Implementation of an octonion algebra** Contrary to quaternion algebras, octonion algebras are not implemented in Sage.
In `implementation_octonion.sage` we implemented octonions as an 8-dimensional vector space equipped with a multiplication as in Section 1.3 and the standard involution. Also we implemented the norm and the inverse.

In `tests_of_implementation_octonion.sage` we verified that the norm is multiplicative, that the multiplication is alternative but not associative. This implies that the implemented multiplication gives us an octonion algebra.

**Implementation of a bioctonion algebra**
In `implementation_OxO.sage` we implemented a bioctonion algebra $O_1 \otimes_k O_2$ as a 64-dimensional vector space over $k$. We implemented the multiplication and the standard involution. We also implemented the Albert quadratic form on skew-elements and the inverse of skew-elements.

In `tests_of_implementation_OxO.sage` we verify if the implemented involution is indeed an involution and if the characterizing properties of skew-elements given in Lemma 3.13 hold. To verify that we implemented a structurable algebra, we need to verify identity (3.2). Verifying this for symbolic elements gives rise to very long computation times. However verifying it for random elements with coefficients in $\mathbb{Q}$ is possible in a couple of minutes.

**The proof of Theorem 5.23** First run `implementation_octonion.sage` and `implementation_OxO.sage`. In `proof_of_theorem_5_22.sage` we verify identity (5.5) with the setup of Theorem 5.23.

Notice that if we prove formula (5.5) in $C_1 \otimes_k C_2$ where $C_1$ and $C_2$ are composition algebras of dimension 8, it is automatically valid for composition algebras of lower dimensions. It is likely that it is easier to find a proof of (5.5) without using Sage in these lower dimensional cases.

Let $O_1$ and $O_2$ be octonion algebras with linkage number one with bases as in Notation 5.17.(i). We let $u \in V$ be a symbolic element and define

$e_0, e_1, r \in \mathcal{S}$ as in Definition 5.19. We verify that Lemma 5.18 holds. Then we define $\bullet$ and $(.,.)$ as in Definition 5.21. We verify the computation in Remark 5.22.(i).

Since $x \in X_0$ should be an arbitrary element, we construct a base of $X_0$ that is efficient to do computations with. We know that $X_0 = e_0 \bullet X$ is 32-dimensional and that

$$e_0 \bullet (\mathbf{i}_1 x \otimes y) = e_0 \bullet (x \otimes \mathbf{i}_2 y),$$

for all $x \in O_1$ and $y \in O_2$.

Let $B_2$ be the base of $O_2$ given in Notation 5.17.(i), it follows that the following set is a base of $X_0$

$$\{ \bigcup_{y \in B_2} \{e_0 \bullet (1 \otimes y), e_0 \bullet (\mathbf{j}_1 \otimes y), e_0 \bullet (\mathbf{k}_1 \otimes y), e_0 \bullet (\mathbf{j}_1 \mathbf{k}_1 \otimes y)\}\}.$$

Since it is not efficient to let Sage calculate with fractions, we multiply each of the above base elements by $2a$. We use this basis to represent a symbolic element of $X_0$ and are able to verify (5.5).

Notice that we never divide by 3 or any other integer, we only divide by $q_A(u)$ which will always be different from 0. Therefore this computation is especially valid in characteristic 3.

# Appendix C

# Dutch summary
# Nederlandstalige samenvatting

Het hoofdthema van dit proefschrift is het bestuderen van verscheidene algebraïsche structuren die gerelateerd zijn aan lineaire algebraïsche groepen van relatieve rang 1 of 2. In het bijzonder, verbinden we de Moufang wereld en de structureerbare wereld, deze twee werelden waren geïsoleerde eilanden tot nu toe.

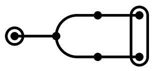
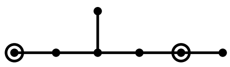
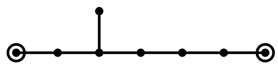
Al sinds het begin van de 20ste eeuw proberen wiskundigen vat te krijgen op lineaire algebraïsche groepen. Dit zijn structuren die in verscheidene deelgebieden van de wiskunde opduiken en reeds aanleiding gaven tot het ontwikkelen van heel wat interessante theorieën. De theorie van de lineaire algebraïsche groepen is een heel actief onderzoeksgebied met nog veel open vragen. Zo is het ondermeer niet duidelijk hoe men bepaalde exceptionele lineaire algebraïsche groepen expliciet kan construeren. Uit classificatieresultaten weet men dat deze bestaan, maar van bepaalde vormen bestaat er geen expliciete beschrijving.

Jacques Tits introduceerde in 1974 bepaalde geometrische structuren die 'gebouwen' genoemd worden. Vanuit iedere isotrope lineaire algebraïsche groep kan een gebouw opgebouwd worden en de lineaire algebraïsche groep heeft dan een natuurlijke actie op het gebouw in kwestie. Het bestuderen van gebouwen kan dus nuttig zijn om lineaire algebraïsche groepen beter te begrijpen. Een voorbeeld hiervan is het feit dat het Kneser-Tits probleem voor groepen van type $E_{8,2}^{66}$ recent is opgelost, zie [PTW12].

Tits classificeerde alle sferische gebouwen van rang groter of gelijk aan 3; hij toonde aan dat elk dergelijk gebouw afkomstig is van een lineaire algebraïsche groep, een gemixte groep, of een klassieke groep.

Moufang veelhoeken zijn sferische Moufang gebouwen van rang 2. Deze werden geclassificeerd in [TW02] in 2002, en opnieuw bleek dat elk dergelijk gebouw afkomstig is van een lineaire algebraïsche groep van relatieve rang 2, een gemixte groep, of een klassieke groep.

We definiëren Moufang veelhoeken in Sectie 2.1; hoewel deze definitie van meetkundige aard is, blijkt al snel dat Moufang veelhoeken gekarakteriseerd worden door bepaalde algebraïsche structuren. In de meeste gevallen zijn dit gekende algebraïsche structuren. Maar in het geval dat komt van de exceptionele groepen van type $E_6$, $E_7$ en $E_8$ weergegeven door de Tits-indices

${}^2E_6$ :  $E_7$ :  $E_8$ : 

bekomt men een zeer artificiële op coördinaten gebaseerde definitie van een ongekende algebraïsche structuur. Deze bevat (in het $E_8$-geval) een 12-dimensionale kwadratische ruimte, een 32-dimensionale vectorruimte en verscheidene afbeeldingen ertussen. In [Wei06b] geeft Weiss een axiomatische aanpak van deze structuren en noemt ze *vierhoekige algebra's* van type $E_6$, $E_7$ en $E_8$ (zie Sectie 2.2). Hoofdstukken 4 en 5 hebben tot doel om een natuurlijkere interpretatie te geven aan deze vierkantige algebras.

Moufang verzamelingen zijn sferische Moufang gebouwen van rang 1. Men kan Moufang verzamelingen construeren uit lineaire algebraïsche groepen van relatieve rang 1. Moufang verzamelingen kunnen beschreven worden zonder gebruik te maken van meetkunde; ze worden beschreven door een verzameling $X$ en een aantal deelgroepen van $\mathrm{Sym}(X)$ (zie Sectie 2.4). Moufang verzamelingen zijn niet geclassificeerd en het is ook niet duidelijk of dit mogelijk is. In karakteristiek verschillend van 2 en 3, ziet de situatie er misschien hoopvoller uit. In Hoofdstuk 6 geven we een nieuwe uniforme beschrijving van vermoedelijk alle gekende Moufang verzamelingen in karakteristiek verschillend van 2 en 3.

In dit proefschrift belichten we een nieuwe invalshoek ten opzichte van deze niet-zo-goed-begrepen Moufang structuren. We passen de theorie van de structureerbare algebra's toe.

Structureerbare algebra's zijn bepaalde niet-associatieve algebra's met involutie, de definitie is gegeven in Hoofdstuk 3. Het zijn veralgemeningen van Jordan algebra's met triviale involutie en van associatieve algebra's met involutie. Men kan structureerbare algebra's enkel definiëren over velden van karakteristiek verschillend van 2 en 3. Ze werden ingevoerd door Bruce Allison in 1978 met als doel om een expliciete constructie te geven van exceptionele Lie algebra's. Allison zijn aanpak blijkt te werken: met behulp van een structureerbare algebra kan men iedere isotrope Lie algebra beschrijven over velden van karakteristiek 0. Hij doet dit door de Tits–Kantor–Koecher constructie die vanuit Jordan algebra's bepaalde Lie algebra's maakt, te veralgemenen.

We geven hieronder de verscheidene hoofdresultaten uit dit proefschrift weer:

**In Hoofdstuk 4** tonen we aan dat iedere vierhoekige algebra op een natuurlijke manier aanleiding geeft tot een Freudenthal triple systeem. Dit zijn vectorruimtes uitgerust met een bilineaire vorm en een trilineair produkt; zij werden geïntroduceerd om Lie algebra's van type $E_7$ beter te begrijpen. We verzijzen naar Sectie 3.6 voor de exacte definitie van Freudenthal triple systems.

Ieder Freudenthal triple systeem is geassocieerd aan een welbepaalde structureerbare algebra van scheve dimensie 1. In de volgende stelling geven we de expliciete vorm van zo een structureerbare algebra:

**Stelling C.1** (Theorem 4.12)**.** *Zij $\Omega$ een vierhoekige algebra van type $E_6$, $E_7$ of $E_8$ over $k$ met karakteristiek verschillend van 2 en 3. Zij $A$ een delingsalgebra als volgt:*

- *$A$ is een quaternionenalgebra $Q$ als $\Omega$ van type $E_6$ is;*
- *$A$ is een tensorproduct $Q \otimes_k L$ met $Q$ een quaternionenalgebra en $L$ een kwadratische velduitbreiding, als $\Omega$ van type $E_7$ is;*
- *$A$ is een biquaternionenalgebra $Q_1 \otimes_k Q_2$ als $\Omega$ van type $E_8$ is.*

*Dan heeft de structureerbare algebra $\mathrm{CD}(A^+, \mathrm{Nrd}, c)$ hetzelfde[1] Freudenthal triple systeem als het Freudenthal triple systeem dat op natuurlijke manier aan $\Omega$ geassocieerd is.*

**In Hoofdstuk 5** geven we een alternatieve beschrijving van vierhoekige algebra's. We geven deze beschrijving hier enkel in het geval van vierhoekige algebra's van type $E_6$, $E_7$ en $E_8$.

Kwadratische vormen van type $E_6$, $E_7$ en $E_8$ bepalen hun respectieve vierhoekige algebra's volledig. De volgende karakterisatie van dit type kwadratische vormen was cruciaal:

*Een kwadratische vorm is van type $E_6$, $E_7$ of $E_8$ als en slechts als ze gelijkvormig is aan het anisotrope deel van de Albert vorm van $C_1 \otimes_k C_2$ voor een bepaalde octonen delingsalgebra $C_1$ en een compositie delingsalgebra $C_2$ van graad 2, 4 of 8 die (op isomorfie na) een kwadratisch deelveld gemeen hebben maar geen quaternionen deelalgebra gemeen hebben.*

We definiëren de involutie $\overline{x_1 \otimes x_2} = \overline{x_1} \otimes \overline{x_2}$ op $C_1 \otimes_k C_2$. We noteren met $\mathcal{S}_1$ de scheve elementen van $C_1$ en met $\mathcal{S}_2$ de scheve elementen van $C_2$. De scheve elementen van $C_1 \otimes_k C_2$ zijn gegeven door $\mathcal{S} = \mathcal{S}_1 \otimes 1 + 1 \otimes \mathcal{S}_2$.

[1] op isometrie na

Zij $(k, L, q, 1, X, \cdot, h)$ een vierhoekige algebra van type $E_6$, $E_7$ of $E_8$. We zetten de dimensies van de verscheidene vectorruimten eens op een rijtje:

| | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|
| $\dim_k \mathcal{S}$ | 8 | 10 | 14 |
| $\dim_k L$ | 6 | 8 | 12 |
| $\dim_k(C_1 \otimes_k C_2)$ | 16 | 32 | 64 |
| $\dim_k X$ | 8 | 16 | 32 |

We bemerken dat er in de drie gevallen geldt dat $\dim_k \mathcal{S} = \dim_k L + 2$ en $\dim_k(C_1 \otimes_k C_2) = 2 \dim_k X$. Geïnspireerd op dit feit en de correspondentie van de kwadratische vormen vonden we volgende expliciete beschrijving van vierhoekige algebra's van type $E_6$, $E_7$ of $E_8$. Dit geeft een alternatief voor de ad-hoc definitie die gegeven wordt in [TW02].

**Stelling C.2** (Theorem 5.23). *Zij $(k, L, q, 1, X, \cdot, h)$ een vierhoekige algebra van type $E_6$, $E_7$ of $E_8$ over een veld $k$ met karakteristiek verschillend van 2. In het bijzonder is $(k, L, q)$ een kwadratische ruimte is van type $E_6$, $E_7$ of $E_8$ met een basispunt 1. Zij $C_1$ en $C_2$ compositie algebra's zoals hierboven, zodanig dat $q$ gelijkvormig is aan de Albert vorm van $C_1 \otimes_k C_2$. Dus er bestaan een $\mathbf{i}_1 \in C_1$ en een $\mathbf{i}_2 \in C_2$ zodanig dat $\mathbf{i}_1^2 = \mathbf{i}_2^2 = a \in k$.*

*We definiëren een deelruimte $V$ van de scheve elementen $\mathcal{S}$ van $C_1 \otimes_k C_2$ als*[2]

$$V := \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^{\perp}.$$

*De vectorruimte $V$ heeft $k$-dimensie 6, 8 of 12, respectievelijk. We kiezen een willekeurig element $u \in V \setminus \{0\}$ en definiëren de kwadratische vorm*

$$Q := \frac{1}{q_A(u)} q_A|_V;$$

*deze vorm heeft als basispunt $u$ en is van respectievelijk type $E_6$, $E_7$ of $E_8$.*

*Nu definiëren we de deelruimte $X_0$ van $C_1 \otimes C_2$ als*

$$X_0 := \Big\langle \big(ax \otimes y + \mathbf{i}_1 x \otimes \mathbf{i}_2 y\big) \mid x \in C_1, y \in C_2 \Big\rangle.$$

*Deze vectorruimte heeft dimensie 8, 16 of 32 respectievelijk. Vervolgens definiëren we een bepaald element $r \in \mathcal{S}$ zoals in Definitie 5.19*(iii) *en we definiëren de bilineaire afbeelding $X_0 \times L_0 \to X_0$ als*

$$x \cdot v = v(r(u(rx)))$$

*en de bilineaire afbeelding $h \colon X_0 \times X_0 \to V$ als*

$$h(x, y) = (u(rx))\overline{y} - y((\overline{x}r)u).$$

[2] We beschouwen het orthogonaal complement t.o.v. de bilineaire vorm geassocieerd aan de Albert vorm.

*Dan is* $(k, V, Q, u, X_0, \cdot, h)$ *een vierhoekige algebra van type respectievelijk* $E_6$*,* $E_7$ *of* $E_8$*.*

We vermelden dat deze constructie gebaseerd is op $J$-ternaire algebra's geïntroduceerd door Bruce Allison in [All76].

**In Hoofdstuk 6** geven we een constructie van Moufang verzamelingen vertrekkende van structureerbare delingsalgebra's. Deze constructie is een veralgemening van de constructie van Moufang verzameling uit Jordan delingsalgebra's beschreven in [DMW06].

**Stelling C.3** (Theorem 6.25)**.** *Zij* $\mathcal{A}$ *een structureerbare delingsalgebra over een veld van karakteristiek verschillend van* 2,3 *of* 5*. Definieer de groep* $\mathcal{U} := \mathcal{A} \times \mathcal{S}$ *met optelling*

$$(x, s) + (y, t) = (x + y, s + t + \overline{x}y - \overline{y}x).$$

*Definieer de afbeelding* $q_x : \mathcal{S} \to \mathcal{S} : s \mapsto \frac{1}{6}x\overline{U_x(sx)} - U_x(sx)\overline{x}$ *en de permutatie* $\tau$ *van* $\mathcal{U} \setminus \{0\}$ *voor alle* $0 \neq x \in \mathcal{A}$ *en* $0 \neq s \in \mathcal{S}$

$$\begin{aligned}
(x, 0) &\mapsto \quad (-\hat{x}, 0),\\
(0, s) &\mapsto \quad (0, -\hat{s}),\\
(x, s) &\mapsto \quad \big(\ \hat{s}\big((q_{\hat{x}}(s) + \hat{s})^{\wedge}\hat{x}\big) + \big(s + q_x(\hat{s})\big)^{\wedge}x,\ -\big(s + q_x(\hat{s})\big)^{\wedge}\big).
\end{aligned}$$

*Dan is* $\mathbb{M}(\mathcal{U}, \tau)$ *een Moufang verzameling.*

Deze behoorlijk ingewikkelde formule is gebaseerd op het één-inverse van elementen van $\mathcal{A} \times \mathcal{S}$. Het concept van een één-inverse van elementen van $\mathcal{A} \times \mathcal{S}$ is ingevoerd in [AF99] in de context van Kantor paren, als een veralgemening van het toegevoegd inverse in een structureerbare algebra. Het bleek dat we precies dit concept nodig hadden voor onze constructie van Moufang verzamelingen.

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