Wigner quantization and Lie superalgebra representations

Wigner kwantisatie en representaties van Lie superalgebra's

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Preface

The theory of quantum mechanics is one of the most successful scientific theories due to its stunning correspondence with experimental results. This, however, does not imply that everything in quantum mechanics is perfectly well understood. Some of the basic assumptions of quantum mechanics remain a moot point and continue to inspire fundamental research. In particular, non-commutative quantum mechanics has been given much attention during the last decade, especially from the mathematical point of view. It is in this context that this thesis focuses on a related question: do the equations of motion determine the quantum mechanical commutation relations?

This question was first addressed by Wigner in 1950 [103]. In quantum mechanics it is usually assumed that the operators describing the position and momentum of the system are subject to the so-called canonical commutation relations. That this is not necessarily true was argued by Wigner. He showed that it is less restrictive to start from the equations of motion, namely the Hamilton equations and the Heisenberg equations. If one assumes both of these equations to be compatible as operator equations, one ends up with a set of compatibility conditions, which form the basis of Wigner quantization.

One characteristic feature of Wigner quantization is the broad variety of mathematical resources needed to fully grasp the subject. Although the importance of partitions, Schur functions, orthogonal polynomials or generating functions can certainly not be underestimated in some research topics related to Wigner quantization, it is mainly the unavoidable connection with Lie superalgebras and their representations that stands out. In essence, the Wigner quantization problem asks for operators satisfying the compatibility conditions. The first step to solving this problem is made by finding Lie superalgebra generators that are subject to the compatibility conditions. Secondly, by considering Lie superalgebra representations, these objects can be seen as operators on a certain representation space. Once this phase is reached one can focus on calculating essential physical quantities like wave functions or the spectrum of an operator.

This thesis can roughly be divided into two parts, each consisting of three distinct chapters. The first part is devoted to the introduction of the mathematical tools required in the second part, where some Hamiltonians are investigated in the context of Wigner quantization. This structure could lead to the indiscriminate conclusion that only Chapters 4, 5 and 6 contain original work. However, some interesting novel results can also be found in Chapters 2 and 3.

A gentle start is taken in **Chapter 1**, where partitions and Schur functions are introduced. Partitions are very accessible and intuitive objects that seem trivial at first. Nevertheless, they will appear in all but one of the remaining chapters. Partitions are mainly used as a labelling device for symmetric functions and representations of Lie algebras or superalgebras. The first of these applications is elucidated already in this introductory chapter, but not before taking a small detour through the world of generating functions. Chapter 5 revolves entirely around generating functions, but since partitions are excellent aids with respect to this subject, they are introduced as early as Chapter 1. Moreover, generating functions can also be constructed for symmetric polynomials, which are also handled in this chapter. Symmetric polynomials remain unchanged when their variables are permuted. Together, the symmetric polynomials in the variables $x_1, \ldots, x_m$ form a ring for which many bases exist. All bases consist of specific families of symmetric polynomials, of which the Schur functions are the most important. They have numerous applications in representation theory and deserve a lot of attention for this reason.

It has been emphasized that partitions and Schur functions are chiefly utilized in representation theory. One of the main purposes of **Chapter 2** is to uncover this connection. First, a comprehensive introduction on Lie algebras is given. Although Lie algebras play a minor role in this thesis compared with Lie superalgebras, they are treated in detail because much of the terminology concerning Lie algebras can be adopted for Lie superalgebras. The theoretical concepts for Lie algebras are clarified by means of an example, namely $\mathfrak{sl}(n;\mathbb{C})$. As an illustration of the Lie superalgebraical theory, the two examples $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(1|2n)$ are worked out. The choice for these superalgebras is made because they are crucial in subsequent chapters.

The treated examples of Lie algebras and Lie superalgebras are all written down as matrix algebras. It is shown that there are other ways of representing elements of these algebras as operators on a vector space. These are representations of Lie algebras and Lie superalgebras, and their significance has already been pointed out. Some important related concepts such as weights and their multiplicities are introduced. Because the weights of a representation carry a lot of information, a generating function for the multiplicities of the weights is a very important tool. Such so-called characters have received a lot of attention in research for this reason, and a lot of related problems remain unsolved. An entire subsection is therefore devoted to character theory.

The highlight of Chapter 2 is the classification of all irreducible $*$-representations of the Lie superalgebra $\mathfrak{osp}(1|2)$. This classification was already obtained many

years ago by Hughes [36]. However, not only can his results be presented in a more accessible way, they can also slightly be improved. The abundant attention for the Lie superalgebra $\mathfrak{osp}(1|2)$ is justified by its many occurrences in the context of Wigner quantization.

In **Chapter 3**, Wigner quantization is finally explained. It is placed in contrast with canonical quantization by considering the one-dimensional harmonic oscillator as an example, which is precisely the system that Wigner considered himself in his famous paper [103]. In order to understand the differences between both systems, one is required to have a certain quantum mechanical background. This is presented in the beginning of Chapter 3. But this is not enough if one wants to analyze the one-dimensional harmonic oscillator, even in the canonical picture. A specific class of orthogonal polynomials describes the wave functions for this physical system. Since orthogonal polynomials reappear on many occasions in future chapters, the need is felt to introduce them all at once. Only then is the one-dimension Wigner harmonic oscillator analyzed and compared to its canonical counterpart.

The one-dimensional Wigner harmonic oscillator is the first example of a Wigner quantum system considered in this thesis. In Chapters 4, 5 and 6, other quantum systems are investigated in the framework of Wigner quantization. A system of coupled harmonic oscillators is the center of attention in **Chapter 4**. In this system, the coupling of the various harmonic oscillators is represented by an interaction matrix. It turns out that the energy spectrum can be written down in terms of the eigenvalues and eigenvectors of the interaction matrix.

Two questions are addressed in this chapter. First, specific types of tridiagonal interaction matrices are described for which the system is analytically solvable. The system is always numerically solvable and involves the eigenvalues and eigenvectors of the interaction matrix. But in some cases analytically closed expressions exist for these objects. Such tridiagonal matrices are found in the context of orthogonal polynomials, which leads to analytically solvable Hamiltonians with Krawtchouk interaction, Hahn interaction and dual q-Krawtchouk interaction. The types of interaction are named after the corresponding orthogonal polynomials describing the system. For a general interaction matrix, the system of coupled harmonic oscillators is described in the context of Wigner quantization. The results are compared to the canonical case in general, and two analytically solvable quantum systems acquire particular attention in this context.

In **Chapter 5**, the system of an n-dimensional harmonic oscillator is studied in depth for $n = 3N$. The Wigner quantization of this system has already been performed in an earlier paper [58]. It was shown that solutions for the operators

exist in terms of $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ generators for this system. Representations for these Lie superalgebras were explored and the spectrum of the Hamiltonian was described by means of generating functions. In Chapter 5, the interest is shifted towards finding the angular momentum content of the system in particular Lie superalgebra representations. The aim is to describe this angular momentum content by generating functions. It is possible to describe a general method for finding such generating functions, but in practice only partial results can be obtained due to demanding computations. Many of the tools introduced earlier, such as Schur functions, partitions, characters and weights find an application in this chapter.

One-dimensional Wigner quantum systems are considered again in the final chapter. The first system is the Berry-Keating-Connes Hamiltonian $\hat{H} = \hat{x}\hat{p}$, notorious for its possible connection with the Riemann hypothesis. The second system is the free particle, the most elementary physical system of them all. A parallel with the one-dimensional Wigner harmonic oscillator from Chapter 3 can easily be drawn. Indeed, the methods used in **Chapter 6** are very similar and the wave functions can also be described in terms of orthogonal polynomials. However, there are also many dissimilarities. For instance, the spectrum of the operators $\hat{H}$, $\hat{x}$ and $\hat{p}$ differ significantly in the considered one-dimensional systems. The similarities and differences are commented on in the conclusions of Chapter 6, where the compatibility with the canonical case is also discussed.

Samenvatting

De theorie der kwantummechanica kan een van de meest succesvolle theorieën aller tijden genoemd worden. Experimentele resultaten hebben immers meermaals aangetoond dat de kwantummechanica de werkelijkheid zeer nauwkeurig beschrijft. Dit betekent echter niet dat alles in de kwantummechanica perfect begrepen is. Tot op heden wordt hevig gedebatteerd over de basisassumpties van deze theorie, wat nog steeds leidt tot fundamenteel onderzoek wereldwijd. In het bijzonder werd veel aandacht besteed aan niet-commutatieve kwantummechanica tijdens het laatste decennium, vooral dan vanuit een wiskundige invalshoek. In die optiek benadert dit proefschrift een prangende vraag: worden de kwantummechanische commutatierelaties eenduidig bepaald door de bewegingsvergelijkingen?

Wigner was in 1950 de eerste die zich deze vraag stelde [103]. In de kwantummechanica gaat men er meestal van uit dat de operatoren die de positie en het impulsmoment van het fysisch systeem beschrijven, voldoen aan de zogeheten canonische commutatierelaties. Dat dit niet noodzakelijk zo hoeft te zijn werd reeds aangekaart door Wigner. Hij toonde aan dat een meer algemene aanpak mogelijk is, waarbij de bewegingsvergelijkingen van Hamilton en Heisenberg als uitgangspunt genomen worden. Als aangenomen wordt dat beide vergelijkingen equivalent zijn als operatorvergelijkingen, bekomt men een stel compatibiliteitscondities die de basis vormen van Wignerkwantisatie.

Een karakteristieke eigenschap van Wignerkwantisatie is het grote aantal wiskundige hulpmiddelen dat nodig is om het onderwerp volledig te bevatten. Hoewel het belang van partities, Schurfuncties, orthogonale veeltermen en genererende functies zeker niet te onderschatten is, valt vooral de rol van Lie-superalgebra's en hun representaties op. Immers, in Wignerkwantisatie wordt in essentie gezocht naar operatoren die voldoen aan de compatibiliteitscondities. Daartoe moeten eerst generatoren van Lie-superalgebra's gevonden worden die zich in overeenstemming met de compatibiliteitscondities gedragen. Vervolgens kunnen die generatoren gezien worden als operatoren op een vectorruimte door representaties van de desbetreffende Lie-superalgebra's te beschouwen. Eenmaal dit punt bereikt is, kan aandacht besteed worden aan het berekenen van essentiële fysische resultaten zoals golffuncties of het spectrum van een operator.

Dit proefschrift kan ruwweg onderverdeeld worden in twee delen die elk bestaan uit drie hoofdstukken. Het eerste deel is gewijd aan de introductie van de wiskundige hulpmiddelen die gebruikt worden in het tweede deel, waarin verschillende Hamiltonianen onderzocht worden in de context van Wignerkwantisatie. Deze structuur

zou kunnen leiden tot de voorbarige conclusie dat al het originele werk gebundeld is in Hoofdstukken 4, 5 en 6. Interessante nieuwe resultaten kunnen echter ook gevonden worden in Hoofdstukken 2 en 3.

Het proefschrift gaat rustig van start in **Hoofdstuk 1**, waarin partities en Schurfuncties geïntroduceerd worden. Partities zijn heel toegankelijke en intuïtieve objecten die op het eerste gezicht triviaal lijken. Desalniettemin duiken ze op in bijna alle overblijvende hoofdstukken. Partities worden voornamelijk gebruikt als identificatiemiddel voor symmetrische functies en representaties van Lie-algebra's of superalgebra's. De eerste van deze applicaties wordt reeds in dit inleidend hoofdstuk toegelicht. Symmetrische veeltermen zijn veeltermen die onveranderd blijven wanneer de variabelen gepermuteerd worden. De symmetrische veeltermen in de variabelen $x_1, \ldots, x_m$ vormen een ring. Voor deze ring bestaan meerdere basissen, die elk corresponderen met een familie van symmetrische veeltermen. De meest belangrijke familie is die van de Schurfuncties. Zij worden genoteerd als $s_\lambda(x)$, waarbij λ een partitie is die de Schurfunctie vastlegt. Schurfuncties hebben talrijke toepassingen in representatietheorie en verdienen daarom veel aandacht.

In Hoofdstuk 1 worden ook genererende functies ingevoerd. Pas in Hoofdstuk 5 wordt het belang van deze objecten duidelijk, maar omdat partities uitstekende hulpmiddelen zijn bij de introductie van genererende functies, worden zij reeds in Hoofdstuk 1 behandeld. Bovendien kunnen ook genererende functies opgesteld worden voor symmetrische polynomen. Ook deze resultaten komen in Hoofdstuk 5 van pas.

Het mag ondertussen wel duidelijk zijn dat partities en Schurfuncties vooral in representatietheorie gebruikt worden. Het blootleggen van dit verband is een van de hoofddoelen van **Hoofdstuk 2**. Eerst wordt een bondige introductie tot Lie-algebra's gegeven. Lie-algebra's spelen in dit proefschrift een ondergeschikte rol in vergelijking met Lie-superalgebra's, maar toch worden ze redelijk gedetailleerd behandeld. De reden hiervoor is dat de terminologie voor Lie-algebra's makkelijk toepasbaar is op Lie-superalgebra's. De theoretische concepten kunnen voor Lie-algebra's eenvoudig verduidelijkt worden met behulp van een voorbeeld, met name $\mathfrak{sl}(n;\mathbb{C})$. Ook voor Lie-superalgebra's worden twee voorbeelden uitgewerkt. Daarbij wordt de keuze gemaakt voor $\mathfrak{gl}(m|n)$ en $\mathfrak{osp}(1|2n)$ omdat zij een cruciale rol spelen in toekomstige hoofdstukken.

De behandelde voorbeelden van Lie-algebra's en Lie-superalgebra's worden alle geformuleerd als matrixalgebra's. Er bestaan echter nog manieren om elementen van deze algebra's voor te stellen als operatoren op een vectorruimte. Dit is precies wat een representatie van een Lie-algebra of Lie-superalgebra genoemd wordt, en

het belang daarvan werd reeds eerder uit de doeken gedaan. Wie over representaties spreekt, moet het zeker ook over gewichten hebben. Omdat representaties volledig bepaald zijn door hun gewichten en diens multipliciteiten, wordt veel waarde gehecht aan een genererende functie voor die multipliciteiten. Zulke zogeheten karakters zijn uitvoerig bestudeerd in het verleden, maar nog steeds blijft een aantal gerelateerde problemen onopgelost. Karakters krijgen daarom ook de nodige aandacht in Hoofdstuk 2.

De blikvanger van het tweede hoofdstuk is de classificatie van alle irreduciebele $*$-representaties van de Lie-superalgebra $\mathfrak{osp}(1|2)$. Deze classificatie werd reeds uitgevoerd door Hughes [36], maar ze wordt hier herbekeken om diverse redenen. Vooreerst is het mogelijk om de resultaten van Hughes op een meer toegankelijke manier af te leiden. Ook blijkt dat de conclusie van Hughes voor verbetering vatbaar is. Immers, hij liet het na om op te merken dat twee klassen van representaties in feite equivalent zijn. Op het einde van Hoofdstuk 2 wordt aangetoond hoe dit ingezien kan worden. De uitgebreide aandacht die alzo naar deze Lie-superalgebra uitgaat, wordt gerechtvaardigd door de veelvuldige optredens van $\mathfrak{osp}(1|2)$ in de context van Wignerkwantisatie.

Dit blijkt voor het eerst in **Hoofdstuk 3**, alwaar Wignerkwantisatie eindelijk echt wordt uitgelegd. Wignerkwantisatie wordt in dit hoofdstuk tegenover canonische kwantisatie geplaatst, de benadering van de kwantummechanica waarin de canonische commutatierelaties aangenomen worden. Het verschil tussen beiden vormen van kwantisatie wordt duidelijk gemaakt door het systeem te beschouwen van een eendimensionale harmonische oscillator. Dit is precies het systeem dat Wigner zelf onderzocht heeft in zijn beroemd artikel [103].

Een zekere achtergrond in de kwantummechanica is vereist om het verschil tussen Wignerkwantisatie en canonische kwantisatie echt te begrijpen. Daarom wordt een beknopte inleiding tot de kwantummechanica gegeven in het begin van het derde hoofdstuk. Dit is op zich onvoldoende om de eendimensionale harmonische oscillator volledig te kunnen onderzoeken, zelfs in het canonische kader. De golffuncties van het fysisch systeem worden immers beschreven aan de hand van bepaalde orthogonale veeltermen. Omdat orthogonale veeltermen nog vaak opduiken in latere hoofdstukken, worden ze allemaal tegelijk geïntroduceerd in Hoofdstuk 3.

Pas dan kan de Wigner-harmonische oscillator onderzocht worden en vergeleken met zijn canonische tegenhanger. Het was reeds bekend dat de golffuncties van de eendimensionale canonische harmonische oscillator beschreven konden worden in termen van Hermiteveeltermen. Het energiespectrum van dit systeem, dit zijn alle

mogelijke waarden die optreden na een meting van de totale energie, blijkt discreet te zijn. De verschillende energieniveaus liggen telkens even ver van elkaar, men spreekt in dit geval van een equidistant energiespectrum. In Wignerkwantisatie liggen veralgemeende Hermiteveeltermen aan de basis van de golffuncties. Het spectrum blijft equidistant, maar het laagste energieniveau wordt beïnvloed door een parameter a. Dit is dezelfde parameter die de representaties van $\mathfrak{osp}(1|2)$ beschrijft, wat geen toeval is aangezien de generatoren van deze Lie-superalgebra inderdaad voldoen aan de compatibiliteitscondities van de eendimensionale Wigner-harmonische oscillator.

Nu het eerste Wigner-kwantumsysteem onderzocht is, is het pad geëffend om andere fysische systemen onder Wigners loep te nemen. Dit is wat gebeurt in Hoofdstukken 4, 5 en 6. Een systeem van gekoppelde harmonische oscillatoren speelt de hoofdrol in **Hoofdstuk 4**. Voor dit systeem wordt de interactie tussen de oscillatoren beschreven door een interactiematrix. Al snel blijkt dat het energiespectrum van zo'n systeem kan beschreven worden in termen van de eigenwaarden en eigenvectoren van de interactiematrix.

Twee problemen komen in dit hoofdstuk aan bod. Eerst wordt onderzocht voor welke tridiagonale interactiematrices het systeem analytisch oplosbaar is. Zoals reeds impliciet werd aangegeven, is het systeem altijd numeriek oplosbaar en zijn de eigenwaarden en eigenvectoren van de interactiematrix van belang. Maar in sommige gevallen bestaan er analytisch gesloten uitdrukkingen voor deze objecten. In dit geval wordt het systeem analytisch oplosbaar genoemd. Tridiagonale matrices waarvan de eigenwaarden en eigenvectoren analytisch bepaald zijn, kunnen gevonden worden in de context van orthogonale veeltermen. De alzo ontstane analytisch oplosbare systemen krijgen de naam mee van de corresponderende familie orthogonale veeltermen. Zo onderscheiden we analytisch oplosbare Hamiltonianen met Krawtchouk-interactie, Hahn-interactie en duale q-Krawtchouk-interactie.

Er wordt teruggegrepen naar een algemene interactiematrix in het tweede deel van Hoofdstuk 4. Ditmaal wordt het systeem bekeken vanuit Wigners oogpunt. Oplossingen voor de compatibiliteitscondities blijken te bestaan in termen van generatoren van de Lie-superalgebra's $\mathfrak{gl}(1|n$ en $\mathfrak{osp}(1|2n)$. De eigenwaarden en eigenvectoren van de interactiematrix spelen opnieuw een belangrijke rol. Het is dan ook nuttig om na te gaan hoe twee analytisch oplosbare systemen zich gedragen wanneer gekeken wordt naar specifieke representaties van beide Lie-superalgebra's. Ook het energiespectrum wordt aan deze vergelijking onderworpen. In de $\mathfrak{gl}(1|n)$-oplossing wordt een eindig spectrum ontdekt waarbij subtiele verschillen gevonden worden tussen de systemen met Krawtchouk-interactie en constante interactie. Het

onderscheid met de $\mathfrak{osp}(1|2n)$-oplossing is echter veel groter, want daar is het energiespectrum oneindig. Uiteindelijk worden de resultaten getoetst aan het canonische geval, dat correspondeert met een zeer specifieke representatie van $\mathfrak{osp}(1|2)$.

In **Hoofdstuk 5** wordt de n-dimensionale harmonische oscillator uitgediept voor $n = 3N$. Dit systeem werd reeds eerder benaderd vanuit Wigners invalshoek [58]. In dit artikel werd aangetoond dat de relevante operatoren kunnen geschreven worden in termen van generatoren van de Lie-superalgebra's $\mathfrak{gl}(1|n)$ en $\mathfrak{osp}(1|2n)$. Er werd gekeken naar representaties van die Lie-superalgebra's en het energiespectrum werd weergegeven met behulp van genererende functies. In Hoofdstuk 5 verschuift de aandacht naar het bepalen van het draai-impulsmoment van het systeem voor verschillende representaties van Lie-superalgebra's. Ook hier worden genererende functies aangewend om het draai-impulsmoment te beschrijven. Een algemene methode om tot zulke genererende functies te komen, kan beschreven worden met behulp van een aantal theoretische argumenten. Het toepassen van deze methode steunt in de praktijk op het bepalen van de constante term van verschillende ingewikkelde machtreeksen, waarvoor verschillende computerpakketten kunnen worden aangewend. Omdat de machtreeksen in Hoofdstuk 5 echter vaak heel complex zijn, faalt de computer dikwijls in het berekenen van constante termen. Om die reden is het maar mogelijk om gedeeltelijke resultaten te behalen. Hoofdstuk 5 is echter ook op theoretisch vlak heel interessant omdat veel van de eerder ingevoerde mathematische hulpmiddelen, zoals Schurfuncties, partities, karakters en gewichten, van pas komen.

Eendimensionale Wigner-kwantumsystemen duiken opnieuw op in **Hoofdstuk 6**. Het eerste systeem wordt beschreven door de Berry-Keating-Connes-Hamiltoniaan $\hat{H} = \hat{x}\hat{p}$, die berucht is omwille van zijn mogelijke relatie met de Riemannhypothese. Het tweede systeem is het meest elementaire dat beschouwd kan worden, namelijk het vrije deeltje. Het duurt niet lang eer de gelijkenissen met de eendimensionale Wigner-harmonische oscillator uit Hoofdstuk 3 komen bovendrijven. Het valt inderdaad op dat exact dezelfde methodes worden gehanteerd in dit laatste hoofdstuk. Bovendien worden de golffuncties hier ook beschreven in termen van orthogonale veeltermen. Bij het bepalen van een formele eigenvector van de Berry-Keating-Connes-Hamiltoniaan komen Meixner-Pollaczekveeltermen aan bod, terwijl de spectra van de positie- en impulsoperator in dit systeem afhankelijk zijn van veralgemeende Hermiteveeltermen. Laguerreveeltermen liggen aan de basis van het spectrum van de Hamiltoniaan van het vrije deeltje. Nog een overeenkomst tussen alle eendimensionale systemen is dat het oplossen van het Wignerprobleem gebeurt in termen van generatoren van de Lie-superalgebra $\mathfrak{osp}(1|2)$.

Om verschillen tussen alle beschouwde systemen waar te nemen, moet naar het spectrum van de operatoren gekeken worden. Het energiespectrum heeft telkens een dubbele multipliciteit, maar voor het vrije deeltje bestaat het spectrum enkel uit positieve reële energiewaarden. De spectra van de positie- en impulsoperatoren hebben een enkelvoudige multipliciteit en ze strekken zich uit over de hele reële as. Zoals blijkt uit de discussie na Hoofdstuk 6 stemmen alle bevindingen overeen met de gekende canonische resultaten.

1

Partitions and Schur functions

Before we start talking about Wigner quantization and its inextricable connection with Lie superalgebra representations, we need to introduce basic concepts like partitions and Schur functions. Partitions play a very important role in the remaining chapters since representations of Lie superalgebras are often labelled by them. Moreover, partitions characterize Schur functions and we will see that these polynomials carry the most important information of some particular Lie superalgebra representations.

Luckily, we do not need to juggle with fancy words in order to understand what partitions and Schur functions are. In fact, they prove to be fairly accessible concepts, nevertheless with a large variety of applications. Most of the information regarding partitions and symmetric functions, of which Schur functions are an example, can be found in the classic manuscript of Macdonald [64]. Another interesting concept that will be introduced in this chapter, is the concept of generating functions. Partitions are nice tools to explain these objects, which is why they are treated in the same chapter.

1.1 Partitions and generating functions

1.1.1 Partitions and diagrams

Partitions will be ubiquitous in all the remaining chapters, so a good understanding is fundamental. In essence, a partition of a positive integer is a way of writing that integer as a sum of positive integers. For future purposes we will give a more formal definition.

Definition 1.1 (Partition of an integer) *A **partition** λ is any sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers λ_i written in non-increasing order, i.e.*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$

*The non-zero λ_i are the **parts** of the partition. The number of parts in λ must be finite and is called the **length** of the partition, denoted by $\ell(\lambda)$. The **weight** of λ is defined by the sum of its parts and is written as $|\lambda| = \sum_i \lambda_i$. When the weight of a partition is equal to N, we write $\lambda \vdash N$ and say that λ is a partition of N.*

Usually, only the non-zero parts of the partition are explicitly written. Hence, a partition with length k is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Any partition $\lambda \vdash N$ with length k can be represented by a **Young diagram**. This is an arrangement of N left-adjusted boxes in k rows, where the ith row contains λ_i boxes. As an example, the Young diagram of the partition $(5,2,2,1)$ is given in Figure 1.1. The weight of this partition is 10, and we have $\ell(\lambda) = 4$.

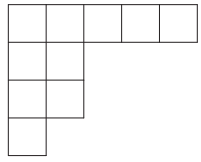

Figure 1.1: Young diagram of the partition $(5,2,2,1)$.

Definition 1.2 (Conjugate partition) *The **conjugate** of a partition λ, denoted by λ', is the partition that arises when the Young diagram of λ is transposed. Otherwise put, the lengths of the columns of the Young diagram of λ form the parts of its conjugate partition λ'.*

For our example $\lambda = (5,2,2,1)$ we have $\lambda' = (4,3,1,1,1)$. We say that λ is **self-conjugate** when $\lambda \equiv \lambda'$. An example of a self-conjugate partition is given by $(4,2,1,1)$, as can be seen from Figure 1.2

There is an alternative way of representing partitions, called the Frobenius notation. To explain what it is, we must introduce the rank of a partition.

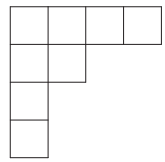

Figure 1.2: The self-conjugate partition $(4,2,1,1)$.

Definition 1.3 (Rank of a partition) *The number of boxes in the **main diagonal** of the Young diagram of a partition, running from northwest to southeast starting in the top-left corner, is called the **rank** of the partition.*

Assume we have a partition of rank r. Let a_i be the number of boxes to the right of the main diagonal in the ith row, and let b_i be the number of boxes below the main diagonal in the ith column. Then we can denote this partition in **Frobenius notation** by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

The interpretation of the a_i and b_i is apparent from Figure 1.3. It is clear that we must have $a_1 > a_2 > \cdots > a_r$ and $b_1 > b_2 > \cdots > b_r$, and that the weight of the partition equals $r + \sum_{i=1}^{r}(a_i + b_i)$.

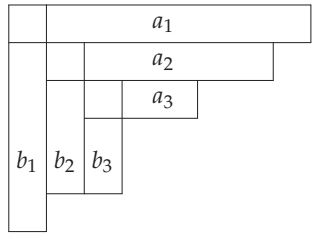

Figure 1.3: Young diagram of a partition with rank 3 in Frobenius notation

By switching the a_i and b_i in the Frobenius notation of a partition λ, one obtains the Frobenius notation of its conjugate λ'.

Now, let us consider two partitions λ and μ such that $\mu_j \leq \lambda_j$ for all j. In other words, the Young diagram of μ is embedded in the Young diagram of λ, a case for

which the notation $\lambda \supset \mu$ is used. The set-theoretic difference $\theta = \lambda - \mu$ is called a **skew diagram** and contains the squares that belong to the Young diagram of λ but not to the Young diagram of μ. If θ contains at most one block per column, i.e. $\theta'_i \leq 1$, the skew diagram is called a **horizontal strip**. A horizontal strip with r blocks is then called a **horizontal r-strip**. In Figure 1.4 we find an example of a horizontal 4-strip.

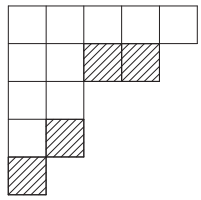

Figure 1.4: The horizontal 4-strip $\lambda - \mu$ with $\lambda = (5,4,2,2,1)$ and $\mu = (5,2,2,1)$.

Obviously, the concept of a vertical strip can be defined analogously, but it is of no use in the present document.

1.1.2 Generating functions

Although the subject of generating functions will remain untouched for many chapters, we mention it here because nice examples related to partitions and Schur functions can be given. Another motive for the early placement of this subject is their ease of handling, a feature they have in common with partitions.

In [104], Herbert Wilf describes generating functions as clotheslines on which we hang up a sequence of numbers for display. We can translate these words to mathematics as follows.

Definition 1.4 (Generating function) *A **generating function** of a sequence a_n is a formal power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

in the variable x, in which the coefficient of x^n is precisely a_n.

The clothesline that was referred to by Wilf does not look elegant as a formal series. Instead, generating functions will be written as (formal) functions for which the series expansion has the numbers a_n as the coefficient of x^n. For example, the constant sequence $1, 1, 1, \ldots$ can be described by the generating function

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

This equality is only valid when $|x| < 1$, yet we do not wish to be concerned with issues of convergence in the theory of generating functions. Therefore, generating functions are defined as formal rather than analytic functions. A less trivial example of a generating function is the generating function for partitions of an integer. The first few terms in the series expansion of this generating function are given by

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots,$$

since there are three partitions of 3, five partitions of 4, seven partitions of 5 and so on. The generating function for partitions of an integer can be written as an infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{1.1}$$

To see why this is true, we note that

$$1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots = \frac{1}{(1-x)}$$

is the generating function for partitions with all parts less than or equal to 1. The same logic shows that

$$\frac{1}{(1-x)(1-x^2)}$$

is the generating function for partitions with all parts less than or equal to 2. Allowing parts of random size in the partition results in the generating function for partitions of an integer, given by equation (1.1).

Generating functions are not confined to number sequences. The coefficient of the variable(s) in the series expansion of the generating function could also be a polynomial or even a generating function in itself. Examples of the latter type will be handled in Chapter 5. We will provide examples of the former type in the context of symmetric functions, a subject which is also closely related to partitions.

1.2 Symmetric polynomials

Our introduction to symmetric polynomials is mainly based on [64], where the ring of symmetric functions is defined for infinitely many variables. We will restrict ourselves to the case where there is only a finite number of variables $x_1, \ldots, x_m$.

Definition 1.5 (Symmetric polynomial) *Consider the ring $\mathbb{Z}[x_1, \ldots, x_m]$ of polynomials in m independent variables $x_1, \ldots, x_m$ with integer coefficients. The symmetric group S_m acts on this ring by permuting the variables. A polynomial in $\mathbb{Z}[x_1, \ldots, x_m]$ is a **symmetric polynomial** if it remains invariant under the action of S_m.*

So a symmetric polynomial is a polynomial that is unchanged under any permutation of its variables. The ring of symmetric polynomials with integer coefficients in $x = x_1, \ldots, x_m$ is denoted by Λ_m. When each monomial of the symmetric polynomial has the same degree k, i.e. the sum of the exponents equals k for each monomial, we speak of a **homogeneous symmetric polynomial of degree** k. We have

$$\Lambda_m = \bigoplus_{k \geq 0} \Lambda_m^k,$$

where Λ_m^k indicates the space of homogeneous symmetric polynomials of degree k. Several bases exist the ring of symmetric polynomials Λ_m.

1. Given a partition λ of weight $|\lambda| = k$ and length $\ell(\lambda) \leq m$. Introducing the notation $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}$, we define the **monomial symmetric function** $m_\lambda(x_1, \ldots, x_m)$ to be the smallest symmetric polynomial that contains the monomial x^λ. We can write

   $$m_\lambda(x) = \sum_\alpha x^\alpha,$$

   where the sum is taken over all distinct permutations α of λ. The m_λ such that $\ell(\lambda) \leq m$ and $|\lambda| = k$ form a $\mathbb{Z}$-basis of the space of homogeneous symmetric polynomials of degree k.

2. For $0 \leq r \leq m$, the rth **elementary symmetric function** $e_r(x_1, \ldots, x_m)$ is defined as the monomial symmetric function that corresponds to the partition

with r 1s, denoted by $\lambda = (1^r)$. We have $e_0 = 1$ and

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r},$$

for $1 \leq r \leq m$. For other values of r, e_r is defined to be 0. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, e_λ is defined as $e_{\lambda_1}e_{\lambda_2}\cdots$. We mention without proof that the set of e_λ, where λ ranges over all partitions of k, is a $\mathbb{Z}$-basis for Λ_m^k. A full proof of this statement can be found in [64].

3. A third basis for Λ_m is supplied by the **complete symmetric functions**. We define h_r as the sum of all monomial symmetric functions of degree r in the variables $x_1, x_2, \ldots$, for each $r \geq 0$. Thus we have

$$h_r = \sum_{\lambda \vdash r} m_\lambda,$$

for $r \geq 0$, while $h_0 = 1$ and $h_r = 0$ for $r \leq 0$. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we extend the homogeneous symmetric functions in the same way as we did for the elementary symmetric functions. Thus, h_λ is defined as $h_{\lambda_1}h_{\lambda_2}\cdots$.

4. The rth **power sum** is defined by

$$p_r = \sum_i x_i^r = m_{(r)},$$

for each $r \geq 1$. Again, we define $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. The power sums are somewhat different in the sense that they form a $\mathbb{Q}$-basis of the symmetric polynomials. This means that any symmetric polynomial can be written as a linear combination of power sums with rational coefficients, not integer coefficients as was the case with any of the previous bases.

Generating functions exist for all of these symmetric polynomials. A generating function for a type of symmetric functions must be interpreted as a formal series in which the coefficient of t^r is the rth symmetric function of that type. As an example we mention the generating function for the elementary symmetric functions, which takes the form

$$\sum_{r=0}^{m} e_r t^r = \prod_{i=1}^{m}(1 + x_i t). \tag{1.2}$$

All of the symmetric functions mentioned above are interesting, and the elementary symmetric functions will even have a role to play in Chapter 5. However, by far the most important basis of the symmetric functions has not been described yet: the Schur functions.

1.3 Schur functions

There are different ways of defining Schur functions. We start with the classical definition in terms of determinants because no extra information is needed to introduce Schur functions in this approach.

Definition 1.6 (Schur function) *Consider a set of indeterminates $x = (x_1, \ldots, x_m)$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of length $\ell(\lambda) \leq m$. Then one can define the* ***Schur function*** *$s_\lambda(x)$ as the following quotient of two alternating polynomials:*

$$s_\lambda(x) = s_\lambda(x_1, \ldots, x_m) = \frac{a_{\lambda+\delta}(x)}{a_\delta(x)},$$

with $\delta = (m-1, m-2, \ldots, 1, 0)$ and

$$a_{\lambda+\delta}(x) = \det(x_i^{\lambda_j+m-j})_{1\leq i,j\leq m}. \tag{1.3}$$

When λ is the zero partition, $a_{\lambda+\delta}(x)$ is nothing more than the Vandermonde determinant $a_\delta(x) = \prod_{1\leq i,j\leq m}(x_i - x_j)$. The Schur function vanishes when the length of the partition λ exceeds the number of variables m.

It is clear from the definition that Schur functions are symmetric polynomials of degree $|\lambda|$. Schur polynomials can be extended to an infinite number of variables, hence the term Schur functions is often used. Another, rather elegant, combinatorial definition of Schur functions can be given in terms of semistandard Young tableaux.

Definition 1.7 (Semistandard Young tableau) *Consider two partitions λ and μ such that $\lambda \supset \mu$. The skew diagram of $\lambda - \mu$ in which each box is filled with a positive integer is called a* ***Young tableau of shape*** *$\lambda - \mu$. The tableau is a* ***semistandard or column-strict Young tableau*** *when the integers are weakly increasing across rows and strictly increasing down columns.*

In principle, any ordered set from a general alphabet, such as $x_1, x_2, x_3, \ldots$, could be used to fill the Young diagram. For reasons of brevity we will restrict ourselves to positive integers. Suppose we have a tableau T in which the integer i occurs a_i times, for $i = 1, \ldots, m$. Then the tableau T gives rise to the monomial

$$x^T = \prod_{i=1}^{m} x_i^{a_i}.$$

A semistandard Young tableau of shape $\lambda = (5,2,2,1)$ (μ is the zero partition in this example) and $m = 5$, for instance, could look like in Figure 1.5.

1	1	3	3	4
2	4			
4	5			
5				

Figure 1.5: Semistandard Young tableau of shape $\lambda = (5,2,2,1)$.

In this example μ is the zero partition, which is not prevented by Definition 1.7. Recording the number of times each integer appears in a tableau results in a sequence known as the **weight of a tableau**. The semistandard Young tableau in Figure 1.5 has weight $(2,1,2,3,2)$.

Without proof we mention that Schur functions can equivalently be defined by

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x^T,$$

where the sum is over all semistandard Young tableaux of shape λ and with entries chosen from the set $\{1, 2, \ldots, m\}$. Let us illustrate this with a simple example. We can form eight semistandard Young tableaux of shape $\lambda = (2,1)$ for $m = 3$. They are all given in Figure 1.6.

1 1	1 1	1 2	1 2	1 3	1 3	2 2	2 3
2	3	2	3	2	3	3	3

Figure 1.6: All semistandard tableaux of shape $\lambda = (2,1)$ with $m = 3$.

Each of these tableaux T corresponds to a monomial x^T as described above, and we find

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Schur functions can be multiplied like ordinary polynomials. Such a product is called the **outer product** of Schur functions. Since the outer product of two Schur functions must also be a symmetric function, and since Schur functions are a $\mathbb{Z}$-basis of the symmetric functions Λ_m [64], we know that the outer product of two Schur functions must be expressible as a linear combination of Schur functions. The coefficients $c_{\mu\nu}^{\lambda}$ in the decomposition

$$s_\mu(x)s_\nu(x) = \sum_\lambda c_{\mu\nu}^{\lambda}\, s_\lambda(x)$$

are known as the Littlewood-Richardson coefficients. A combinatorial rule for evaluating them was given in [60], and a proof of this rule can be found in [64, §I.9]. We will briefly state the rule in terms of semistandard Young tableaux, for which we need the concepts of a **word of a tableau** and a **lattice word**.

Definition 1.8 (Word of a tableau) *The word of a Young tableau T, denoted by $w(T)$, is derived from T by reading the symbols in T from right to left in successive rows, starting in the top-right row and proceeding to the bottom row.*

For the tableau T in Figure 1.5 we have

$$w(T) = 4331142545.$$

This word is not a lattice word, as we conclude from the following definition.

Definition 1.9 (Lattice word) *A word $w(T)$ is a lattice word or lattice permutation if at any point in the word - reading from left to right - the symbol i has occurred at least as often as the symbol $i+1$.*

The word $w(T) = 1121221$ is a lattice permutation, while the word $w(T) = 11232211231$ is not because the subword 112322 contains more 2s than 1s.

Proposition 1.10 *Let λ, μ and ν be partitions such that $|\mu| + |\nu| = |\lambda|$ and $\lambda \supset \mu, \nu$. The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is then equal to the number of semistandard Young tableaux T of shape $\lambda - \mu$ and weight ν such that $w(T)$ is a lattice word.*

When the necessary conditions on λ, μ and ν are not fulfilled, the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu} = 0$. Proposition 1.10 can be applied to obtain that $c^{\lambda}_{\mu\nu} = 2$ when $\lambda = (5,3,3,1)$, $\mu = (2,2,1)$ and $\nu = (4,2,1)$. Figure 1.7 shows the two semistandard Young tableaux T of shape $\lambda - \mu$ with weight ν for which $w(T)$ is a lattice word.

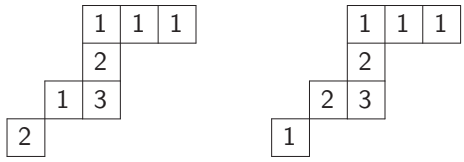

Figure 1.7: The lattice words $w(T)$ with T of shape $\lambda - \mu$ and weight ν.

Many special cases of the Littlewood-Richardson coefficients have been identified. We mention Pieri's formula since it will reappear in Chapter 5. Pieri's formula describes the Littlewood-Richardson coefficients when the length of either of the partitions μ or ν is equal to one. When $\nu = (r)$ we have

$$s_{\mu}(x)s_{(r)}(x) = \sum_{\lambda} s_{\lambda}(x) \tag{1.4}$$

where the sum is over all partitions λ for which $\lambda - \mu$ is a horizontal r-strip.

The outer product is not the only product defined for the Schur functions. There is also an **inner or Kronecker product** of Schur functions. This is defined by

$$s_{\lambda}(x) \times s_{\mu}(x) = \sum_{\nu} g_{\lambda\mu\nu}\, s_{\nu}(x),$$

where $g_{\lambda\mu\nu}$ are integers that can best be introduced by considering the **inner coproduct** of Schur functions. This is a Schur function with variables $xy = x_1y_1, x_1y_2, \ldots, x_my_n$. Such a Schur function is symmetric in x and in y, so it must be decomposable in the Schur function bases of Λ_m and Λ_n. Thus we can

write

$$s_\lambda(xy) = \sum_{\mu,\nu} g_{\lambda\mu\nu} s_\mu(x) s_\nu(y).$$

The coefficients $g_{\lambda\mu\nu}$ are called the **Kronecker coefficients** and they are used to define the inner product of Schur functions. Likewise, the Littlewood-Richardson coefficients feature in the **outer coproduct** of Schur functions, defined by

$$s_\lambda(x+y) = s_\lambda(x_1,\ldots,x_m,y_1,\ldots,y_n) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_\nu(y).$$

The Kronecker coefficients are crucial in Chapter 5, where some more context is given. We will see that these coefficients are used in representation theory, a topic in which also Schur functions are omnipresent. The basics of representation theory will be explained in the next chapter, but as a sneak-peak we wish to mention that Schur functions are so important because they are the characters of specific representations of groups and algebras.

All symmetric functions have a so-called supersymmetric counterpart. We will not pay much attention to these supersymmetric functions, but the supersymmetric Schur functions are rather important so we briefly explain how these are defined.

Let $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_n)$ be two sets of independent variables. We define the **complete supersymmetric functions** $h_r(x|y)$, with r a non-negative integer, as

$$h_r(x|y) = \sum_{k=0}^{r} h_k(x) e_{r-k}(y),$$

where $h_i(x)$ and $e_i(y)$ are the complete and elementary symmetric functions defined in Section 1.2 respectively. The **supersymmetric Schur functions** indexed by a partition λ can then be defined by the following determinantal formula:

$$s_\lambda(x|y) = \det\big(h_{\lambda_i-i+j}(x|y)\big)_{1\le i,j\le \ell(\lambda)}.$$

Particularly interesting to us is the expression of these supersymmetric Schur functions in terms of ordinary Schur polynomials, given by

$$s_\lambda(x|y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_{\nu'}(y), \tag{1.5}$$

in which we find a new encounter with the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$. Supersymmetric Schur functions are so interesting because they turn out to be

characters of representations of superalgebras. As mentioned in the very first lines of this chapter, superalgebra representations will play a very important role in the remainder of this thesis.

2

Lie superalgebras and their representations

The partitions described in the previous chapter come in very handy in the theory of representations. Schur functions are also an important tool in this area. So what is it that makes these representations so important to us? It is their power to describe certain objects as operators on a vector space.

In the next chapter we will explain how we will be looking for operators subject to specific commutation relations in the context of Wigner quantization. We will find such operators in two steps. First, we consider objects that satisfy the given commutation relations, which will lead us to Lie superalgebras. After we have identified the objects with elements of a certain Lie superalgebra, we will make operators out of them by considering representations of that superalgebra. This is why a thorough knowledge of the relevant Lie superalgebras and their representations is needed. This chapter is entirely aimed at achieving this goal.

2.1 Lie algebras

We have suggested many times that Lie superalgebras and their representations will play a leading role in this thesis. However, the importance of some Lie algebras may not be underestimated. We devote this section to the introduction of the most important notions concerning Lie algebras. We will only consider Lie algebras over the field K, where K is $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 (Lie algebra) *A vector space $\mathfrak{g}$ over the field K, together with an operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, is called a **Lie algebra** over K if*

- $[\cdot,\cdot]$ *is bilinear,*
- $[x,y] = -[y,x]$,
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ ***(Jacobi identity)***

for all $x,y,z \in \mathfrak{g}$.

The operation $[\cdot,\cdot]$ is called the **Lie bracket**. When $\mathfrak{g}$ is an ordinary associative algebra over K it can be made into a Lie algebra by taking

$$[x,y] = xy - yx, \tag{2.1}$$

where the product of two elements $x,y \in \mathfrak{g}$ has been denoted by $x \cdot y \equiv xy$. It can easily be checked that for this interpretation of the Lie bracket all of the properties in Definition 2.1 are satisfied. Because of this observation the Lie bracket is often referred to as the **commutator**.

A **subalgebra** of a Lie algebra $\mathfrak{g}$ is defined in the obvious way as a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[x,y] \in \mathfrak{h}$ whenever $x,y \in \mathfrak{h}$.

The simplest example of a Lie algebra is $\mathfrak{gl}(n;\mathbb{C})$. As a vector space, this Lie algebra consists of all complex $n \times n$ matrices. The product of the algebra is the usual matrix product, and the Lie bracket is defined by (2.1). The dimension of $\mathfrak{gl}(n;\mathbb{C})$ is n^2. Since the trace of square matrices satisfies the property $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ we have $\mathrm{tr}([x,y]) = 0$. Thus, the space of all traceless $n \times n$ matrices forms a Lie subalgebra of $\mathfrak{gl}(n;\mathbb{C})$, which is denoted by $\mathfrak{sl}(n;\mathbb{C})$. This subalgebra has dimension $n^2 - 1$. In the future we will sometimes write $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$ as shorthand for $\mathfrak{gl}(n;\mathbb{C})$ and $\mathfrak{sl}(n;\mathbb{C})$ respectively.

The theory of Lie algebras is very elaborate and it is impossible to give a full overview in the limited space we have. A very nice introduction on Lie algebras can be found in [32], which is based on the classic [37]. The objective here is merely to introduce some essential notions concerning Lie algebras that are needed at a later stage, either in this chapter or in one of the following. We will start this introduction with a theoretical approach. For clarification, the theory will be applied on $\mathfrak{sl}(n;\mathbb{C})$ as an example.

2.1.1 Theoretical overview

Although Lie algebras are interesting on their own, a large part of our attention will go to representations of Lie algebras (and Lie superalgebras later on). Representations, as we will see in Section 2.3, associate an operator acting on a vector space V with every element of the algebra. The role of operators is made clear in the next chapter on Wigner quantization, and in order to find operators satisfying specific properties we must have a deep understanding of representations. Studying representations becomes much cleaner for semisimple Lie algebras, and therefore these are the only ones we will consider. We will also assume that any Lie algebra we study is finite-dimensional.

Whether a Lie algebra is semisimple or not depends on the nature of its subspaces. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is an **ideal** in $\mathfrak{g}$ when $[x,h] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Note that this restriction is stronger than that of a subalgebra, where x only needed to be an element of $\mathfrak{h}$. Any Lie algebra has two trivial ideals: the full algebra itself and the zero ideal $\mathfrak{h} = \{0\}$.

Definition 2.2 *A complex Lie algebra $\mathfrak{g}$ is called **simple** if $\dim(\mathfrak{g}) \geq 2$ and if it has no ideals except for $\mathfrak{g}$ and $\{0\}$. If a complex Lie algebra is isomorphic to a direct sum of simple Lie algebras, it is called **semisimple**.*

Let us look at $\mathfrak{gl}(n;\mathbb{C})$ for an example. Since $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all matrices (i.e. for all $x,y \in \mathfrak{gl}(n;\mathbb{C})$), the subalgebra of traceless matrices $\mathfrak{sl}(n;\mathbb{C})$ is an ideal of $\mathfrak{gl}(n;\mathbb{C})$ and the latter Lie algebra is not simple. In fact, $\mathfrak{gl}(n;\mathbb{C})$ is not even semisimple. To see why this is true, notice that since $\mathfrak{sl}(n;\mathbb{C})$ has dimension n^2-1, $\mathfrak{gl}(n;\mathbb{C})$ can be written as a direct sum of two Lie algebras, one of which must be one-dimensional and therefore not simple. In contrast it is possible to show that $\mathfrak{sl}(n;\mathbb{C})$ is semisimple. We note that even though $\mathfrak{gl}(n;\mathbb{C})$ is not semisimple, it is still important to us. Many properties of semisimple Lie algebras also apply for $\mathfrak{gl}(n;\mathbb{C})$.

Having motivated the importance of semisimple Lie algebras from our interest in representations, the same can be said about Cartan subalgebras, roots and the Weyl group.

Definition 2.3 (Cartan subalgebra) *If $\mathfrak{g}$ is a complex Lie algebra, then a **Cartan subalgebra** of $\mathfrak{g}$ is a subalgebra H of $\mathfrak{g}$ with the following properties:*

- *$H_{k+1} \equiv [H, H_k]$, with $H_0 = H$ vanishes for some k.*

- *Let x be an element of $\mathfrak{g}$. If $[h, x] \in H$ for all $h \in H$, then x is an element of H.*

A Lie algebra satisfying the first of these properties is said to be **nilpotent**. The Cartan subalgebra is also **self-normalizing** due to the second property.

When $\mathfrak{g}$ is a semisimple Lie algebra over a field K of characteristic zero, it can be proved that the Cartan subalgebra is a maximal commutative subalgebra of $\mathfrak{g}$. This means that it is commutative and not contained in any larger commutative subalgebra. Moreover, a Cartan subalgebra exists for each complex semisimple Lie algebra $\mathfrak{g}$ and two Cartan subalgebras H and H' are related by

$$\phi(H) = H'$$

for some automorphism ϕ of $\mathfrak{g}$.

When a Cartan subalgebra H is chosen, we define the dual space H^* as

$$H^* = \{\alpha | \alpha : H \to \mathbb{C} \text{ is linear}\}.$$

So H^* is the space of **linear functionals** on H.

Definition 2.4 (Roots) *Consider a Lie algebra $\mathfrak{g}$ with Cartan subalgebra H. A nonzero linear functional α on H is a **root** of $\mathfrak{g}$ when there exists a nonzero element x of $\mathfrak{g}$ such that*

$$[h, x] = \alpha(h)x \tag{2.2}$$

for all $h \in H$. The set of all roots is denoted by R.

With every element x of $\mathfrak{g}$ one can associate a linear transformation ad_x which acts on $\mathfrak{g}$ as follows:

$$\mathrm{ad}_x(y) = [x, y], \qquad x, y \in \mathfrak{g}. \tag{2.3}$$

In fact, since we can associate an operator with every element of the Lie algebra $\mathfrak{g}$ in this manner, what we have here is a representation of $\mathfrak{g}$ called the **adjoint representation**. We shall postpone an accurate definition of representations to Section 2.3. We mention the adjoint representation here because the condition (2.2) says that x is an eigenvector of ad_h with eigenvalue $\alpha(h)$. The space of all eigenvectors of ad_h with eigenvalue $\alpha(h)$ is called the **root space** $\mathfrak{g}_\alpha$. An element of the root space is called a **root vector**. We state an interesting result about roots and root spaces without proof.

Theorem 2.5 *Consider a complex semisimple Lie algebra $\mathfrak{g}$. Then we have*

1. *If α is a root, then so is $-\alpha$. Moreover, $-\alpha$ is the only multiple of α that is a root.*
2. *If α is a root, then the root space $\mathfrak{g}_\alpha$ is one-dimensional.*

Roots are very important as they help to classify all semisimple Lie algebras. It is not our ambition to present this classification here, but roots are also important because they have an impact on representation theory. For example, the Weyl character formula (2.15) is written in terms of the positive roots of the Lie algebra.

Definition 2.6 *For any set of roots Δ one can choose a set of **positive roots** Δ^+ as follows:*

- *For each root $\alpha \in R$ exactly one of the roots α or $-\alpha$ is in Δ^+.*
- *For any two distinct roots $\alpha, \beta \in \Delta^+$, we have that if $\alpha + \beta$ is a root, it must be inside Δ^+.*

*If an element of Δ^+ cannot be written as a sum of two elements of Δ^+, it is called a **simple root**.*

Note that, although some authors [12, 32] give algorithms to find a set of positive roots, the choice is completely arbitrary. When the positive roots are chosen, the simple roots are uniquely determined.

Now let us choose an **invariant inner product** $\langle .,. \rangle$ on $\mathfrak{g}$, that is

$$\langle [x,y],z \rangle = \langle x,[y,z] \rangle$$

for all x, y, z in $\mathfrak{g}$. For any root $\alpha \in H^*$ there then exists a unique element h_α of the Cartan subalgebra H such that

$$\alpha(h) = \langle h_\alpha, h \rangle \tag{2.4}$$

for all h in H. This means we can write

$$[h,x] = \langle h_\alpha, h \rangle x$$

for a root vector x corresponding to the root α. In fact, this correspondence allows us to use the inner product to define an inner product on the dual Cartan subalgebra H^* as follows:

$$\langle \alpha, \beta \rangle = \langle h_\alpha, h_\beta \rangle \tag{2.5}$$

for all $\alpha, \beta \in H^*$. The inner product of the root space is a powerful tool. For example, it is now possible to obtain a geometrical picture of the roots. We can also introduce a finite group that leaves the set of roots R and the inner product $\langle .,. \rangle$ invariant. This group is called the **Weyl group** W and it is generated by the elements w_α whose action on H^* is defined by

$$w_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \tag{2.6}$$

for all $\beta \in H^*$. This means that the element w_α acts on H^* as a reflection about the hyperplane perpendicular to α. Indeed, we have $w_\alpha(\alpha) = -\alpha$ and $w_\alpha(\beta) = \beta$ if $\langle \alpha, \beta \rangle = 0$.

The theoretical picture is starting to look rather abstract, so it is useful to consider an accessible example. All of the previously introduced notions will be worked out for $\mathfrak{sl}(n;\mathbb{C})$, the space of traceless complex matrices.

2.1.2 An example involving traceless matrices

The semisimple Lie algebra $L = \mathfrak{sl}(n;\mathbb{C})$ is a good example because the involved matrices are traceless, a condition that is not too restrictive and easy to handle. First, we choose a basis for the space of all matrices consisting of the n^2 matrices $e_{ij}, (i, j = 1, \ldots, n)$. The elements of these matrices are all zero, except for a 1 on position (i, j). The commutation relations of these elements are easily computed and given by

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \tag{2.7}$$

The elements e_{ii} are not traceless and therefore they do not belong to L. However, the elements $h_i = e_{ii} - 1/n$, with $i = 1, \ldots, n$, have trace zero. We note that the h_i are not linearly independent since we have

$$h_1 + \cdots + h_n = 0.$$

The space spanned by the matrices h_i, denoted by

$$H = \text{span}\{h_1, \ldots, h_n\}$$

is the space of all complex diagonal matrices with trace zero. It is a maximal commutative subalgebra of L with dimension $n-1$, and due to the fact that $\mathfrak{sl}(n;\mathbb{C})$ is semisimple this is precisely the Cartan subalgebra of L. The total algebra L has

dimension n^2-1 because there are n^2-n basis elements e_{ij} outside the Cartan subalgebra.

The roots of L are elements of the dual space H^*, the space of linear functionals on the Cartan subalgebra H. The space H^* is generated by the elements ϵ_i whose action on the space of diagonal matrices is given by

$$\epsilon_i(D) = D_{ii}, \qquad (i = 1, \ldots, n)$$

for any diagonal matrix D. In other words, ϵ_i maps a diagonal matrix onto its ith diagonal entry. Restricting the action of the ϵ_i to H implies that they are related by

$$\epsilon_1 + \cdots + \epsilon_n = 0.$$

So the dual space H^* also has dimension $n-1$. The roots are those elements of H^* for which a root vector $x \in L$ exists such that the condition (2.2) is satisfied. With the help of equation (2.7) we find

$$[h_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} = (\epsilon_j - \epsilon_k)(h_i)e_{jk},$$

where we have used

$$\epsilon_i(h_j) = \delta_{ij} - \frac{1}{n}.$$

So the roots are $\alpha_{jk} = \epsilon_j - \epsilon_k$ with corresponding root vectors e_{jk}, with $j,k = 1,\ldots,n$ and $j \neq k$. Note that, in agreement with Theorem 2.5, $\alpha_{jk} = -\alpha_{kj}$ but no other multiple of α_{jk} is a root.

We choose a set of positive roots as the one containing the elements α_{jk} with $j < k$. The $n-1$ simple roots are then given by

$$\begin{aligned}
\alpha_{12} &= \epsilon_1 - \epsilon_2, \\
\alpha_{23} &= \epsilon_2 - \epsilon_3, \\
&\vdots \\
\alpha_{n-1,n} &= \epsilon_{n-1} - \epsilon_n.
\end{aligned}$$

The Hilbert-Schmidt inner product

$$\langle x, y \rangle = \mathrm{tr}(x^* y),$$

where $x, y \in L$ and x^* is the conjugate transpose of the matrix x, provides an invariant inner product on L and therefore also on H. An inner product on H^*

is constructed from the Hilbert-Schmidt inner product using the one-to-one correspondence (2.4) of a root with an element of H. For $\epsilon_i \in H^*$ for instance, we have $h_{\epsilon_i} = h_i$ because

$$\langle h_{\epsilon_i}, h_j \rangle = \langle h_i, h_j \rangle = \delta_{ij} - \frac{1}{n} = \epsilon_i(h_j).$$

An inner product of two elements in H^* is then defined by equation (2.5). We have for example

$$\langle \epsilon_i, \epsilon_j \rangle = \langle h_{\epsilon_i}, h_{\epsilon_j} \rangle = \langle h_i, h_j \rangle = \delta_{ij} - \frac{1}{n}.$$

With this inner product, the roots have length $\sqrt{2}$ as we see from

$$\langle \alpha_{jk}, \alpha_{jk} \rangle = \langle \epsilon_j - \epsilon_k, \epsilon_j - \epsilon_k \rangle = 2$$

because $j \neq k$. Also, the angle between two different roots α_{jk} and α_{lm} is calculated to be $60°$, $90°$ or $120°$ depending on the value of the inner product of both roots. The geometrical interpretation of the roots is depicted in Figure 2.1 for the case $\mathfrak{sl}(3;\mathbb{C})$.

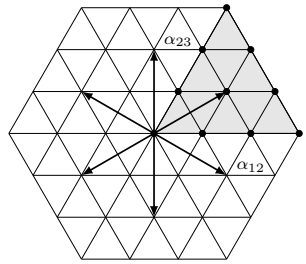

Figure 2.1: The roots of $\mathfrak{sl}(3;\mathbb{C})$. The simple roots α_{12} and α_{23} stand in an angle of $120°$. The gray area represents the dominant Weyl chamber and the dots are the dominant integral elements.

The two simple roots α_{12} and α_{23} generate a real vector space E for which some coordinates are drawn by the triangular grid. The gray area embodies the following subset of E:

$$\{\lambda \in E | \langle \lambda, \alpha_{jk} \rangle \geq 0 \text{ for all } \alpha_{jk} \in \Delta^+\},$$

where Δ^+ is the set of positive roots α_{12}, α_{23} and α_{13}. This subset of E is called the **dominant Weyl chamber** relative to the given set of positive roots. For an

element of E written as

$$\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3$$

one can check that the inner product with the simple roots α_{12} and α_{23} equals

$$\langle \lambda, \alpha_{12} \rangle = \lambda_1 - \lambda_2, \qquad \langle \lambda, \alpha_{23} \rangle = \lambda_2 - \lambda_3.$$

Saying that λ is in the dominant Weyl chamber of $\mathfrak{sl}(3;\mathbb{C})$ is equivalent to the condition

$$\lambda_1 \geq \lambda_2 \geq \lambda_3.$$

When λ_1, λ_2 and λ_3 are non-negative integers, this means that $(\lambda_1, \lambda_2, \lambda_3)$ must be a partition. Such elements of the dominant Weyl chamber are called the **dominant integral elements**. One can show in general for $\mathfrak{sl}(n;\mathbb{C})$ that the dominant integral elements are of the form

$$\lambda = \sum_{j=1}^{n} \lambda_j \epsilon_j$$

where $(\lambda_1, \ldots, \lambda_n)$ is a partition.

The Weyl group of $L = \mathfrak{sl}(n, \mathbb{C})$ is generated by the reflections $w_{\alpha_{jk}}$ about the hyperplane perpendicular to α_{jk}, where α_{jk} is a root in R. Since $\alpha_{jk} = \epsilon_j - \epsilon_k$, the elements $w_{\alpha_{jk}}$ are just transpositions of indices j and k, with $j, k \in \{1, \ldots, n\}$. The group generated by such transpositions is the symmetric group S_n, and it acts on H by permuting the diagonal entries.

In the next chapters we will encounter a variety of Lie algebras, like $\mathfrak{gl}(n)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$. It would be overzealous to describe all these Lie algebras in detail here. Moreover, this would generate an abundance of superfluous information. Hence, we choose to introduce these specific Lie algebras at the point where they are needed and we will confine ourselves to the strictly necessary information about them.

2.2 Lie superalgebras

In this section Lie superalgebras will be introduced by the same principles as the previous section about Lie algebras. We will first give some theoretical definitions and illustrate the theory with an example. An extensive introduction to Lie superalgebras can be found in [91].

Many notions of Lie algebras, such as roots or the Cartan subalgebra, have a generalization in the theory of Lie superalgebras. Since we have already discussed

them in detail in the previous section, and since our main focus will be on representations of Lie superalgebras, we will be rather curt on the theoretical part here. The lack of depth in this section will be compensated by more elaborate explanations in the next.

Definition 2.7 *Let V be a vector space over a field K, and let Γ be any ring. A Γ-**grading** of the vector space V is a family $(V_\gamma)_{\gamma\in\Gamma}$ of subspaces of V such that*

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma.$$

The vector space V is said to be Γ-graded if it is equipped with a Γ-grading.

Any element of V_γ, with $\gamma \in \Gamma$ is called **homogeneous** of degree γ. An algebra A is said to be Γ-graded if its underlying vector space is Γ-graded and if

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta}$$

for all $\alpha, \beta \in \Gamma$. In what follows, we will only consider the case when $\Gamma = \mathbb{Z}_2$ and we denote its two elements by $\bar{0}$ and $\bar{1}$.

Definition 2.8 (Superalgebra) *A $\mathbb{Z}_2$-graded algebra is called a **superalgebra**.*

For a superalgebra A, the elements of $A_{\bar{0}}$ and $A_{\bar{1}}$ are called the **even** and **odd elements** respectively, and $A_{\bar{0}}$ and $A_{\bar{1}}$ are the even and odd subalgebra of A. An even element $x \in A_{\bar{0}}$ is said to have **degree** 0, denoted by $\deg(x) = 0$, while an odd element has degree 1.

Definition 2.9 (Lie superalgebra) *A $\mathbb{Z}_2$-graded algebra $\mathfrak{g}$ over the field K of characteristic 0, together with a bilinear operation $[\![\cdot,\cdot]\!] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, is called a **Lie superalgebra** if*

- $[\![x,y]\!] \in \mathfrak{g}_{\alpha+\beta}$,
- $[\![x,y]\!] = -(-1)^{\deg(x)\deg(y)}[\![y,x]\!]$,
- $(-1)^{\deg(x)\deg(z)}[\![x,[\![y,z]\!]]\!] + (-1)^{\deg(x)\deg(y)}[\![y,[\![z,x]\!]]\!] + (-1)^{\deg(y)\deg(z)}[\![z,[\![x,y]\!]]\!] = 0$

for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $z \in \mathfrak{g}_\gamma$, with $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

The operator $[\![\cdot,\cdot]\!]$ is called the **Lie superbracket** or supercommutator. The third property of the Lie superbracket is called the super Jacobi identity, in analogy with the comparable property of the Lie bracket. Given any associative superalgebra A, one can make A into a Lie superalgebra by defining the Lie superbracket for the homogeneous elements of A as

$$[\![x,y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx, \tag{2.8}$$

and extending this definition to all elements of A by linearity. In this definition we have written the product of two elements $x, y \in A$ more briefly as $x \cdot y \equiv xy$. The similarity with the Lie algebra case is apparent here, and for the familiar reasons the superbracket is often referred to as the **supercommutator**.

It is easy to see that the even subalgebra $\mathfrak{g}_{\bar{0}}$ of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie algebra. The Lie superalgebra $\mathfrak{g}$ is not a Lie algebra unless $\mathfrak{g}_{\bar{1}} = 0$.

The easiest example of a Lie superalgebra is $\mathfrak{gl}(m|n)$, where m and n are positive integers. It is the algebra of all complex square matrices of order $m+n$. In other words

$$\mathfrak{gl}(m|n) = \left\{ x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_{m\times m}, B \in M_{m\times n}, C \in M_{n\times m}, D \in M_{n\times n} \right\},$$

where $M_{p\times q}$ denotes the space of complex $p \times q$ matrices. The even and odd subalgebras are given by

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A \in M_{m\times m}, D \in M_{n\times n} \right\}$$

and

$$\mathfrak{gl}(m|n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} | B \in M_{m\times n}, C \in M_{n\times m} \right\}.$$

The superbracket is defined by equation (2.8) for all $x, y \in \mathfrak{gl}(m|n)$. The juxtaposition in the right hand side stands for ordinary matrix multiplication. It can be verified that all three properties of a Lie superbracket are satisfied for this definition. A basis for $\mathfrak{gl}(m|n)$ is given by the elements e_{jk} with $j,k = 0,1,\ldots,m+n-1$, which are all zero matrices except for a 1 on position (j,k). For this basis the Lie superbracket becomes

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})} \delta_{il} e_{kj}. \tag{2.9}$$

Note that when x and y are odd elements, the superbracket becomes

$$[\![x,y]\!] = xy + yx = \{x,y\},$$

where we have introduced the notation $\{x,y\} = xy + yx$ for the **anticommutator** of two elements. When either x or y is even, the superbracket takes the form of a commutator, indicating that $\mathfrak{gl}(m|n)_{\bar{0}}$ is indeed a Lie algebra. This even subalgebra of $\mathfrak{gl}(m|n)$ can be written as $\mathfrak{gl}(m;\mathbb{C}) \oplus \mathfrak{gl}(n;\mathbb{C})$.

In analogy with the Lie algebra case, there is a subalgebra $\mathfrak{sl}(m|n)$ of $\mathfrak{gl}(m|n)$ given by

$$\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) | \mathrm{str}(x) = 0\},$$

where $\mathrm{str}(x)$ denotes the **supertrace** of a matrix in $\mathfrak{gl}(m|n)$, which is defined by

$$\mathrm{str}(x) = \mathrm{tr}(A) - \mathrm{tr}(D).$$

We have that

$$\mathrm{str}([\![x,y]\!]) = 0$$

for all $x,y \in \mathfrak{gl}(m|n)$, so the algebra $\mathfrak{sl}(m|n)$ of all matrices with supertrace zero is indeed a subalgebra of $\mathfrak{gl}(m|n)$.

For our purposes two Lie superalgebras are of paramount importance. We already had the chance to get acquainted with the first of them, namely $\mathfrak{gl}(1|n)$. The second one is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$. The orthosymplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ forms a subalgebra of $\mathfrak{sl}(m|2n)$ and it is spanned by the elements $x \in \mathfrak{sl}(m|2n)$ such that

$$Jx + (-1)^{\deg(x)} x^T J = 0,$$

where x^T is the transpose of x and J is the matrix

$$J = \left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & 0 & I_n \\ 0 & -I_n & 0 \end{array}\right).$$

Herein, I_n denotes the identity matrix of order n. One can show that the general form of a matrix of $\mathfrak{osp}(m|2n)$ must be

$$\left(\begin{array}{c|cc} a & b & c \\ \hline c^T & d & e \\ -b^T & f & -d^T \end{array}\right), \tag{2.10}$$

where a, b, c, d, e and f are matrices with appropriate dimensions. The matrices b, c and d are random and the limitations for the other three matrices are that e and f must be symmetric matrices and $a + a^T = 0$.

As we have already mentioned, we are chiefly interested in the $\mathfrak{osp}(1|2n)$ Lie superalgebra. In this case the matrices b and c in equation (2.10) are row vectors of dimension n, and $a = 0$. Let us agree on row and column indices running from 0 to $2n$ instead of 1 to $2n+1$. We also define the matrices e_{ij} as the matrix with zeros everywhere except for a 1 on position (i, j). Note that the matrices e_{ij} are not inside $\mathfrak{osp}(1|2n)$ because they are not of the form (2.10). On the other hand we have that

$$b_j^+ = \sqrt{2}\,(e_{0,n+j} + e_{j,0}), \qquad b_j^- = \sqrt{2}\,(e_{0,j} - e_{n+j,0}) \qquad (j = 1, \ldots, n)$$

are elements of the odd subspace of $\mathfrak{osp}(1|2n)$. These are the matrices of the form (2.10) with $d, e, f = 0$. The even elements of $\mathfrak{osp}(1|2n)$ have $b, c = 0$ in (2.10) and are obtained by taking superbrackets of the odd elements, namely $[\![b_j^+, b_k^+]\!]$, $[\![b_j^+, b_k^-]\!]$ and $[\![b_j^-, b_k^-]\!]$. One can then calculate the result of the superbracket of an even and an odd element, which can concisely be written as

$$[\![[\![b_j^\xi, b_k^\eta]\!], b_l^\epsilon]\!] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi,$$

where j, k and l are elements from the set $\{1, 2, \ldots, n\}$ and $\eta, \xi, \epsilon \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $(\epsilon - \xi)$ and $(\epsilon - \eta)$). Since in reality a superbracket is either a commutator or an anticommutator, we have obtained an important result of Ganchev and Palev [23].

Theorem 2.10 (Ganchev and Palev) *The $2n$ odd elements $b_j^\pm$ subject to the following relations:*

$$\left[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon\right] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi \qquad (2.11)$$

generate the Lie superalgebra $\mathfrak{osp}(1|2n)$.

Thus, $\mathfrak{osp}(1|2n)$ can be interpreted more generally as a Lie superalgebra defined by generators and relations, rather than a superalgebra of matrices.

The relations (2.11) are called the **defining triple relations of** $\mathfrak{osp}(1|2n)$. They are often referred to as the paraboson relations in literature. We will encounter the

defining relations of $\mathfrak{osp}(1|2n)$ (and other Lie superalgebras) on many different occasions in the context of Wigner quantization, which is described in the next chapter. As we pointed out in the introduction to the present chapter, our next goal will be to obtain operators from the elements of the given Lie superalgebra. This is when Lie superalgebra representations come into the picture.

2.3 Some notes on representation theory

Without explicitly saying so, we have already given some example of representations. All Lie algebras and Lie superalgebras have been introduced as matrix algebras. However, as the $\mathfrak{osp}(1|2n)$ example shows, the elements of a Lie algebra or Lie superalgebra are not necessarily matrices. The matrices are nothing more than *some* realization of the elements of the algebra. This does not exclude other possible realizations. What we have seen is the **standard representation** of the considered algebras, but many more representations exist.

2.3.1 Lie algebra representations

Our introduction on representation theory will be from the Lie algebra point of view. Here, we will talk about homomorphisms, weights, the adjoint representation and highest weight representations. Most concepts are easily generalized to Lie superalgebras, so our focus in the Lie superalgebra case will be on presenting an example of a representation. More precisely, we will classify all irreducible $*$-representations of $\mathfrak{osp}(1|2)$ in Section 2.4. This is the first novel result in this thesis.

Up till now we have written all considered Lie algebras as a set of matrices, where the Lie bracket was given by the commutator. This is precisely the definition of what a Lie algebra representation is.

Definition 2.11 (Lie algebra representation) *A **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ with the property*

$$\phi([x,y]) = [\phi(x),\phi(y)],$$

*for all $x,y \in \mathfrak{g}$. A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the Lie algebra of all linear transformations of a vector space V.*

The vector space V is called the **representation space**. It has the structure of a $\mathfrak{g}$-**module** by setting $x \cdot v = \pi(x)v$, with $x \in \mathfrak{g}$ and $v \in V$. Our preference will go to the former terminology in the future. In essence, a Lie algebra representation associates an operator acting on a vector space with every element of the Lie algebra. Because of this we will think of a representation as a linear action on the representation space V. The dimension of V is called the **dimension** of the representation.

In the future we will mostly be studying unitary irreducible representations of Lie algebras and Lie superalgebras.

Definition 2.12 *Let π be a representation of a Lie algebra $\mathfrak{g}$ acting on a vector space V. A subspace W of V is called **invariant** if $\pi(x)w \in W$ for all $w \in W$ and all $x \in \mathfrak{g}$. The invariant subspaces $W = V$ and $W = \{0\}$ are called trivial. A representation without nontrivial invariant subspaces is said to be **irreducible**.*

Irreducible representations can be viewed as the building blocks of representations, since many reducible representations arise as direct sums of irreducible representations. For the definition of **unitary representations** we need the notion of a $*$-**structure** on a complex algebra $\mathfrak{g}$. This is an anti-linear anti-multiplicative involution $x \mapsto x^*$. So for $x, y \in \mathfrak{g}$ and $a, b \in \mathbb{C}$ we have that $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$ and $(xy)^* = y^*x^*$, where $\bar{a}$ denotes complex conjugation. The algebra $\mathfrak{g}$ equipped with a $*$-structure is said to be a $*$-**algebra**.

Definition 2.13 ($*$-representations) *Let $\mathfrak{g}$ be a $*$-algebra, let $\mathcal{H}$ be a Hilbert space and let V be a dense subspace of $\mathcal{H}$. A $*$-**representation** of $\mathfrak{g}$ on V is a map π from $\mathfrak{g}$ into the linear operators on V such that π is a representation of $\mathfrak{g}$ regarded as a Lie algebra, together with the condition*

$$\langle \pi(x)v, w \rangle = \langle v, \pi(x^*)w \rangle \tag{2.12}$$

for all $x \in \mathfrak{g}$ and $v, w \in V$. The subspace V is the representation space of the $*$-algebra.

Such $*$-representations are sometimes called **unitarizable representations**, to which we will often refer as unitary representations. The latter notion has another meaning in standard literature, so it is important to note that we use it in a different sense in this thesis.

Given a $*$-structure for an algebra $\mathfrak{g}$, it is possible to associate a **real form** of $\mathfrak{g}$ with this $*$-structure by putting

$$\mathfrak{g}_{\mathbb{R}} = \{x \in \mathfrak{g} \mid x^* + x = 0\}.$$

Let us clarify these notions by considering $\mathfrak{gl}(n;\mathbb{C})$ as an example. We will show that the real form of $\mathfrak{gl}(n;\mathbb{C})$ under the $*$-condition

$$e_{jk}^* = e_{kj}, \qquad (j,k = 1,\ldots,n) \tag{2.13}$$

is the Lie algebra $\mathfrak{u}(n)$. This Lie algebra consists of all complex $n \times n$ matrices x such that $\bar{x}_{jk} = -x_{kj}$, where x_{jk} represents the element at position (j,k) of the matrix x. Any element x of the real form of $\mathfrak{gl}(n;\mathbb{C})$ under the $*$-condition (2.13) must satisfy

$$x^* = \left(\sum_{j,k} c_{jk}\, e_{jk}\right)^* = \sum_{j,k} \bar{c}_{jk}\, e_{kj} = -x,$$

where the second equality uses the anti-linearity of the $*$-structure. This shows that $\mathfrak{u}(n)$ is the real form of $\mathfrak{gl}(n;\mathbb{C})$ under the given $*$-condition. The Lie algebra $\mathfrak{u}(n)$ will return on several occasions in the future.

The Lie algebras we have considered before were all Lie algebras of matrices. In other words, they were subalgebras of $\mathfrak{gl}(n;\mathbb{C})$. As the following theorem proves, there is no loss of generality by considering only such matrix Lie algebras.

Theorem 2.14 (Ado) *Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of* $\mathfrak{gl}(n;\mathbb{C})$.

So we can assume for every finite-dimensional complex Lie algebra $\mathfrak{g}$ that $\mathfrak{g} \subset \mathfrak{gl}(n;\mathbb{C})$. The linear map

$$\begin{aligned} \pi : \quad & \mathfrak{g} \to \mathfrak{gl}(n;\mathbb{C}) \\ & x \mapsto x \end{aligned}$$

is a representation of $\mathfrak{g}$ called the **standard representation**.

The simplest example of a representation, however, is the **trivial representation**. It is defined as the homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(1;\mathbb{C})$, with

$$\pi(x) = 0$$

for all $x \in \mathfrak{g}$. This representation has no nontrivial subspaces and is therefore irreducible.

Much more interesting is the **adjoint representation** of a Lie algebra, which we have briefly mentioned already in the context of equation (2.3). The adjoint representation maps an element $x \in \mathfrak{g}$ to $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$, which acts on $\mathfrak{g}$ by

$$\mathrm{ad}_x(y) = [x, y], \qquad x, y \in \mathfrak{g}.$$

To see why this is a representation, we need to show that ad_x is a Lie algebra homomorphism. This follows from the Jacobi identity for the Lie bracket as follows:

$$\begin{aligned} \mathrm{ad}_{[x,y]}(z) &= [[x, y], z] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= [\mathrm{ad}_x, \mathrm{ad}_y](z), \end{aligned}$$

with $x, y, z \in \mathfrak{g}$.

A typical problem in representation theory is trying to classify the representations of a certain Lie algebra or Lie superalgebra. In Section 2.4, we will perform such a classification for the Lie superalgebra $\mathfrak{osp}(1|2)$. Crucial in classifying these representations in particular and representations in general is the concept of weights.

Definition 2.15 *Consider a Lie algebra $\mathfrak{g}$ with Cartan subalgebra H, and let π be a representation of $\mathfrak{g}$ acting on a vector space V. A linear functional $\lambda \in H^*$ is called a **weight** for π if there exists a nonzero vector $v \in V$ such that*

$$\pi(h)v = \lambda(h)v \tag{2.14}$$

*for all $h \in H$. The vector v is called a **weight vector** with weight λ. All weight vectors satisfying (2.14) form the **weight space** with weight λ, of which the dimension is called the **multiplicity** of the weight.*

By this definition it is clear that the roots are precisely the nonzero weights of the adjoint representation. It is possible to show that two equivalent representations have the same weights and multiplicities, thus representations can be classified by their weights. The key to finding all weights and weight vectors lies in the following lemma.

Lemma 2.16 *Consider a representation π of $\mathfrak{g}$ with representation space V. Suppose that $v \in V$ is a weight vector with weight λ and suppose that x is a root vector for the root α. Then, for all h in the Cartan subalgebra H, we have*

$$\pi(h)\pi(x)v = \big(\lambda(h) + \alpha(h)\big)\,\pi(x)v.$$

In other words, $\pi(x)v$ is either zero or a new weight vector with weight $\lambda + \alpha$.

Proof. The proof of this lemma is straightforward and follows from the definition of a root, namely $[h, x] = \alpha(h)x$. We have

$$\begin{aligned}\pi(h)\pi(x)v &= \big(\pi(x)\pi(h) + \pi([h,x])\big)\,v \\ &= \big(\pi(x)\pi(h) + \alpha(h)\pi(x)\big)\,v \\ &= \big(\lambda(h) + \alpha(h)\big)\,\pi(x)v,\end{aligned}$$

where the first equality is based on $[\pi(h), \pi(x)] = \pi[h, x]$. □

It is possible to define a partial ordering on the set of weights relative to a chosen set of positive roots. Using such an ordering we can define a **highest weight** or a **lowest weight**. We will only use these concepts in the classification of $\mathfrak{osp}(1|2)$ representations, where it is very intuitive. Therefore, we will not explain explicitly how a partial ordering on the set of weights can be defined. More details can be found in [32, 37].

The importance of the notion of a highest weight lies in the following theorem:

Theorem 2.17 (Theorem of the highest weight) *Every finite-dimensional irreducible representation of a semisimple Lie algebra has a highest weight, and this highest weight is necessarily a dominant integral element. Moreover, two irreducible representations with the same highest weight are equivalent.*

Sometimes, we will specify a highest weight λ of a finite-dimensional irreducible representation of a simple Lie-algebra by its **Dynkin labels**. These are defined by

$$a_i = 2\frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle},$$

where α_i is a simple root and i runs from 1 to the **rank** of the Lie-algebra, which is defined as the dimension of the root space. It is possible to show that the Dynkin

labels are nonnegative integers. For the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ this is easy to see. Indeed, in this case a highest weight is a partition λ and using the results of Section 2.1.2 we find

$$a_i = \lambda_i - \lambda_{i+1},$$

with $i = 1,\dots,n-1$. Thus the Dynkin labels are nonnegative integers in this case.

Since representations can be classified by their weights and multiplicities, a function keeping track of this information should be very useful. Such a function is provided by the character of a representation.

2.3.2 Character theory

Most references introduce characters in the context of group representations as a function that maps a group element to the trace of its matrix representation. However, we will mostly be interested in characters of algebra or superalgebra representations.

Definition 2.18 (Character of a representation) *Consider a representation π of a Lie algebra or a Lie superalgebra with representation space V. The **character** of the representation V is defined by*

$$\mathrm{char}V = \sum_{\lambda} m_\lambda\, e^\lambda,$$

where the summation runs over all weights λ and m_λ denotes the multiplicity of λ. Thus, the character is a generating function for the multiplicities of the weights of a representation.

In the definition of a character of a representation, e^λ is a formal exponent. In particular for $\mathfrak{gl}(n;\mathbb{C})$ or $\mathfrak{sl}(n;\mathbb{C})$, when the weight λ is decomposed in a basis ϵ_i of the dual Cartan subalgebra H^*, the formal exponent can be written as

$$e^\lambda = e^{\sum_i \lambda_i \epsilon_i}.$$

In practice we will write $x_i = e^{\epsilon_i}$ for all i, so that the character of a representation becomes

$$\mathrm{char}V = \sum_{\lambda} m_\lambda\, x^\lambda,$$

where the notation $x^\lambda = \prod_i x_i^{\lambda_i}$ has been utilized.

A powerful result involving characters is the Weyl character formula, which describes the character of an irreducible representation of a semisimple Lie-algebra in terms of its highest weight. The Weyl character formula uses the elements of the Weyl group. It is useful in this context to define the **length of the Weyl group element** $\ell(w)$ as the minimum number of reflections w_α with respect to simple roots α (see equation (2.6)) such that the Weyl group element w equals the product of those reflections. With this definition we have:

Theorem 2.19 (Weyl character formula) *Let $\mathfrak{g}$ be a semisimple Lie algebra and $V(\lambda)$ an irreducible representation of $\mathfrak{g}$ with highest weight λ. Then the character of $V(\lambda)$ is given by*

$$\mathrm{char} V(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}, \tag{2.15}$$

*where W is the Weyl group, $\ell(w)$ is the length of the Weyl group element w and the **Weyl tool** ρ denotes half of the sum of the positive roots of $\mathfrak{g}$.*

When the highest weight λ of a representation is zero, we obtain the trivial representation which has character 1. As a result the denominator of the Weyl character formula, called the **Weyl denominator**, can be rewritten in a more convenient way.

Theorem 2.20 (Weyl denominator formula) *The Weyl denominator can be rewritten as*

$$e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)},$$

where Δ^+ denotes the set of positive roots.

Let us interpret the Weyl character formula for $\mathfrak{sl}(n)$ representations using the results of Section 2.1.2. We have seen that the Weyl group is the symmetric group S_n in this case, so the elements $w \in S_n$ are permutations. We can write a permutation as a product of transpositions, and the minimum amount of transpositions needed for this is the length of our permutation. This means that $(-1)^{\ell(w)}$ is what is called the **sign of the permutation**, denoted by $\epsilon(w)$.

The highest weight of a finite-dimensional irreducible $\mathfrak{sl}(n)$ representation must be a partition λ, which follows from the theorem of the highest weight and the fact

that a dominant integral element for $\mathfrak{sl}(n)$ must be a partition. We write the highest weight λ as $(\lambda_1, \ldots, \lambda_n)$ in which the λ_i are the components of the elements ϵ_i that generate the dual space H^* of the Cartan subalgebra of $\mathfrak{sl}(n)$. By this convention, the Weyl tool can be written as

$$\rho = \frac{1}{2} \sum_{j<k} \alpha_{jk} = \frac{1}{2}(n-1, n-3, \ldots, -n+1).$$

We will modify the formal exponent in the Weyl character formula (2.15) by setting $x_i = e^{\epsilon_i}$ $(i = 1, \ldots, n)$ again. Using the Weyl denominator formula, the Weyl character formula for a $\mathfrak{sl}(n)$ representation with highest weight λ then becomes

$$\mathrm{char}V(\lambda) = \frac{\sum_{w \in S_n} \epsilon(w)\, x^{w(\lambda+\rho)}}{\sum_{w \in S_n} \epsilon(w)\, x^{w(\rho)}}.$$

Multiplying numerator and denominator by $x^{(n-1,\ldots,n-1)/2}$ this can be rewritten as

$$\mathrm{char}V(\lambda) = \frac{\sum_{w \in S_n} \epsilon(w)\, x^{w(\lambda+\delta)}}{\sum_{w \in S_n} \epsilon(w)\, x^{w(\delta)}},$$

where $\delta = (n-1, n-2, \ldots, 0)$. By the Leibniz formula for a determinant, we see that numerator and denominator of this character are the determinants $a_{\lambda+\delta}(x)$ and $a_\delta(x)$ respectively (see equation (1.3)). Thus, the character of a $\mathfrak{sl}(n)$ representation where the highest weight is given by a partition λ is the Schur function

$$\mathrm{char}V(\lambda) = s_\lambda(x).$$

This is a crucial observation and it will frequently be used in Chapter 5.

At this point we have introduced most of the basic concepts about Lie algebras, Lie superalgebras and their representations. In order to grasp the subject even more, an example of a representation would be more than welcome. In the next section we will build up a representation space from scratch. We will do this for the Lie superalgebra $\mathfrak{osp}(1|2)$, which will play a big role in the remaining of this thesis.

2.4 Classification of $\mathfrak{osp}(1|2)$ representations

For the first time in this thesis we will present some original results involving representations of the Lie superalgebra $\mathfrak{osp}(1|2)$. Following Theorem 2.10, this Lie

superalgebra is generated by two elements b^+ and b^- subject to the relations

$$\left[\{b^-, b^+\}, b^\pm\right] = \pm 2b^\pm. \tag{2.16}$$

The omnipresence of $\mathfrak{osp}(1|2)$ in the context of Wigner quantization forces us to investigate the representations of this Lie superalgebra. In this section we will classify all irreducible $*$-representations of $\mathfrak{osp}(1|2)$, the results of which have been published in [86]. Although we are aware of the classification by Hughes in [36], we think it is possible to achieve his results in a more accessible way, based on [31]. In addition we will be able to identify some equivalent representation classes. Before giving the details of our classification, we provide the readers with the necessary definitions and a general outline of how we will construct all irreducible $*$-representations of $\mathfrak{osp}(1|2)$.

2.4.1 The Lie superalgebra $\mathfrak{osp}(1|2)$ and $*$-representations

We will be dealing with the Lie superalgebra $\mathfrak{osp}(1|2)$, generated by two operators b^+ and b^- that are subject to the relations (2.16). The generating operators b^+ and b^- are the odd elements of the algebra, while the even elements are

$$h = \frac{1}{2}\{b^-, b^+\}, \qquad e = \frac{1}{4}\{b^+, b^+\}, \qquad f = -\frac{1}{4}\{b^-, b^-\}. \tag{2.17}$$

Among others, the following commutation relations can now be computed from the defining relations (2.16):

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

When we encounter $\mathfrak{osp}(1|2)$ in future chapters, a dagger operation $x \mapsto x^\dagger$ will usually be defined on the operators b^+ and b^-. This dagger operation relates b^+ and b^- by

$$(b^\pm)^\dagger = b^\mp.$$

So we can define a $*$-structure on $\mathfrak{osp}(1|2)$ by this dagger operation and we have $\left(b^\pm\right)^* = b^\mp$. As a consequence, the relations $h^* = h$, $e^* = -f$ and $f^* = -e$ hold. The even operators h, e and f, together with this $*$-structure, form the Lie algebra $\mathfrak{su}(1,1)$. Both $\mathfrak{su}(1,1)$ and $\mathfrak{osp}(1|2)$ possess a **Casimir operator**, denoted by Ω and C respectively:

$$\Omega = -\frac{1}{4}(4fe + h^2 + 2h), \qquad C = -4\Omega + \frac{1}{2}(b^-b^+ - b^+b^-).$$

The Casimir elements generate the center of the respective algebras. So Ω commutes with every element of $\mathfrak{su}(1,1)$ and similary for C. Moreover, we have $\Omega^* = \Omega$ and $C^* = C$.

We will construct all possible irreducible $*$-representations of $\mathfrak{osp}(1|2)$ starting from one assumption: h has at least one eigenvector in the representation space with eigenvalue 2μ, or

$$\pi(h)v_0 = 2\mu\, v_0. \tag{2.18}$$

Starting from this one vector, we will build other basis vectors of the representation space V by letting operators of $\mathfrak{osp}(1|2)$ act on it. After having determined the actions of all $\mathfrak{osp}(1|2)$ operators on all basis vectors of V, we will extend the representation π to a $*$-representation. This is done by defining a sesquilinear form $\langle .,. \rangle : V \to \mathbb{C}$, which is to be an inner product that satisfies (2.12).

The stipulation that $\langle .,. \rangle$ should be an inner product will be crucial in limiting the possible representation spaces. However, we will postpone the details of this discussion to the point where we have enough arguments for this end. So let us start with the actual construction of the representation space V.

2.4.2 Construction of the representation space

In this section, the $*$-structure is of no importance. We will construct an ordinary $\mathfrak{osp}(1|2)$ representation space that we will extend to a $*$-representation in the next section.

The embedding of $\mathfrak{su}(1,1)$ in $\mathfrak{osp}(1|2)$ implies that any irreducible representation of $\mathfrak{osp}(1|2)$ is a representation of $\mathfrak{su}(1,1)$, the latter being not necessarily irreducible. V can therefore be written as a direct sum of irreducible representation spaces of $\mathfrak{su}(1,1)$, or

$$V = \bigoplus_i W_i.$$

Without loss of generality, we can regard v_0 as an element of W_0. Since W_0 is a representation space of $\mathfrak{su}(1,1)$, we know that

$$v_{2k} = \pi(e)^k v_0 \quad \text{and} \quad v_{-2k} = \pi(f)^k v_0$$

must either be zero or elements of W_0. All these vectors span the space W_0, which is generated by a single vector v_0.

The action of b^+ on any vector of W_0 must be a vector outside W_0, provided that this action differs from zero. Let us define

$$v_1 = \pi(b^+)v_0.$$

We can say that v_1 is an element of W_1. Similarly, we can look at the action of b^- on v_0:

$$v_{-1} = \pi(b^-)v_0.$$

We observe from the definition of the Casimir operator C that $\pi(b^-b^+)$ is a **diagonal operator** on V. This means that $\pi(b^-b^+)v$ is proportional to v for all $v \in V$. So $\pi(b^-)v_1$ must be a certain multiple of v_0. At this point however, we cannot be sure that $\pi(b^-)v_1$ is different from zero. Likewise, it is impossible to tell whether $\pi(b^+)v_{-1} \neq 0$. Since we can neither say that $\pi(f)v_1$ is a nonzero multiple of v_{-1}, nor that $\pi(e)v_{-1}$ is a multiple of v_1, we must regard v_{-1} as an element of a different subspace W_{-1}. Note that W_1 and W_{-1} are the same spaces when either $\pi(b^-)v_1$ or $\pi(b^+)v_{-1}$ differs from zero. These actions are zero simultaneously only when $\mu = 0$.

We denote the generating vectors of W_{-1} as $v_{-2k-1} = \pi(f)^k v_{-1}$ and the generating vectors of W_1 as $v_{2k+1} = \pi(e)^k v_1$.

Lemma 2.21 *The vectors of W_0, W_{-1} and W_1 are connected by the actions of b^+ and b^- in the following manner*

$$v_{2k+1} = \pi(b^+)v_{2k} \qquad \text{and} \qquad v_{-2k-1} = \pi(b^-)v_{-2k},$$

for every positive integer value of k.

Proof. Letting $\pi(b^+)$ act on the vector v_1 results in a vector of W_0 because $\pi(b^+)v_1 = 2\pi(e)v_0 = 2v_2$. Thus we find $\pi(b^+)v_2 = \frac{1}{2}\pi(b^+)^2 v_1 = \pi(e)v_1 = v_3$. It is clear that this can be generalized to the stated formula for v_{2k+1}. The result for v_{-2k-1} can be found analogously. □

Figure 2.2 helps to visualize how the representation space is constructed. We emphasize that the relationship between v_1 and v_{-1} is not yet determined.

The action of h on the entire representation space V can already be determined.

Lemma 2.22 *The action of h on V is given by*

$$\pi(h)v_k = (2\mu + k)v_k,$$

for all $k \in \mathbb{Z}$.

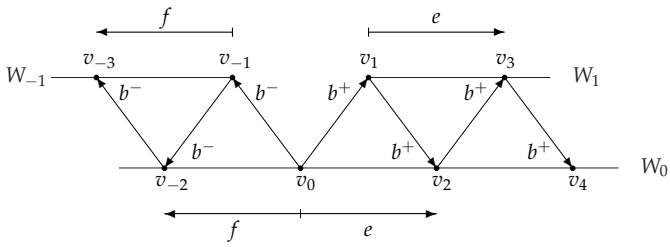

Figure 2.2: The representation $V = W_{-1} \oplus W_0 \oplus W_1$

Proof. For even values of k, this follows just from the relations

$$[h, e^k] = 2ke^k, \qquad [h, f^k] = -2kf^k.$$

For $k = 1$, these are the commutation relations $[h, e] = 2e$ and $[h, f] = -2f$, and the required identities follow by induction. We then obtain

$$\pi(h)\, v_{2k} \;=\; \pi(h)\pi(e)^k\, v_0 \;=\; (2\mu + 2k)\, v_{2k}.$$

For the odd values of k, we need $[h, b^\pm] = \pm b^\pm$, which is an instant consequence of equation (2.16). From this, we obtain

$$\pi(h)\, v_{2k+1} \;=\; \pi(h)\pi(b^+)\, v_{2k} \;=\; (2\mu + 2k + 1)\, v_{2k+1},$$

and similarly for v_{-2k-1}. □

We would like to determine the actions of b^+ and b^- on every vector of W_0, W_{-1} and W_1. Our method involves defining the action of the Casimir operators on the representation space. We write the respective diagonal actions as

$$\begin{aligned} \pi(C)\, v &= \lambda v && (\forall\, v \in V), \\ \pi(\Omega)\, v_{2k} &= -\delta(\delta + 1)\, v_{2k} && (\forall\, k \in \mathbb{Z}). \end{aligned}$$

We will argue that the choice of λ is not independent of δ. It is a nice exercise to show with the help of equation (2.16) that

$$(b^-b^+ - b^+b^-)^2 = 4(b^-b^+ - b^+b^-) - 16\Omega.$$

This can be used to show that $C^2 = (1 - 4\Omega)(2C + 4\Omega)$. If we let both sides of this equation act on a vector v_{2k}, we get a quadratic equation in λ. The two possible solutions are

$$\lambda_1 = 2\delta(2\delta + 1) \quad \text{and} \quad \lambda_2 = 2(\delta + 1)(2\delta + 1).$$

We choose $\lambda = \lambda_1$ and remark that the results for the choice $\lambda = \lambda_2$ can be reproduced with the transformation $\delta \to -\delta - 1$.

In order to be able to determine the actions of b^+ and b^- on every vector of V, we still need the action of the $\mathfrak{su}(1,1)$ Casimir operator Ω on W_{-1} and W_1.

Lemma 2.23 *The Casimir operator Ω acts on W_{-1} and W_1 as given by*

$$\pi(\Omega)v_{2k+1} = -(\delta - \frac{1}{2})(\delta + \frac{1}{2})v_{2k+1}, \qquad (k \in \mathbb{Z}). \tag{2.19}$$

As desired, the $\mathfrak{su}(1,1)$-Casimir operator is constant on the subspaces W_{-1} and W_1 as well. Moreover, the actions on both subspaces are the same.

Proof. To prove equation (2.19), we will calculate $\pi(\Omega)v_{2k+1}$ as $\pi(\Omega b^+)v_{2k}$. From (2.16) we can immediately derive that

$$[b^-, b^+]b^+ = 2b^+ - b^+[b^-, b^+].$$

Using this and twice the definition of the Casimir element C, we obtain

$$4\Omega b^+ = b^+(1 - 2C - 4\Omega).$$

The same formula holds if we change b^+ into b^- in both sides of the equation. All of the operators on the right hand side can be applied to vectors of W_0. So now $\pi(\Omega\, b^+)v_{2k}$ can be easily calculated, with equation (2.19) as a result. □

It has now become straightforward to find the actions of b^+ and b^- on all the vectors of V.

Proposition 2.24 *The actions of the operators b^+ and b^- on the vectors of V are given by*

$$\begin{aligned} \pi(b^-)v_{2k} &= (\mu + k + \delta)v_{2k-1}, \\ \pi(b^-)v_{2k+1} &= 2(\mu + k - \delta)v_{2k}, \\ \pi(b^+)v_{-2k} &= -(\mu - k - \delta)v_{-2k+1}, \\ \pi(b^+)v_{-2k-1} &= 2(\mu - k + \delta)v_{-2k}. \end{aligned} \tag{2.20}$$

After the choice $\lambda = \lambda_2$ one would find these actions by means of the transformation $\delta \to -\delta - 1$.

Since the actions of h, e and f follow directly from these relations, we have now constructed all representations of $\mathfrak{osp}(1|2)$ generated by a weight vector v_0. It remains to investigate irreducibility and the $*$-condition.

2.4.3 Extension to $*$-representations

Recall that V is the space spanned by all the vectors v_k, $k \in \mathbb{Z}$. We introduce a sesquilinear form $\langle .,. \rangle : V \to \mathbb{C}$ such that

$$\langle \pi(X)v, w\rangle = \langle v, \pi(X^*)w\rangle$$

for all $X \in \mathfrak{osp}(1|2)$ and for all $v, w \in V$. We see that $h^* = h$ implies that $\langle v_k, v_l\rangle = 0$ for $k \neq l$. This means that the set $\mathcal{S} = \{v_k | k \in \mathbb{Z}, v_k \neq 0\}$ forms an orthogonal basis for V. We denote by $\mathcal{I}$ the index set such that $v_k \in \mathcal{S}$ for all $k \in \mathcal{I}$.

The form $\langle .,. \rangle$ is defined by putting

$$\langle v_k, v_l\rangle = a_k \delta_{kl}, \qquad k, l \in \mathcal{I},$$

with a_k to be determined and $a_0 = 1$. The definition of a $*$-representation requires that the representation space is a Hilbert space, so our sesquilinear form needs to be an inner product. Hence, we want $a_k > 0$ for $k \in \mathcal{I}$. This also guarantees that the representation is irreducible, which was not necessarily the case up to this point. From the action of h and from $h^* = h$ we obtain

$$2\mu = \langle \pi(h)v_0, v_0\rangle = \langle v_0, \pi(h)v_0\rangle = 2\bar{\mu},$$

so μ must be a real number. Similar calculations for the actions of Ω and C reveal that both $\delta(\delta + 1)$ and $\delta(2\delta + 1)$ are real. These two conditions together imply that δ must be real.

From the actions of b^+ and b^- and from $(b^\pm)^* = b^\mp$, we derive

$$a_{2k+1} = \langle v_{2k+1}, \pi(b^+)v_{2k}\rangle = \langle \pi(b^-)v_{2k+1}, v_{2k}\rangle = 2(\mu + k - \delta)a_{2k}.$$

In the same way we find

$$a_{2k} = \frac{1}{2}(\mu + k + \delta)a_{2k-1}.$$

We wish to determine under which conditions $\langle .,. \rangle$ is an inner product. Alternatively put, for which parameter values is $a_k > 0$ for all $k \in \mathcal{I}$? Starting from $a_0 = 1$ this

can be derived inductively using the two previous equations. We find that all a_k can be positive only if $\mu - \delta > 0$ and $\mu + \delta + 1 > 0$.

A similar reasoning should yield a positivity condition for the a_k for negative k. However, the resulting conditions $\mu \pm \delta + k > 0$ can never be satisfied for all negative values of k. Hence, the representation π must have a lowest weight vector, because otherwise it would not be possible to define an inner product on the entire representation space. In this case, the restriction of π to an $\mathfrak{su}(1,1)$ subspace is known as a positive discrete series representation [4, 31].

There are two choices for δ to obtain a lowest weight representation. One choice is to have v_0 as a lowest weight vector, which will arise when $\delta = -\mu$ as one sees from the actions (2.20). For $\delta = \mu - 1$ we obtain $\pi(b^+)v_{-2} = 0$, in which case v_{-1} is the lowest weight vector. After one of these choices Proposition 2.24 must obviously be rewritten. Before we do this, let us make use of the inner product $\langle .,. \rangle$ to construct an orthonormal basis $\{e_k\}$:

$$e_{2k} = \frac{v_{2k}}{\|v_{2k}\|} \quad (k \geq 0), \qquad e_{2k} = (-1)^k \frac{v_{2k}}{\|v_{2k}\|} \quad (k < 0),$$

and

$$e_{2k+1} = \frac{v_{2k+1}}{\|v_{2k+1}\|} \quad (k \geq 0), \qquad e_{2k+1} = (-1)^{k-1} \frac{v_{2k+1}}{\|v_{2k+1}\|} \quad (k < 0),$$

for $k \in \mathcal{I}$. We are now ready to investigate all irreducible $*$-representations of $\mathfrak{osp}(1|2)$.

Proposition 2.25 *The only class of irreducible $*$-representations of $\mathfrak{osp}(1|2)$ is a direct sum of two positive discrete series representations of $\mathfrak{su}(1,1)$, determined by a parameter μ. For $0 < \mu \leq \frac{1}{2}$, there is only one irreducible $*$-representation of $\mathfrak{osp}(1|2)$. The actions of the generators on the basis vectors $\{e_k | \, k = 0, 1, 2, \ldots\}$ of the representation space are determined by*

$$\begin{aligned} \pi(b^+)e_{2k} &= \sqrt{2(2\mu + k)}\, e_{2k+1}, \\ \pi(b^-)e_{2k} &= \sqrt{2k}\, e_{2k-1}, \\ \pi(b^+)e_{2k+1} &= \sqrt{2(k+1)}\, e_{2k+2}, \\ \pi(b^-)e_{2k+1} &= \sqrt{2(2\mu + k)}\, e_{2k}. \end{aligned} \tag{2.21}$$

For $\mu > \frac{1}{2}$, this representation can occur alongside another one, for which the actions of the generators on the basis vectors $\{e_k | \, k = -1, 0, 1, 2, \ldots\}$ are given

by

$$\begin{aligned} \pi(b^+)e_{2k} &= \sqrt{2(k+1)}\, e_{2k+1}, \\ \pi(b^-)e_{2k} &= \sqrt{2(2\mu+k-1)}\, e_{2k-1}, \\ \pi(b^+)e_{2k+1} &= \sqrt{2(2\mu+k)}\, e_{2k+2}, \\ \pi(b^-)e_{2k+1} &= \sqrt{2(k+1)}\, e_{2k}. \end{aligned} \tag{2.22}$$

The actions of the other generators follow immediately from these relations and are left for the reader to calculate.

Proof. For $\delta = -\mu$, we get the first representation, which is a lowest weight representation since $\pi(b^-)e_0 = 0$. It is clear that μ must be strictly positive so that all the given actions are well defined. The case $\mu = 0$ is excluded to be sure that $\pi(b^+)e_{2k}$ differs from zero.

In the case of the second representation, for $\delta = \mu - 1$, we must add the condition $\mu > \frac{1}{2}$ to guarantee that $\pi(b^+)e_{-1}$ is well defined and different from zero. We end up with the desired classification. □

Note that if we were to choose $\lambda = \lambda_2$ in the discussion preceding Lemma 2.23, we would find exactly the same class of irreducible $*$-representations. Indeed, these two representations would arise for the choices $-\delta - 1 = -\mu$ or $-\delta - 1 = \mu - 1$. An immediate consequence of this observation is that the other actions remain the same in this case.

The results of Proposition 2.25 were already obtained by Hughes using a different approach [36]. However, we notice that there is an equivalence between both representation classes in Proposition 2.25. Thus, we end up with only one class of irreducible representations of $\mathfrak{osp}(1|2)$.

Proposition 2.26 *The only class of irreducible $*$-representations of $\mathfrak{osp}(1|2)$ is a direct sum of two positive discrete series representations of $\mathfrak{su}(1,1)$, determined by a parameter $\mu > 0$. The actions of the generators on the basis vectors $\{e_k | \, k = 0, 1, 2, \ldots\}$ of the representation space are determined by (2.21).*

Proof. If we replace e_k by $\bar{e}_k$ and μ by $\bar{\mu}$ in (2.22), then we may recover (2.21) from (2.22) by setting $\bar{e}_k = e_{k+1}$ and $\bar{\mu} = \mu + \frac{1}{2}$. Hence, both representations are equivalent. □

Combining all of the results above, we find that the $\mathfrak{osp}(1|2)$ representation space can be depicted as in Figure 2.3.

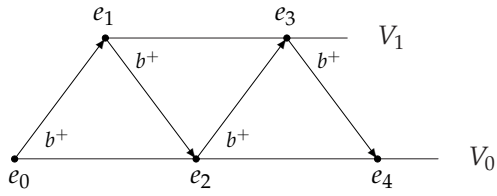

Figure 2.3: The $\mathfrak{osp}(1|2)$ representation space $V = V_0 \oplus V_1$

Separately, V_0 and V_1 are both lowest weight representation spaces of $\mathfrak{su}(1,1) \subset \mathfrak{osp}(1|2)$. The subspaces V_0 and V_1 will be referred to as the **even subspace** and the **odd subspace** respectively from now on.

We have mentioned in the beginning of this chapter that representations are important because they can be seen as actions on a vector space. Operators acting on a Hilbert space are of paramount importance in the field of quantum mechanics. An introduction to quantum mechanics is presented in the next chapter, where a first encounter with $\mathfrak{osp}(1|2)$ representations can also be found.

3

The harmonic oscillator in canonical and Wigner quantization

The title of this chapter does not immediately reveal the main purpose of the following sections, namely the introduction of Wigner quantum systems. One cannot speak of Wigner quantization without referring to canonical quantization. In fact, in many ways Wigner quantization can be seen as a more general approach to quantization than the widely used canonical approach. Hence, it is logical to introduce both concepts simultaneously. In this chapter, this is realized by considering the one-dimensional harmonic oscillator both in the context of canonical quantization and Wigner quantization. This particular system is chosen for many of its characteristic features, such as its simplicity, physical importance and exact solvability.

3.1 The one-dimensional canonical harmonic oscillator

In this section we will consider the one-dimensional harmonic oscillator in the framework of canonical quantization. This physical system is very-well known and has been extensively studied for many decades in an attempt to grasp the deep intricacies of quantum mechanics. Before we move on to the analysis of the canonical harmonic oscillator, we will give a concise introduction to the basics of quantum mechanics.

3.1.1 Quantum mechanics in a nutshell

The motion of large, so-called macroscopic objects can be described by classical mechanics. The theory of classical mechanics predicts reality very accurately, as long as the systems of interest include large objects and the velocities involved do not approach the speed of light. However, the situation is very different when the objects become sufficiently small. Experiments have shown that objects of atomic or sub-atomic scale exhibit both wave and particle properties. This implies that one can no longer make an appeal on the classical concepts of "particle" or "wave" to fully describe a quantum-scale system. Instead, one needs a new theoretical structure in order to meet with the peculiarities encountered in various experiments on a microscopic scale. Such a theory was developed between the years 1925 and 1930 and now goes by the name of quantum mechanics.

Quantum mechanics is a theory in the sense that it is based on a number of assumptions or postulates in order to describe reality as accurately as possible. It is not our intention to give a deep introduction to quantum mechanics and to analyze the postulates profoundly. Many books have been written on this subject by very well qualified authors. We will mainly base ourselves on [10] to introduce the concepts that will play a major role in the next chapters, such as Hamiltonians, operators and their spectrum, wave functions and representation spaces.

In classical mechanics, the knowledge of physical observables such as position vectors and momenta of the objects allows us to predict the dynamical state of a system. Instead, in quantum mechanics all information that can be known about a physical system is incorporated in a **wave function**, which plays the role of a probability amplitude. For a system with a single structureless particle, the wave function in configuration space is denoted by $\Psi(\mathbf{r},t)$. The first postulate of quantum mechanics then states that the square of the modulus of this wave function gives us the probability of finding the particle within the volume element $d\mathbf{r} \equiv dxdydz$ about the point $\mathbf{r} \equiv (x,y,z)$ at time t. In other words

$$P(\mathbf{r},t)d\mathbf{r} = |\Psi(\mathbf{r},t)|^2 \, d\mathbf{r},$$

where

$$P(\mathbf{r},t) = |\Psi(\mathbf{r},t)|^2 = \Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t)$$

is the **position probability density**, with $\Psi^*(\mathbf{r},t)$ being the complex conjugate of $\Psi(\mathbf{r},t)$. Thus, the theory of quantum mechanics has a statistical nature. The position probability density must always be a real value between 0 and 1. Moreover,

we are certain to find the particle somewhere in space, so we have

$$\int |\Psi(\mathbf{r},t)|^2 \, d\mathbf{r} = 1, \tag{3.1}$$

where the integral is taken over all space. So we see that our wave function needs to be **square integrable**, and we like it to be normalized to unity so that the previous relation holds. The wave function is generalized in a straightforward manner for a system of N structureless particles. In configuration space, this wave function is denoted by $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)$ and it has a similar probabilistic interpretation. More concretely, we have that

$$P(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)d\mathbf{r}_1\cdots d\mathbf{r}_N = |\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)|^2 \, d\mathbf{r}_1\cdots d\mathbf{r}_N$$

is the probability of finding, at time t, particle 1 in the volume element $d\mathbf{r}_1$ about $\mathbf{r}_1$, particle 2 in the volume element $d\mathbf{r}_2$ about $\mathbf{r}_2$, and so on.

The wave function $\Psi(\mathbf{r},t)$ says something about the position of the particle. Similarly, one could encapsulate the momentum probabilities in a momentum space wave function $\Phi(\mathbf{p},t)$. The square of the modulus of such a wave function would give us a **momentum probability density**. Yet, both wave functions $\Psi(\mathbf{r},t)$ and $\Phi(\mathbf{p},t)$ must describe the same physical system. This suggests that more general machinery exists in which the particular coordinate representation we are using is not immediately specified.

Matrix mechanics

In **matrix mechanics**, an equivalent formulation of quantum mechanics created by Werner Heisenberg, Max Born and Pascual Jordan, the wave function is considered to be a vector – the so-called **state vector** – in a certain Hilbert space. Following Dirac's bra-ket notation [20], the state vector is denoted by $|\Psi\rangle$. It is part of a (usually infinite-dimensional) Hilbert space, for which the basis vectors are chosen to be associated with the possible outcomes for measurements of a certain physical observable. We call these basis vectors the **eigenstates of the observable**. Each observable creates a different basis and thus a different **representation space**. The state vector represents the state of a system in a particular representation space. All of the basis vectors are themselves possible states of the physical system, since when a certain value is measured for a given observable, the system must necessarily be in the state corresponding to that measurement.

Let us denote the basis vectors of the representation space by $|\Psi_n\rangle$. The state vector $|\Psi\rangle$ can then be written as a linear combination of its basis vectors:

$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle . \tag{3.2}$$

From this decomposition one sees that the state vector $|\Psi\rangle$ is entirely determined by its components c_n. The squares of the moduli of these components c_n are to be interpreted as the probability that a measurement of the considered observable is the value corresponding to the basis vector $|\Psi_n\rangle$.

We follow an analogy made by Henry in [33] to make things a bit more concrete. Consider the experiment of tossing a coin. When the result of the experiment is "measured", there are two possible outcomes: heads or tails. Before the measurement, there is no way of telling in which of these two states the coin is. In a way, the spinning coin is in a state of heads and tails at the same time. One can only say something concrete about the probability of measuring heads or tails. In Figure 3.1 we see that we have an honest coin, with equal probabilities of finding heads or tails. Indeed, the **probability amplitudes** $1/\sqrt{2}$ are read off as the components of the state vector on both axes. This means that the probability of measuring heads or tails is equal to the square of these probability amplitudes, namely $1/2$.

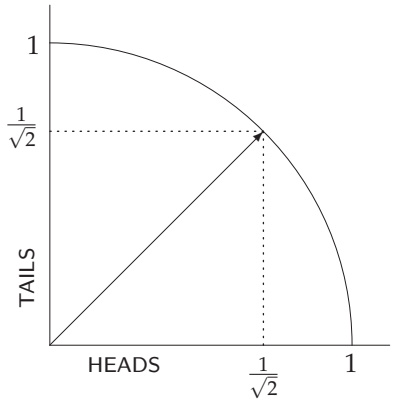

Figure 3.1: In the case of a coin-toss experiment, the Hilbert space is two-dimensional. The probability amplitudes are the components of the state vector, which is normalized to unity.

Notice that we have drawn our axes orthogonally. This is by no means a coincidence, for if this were not the case, then there would still be a non-zero probability

of measuring tails after heads was already measured and vice-versa. Indeed, when heads is being measured, the state vector lies along the heads-axis and the component of the state vector along the tails-axis would not be zero. However, when the coin is in a heads upward position there is obviously no chance of it being tails upward, so an inconsistency arises. Going back to our general description of matrix mechanics, this implies that all basis vectors of the Hilbert space need to be mutually orthogonal.

We wish to stress that the previous analogy is not ideal by any means. First of all, the Hilbert space drawn here is finite-dimensional, which is atypical. Moreover, in the coin-toss analogy the Hilbert space is real, while in matrix mechanics we must be working with a complex Hilbert space. So in fact it is impossible to draw the Hilbert space at all, we cannot even draw a small chunk of it. The example given above merely helps us to grasp the concepts of state vector and eigenstates, and it emphasizes the probabilistic character of quantum mechanics. We refer to [33] for a very clear explanation on how complex Hilbert spaces arise in quantum mechanics.

A Hilbert space is a vector space with an inner product defined on it, so our representation space needs an inner product in order to be a Hilbert space. Assume that we have two state vectors $|\Phi\rangle$ and $|\Psi\rangle$ in the position representation space. Then their inner product is denoted by the symbol $\langle\Phi|\Psi\rangle$, namely

$$\langle\Phi|\Psi\rangle = \int \Phi^*(\mathbf{r})\Psi(\mathbf{r})d\mathbf{r}.$$

The symbol $|\Psi\rangle$ is known as a **ket**, while $\langle\Phi|$, an element of the dual vector space, is called a **bra**. From the above definition we see that the inner product is linear in its second argument and anti-linear in its first argument, and that

$$\langle\Phi|\Psi\rangle = \langle\Psi|\Phi\rangle^*.$$

We have constructed our Hilbert space in such a way that the state vectors are normalized to unity, i.e.

$$\langle\Psi|\Psi\rangle = 1.$$

This is equivalent to the normalization condition given in equation (3.1). All basis vector $|\Psi_n\rangle$ are also normalized to unity and they are orthogonal by construction. Thus we have

$$\langle\Psi_m|\Psi_n\rangle = \delta_{mn},$$

where δ_{mn} is the Kronecker delta function defined by

$$\delta_{mn} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n. \end{cases}$$

This relation, together with equation (3.2), shows that the components of a state vector $|\Psi\rangle$ are given by

$$\langle \Psi_n | \Psi \rangle = \sum_m c_m \langle \Psi_n | \Psi_m \rangle = \sum_m c_m \delta_{mn} = c_n,$$

for all n. Thus a vector is completely determined by the values of $\langle \Psi_n | \Psi \rangle$ for all n.

Observables as operators

A second postulate in quantum mechanics is that a linear operator $\hat{A}$ is associated with every physical quantity or observable $\mathcal{A}$. Moreover, it is assumed that the only result of a precise measurement of this observable is one of the **eigenvalues of the operator** $\hat{A}$. Thus, for any observable $\mathcal{A}$ we have a representation space for which the basis vectors $|\Psi_n\rangle$ are subject to the relation

$$\hat{A} |\Psi_n\rangle = a_n |\Psi_n\rangle ,$$

where a_n is an eigenvalue of the operator $\hat{A}$. The totality of the eigenvalues of $\hat{A}$ is called the **spectrum** of $\hat{A}$. The spectrum can be discrete or continuous and assembles all possible measurements of the observable $\mathcal{A}$. Since such measurements must always be real numbers, the spectrum of $\hat{A}$ must also be real. To overcome this issue, we will associate a so-called self-adjoint operator with every observable because they have the nice feature that their spectrum is necessarily real. A self-adjoint operator is an operator that is its own adjoint. The **adjoint** $\hat{A}^\dagger$ of an operator $\hat{A}$ is defined by the relation

$$\langle \Psi | \hat{A} | \Phi \rangle^* \equiv \langle \Phi | \hat{A}^\dagger | \Psi \rangle. \tag{3.3}$$

It takes a small explanation to see how this relation defines the entire operator $\hat{A}^\dagger$. The inner product of the vector $\hat{A}^\dagger |\Psi\rangle$ with a random vector $|\Phi\rangle$ is well defined by equation (3.3). Therefore, the values $\langle \Psi_n | \hat{A}^\dagger | \Psi \rangle$ are well-defined for all basis vectors $|\Psi_n\rangle$, which determines the vector $\hat{A}^\dagger |\Psi\rangle$ completely. Thus, the relation

(3.3) defines the action of $\hat{A}^\dagger$ on all vectors $|\Psi\rangle$ of the Hilbert space. We note that we have the implication

$$|\Xi\rangle = \hat{A}\,|\Phi\rangle \quad \Rightarrow \quad \langle\Xi| = \langle\Phi|\hat{A}^\dagger. \tag{3.4}$$

This follows directly from

$$\langle\Xi|\Psi\rangle = \langle\Psi|\Xi\rangle^* = \left\langle\Psi|\hat{A}|\Phi\right\rangle^* = \langle\Phi|\hat{A}^\dagger|\Psi\rangle.$$

Equation (3.4) in turn indicates that when a_n is an eigenvalue of $\hat{A}$, i.e. $\hat{A}\,|\Psi_n\rangle = a_n\,|\Psi_n\rangle = |a_n\Psi_n\rangle$, there follows that a_n^* is an eigenvalue of $\hat{A}^\dagger$ with the same eigenvector. Indeed, we must have $\langle\Psi_n|\hat{A}^\dagger = \langle a_n\Psi_n|$ which is equal to $a_n^*\langle\Psi_n|$ due to the anti-linearity of the inner product in its first argument.

A **self-adjoint operator** $\hat{A}$ is defined as an operator that satisfies $\hat{A} = \hat{A}^\dagger$. Note that physicists often use the word **Hermitian operator** to indicate a self-adjoint operator. Mathematically, these two terms are synonyms only for bounded operators. Since we will encounter many unbounded self-adjoint operators, we will not adopt the physicist convention.

In our normalized Hilbert space we deduce

$$a_n = a_n\,\langle\Psi_n|\Psi_n\rangle = \left\langle\Psi_n|\hat{A}|\Psi_n\right\rangle = \langle\Psi_n|\hat{A}^\dagger|\Psi_n\rangle = a_n^*\,\langle\Psi_n|\Psi_n\rangle = a_n^*$$

for any self-adjoint operator. Thus the spectrum of a self-adjoint operator must be real. This is the reason why only self-adjoint operators are associated with observables.

The measuring process in quantum mechanics can now be summarized as follows. We start with a physical system described by a state vector $|\Psi\rangle$. Measuring the value of an observable $\mathcal{A}$ for this system will amount to finding an eigenvalue a_n of the operator $\hat{A}$ associated with $\mathcal{A}$. After the measurement, the system is in an eigenstate $|\Psi_n\rangle$, which is an eigenvector of $\hat{A}$ with corresponding eigenvalue a_n. The probability of measuring a_n is given by $|\langle\Psi|\Psi_n\rangle|^2$.

Although every physical quantity can be considered, the most prominent ones are position, momentum and total energy. Position is so important because it is a very natural quantity to measure, and momentum and total energy owe their significance to the fact that they are conserved in classical mechanics. The position and momentum operators are denoted by $\hat{Q}$ and $\hat{P}$ respectively. The operator belonging to the total energy of the system is called the **Hamiltonian** and is denoted by $\hat{H}$. Each of these observables can be used to define a representation space. Consider for example the position representation space for a particle moving in

three dimensions. Let us denote the eigenstates of the position operator by $|\mathbf{r}\rangle$. In other words, $|\mathbf{r}\rangle$ is an eigenvector of the three components $\hat{Q}_x, \hat{Q}_y$ and $\hat{Q}_z$ of the position operator $\hat{Q}$, or

$$\hat{Q}_x |\mathbf{r}\rangle = q_1 |\mathbf{r}\rangle , \qquad \hat{Q}_y |\mathbf{r}\rangle = q_2 |\mathbf{r}\rangle \qquad \text{and} \qquad \hat{Q}_z |\mathbf{r}\rangle = q_3 |\mathbf{r}\rangle .$$

The components of the state vector $|\Psi\rangle$ are then given by

$$\Psi(\mathbf{r}) = \langle \mathbf{r}|\Psi\rangle . \tag{3.5}$$

Due to the continuous nature of the position label, we end up with a function of the position $\mathbf{r}$, the spatial wave function. We can write down the wave function in momentum space in a similar manner.

Continuous spectrum

In the previous discussion we have always silently assumed that all considered operators have a discrete spectrum. Otherwise, it would not have been possible to find a countable basis of the representation space. However, some operators have a (partly) continuous spectrum, so it is important to realize that our description is not sufficiently general. Fortunately it is possible to present a more general picture involving a Dirac delta function normalization of the basis vectors.

Consider an observable A with a spectrum consisting of both discrete eigenvalues a_n and a continuous range of eigenvalues denoted by a. The corresponding eigenfunctions are Ψ_n and Ψ_a respectively. In other words we have

$$A\Psi_n = a_n \Psi_n, \qquad A\Psi_a = a\Psi_a.$$

It is postulated that an arbitrary wave function Ψ must be expandable in the complete set $\{\Psi_n, \Psi_a\}$. This means

$$\Psi = \sum_n c_n \Psi_n + \int c(a)\Psi_a da,$$

where the integral is taken over all values of a. The eigenfunctions then have the following orthonormality relations:

(1) The discrete eigenfunctions are taken to be orthonormal as before:

$$\langle \Psi_m | \Psi_n \rangle = \delta_{mn}.$$

(2) All the eigenfunctions belonging to the continuous spectrum must be orthogonal to all those belonging to the discrete spectrum:

$$\langle \Psi_n | \Psi_a \rangle = 0.$$

(3) The orthonormality conditions for the eigenfunctions belonging to the continuum spectrum are

$$\langle \Psi_{a'} | \Psi_a \rangle = \delta(a - a'),$$

where $\delta(a - a')$ is the **Dirac delta function**.

Strictly speaking, the Dirac delta function is not a function but a distribution. It is defined by the following property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0). \tag{3.6}$$

It is possible to write the delta function as a non-converging integral, emphasizing that it is not a function. We have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-x_0)}dk = \delta(x - x_0). \tag{3.7}$$

The delta function can be seen as the limit for $\epsilon \to 0$ of functions $\delta^{(\epsilon)}(x - x_0)$ that are defined by

$$\delta^{(\epsilon)}(x - x_0) = \begin{cases} \frac{1}{\epsilon} & \text{for } -\frac{\epsilon}{2} \leq x - x_0 \leq \frac{\epsilon}{2} \\ 0 & \text{for } |x - x_0| > \frac{\epsilon}{2}. \end{cases}$$

Intuitively the Dirac delta function is zero anywhere and equals infinity at $x = x_0$.

Commuting operators

Whereas in classical mechanics every quantity obeys the rules of ordinary algebra, we have just postulated that we will be working with operators in quantum mechanics, which in general do not commute with each other. For two operators $\hat{A}$ and $\hat{B}$ we define the **commutator** of $\hat{A}$ and $\hat{B}$ as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

The statement that such a commutator usually differs from zero will be immediately clear after we will have explained that operators can in fact be represented as

matrices. This in itself is rather intuitive since we described these operators as acting on the state vectors of our Hilbert space. Any linear transformation of one vector to another can be represented by a matrix.

Assume we have a representation space corresponding to a physical observable for which the associated operator has a discrete spectrum. Let us denote the basis vectors of our representation space by $|n\rangle$. Any state vector $|\Psi\rangle$ is then fully determined by its components

$$c_n = \langle n|\Psi\rangle .$$

We define an operator $\hat{I}$ by

$$\hat{I} = \sum_n |n\rangle \langle n|.$$

This operator works as the identity operator, as one sees from

$$\hat{I}\,|\Psi\rangle = \sum_n |n\rangle \langle n|\Psi\rangle = \sum_n c_n\,|n\rangle = |\Psi\rangle .$$

Now consider an operator $\hat{A}$ that maps a vector $|\Psi\rangle$ onto a vector $|\Phi\rangle$, or $|\Phi\rangle = \hat{A}\,|\Psi\rangle$. Both $|\Phi\rangle$ and $|\Psi\rangle$ are fully determined by their components, so let us try to determine the action of the operator $\hat{A}$ by expressing the components of $|\Phi\rangle$ in terms of the components of $|\Psi\rangle$. We find

$$\langle m|\Phi\rangle = \langle m|\hat{A}|\Psi\rangle = \sum_n \langle m|\hat{A}|n\rangle \langle n|\Psi\rangle . \tag{3.8}$$

Clearly, the action of the operator $\hat{A}$ is completely determined by the values $A_{mn} = \langle m|\hat{A}|n\rangle$. They can be interpreted as matrix elements of a matrix A, and equation (3.8) can be seen as the product of a matrix with a vector. A product of operators is then nothing more than a matrix product, which is obviously not commutative.

The commutation relations of two operators are of paramount importance in quantum mechanics. It is possible to prove that two operators share a complete set of common eigenvectors if and only if they commute. We use our intuitive picture of the Hilbert space to clarify the implications of this statement. Consider two observables for which the associated operators do not commute, and consider their representation spaces. Performing a measurement of one of the operators moves the state vector to an eigenstate. Now place the origins of both Hilbert spaces on top of each other and let the state vectors coincide. Since both operators do not share any eigenvectors, the state vector of the second operator - the one that has not been measured yet - is not an eigenstate, so we do not know the value of the

second observable. This means that two such observables, for which the associated operators do not commute, cannot be measured simultaneously to an arbitrary precision. This phenomenon is called the **Heisenberg uncertainty principle**.

In **canonical quantization**, two operators that are known not to commute are the position and momentum operator. In one space dimension x, they operate on the wave function as follows:

$$\hat{X}\Psi(x) = x\Psi(x) \qquad \text{and} \qquad \hat{P}\Psi(x) = -i\hbar\frac{\partial}{\partial x}\Psi(x), \tag{3.9}$$

where we have denoted the position operator by $\hat{X}$ instead of $\hat{Q}$ since we have assumed to have only one spatial dimension for simplicity. The quantity $\hbar$ is defined by $h/2\pi$, with h being a fundamental physical constant called the **Planck constant**. Again, we emphasize that equation (3.9) gives the operator realizations in canonical quantization. We will see that different operator solutions exist in Wigner quantization. From the relations (3.9) it is not hard to calculate the commutator between the position and momentum operator. This way we obtain the one-dimensional **canonical commutation relation** which is imposed in canonical quantization. It can be written as

$$[\hat{X}, \hat{P}] = i\hbar. \tag{3.10}$$

Equations of motion

The time evolution of a wave function of a quantum system is addressed in another postulate of quantum mechanics. It is assumed to be determined by the time-dependent Schrödinger equation given by

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = \hat{H}\Psi(t),$$

where $\hat{H}$ is the Hamiltonian of the system. Since this equation is essentially a first-order differential equation in time, the wave function $\Psi(t)$ is determined for all t once it is known at a given time t_0. Hence we can introduce an evolution operator $U(t, t_0)$ which represents the time-evolution of the Hamiltonian:

$$\Psi(t) = U(t, t_0)\Psi(t_0).$$

Using this, we can also write

$$\Psi(t_0) = U(t_0, t)\Psi(t) = \Psi_H$$

and view the wave function Ψ_H as a time-independent wave function. This is the **Heisenberg picture** of quantum mechanics and Ψ_H is called the Heisenberg wave function. In the Heisenberg picture, operators depend on time and therefore a time variation can be determined for them. Without proof, we mention the **Heisenberg equations** of motion for the position and momentum operators for a one-dimensional system:

$$\frac{\partial \hat{X}}{\partial t} = \frac{i}{\hbar}[\hat{H}, \hat{X}], \qquad \frac{\partial \hat{P}}{\partial t} = \frac{i}{\hbar}[\hat{H}, \hat{P}],$$

where $\hat{H}$ is the Hamiltonian in the Heisenberg picture. For Hamiltonians that are written as polynomials or series in the operators $\hat{x}$ and $\hat{p}$, one can use the canonical commutation relations to prove the compatibility of the Heisenberg equations with **Hamilton's equations** of motion from classical mechanics, given by

$$\frac{\partial \hat{X}}{\partial t} = \mathrm{op}\left(\frac{\partial H}{\partial P}\right), \qquad \frac{\partial \hat{P}}{\partial t} = -\mathrm{op}\left(\frac{\partial H}{\partial X}\right).$$

The right-hand side of these equations is to be interpreted as a formal derivation of the Hamiltonian H as a function of X and P, after which X and P are changed back to operators. This is done in accordance with the Weyl ordering of operators [102], in which we have for instance

$$xp \to \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

It is also possible to show that the Heisenberg equations follow from the Hamilton equations together with the canonical commutation relations. One might then ask the question if the canonical commutation relations can be derived from the Hamilton and Heisenberg equations. This question, which has a negative answer, will lead us to Wigner quantization. We postpone a further discussion of this problem to Section 3.3.

3.1.2 Solving the 1D canonical harmonic oscillator

It is time to apply some of the concepts we have introduced to an actual physical system. We will be considering the one-dimensional harmonic oscillator in canonical quantization. The system consists of a particle moving in one dimension, which is attracted to a fixed centre by a force proportional to the displacement from that centre. This force can be written according to Hooke's law as

$$F = -kx,$$

where k is the positive force constant. This is such an important system for many reasons. First of all, it is the simplest system one can imagine besides a free particle. Secondly, they occur in various forms in nature and can serve as a model for systems in which there exist small vibrations around a point of stable equilibrium. So any result concerning the harmonic oscillator can potentially be applied to a wide variety of physical problems. We mention for instance the vibration of atoms of a molecule around their equilibrium state, or the oscillations of atoms or ions in crystal lattices [16]. Finally, the harmonic oscillator is exactly solvable. This implies that the wave function, energy spectrum, representation spaces and so on can be determined analytically, without numerical computations. Therefore, it is the perfect example to use in textbooks for students.

The total energy of a harmonic oscillator with mass m can be written classically as the sum of its kinetic and potential energy:

$$\frac{p^2}{2m} + \frac{1}{2}kx^2.$$

As a quantum system, where p and x are made into operators, this corresponds to the Hamiltonian of the one-dimensional harmonic oscillator, given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \tag{3.11}$$

where we have introduced a notation for the **angular frequency** of the oscillator:

$$\omega = \sqrt{\frac{k}{m}}.$$

From now on, we will write the position and momentum operators with small letters for aesthetic reasons. As we already know, the Hamiltonian is the operator describing the total energy of the system, split up as the sum of the kinetic and the potential energy. Our goal is to find the energy eigenvalues and eigenstates and decompose the eigenstates in the position representation. This way, we are able to find the position probability distribution of the oscillator when the system is in an energy eigenstate.

We start by introducing new operators a^+ and a^- given by

$$a^{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} \mp \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}.$$

We have $(a^{\pm})^{\dagger} = a^{\mp}$ because the operators $\hat{x}$ and $\hat{p}$ are Hermitian. The position and momentum operators can be rewritten in terms of a^{+} and a^{-} as follows:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^{+} + a^{-}) \tag{3.12}$$

and

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(a^{+} - a^{-}).$$

Using the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ given in equation (3.10), we find

$$[a^{-}, a^{+}] = 1. \tag{3.13}$$

This allows us to rewrite the Hamiltonian in terms of the new operators as

$$\hat{H} = \hbar\omega\left(a^{+}a^{-} + \frac{1}{2}\right).$$

It is possible to find the commutator of $\hat{H}$ with a^{+} and a^{-} using the commutation relation (3.13). We have

$$[\hat{H}, a^{\pm}] = \pm\hbar\omega a^{\pm}, \tag{3.14}$$

Now assume we have an eigenvector $|E\rangle$ corresponding to the energy eigenvalue E, so that

$$\hat{H}\,|E\rangle = E\,|E\rangle\,.$$

A small calculation then learns us that the vectors $a^{\pm}\,|E\rangle$ are also eigenvectors of the Hamiltonian with energy eigenvalue $E \pm \hbar\omega$. Indeed, supported by the relations (3.14) we find

$$\hat{H}a^{\pm}\,|E\rangle = (a^{\pm}\hat{H} \pm \hbar\omega a^{\pm})\,|E\rangle = (E \pm \hbar\omega)a^{\pm}\,|E\rangle\,.$$

As a^{+} raises and a^{-} lowers the value of E by $\hbar\omega$, we call them **raising** and **lowering operators** respectively. The Hamiltonian is a positive operator because it contains the squares of $\hat{x}$ and $\hat{p}$. Hence, $\hat{H}$ can only have non-negative eigenvalues so it must be possible to assign a smallest eigenvalue, denoted by E_0. This must satisfy the property

$$a^{-}\,|E_0\rangle = 0.$$

By letting $\hbar\omega a^{+}$ act on the both sides of this equation, we obtain

$$0 = \hbar\omega a^{+}a^{-}\,|E_0\rangle = (\hat{H} - \frac{\hbar\omega}{2})\,|E_0\rangle = (E_0 - \frac{\hbar\omega}{2})\,|E_0\rangle\,.$$

We conclude that the lowest energy eigenvalue is given by $E_0 = \frac{\hbar\omega}{2}$, and the action of the raising operator on $|E_0\rangle$ creates the higher energy levels

$$|E_n\rangle = (a^+)^n |E_0\rangle .$$

We normalize these vectors to unity and obtain

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$$

as the eigenstates of the energy representation space. The actions of the relevant operators on the normalized energy eigenstates are given by

$$a^+ |n\rangle = \sqrt{n+1}\,|n+1\rangle , \qquad a^- |n\rangle = \sqrt{n}\,|n-1\rangle$$

and

$$\hat{H} |n\rangle = \hbar\omega(n + \frac{1}{2}) |n\rangle . \tag{3.15}$$

So we have a discrete energy spectrum with energy levels $\hbar\omega(n + \frac{1}{2})$ and $n \geq 0$. The spectrum is equidistant with spacing $\hbar\omega$.

We will now demonstrate how we can find the position probability distribution when the system is in an energy eigenstate $|n\rangle$. We emphasize that the method given below is not a standard approach to find the position probability distribution. Usually, one substitutes the canonical operator solutions (3.9) for $\hat{x}$ and $\hat{p}$ into the Hamiltonian (3.11) to end up with a differential equation. Solving this differential equation then gives the desired wave function. Here, we use a different strategy which is more generally valid. The method is particularly interesting when one cannot exploit the canonical interpretation of the position and momentum operators, as is the case in Wigner quantization.

First, we use equation (3.12) to find

$$\hat{x} |n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle). \tag{3.16}$$

Now consider a formal eigenvector $|x\rangle$ of the position operator, and decompose it in the energy representation basis as follows:

$$|x\rangle = \sum_{n=0}^{\infty} \alpha_n(x) |n\rangle .$$

Because $|x\rangle$ is not a finite linear combination of the basis vectors $|n\rangle$, it is not really a vector of the representation space. However, as a formal vector it proves to be a powerful tool for finding the coefficients $\alpha_n(x)$ that are as yet unknown. We can find a relation for these coefficients starting from $\hat{x}\,|x\rangle = x\,|x\rangle$. Using equation (3.16) and comparing the coefficients of $|n\rangle$ on both sides of the resulting equation, one finds

$$\sqrt{\frac{2m\omega}{\hbar}}\,x\,\alpha_n(x) = \sqrt{n}\,\alpha_{n-1}(x) + \sqrt{n+1}\,\alpha_{n+1}(x). \tag{3.17}$$

This is a three term recurrence relation for the coefficients $\alpha_n(x)$. Such recurrence relations appear in abundance in the theory of orthogonal polynomials. In fact, we will see that the $\alpha_n(x)$ can be written in terms of Hermite polynomials, and that the position spectrum is equal to the support of the weight function of these polynomials.

In principle, it would be possible to introduce the Hermite polynomials quickly at this stage. In future chapters, however, we will encounter many other orthogonal polynomials, so we choose to interrupt our analysis of the canonical harmonic oscillator and dedicate an entire section to the special functions and polynomials we will come across later on.

3.2 Special functions and orthogonal polynomials

Many eigenvalue problems in canonical and Wigner quantization result in a three term recurrence relation of some unknown coefficient. We have seen such an example already for the canonical harmonic oscillator, but we will encounter many more in the future. Often it is possible to modify the recurrence relation in such a way that we are able to identify the objects satisfying this relation with orthogonal polynomials. It is therefore necessary to introduce some of the polynomials of relevance in the future.

Koekoek and Swarttouw have listed all orthogonal polynomials that appear in the so-called Askey-scheme, as well as their q-analogues in [50, 51]. These are precisely all known orthogonal polynomials that can be defined in terms of hypergeometric functions. The **hypergeometric series** ${}_rF_s$ [97] is defined by

$${}_rF_s\left(\left.\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!}, \tag{3.18}$$

where $(a)_k$ is the **Pochhammer symbol** defined by

$$(a)_0 = 1 \qquad \text{and} \qquad (a)_k = a(a+1)(a+2)\cdots(a+k-1)$$

for all integers $k \geq 1$. Obviously, the parameters of the hypergeometric function must be such that none of the denominator factors in any term of the hypergeometric series is zero. When one of the numerator parameters a_i equals a negative integer $-n$ the function ${}_rF_s$ is a polynomial in z.

Hypergeometric series might look very exotic at first, but some special cases are very well-known. One of these special cases, for instance, is the exponential function which can be written as

$$\exp(z) = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}\middle| z\right).$$

Slightly more complicated are the trigonometric functions

$$\sin(z) = z\ {}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix}\middle| -\frac{z^2}{4}\right)$$
$$\cos(z) = {}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix}\middle| -\frac{z^2}{4}\right).$$

A whole bunch of identities exist for hypergeometric functions, many of which can be found in the introductory chapter of Koekoek and Swarttouw. We will give some of these formulae at the point where they are needed. Right now, we focus on using hypergeometric functions to describe orthogonal polynomials.

Definition 3.1 *The polynomials of the set $\{\phi_k(x)\}, k = 0, 1, 2, \ldots$ are called **orthogonal polynomials** with respect to a **weight function** $w(x)$ with support I, when the orthogonality relation*

$$\int_I w(x)\phi_k(x)\phi_j(x)dx = \gamma_k\delta_{kj}$$

*is valid for each of these polynomials, with $\gamma_k > 0$ for all k. When $\gamma_k = 1$ for all $k = 0, 1, 2, \ldots$ the polynomials are said to be normalized and we speak of **orthonormal polynomials**.*

In this definition, the function $w(x)$ is called a weight function if it is strictly positive almost everywhere in I, and if the moments $\int_I w(x)x^n dx$ are finite for all $n = 0, 1, 2, \ldots$.

We devote the rest of this section to the introduction of the orthogonal polynomials that are important to us, and we supply useful formulae for each of them. The word 'useful' here needs to be interpreted in the sense that only formulae that will be used in the remaining of this text will be given.

3.2.1 Hermite polynomials

In terms of hypergeometric series, the **Hermite polynomials** [50] are defined by

$$H_n(x) = (2x)^n \, {}_2F_0\left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix} \middle| -\frac{1}{x^2} \right).$$

The Hermite polynomials are orthogonal over $\mathbb{R}$ for the weight function $w(x) = e^{-x^2}$, so we have

$$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi} \delta_{mn}.$$

We can define the normalized Hermite functions by

$$\tilde{H}_n(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2^n n!} \pi^{1/4}} H_n(x).$$

These normalized Hermite functions satisfy the following three term recurrence relation:

$$\sqrt{2} x \tilde{H}_n(x) = \sqrt{n} \tilde{H}_{n-1}(x) + \sqrt{n+1} \tilde{H}_{n+1}(x). \tag{3.19}$$

The connection with the recurrence relation (3.17) is already apparent. However, we postpone the rest of the discussion of the canonical harmonic oscillator to the next section.

3.2.2 Laguerre polynomials

Laguerre polynomials will only be important in Chapter 6, but they are also needed to define the generalized Hermite polynomials that will make an appearance in the next section. The **Laguerre polynomials** [50] are defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right). \tag{3.20}$$

The orthogonality of the Laguerre polynomials is expressed by

$$\int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}, \qquad \alpha > -1,$$

where $\Gamma(z)$ is the **Gamma function** defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

One of the most well-known properties of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z),$$

which implies that when z is an integer, the Gamma function is nothing more than a factorial.

We define a modified version of the Laguerre polynomials by

$$\tilde{L}_n^{(\alpha)}(x) = \sqrt{2}\, e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} L_n^{(\alpha)}(x). \tag{3.21}$$

These functions are not really the normalized Laguerre functions. A factor of $\sqrt{2}$ is added in order to obtain

$$\int_0^\infty \tilde{L}_m^{(\alpha)}(x)\, \tilde{L}_n^{(\alpha)}(x) dx = 2\,\delta_{mn}.$$

The reason for this choice will be clarified in Chapter 6. The three term recurrence relation of the functions (3.21) can be written as

$$\begin{aligned}(2n - x + \alpha + 1)\, \tilde{L}_n^{(\alpha)}(x) = &\sqrt{n(n+\alpha)}\, \tilde{L}_{n-1}^{(\alpha)}(x) \\ &+ \sqrt{(n+1)(n+\alpha+1)}\, \tilde{L}_{n+1}^{(\alpha)}(x)\end{aligned} \tag{3.22}$$

and will come back when we try to find the energy spectrum of the free particle.

3.2.3 Generalized Hermite polynomials

The **generalized Hermite polynomials** $T_n^{(a)}(x)$ form the third family of orthogonal polynomials that is of importance to us. They are not listed in the Askey-scheme,

but they are closely related to the Laguerre polynomials $L_n^{(\alpha)}(x)$, defined previously by equation (3.20). For positive n, the definition of the generalized Hermite polynomials [14] is given by

$$T_{2n}^{(a)}(x) = (-1)^n L_n^{(a-1)}(x^2), \qquad T_{2n+1}^{(a)}(x) = (-1)^n x L_n^{(a)}(x^2).$$

The classical Hermite polynomials are found by choosing $a = 1/2$. Of course, there is also a set of orthogonality relations and a pair of recurrence relations for the generalized Hermite polynomials. Instead of giving them here, we immediately define the normalized version of these polynomials:

$$\tilde{T}_{2n}^{(a)}(x) = |x|^{a-\frac{1}{2}} e^{-\frac{x^2}{2}} \sqrt{\frac{n!}{\Gamma(n+a)}}\, T_{2n}^{(a)}(x),$$
$$\tilde{T}_{2n+1}^{(a)}(x) = |x|^{a-\frac{1}{2}} e^{-\frac{x^2}{2}} \sqrt{\frac{n!}{\Gamma(n+a+1)}}\, T_{2n+1}^{(a)}(x).$$

The functions $\tilde{T}_n^{(a)}(x)$ satisfy the following orthogonality relation:

$$\int_{-\infty}^{+\infty} \tilde{T}_m^{(a)}(x) \tilde{T}_n^{(a)}(x) dx = \delta_{mn}.$$

The pair of recurrence relations corresponding to the normalized generalized Hermite functions $\tilde{T}_n^{(a)}(x)$ is given by

$$\begin{aligned} x\,\tilde{T}_{2n}^{(a)}(x) &= \sqrt{n}\,\tilde{T}_{2n-1}^{(a)}(x) + \sqrt{n+a}\,\tilde{T}_{2n+1}^{(a)}(x), \\ x\,\tilde{T}_{2n+1}^{(a)}(x) &= \sqrt{n+a}\,\tilde{T}_{2n}^{(a)}(x) + \sqrt{n+1}\,\tilde{T}_{2n+2}^{(a)}(x). \end{aligned} \tag{3.23}$$

This recurrence relation is found when the position spectrum of the Wigner harmonic oscillator is being studied.

3.2.4 Meixner-Pollaczek polynomials

The classical **Meixner-Pollaczek polynomials** [50] are defined by

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right).$$

Although it does not appear so at first sight, the Meixner-Pollaczek polynomials are real polynomials of the real variable x, when λ and ϕ are real. This can be seen with the help of the following transformation formula for the ${}_2F_1$ series [50, 51]:

$$ {}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|z\right) = (1-z)^{-a}\,{}_2F_1\left(\left.\begin{matrix}a,c-b\\c\end{matrix}\right|\frac{z}{z-1}\right). $$

Indeed, from this equation it follows that the Meixner-Pollaczek polynomials are equal to their complex conjugate.

As all orthogonal polynomials, the Meixner-Pollaczek polynomials satisfy a certain orthogonality relation and a three term recurrence relation. Both formulas are quite involved and can be found in [51]. Instead, we will use the notation $P_n(E)$ for $P_n^{(\frac{a}{2})}(E;\frac{\pi}{2})$ [1], so we have

$$ P_n(E) = i^n \frac{(a)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n,\frac{a}{2}+iE\\a\end{matrix}\right|2\right). $$

These specific Meixner-Pollaczek polynomials are the only ones we will use later on. For these polynomials, the orthogonality relation takes the form

$$ \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\,\Gamma\left(\frac{a}{2}+iE\right)|^2 P_m(E)\,P_n(E)dE = \frac{\Gamma(n+a)}{2^a\,n!}\,\delta_{mn}, \tag{3.24} $$

and the recurrence formula is

$$ 2E\,P_n(E) = (n+a-1)\,P_{n-1}(E) + (n+1)\,P_{n+1}(E). $$

We need a normalized version of our Meixner-Pollaczek polynomials, which we will denote by $\tilde{P}_n(E)$:

$$ \tilde{P}_n(E) = \frac{|\Gamma(\frac{a}{2}+iE)|}{2}\sqrt{\frac{2^a\,n!}{\pi\,\Gamma(n+a)}}\,P_n(E). $$

Actually, we should speak about the pseudo-normalized Meixner-Pollaczek functions again, similar to the Laguerre polynomials $\tilde{L}_n^{(\alpha)}(x)$ we have defined in equation (3.21). For reasons that will become clear later, we like our normalized Meixner-Pollaczek functions to satisfy the following orthogonality relation:

$$ \int_{-\infty}^{+\infty} \tilde{P}_m(E)\,\tilde{P}_n(E)dE = \frac{1}{2}\,\delta_{mn}. \tag{3.25} $$

[1] The choice for E as a variable rather than x is motivated in Chapter 6, when the Meixner-Pollaczek polynomials appear in the context of an energy spectrum.

Of course, the recurrence relation is also different for the $\tilde{P}_n(E)$. It is given by

$$2E\,\tilde{P}_n(E) = \sqrt{n(n+a-1)}\,\tilde{P}_{n-1}(E) + \sqrt{(n+1)(n+a)}\,\tilde{P}_{n+1}(E). \qquad (3.26)$$

This equation will come back when we determine the energy spectrum of the Berry-Keating-Connes Hamiltonian $\hat{H}_b = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ in Chapter 6.

3.2.5 Other orthogonal polynomials

All of the previously mentioned orthogonal polynomials are important for the same reason: their recurrence relation is obtained in the analysis of the spectrum of a certain operator for some quantum system. Not so for the next three families of orthogonal polynomials. They are all used to create new analytically solvable systems in Chapter 4. Recall that we have already considered such a system, namely the harmonic oscillator.

The Krawtchouk, Hahn and dual q-Krawtchouk polynomials are much more complex than the polynomials we have seen so far. In fact, a lot of temporary variables need to be introduced in order to present their recurrence relations and orthogonality conditions in a decent way. We do not wish to overload this section with content of this type, so we postpone a detailed introduction of these orthogonal polynomials to Chapter 4. We will confine ourselves to giving the definition of the Krawtchouk, Hahn and dual q-Krawtchouk polynomials in terms of hypergeometric or q-generalized hypergeometric series.

For a fixed positive integer parameter N and a real parameter p $(0 < p < 1)$, the **Krawtchouk polynomial** of degree n $(n = 0, 1, \ldots, N)$ in the variable x is defined by [39, 50, 71]

$$K_n(x) \equiv K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -x, -n \\ -N \end{matrix}\middle| \frac{1}{p}\right).$$

Hahn polynomials are characterized by a positive integer parameter N and two real parameters α and β. For orthogonality, one should have $\alpha > -1$ and $\beta > -1$, or $\alpha < -N$ and $\beta < -N$. The Hahn polynomial of degree n $(n = 0, 1, \ldots, N)$ in the variable x is defined by [39, 50]

$$Q_n(x) \equiv Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}\middle| 1\right). \qquad (3.27)$$

Both the Krawtchouk and Hahn polynomials satisfy a discrete orthogonality relation. This means that their weight function has a discrete spectrum and the orthogonality

is expressed as a sum rather than as an integral. The dual q-Krawtchouk polynomials are part of the so-called q-**scheme**, a q-analogue – or a q-extension if you will – of the Askey-scheme. Such q-extensions of classical formulae and functions are based on the observation that

$$\lim_{q\to 1}\frac{1-q^a}{1-q}=a.$$

The number $(1-q^a)/(1-q)$ is called the q-number. The q-analogue of the Pochhammer symbol [97] is defined by

$$(a;q)_0=1, \qquad \text{and} \qquad (a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1}).$$

This is used to define the q-**hypergeometric series** as follows:

$${}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\middle|\, q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k}.$$

Note that the q-hypergeometric series is a polynomial in z if one of the numerator parameters $a_i=q^{-n}$, where n is a non-negative integer. This function forms the core of the q-scheme. The orthogonal polynomials of the Askey-scheme can be generalized – often in several ways – using the q-hypergeometric series. More information on q-generalizations can be found in [26].

For a fixed positive integer parameter N, and real parameters $q>0$ and $\bar{c}<0$, the **dual** q-**Krawtchouk polynomial** of degree n in the variable $\lambda(x)=q^{-x}+\bar{c}q^{x-N}$ is defined by [39, 50]

$$K_n(\lambda(x);q)\equiv K_n(\lambda(x);\bar{c},N|q)={}_3\phi_2\left(\begin{matrix}q^{-n},q^{-x},\bar{c}q^{x-N}\\ q^{-N},0\end{matrix}\middle|\, q;q\right), \tag{3.28}$$

which is terminating here due to the numerator parameter q^{-n}. In standard literature, the parameter $\bar{c}$ is usually denoted by c, but we replace it by $\bar{c}$ in order not to have a notation conflict at a later stage.

3.3 The 1D Wigner harmonic oscillator

It is time to pick up the story on the canonical harmonic oscillator where we left it, namely at the three term recurrence relation (3.17) for the coefficients $\alpha_n(x)$. Comparing this with equation (3.19), we see that the coefficients $\alpha_n(x)$ are proportional with the normalized Hermite polynomials evaluated in $\sqrt{m\omega/\hbar}\,x$. We

have defined the normalized Hermite polynomials $\tilde{H}_n(x)$ in such a way that we will automatically end up with normalized position eigenstates, so we have

$$\alpha_n(x) = \tilde{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right).$$

So we can write the position eigenstates as

$$|x\rangle = \sum_{n=0}^{\infty} \tilde{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)|n\rangle.$$

Using the orthogonality of the Hermite polynomials we find

$$|n\rangle = \int_{-\infty}^{+\infty} \sqrt{\frac{m\omega}{\hbar}}\,\tilde{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)|x\rangle\,dx. \tag{3.29}$$

When the system is in an energy eigenstate $|n\rangle$, its decomposition in the position basis is given by equation (3.29). The square of the modulus of the coefficient of $|x\rangle$ in (3.29) is exactly the position probability distribution when the system is in an energy eigenstate $|n\rangle$.

It is important to realize that we have supported on the canonical commutation relation (3.10) to construct the energy representation space of the one-dimensional canonical harmonic oscillator. They were used to derive equation (3.13) which was at the basis of future calculations. However, we have already hinted at the end of section 3.1.1 that the canonical commutation relations are no necessary assumption. It is instead much more natural to assume that Hamilton's and Heisenberg's equations of motion are valid because of their physical significance. This is the starting point of Wigner quantization.

3.3.1 Wigner quantization

Do the equations of motion determine the quantum mechanical commutation relations? When Wigner discovered in [103] that the answer to this question was negative he created the first example of a **Wigner quantum system** (WQS), a term which was only coined much later by Palev in [42]. Inspired by Wigner, the same question was addressed in the context of quantized field theories [93]. Quickly some papers followed trying to describe some physical implications or giving new examples to accompany Wigner's question [82, 107].

Wigner's example was the one-dimensional harmonic oscillator with Hamiltonian (3.11). Using this specific system, he discovered that the compatibility of Hamilton's and Heisenberg equations of motion does not imply that the canonical commutation relations are valid. So instead of using the canonical commutation relations as a starting point, one can express the compatibility of the Hamilton and Heisenberg equations. This results in a set of **compatibility conditions**. When the canonical commutation relations are postulated, both equations of motion are automatically equivalent. In other words, the canonical commutation relations are a sufficient but not a necessary condition for the compatibility conditions.

Postulating the compatibility of the Hamilton and Heisenberg equations leads – for many physical systems – to compatibility conditions containing commutators and anticommutators. It is therefore no surprise that Lie superalgebras will be involved most of the time. However, the theory of Lie superalgebras had not been developed yet at the time of Wigner's paper. Hence, its impact was overlooked for many years.

Wigner found an infinite amount of different commutation relations for the position and momentum operator. Each of these solutions is now known to correspond to a representation of $\mathfrak{osp}(1|2)$. Wigner's relations were later generalized by Green [29] who was the first to write down the paraboson relations, connected to $\mathfrak{osp}(1|2n)$ explicitly. This created a high interest in parastatistics and the theory of parabosons and parafermions in quantum field theory [29, 30, 41, 90].

Palev introduced Lie superalgebras in the theory of Wigner quantization [74, 75] and found solutions for many different systems [42, 77, 78]. Many other researchers joined the subject and produced a variety of interesting results [34, 35, 43, 46, 47, 65, 76]. Some recent examples where the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ and their representations take an important role can be found in [55, 56, 61, 84].

Wigner's paper has had a great impact in many different directions of quantum physics, and a lot of complicated Wigner quantum systems have been investigated so far. It is useful to get acquainted with the subject slowly by reconsidering the one-dimensional harmonic oscillator and comparing the Wigner harmonic oscillator to its canonical brother discussed previously.

3.3.2 Solving the 1D Wigner harmonic oscillator

As our very first example of a Wigner quantum system, we will examine the Wigner harmonic oscillator. Coincidentally this was the example that was chosen by Wigner in his famous paper [103]. For us, it is a good system to analyze for different reasons. For one, it is easy to compare with the canonical case given previously in

this chapter. Secondly, the Lie superalgebra solutions that arise in this case are in terms of $\mathfrak{osp}(1|2)$ generators. We know this Lie superalgebra and its representations very well from our classification in Section 2.4.

Consider once again the Hamiltonian of the harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

In the context of Wigner quantization one requires that Hamilton's equations

$$\dot{\hat{p}} = -m\omega^2\hat{x}, \qquad \dot{\hat{x}} = \frac{1}{m}\hat{p}$$

and the Heisenberg equations

$$\dot{\hat{p}} = \frac{i}{\hbar}[\hat{H}, \hat{p}], \qquad \dot{\hat{x}} = \frac{i}{\hbar}[\hat{H}, \hat{x}]$$

are equivalent as operator equations. We have introduced the notation $\dot{\hat{x}}$ and $\dot{\hat{p}}$ for the time derivative of the position and momentum operator respectively. The compatibility conditions for this system can then be written as

$$[\hat{H}, \hat{p}] = i\hbar m\omega^2\hat{x}, \qquad [\hat{H}, \hat{x}] = -\frac{i\hbar}{m}\hat{p}. \tag{3.30}$$

The goal is to find self-adjoint operator solutions for $\hat{x}$ and $\hat{p}$ without making any assumptions about $[\hat{x}, \hat{p}]$. These solutions can then be used to find the spectra of the relevant operators and the position and momentum probability distributions. As in the canonical case, we introduce new unknown operators $a^{\pm}$ as follows:

$$a^{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} \mp i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}. \tag{3.31}$$

The Hamiltonian $\hat{H}$ can then be rewritten in terms of these unknown operators as

$$\hat{H} = \frac{\hbar\omega}{2}\{a^+, a^-\},$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the **anticommutator** of two operators $\hat{A}$ and $\hat{B}$. The compatibility conditions (3.30) are equivalent to $[\hat{H}, a^{\pm}] = \pm\hbar\omega a^{\pm}$, which in turn can be written as

$$[\{a^+, a^-\}, a^{\pm}] = \pm 2a^{\pm}.$$

These relations are recognized as the defining relations (2.16) of the Lie superalgebra $\mathfrak{osp}(1|2)$. So we have a Lie superalgebra solution for our problem, namely $a^{\pm} = b^{\pm}$, where b^{+} and b^{-} are the generating elements of $\mathfrak{osp}(1|2)$. Furthermore, because the operators $\hat{H}$, $\hat{x}$ and $\hat{p}$ need to be self-adjoint, we have the following star conditions:

$$(b^{\pm})^{\dagger} = b^{\mp}.$$

The classification of irreducible $*$-representations of $\mathfrak{osp}(1|2)$ [86] given in Section 2.4 comes into the picture. Proposition 2.26 tells us that there is only one such class of representations. The positive discrete series representations are characterized by a positive parameter a and a lowest weight vector $|0\rangle$ such that

$$b^{-}|0\rangle = 0, \qquad \{b^{+}, b^{-}\}|0\rangle = 2a|0\rangle .$$

Notice that, compared to Proposition 2.26, we will use the notation $|n\rangle = e_n$ for all n in order to be consistent with the bra-ket notation. Moreover, we prefer to characterize our $\mathfrak{osp}(1|2)$ representations by $a = 2\mu$ in the present context. We then have

$$\begin{array}{rcl} b^{+}|2n\rangle & = & \sqrt{2(n+a)}\,|2n+1\rangle \\ b^{-}|2n\rangle & = & \sqrt{2n}\,|2n-1\rangle \\ b^{+}|2n+1\rangle & = & \sqrt{2(n+1)}\,|2n+2\rangle \\ b^{-}|2n+1\rangle & = & \sqrt{2(n+a)}\,|2n\rangle \end{array} \tag{3.32}$$

The action of the Hamiltonian on the basis vectors is easily determined using (3.32). We have

$$\hat{H}|n\rangle = \hbar\omega(n+a)|n\rangle .$$

Thus the energy spectrum is equidistant with spacing $\hbar\omega$, but the lowest energy level depends on the representation parameter a. Comparing this with equation (3.15), we see that the results of the canonical case are retrieved for $a = \frac{1}{2}$. This was already obtained by Wigner in [103]. Another interesting result of Wigner is the calculation of the action of the commutator $[\hat{x}, \hat{p}]$ on the energy eigenstates $|n\rangle$. Using (3.31) one finds

$$[\hat{x}, \hat{p}] = i\hbar[b^{-}, b^{+}]$$

in the $\mathfrak{osp}(1|2)$ solution $a^{\pm} = b^{\pm}$. The actions (3.32) then yield

$$[\hat{x}, \hat{p}]|2n\rangle = 2ai\hbar|2n\rangle , \qquad [\hat{x}, \hat{p}]|2n+1\rangle = 2(1-a)i\hbar|2n+1\rangle .$$

Again, one finds back the canonical commutation relations $[\hat{x}, \hat{p}] = i\hbar$ when $a = \frac{1}{2}$. But the previous relations also show that a lot of non-canonical solutions for the Wigner harmonic oscillator exist.

We will calculate the position probability distributions for the Wigner harmonic oscillator when the system is in an energy eigenstate $|n\rangle$. In the $\mathfrak{osp}(1|2)$ solution we have

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(b^+ + b^-).$$

Together with the relations (3.32) this yields

$$\hat{x}\,|2n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{2n}\,|2n-1\rangle + \sqrt{2(n+a)}\,|2n+1\rangle\right),$$
$$\hat{x}\,|2n+1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{2(n+a)}\,|2n\rangle + \sqrt{2(n+1)}\,|2n+2\rangle\right).$$

Consider a formal eigenvector of the position operator $\hat{x}$ with eigenvalue x and write it as $|x\rangle = \sum_n \alpha_n(x)\,|n\rangle$. Using the same technique as for the canonical harmonic oscillator one uses the eigenvalue equation $\hat{x}\,|x\rangle = x\,|x\rangle$ to find a recurrence relation for the unknown coefficients $\alpha_n(x)$. We obtain

$$\sqrt{\frac{m\omega}{\hbar}}x\,\alpha_{2n}(x) = \sqrt{n}\,\alpha_{2n-1}(x) + \sqrt{n+a}\,\alpha_{2n+1}(x),$$
$$\sqrt{\frac{m\omega}{\hbar}}x\,\alpha_{2n+1}(x) = \sqrt{n+a}\,\alpha_{2n}(x) + \sqrt{n+1}\,\alpha_{2n+2}(x).$$

Comparison with equation (3.23) tells us that we can write the unknown coefficients $\alpha_n(x)$ as generalized Hermite polynomials:

$$\alpha_n(x) = \tilde{T}_n^{(a)}\left(\sqrt{\frac{m\omega}{\hbar}}x\right).$$

Thus we have

$$|x\rangle = \sum_{n=0}^{\infty} \tilde{T}_n^{(a)}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)|n\rangle$$

and using the orthonormality of the normalized polynomials $\tilde{T}_n^{(a)}(x)$

$$|n\rangle = \int_{-\infty}^{+\infty} \sqrt{\frac{m\omega}{\hbar}}\tilde{T}_n^{(a)}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)|x\rangle\,dx.$$

It is again the square of the modulus of the coefficient of $|x\rangle$ in this integral that gives the position probability distribution of the system when it is in an energy

eigenstate $|n\rangle$. All of the results given above are compatible with the canonical case when $a = \frac{1}{2}$.

The wave functions $\sqrt{m\omega/\hbar}\,\tilde{T}_n^{(a)}(\sqrt{m\omega/\hbar}\,x)$ were obtained for the first time in [70]. The authors of that paper rely on specific realizations of the position and momentum operators [72, Chapter 3] to solve the time-independent Schrödinger equation. Here, we have found the same wave function using a technique that can easily be applied to different quantum systems, as we will see in Chapter 6.

3.3.3 Conclusion

It is clear that Wigner quantization, even for a very simple system, leads to a variety of new solutions. In the words of Martin Cederwall [2], we can say that "Wigner quantization draws a whole new set of arrows from classical to quantum mechanics". However, the one-dimensional harmonic oscillator is only a very simple example of a Wigner quantum system. The energy spectrum of the Wigner harmonic oscillator very much resembles that of the canonical case. Indeed, the parameter a only influences the ground energy level, not the discreteness and spacing of the spectrum. But even in this simple case, significant differences can already be perceived. As depicted in Figure 3.2, the wave functions $\sqrt{m\omega/\hbar}\,\tilde{T}_n^{(a)}(\sqrt{m\omega/\hbar}\,x)$ differ considerably as a varies.

In the next chapters, we will consider many different Wigner quantum systems. They vary from quasi-trivial to very complicated or highly unusual. In Chapter 4, we consider a system of harmonic oscillators coupled by springs. For such systems the interaction can be described by an interaction matrix. Our first objective is to find interaction matrices for which the quantum system is exactly solvable. The second part of the chapter is devoted to Wigner quantization of harmonic oscillators coupled by a general interaction matrix.

To illustrate that many interesting open questions remain in Wigner quantization, we consider a $3N$-dimensional harmonic oscillator in Chapter 5 and look for the angular momentum and energy content of this system. Moreover, we try to describe this content with generating functions, a challenging computational question.

In the last chapter of this thesis, we ditch the harmonic oscillator and consider two different one-dimensional systems. The Berry-Keating-Connes Hamiltonian $\hat{H}_b = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ is possibly connected to the Riemann hypothesis and it

[2] Informal conversation, VIII international workshop on Lie theory and its applications in physics, Varna, June 17, 2009.

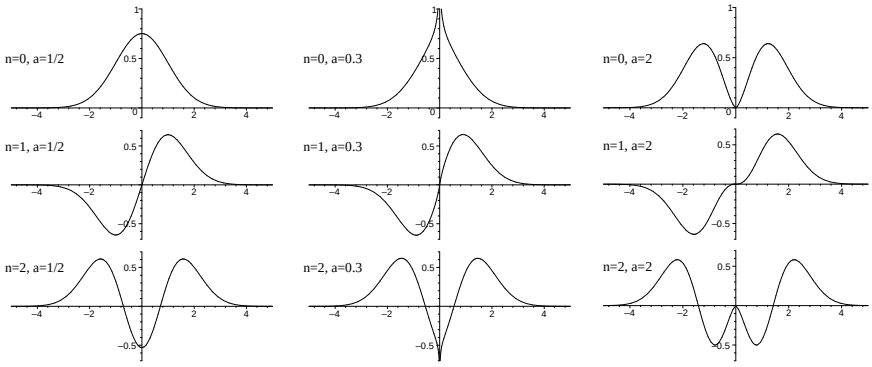

Figure 3.2: Comparison of canonical ($a = 1/2$) and non-canonical ($a = 0.3$ and $a = 2$) position wave functions when the system is in the energy eigenstate $|n\rangle$, with $n = 0, 1, 2$. The left column represents the canonical case. All plots are in units $m = \hbar = \omega = 1$.

is therefore a popular research object. Finally, we consider the simplest possible system, the free particle, which stunningly had not yet been investigated before. It will be pleasing to notice that all of the mentioned systems are very different in many aspects, and that a lot of intriguing properties can be derived for them.

4

Harmonic oscillators coupled by an interaction matrix

Wigner quantization can give rise to many interesting features such as new operator forms for the observables, altered spectra and deformed wave functions. Also, the unexpected connection with Lie superalgebras faces researchers with many new challenges that are at first sight unrelated to quantum mechanics. Understanding the implications of Wigner quantization and the impact of Lie superalgebras and their representations in this context requires the investigation of several quantum systems. The upcoming chapters are entirely devoted to such explorations.

In the previous chapter, emphasis was also placed on the importance of the harmonic oscillator in particular and **analytically solvable quantum systems** in general. These are quantum systems for which all relevant results can be obtained analytically, without using numerical computations. We are therefore steered towards finding new analytically solvable systems involving harmonic oscillators and investigating the Wigner quantization of such systems.

This chapter is quite bulky and can be seen as a composition of two main parts. In the first part, running from Section 4.2 to Section 4.6, we consider quantum systems consisting of a linear chain of n harmonic oscillators coupled by a nearest-neighbour interaction of the form $-\hat{q}_r\hat{q}_{r+1}$, where $\hat{q}_r$ refers to the position of the rth oscillator. We will describe the coupling of the oscillators by means of a real, symmetric and positive definite interaction matrix, which has a *tridiagonal* form reflecting the nearest-neighbour interaction. Solving such a system is always numerically possible and requires the eigenvalues and eigenvectors of the interaction matrix. This implies that the system is analytically solvable whenever a closed ex-

pression for the eigenvalues and eigenvectors of the interaction matrix exists. We will look for such interaction matrices in the context of orthogonal polynomials, thus creating three new types of analytically solvable Hamiltonians: with a Krawtchouk interaction, a Hahn interaction or a q-Krawtchouk interaction.

While the analysis in the first part of this chapter is situated in the context of canonical quantization, Sections 4.7 through 4.10 revolve around Wigner quantization. The Wigner quantum system is first considered for a *general* interaction matrix, and solutions are found in terms of the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$. Afterwards, we determine the spectrum of the Hamiltonian in specific representations of these Lie superalgebras and compare the results for two analytically solvable quantum systems, namely a chain of coupled harmonic oscillators with a constant interaction and a Krawtchouk interaction.

The results of this chapter have been published in [85], [83] and in [84].

4.1 Introduction

It has already been pointed out that harmonic oscillator models have been thoroughly investigated in the past because of their analytic solvability as quantum systems and their numerous applications. Systems of interacting harmonic oscillators are among these well-known models. A system of n one-dimensional harmonic oscillators interacting with each other can be described in its most general form as [19]

$$\hat{H} = \hat{r}^\dagger V \hat{r}.$$

In this equation, $\hat{r}^\dagger$ is the vector $(\hat{p}_1^\dagger, \ldots, \hat{p}_n^\dagger, \hat{q}_1^\dagger, \ldots, \hat{q}_n^\dagger)$, with $\hat{q}_r$ and $\hat{p}_r$ respectively the position and momentum operator of the oscillator at location r. More precisely $\hat{q}_r$ measures the displacement of the rth mass point with respect to its equilibrium position. V is a positive definite matrix describing the coupling in position and momentum coordinates. In the present chapter, we will assume that there is no coupling involving the momentum operators. Following this approach, we can write the Hamiltonian in the following manner:

$$\hat{H} = \frac{1}{2m} \begin{pmatrix} \hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} + \frac{m}{2} \begin{pmatrix} \hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger \end{pmatrix} A \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix}. \tag{4.1}$$

The matrix A is called the **interaction matrix** [38, 44], and it is assumed to be real, symmetric and positive definite. In order to connect the physical context of

harmonic oscillators coupled by 'springs' obeying Hooke's law to this Hamiltonian, we can rewrite A as $\omega^2 I + cM$. All oscillators then have mass m and natural frequency ω, and the **coupling strength** is called c ($c \geq 0$). The $n \times n$ identity matrix is denoted I and M is a general real and symmetric matrix. For $c = 0$ one is simply dealing with a set of identical uncoupled harmonic oscillators.

The system (4.1) will be handled in its most general form in Section 4.7 and onwards. Until then, we will restrict ourselves to tridiagonal interaction matrices, i.e. matrices with nonzero entries only on the diagonal, the subdiagonal and the superdiagonal. The quantum system arising as such is a linear chain of harmonic oscillators with a nearest-neighbour coupling, as shown in Figure 4.1.

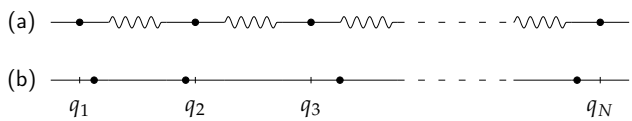

Figure 4.1: A chain of coupled harmonic oscillators. The N masses stand in equilibrium position in (a) and the displacements q_k are shown in (b).

4.1.1 Historical context

In classical mechanics, one-dimensional systems (or lattices) consisting of mass points with some nearest-neighbour interaction have a long history. A typical system is a lattice of n particles with masses $m_1, m_2, \ldots, m_n$, and a harmonic coupling with spring constants $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ leading to the Hamiltonian

$$H(p,x) = \sum_{r=1}^{n} \frac{p_r^2}{2m_r} + \sum_{r=1}^{n-1} \frac{\kappa_r}{2}(x_r - x_{r+1})^2.$$

Such classical systems (or variations, with an infinite number of mass points, or with various boundary conditions) were already considered by Schrödinger [92]. The equations of motion of such a system can be solved by numerically diagonalizing the interaction matrix, the eigenvalues of which yield the normal modes of the system. Alternatively, the system can be solved using orthogonal polynomials whose recurrence relations are derived from the equations of motion [15, 53, 66]. In that case, the normal modes are obtained from the zeros of the nth orthogonal polynomial.

One-dimensional systems with a different type of nearest-neighbour interaction received a lot of attention, especially those that are still exactly solvable. Among the most famous, we mention the Toda system [100] and the Calogero-Sutherland-Moser models [13, 68, 98]. In this context, the emphasis was shifted from physics to mathematical aspects such as integrability and the underlying algebraic structures.

Also the quantum versions of many of these systems or models were investigated from various points of view during the last decades. Quantum Calogero-Moser systems for any root system were studied by Olshanetsky and Perelomov [73]; for a review, see [18]. In such quantum systems, the emphasis – from the physics point of view – is on a construction of ground wave states, formulae for the excitation spectrum, a description of stationary states, etc. Several other quantum systems with a nearest-neighbour interaction closely related to Calogero-Sutherland-Moser models were explored, see [3, 5, 40, 44] to cite a few.

In the present chapter we consider yet another quantum system given in equation (4.1). When the interaction matrix A is considered to be tridiagonal, the system consists of a one-dimensional chain of particles with a certain nearest-neighbour interaction, which is quadratic in the position operators. Our emphasis is on the investigation of analytical solvability of the quantum system, i.e. on obtaining analytically closed expressions of the spectrum of the Hamiltonian and of its eigenstates, the stationary states of the system. In such a context, the physical significance of the interaction introduced here is less clear: it can, in a sense, be considered as a deviation from a vibrating quantum system.

4.1.2 Nearest-neighbour interaction

To introduce the system we will be considering in Sections 4.2 through 4.6, let us first consider one of the most common quantum systems, consisting of a chain of harmonic oscillators coupled by a constant nearest-neighbour interaction [16, 55, 56, 80]. In this popular model the particles are described as identical harmonic oscillators which are moreover coupled by springs obeying Hooke's law. Then the Hamiltonian of the system is given by:

$$\hat{H}_{\mathsf{cst}} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) + \sum_{r=0}^{n} \frac{cm}{2} (\hat{q}_r - \hat{q}_{r+1})^2, \tag{4.2}$$

where $\hat{q}_0 = \hat{q}_{n+1} \equiv 0$, representing fixed wall boundary conditions. In other words, the quantum system consists of a string or chain of n identical harmonic oscillators, each having the same mass m and natural frequency ω. The last term in (4.2)

represents the nearest-neighbour coupling by means of 'springs' with a coupling strength c.

With $\tilde{\omega}^2 = \omega^2 + 2c$, the Hamiltonian (4.2) can be rewritten as

$$\hat{H}_{\text{cst}} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\tilde{\omega}^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} (\hat{q}_r \hat{q}_{r+1} + \hat{q}_{r+1} \hat{q}_r). \tag{4.3}$$

The fact that the nearest-neighbour interaction is the same everywhere in the chain, thus independent of the position r, implies that all coefficients of $\hat{q}_r \hat{q}_{r+1}$ and $\hat{q}_{r+1} \hat{q}_r$ are the same. Note that equation (4.3) is obtained from (4.2) without making assumptions on the commutation relations between the position and momentum operators. Obviously, the expression (4.3) simplifies a lot in the context of canonical quantization, when the operators $\hat{q}_r$ and $\hat{q}_{r+1}$ commute.

The Hamiltonian (4.3) can be written in matrix form as

$$\hat{H}_{\text{cst}} = \frac{1}{2m} \begin{pmatrix} \hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} + \frac{m}{2} \begin{pmatrix} \hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger \end{pmatrix} A_{\text{cst}} \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix}, \tag{4.4}$$

where A_{cst} is a symmetric tridiagonal matrix of the form

$$A_{\text{cst}} = \omega^2 I + c M_{\text{cst}},$$

with I the $n \times n$ identity matrix and

$$M_{\text{cst}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{4.5}$$

We shall refer to A_{cst} as the interaction matrix for constant interaction.

In Sections 4.2 through 4.6, we shall study deviations of (4.3), where the interaction is not constant in the chain but depends on the position r. We will do this in the context of canonical quantization, i.e. the $\hat{q}_r$ and $\hat{p}_r$ are self-adjoint operators ($\hat{q}_r^\dagger = \hat{q}_r$ and $\hat{p}_r^\dagger = \hat{p}_r$) satisfying the commutation relations

$$[\hat{q}_r, \hat{q}_s] = 0, \qquad [\hat{p}_r, \hat{p}_s] = 0, \qquad [\hat{q}_r, \hat{p}_s] = i\hbar\delta_{rs}, \qquad (r, s = 1, \ldots, n). \tag{4.6}$$

In this case the Hamiltonian of our system, consisting of a linear chain of harmonic oscillators coupled by a non-constant nearest-neighbour interaction, can be written as

$$\hat{H} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} \gamma_r \, \hat{q}_r \hat{q}_{r+1}. \tag{4.7}$$

The physical interpretation of such a system is not obvious, because the interaction is in general no longer harmonic. It could be seen as a chain still consisting of identical oscillators but with a quadratic nearest-neighbour interaction of the form $-\hat{q}_r\hat{q}_{r+1}$. This interaction is not homogeneous in the chain but depends on the location r in the linear system. For a more general context in which quantum systems of the form (4.7) appear as a special case, see the notion of 'harmonic systems on general lattices' in [1, 19] in the study of entanglement in many-body systems.

It is well known that the Hamiltonian (4.4) is completely solvable (see also Section 4.2), and the complete energy spectrum can be described using the eigenvalues and eigenvectors of the interaction matrix. In fact, it is analytically solvable in the sense that one has an analytically closed expression for the eigenvalues and eigenvectors of the interaction matrix A_{cst} in (4.4). We wish to investigate more cases for which $\hat{H}$ is analytically solvable. Some examples of Hamiltonians for which this is the case are:

$$\hat{H}_K = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} \sqrt{r(n-r)} \, \hat{q}_r \hat{q}_{r+1},$$

$$\hat{H}_Q^{(1)} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} \frac{\sqrt{(n-r)(n+r+1)}}{2} \, \hat{q}_r \hat{q}_{r+1},$$

$$\hat{H}_{Kq} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} 2\sqrt{q^{r+1-2n}(1-q^r)(1-q^{n-r})} \, \hat{q}_r \hat{q}_{r+1},$$

where q is some positive parameter.

In Section 4.2, we shall consider (4.4) again, and describe a method to solve this Hamiltonian. This method can best be formulated in terms of a general interaction matrix $A = \omega^2 I + cM$, and we shall determine the conditions for M to be analytically solvable. In particular, systems of the form (4.7) correspond to tridiagonal interaction matrices with a constant diagonal. As we shall see, this leads us to the area of discrete orthogonal polynomials. We have investigated the families of discrete orthogonal polynomials that lead to solutions. These are described in

the following sections: an interaction based upon Krawtchouk polynomials (Section 4.3), on Hahn polynomials (Section 4.4), or on dual q-Krawtchouk polynomials (Section 4.5). For each of these cases, we give the corresponding Hamiltonian and its solution. In Section 4.6 we also briefly describe some interesting features of the energy levels of these Hamiltonians.

4.1.3 Wigner quantization

From Section 4.7 onwards we look at the Wigner quantization of a system of harmonic oscillators for which the coupling is described by a general interaction matrix $A = \omega^2 I + cM$. The Hamiltonian of such a system is given by equation (4.1) and can be rewritten explicitly as

$$\hat{H} = \frac{1}{2m} \sum_{r=1}^{n} \hat{p}_r^2 + \frac{m}{2} \sum_{r,s=1}^{n} a_{rs} \hat{q}_r \hat{q}_s,$$

where the elements of the interaction matrix A on position (r,s) are denoted by a_{rs}. The Wigner quantization of this system leads to Lie superalgebra solutions for $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$. A light is shed on both solutions by means of two specific examples: constant interaction and Krawtchouk interaction. We postpone the further analysis to Sections 4.7 through 4.10.

4.2 General method

A general method of solving the Hamiltonian (4.4) in canonical quantization can best be understood by considering a system where the coupling is described by a more general interaction matrix A. A Hamiltonian belonging to such a system is given by equation (4.1), which can be written as

$$\hat{H} = \frac{1}{2m} \begin{pmatrix} \hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} + \frac{m}{2} \begin{pmatrix} \hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger \end{pmatrix} (\omega^2 I + cM) \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix}. \quad (4.8)$$

In (4.8), M is a *real and symmetric* matrix. In order to be physically meaningful, $\omega^2 I + cM$ should be a positive definite matrix [1, 19]. A general method to deal with such Hamiltonians was described in [19, Section 2.1]. Since M is real and symmetric, the spectral theorem [27] implies

$$M = UDU^T \quad (4.9)$$

where D is the diagonal matrix

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{4.10}$$

and U is an orthogonal matrix satisfying

$$UU^T = U^T U = I. \tag{4.11}$$

The entries of the diagonal matrix D are the (real) eigenvalues λ_i of M, in some order, and the columns of the real orthogonal matrix U are eigenvectors of M (in the same order); U^T stands for the transpose of U.

Introducing new operators, the so-called **normal coordinates and momenta**, as follows:

$$\begin{pmatrix} \hat{Q}_1 \\ \vdots \\ \hat{Q}_n \end{pmatrix} = U^T \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix}, \qquad \begin{pmatrix} \hat{P}_1 \\ \vdots \\ \hat{P}_n \end{pmatrix} = U^T \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix}, \tag{4.12}$$

the Hamiltonian (4.8) reads

$$\begin{aligned} \hat{H} &= \frac{1}{2m} \begin{pmatrix} \hat{P}_1^\dagger & \cdots & \hat{P}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{P}_1 \\ \vdots \\ \hat{P}_n \end{pmatrix} + \frac{m}{2} \begin{pmatrix} \hat{Q}_1^\dagger & \cdots & \hat{Q}_n^\dagger \end{pmatrix} (\omega^2 I + cD) \begin{pmatrix} \hat{Q}_1 \\ \vdots \\ \hat{Q}_n \end{pmatrix} \\ &= \frac{1}{2m} \sum_{j=1}^{n} \hat{P}_j^2 + \frac{m}{2} \sum_{j=1}^{n} (\omega^2 + c\lambda_j) \hat{Q}_j^2. \end{aligned} \tag{4.13}$$

By the transformation (4.12), the new operators also satisfy the canonical commutation relations:

$$[\hat{Q}_j, \hat{Q}_k] = 0, \qquad [\hat{P}_j, \hat{P}_k] = 0, \qquad [\hat{Q}_j, \hat{P}_k] = i\hbar\delta_{jk} \qquad (j, k = 1, \ldots, n).$$

In (4.13), the values of $\omega^2 + c\lambda_j$ are all positive since the interaction matrix $\omega^2 I + cM$ is assumed to be positive definite. So one can introduce

$$\omega_j = \sqrt{\omega^2 + c\lambda_j}. \tag{4.14}$$

Then we can write

$$\hat{H} = \frac{1}{2m} \sum_{j=1}^{n} \hat{P}_j^2 + \frac{m}{2} \sum_{j=1}^{n} \omega_j^2 \hat{Q}_j^2.$$

This expression is just like the Hamiltonian of an n-dimensional non-isotropic oscillator, which we have encountered in Section 3.1.2 for $n = 1$. We can use the commonly known method for the solution of the n-dimensional case [69, 106]. Introducing boson operators

$$a_j^{\pm} = \sqrt{\frac{m\omega_j}{2\hbar}}\hat{Q}_j \mp \frac{i}{\sqrt{2\hbar m\omega_j}}\hat{P}_j,$$

these satisfy

$$[a_j^-, a_k^-] = [a_j^+, a_k^+] = 0, \qquad [a_j^-, a_k^+] = \delta_{jk}, \qquad (j,k = 1,\ldots,n)$$

and $\hat{H}$ can be written as

$$\hat{H} = \sum_{j=1}^{n} \frac{\hbar\omega_j}{2}\{a_j^+, a_j^-\} = \sum_{j=1}^{n} \frac{\hbar\omega_j}{2}(2a_j^+ a_j^- + 1).$$

Furthermore,

$$[\hat{H}, a_j^{\pm}] = \pm\hbar\omega_j a_j^{\pm} \qquad (j = 1,\ldots,n).$$

So if we assume that there is a lowest $\hat{H}$-eigenvalue (lowest energy), say for the state $|0\rangle$, then we have the usual n-**boson Fock space** in which the action of $\hat{H}$ is diagonal. The vacuum vector $|0\rangle$ satisfies

$$\langle 0|0\rangle = 1, \qquad a_j^- |0\rangle = 0;$$

the other (orthogonal and normalized) basis vectors are then defined by

$$|k_1,\ldots,k_n\rangle = \frac{(a_1^+)^{k_1}\ldots(a_n^+)^{k_n}}{\sqrt{k_1!\ldots k_n!}}|0\rangle, \qquad (k_j = 0,1,\ldots). \tag{4.15}$$

The spectrum of $\hat{H}$ is now determined by

$$\hat{H}|k_1,\ldots,k_n\rangle = \sum_{j=1}^{n} \hbar\omega_j(k_j + \frac{1}{2})|k_1,\ldots,k_n\rangle. \tag{4.16}$$

This analysis is well known, and it seems to indicate that a Hamiltonian of the form (4.1) with a general interaction matrix $A = \omega^2 I + cM$ is exactly solvable as a quantum system. Note, however, that the solution we have described involves also a

numerical process, namely the determination of the eigenvalues and eigenvectors of M in (4.10) and (4.11). We shall say that the Hamiltonian $\hat{H}$ is **analytically solvable** if we have an analytically closed expression for the eigenvalues and eigenvectors of M (and of A by extension), for arbitrary n.

One example of an analytically solvable Hamiltonian is (4.4), with interaction matrix $A_{\mathsf{cst}} = \omega^2 I + cM_{\mathsf{cst}}$ determined by (4.5). In this case, the decomposition (4.9) is determined by [56, 80]

$$U = \sqrt{\frac{2}{n+1}} \left(\sin(\frac{ij\pi}{n+1}) \right)_{1 \leq i,j \leq n}$$

and the eigenvalues in (4.10) by

$$\lambda_j = 2 - 2\cos\left(\frac{j\pi}{n+1}\right). \tag{4.17}$$

So we have

$$\omega_j^2 = \omega^2 + 2c - 2c\cos\left(\frac{j\pi}{n+1}\right) = \omega^2 + 4c\sin^2\left(\frac{j\pi}{2(n+1)}\right). \tag{4.18}$$

Apparently, there are not so many examples of analytically solvable Hamiltonians of the type (4.1) known in the literature. One paper dealing with this problem (and closely related ones) is [38]. In that paper, some examples of analytically solvable Hamiltonians are given.

In the current chapter, we present some new examples. Since we are studying Hamiltonians of the form (4.7), we are dealing with tridiagonal interaction matrices A with constant entries on the diagonal. For tridiagonal matrices, an explicit spectral decomposition (4.9) can be found by relating these matrices to Jacobi matrices of discrete orthogonal polynomials. So it is natural to look for new examples in that area. We shall first describe the example of 'Krawtchouk interaction', and then indicate how to find other examples.

4.3 Krawtchouk interaction

The Krawtchouk polynomials were already introduced briefly in Chapter 3. In this section, let us first collect some known properties of Krawtchouk polynomials and then use these to describe the spectrum of a Hamiltonian with a Krawtchouk interaction term. For a list of hypergeometric orthogonal polynomials, see [51] or [39].

4.3.1 Krawtchouk polynomials

For a fixed positive integer parameter N and a real parameter $\tilde{p}$ $(0 < \tilde{p} < 1)$, the Krawtchouk polynomial of degree i $(i = 0, 1, \ldots, N)$ in the variable x is defined by [39, 51, 71]

$$K_i(x) \equiv K_i(x; \tilde{p}, N) = {}_2F_1\left(\begin{matrix} -x, -i \\ -N \end{matrix} \middle| \frac{1}{\tilde{p}}\right), \tag{4.19}$$

where, ${}_2F_1$ is the Gauss hypergeometric series defined by equation (3.18). We have used $\tilde{p}$ instead of the usual parameter p for the Krawtchouk polynomials in order to avoid a notation conflict with the $\mathfrak{gl}(1|n)$ and $\mathfrak{osp}(1|2n)$ representations $V(p)$ described in Section 4.9. In (4.19), the series is terminating because one of the numerator parameters is a negative integer. The Krawtchouk polynomials satisfy a discrete orthogonality relation of the form

$$\sum_{x=0}^{N} w(x) K_i(x) K_j(x) = h_i \delta_{ij}, \tag{4.20}$$

where $w(x)$ is a weight function in x given by

$$w(x) = \binom{N}{x} \tilde{p}^x (1-\tilde{p})^{N-x} \qquad (x = 0, 1, \ldots, N)$$

and h_i is a function depending on i:

$$h_i = \frac{1}{\binom{N}{i}} \left(\frac{1-\tilde{p}}{\tilde{p}}\right)^i.$$

The recurrence relation for Krawtchouk polynomials is given by

$$\begin{aligned} -xK_i(x) = \quad & i(1-\tilde{p})\, K_{i-1}(x) \\ & - \left[\tilde{p}(N-i) + i(1-\tilde{p})\right] K_i(x) \\ & + \tilde{p}(N-i)\, K_{i+1}(x). \end{aligned} \tag{4.21}$$

For future purposes we will however be interested in an orthonormality condition, so we define the orthonormal Krawtchouk functions by

$$\tilde{K}_i(x) \equiv \tilde{K}_i(x; \tilde{p}, N) = \frac{\sqrt{w(x)}\, K_i(x)}{\sqrt{h_i}}, \qquad i = 0, 1, 2, \ldots, N.$$

Now we can state the following property:

Lemma 4.1 *Let M_K be the tridiagonal $(N+1)\times(N+1)$-matrix*

$$M_K = \begin{pmatrix} F_0 & -E_1 & 0 & & \\ -E_1 & F_1 & -E_2 & \ddots & \\ 0 & -E_2 & F_2 & \ddots & 0 \\ & \ddots & \ddots & \ddots & -E_N \\ & & 0 & -E_N & F_N \end{pmatrix}, \tag{4.22}$$

where

$$E_i = \sqrt{\tilde{p}(1-\tilde{p})}\sqrt{i(N-i+1)}, \qquad F_i = N\tilde{p} + (1-2\tilde{p})i, \tag{4.23}$$

and let U be the $(N+1)\times(N+1)$-matrix with matrix elements

$$U_{ij} = \tilde{K}_i(j), \qquad (i,j = 0,1,\ldots,N).$$

Then

$$UU^T = U^TU = I \qquad \text{and} \qquad M_K = UDU^T$$

where $D = \mathrm{diag}(0,1,2,\ldots,N)$.

Proof. We have that

$$(UU^T)_{ij} = \sum_{k=0}^{N} U_{ik}U_{jk} = \sum_{k=0}^{N} \tilde{K}_i(k)\tilde{K}_j(k) = \delta_{ij}$$

by the orthogonality relations (4.20). So $UU^T = I$, hence U^T is the inverse of U and thus also $U^TU = I$. Furthermore, notice that

$$E_i = i(1-\tilde{p})\sqrt{\frac{h_{i-1}}{h_i}} \qquad \text{and} \qquad E_{i+1} = \tilde{p}(N-i)\sqrt{\frac{h_{i+1}}{h_i}}.$$

Equation (4.21) can then be rewritten as a recurrence relation for the orthonormal Krawtchouk functions $\tilde{K}_i(x)$:

$$x\tilde{K}_i(x) = -E_i\,\tilde{K}_{i-1}(x) + F_i\,\tilde{K}_i(x) - E_{i+1}\,\tilde{K}_{i+1}(x).$$

Then we have, using $E_0 = E_{N+1} = 0$,

$$\begin{aligned}(M_K U)_{ij} &= \sum_{k=0}^{N} (M_K)_{ik} U_{kj} \\ &= -E_i \tilde{K}_{i-1}(j) + F_i \tilde{K}_i(j) - E_{i+1} \tilde{K}_{i+1}(j) \\ &= j \tilde{K}_i(j) \\ &= (UD)_{ij},\end{aligned}$$

so $M_K U = UD$ or $M_K = UDU^T$. □

So we now have a good candidate interaction matrix using the matrix M_K. In order to describe systems of the form (4.7), however, the diagonal entries F_i of M_K should be constants (i.e. independent of i). We see from (4.23) that this is the case for $\tilde{p} = 1/2$. So this leads to a new analytically solvable Hamiltonian of the form (4.7).

4.3.2 Hamiltonian with Krawtchouk interaction

Consider a linear chain of n identical harmonic oscillators, with a nearest-neighbour interaction that is given by

$$\hat{H}_K = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) - \frac{cm}{2} \sum_{r=1}^{n-1} \sqrt{r(n-r)}\, \hat{q}_r \hat{q}_{r+1}. \tag{4.24}$$

We shall refer to the interaction term as "Krawtchouk interaction". The purpose is to find the analytic solution for the spectrum of H_K. It is easy to see that this Hamiltonian can be written in matrix form, like (4.1):

$$\hat{H}_K = \frac{1}{2m} \begin{pmatrix} \hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} + \frac{m}{2} \begin{pmatrix} \hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger \end{pmatrix} A_K \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix},$$

where A_K is given by

$$A_K = \left(\omega^2 - \frac{c(n-1)}{2} \right) I + cM_K \tag{4.25}$$

and M_K is the matrix (4.22) with $N = n-1$ and $p = 1/2$. For the matrix M_K we have an explicit spectral decomposition, given in Lemma 4.1, and the eigenvalues

of M_K are given by $0, 1, \ldots, n-1$. Using this result, and following the general procedure described in Section 4.2, one introduces here the following quantities:

$$\omega_j = \sqrt{\omega^2 - \frac{c(n-1)}{2} + c(j-1)} = \sqrt{\omega^2 - \frac{c}{2}(n-2j+1)}, \tag{4.26}$$

with $j = 1, 2, \ldots, n$. The interaction matrix $(\omega^2 - \frac{c(n-1)}{2})I + cM_K$ is positive definite if all quantities under the square root symbol are positive. Since c and ω^2 are positive, $\omega^2 - \frac{c(n-1)}{2} + c(j-1)$ is an increasing sequence as j increases, with $j = 1, 2, \ldots, n$. So this condition leads to $c < 2\omega^2/(n-1)$, or the "coupling strength" should be sufficiently small. Now we have:

Proposition 4.2 *The Hamiltonian $\hat{H}_K$ given by (4.24) is analytically solvable. The explicit spectrum of $\hat{H}_K$ follows from (4.16):*

$$\hat{H}_K \left| k_1, \ldots, k_n \right\rangle = \sum_{j=1}^{n} \hbar\omega_j (k_j + \frac{1}{2}) \left| k_1, \ldots, k_n \right\rangle, \tag{4.27}$$

where the constants ω_j are given by $\omega_j = \sqrt{\omega^2 - c(n-2j+1)/2}$.

Finally, notice that the interaction term in equation (4.24) is invariant under the reflection $r \to n - r$.

4.3.3 Remark

It is clear that the general procedure worked out here for the Krawtchouk polynomials works in general for discrete orthogonal polynomials. So in order to find other interesting examples, one can go through the list of discrete orthogonal polynomials [39, 51, 71] and their q-analogues. The basic restriction, in order to have Hamiltonians of the form (4.7), is that the diagonal elements in the interaction matrix are constant (for specific values of the parameters). An investigation of this restriction has shown that, apart from the Krawtchouk polynomials with $\tilde{p} = 1/2$, only the following cases are to be considered: the Hahn polynomials with $\beta = \alpha$ and the dual q-Krawtchouk polynomials with $\bar{c} = -1$. We shall now study these cases and the corresponding Hamiltonians.

4.4 Hahn interaction

4.4.1 Hahn polynomials

The Hahn polynomial of degree i ($i = 0, 1, \ldots, N$) in the variable x is defined by [39, 51]

$$Q_i(x) \equiv Q_i(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -i, i+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix} \middle| 1 \right)$$

where ${}_3F_2$ is the hypergeometric series which is terminating here due to the numerator parameter $-i$. The Hahn polynomials satisfy a discrete orthogonality relation:

$$\sum_{x=0}^{N} w(x) Q_i(x) Q_j(x) = h_i \delta_{ij}, \tag{4.28}$$

where

$$w(x) = \binom{\alpha+x}{x} \binom{N+\beta-x}{N-x},$$

with $x = 0, 1, \ldots, N$, and

$$h_i = \frac{i!(N-i)!}{(N!)^2} \frac{(i+\alpha+\beta+1)_{N+1} (\beta+1)_i}{(2i+\alpha+\beta+1)(\alpha+1)_i}.$$

The recurrence relation for Hahn polynomials is given by

$$-x Q_i(x) = A_i\, Q_{i+1}(x) - (A_i + C_i)\, Q_i(x) + C_i\, Q_{i-1}(x), \tag{4.29}$$

where

$$A_i = \frac{(i+\alpha+\beta+1)(i+\alpha+1)(N-i)}{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}$$

and

$$C_i = \frac{i(i+\alpha+\beta+N+1)(i+\beta)}{(2i+\alpha+\beta)(2i+\alpha+\beta+1)}.$$

The orthonormal Hahn functions are defined by

$$\tilde{Q}_i(x) = \frac{\sqrt{w(x)}\, Q_i(x)}{\sqrt{h_i}}, \qquad i = 0, 1, 2, \ldots, N.$$

Then we have a similar lemma as Lemma 4.1:

Lemma 4.3 *Let M_Q be the tridiagonal $(N+1)\times(N+1)$-matrix*

$$M_Q = \begin{pmatrix} F_0 & -E_1 & 0 & & \\ -E_1 & F_1 & -E_2 & \ddots & \\ 0 & -E_2 & F_2 & \ddots & 0 \\ & \ddots & \ddots & \ddots & -E_N \\ & & 0 & -E_N & F_N \end{pmatrix}, \tag{4.30}$$

where

$$E_i = \sqrt{\frac{i(i+\alpha)(i+\beta)(i+\alpha+\beta)(i+\alpha+\beta+N+1)(N-i+1)}{(2i+\alpha+\beta)^2(2i+\alpha+\beta-1)(2i+\alpha+\beta+1)}}$$

and

$$F_i = \frac{N}{2} + \frac{(\alpha-\beta)\big[(\alpha+\beta)(N-2i)-2i(i+1)\big]}{2(2i+\alpha+\beta)(2i+\alpha+\beta+2)}, \tag{4.31}$$

and let U be the $(N+1)\times(N+1)$-matrix with matrix elements

$$U_{ij} = \tilde{Q}_i(j)$$

where $i,j = 0,1,\ldots,N$. Then

$$UU^T = U^TU = I \qquad \text{and} \qquad M_Q = UDU^T$$

where $D = diag(0,1,2\ldots,N)$.

Proof. The proof is essentially the same as that of Lemma 4.1:

$$(UU^T)_{ij} = \sum_{k=0}^{N} U_{ik}U_{jk} = \sum_{k=0}^{N} \tilde{Q}_i(k)\tilde{Q}_j(k) = \delta_{ij}$$

by the orthogonality relations (4.28), hence $UU^T = U^TU = I$. Furthermore, equation (4.29) can then be rewritten as a recurrence relation for the orthonormal Hahn functions $\tilde{Q}_i(x)$:

$$x\tilde{Q}_i(x) = -E_i\tilde{Q}_{i-1}(x) + F_i\tilde{Q}_i(x) - E_{i+1}\tilde{Q}_{i+1}(x),$$

and this implies $M_Q U = UD$ or $M_Q = UDU^T$. □

It remains to be determined when the diagonal of (4.30) is constant, in other words when F_i is independent of i. Following (4.31), this happens when $\beta = \alpha$. Note that in that case the parameter should satisfy $\alpha > -1$ or $\alpha < -N$. It is worthwhile mentioning that there are two other cases where the diagonal of (4.30) is "almost constant":

- When $\beta = -\alpha$ (with $-1 < \alpha < 1$) one finds that all $F_i = (N-\alpha)/2$ for $i = 1, 2, \ldots, N$, but $F_0 = N(\alpha+1)/2$.
- When $\beta = -2N-2-\alpha$ (with $-2-N < \alpha < -N$) one finds that all $F_i = -(\alpha+1)/2$ for $i = 0, 1, \ldots, N-1$, but $F_N = N(N+\alpha+2)/2$.

For these cases, it could still be interesting to consider the Hamiltonian (4.1) built from the corresponding interaction matrix. The Hamiltonian, however, is not of the form (4.7) as either the first or last oscillator in the chain would play a special role and give rise to an extra term (either in $\hat{q}_1^2$ or else in $\hat{q}_n^2$); so the chain would no longer consist of identical oscillators, but have one of them different from the others.

The Hamiltonians corresponding to $\beta = \alpha$ will now be considered in the next subsection.

4.4.2 Hamiltonian with Hahn interaction

Consider a linear chain of n identical harmonic oscillators with a nearest-neighbour interaction given by

$$\begin{aligned} \hat{H}_Q = & \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) \\ & - \frac{cm}{2} \sum_{r=1}^{n-1} \sqrt{\frac{r(n-r)(r+2\alpha)(r+2\alpha+n)}{(2r+2\alpha-1)(2r+2\alpha+1)}}\, \hat{q}_r \hat{q}_{r+1}, \end{aligned} \tag{4.32}$$

where the interaction term is to be referred to as the "Hahn interaction", and where α is some parameter with $\alpha > -1$ or $\alpha < -n+1$. The restriction on α guarantees that the expression under the square root is positive. This Hamiltonian can be

written in matrix form:

$$\hat{H}_Q = \frac{1}{2m}\begin{pmatrix}\hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger\end{pmatrix}\begin{pmatrix}\hat{p}_1\\ \vdots \\ \hat{p}_n\end{pmatrix} + \frac{m}{2}\begin{pmatrix}\hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger\end{pmatrix} A_Q \begin{pmatrix}\hat{q}_1\\ \vdots \\ \hat{q}_n\end{pmatrix},$$

where A_Q is given by

$$A_Q = \left(\omega^2 - \frac{c(n-1)}{2}\right) I + cM_Q$$

and M_Q is the matrix (4.30) with $\beta = \alpha$ and $N = n-1$. Since the diagonal matrix D in the spectral decomposition of M_Q is again $\mathrm{diag}(0, 1, 2 \ldots, n-1)$, it follows that:

Proposition 4.4 *The Hamiltonian $\hat{H}_Q$ given by (4.32) is analytically solvable. The spectrum of $\hat{H}_Q$ is exactly the same as that of $\hat{H}_K$, and given by (4.27) and (4.26).*

Also the condition for positive definiteness is the same as in the Krawtchouk case, namely $c < 2\omega^2/(n-1)$. Note that for $\alpha = 1/2$ the form of the interaction term is considerably simpler:

$$\hat{H}_Q^{(1)} = \sum_{r=1}^{n}\left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2}\hat{q}_r^2\right) - \frac{cm}{2}\sum_{r=1}^{n-1}\frac{\sqrt{(n-r)(n+r+1)}}{2}\hat{q}_r\hat{q}_{r+1}.$$

Observe that for $\alpha \to +\infty$, the interaction (4.32) reduces to the Krawtchouk interaction (4.24). Also note that under the reflection $r \to n-r$ in the interaction term, the Hamiltonian reads

$$\hat{H}_Q^{(2)} = \sum_{r=1}^{n}\left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2}\hat{q}_r^2\right) - \frac{cm}{2}\sum_{r=1}^{n-1}\frac{\sqrt{r(2n-r+1)}}{2}\hat{q}_r\hat{q}_{r+1}.$$

This can also be obtained by taking $\alpha = -n - \frac{1}{2}$ in the general Hamiltonian with Hahn interaction (4.32).

4.5 q-Krawtchouk interaction

4.5.1 The dual q-Krawtchouk polynomials

For a fixed positive integer parameter N, and real parameters $q > 0$ and $\bar{c} < 0$, the dual q-Krawtchouk polynomial of degree i in the variable $\lambda(x) = q^{-x} + \bar{c}q^{x-N}$ is defined by [39, 51]

$$K_i(\lambda;q) \equiv K_i(\lambda(x);\bar{c},N|q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-i}, q^{-x}, \bar{c}q^{x-N} \\ q^{-N}, 0 \end{matrix}\right| q;q\right),$$

where ${}_3\phi_2$ is the q-hypergeometric series (which is terminating here due to the numerator parameter q^{-i}). In standard literature, the parameter $\bar{c}$ is usually denoted by c, but we replace it by $\bar{c}$ in order not to confuse with the notation for the coupling constant c. The dual q-Krawtchouk polynomials satisfy the discrete orthogonality relation:

$$\sum_{x=0}^{N} w(x) K_i(\lambda(x);q)\, K_j(\lambda(x);q) = h_i \delta_{ij},$$

where

$$w(x) = \frac{(\bar{c}q^{-N};q)_x\,(q^{-N};q)_x\,(1-\bar{c}q^{2x-N})}{(q;q)_x\,(\bar{c}q;q)_x\,(1-\bar{c}q^{-N})}\,\bar{c}^{(-x)}q^{x(2N-x)},$$

with $x = 0, 1, \ldots, N$, and

$$h_i = (\bar{c}^{-1};q)_N\,\frac{(q;q)_i}{(q^{-N};q)_i}(\bar{c}q^{-N})^i.$$

The recurrence relation for dual q-Krawtchouk polynomials is given by

$$\begin{aligned} -(1-q^{-x})(1-\bar{c}q^{x-N})K_i(\lambda;q) = \quad & \bar{c}q^{-N}(1-q^i)K_{i-1}(\lambda;q) \\ & - \left[(1-q^{i-N}) + \bar{c}q^{-N}(1-q^i)\right]K_i(\lambda;q) \\ & + (1-q^{i-N})K_{i+1}(\lambda;q), \end{aligned}$$

The orthonormal dual q-Krawtchouk functions are defined by

$$\tilde{K}_i(\lambda;q) = \frac{\sqrt{w(x)}\,K_i(\lambda;q)}{\sqrt{h_i}}, \qquad i = 0, 1, 2, \ldots, N.$$

Then we have the familiar lemma:

Lemma 4.5 *Let M_{Kq} be the tridiagonal $(N+1)\times(N+1)$-matrix*

$$M_{Kq} = \begin{pmatrix} F_0 & -E_1 & 0 & & \\ -E_1 & F_1 & -E_2 & \ddots & \\ 0 & -E_2 & F_2 & \ddots & 0 \\ & \ddots & \ddots & \ddots & -E_N \\ & & 0 & -E_N & F_N \end{pmatrix}, \tag{4.33}$$

where

$$E_i = \sqrt{\bar{c}q^{-N}(1-q^i)(1-q^{i-1-N})}, \quad F_i = (1-q^{i-N}) + \bar{c}q^{-N}(1-q^i),$$

and let U be the $(N+1)\times(N+1)$-matrix with matrix elements

$$U_{ij} = \tilde{K}_i(\lambda(j);q)$$

where $i,j = 0,1,\ldots,N$. Then

$$UU^T = U^TU = I \qquad \text{and} \qquad M_{Kq} = UDU^T$$

where $D = diag((1-q^{-j})(1-\bar{c}q^{j-N}))$, $(j = 0,1,2\ldots,N)$.

The proof is the same as those given for the ordinary Krawtchouk polynomials and the Hahn polynomials. In it, the orthonormality of the normalized dual q-Krawtchouk functions $\tilde{K}_i(\lambda(j);q)$ is used. The recurrence relation for the $\tilde{K}_i(\lambda(j);q)$ is given by

$$\begin{aligned}(1-q^{-x})(1-\bar{c}q^{x-N})\tilde{K}_i(\lambda(x);q) = &- E_i\,\tilde{K}_{i-1}(\lambda(x);q) \\ &+ F_i\,\tilde{K}_i(\lambda(x);q) \\ &- E_{i+1}\,\tilde{K}_{i+1}(\lambda(x);q).\end{aligned}$$

This case is interesting because for $\bar{c} = -1$ the diagonal of (4.33) is constant, in other words then F_i is independent of i. This leads again to a Hamiltonian of the type (4.7).

4.5.2 Hamiltonian with dual q-Krawtchouk interaction

Now we consider a linear chain of n identical harmonic oscillators with a nearest-neighbour interaction given by

$$\begin{aligned}\hat{H}_{Kq} = & \sum_{r=1}^{n}\left(\frac{\hat{p}_r^2}{2m}+\frac{m\omega^2}{2}\hat{q}_r^2\right)\\ & -\frac{cm}{2}\sum_{r=1}^{n-1}2\sqrt{q^{r+1-2n}(1-q^r)(1-q^{n-r})}\,\hat{q}_r\hat{q}_{r+1},\end{aligned} \tag{4.34}$$

the interaction term to be referred to as "dual q-Krawtchouk interaction", where $q>0$. This Hamiltonian can be written in matrix form:

$$\hat{H}_Q=\frac{1}{2m}\begin{pmatrix}\hat{p}_1^\dagger & \cdots & \hat{p}_n^\dagger\end{pmatrix}\begin{pmatrix}\hat{p}_1\\ \vdots\\ \hat{p}_n\end{pmatrix}+\frac{m}{2}\begin{pmatrix}\hat{q}_1^\dagger & \cdots & \hat{q}_n^\dagger\end{pmatrix}A_{Kq}\begin{pmatrix}\hat{q}_1\\ \vdots\\ \hat{q}_n\end{pmatrix},$$

where A_{Kq} is given by

$$A_{Kq}=\left(\omega^2-c(1-q^{1-n})\right)I+cM_{Kq}$$

and M_{Kq} is the matrix (4.33) with $\bar{c}=-1$ and $N=n-1$. It follows that the spectrum of $\hat{H}_{Kq}$ is given by (4.27), with

$$\begin{aligned}\omega_j&=\sqrt{\omega^2-c(1-q^{1-n})+c(1-q^{-j})(1+q^{j-n+1})}\\ &=\sqrt{\omega^2+c(q^{j-n+1}-q^{-j})},\end{aligned} \tag{4.35}$$

with $j=1,2,\ldots,n$. For positive definiteness of the interaction matrix, all quantities under the square root must be positive. It is easy to see that $\omega^2+c(q^{j-n+1}-q^{-j})$ $(j=1,2,\ldots,n)$ is an increasing sequence of j when $q>1$ and a decreasing sequence of j when $q<1$. So for $q>1$ the condition means $c<q\omega^2/(1-q^{3-n})$, while for $0<q<1$ we need $c<q^n\omega^2/(1-q^{n+1})$. To conclude, we have

Proposition 4.6 *The Hamiltonian $\hat{H}_{Kq}$ given by (4.34) is analytically solvable. The explicit spectrum of $\hat{H}_{Kq}$ is given by (4.16), where the constants ω_j are given by $\omega_j=\sqrt{\omega^2+c(q^{j-n+1}-q^{-j})}$.*

4.6 Some properties of the spectra

The spectrum of each of the Hamiltonians given here is of the form (4.16), thus it is discrete but infinite dimensional. In order to appreciate the differences of the various examples given here, we shall plot the energy levels of the **singly excited states** (the single phonons, or the simple vibrations) of the system. These are the levels of the n states $|1,0,\ldots,0\rangle$, $|0,1,0,\ldots,0\rangle$, $\ldots$, $|0,\ldots,0,1\rangle$ (in the notation of (4.16)). So, following (4.16), these levels are given by

$$E_0+\hbar\omega_1,\ E_0+\hbar\omega_2,\ldots,\ E_0+\hbar\omega_n,$$

where

$$E_0=\frac{1}{2}\sum_{j=1}^{n}\hbar\omega_j.$$

In order to illustrate the spacing of the energy levels of the singly excited states, it is sufficient to plot the values of $(\omega_1,\omega_2,\ldots,\omega_n)$. We plot these values in Figure 4.2, for $n=12$, in four different cases:

(a) The Hamiltonian (4.3) with constant nearest-neighbour interaction, where the values ω_j are given by (4.18), plotted in Figure 4.2(a).

(b) The Hamiltonian with Krawtchouk interaction (4.24) or with Hahn interaction (4.32), which have the same spectrum and where the values ω_j are given by (4.26), plotted in Figure 4.2(b).

(c) The Hamiltonian with q-Krawtchouk interaction (4.34) where $q>1$, for which the ω_j are given by (4.35), plotted in Figure 4.2(c).

(d) The same Hamiltonian (4.34) but with $q<1$, for which the ω_j are also given by (4.35), plotted in Figure 4.2(d).

The values of c and q are appropriately chosen (see the figure caption for actual values) in order to illustrate the typical energy level spacing properties for each case.

For a Hamiltonian with constant nearest-neighbour interaction like (4.2) or (4.3), the levels are wider apart in the middle of the spectrum, and closer to each other near the top and the bottom of the spectrum, see Figure 4.2(a). The property is known, and was e.g. also observed in [38]. For a Hamiltonian with a Krawtchouk interaction or a Hahn interaction like (4.24) or (4.32), the energy level spacing decreases as the energy increases, a phenomenon also typical for molecular spectra, see Figure 4.2(b).

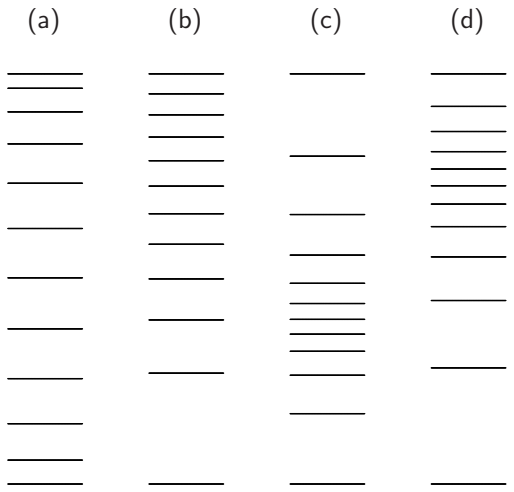

Figure 4.2: Energy levels of the n single phonon states $|0,\ldots,0,1,0,\ldots,0\rangle$, for $n = 12$ and $\hbar = \omega = 1$. The four cases correspond to a Hamiltonian with:
(a) constant nearest-neighbour interaction (4.3), for $c = 0.5$;
(b) Krawtchouk interaction (4.24) or Hahn interaction (4.32), for $c = 0.18$;
(c) q-Krawtchouk interaction (4.34), for $q = 1.6$ and $c = 1.0$;
(d) q-Krawtchouk interaction (4.34), for $q = 0.7$ and $c = 0.01$.
The levels are rescaled, so that the lowest and highest levels match in the four cases.

Finally, for a Hamiltonian with a q-Krawtchouk interaction like (4.34), the energy level properties depend on whether $0 < q < 1$ or $q > 1$. For $q > 1$, one observes just the opposite of a constant nearest-neighbour interaction: the energy level spacing is small near the middle of the spectrum, and larger near the top and bottom of the spectrum (but they are wider apart near the top than near the bottom), see Figure 4.2(c). For $0 < q < 1$, the levels behave similarly: the energy level spacing is small near the middle of the spectrum, and larger near the top and bottom of the spectrum, but now they are wider apart near the bottom than near the top, see Figure 4.2(d). It should be noted that for the cases (a), (b) and (c) the order of the levels from bottom to top correspond to the states $|1,0,\ldots,0\rangle$, $|0,1,\ldots,0\rangle$, $\ldots$, $|0,0,\ldots,1\rangle$ in this order, whereas for (d) it is just the opposite. This is related to the fact that the sequence ω_j $(j = 1,2,\ldots,n)$ is an increasing sequence in the

cases (a), (b) and (c), but a decreasing sequence in the case (d) (see the remark following equation (4.35)).

Our original interest in Hamiltonians with a nearest-neighbour interaction of the form (4.2) stems from the fact that this Hamiltonian can also be quantized as a Wigner quantum system, leading in particular to finite spectra and non-commutative coordinates [46, 47, 55, 56, 77, 78]. Our next aim is to investigate when Hamiltonians with a general interaction matrix, of the form (4.7), can still be solved as a Wigner quantum system, and investigate the properties of such a solution.

4.7 The Wigner quantization procedure

For the second major part of this chapter, we will consider systems of the type (4.1) again, but this time we will work in the framework of Wigner quantization. After the one-dimensional Wigner harmonic oscillator of Section 3.3, the quantum system of n coupled one-dimensional harmonic oscillators will be the second system that we treat as a Wigner quantum system. We will demonstrate how the analysis of the spectrum of this system is connected to the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$. The Wigner quantization of these coupled harmonic oscillators deviates from its standard quantum mechanical counterpart by a certain parameter, corresponding to the parameter characterizing unitary irreducible representations of these Lie superalgebras. The representations that we will consider in this chapter are characterized by a parameter p and $\mathfrak{p}$ respectively. For $\mathfrak{osp}(1|2n)$ one finds back the canonical case by choosing $\mathfrak{p} = 1$.

Already in [55] a system of coupled harmonic oscillators with periodic boundary conditions has been studied, where solutions for the position and momentum operators are found in terms of generators of the Lie superalgebra $\mathfrak{gl}(1|n)$. Analysis of the properties of this quantum system has been done in a Fock type representation space of $\mathfrak{gl}(1|n)$ and the authors found a discrete and finite spectrum of the coordinate and energy operators. Another quantum system consisting of coupled harmonic oscillaters with a fixed wall boundary condition has been the subject of investigation in [56]. Here the authors present solutions in another class of representation spaces of $\mathfrak{gl}(1|n)$, called the ladder representations. These systems of coupled harmonic oscillators, however, correspond to specific interaction matrices A. We wish to extend the performed analysis to a general interaction matrix and investigate properties of the system in specific representations of $\mathfrak{gl}(1|n)$ and $\mathfrak{osp}(1|2n)$.

First we will translate the Wigner quantization problem into a different form,

which is connected to Lie superalgebras. This analysis works for an interaction matrix in its most general form. Next, the problem is restated in terms of Lie superalgebra generators after which one can determine the actual spectrum in specific representations of the Lie superalgebras $\mathfrak{gl}(1|n)$ and $\mathfrak{osp}(1|2n)$. This is done in Section 4.9, where the reader is supplied with general formulae to determine the spectrum and a detailed analysis including some plots. The results up to this point are exemplified by the Hamiltonians with constant interaction and Krawtchouk interaction. Finally, we go back to the known canonical quantization and find connections with the results from the previous sections.

We start with the Hamiltonian (4.1), which can be rewritten as

$$\hat{H} = \frac{1}{2m}\sum_{r=1}^{n}\hat{p}_r^2 + \frac{m}{2}\sum_{r,s=1}^{n}a_{rs}\hat{q}_r\hat{q}_s, \tag{4.36}$$

where a_{rs} denotes the element on position (r,s) of the interaction matrix A. We also assume that the position and momentum operators are self-adjoint, that is $\hat{q}_r^\dagger = \hat{q}_r$ and $\hat{p}_r^\dagger = \hat{p}_r$. Instead of imposing the canonical commutation relations (4.6), we just require the equivalence of the Hamilton equations and the Heisenberg equations. The resulting compatibility conditions, applied for the Hamiltonian (4.36), become

$$[\hat{H},\hat{q}_r] = -\frac{i\hbar}{m}\,\hat{p}_r, \qquad [\hat{H},\hat{p}_r] = i\hbar m\sum_{s=1}^{n}a_{rs}\hat{q}_s, \tag{4.37}$$

with $r = 1,2,\ldots,n$. We are now looking for operator solutions for $\hat{q}_r$ and $\hat{p}_r$ satisfying the compatibility conditions (4.37), with Hamiltonian (4.36). In analogy with the canonical case, we will write the interaction matrix as

$$A = \omega^2 I + cM. \tag{4.38}$$

Although the Wigner quantization procedure presented here does not require this form, we impose it because the procedure of Section 4.2 can then simply be copied here. For example, one can rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{n}\hat{P}_j^2 + \frac{m}{2}\sum_{j=1}^{n}\omega_j^2\,\hat{Q}_j^2 \tag{4.39}$$

using the spectral decomposition of M and the definition (4.14) of ω_j, i.e.

$$\omega_j = \sqrt{\omega^2 + c\lambda_j}.$$

In essence, ω_j is the square root of the jth eigenvalue of the interaction matrix A, which clarifies how the procedure given here can be done for a general interaction matrix.

The operators $\hat{P}_j$ and $\hat{Q}_j$ are the normal coordinates and momenta (4.12), but in this case they do not satisfy the canonical commutation relations, just like the operators $\hat{p}_r$ and $\hat{q}_r$. Instead, the compatibility conditions translate into

$$[\hat{H}, \hat{Q}_j] = -\frac{i\hbar}{m}\,\hat{P}_j, \qquad [\hat{H}, \hat{P}_j] = i\hbar m\omega_j^2\,\hat{Q}_j. \tag{4.40}$$

This can be obtained by substituting the transformations (4.12) in the compatibility conditions (4.37).

It turns out that we will be able to find solutions for $\hat{Q}_j$ and $\hat{P}_j$ satisfying the compatibility conditions (4.40) and the Hamiltonian in equation (4.39) in terms of Lie superalgebra generators. The easiest way to establish such a result, is again by introducing boson operators

$$a_j^{\pm} = \sqrt{\frac{m\omega_j}{2\hbar}}\,\hat{Q}_j \mp \frac{i}{\sqrt{2\hbar m\omega_j}}\,\hat{P}_j.$$

In terms of the operators $a_j^{\pm}$, which satisfy the adjointness relations $(a_j^{\pm})^\dagger = a_j^{\mp}$, the Hamiltonian (4.39) can be rewritten as

$$\hat{H} = \sum_{j=1}^{n} \frac{\hbar\omega_j}{2}\{a_j^+, a_j^-\} \tag{4.41}$$

Again, we need to have the compatibility conditions in terms of the newly introduced operators. These follow from (4.40) and are

$$\left[\hat{H}, a_j^{\pm}\right] = \pm\hbar\omega_j\, a_j^{\pm}, \qquad (j = 1, 2, \ldots, n). \tag{4.42}$$

Thus we have:

Theorem 4.7 *The Wigner quantization of the system (4.36) has been reduced to the problem of finding $2n$ operators $a_j^{\pm}$ $(j = 1, \ldots, n)$ acting in a certain Hilbert space. These operators must satisfy $(a_j^{\pm})^\dagger = a_j^{\mp}$ and*

$$\sum_{j=1}^{n} \left[\omega_j \{a_j^+, a_j^-\}, a_k^{\pm}\right] = \pm 2\omega_k\, a_k^{\pm}, \qquad (k = 1, 2, \ldots, n). \tag{4.43}$$

The Wigner quantization procedure is reversible, so that the knowledge of the operators $a_j^{\pm}$ allows us to reconstruct the observables $\hat{p}_r$ and $\hat{q}_r$. The Hamiltonian is given by equation (4.41).

Equation (4.43) is equivalent to a quantum system describing an n-dimensional non-isotropic oscillator [58, Section 2]. For such systems, it is known that solutions in terms of Lie superalgebra generators exist [58]. Some specific solutions are related to the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$, but not all solutions are known for $n > 1$. We will focus on these two solutions and investigate the spectrum of our system in representations of these Lie superalgebras. However, before moving on to this analysis, we will give some explicit examples of interaction matrices.

The Wigner quantization procedure only requires the spectral decomposition of the interaction matrix. Since we assume that A is real and symmetric, the spectral theorem is always applicable. Hence, for a given interaction matrix, the Wigner quantization procedure always works as above. However, the Hamiltonian is not necessarily analytically solvable for a general interaction matrix A. We will consider two examples of analytically solvable quantum systems for comparison in the next sections, namely the Hamiltonians with constant interaction and Krawtchouk interaction.

The constant interaction matrix can be written as $A_{\mathsf{cst}} = \omega^2 I + M_{\mathsf{cst}}$, where the matrix M_{cst} is given by equation (4.5). The corresponding Hamiltonian is found in (4.2) and the values ω_j are determined by equation (4.18).

We will slightly adapt the Krawtchouk interaction matrix A_K as it was defined in equation (4.25). Our new Krawtchouk interaction matrix $A_{K'}$ will be of the same form as A_{cst}, namely

$$A_{K'} = \omega^2 I + cM_K,$$

where M_K is the matrix defined by (4.22). This new interaction matrix simply amounts to a rescaling of the natural frequency ω in (4.25), but it is useful for future purposes to write the interaction matrix A in a standard form like (4.38). The interaction matrix $A_{K'}$ brings forth the Hamiltonian

$$\begin{aligned}\hat{H}_{K'} = \frac{1}{2m}\sum_{r=1}^{n}\hat{p}_r^2 + \frac{m}{2}\sum_{r=1}^{n}\left(\omega^2 + \frac{c\,(n-1)}{2}\right)\hat{q}_r^2 \\ - \frac{cm}{4}\sum_{r=1}^{n-1}\sqrt{r(n-r)}\,(\hat{q}_r\hat{q}_{r+1} + \hat{q}_{r+1}\hat{q}_r),\end{aligned}$$

where the parameter $\tilde{p}$ for the Krawtchouk polynomials has been set to $1/2$. The Wigner quantization of the Hamiltonian with Krawtchouk interaction reduces to finding $2n$ operators $a_j^{\pm}$ that are subject to $(a_j^{\pm})^\dagger = a_j^{\mp}$ and equation (4.43), with

$$\omega_j^2 = \omega^2 + c\lambda_j = \omega^2 + c(j-1) \qquad (j = 1, 2, \ldots, n).$$

These two leading examples will come back throughout the rest of this chapter. Finding operator solutions for $a_j^{\pm}$, however, can be done for a general Hamiltonian of the form (4.1). We will see that the Lie superalgebras $\mathfrak{gl}(1|n)$ and $\mathfrak{osp}(1|2n)$ introduced in Section 2.2 are the key to solving the Wigner quantization problem given in Theorem 4.7.

4.8 Lie superalgebra solutions

4.8.1 The $\mathfrak{gl}(1|n)$ solution

The Lie superalgebra $\mathfrak{gl}(1|n)$ has been introduced in Section 2.2 as the Lie superalgebra with basis elements e_{jk}, with $j, k = 0, 1, \ldots, n$, subject to the relations (2.9). We can use a star condition for $\mathfrak{gl}(1|n)$ that is fixed by a signature $\sigma = (\sigma_1, \ldots, \sigma_n)$, a sequence of plus or minus signs, and by

$$(e_{0j})^\dagger = \sigma_j e_{j0}, \qquad (j = 1, \ldots, n). \tag{4.44}$$

We will restrict ourselves to the case where all σ_j's are equal to $+1$, since this corresponds to the real form $\mathfrak{u}(1|n)$. In this case it is known that finite-dimensional unitary representations exist [28].

Solutions of (4.43) in terms of generators of $\mathfrak{gl}(1|n)$ are known. They have been constructed for a fixed interaction matrix in [55]. Therefore, it is not necessary to copy the entire analysis here. The solutions can be written in terms of the odd elements e_{j0} and e_{0j} of $\mathfrak{gl}(1|n)$ and are of the form

$$a_j^- = \sqrt{\frac{2|\beta_j|}{\omega_j}}\, e_{j0}, \qquad a_j^+ = \operatorname{sign}(\beta_j)\sqrt{\frac{2|\beta_j|}{\omega_j}}\, e_{0j}, \qquad (j = 1, \ldots, n), \tag{4.45}$$

where the β_j's are given by $(n > 1)$

$$\beta_j = -\omega_j + \frac{1}{n-1}\sum_{k=1}^{n} \omega_k. \tag{4.46}$$

It is straightforward to verify that the Hamiltonian (4.41) can be written as

$$\hat{H} = \hbar \left(\beta e_{00} + \sum_{j=1}^{n} \beta_j e_{jj} \right), \tag{4.47}$$

where we have used the notation $\beta = \sum_{k=1}^{n} \beta_k$ to indicate that this is a constant. We want all σ_j's in equation (4.44) to be equal to 1, which is, together with the adjointness condition $(a_j^{\pm})^{\dagger} = a_j^{\mp}$, equivalent to saying that the values β_j need to be positive. Examining the form of β_j we see that it is equal to $-\omega_j$ plus some average value of the quantities ω_k. Thus, it seems reasonable that half of the β_j's will be positive and half of them will be negative. However, it is possible to prove that this is not always the case. More concretely, we will need to assume that the coupling strength c is small enough. We shall refer to this as **weak coupling**.

Note that the condition that all β_j's are positive does not stem from the algebraic solution of (4.43) by means of (4.45), but only from the requirement of the star condition $(e_{0j})^{\dagger} = e_{j0}$, since we are primarily interested in unitary representations of the real form $\mathfrak{u}(1|n)$ of $\mathfrak{gl}(1|n)$.

Krawtchouk interaction

Remember that we are considering a system of n harmonic oscillators that are coupled by a certain interaction matrix. Finding operator solutions for the Hamiltonian of this system treated as a Wigner quantum system was proved to be equivalent to finding operators $a_j^{\pm}$ that satisfy the relations (4.43). This equation contains the eigenvalues of A and the operators $a_j^{\pm}$ are dependent on the eigenvectors of A. In the specific case of Krawtchouk interaction, we know that the jth eigenvalue λ_j of M_K is equal to $j-1$, with $j = 1, 2, \ldots, n$. We will find an upper bound for the coupling strength c so that in this case all the β_j's are positive.

In order to find an upper bound for the value of c, we need the following property.

Lemma 4.8 *For $C > \frac{(n-4)^2}{16}$, we have the inequality*

$$\sum_{j=0}^{n} \sqrt{C+j} > (n+1) \sqrt{C + \frac{n}{2} - 1}.$$

Note that C denotes an arbitrary positive constant in this lemma, it is not the coupling strength c.

Proof. The proof goes by induction on n. For $n = 1$ and $n = 2$ the property is trivial, as one sees for example from

$$\sqrt{C} + \sqrt{C+1} + \sqrt{C+2} > 3\sqrt{C}.$$

For larger n, we first notice that

$$\sqrt{C} + \sqrt{C+n} > 2\sqrt{C + \frac{n}{2} - 1}$$

if $C > (\frac{n}{4} - 1)^2$. This can be verified by solving this inequality for C as if it were an equality. Taking the square of both sides twice results in the given boundary for C. Consequently, one finds

$$\begin{aligned}\sum_{j=0}^{n} \sqrt{C+j} &= \sqrt{C} + \sqrt{C+n} + \sum_{j=0}^{n-2} \sqrt{(C+1)+j} \\ &> 2\sqrt{C + \frac{n}{2} - 1} + (n-1)\sqrt{(C+1) + \frac{n-2}{2} - 1} \\ &= (n+1)\sqrt{C + \frac{n}{2} - 1},\end{aligned}$$

where induction is used to justify the inequality. This is possible because if $C > \frac{(n-4)^2}{16}$, then surely $C + 1 > \frac{(n-6)^2}{16}$ as long as $n \geq 1$. □

It is then possible to construct an upper bound for the coupling strength c, as is shown in the following proposition.

Proposition 4.9 *Assume that the eigenvalues of the interaction matrix A are equal to $\omega^2 + c\lambda_j$, with $\lambda_j = j - 1$ $(j = 1, 2, \ldots, n)$. An upper bound for the coupling strength c is then given by*

$$c < \frac{2(2n-3)\omega^2}{(n-1)(n^2-3n+4)}.$$

If c satisfies this condition, then all the β_j's given in equation (4.46) are positive.

Proof. First of all, since $\beta_j - \beta_{j-1} = \sqrt{\omega^2 + c\,(j-2)} - \sqrt{\omega^2 + c\,(j-1)} < 0$, the sequence β_j $(j = 1, \ldots, n)$ is decreasing. All of the β_j's will thus be positive if

and only if β_n is positive. An equivalent condition is determined by

$$\beta_n > 0 \quad \Leftrightarrow \quad \sum_{j=0}^{n-1} \sqrt{\frac{\omega^2}{c} + j} > (n-1)\sqrt{\frac{\omega^2}{c} + n - 1}. \tag{4.48}$$

To prove this inequality, we want to use Lemma 4.8 for $n-2$. Therefore, we need to check if the condition of the lemma is satisfied. For $n \geq 2$ we have that

$$\frac{\omega^2}{c} > \frac{(n-1)(n^2-3n+4)}{2(2n-3)} \geq \frac{(n-6)^2}{16}.$$

Since we will only consider systems with at least two coupled harmonic oscillators ($n \geq 2$), Lemma 4.8 is applicable:

$$\sum_{j=0}^{n-1} \sqrt{\frac{\omega^2}{c} + j} > (n-1)\sqrt{\frac{\omega^2}{c} + \frac{n}{2} - 2} + \sqrt{\frac{\omega^2}{c} + n - 1}.$$

By demanding that the right-hand side of this equation is larger than or equal to $(n-1)\sqrt{\frac{\omega^2}{c} + n - 1}$, we ensure that β_n is positive. A simple calculation shows that this is true for values of c that are smaller than or equal to the upper bound given in this proposition. □

The upper bound for c/ω^2 is of the order $4/n^2$. An idea of how accurate our approximation of the boundary value is, can be found in Table 4.1. In this table, c_n denotes the highest value for the coupling strength c for which β_n and hence all the β_j's are positive. These values can be found by solving equation (4.48) numerically for c. The boundary value as proposed in Proposition 4.9 is denoted by $\tilde{c}_n$.

For example, if $n = 8$, all β_j's will be positive if the coupling strength $c < 0.11887\,\omega^2$. The boundary value from Proposition 4.9 is a little more pessimistic ($c < 0.08442\,\omega^2$), but not too much. One might believe from these observations that the fraction $\tilde{c}_n/c_n$ tends to 1 for large n.

Constant interaction

The problem of finding a boundary value for the coupling strength c so that all the β_j's are positive was discussed for constant interaction in [56]. The authors propose an estimation of the upper bound for the coupling strength. Moreover, for $n = 4, \ldots, 21$ they give exact values of this upper bound [56, Table 1, page 22].

Table 4.1: Critical values c_n/ω^2 in the case $\lambda_j = j-1$

n	$\frac{\tilde{c}_n}{\omega^2}$	$\frac{c_n}{\omega^2}$	$\frac{\tilde{c}_n}{c_n}$	n	$\frac{\tilde{c}_n}{\omega^2}$	$\frac{c_n}{\omega^2}$	$\frac{\tilde{c}_n}{c_n}$
4	0.41667	1.27357	0.32717	9	0.06466	0.08639	0.74843
5	0.25000	0.51723	0.48334	10	0.05105	0.06562	0.77802
6	0.16364	0.27857	0.58742	20	0.01132	0.01259	0.89893
7	0.11458	0.17391	0.65886	50	0.00168	0.00175	0.96186
8	0.08442	0.11887	0.71013	100	0.00041	0.00042	0.98130

4.8.2 The $\mathfrak{osp}(1|2n)$ solution

Apart from the solution in terms of generators of the Lie superalgebra $\mathfrak{gl}(1|n)$, we can also express a class of solutions of (4.43) by means of $\mathfrak{osp}(1|2n)$ generators. Theorem 2.10 states that this Lie superalgebra is generated by a set of $2n$ paraboson operators $b_j^{\pm}$ $(j = 1, 2, \ldots, n)$ that satisfy the relations (2.11), i.e.

$$\left[\{b_j^{\xi}, b_k^{\eta}\}, b_l^{\epsilon}\right] = (\epsilon - \xi)\delta_{jl} b_k^{\eta} + (\epsilon - \eta)\delta_{kl} b_j^{\xi}.$$

Using these defining triple relations it is then easy to check (see also [58]) that the operators

$$a_j^- = b_j^-, \qquad a_j^+ = b_j^+,$$

with $(j = 1, 2, \ldots, n)$ indeed satisfy equation (4.43). The Hamiltonian (4.41) then takes the following form:

$$\hat{H} = \sum_{j=1}^{n} \frac{\hbar}{2} \omega_j \{a_j^+, a_j^-\} = \hbar \sum_{j=1}^{n} \omega_j h_j, \tag{4.49}$$

where we have introduced the notation $h_j = \{a_j^+, a_j^-\}/2 = \{b_j^+, b_j^-\}/2$. The Cartan subalgebra of $\mathfrak{osp}(1|2n)$ is spanned by the n elements h_j $(j = 1, 2, \ldots, n)$.

4.9 The spectrum of $\hat{H}$ in a class of representations

In order to study the spectrum of the Hamiltonian $\hat{H}$ in terms of the $\mathfrak{gl}(1|n)$ or $\mathfrak{osp}(1|2n)$ solutions, it is necessary to work with specific representations of those Lie superalgebras. An approach to this problem with respect to the Fock-type representations $W(p)$ of $\mathfrak{gl}(1|n)$ was given in [55]. Here, we will work with other representations $V(p)$.

4.9.1 The $\mathfrak{gl}(1|n)$ representations $V(p)$

Before analyzing the spectrum of $\hat{H}$, we will summarize the main features of the representations $V(p)$ of $\mathfrak{gl}(1|n)$. First of all, they are finite-dimensional, unitary representations. For any natural number p, the basis vectors of $V(p)$ are given by [48]

$$v(\theta;\mathbf{r}) \equiv v(\theta;r_1,r_2,\ldots,r_n),$$

with $\theta \in \{0,1\}$, $r_i \in \{0,1,2,\ldots\}$ and $\theta + r_1 + \ldots + r_n = p$. The dimension of the vector space $V(p)$ equals

$$\binom{p+n-1}{n-1} + \binom{p+n-2}{n-1},$$

in which the two terms represent the number of basis vectors for $\theta = 0$ and $\theta = 1$ respectively. The action of the $\mathfrak{gl}(1|n)$ generators on these basis vectors can be determined [48]. We will give the actions of the diagonal elements e_{00} and e_{kk} since only these actions will be needed to find the spectrum of the Hamiltonian:

$$e_{00}v(\theta;\mathbf{r}) = \theta\, v(\theta;\mathbf{r}),$$
$$e_{kk}v(\theta;\mathbf{r}) = r_k\, v(\theta;\mathbf{r}).$$

In terms of the $\mathfrak{gl}(1|n)$ generators, the Hamiltonian takes the form (4.47):

$$\hat{H} = \hbar\left(\beta e_{00} + \sum_{j=1}^{n} \beta_j e_{jj}\right).$$

Clearly, looking at the actions of the elements e_{00} and e_{kk}, the vectors $v(\theta;\mathbf{r})$ are eigenvectors for $\hat{H}$:

$$\hat{H}v(\theta;\mathbf{r}) = \hbar E_{\mathbf{r}}\, v(\theta;\mathbf{r}),$$

where the eigenvalues $\hbar E_{\mathbf{r}}$ are determined by

$$E_{\mathbf{r}} = \beta p - \sum_{j=1}^{n} \omega_j r_j. \tag{4.50}$$

This can be established by noting that

$$\beta = \sum_{j=1}^{n} \beta_j = \frac{1}{n-1} \sum_{j=1}^{n} \omega_j$$

and by using the fact that $\theta + r_1 + \ldots + r_n = p$.

In the case where there is no coupling ($c = 0$), all the β_j's become the same. It follows that $\beta_j = \omega/(n-1)$ and $\beta = n\omega/(n-1)$. In this case, we thus see that the eigenvalues of $\hat{H}$ are

$$\hbar\omega \left(\frac{p}{n-1} + \theta \right).$$

So in fact, there are two eigenvalues. The lowest one, for $\theta = 0$, has multiplicity $\binom{p+n-1}{n-1}$. The highest eigenvalue has multiplicity $\binom{p+n-2}{n-1}$.

Our main interest lies in the weak coupling case, where $0 < c < c_n$. The energy levels are easily computed through equation (4.50). The result for $n = 4, p = 2$ and $\omega = \hbar = 1$ can be seen in Figure 4.3, where we have chosen to compare the systems with constant and Krawtchouk interaction.

Both figures look quite similar, but there are some differences. We see that in general all eigenvalues are different, but for specific values of c some of the energy levels cross each other. For these values of c, the multiplicity of some of the eigenvalues is higher than 1. In the constant interaction case, we see that energy levels can cross if we restrict ourselves to, say, $\theta = 0$. Also, note that there are indeed only two eigenvalues in the case without coupling ($c = 0$).

Figure 4.3 also suggests that the lowest energy level tends to zero as the coupling strength reaches c_n. In order to prove this, we need to know the lowest energy level. First, we note that $\beta_n \leq \beta_j$ for all j as soon as $\lambda_n \geq \lambda_j$ for all j. We can always choose λ_n to be the largest eigenvalue, so we can assume that β_n is the smallest of all β_j. Next, θ must be zero for the eigenvector corresponding to the lowest energy eigenvalue. So we can write

$$E_{\mathbf{r}} = \sum_{j=1}^{n} \beta_j r_j$$

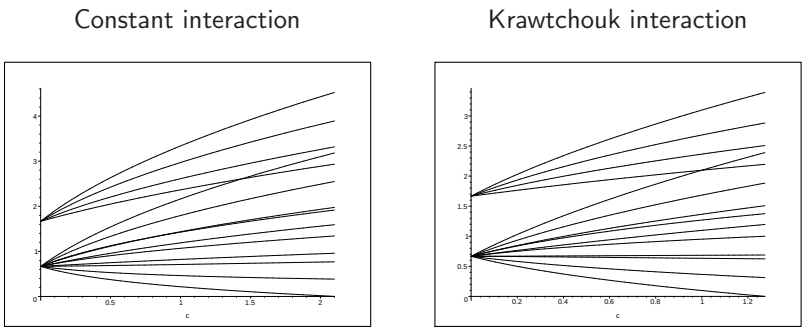

Figure 4.3: Spectrum of the Hamiltonian (4.47) in the $\mathfrak{gl}(1|n)$ representation $V(p)$ for $n = 4, p = 2$ and $\omega = \hbar = 1$, as a function of the coupling constant c. The left plot belongs to the system with constant interaction, where the λ_j are given by equation (4.17). The right plot represents the Krawtchouk case, with $\lambda_j = j - 1$.

Thus, the lowest energy level arises when all terms are equal to β_n, and is therefore equal to $p\beta_n$. The definition of c_n tells us that $\beta_n = 0$ for $c = c_n$, so the lowest energy level $p\beta_n$ tends to zero when c approaches c_n.

4.9.2 The $\mathfrak{osp}(1|2n)$ representations $V(\mathfrak{p})$

We will also take a look at the infinite-dimensional representations $V(\mathfrak{p})$ of $\mathfrak{osp}(1|2n)$, with lowest weight $(\frac{\mathfrak{p}}{2}, \ldots, \frac{\mathfrak{p}}{2})$. Such a representation is a unitary, irreducible representation if and only if $\mathfrak{p} \in \{1, 2, \ldots, n-1\}$ or $\mathfrak{p} > n - 1$ [57, Theorem 7], see also Theorem 5.1. In literature, where $\mathfrak{osp}(1|2n)$ is related to the n-paraboson algebra, the parameter $\mathfrak{p}$ is sometimes referred to as the order of the parastatistics. A basis for the representations $V(\mathfrak{p})$ was given in [57], and consists of all Gelfand-Zetlin patterns for partitions of length at most n. These GZ-vectors have the following form:

$$|m) \equiv |m)^n \equiv \left| \begin{array}{cccc} m_{1\,n} & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & m_{n-1,n-1} & \\ \vdots & \iddots & & \\ m_{11} & & & \end{array} \right) \equiv \left| \begin{array}{c} [m]^n \\ |m)^{n-1} \end{array} \right). \qquad (4.51)$$

The top line of the GZ-vectors is any partition into at most $\mathfrak{p}$ parts, where $\mathfrak{p}$ is the label of the representation. This partition is denoted by $[m]^n$. The other elements of the GZ-vectors, denoted by $|m)^{n-1}$, satisfy the so-called **betweenness conditions**

$$m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1} \qquad (1 \leq i \leq j \leq n-1).$$

The actions of the $\mathfrak{osp}(1|2n)$ generators on these basis vectors are known [57]. In particular, the action of the diagonal elements h_j is given by

$$h_j\,|m) = \left(\frac{\mathfrak{p}}{2} + \sum_{r=1}^{j} m_{rj} - \sum_{r=1}^{j-1} m_{r,j-1}\right)|m).$$

The Hamiltonian of the system in terms of $\mathfrak{osp}(1|2n)$ generators was given by equation (4.49):

$$\hat{H} = \hbar \sum_{j=1}^{n} \omega_j h_j.$$

From the action of the diagonal elements h_j it is clear that the vectors $|m)$ are eigenvectors of the Hamiltonian. We can write

$$\hat{H}\,|m) = \hbar\, E_m\,|m),$$

in which E_m stands for

$$E_m = \sum_{j=1}^{n} \omega_j \left(\frac{\mathfrak{p}}{2} + \sum_{r=1}^{j} m_{rj} - \sum_{r=1}^{j-1} m_{r,j-1}\right). \tag{4.52}$$

In the case without coupling we have that $c = 0$ and consequently $\omega_j = \omega$ for all j. The eigenvalues simplify significantly in this case and they can be written in the form

$$\hbar\omega \left(\frac{n\mathfrak{p}}{2} + \sum_{r=1}^{n} m_{rn}\right).$$

The summation in this expression is in fact the weight of the partition $[m]^n$. This weight can be any positive integer k, which we shall call the **height of the eigenvalue** $E_k^{(\mathfrak{p})}$. This means that for $c = 0$ there is an infinite number of eigenvalues, that can be written as

$$E_k^{(\mathfrak{p})} = \hbar\omega \left(\frac{n\mathfrak{p}}{2} + k\right), \qquad k = 0, 1, 2, \ldots.$$

The multiplicity $\mu(E_k^{(\mathfrak{p})})$ of each eigenvalue can be determined with the help of some theoretical arguments. First of all, $\mu(E_k^{(\mathfrak{p})})$ will be equal to the total number of GZ-vectors with a partition ν in the top row, where ν is any partition of k into at most $\mathfrak{p}$ parts. Let ν' be the conjugate partition of ν [64]. It is known (see for example [105, Section 4.6]) that the representation of $\mathfrak{gl}(n)$ that is labelled by the partition ν has dimension $\binom{n}{\nu'}$, where we have used the generalization of the binomial coefficient for a partition [64, page 45]. This is defined by

$$\binom{X}{\nu} = \prod_{(i,j)\in\nu} \frac{X - c(i,j)}{h(i,j)},$$

where $c(i,j) = j - i$ and $h(i,j) = \nu_i + \nu'_j - i - j + 1$ are the content and the hook length of (i,j) respectively. So for a given partition ν, the number of GZ-patterns that have ν in the top row equals $\binom{n}{\nu'}$. This implies that the multiplicity of each eigenvalue is equal to

$$\mu(E_k^{(\mathfrak{p})}) = \sum_{\nu,\,|\nu|=k,\,l(\nu)\leq\lceil\mathfrak{p}\rceil} \binom{n}{\nu'}. \tag{4.53}$$

The ceiling function $\lceil\mathfrak{p}\rceil$ is used to cover the cases where $n-1 < \mathfrak{p} < n$. So we have found that the energy levels for $c = 0$ are equidistant with spacing $\hbar\omega$ and the multiplicities of the eigenvalues can be computed through equation (4.53).

In the case with actual coupling ($c \neq 0$) the eigenvalues can be found by equation (4.52). Unlike the weak coupling case in the representations $V(p)$ of $\mathfrak{gl}(1|n)$, the multiplicities of the eigenvalues are not all equal to one. Any two basis vectors $|m)$ and $|m')$ that are subject to

$$\sum_{r=1}^{j} m_{rj} = \sum_{r=1}^{j} m'_{rj} \qquad \forall j = 1,2,\ldots,n \tag{4.54}$$

yield the same eigenvalue $\hbar E_m$. So the multiplicity of an eigenvalue $\hbar E_m$ is equal to the number of basis vectors for which the sum of the elements on row j is equal to the sum of the elements on row j of $|m)$, for every j. For example, the vectors

$$|m) \equiv \begin{vmatrix} 5 & 0 & 0 & 0 \\ 4 & 0 & 0 & \\ 2 & 0 & & \\ 1 & & & \end{vmatrix} \quad \text{and} \quad |m') \equiv \begin{vmatrix} 3 & 2 & 0 & 0 \\ 3 & 1 & 0 & \\ 2 & 0 & & \\ 1 & & & \end{vmatrix}$$

yield the same eigenvalue. From this it is also clear that the total number of distinct eigenvalues at height k is equal to that number for $\mathfrak{p} = 1$. Indeed, every vector with $\mathfrak{p} > 1$ can be associated with a vector with $\mathfrak{p} = 1$ for which equation (4.54) holds, as can be seen in the previous example. Moreover, all eigenvalues in the case $\mathfrak{p} = 1$ clearly have multiplicity 1 (in the generic case when all λ_j are distinct). We know what the number of eigenvalues at height k for $\mathfrak{p} = 1$ is, namely

$$\sum_{\nu,\,|\nu|=k,\,l(\nu)\leq 1} \binom{n}{\nu'} = \binom{n}{(k)'} = \prod_{j=1}^{k} \frac{n+j-1}{j},$$

where the default value for $k = 0$ is equal to 1. The latter product is nothing more than the binomial coefficient $\binom{n+k-1}{n-1}$, which shows that it is an integer.

It is now clear that some eigenvalues have multiplicity greater than 1. Furthermore it is possible that some of the energy levels cross each other, just as in the $\mathfrak{gl}(1|n)$ case. This means that for specific values of c there are some eigenvalues for which the multiplicity is even higher. It would be inappropriate to try to compute these values of c. Let us instead look at Figure 4.4, where we have plotted a part of the energy spectrum for $n = 4, \mathfrak{p} = 2$, $\omega = \hbar = 1$ and $\lambda_j = j - 1$, to visualize things. Recall that we are dealing with an infinite spectrum. Therefore, we will only plot the spectrum up to height k, for $k = 1$ and $k = 2$.

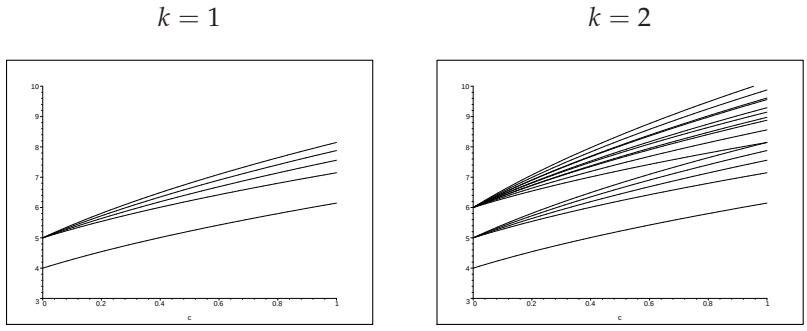

Figure 4.4: Spectrum of the Hamiltonian (4.49) in the $\mathfrak{osp}(1|2n)$ representation $V(\mathfrak{p})$ for $n = 4, \mathfrak{p} = 2$, $\omega = \hbar = 1$ and $\lambda_j = j - 1$, as a function of the coupling constant c. The left figure gives the spectrum up to height $k = 1$, the image on the right goes one step higher ($k = 2$). The total spectrum is infinite.

The eigenvalues on height 0 and 1 all have multiplicity 1 for $c > 0$ and they never cross. The figure on the right shows the energy values for $k = 2$ as well, where we both have higher multiplicities and crossing energy levels. Six of the ten distinct energy levels at height 2 have multiplicity 2.

4.10 Relation to canonical quantization

Having treated the quantum mechanical system (4.1) as a Wigner quantum system in general, it is interesting to compare our results with the canonical case as a special case. This case was treated in Section 4.2 and resulted in the n-boson Fock space, for which the basis vectors (see equation (4.15)) were built up from a vacuum vector $|0\rangle$ by

$$|k_1,\dots,k_n\rangle = \frac{(a_1^+)^{k_1}\dots(a_n^+)^{k_n}}{\sqrt{k_1!\dots k_n!}}\,|0\rangle\,, \qquad (k_j = 0,1,\dots).$$

Since the canonical case is retrieved from the $\mathfrak{osp}(1|2n)$ solution by looking at the representation $V(\mathfrak{p})$ with $\mathfrak{p} = 1$, we need to find a correspondence between the vectors $|k_1,\dots,k_n\rangle$ and the Gelfand-Zetlin basis vectors of the representation $V(1)$ of $\mathfrak{osp}(1|2n)$, generally denoted by

$$\begin{pmatrix} m_n & 0 & \cdots & 0 & 0 \\ m_{n-1} & 0 & \cdots & 0 & \\ \vdots & \vdots & \iddots & & \\ m_2 & 0 & & & \\ m_1 & & & & \end{pmatrix}. \tag{4.55}$$

We find such a connection by looking at the spectrum of $\hat{H}$, which is determined by equation (4.16):

$$\hat{H}\,|k_1,\dots,k_n\rangle = \sum_{j=1}^{n} \hbar\omega_j(k_j + \frac{1}{2})\,|k_1,\dots,k_n\rangle\,.$$

By comparison with equation (4.52) for $\mathfrak{p} = 1$, one finds that

$$\begin{aligned} k_j &= \sum_{r=1}^{j} m_{rj} - \sum_{r=1}^{j-1} m_{r,j-1} \\ &= m_j - m_{j-1}. \end{aligned}$$

Thus we have:

Proposition 4.10 *The n-boson Fock space and the $\mathfrak{osp}(1|2n)$ representation space $V(1)$ are equivalent and their basis vectors (4.15) and (4.55) are related by $k_j = m_j - m_{j-1}$.*

4.11 Summary

A system of coupled harmonic oscillators was considered in this chapter, both in canonical and in Wigner quantization. The quadratic coupling has been represented by an interaction matrix A which is to be real, symmetric and positive definite. We have seen that the system can completely be solved once the eigenvalues and eigenvectors of the interaction matrix are known, which led to the distinction between numerically solvable and analytically solvable systems. The difference between the two concepts depends on whether or not an analytically closed expression for the eigenvalues and eigenvectors exists.

At first the system was considered in canonical quantization, and the interaction matrix was assumed to be tridiagonal, resulting in a chain of coupled harmonic oscillators. The question was addressed whether analytically solvable systems of this kind existed that were as yet unknown. Such systems where found in the context of orthogonal polynomials, and resulted in analytically solvable Hamiltonians with Krawtchouk interaction, Hahn interaction and dual q-Krawtchouk interaction. For each of these systems the spectrum was discrete and infinite-dimensional. Substantial spectral differences between the considered systems were established and depicted in Figure 4.2.

The system of coupled harmonic oscillators was also contemplated as a Wigner quantum system. We have shown that the Wigner quantization procedure can be performed completely, leading to a set of algebraic triple relations as compatibility conditions. These relations have particular solutions in terms of generators of the Lie superalgebras $\mathfrak{gl}(1|n)$ or $\mathfrak{osp}(1|2n)$. Then the unitary representations of these Lie superalgebras play an important role: the algebraic generators, and thus also the physical operators corresponding to observables, act in these representations. For some classes of representations, the spectrum of the Hamiltonian operator was determined explicitly, and discussed.

In the $\mathfrak{gl}(1|n)$ solution, with representation space $V(\mathfrak{p})$, the spectrum of the Hamiltonian turned out to be finite. The energy levels turned out to show some

interesting and unexpected behaviour when plotted as a function of the coupling constant c, see Figure 4.3. The spectrum of the Hamiltonian in the $\mathfrak{osp}(1|2n)$ solution is infinite-dimensional and many energy-eigenvalues proved to have a multiplicity greater than one. Figure 4.4 shows the energy levels as a function of the coupling constant. For the $\mathfrak{osp}(1|2n)$ representation $V(1)$, the canonical case was retrieved and a connection between the basis vectors of the n-boson Fock space and the Gelfand-Zetlin basis vectors of the representation space $V(1)$ was detected.

5

The n-dimensional Wigner harmonic oscillator

We have seen in the previous chapter how Wigner quantization works for certain systems involving harmonic oscillators and how this procedure is related to Lie superalgebra representations. We will consider yet another harmonic oscillator model in this chapter, namely the n-dimensional harmonic oscillator. This system has been studied extensively before, even in the context of Wigner quantization. The n-dimensional non-isotropic harmonic oscillator considered as a Wigner quantum system has solutions in terms of the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$, which can be found in [58] together with many other interesting results. The system, however, still proves to be fertile ground for research. For $n = 3N$ for example, the question arises as to how the angular momentum decomposition of representations of the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ is computed.

In this chapter we will construct generating functions for the angular momentum decomposition of specific series of representations of $\mathfrak{osp}(1|6N)$ and $\mathfrak{gl}(1|3N)$, with $N = 1$ and $N = 2$. This problem can be completely solved for $N = 1$. However, for $N = 2$ only some classes of representations allow executable computations. We describe how to find the generating functions for higher values of N theoretically, but conclude that the computations become too hard in practice. Most of the results that are needed in this chapter are only at hand for the n-dimensional isotropic harmonic oscillator. Therefore, we will restrict ourselves to the isotropic case.

The main results of this chapter have been published in [88].

5.1 Introducing the system

The Hamiltonian of the n-dimensional non-isotropic harmonic oscillator with mass m and frequencies ω_j $(j = 1, \ldots, n)$ is given by

$$\hat{H} = \frac{1}{2m} \sum_{j=1}^{n} \hat{p}_j^2 + \frac{m}{2} \sum_{j=1}^{n} \omega_j^2 \hat{q}_j^2, \tag{5.1}$$

where the position and momentum operators are denoted by $\hat{p}_j$ and $\hat{q}_j$ respectively. We will look at this Hamiltonian in the framework of Wigner quantization. Since the Wigner perspective has already been considered in detail by Lievens and Van der Jeugt in [58], we will only present their results succinctly. For a more thorough deduction of the results, we refer to the aforementioned paper.

For the n-dimensional Wigner harmonic oscillator, the compatibility conditions are given by

$$[\hat{H}, \hat{q}_j] = -\frac{i\hbar}{m} \hat{p}_j, \qquad [\hat{H}, \hat{p}_j] = i\hbar m \omega_j^2 \hat{q}_j,$$

for $j = 1, \ldots, n$. By introducing new operators $a_j^{\pm}$ by

$$a_j^{\pm} = \sqrt{\frac{m\omega_j}{2\hbar}} \hat{q}_j \mp \frac{i}{\sqrt{2m\hbar\omega_j}} \hat{p}_j \tag{5.2}$$

we can rewrite the Hamiltonian as

$$\hat{H} = \frac{\hbar}{2} \sum_{j=1}^{n} \omega_j \{a_j^+, a_j^-\}.$$

In terms of these new operators, the compatibility conditions take the form

$$\sum_{j=1}^{n} \omega_j \left[\{a_j^+, a_j^-\}, a_k^{\pm}\right] = \pm 2\omega_k a_k^{\pm}, \tag{5.3}$$

for $k = 1, \ldots, n$. Since the position and momentum operators are self-adjoint, we have $(a_j^{\pm})^{\dagger} = a_j^{\mp}$. It turns out that we can find operators $a_j^{\pm}$ subject to the latter hermiticity conditions and to the compatibility conditions (5.3) in terms of Lie superalgebra generators.

There are two known classes of solutions for the non-isotropic case; we can express $a_j^{\pm}$ in terms of elements of $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$. For each of these solutions,

the spectrum of the Hamiltonian in specific Lie superalgebra representations was found in [58]. In their paper the authors give a nice overview of the relevant representations and their characters, and they present the energy spectrum by means of spectrum generating functions. For a detailed analysis, we refer the reader to that paper and the references therein. In this chapter we will summarize the elements of this paper that are useful for our purposes. It should be noted that we will restrict ourselves to the isotropic case where $\omega_j = \omega$ for all $j = 1, \ldots, n$.

When both Lie superalgebra solutions are examined, it is time to move forward to the main objective of this chapter. The solution of the Wigner quantum system under consideration depends on the Lie superalgebra representation V in which the operators $a_j^{\pm}$ act. So the purpose is to study properties of the Wigner oscillator in different representations, one of which will correspond to the canonical case. For the three-dimensional N-particle Wigner harmonic oscillator, i.e. the case $n = 3N$, we want to find the angular momentum and energy content of Lie superalgebra representations of $\mathfrak{osp}(1|6N)$ and $\mathfrak{gl}(1|3N)$. In particular, this means we will try to discover which representations of the Lie algebra $\mathfrak{so}(3)$ occur in the decomposition of specific Lie superalgebra representations at a given energy level. We will use the tool of generating functions to achieve this aim. The obtained results are known in the canonical case, and we will compare this case to the new solutions offered by Wigner quantization.

First, we discuss the orthosymplectic case in Sections 5.2, 5.3 and 5.4. The $\mathfrak{osp}(1|2n)$ solution to the Wigner problem is discussed in Section 5.2. All of the results in this introductory section stem from [58]. In Section 5.3 we explain how the angular momentum content of the $\mathfrak{osp}(1|6N)$ representations $V(\mathfrak{p})$ can be found by means of generating functions, both theoretically as practically. The actual generating functions for representations of $\mathfrak{osp}(1|6)$ and $\mathfrak{osp}(1|12)$ are computed in Section 5.4. We also plot the angular momentum and energy content in so-called (E, j)-diagrams and compare with the canonical case in this section. The Lie superalgebra solution $\mathfrak{gl}(1|n)$ is handled in Sections 5.5, 5.6 and 5.7. In the concluding section, we summarize our main results.

5.2 The $\mathfrak{osp}(1|2n)$ solution

By the theorem of Ganchev and Palev (see Theorem 2.10), the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ is generated by its odd elements $b_j^{\pm}$ $(j = 1, \ldots, n)$. These paraboson operators are subject to defining triple relations given by equation (2.11),

or

$$\left[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon\right] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi.$$

In these triple relations, j,k and l are elements from the set $\{1,2,\ldots,n\}$ and $\eta,\xi,\epsilon \in \{+,-\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $(\epsilon - \xi)$ and $(\epsilon - \eta)$).

We can use the paraboson operators to find solutions for the Wigner quantization discussed earlier. Indeed, writing $a_j^\pm$ as

$$a_j^- = b_j^-, \qquad a_j^+ = b_j^+,$$

with $(j = 1,2,\ldots,n)$, we see that the compatibility conditions (5.3) are satisfied using the defining triple relations (2.11). The Hamiltonian (5.1) then takes the following form:

$$\hat{H} = \frac{\hbar\omega}{2} \sum_{j=1}^{n} \{b_j^+, b_j^-\}.$$

In order to obtain that $(a_j^\pm)^\dagger = a_j^\mp$, we need to work with suitable representations of $\mathfrak{osp}(1|2n)$. In the paraboson Fock space $V(\mathfrak{p})$, described in Section 4.9.2 we automatically have $(b_j^\pm)^\dagger = b_j^\mp$, which makes this unitary irreducible representation of $\mathfrak{osp}(1|2n)$ an appropriate choice. In [57] the representation $V(\mathfrak{p})$ was thoroughly investigated, resulting in an explicit basis, matrix elements and character formulas. The main theorem of that paper gives the conditions on $\mathfrak{p}$ for $V(\mathfrak{p})$ to be a unitary irreducible representation and it states the character of the representation [57, Theorem 7].

Theorem 5.1 *The $\mathfrak{osp}(1|2n)$ representation $V(\mathfrak{p})$ with lowest weight $(\frac{\mathfrak{p}}{2},\ldots,\frac{\mathfrak{p}}{2})$ is a unirrep if and only if $\mathfrak{p} \in \{1,\ldots,n-1\}$ or $\mathfrak{p} > n-1$. The character of $V(\mathfrak{p})$ is given by*

$$\mathrm{char}V(\mathfrak{p}) = (x_1 \cdots x_n)^{\mathfrak{p}/2} \sum_{\lambda,\, \ell(\lambda) \le \lceil \mathfrak{p} \rceil} s_\lambda(x) \tag{5.4}$$

The ceiling function $\lceil \mathfrak{p} \rceil$ is there to cover the cases where $n-1 < \mathfrak{p} < n$.

As usual, $s_\lambda(x) = s_\lambda(x_1,\ldots,x_n)$ denotes the symmetric Schur function, which vanishes when the length of the partition λ exceeds the number of variables n.

For our purposes, the character formula (5.4) is inadequate. Instead the following equivalent formula [45, 57] for $\mathfrak{p} \in \{1, 2, \ldots, n-1\}$ will be more practical:

$$\mathrm{char}V(\mathfrak{p}) = (x_1 \cdots x_n)^{\mathfrak{p}/2} \frac{\mathbf{E}_{(0,\mathfrak{p})}}{\prod_i (1-x_i) \prod_{j<k} (1-x_j x_k)}, \tag{5.5}$$

with

$$\mathbf{E}_{(0,\mathfrak{p})} = \sum_{\eta} (-1)^{c_\eta} s_\eta(x_1, \ldots, x_n).$$

In this expression for $\mathbf{E}_{(0,\mathfrak{p})}$, the sum is over all partitions of the form

$$\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 + \mathfrak{p} & a_2 + \mathfrak{p} & \cdots & a_r + \mathfrak{p} \end{pmatrix}$$

in Frobenius notation (see page 23), and

$$c_\eta = a_1 + a_2 + \cdots + a_r + r.$$

For the partition η, a typical shape of the Young diagram is given in Figure 5.1.

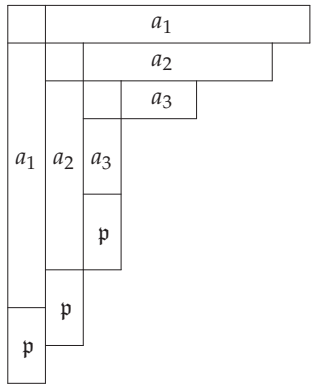

Figure 5.1: Young diagram of the partition η in Frobenius notation for $r = 3$.

A simple expression for $\mathbf{E}_{(0,\mathfrak{p})}$ exists when $\mathfrak{p} = 1$ and $\mathfrak{p} = n-1$. In these cases we have

$$\mathrm{char}V(1) = (x_1 \cdots x_n)^{1/2} \frac{1}{\prod_i (1-x_i)} \tag{5.6}$$

and

$$\mathrm{char}V(n-1) = (x_1 \cdots x_n)^{(n-1)/2} \frac{(1 - x_1 \cdots x_n)}{\prod_i (1-x_i) \prod_{j<k} (1 - x_j x_k)}. \tag{5.7}$$

It is possible to find the spectrum of the Hamiltonian $\hat{H}$ in the representation $V(\mathfrak{p})$. In fact, a straightforward technique described in [58] delivers the spectrum generating function $\mathrm{spec}\,\hat{H}$, which assigns all eigenvalues of $\hat{H}$ to a power of t. The multiplicity of an energy level E is then given by the coefficient of t^E. For the $\mathfrak{osp}(1|2n)$ solution, the spectrum generating function takes the form

$$\mathrm{spec}\,\hat{H} = \sum_{k \geq 0} \sum_{\lambda,\, |\lambda| = k,\, \ell(\lambda) \leq \lceil \mathfrak{p} \rceil} s_\lambda(1, \ldots, 1)\, t^{\hbar\omega(\frac{n\mathfrak{p}}{2} + k)}, \tag{5.8}$$

where $|\lambda|$ is the order of the partition. The energy levels are equidistant and can be written as

$$E_k^{(\mathfrak{p})} = \hbar\omega(\frac{n\mathfrak{p}}{2} + k),$$

with $k = 0, 1, 2, \ldots$. The multiplicities of these energy levels are equal to

$$\mu(E_k^{(\mathfrak{p})}) = \sum_{\lambda,\, |\lambda| = k,\, \ell(\lambda) \leq \lceil \mathfrak{p} \rceil} s_\lambda(1, \ldots, 1).$$

For $\mathfrak{p} = 1$ all of the results above coincide with the canonical results. Indeed, as noted in Section 4.10, $\mathfrak{p} = 1$ represents canonical quantization.

5.3 Angular momentum content for $\mathfrak{osp}(1|2n)$

The main objective of the present chapter is to find the angular momentum content of Lie superalgebra representations related to the Wigner quantization of the $3D$ Wigner harmonic oscillator, both for $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ with $n = 3N$. Both cases are dissimilar with respect to the dimension of the representation spaces, so a proper approach is needed to tackle both problems. This asks for a small clarification.

We would like to describe the angular momentum content with a generating function. The representation $V(\mathfrak{p})$ of $\mathfrak{osp}(1|2n)$ is infinite-dimensional, which implies that it is impossible to construct a generating function comprising all $\mathfrak{osp}(1|2n)$ representations. Therefore, our objective in the $\mathfrak{osp}(1|2n)$ case is to construct a generating function for every representation $V(\mathfrak{p})$ separately. In the $\mathfrak{gl}(1|n)$ solution,

examined from Section 5.5 onwards, the representations will be finite-dimensional. In that case, the generating function will contain variables characterizing the $\mathfrak{gl}(1|n)$ representation.

We would like to find how the Hilbert space in which the Hamiltonian acts decomposes to $\mathfrak{so}(3)$ representations [22]. In this section we will discuss the $\mathfrak{osp}(1|2n)$ case, where $n = 3N$. In that case we can rely on the embedding

$$\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{u}(3N) \supset \mathfrak{u}(3) \oplus \mathfrak{u}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1) \tag{5.9}$$

to come up with a generating function that represents the angular momentum decomposition of the $\mathfrak{osp}(1|6N)$ representation $V(\mathfrak{p})$. Note that the $*$-condition $(b_j^{\pm})^\dagger = b_j^{\mp}$ makes that the unitary irreducible representations of $\mathfrak{gl}(n)$ correspond to the irreducible $*$-representations of its real form $\mathfrak{u}(n)$. Therefore it does not matter if we write $\mathfrak{gl}(n)$ or $\mathfrak{u}(n)$ in this branching.

Some explanation is needed to see why the subalgebra chain (5.9) is the correct one to use, i.e. to see why the angular momentum operators are elements of this particular $\mathfrak{so}(3)$ subalgebra of $\mathfrak{osp}(1|6N)$.

5.3.1 Angular momentum

For $n = 3$, our physical system is a three-dimensional harmonic oscillator. In a canonical context, the angular momentum operators are defined by $\mathbf{M} = \hat{\mathbf{q}} \times \hat{\mathbf{p}}$, or

$$M_j = \sum_{k,l=1}^{3} \epsilon_{jkl}\, \hat{q}_k \hat{p}_l, \qquad (j = 1,2,3),$$

where ϵ_{jkl} is the Levi-Civita symbol. Since the position and momentum operator cannot be assumed to commute in Wigner quantization, a logical definition of the angular momentum operators is

$$M_j = \frac{1}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} \{\hat{q}_k, \hat{p}_l\}, \qquad (j = 1,2,3),$$

which, by means of (5.2), can be written as

$$M_j = \frac{-i\hbar}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} \{a_k^+, a_l^-\}, \qquad (j = 1,2,3). \tag{5.10}$$

The compatibility conditions (5.3) do not contain enough information to lead to commutation relations between M_1, M_2 and M_3. However, in the $\mathfrak{osp}(1|6)$ solution $a_j^{\pm} = b_j^{\pm}$ one finds

$$[M_i, M_j] = i\hbar\,\epsilon_{ijk} M_k, \qquad (i,j,k = 1,2,3).$$

These are the commutation relations of $\mathfrak{so}(3)$. Now, since the operators M_j are in essence elements $\{b_k^+, b_l^-\}$ of $\mathfrak{osp}(1|6)$, they belong to the $\mathfrak{osp}(1|6)$ subalgebra $\mathfrak{u}(3)$ from the embedding

$$\mathfrak{osp}(1|6) \supset \mathfrak{sp}(6) \supset \mathfrak{u}(3) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1),$$

which follows from [57, Proposition 3]. The generalization to $\mathfrak{osp}(1|6N)$ is rather straightforward. The physical system is now an N-particle three-dimensional harmonic oscillator. The position and momentum operators have a second index α, with $\alpha = 1,\ldots,N$. The angular momentum operators of the particle α can be written as

$$M_{j,\alpha} = \frac{1}{2}\sum_{k,l=1}^{3} \epsilon_{jkl}\{\hat{q}_{k,\alpha}, \hat{p}_{l,\alpha}\} = \frac{-i\hbar}{2}\sum_{k,l=1}^{3} \epsilon_{jkl}\{a_{k,\alpha}^+, a_{l,\alpha}^-\}, \qquad (j = 1,2,3).$$

The total angular momentum is obtained by adding all the angular momenta of the individual particles. Thus we have

$$M_j = \sum_{\alpha=1}^{N} M_{j,\alpha}, \qquad (j = 1,2,3). \tag{5.11}$$

These M_j are elements of the $\mathfrak{u}(3)$ subalgebra of $\mathfrak{osp}(1|6N)$ and satisfy the $\mathfrak{so}(3)$ commutation relations. Therefore the angular momentum components generate the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{u}(3)$ in the following chain of subalgebras:

$$\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{u}(3N) \supset \mathfrak{u}(3) \oplus \mathfrak{u}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1).$$

The question now is how the $\mathfrak{osp}(1|6N)$ representation $V(\mathfrak{p})$ decomposes with respect to these subalgebras.

5.3.2 Decomposing the $\mathfrak{osp}(1|6N)$ representation $V(\mathfrak{p})$

The starting point of the decomposition of $V(\mathfrak{p})$ is the character of the Lie superalgebra representation given by equation (5.4). Each Schur-function $s_\lambda(x_1,\ldots,x_n)$,

with $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the character of an irreducible covariant tensor representation of $\mathfrak{u}(n)$ [59] and corresponds to the $\mathfrak{u}(n)$ representation with highest weight λ, where $n = 3N$. In other words, equation (5.4) is a $\mathfrak{u}(3N)$ character generating function. In contrast, we want the result of our analysis to be a representation generating function. In other words, the generating function returns all representations of $\mathfrak{so}(3)$ that appear in the decomposition of a fixed representation $V(\mathfrak{p})$ of $\mathfrak{osp}(1|6N)$. By means of an example we will try to avoid confusion between both concepts.

The character of $V(\mathfrak{p})$ given by equation (5.4) is, as explained earlier, a $\mathfrak{u}(3N)$ character generating function. Indeed, it contains the characters of all $\mathfrak{u}(3N)$ representations in the decomposition of the $\mathfrak{osp}(1|6N)$ representation $V(\mathfrak{p})$. Such a $\mathfrak{u}(3N)$ character however, consists of many superfluous terms if one only wishes to know which $\mathfrak{u}(3N)$ representations appear. After all, a $\mathfrak{u}(3N)$ representation is characterized by a partition λ, so a term $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$ would suffice instead of $s_\lambda(x_1, \ldots, x_n)$.

Consider the representation $V(2)$ of $\mathfrak{osp}(1|6)$ for example. Following equation (5.7) we see that the $\mathfrak{u}(3)$ character generating function takes the form

$$\frac{x_1x_2x_3(1 - x_1x_2x_3)}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_1x_2)(1 - x_1x_3)(1 - x_2x_3)}. \tag{5.12}$$

The expansion of this function contains all $\mathfrak{u}(3)$ characters in the decomposition of the $\mathfrak{osp}(1|6)$ representation $V(2)$. The $\mathfrak{u}(3)$ character generating function could just as well have been derived directly from equation (5.4). The partitions λ in this equation have a maximum of two parts, so the $\mathfrak{u}(3)$ representation generating function is created by replacing every Schur function in equation (5.4) by its leading term $x_1^{\lambda_1}x_2^{\lambda_2}$. We obtain

$$x_1x_2x_3 \sum_{\lambda_2=0}^{\infty} \sum_{\lambda_1=\lambda_2}^{\infty} x_1^{\lambda_1}x_2^{\lambda_2} = \frac{x_1x_2x_3}{(1 - x_1)(1 - x_1x_2)}. \tag{5.13}$$

Every monomial $x_1^{\lambda_1}x_2^{\lambda_2}$ in the expansion of this easier looking function corresponds to a $\mathfrak{u}(3)$ representation characterized by the partition λ. One can verify that the method described in the next paragraph, applied to the $\mathfrak{u}(3)$ character generating function (5.12), will indeed give (5.13) as a result.

Let us now return to the general case and consider the branching to $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$ in (5.9). The substitution

$$x_i := u_j v_l z, \qquad (j = 1, 2, 3 \text{ and } l = 1, \ldots, N) \tag{5.14}$$

in equation (5.5) yields a character generating function for $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$. The factor

$$z = t^{\hbar\omega}$$

keeps track of the energy, since the power of z after the substitution (5.14) in (5.4) equals $|\lambda|$ and the order of the partition λ determines the energy level $E_k^{(\mathfrak{p})}$, as can be seen from equation (5.8). We will keep using this notation throughout the rest of the chapter.

By now it should be clear that the $\mathfrak{u}(3N)$ character generating function (5.5) is not a representation generating function. Likewise, after the substitution (5.14) one does not obtain a $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$ representation generating function. Therefore, we need to describe a technique for changing a character generating function into a representation generating function.

From character generating function to representation generating function

A character generating function for a simple Lie algebra $\mathfrak{g}$ can generally be written as:

$$F(\eta) = \sum_{\lambda} \chi_\lambda(\eta) N_\lambda,$$

where the sum runs over a fixed set of dominant integral highest weights of $\mathfrak{g}$, and where each character $\chi_\lambda(\eta)$ is the coefficient of a variable N_λ of some sort. Suppose the vector $\eta = (\eta_1, \ldots, \eta_m)$ has m components, corresponding to the number of nonzero Dynkin labels of λ. The Weyl character formula (2.15) allows us to write the characters as

$$\chi_\lambda(\eta) = \frac{\sum_{w \in W} (-1)^{\ell(w)} \eta^{w(\lambda+\rho)}}{\eta^\rho \prod_{\alpha \in \Delta_+} (1 - \eta^{-\alpha})},$$

where W is the Weyl group, ρ is the Weyl tool and Δ_+ is the set of positive roots of the Lie algebra. In order to transform $F(\eta)$ into a representation generating function, one has to multiply $F(\eta)$ by $\prod_{\alpha \in \Delta_+}(1 - \eta^{-\alpha})$ and keep the terms in the dominant Weyl sector. All of this applied to our situation, where the Lie algebra $\mathfrak{u}(n)$ has the symmetric group as its Weyl group, means that we need to maintain the terms in η^λ, where λ is a partition. One method of doing this, is by making the substitution

$$\eta_1 = c_1\eta_1, \quad \eta_2 = \frac{c_2}{c_1}\eta_2, \quad \ldots \quad \eta_{m-1} = \frac{c_{m-1}}{c_{m-2}}\eta_{m-1}, \quad \eta_m = \frac{1}{c_{m-1}}\eta_m$$

in $F(\eta)\prod_{\alpha\in\Delta_+}(1-\eta^{-\alpha})$ and keep all positive powers of $c_1,\ldots,c_{m-1}$. This comes down to finding the term in $c_1^0 c_2^0\ldots c_{m-1}^0$ in the power series expansion of the function

$$F(\eta)\prod_{\alpha\in\Delta_+}(1-\eta^{-\alpha})\frac{1}{(1-c_1^{-1})\cdots(1-c_{m-1}^{-1})}.$$

Several computational software packages have specific methods of finding constant terms in an expression. We have used Maple for this, which allows the computation of algebraic residues of a function around a given point. The term in $c_1^0 c_2^0\ldots c_{m-1}^0$ in the power series expansion of the previous function is found by dividing this function by $\prod_i c_i$ and calculating the sum of residues of poles for the variables $c_1,\ldots,c_{m-1}$, see [79]. Also, the Omega package for Mathematica, developed by Axel Riese, George E. Andrews and Peter Paule, provides interesting tools for finding constant terms.

For our $\mathfrak{u}(3)\oplus\mathfrak{u}(N)$ character generating function, we would have to perform the substitution described above for the variables u_i, belonging to $\mathfrak{u}(3)$, and v_l, belonging to $\mathfrak{u}(N)$. Since the next step in the decomposition (5.9) is from $\mathfrak{u}(3)\oplus\mathfrak{u}(N)$ to $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$, we want the $\mathfrak{u}(N)$ representation labels to be replaced by the dimension of the corresponding $\mathfrak{u}(N)$ representation in the obtained representation generating function for $\mathfrak{u}(3)\oplus\mathfrak{u}(N)$.

Introducing the dimensions of the $\mathfrak{u}(N)$ representations

Replacing a term $v^\nu = v_1^{\nu_1}\ldots v_N^{\nu_N}$ by the dimension of the $\mathfrak{u}(N)$ representation labeled by $\nu=(\nu_1,\ldots,\nu_N)$ demands knowledge of a so-called dimension generating function, in which the coefficient of v^ν is the dimension of the corresponding $\mathfrak{u}(N)$ representation. The dimension of such a representation is known [64] and equals $s_\nu(1,\ldots,1)$. Thus, the dimension generating function we need is of the form

$$\sum_\nu s_\nu(1,\ldots,1)v^\nu.$$

An expression for this $\mathfrak{u}(N)$ dimension generating function for general N is not known. However, for our purposes the $\mathfrak{u}(2)$ dimension generating function will be enough. In this $N=2$ case we have

$$\sum_\nu s_\nu(x_1,x_2)v^\nu = \frac{1}{(1-x_1v_1)(1-x_2v_1)(1-x_1x_2v_1v_2)}. \tag{5.15}$$

Indeed, the summation on the left-hand side can be written as

$$\sum_{\lambda_1=\lambda_2}^{\infty}\sum_{\lambda_2=0}^{\infty}\sum_{k=0}^{\lambda_1-\lambda_2} x_1^{\lambda_1-k}x_2^{\lambda_2+k}v_1^{\lambda_1}v_2^{\lambda_2},$$

which simplifies to the right-hand side of (5.15). Therefore, the dimension generating function of $\mathfrak{u}(2)$ is

$$\frac{1}{(1-v_1)^2(1-v_1v_2)}. \tag{5.16}$$

We now want this $\mathfrak{u}(2)$ dimension generating function and the previously obtained $\mathfrak{u}(3)\oplus\mathfrak{u}(2)$ generating function – let us denote this by $H_2(u,v)$ – to be "substituted" in one another. Saying that two generating functions $F_1(X,Y)$ and $F_2(X,Z)$, with common variables X, are substituted in each other, means that X^x is replaced in either of these generating functions by the coefficient of X^x in the other. This is achieved by finding the term in X^0 in either $F_1(X,Y)F_2(X^{(-1)},Z)$ or $F_1(X^{(-1)},Y)F_2(X,Z)$, whichever is more easily calculated. Indeed, if $F_1(X,Y)$ contains a term $p_1(Y)X^x$ and $F_2(X^{(-1)},Z)$ includes a term $p_2(Z)X^{(-x)}$, then we find a term $p_1(Y)p_2(Z)$ in the product of both functions.

Here, we have $H_2(u,v)$ on the one hand and the $\mathfrak{u}(2)$ dimension generating function (5.16) on the other hand. Substituting these generating functions in each other is done by taking the constant term in the variables v_1 and v_2 in the expansion of

$$H_2(u,v)\frac{1}{(1-v_1^{-1})^2(1-v_1^{-1}v_2^{-1})}.$$

This replaces the variables v_1 and v_2 in $H_2(u,v)$ by the dimensions of the corresponding $\mathfrak{u}(2)$ representations. Again, finding the constant term can be done by various mathematical software packages.

Angular momentum content

What is left now is a generating function which models the decomposition of a $\mathfrak{u}(3N)$ representation into $\mathfrak{u}(3)$ representations. These representations of $\mathfrak{u}(3)$ are labeled by the variables u_1,u_2,u_3, the powers of which represent the partition $(\lambda_1,\lambda_2,\lambda_3)$ that characterizes the representation. Further decomposition to $\mathfrak{so}(3)$ is brought about by the known generating function for $\mathfrak{su}(3)\supset\mathfrak{so}(3)$ [25], given here in Dynkin label notation:

$$\frac{1+PQJ}{(1-PJ)(1-QJ)(1-P^2)(1-Q^2)}, \tag{5.17}$$

in which J is the $\mathfrak{so}(3)$ label. Hence, the generating function for $\mathfrak{u}(3) \supset \mathfrak{so}(3)$ in partition notation can be written as

$$G(u_1, u_2, u_3) = \frac{1 + u_1^2 u_2 J}{(1 - u_1 u_2 u_3)(1 - u_1 J)(1 - u_1 u_2 J)(1 - u_1^2)(1 - u_1^2 u_2^2)}. \quad (5.18)$$

All factors from equation (5.17) appear in (5.18) in accordance with the relation $[p, q] = [\lambda_1 - \lambda_2, \lambda_2 - \lambda_3]$, except for $(1 - u_1 u_2 u_3)$ in the denominator of (5.18). This is explained by the fact that the Dynkin-label $[p, q]$ is not influenced when a random integer is added to every part of the partition λ.

Substituting one of these generating functions into the other is done by a similar technique as before. One just has to multiply the first generating function, embodying the embedding of $\mathfrak{u}(3)$ in $\mathfrak{osp}(1|6N)$, by $G(u_1^{-1}, u_2^{-1}, u_3^{-1})$ and take the term in $u_1^0 u_2^0 u_3^0$ in this expression. The resulting generating function describes the angular momentum content of the $\mathfrak{osp}(1|6N)$ representation $V(\mathfrak{p})$.

5.4 Generating functions for $\mathfrak{osp}(1|6)$ and $\mathfrak{osp}(1|12)$

Remember that the goal in the orthosymplectic case is to derive a generating function for each representation separately. Such a generating function will be a function of two variables, J and z. The former labels the $\mathfrak{so}(3)$ content of the representation $V(\mathfrak{p})$, while the latter accounts for the $\mathfrak{u}(1)$ part. In fact, each power of z stands for an energy level.

We can now apply the techniques described in the previous section to derive generating functions for the angular momentum decomposition of the representations $V(\mathfrak{p})$ of $\mathfrak{osp}(1|6)$ and $\mathfrak{osp}(1|12)$. However, the $\mathfrak{osp}(1|6)$ case simplifies drastically as each representation $V(\mathfrak{p})$ decomposes to $\mathfrak{u}(3)$ right away. Therefore we can use a different logic to find the desired generating functions.

5.4.1 Generating functions for $\mathfrak{osp}(1|6) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1)$

From the character formula in equation (5.4) one can see that the representation $V(\mathfrak{p})$ of $\mathfrak{osp}(1|6)$ decomposes as a direct sum of $\mathfrak{u}(3)$ representations labeled by a partition λ, where λ has at most three parts. The branching of these $\mathfrak{u}(3)$ representations can immediately be obtained with the help of equation (5.18). We separate three cases: $\mathfrak{p} = 1$, $\mathfrak{p} = 2$ and $\mathfrak{p} > 2$.

$\mathfrak{p} = \mathbf{1}$: All partitions in the character formula (5.4) have length 1, so $\lambda_2 = \lambda_3 = 0$. It is then obvious that the generating function for $\mathfrak{p} = 1$ is simply $G(z,0,0)$, where $G(u_1,u_2,u_3)$ is the generating function (5.18). Not forgetting the factor $z^{3\mathfrak{p}/2}$ for the energy we obtain

$$\frac{z^{3/2}}{(1-zJ)(1-z^2)}. \tag{5.19}$$

One can use this generating function to derive the $\mathfrak{so}(3)$ representations that emerge at energy level $E_k^{(1)}$. This information can be made accessible by means of a table in which the element in row $k+1$ and column $j+1$ (counted from the bottom) marks the number of representations J^j at energy level $E_k^{(1)}$ in the angular momentum decomposition of $\mathfrak{osp}(1|6)$. We call this the (E,j)-diagram of $\mathfrak{osp}(1|6)$ for $\mathfrak{p} = 1$.

$\vdots$						$\iddots$
11/2	1		1		1	
9/2		1		1		
7/2	1		1			
5/2		1				
3/2	1					
E_k						
j	0	1	2	3	4	$\cdots$

Indeed, the first few terms in the expansion of (5.19) are

$$z^{3/2} + J\,z^{5/2} + (1+J^2)\,z^{7/2} + (J+J^3)\,z^{9/2} + (1+J^2+J^4)\,z^{11/2} + \cdots .$$

We see for example that at energy level $E_k^{(1)} = 9\hbar\omega/2$, there are two $\mathfrak{so}(3)$ representations in the decomposition of the $\mathfrak{osp}(1|6)$ representation $V(1)$, characterized by $j = 1$ and $j = 3$. Of course, these results were already known because $\mathfrak{p} = 1$ represents the canonical case. This (E,j)-diagram for instance also appears in [106].

$\mathfrak{p} = \mathbf{2}$: The partition λ now has at most two parts, so $\lambda_3 = 0$. The $\mathfrak{so}(3) \oplus \mathfrak{u}(1)$ decomposition of $\mathfrak{u}(3)$ representations labeled by such partitions is given by $G(z,z,0)$. Therefore, we can write the generating function for the angular momentum decomposition of $\mathfrak{osp}(1|6)$ for $\mathfrak{p} = 2$ as

$$\frac{(1+z^3J)\,z^3}{(1-zJ)(1-z^2J)(1-z^2)(1-z^4)}. \tag{5.20}$$

As in the previous case, we can generate the (E,j)-diagram of $\mathfrak{osp}(1|6)$ for $\mathfrak{p}=2$.

$\vdots$						$\iddots$
7	2	1	3	1	1	
6		2	1	1		
5	1	1	1			
4		1				
3	1					
E_k						
j	0	1	2	3	4	$\cdots$

Let us look at the case $E_k = 7\hbar\omega$, i.e. $k=4$ as an example. There are three partitions with two parts of order 4, namely $(4,0,0)$, $(3,1,0)$, and $(2,2,0)$. The $\mathfrak{so}(3)$ representations that emerge in these cases can be found by equation (5.18). In total we have

$$(1+J^2+J^4)+(J+J^2+J^3)+(1+J^2).$$

This is in accordance with the coefficient of z^7 in equation (5.20), as can be seen from the (E,j)-diagram as well.

$\mathfrak{p}>\mathbf{2}$: Since the length of the partitions in (5.4) cannot exceed the number of variables, we are looking in this case at partitions of length at most 3. So the generating function for $\mathfrak{p}>2$ is $z^{3\mathfrak{p}/2}\,G(z,z,z)$, or

$$\frac{(1+z^3J)\,z^{3\mathfrak{p}/2}}{(1-zJ)(1-z^2J)(1-z^2)(1-z^3)(1-z^4)}. \tag{5.21}$$

The (E,j)-diagram for $\mathfrak{p}>2$ is given by.

$\vdots$						$\iddots$
$3\mathfrak{p}/2+4$	2	2	3	1	1	
$3\mathfrak{p}/2+3$	1	2	1	1		
$3\mathfrak{p}/2+2$	1	1	1			
$3\mathfrak{p}/2+1$		1				
$3\mathfrak{p}/2$	1					
E_k						
j	0	1	2	3	4	$\cdots$

Notice that for the lower energy levels, the cases $\mathfrak{p}=2$ and $\mathfrak{p}>2$ do not differ very much from the canonical case. The larger discrepancies are found in higher energy regions.

5.4.2 Generating functions for $\mathfrak{osp}(1|12) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1)$

The previous case might have been very elementary, for $\mathfrak{osp}(1|12)$ the computations are much harder. In fact, it is not practically possible to find generating functions for all $\mathfrak{osp}(1|12)$ representation $V(\mathfrak{p})$. For $V(1)$ and $V(2)$ however, we are able to follow all the steps from Section 5.3.2 to construct the generating function for the angular momentum decomposition. We were unable to compute these generating functions for the representations $V(\mathfrak{p})$ with $\mathfrak{p} \geq 3$.

The representation $V(1)$

We start with the character of $V(1)$, given by equation (5.6), and perform the substitution (5.14), thus creating a $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ character generating function:

$$\frac{u_1 u_2 u_3 \, (v_1 v_2)^{3/2} \, z^3}{(1-u_1 v_1 z)(1-u_2 v_1 z)(1-u_3 v_1 z)(1-u_1 v_2 z)(1-u_2 v_2 z)(1-u_3 v_2 z)}.$$

We need to change this into a representation generating function. To this end, we multiply the previous function by

$$\prod_{\alpha \in \Delta_+} (1-u^{-\alpha}) \prod_{\alpha' \in \Delta'_+} (1-v^{-\alpha'}),$$

where Δ_+ and Δ'_+ are the positive roots of $\mathfrak{u}(3)$ and $\mathfrak{u}(2)$ respectively. Thus we have

$$\Delta_+ = \{(1,-1,0),(1,0,-1),(0,1,-1)\} \qquad \text{and} \qquad \Delta'_+ = \{(1,-1)\}.$$

Therefore we need to multiply our $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ character generating function by

$$(1-\frac{u_2}{u_1})(1-\frac{u_3}{u_1})(1-\frac{u_3}{u_2})(1-\frac{v_2}{v_1})$$

and perform the substitutions

$$u_1 = a u_1, \quad u_2 = \frac{b u_2}{a}, \quad u_3 = \frac{u_3}{b}, \qquad v_1 = c v_1, \quad v_2 = \frac{v_2}{c}.$$

We want to keep all positive powers of a, b and c, so we multiply our function by

$$\frac{1}{(1-a^{-1})(1-b^{-1})(1-c^{-1})}$$

and find the constant term in a, b and c. This is the hardest step to compute. The term in $a^0b^0c^0$ factorizes nicely as

$$\frac{u_1u_2u_3\,(v_1v_2)^{3/2}\,z^3}{(1-u_1v_1z)(1-u_1u_2v_1v_2z^2)}. \tag{5.22}$$

This is the $\mathfrak{u}(3)\oplus\mathfrak{u}(2)$ representation generating function, in which we want to change the $\mathfrak{u}(2)$ labels v_1 and v_2 by the dimensions of the corresponding $\mathfrak{u}(2)$ representations. Following equation (5.16), the $\mathfrak{u}(2)$ dimension generating function is

$$\frac{1}{(1-v_1)^2(1-v_1v_2)}.$$

We want to substitute this into the previously obtained $\mathfrak{u}(3)\oplus\mathfrak{u}(2)$ representation generating function (5.22). However, one first needs to take out a factor $(v_1v_2)^{3/2}$ in (5.22) before this can be done. To see why this factor can be left out, we note that

$$s_{(\nu_1+k,\nu_2+k)}(x_1,x_2) = x_1^k x_2^k\, s_{(\nu_1,\nu_2)}(x_1,x_2)$$

so the dimension $s_\nu(1,1)$ of the $\mathfrak{u}(2)$ representation labeled by the partition ν is unchanged when the same term is added to each part of ν. The result after substitution of (5.16) into the modified (5.22) is then given by

$$\frac{u_1u_2u_3\,z^3}{(1-u_1z)^2(1-u_1u_2z^2)}.$$

The angular momentum content is then found by substituting this function and $G(u_1,u_2,u_3)$ from equation (5.18) into each other. The resulting angular momentum generating function for the representation $V(1)$ of $\mathfrak{osp}(1|12)$ is

$$\frac{(1+Jz^2)\,z^3}{(1-z^2)^3(1-Jz)^2}.$$

The (E,j)-diagram shows the result for the first few energy levels.

$\vdots$						$\iddots$
7	6	3	9	3	5	
6		6	2	4		
5	3	1	3			
4		2				
3	1					
E_k						
j	0	1	2	3	4	$\cdots$

The representation $V(2)$

The character of $V(2)$, obtained from (5.5), does not factorize nicely. The angular momentum generating function for $V(2)$ can be constructed in precisely the same manner as for $V(1)$. The $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ representation generating function in this case equals N/D, with

$$\begin{aligned} N = 1 &+ u^{21}v^{21}z^3 + u^{221}v^{32}z^5 + u^{321}v^{33}z^6 - u^{321}v^{42}z^6 - u^{421}v^{43}z^7 \\ &- u^{421}v^{52}z^7 - u^{422}v^{53}z^8 - u^{431}v^{53}z^8 - u^{432}v^{63}z^9 - u^{432}v^{54}z^9 \\ &- u^{532}v^{64}z^{10} + u^{532}v^{73}z^{10} + u^{632}v^{74}z^{11} + u^{643}v^{85}z^{13} \\ &+ u^{853}v^{10,6}z^{16} \end{aligned} \tag{5.23a}$$

and

$$\begin{aligned} D = &(1-u^1v^1z)(1-u^2v^{11}z^2)(1-u^{11}v^2z^2)(1-u^{11}v^{11}z^2) \\ &\times (1-u^{111}v^{21}z^3)(1-u^{22}v^{22}z^4)(1-u^{211}v^{31}z^4) \\ &\times (1-u^{211}v^{22}z^4)(1-u^{222}v^{33}z^6), \end{aligned} \tag{5.23b}$$

where we have used the notation $u^\lambda = u_1^{\lambda_1}u_2^{\lambda_2}u_3^{\lambda_3}$, and similarly for v. The powers of z are integers, not partitions. The angular momentum generating function, also quite cumbersome, has a numerator equal to

$$\begin{aligned} z^6\Big(1 &- 2z + 3z^2 + Jz^2 - 2z^3 + 4Jz^3 + 6z^4 - 7Jz^4 - 3J^2z^4 - J^3z^4 - 6z^5 + 4Jz^5 \\ &- 2J^2z^5 - 4J^3z^5 + 6z^6 - 8Jz^6 - 3J^2z^6 + 10J^3z^6 + 3J^4z^6 - 2z^7 + 10Jz^7 \\ &+ 4J^2z^7 - 8J^3z^7 + 3z^8 - 14Jz^8 - 16J^2z^8 + 13J^3z^8 - 3J^4z^8 - 3J^5z^8 - 2z^9 \\ &+ 2Jz^9 + 6J^2z^9 - 12J^3z^9 + 6J^4z^9 + 4J^5z^9 + z^{10} - 5Jz^{10} + 2J^2z^{10} + 28J^3z^{10} \\ &+ 2J^4z^{10} - 5J^5z^{10} + J^6z^{10} + 4Jz^{11} + 6J^2z^{11} - 12J^3z^{11} + 6J^4z^{11} + 2J^5z^{11} \\ &- 2J^6z^{11} - 3Jz^{12} - 3J^2z^{12} + 13J^3z^{12} - 16J^4z^{12} - 14J^5z^{12} + 3J^6z^{12} \\ &- 8J^3z^{13} + 4J^4z^{13} + 10J^5z^{13} - 2J^6z^{13} + 3J^2z^{14} + 10J^3z^{14} - 3J^4z^{14} \\ &- 8J^5z^{14} + 6J^6z^{14} - 4J^3z^{15} - 2J^4z^{15} + 4J^5z^{15} - 6J^6z^{15} - J^3z^{16} - 3J^4z^{16} \\ &- 7J^5z^{16} + 6J^6z^{16} + 4J^5z^{17} - 2J^6z^{17} + J^5z^{18} + 3J^6z^{18} - 2J^6z^{19} + J^6z^{20}\Big), \end{aligned}$$

while the denominator is

$$(1-z^4)^4(1-z^2)^2(1-z)^2(1-J^2z^2)(1-Jz^2)^3(1-Jz)^2.$$

Schematically thrown into a (E, j)-diagram this gives

$\vdots$						$\iddots$
10	19	22	34	15	9	
9	2	14	8	6		
8	4	4	4			
7		2				
6	1					
E_k						
j	0	1	2	3	4	$\cdots$

for the lower energies.

5.5 The $\mathfrak{gl}(1|n)$ solution

The elements of the general linear Lie superalgebra $\mathfrak{gl}(1|n)$, introduced in Section 2.2, are denoted by e_{jk}, $j,k = 0,1,\ldots,n$. The odd generators of $\mathfrak{gl}(1|n)$ are given by e_{0j} and e_{j0} and all elements e_{jk} are subject to the relations (2.9). In terms of $\mathfrak{gl}(1|n)$ generators, solutions of the compatibility conditions (5.3) can be written as

$$a_j^- = \sqrt{\frac{2}{n-1}}\, e_{j0}, \qquad a_j^+ = \sqrt{\frac{2}{n-1}}\, e_{0j}.$$

The hermiticity condition $(a^\pm)^\dagger = a^\mp$ implies the star condition

$$(e_{0j})^\dagger = e_{j0}. \tag{5.24}$$

The Hamiltonian can then be rewritten as

$$\hat{H} = \frac{\hbar\omega}{n-1}\left(n e_{00} + \sum_{j=1}^{n} e_{jj}\right).$$

The unitary representations of $\mathfrak{gl}(1|n)$ compatible with the star condition (5.24) are known [28]: aside from the **typical representations**, we have the **covariant** and **contravariant tensor representations**. Here we are going to work with the covariant tensor representations V_λ, labeled by a partition λ with $\lambda_2 \leq n$. The character of this representation was given by Berele and Regev in [6]. It is a

supersymmetric Schur function $s_\lambda(x_1|y_1,\ldots,y_n)$ that, following equation (1.5), can be written as

$$\begin{aligned}\mathrm{char}V_\lambda &= s_\lambda(x_1|y_1,\ldots,y_n)\\ &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(x_1) s_{\nu'}(y_1,\ldots,y_n),\end{aligned} \tag{5.25}$$

where the coefficients $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients and ν' is the conjugate partition of ν.

Equation (5.25) is worth a closer look. First, we note that $s_\mu(x_1)$ vanishes unless the length of the partition μ is equal to one, the number of variables of the Schur function. Thus, only partitions of the form $\mu = (r)$ are allowed. In this case we have

$$s_{(r)}(x_1) = x_1^r.$$

Pieri's formula (1.4) then tells us that the Littlewood-Richardson coefficients can be written as

$$c^\lambda_{(r)\nu} = \begin{cases} 1 & \text{if } \lambda - \nu \text{ is a horizontal } r\text{-strip} \\ 0 & \text{otherwise.}\end{cases}$$

Combining these results, we find that equation (5.25) can conveniently be rewritten as

$$\mathrm{char}V_\lambda = s_\lambda(x_1|\,y) = \sum_{r\geq 0} x_1^r \sum_\nu s_{\nu'}(y),$$

where the second summation runs over all partitions ν such that $\lambda - \nu$ is a horizontal r-strip. The first summation is not infinite. Since the Young diagram of the partition λ has λ_1 columns, the horizontal strip $\lambda - \nu$ can only have a maximum of λ_1 parts. So r will never exceed the value λ_1. Another observation is that r cannot be too small either. If $\lambda - \nu$ would be an r-strip with $r < \lambda_1 - n$, then ν would have to be a partition with $\nu_1 > n$. This would imply that the length of the conjugate partition ν' is larger than n, which means that the Schur function $s_{\nu'}(y)$ vanishes. Since $\lambda_1 - n$ could be negative, we say that r takes values between r^* and λ_1, where r^* is given by

$$r^* = \begin{cases} 0 & \text{if } \lambda_1 \leq n \\ \lambda_1 - n & \text{otherwise.}\end{cases}$$

Since $\lambda_2 \leq n$, horizontal r-strips can always be formed for each r between r^* and

λ_1. In conclusion, the character can be simplified further as follows:

$$\mathrm{char}V_\lambda = s_\lambda(x_1|\,y) = \sum_{r=r^*}^{\lambda_1} x_1^r \sum_\nu s_{\nu'}(y), \tag{5.26}$$

where the second sum is again taken over all partitions ν such that $\lambda - \nu$ is a horizontal r-strip.

As in the $\mathfrak{osp}(1|2n)$ case, a spectrum generating function can be produced. We confine ourselves to giving the energy levels of the system. They can be written as

$$E_k^{(\lambda)} = \hbar\omega\big(\frac{|\lambda|}{n-1} + r^* + k\big), \qquad (k = 0, \ldots, \min(\lambda_1, n)).$$

The total number of energy levels depends on λ_1 and is equal to $\min(\lambda_1, n) + 1$.

5.6 Angular momentum decomposition of $\mathfrak{gl}(1|n)$

We will use many of the principles of the orthosymplectic case to find generating functions for the angular momentum decomposition of $\mathfrak{gl}(1|n)$ for $n = 3N$. One must always bear in mind, however, that the $\mathfrak{gl}(1|n)$ representations V_λ are finite-dimensional. Therefore our goal will be to create a generating function in which the coefficient of $A^\lambda = A_1^{\lambda_1'} \ldots A_n^{\lambda_n'}$ represents the angular momentum decomposition of V_λ. We will thus construct a generating function comprising the angular momentum decomposition of every $\mathfrak{gl}(1|n)$ representation V_λ. This is different than our approach for $\mathfrak{osp}(1|2n)$, where the generating functions applied to just one representation $V(\mathfrak{p})$.

As before, the angular momentum operators will be part of the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{gl}(1|3N)$ in the chain of subalgebras

$$\mathfrak{gl}(1|3N) \supset \mathfrak{u}(3N) \supset \mathfrak{u}(3) \oplus \mathfrak{u}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1) \tag{5.27}$$

However, a little caution is required since it turns out that the angular momentum operators do not immediately satisfy the commutation relations of $\mathfrak{so}(3)$.

5.6.1 Angular momentum

The angular momentum operators must obviously be defined independently from the choice of the Lie superalgebra representation. Thus, just as in equation (5.10),

we have

$$M_j = \frac{-i\hbar}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} \{a_k^+, a_l^-\}, \qquad (j = 1,2,3)$$

for $n = 3$, i.e. the three-dimensional Wigner harmonic oscillator. In the $\mathfrak{gl}(1|3)$ solution $a_j^+ = e_{0j}$ and $a_j^- = e_{j0}$ one finds

$$[M_i, M_j] = \frac{i\hbar}{2} \epsilon_{ijk} M_k, \qquad (i,j,k = 1,2,3).$$

So the operators $L_j = 2M_j$ generate $\mathfrak{so}(3)$. The purpose of angular momentum decomposition is mainly finding the spectrum of operators like e.g. M^2 or M_3. Obviously, this spectrum only differs by a factor from the spectrum of L^2 and L_3, so finding the $\mathfrak{so}(3)$ content is again a useful problem to tackle in this case.

For $n = 3N$, the N-particle three-dimensional Wigner harmonic oscillator, the angular momentum operators M_j are defined as in equation (5.11). Again, apart from a factor 2 these operators generate the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{gl}(1|3N)$ in the $\mathfrak{gl}(1|3N)$ solution $a_j^+ = \sqrt{2/(3N-1)}\, e_{0j}$ and $a_j^- = \sqrt{2/(3N-1)}\, e_{j0}$. Therefore we wish to know how the $\mathfrak{gl}(1|3N)$ representation V_λ decomposes with respect to the chain of subalgebras (5.27).

5.6.2 Decomposing the $\mathfrak{gl}(1|3N)$ representation V_λ

As in the $\mathfrak{osp}(1|6N)$ case, the starting point of this decomposition is the character of V_λ, given by equation (5.26). The Schur-functions $s_{\nu'}(y_1, \ldots, y_{3N})$ in this character are characters of the $\mathfrak{u}(3N)$ representations that occur in this decomposition. The branching to $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$ representations is done by the substitution

$$y_i := u_j v_l z, \qquad (j = 1,2,3 \text{ and } l = 1, \ldots, N). \tag{5.28}$$

The $\mathfrak{u}(3)$ and $\mathfrak{u}(N)$ representations that occur in this branching can be deduced from the following known relation:

$$s_{\nu'}(u_1 v_1, \ldots, u_3 v_N) = \sum_{\sigma,\tau} g_{\nu',\sigma,\tau}\, s_\sigma(u_1, u_2, u_3) s_\tau(v_1, \ldots, v_N).$$

This inner coproduct of Schur functions features the Kronecker coefficients introduced in Chapter 1. The Kronecker coefficients were initially handled in the context

of inner coproducts, but they also appear when the product of two characters [54] of the symmetric group S_n are expanded in terms of S_n characters:

$$\chi^{\sigma}_{\rho}\chi^{\tau}_{\rho} = \sum_{\nu'} g_{\nu',\sigma,\tau}\,\chi^{\nu'}_{\rho}\,,$$

where σ, τ, ν' and ρ are partitions of n. Recently King and Welsh [49] developed a so-called 'grand generating function' for the Kronecker coefficients. Applied to our context, we can say that when ν', σ and τ are partitions of n, the Kronecker coefficients are the coefficients of the term in $z^n y^{\nu'} u^{\sigma} v^{\tau}$ in the expansion of

$$\prod_{i,j,k} \frac{1}{(1 - y_i u_j v_k z)} \prod_{i<j} \left(1 - \frac{y_j}{y_i}\right) \prod_{i<j} \left(1 - \frac{u_j}{u_i}\right) \prod_{i<j} \left(1 - \frac{v_j}{v_i}\right). \tag{5.29}$$

All products in this grand generating function run from 1 to the length of the corresponding partitions, which in our case would be $3N$, 3 and N for ν', σ and τ respectively. This is already a generating function for the decomposition of $\mathfrak{u}(3N)$ to $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$. But the grand generating function is not a $\mathfrak{u}(3) \oplus \mathfrak{u}(N)$ representation generating function since its expansion contains many terms of the form $y^{\mu_1} u^{\mu_2} v^{\mu_3} z^n$ in which μ_1, μ_2 and μ_3 are not partitions. However, the same technique as for the orthosymplectic case will turn the grand generating function into a representation generating function. This step is computationally very demanding and can only be performed for specific types of $\mathfrak{u}(3N)$ representations.

Once this step is done, the rest is easy. The $\mathfrak{u}(N)$ labels $v_1, \ldots, v_N$ need to be replaced by the dimensions of their corresponding $\mathfrak{u}(N)$ representations as before. The resulting generating function then needs to be substituted into the generating function (5.18) for the branching $\mathfrak{u}(3) \supset \mathfrak{so}(3)$.

Generating functions for the angular momentum decomposition of the $\mathfrak{gl}(1|3N)$ representations V_λ, where $\lambda_1 = 1$, have been constructed for general values of N in [48] using a different group theoretical method. We were able to extend these results to other forms of λ, but only for $\mathfrak{gl}(1|3)$ and $\mathfrak{gl}(1|6)$. For other values of N the computations prove to be too hard.

5.7 Generating functions for $\mathfrak{gl}(1|3)$ and $\mathfrak{gl}(1|6)$

Since the length of the partition λ is arbitrary, a generating function in which the coefficient of $A^\lambda = A_1^{\lambda_1} A_2^{\lambda_2} \ldots$ is the angular momentum decomposition of the $\mathfrak{gl}(1|3N)$ representation V_λ would have an infinite amount of variables A_i.

Therefore we choose this angular momentum decomposition to be accompanied by $A_1^{\lambda'_1} \dots A_n^{\lambda'_n}$, thus creating a generating function with $n = 3N$ variables. This is possible because only the values of $\lambda'_1, \dots, \lambda'_n$ affect the angular momentum decomposition of V_λ, as can be seen from the character formula (5.26).

In the previous section we explained how a generating function for the decomposition of a $\mathfrak{u}(3N)$ representation in accordance with the chain of subalgebras

$$\mathfrak{u}(3N) \supset \mathfrak{u}(3) \oplus \mathfrak{u}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1)$$

can be created. Let us denote this generating function by $H(J, A_1, \dots, A_n)$. The angular momentum decomposition of the $\mathfrak{gl}(1|3N)$ representation V_λ will then be described by the following generating function:

$$H(J, A_1, \dots, A_n)\, z^{\frac{|\lambda|}{n-1} + r^*} \prod_{i=1}^{\min(\lambda_1, n)} (1 + A_i z). \tag{5.30}$$

To see why this is true, we first note that the bottom energy level equals $\frac{|\lambda|}{n-1} + r^*$, which explains the power of z in (5.30). Each value of $r = r^* + k$ then defines a new energy level and is thus responsible for an extra factor z. For reasons of clarity, the rest of the analysis will be done for a typical representation, characterized by a partition λ with $\lambda_1 \geq n$. The same ideas can be adopted in the atypical cases.

For $r = \lambda_1 - n$, i.e. on the ground energy level, there is only one partition ν for which $\lambda - \nu$ is a horizontal r-strip. Its conjugate can be written as $\nu' = (\lambda'_1, \dots, \lambda'_n)$. The corresponding representation of $\mathfrak{u}(3N)$ decomposes to $\mathfrak{so}(3)$ as described by the generating function $H(J, A_1, \dots, A_n)$, so there must be at least one term

$$H(J, A_1, \dots, A_n)$$

in the generating function we are trying to describe. In general there are n partitions ν such that $\lambda - \nu$ is a horizontal $(\lambda_1 - n + 1)$-strip. Their conjugate partitions are of the form

$$\nu' = (\lambda'_1, \dots, \lambda'_i - 1, \dots, \lambda'_n),$$

where $i = 1, \dots, n$. The angular momentum decomposition of the $\mathfrak{u}(3N)$ representations characterized by these partitions will be in the coefficient of $A_1^{\lambda'_1} \dots A_n^{\lambda'_n}$ in $A_i\, H(J, A_1, \dots, A_n)$. Therefore, our generating function must also contain the term

$$H(J, A_1, \dots, A_n)\, z\, (A_1 + \dots + A_n).$$

Note that it is possible that $\lambda'_i = \lambda'_{i+1}$, in which case ν' would not be a partition. However, in this case the coefficient of $A^{\nu'}$ in $H(J, A_1, \ldots, A_n)$ will be zero. In other words, this 'non-partition' will not be counted at all.

The factor $(A_1 + \cdots + A_n)$ in the term for $r = \lambda_1 - n + 1$ is in fact the elementary symmetric polynomial $e_1(A_1, \ldots, A_n)$ which was defined in Section 1.2. For $r = \lambda_1 - n + 2$ there are typically $n(n-1)/2$ partitions ν for which $\lambda - \nu$ is a horizontal r-strip. An analogous reasoning as before shows that they are responsible for a term

$$H(J, A_1, \ldots, A_n)\, z^2 \sum_{1 \leq i < j \leq n} A_i A_j$$

in our generating function. This term contains the elementary symmetric function $e_2(A_1, \ldots, A_n)$. Taking all values of r into account, it is not so hard to see that the generating function for the angular momentum decomposition of the $\mathfrak{gl}(1|3N)$ representation V_λ equals

$$H(J, A_1, \ldots, A_n)\, z^{\frac{|\lambda|}{n-1} + \lambda_1 - n} \sum_{i=0}^{n} e_i(A_1, \ldots, A_n) z^i$$

in the typical case where $\lambda_1 \geq n$. In this expression we find back the generating function for elementary symmetric functions, which, following equation (1.2), can be rewritten as

$$\sum_{i=0}^{n} e_i(A_1, \ldots, A_n) z^i = \prod_{i=1}^{n} (1 + A_i z).$$

For the atypical cases, where $\lambda_1 < n$, we can build up a similar analysis to eventually obtain the generating function in equation (5.30).

Clearly, the most important part of our problem is finding the $\mathfrak{u}(3N) \supset \mathfrak{so}(3)$ generating function $H(J, A_1, \ldots, A_n)$. However, for $\mathfrak{gl}(1|3)$ this step is trivial, so the results can be written down immediately.

5.7.1 Generating functions for $\mathfrak{gl}(1|3) \supset \mathfrak{so}(3)$

For the main part, the decomposition of the representation V_λ of $\mathfrak{gl}(1|3)$ following the branching

$$\mathfrak{gl}(1|3) \supset \mathfrak{u}(3) \supset \mathfrak{so}(3)$$

is described by the generating function $G(A_1, A_2, A_3)$ given by equation (5.18), where A_1, A_2 and A_3 label the first three parts of the conjugate partition of λ.

The rest of the generating function follows from the previous discussion, equation (5.30) in particular, and depends on λ_1. For $\lambda_1 \geq 3$ we find

$$z^{\frac{|\lambda|}{2}+r^*}\frac{(1+A_1z)(1+A_2z)(1+A_3z)(1+A_1^2A_2\,J)}{(1-A_1A_2A_3)(1-A_1\,J)(1-A_1A_2\,J)(1-A_1^2)(1-A_1^2A_2^2)}. \tag{5.31}$$

The cases where $\ell(\lambda')=2$ and $\ell(\lambda')=1$ are easily deduced from this equation by setting $A_3=0$ and $A_2=A_3=0$ respectively. For $\ell(\lambda')=1$ we find back the results from King, Stoilova and Van der Jeugt in [48].

The generating function (5.31) allows us to construct (E,j)-diagrams for any $\mathfrak{gl}(1|3)$ representation V_λ. Some examples for the typical case are given below, with $\lambda_a=(3,1,0)$ and $\lambda_b=(3,2,2,1,1)$.

5		1	
4	1	1	1
3	1	1	1
2		1	
$E_k^{\lambda_a}$			
j	0	1	2

and

15/2	1		2	1	1	
13/2		3	3	4	2	1
11/2		3	3	4	2	1
9/2	1		2	1	1	
$E_k^{\lambda_b}$						
j	0	1	2	3	4	5

The information of these (E,j)-diagrams can be obtained from the coefficient of $A^{\nu_a'}$ and $A^{\nu_b'}$ in the expansion of (5.31), for $\nu_a'=(2,1,1)$ and $\nu_b'=(5,3,1)$ respectively. As a primary difference with the canonical case (and the $\mathfrak{osp}(1|6N)$ case in general) we see that there is a finite number of energy levels. Also, at the bottom energy level we see more than one $\mathfrak{so}(3)$-multiplet. The first (E,j)-diagram, where $\lambda=(3,1,0)$ shows an exception to this remark. We also note that we still have equidistant energy levels, and that there are again higher multiplicities of $\mathfrak{so}(3)$ representations.

In the atypical cases, the number of energy levels decreases as the length of λ becomes smaller. For $\lambda=(2,1,1)$ we have

4	1		1	
3		2	1	1
2		1	1	1
E_k^{λ}				
j	0	1	2	3

There are only three energy levels in this case, and we observe that the vertical symmetry of the (E,j)-diagram is now gone.

5.7.2 Generating functions for $\mathfrak{gl}(1|6) \supset \mathfrak{so}(3)$

The case $\mathfrak{gl}(1|3)$ is deceivingly simple compared to $\mathfrak{gl}(1|6)$. In fact, it will no longer be possible to construct the generating function $H(J, A_1, \ldots, A_6)$ for all representations of $\mathfrak{u}(6)$. More precisely, we will only be able to handle the cases $\ell(\lambda') = 1$ and $\ell(\lambda') = 2$ completely, where λ is the partition that characterizes the $\mathfrak{gl}(1|6)$ representation V_λ.

The representation V_λ with $\ell(\lambda') = 1$

We follow the procedure described in Section 5.6.2 to obtain a generating function $H(J, A_1)$ which will describe the angular momentum generating function for the $\mathfrak{u}(6)$ representation characterized by a partition ν', where ν' consists of one part only. The starting point is the grand generating function (5.29) for $N = 2$:

$$\frac{(1-\frac{u_2}{u_1})(1-\frac{u_3}{u_1})(1-\frac{u_3}{u_2})(1-\frac{v_2}{v_1})}{(1-u_1v_1A_1)(1-u_2v_1A_1)(1-u_3v_1A_1)(1-u_1v_2A_1)(1-u_2v_2A_1)(1-u_3v_2A_1)},$$

where we have taken into account the fact that $\lambda_1 = \ell(\lambda') = 1$ and therefore only one parameter A_1 is necessary to describe the $\mathfrak{u}(6)$ representation. We recognize this function from the $V(1)$ representation of $\mathfrak{osp}(1|12)$, where the starting function was the same. Thus, the rest of the analysis can be adopted from that case and eventually we find

$$H(J, A_1) = \frac{(1 + JA_1^2)}{(1 - A_1^2)^3(1 - JA_1)^2}.$$

The generating function for the angular momentum decomposition of the $\mathfrak{gl}(1|6)$ representation V_λ, with $\ell(\lambda') = 1$, then follows from equation (5.30) and is equal to

$$z^{\frac{|\lambda|}{5}} \frac{(1 + A_1 z)\,(1 + JA_1^2)}{(1 - A_1^2)^3\,(1 - JA_1)^2}. \tag{5.32}$$

This is again confirmed by the results in [48].

The representation V_λ with $\ell(\lambda') = 2$

Compared to the case where $\ell(\lambda') = 1$, the denominator of the grand generating function will have six extra factors containing A_2, and the numerator must have an extra factor $(1 - A_2/A_1)$. From that point on, essentially all computations run along previously traveled paths. Yet, the computer has a much harder time

performing these computations, and we have the end result (see Section 5.A) as a witness. Some intermediate results are however interesting. The $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ representation generating function in this case equals N/D with

$$\begin{aligned} N = 1 &+ u^{21}v^{21}A^{21} + u^{221}v^{32}A^{32} + u^{321}v^{42}A^{33} + u^{321}v^{33}A^{42} \\ &- 2u^{321}v^{42}A^{42} - u^{421}v^{52}A^{43} - u^{421}v^{43}A^{52} - u^{422}v^{53}A^{53} \\ &- u^{431}v^{53}A^{53} - u^{432}v^{63}A^{54} - u^{432}v^{54}A^{63} - 2u^{532}v^{64}A^{64} \\ &+ u^{532}v^{73}A^{64} + u^{532}v^{64}A^{73} + u^{632}v^{74}A^{74} + u^{643}v^{85}A^{85} \\ &+ u^{853}v^{10,6}A^{10,6} \end{aligned} \tag{5.33a}$$

and

$$\begin{aligned} D = &(1-u^1v^1A^1)(1-u^2v^{11}A^{11})(1-u^{11}v^2A^{11})(1-u^{11}v^{11}A^2) \\ &\times (1-u^{111}v^{21}A^{21})(1-u^{22}v^{22}A^{22})(1-u^{211}v^{31}A^{22}) \\ &\times (1-u^{211}v^{22}A^{31})(1-u^{222}v^{33}A^{33}), \end{aligned} \tag{5.33b}$$

where we have used the notation $u^\sigma = u_1^{\sigma_1}u_2^{\sigma_2}u_3^{\sigma_3}$, and similarly for v and A. This generating function was first obtained by Patera and Sharp [79] as a plethysm generating function for two-rowed representations of $SU(n)$. In this chapter, we have already computed N/D given by equation (5.33) independently, not using the grand generating function of King and Welsh. Indeed, this generating function and the function N/D, with numerator and denominator given by equation (5.23), are similar. The only difference is that A^λ in (5.33) has been changed into $z^{|\lambda|}$ in equation (5.23). The functions N/D given by equations (5.23) and (5.33) represent the $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ branching of all $\mathfrak{u}(6)$ representations occurring in the representation $V(2)$ of $\mathfrak{osp}(1|12)$ and V_λ (with $\ell(\lambda') = 2$) of $\mathfrak{gl}(1|6)$ respectively. The $\mathfrak{u}(6)$ representations that occur in both cases are characterized by a partition with a maximum length of two, so both generating functions must be equal.

Introducing the dimensions of the $\mathfrak{u}(2)$ representations in our $\mathfrak{u}(3) \oplus \mathfrak{u}(2)$ representation generating function, and then substituting the result into $G(u_1, u_2, u_3)$ gives us the function $H(J, A_1, A_2)$. The unappealing sight of this function forces us to relocate its full expression to Section 5.A. We can still write our angular momentum content generating function as

$$z^{\frac{|\lambda|}{5}}\,(1 + A_1 z)(1 + A_2 z)\,H(J, A_1, A_2).$$

Note that for $A_2 = 0$ we must find back equation (5.32), which is indeed the case. It is interesting to see what happens when the powers of A_1 and A_2 are equal. This

means that we are looking at $\mathfrak{u}(6)$ representations with character $s_{\nu'}(A_1, A_2)$, where ν' is a partition for which both rows have equal length. These representations are of interest in complexity theory and in the study of qubits [62, 63]. The generating function in this case can be computed out of the previous one by making the substitution

$$A_1 = aA_1, \qquad A_2 = \frac{A_2}{a}$$

and then looking for the constant term in a. Alternatively, all terms involving $A^{\lambda_1\lambda_2}$ with $\lambda_1 > \lambda_2$ can simply be deleted in N and D given by equation (5.33). Either way, we find that the $\mathfrak{u}(6) \supset \mathfrak{u}(3) \oplus \mathfrak{u}(2)$ branching is represented by the generating function

$$\frac{(1 + u^{321}v^{42}A^{33})}{(1 - u^2v^{11}A^{11})(1 - u^{11}v^2A^{11})(1 - u^{22}v^{22}A^{22})(1 - u^{211}v^{31}A^{22})(1 - u^{222}v^{33}A^{33})},$$

a result which was obtained recently by King and Welsh [49]. This confirms earlier observations concerning inner products of Schur functions [11, 24]. The generating function $H_{\lambda'_1=\lambda'_2}(J, A_1, A_2) = N/D$, with

$$\begin{aligned} N = 1 - {} & A^{11} + 2A^{22} + 3JA^{22} + 3J^2A^{22} - A^{33} - 3JA^{33} - 4J^2A^{33} - 3J^3A^{33} \\ + {} & A^{44} + 3JA^{44} - 2J^2A^{44} + 3J^3A^{44} + J^4A^{44} - 3JA^{55} - 4J^2A^{55} \\ - {} & 3J^3A^{55} - J^4A^{55} + 3J^2A^{66} + 3J^3A^{66} + 2J^4A^{66} - J^4A^{77} + J^4A^{88} \end{aligned}$$

and

$$D = (1 - A^{11})^2 (1 - JA^{11})^3 (1 - J^2A^{11}) (1 - A^{22})^4,$$

describes the further branching of a two-rowed $\mathfrak{u}(6)$ representation to $\mathfrak{so}(3)$. Note that we cannot use this generating function to describe the angular momentum decomposition of the $\mathfrak{gl}(1|6)$ representation V_λ, for which λ is a partition with two columns of equal length in the Young diagram. Indeed, solving such a problem would require the angular momentum decomposition of all $\mathfrak{u}(6)$ representations characterized by a partition ν', such that $\lambda - \nu$ is a horizontal r-strip, with $r = 0, 1, 2$. For $r = 1$, the partition ν' does not consist of two rows of equal length (in fact, $\nu'_1 = \nu'_2 + 1$), thus the generating function $H_{\lambda'_1=\lambda'_2}(J, A_1, A_2)$ is of no use for $\mathfrak{u}(6)$ representations characterized by this particular partition ν'.

The representation V_λ with $\ell(\lambda') = 3$

Computationally, this case can only be worked out when $\lambda'_1 = \lambda'_2 = \lambda'_3$. A three-rowed $\mathfrak{u}(6)$ representation where all rows are of equal length decomposes to $\mathfrak{u}(3) \oplus$

$\mathfrak{u}(2)$ in accordance with the generating function of the form N/D, with

$$\begin{aligned} N = 1 - & \; u^{111}v^{21}A^{111} - u^{321}v^{33}A^{222} + u^{222}v^{42}A^{222} + u^{321}v^{42}A^{222} \\ & + 2u^{432}v^{54}A^{333} + u^{531}v^{54}A^{333} - u^{432}v^{63}A^{333} - v^{75}u^{543}A^{444} \\ & + u^{642}v^{66}A^{444} - 2u^{642}v^{75}A^{444} - u^{753}v^{87}A^{555} - u^{852}v^{87}A^{555} \\ & + u^{753}v^{96}A^{555} + u^{963}v^{10,8}A^{666} - u^{10,7,4}v^{12,9}A^{777} \end{aligned}$$

and

$$\begin{aligned} D = & (1 - u^{111}v^{21}A^{111})(1 - u^{21}v^{21}A^{111})(1 - u^{111}v^{3}A^{111})(1 - u^{321}v^{33}A^{222}) \\ & \times (1 - u^{33}v^{33}A^{222})(1 - u^{411}v^{33}A^{222})(1 - u^{444}v^{66}A^{444}). \end{aligned}$$

The same representation of $\mathfrak{u}(6)$ decomposes to $\mathfrak{so}(3)$ in accordance with the generating function $H_{\lambda'_1=\lambda'_2=\lambda'_3}(J, A_1, A_2, A_3) = N/D$, with

$$\begin{aligned} N = 1 + & \; 2JA^{222} + 3J^2A^{222} + J^3A^{222} + 2JA^{333} - 2J^2A^{333} - 2J^3A^{333} - 2J^4A^{333} \\ & + A^{444} - 4J^2A^{444} - 4J^3A^{444} + J^5A^{444} - 2JA^{555} - 2J^2A^{555} - 2J^3A^{555} \\ & + 2J^4A^{555} + J^2A^{666} + 3J^3A^{666} + 2J^4A^{666} + J^5A^{888} \end{aligned}$$

and

$$D = (1 - A^{111})^4 \, (1 - JA^{111})^2 \, (1 - J^2A^{111})^2 \, (1 - A^{222})^3.$$

The argument given in the previous section implies again that this generating function does not contain sufficient information for the angular momentum decomposition of the $\mathfrak{gl}(1|6)$ representation V_λ, with $\lambda'_1 = \lambda'_2 = \lambda'_3$.

5.8 Conclusions

For a 3D N-particle Wigner harmonic oscillator, operator solutions exist in terms of generators of the Lie superalgebras $\mathfrak{osp}(1|6N)$ and $\mathfrak{gl}(1|3N)$. These operators act in representation spaces of these Lie superalgebras. Our goal was to find the angular momentum and energy content of the representations $V(\mathfrak{p})$ of $\mathfrak{osp}(1|6N)$ and V_λ of $\mathfrak{gl}(1|3N)$. For $N = 1$, we have been able to construct generating functions representing the angular momentum decomposition for all of these representations. For $N = 2$, the computer allowed us to construct only partial results. For $\mathfrak{osp}(1|12)$ we have generating functions for the representations $V(1)$ and $V(2)$, but for other representations the results proved computationally too hard. In the $\mathfrak{gl}(1|6)$ case,

we had to restrict ourselves to representations V_λ for which λ had a maximum of two columns.

By means of the obtained generating functions, we were able to plot the angular momentum and energy content in so-called (E, j)-diagrams. These are tables showing the multiplicities of all angular momentum values at each energy level. These tables are a practical tool to compare the results of the new Wigner solutions to the well-known canonical case.

For the 1-dimensional Wigner harmonic oscillator, investigated by Wigner in [103], the energy levels in the non-canonical solutions are shifted in height but remain equidistant. The (E, j)-diagrams in the $\mathfrak{osp}(1|6N)$ solution show that for all representations $V(\mathfrak{p})$ we have a similar behavior for the angular momentum content. Apart from the shifted energy levels, the structure of the (E, j)-diagrams in the non-canonical solutions is the same as that for the representation $V(1)$ of $\mathfrak{osp}(1|6)$. The main difference is that the multiplicities of the angular momentum representations can be higher than 1, a feature not observed in the canonical case.

In the $\mathfrak{gl}(1|3N)$ solution, the situation is drastically different due to the finite-dimensional nature of the representations. There is a finite amount of energy levels and the number of $\mathfrak{so}(3)$-multiplets, the angular momentum content, does not increase when the energy gets higher. On the contrary, for higher (and lower) energy levels we see less $\mathfrak{so}(3)$-multiplets than in the bulk of the energy spectrum.

5.A The function $H(J, A_1, A_2)$

The generating function $H(J, A_1, A_2)$ for the angular momentum decomposition of a two-rowed representation of $\mathfrak{u}(6)$, has the form N/D, with

$$
\begin{aligned}
N = 1 &- A^{11} + JA^2 + 2A^{21} + 6JA^{21} + 2A^{22} + 3JA^{22} + 3J^2A^{22} + 3A^{31} \\
&- 2JA^{31} - 6J^2A^{31} - J^3A^{31} + 2A^{32} - 6JA^{32} - 8J^2A^{32} - 6J^3A^{32} - A^{33} \\
&- 3JA^{33} - 4J^2A^{33} - 3J^3A^{33} - 6JA^{41} - 3A^{42} - 6JA^{42} - 6J^2A^{42} \\
&+ 5J^3A^{42} + 3J^4A^{42} - 2JA^{43} + 4J^2A^{43} + 8J^3A^{43} + 6J^4A^{43} + A^{44} \\
&+ 3JA^{44} - 2J^2A^{44} + 3J^3A^{44} + J^4A^{44} - A^{51} + 3J^2A^{51} + 4JA^{52} + 6J^3A^{52} \\
&+ 3A^{53} + 2JA^{53} + J^2A^{53} + 3J^3A^{53} - 4J^4A^{53} - 3J^5A^{53} - 2JA^{54} \\
&- 8J^2A^{54} - 6J^4A^{54} - 2J^5A^{54} - 3JA^{55} - 4J^2A^{55} - 3J^3A^{55} - J^4A^{55} \\
&+ 2JA^{61} + A^{62} + J^2A^{62} - 2A^{63} - 10JA^{63} - 10J^2A^{63} - 3A^{64} - 11JA^{64} \\
&+ 4J^2A^{64} + 5J^3A^{64} + 6J^4A^{64} + 3J^5A^{64} + J^6A^{64} + 4J^2A^{65} + 6J^3A^{65}
\end{aligned}
$$

$$
\begin{aligned}
&+ 8J^4A^{65} + 2J^5A^{65} + 3J^2A^{66} + 3J^3A^{66} + 2J^4A^{66} - J^2A^{71} - 2JA^{72} \\
&- 2J^3A^{72} - 2A^{73} + JA^{73} + 8J^2A^{73} + 8J^3A^{73} + 6J^4A^{73} - 2A^{74} + 8JA^{74} \\
&+ 30J^2A^{74} + 18J^3A^{74} + 8J^4A^{74} - 4J^5A^{74} + 8JA^{75} + 15J^2A^{75} + 12J^3A^{75} \\
&+ 3J^4A^{75} - 7J^5A^{75} - J^6A^{75} - 2J^3A^{76} - 4J^4A^{76} - 4J^5A^{76} - J^4A^{77} \\
&+ J^2A^{82} + 4JA^{83} + 4J^2A^{83} + 2J^3A^{83} + A^{84} + 7JA^{84} - 3J^2A^{84} \\
&- 12J^3A^{84} - 15J^4A^{84} - 8J^5A^{84} + 4JA^{85} - 8J^2A^{85} - 18J^3A^{85} - 30J^4A^{85} \\
&- 8J^5A^{85} + 2J^6A^{85} - 6J^2A^{86} - 8J^3A^{86} - 8J^4A^{86} - J^5A^{86} + 2J^6A^{86} \\
&+ 2J^3A^{87} + 2J^5A^{87} + J^4A^{88} - 2J^2A^{93} - 3J^3A^{93} - 3J^4A^{93} - 2JA^{94} \\
&- 8J^2A^{94} - 6J^3A^{94} - 4J^4A^{94} - A^{95} - 3JA^{95} - 6J^2A^{95} - 5J^3A^{95} \\
&- 4J^4A^{95} + 11J^5A^{95} + 3J^6A^{95} + 10J^4A^{96} + 10J^5A^{96} + 2J^6A^{96} - J^4A^{97} \\
&- J^6A^{97} - 2J^5A^{98} + J^2A^{10,4} + 3J^3A^{10,4} + 4J^4A^{10,4} + 3J^5A^{10,4} + 2JA^{10,5} \\
&+ 6J^2A^{10,5} + 8J^4A^{10,5} + 2J^5A^{10,5} + 3JA^{10,6} + 4J^2A^{10,6} - 3J^3A^{10,6} \\
&- J^4A^{10,6} - 2J^5A^{10,6} - 3J^6A^{10,6} - 6J^3A^{10,7} - 4J^5A^{10,7} - 3J^4A^{10,8} \\
&+ J^6A^{10,8} - J^2A^{11,5} - 3J^3A^{11,5} + 2J^4A^{11,5} - 3J^5A^{11,5} - J^6A^{11,5} \\
&- 6J^2A^{11,6} - 8J^3A^{11,6} - 4J^4A^{11,6} + 2J^5A^{11,6} - 3J^2A^{11,7} - 5J^3A^{11,7} \\
&+ 6J^4A^{11,7} + 6J^5A^{11,7} + 3J^6A^{11,7} + 6J^5A^{11,8} + 3J^3A^{12,6} + 4J^4A^{12,6} \\
&+ 3J^5A^{12,6} + J^6A^{12,6} + 6J^3A^{12,7} + 8J^4A^{12,7} + 6J^5A^{12,7} - 2J^6A^{12,7} \\
&+ J^3A^{12,8} + 6J^4A^{12,8} + 2J^5A^{12,8} - 3J^6A^{12,8} - 3J^4A^{13,7} - 3J^5A^{13,7} \\
&- 2J^6A^{13,7} - 6J^5A^{13,8} - 2J^6A^{13,8} - J^5A^{13,9} + J^6A^{14,8} - J^6A^{15,9}
\end{aligned}
$$

and

$$D = (1 - A^{11})^2(1 - A^2)^3(1 - A^{22})^4(1 - JA)^2(1 - JA^{11})^3(1 - J^2A^{11}).$$

We have used the notation $A^{\nu'}$ to denote $A_1^{\nu'_1}A_2^{\nu'_2}$ for practical reasons.

6

Wigner quantization of one-dimensional Hamiltonians

Back in Chapter 3, when we spoke of Wigner quantization for the first time, we considered the one-dimensional harmonic oscillator as our first example of a Wigner quantum system. We made further investigations in the context of Wigner quantum systems by looking at various harmonic oscillator models. Chapters 4 and 5 were aimed at convincing the reader that many interesting features are to be discovered in this area.

The harmonic oscillator is left aside in this chapter and we will walk along the path of one-dimensional systems instead. First we will consider a rather unconventional yet fairly popular Hamiltonian, namely the Berry-Keating-Connes Hamiltonian $\hat{H} = \hat{x}\hat{p}$. Next we consider the simplest of all physical systems: the free particle. This does not imply that we are completely stepping out of our comfort zone. Both of these one-dimensional systems have Lie superalgebra solutions in terms of $\mathfrak{osp}(1|2)$ generators when considered as a Wigner quantum system.

The Berry-Keating-Connes Hamiltonian is the first of the two one-dimensional systems we are going to consider. At first sight it seems odd to deal with the less known Hamiltonian first. However, the calculations are more accessible for $\hat{H} = \hat{x}\hat{p}$ than for the free particle. Therefore, we treat the former, less difficult case in detail and we will be more succinct throughout the second part of the chapter.

Both parts of this chapter, each associated with one of the Hamiltonians, are roughly built up in the same way. First we solve the Wigner quantization problem in terms of $\mathfrak{osp}(1|2)$ generators. In the $\mathfrak{osp}(1|2)$ representation spaces described in Proposition 2.26, we can determine the action of $\hat{H} = \hat{x}\hat{p}$, $\hat{x}$ and $\hat{p}$, and they turn

out to be related to orthogonal polynomials in analogy with the one dimensional Wigner harmonic oscillator. The techniques used in this problem, see Section 3.3.2, are adopted here to find the spectrum of the operators $\hat{H}_b$, $\hat{x}$ and $\hat{p}$ and the wave functions of the system. Finally, we compare our results with the canonical case by taking $a = 1/2$, where $a = 2\mu$ is the parameter characterizing the $\mathfrak{osp}(1|2)$ representations. In the end, our results prove to be compatible with what is known from the canonical setting.

The main results of this chapter have appeared in [87].

6.1 The Berry-Keating-Connes Hamiltonian

The recent popularity of the Hamiltonian $\hat{H} = \hat{x}\hat{p}$ must be attributed to its possible connection with the Riemann hypothesis [2, 89, 95, 96]. This conjecture states that the non-trivial zeros of the Riemann zeta function can all be written as $1/2 + it_n$, where the t_n are real numbers. The first speculations about a potential relation between the Hamiltonian $\hat{H}$ and the Riemann zeros have been made in 1999, by Berry and Keating on the one hand [8, 9] and Connes on the other hand [17]. However, the idea of linking a certain Hamiltonian to the Riemann hypothesis is much older.

The origin of this suggestion lies almost a century behind us, when Pólya proposed that the imaginary parts of the non-trivial Riemann zeros could correspond to the (real) eigenvalues of some self-adjoint operator. This statement is known as the Hilbert-Pólya conjecture, although Hilbert's contribution to this is unclear. For a long time the Hilbert-Pólya conjecture was regarded as a bold speculation, but it gained in credibility due to papers by Selberg [94] and Montgomery [67].

The historical commotion around the Hamiltonian $\hat{H} = \hat{x}\hat{p}$ inspired us to perform the Wigner quantization of this one-dimensional system. As we shall see, a lot of interesting results emerge.

6.1.1 Wigner quantization and $\mathfrak{osp}(1|2)$ solutions

In this section, we will perform the Wigner quantization of the Hamiltonian $\hat{H} = \hat{x}\hat{p}$. The easiest Hermitian operator corresponding to $\hat{x}\hat{p}$, without assuming any commutation relations between $\hat{x}$ and $\hat{p}$, is

$$\hat{H}_b = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

As always in Wigner quantization, one starts from the operator form of Hamilton's equations and the equations of Heisenberg for this particular Hamiltonian. One then expresses the compatibility between these operator equations, thus creating a pair of compatibility conditions. For simplicity of notation, we set $\hbar = 1$ and we find

$$[\hat{H}_b, \hat{x}] = -i\hat{x}, \qquad [\hat{H}_b, \hat{p}] = i\hat{p},$$

or equivalently

$$[\{\hat{x}, \hat{p}\}, \hat{x}] = -2i\hat{x}, \qquad [\{\hat{x}, \hat{p}\}, \hat{p}] = 2i\hat{p}, \tag{6.1}$$

where $\{\hat{x}, \hat{p}\}$ denotes the anticommutator between $\hat{x}$ and $\hat{p}$ as usual.

The goal is now to find Hilbert spaces in which the operators $\hat{x}$ and $\hat{p}$ act as self-adjoint operators in such a way that they satisfy the compatibility conditions (6.1). The strategy will be to identify the algebra generated by $\hat{x}$ and $\hat{p}$ subject to the compatibility conditions.

Proposition 6.1 *The operators $\hat{x}$ and $\hat{p}$, subject to the relations (6.1), generate the Lie superalgebra $\mathfrak{osp}(1|2)$.*

Proof. Let us define new operators b^+ and b^-:

$$b^\pm = \frac{\hat{x} \mp i\hat{p}}{\sqrt{2}}. \tag{6.2}$$

These operators should satisfy $(b^\pm)^\dagger = b^\mp$, where the dagger operation stands for the ordinary Hermitian conjugate. In terms of the $b^\pm$, the operators $\hat{H}_b$, $\hat{x}$ and $\hat{p}$ take the form

$$\hat{H}_b = \frac{i}{2}\left((b^+)^2 - (b^-)^2\right), \qquad \hat{x} = \frac{b^+ + b^-}{\sqrt{2}}, \qquad \hat{p} = \frac{i(b^+ - b^-)}{\sqrt{2}}.$$

The compatibility conditions (6.1) are equivalent to the equations $[\hat{H}_b, b^\pm] = -ib^\mp$, which in turn can be written as

$$[(b^-)^2, b^+] = 2b^- \qquad \text{and} \qquad [(b^+)^2, b^-] = -2b^+,$$

using the previous expression of $\hat{H}_b$ in terms of b^+ and b^-. By writing down the commutators and anticommutators explicitly, one sees that the latter two relations are equivalent to

$$\left[\{b^-, b^+\}, b^\pm\right] = \pm 2b^\pm.$$

These equations should look familiar by now. Indeed they are the defining relations (2.16) of the Lie superalgebra $\mathfrak{osp}(1|2)$. □

Theorem 6.1 states that the operators $\hat{H}_b$, $\hat{x}$ and $\hat{p}$ are operators on a certain representation space of $\mathfrak{osp}(1|2)$. Because the position and momentum operator need to be self-adjoint, we have that $(b^{\pm})^{\dagger} = b^{\mp}$. The dagger operation defines a $*$-condition on $\mathfrak{osp}(1|2)$, so we are interested in the $*$-representations classified in Section 2.4. The $\mathfrak{osp}(1|2)$ representation space is therefore a direct sum of two subspaces $V = V_0 \oplus V_1$. The action of the even elements of $\mathfrak{osp}(1|2)$, defined by equation (2.17), is confined to one or the other subspace in this representation.

The Hamiltonian $\hat{H}_b$ is an even operator that can be written as

$$\hat{H}_b = i(e + f).$$

The position and momentum operators are odd operators on this representation space because they are written in terms of b^+ and b^-. These odd elements of $\mathfrak{osp}(1|2)$ map an element of the $\mathfrak{osp}(1|2)$ representation space V from one subspace to the other, so their action is not confined to either V_0 or V_1.

The $\mathfrak{osp}(1|2)$ representation space V was constructed in Section 2.4. It can be visualized as a direct sum of two representation spaces shown in Figure 2.3, repeated here for convenience:

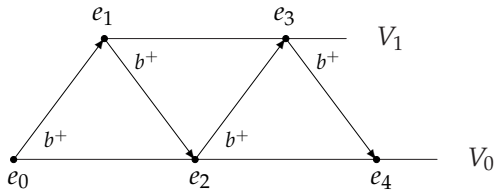

Figure 6.1: The $\mathfrak{osp}(1|2)$ representation space $V = V_0 \oplus V_1$

The representation with parameter $a = 2\mu$ will be called ρ_a in this chapter, and the action of the odd operators is then given by

$$\begin{aligned}
\rho_a(b^+)e_{2n} &= b^+ e_{2n} = \sqrt{2(n+a)}\, e_{2n+1} \\
\rho_a(b^-)e_{2n} &= b^- e_{2n} = \sqrt{2n}\, e_{2n-1} \\
\rho_a(b^+)e_{2n+1} &= b^+ e_{2n+1} = \sqrt{2(n+1)}\, e_{2n+2} \\
\rho_a(b^-)e_{2n+1} &= b^- e_{2n+1} = \sqrt{2(n+a)}\, e_{2n}
\end{aligned} \tag{6.3}$$

Following the analysis of Section 3.3.2, we have that the representation corresponding to the canonical case is the one with $a = 1/2$.

6.1.2 Spectrum of the operators $\hat{H}_b$, $\hat{x}$ and $\hat{p}$

The first main goal of this chapter is to determine the spectrum of the essential operators in the $\mathfrak{osp}(1|2)$ representation space $V = V_0 \oplus V_1$. The operators $\hat{x}$ and $\hat{p}$ have comparable expressions in terms of $\mathfrak{osp}(1|2)$ generators and can thus be handled in a similar way. The spectrum of $\hat{H}_b$ on the other hand is an entirely different issue. Since $\hat{H}_b = i(e + f)$ is an even operator, the spectrum of $\hat{H}_b$ can be considered in both subspaces V_0 or V_1 separately. The operator $\hat{x}$ is clearly an odd operator, so in this case one cannot look at both subspaces individually.

Determining the spectrum of the operators $\hat{H}_b$ and $\hat{x}$ is done using the same method as for the one-dimensional Wigner harmonic oscillator (see Section 3.3.2). One starts by defining a formal eigenvector of $\hat{H}_b$ and $\hat{x}$, with respective eigenvalues E and x. These eigenvectors are unknown linear combinations of the basis vectors e_n of V. A three term recurrence relation for the coefficients can then be calculated, and this will enable us to identify these coefficients with orthogonal polynomials. From the spectral theorem for unbounded self-adjoint operators [7, 21] one can then derive that the spectrum of the operators is equal to the support of the weight function of the corresponding orthogonal polynomials. It is then easy to find the spectrum of $\hat{H}_b$ and $\hat{x}$.

The orthogonal polynomials that occur in the remaining of this chapter have all been defined in Section 3.2. The spectrum of $\hat{H}_b$ will be determined using Meixner-Pollaczek polynomials (Section 3.2.4), while generalized Hermite polynomials (Section 3.2.3) will be used to find the spectrum of $\hat{x}$ and $\hat{p}$. We will encounter the Laguerre polynomials (Section 3.2.2) when the spectrum of the Hamiltonian of the free particle is being studied.

Spectrum of $\hat{H}_b = i(e + f)$

For the spectrum of the Hamiltonian $\hat{H}_b$, we start by defining a formal eigenvector $u_0(E)$, for the eigenvalue E:

$$u_0(E) = \sum_{n=0}^{\infty} \alpha_{2n}(E) e_{2n}, \tag{6.4}$$

where $\alpha_{2n}(E)$ are some unknown functions to be determined. We have already argued that the spectrum of $\hat{H}_b$ can be considered in both subspaces V_0 and V_1 separately. The formal vector $u_0(E)$ is an infinite linear combination of basis vectors of the even subspace V_0 and the action of $\hat{H}_b$ on this vector results in a similar form, as one sees from

$$\hat{H}_b\, u_0(E) = \sum_{n=0}^{\infty} \alpha_{2n}(E) \left(i\sqrt{(n+1)(n+a)}\, e_{2n+2} - i\sqrt{n(n+a-1)}\, e_{2n-2} \right).$$

This has been established by rewriting $\hat{H}_b = i(e+f)$ with the help of (2.17), which makes it possible to use equations (6.3) in order to determine the action of $\hat{H}_b$ on $u_0(E)$. We can try to find the unknown coefficients $\alpha_{2n}(E)$ by collecting the coefficients of e_{2n} in the expression $\hat{H}_b u_0(E) = E u_0(E)$. We end up with the following recurrence relation:

$$E\,\alpha_{2n} = i\sqrt{n(n+a-1)}\,\alpha_{2n-2} - i\sqrt{(n+1)(n+a)}\,\alpha_{2n+2}.$$

We then define $\tilde{\alpha}_{2n}(E) = (-i)^n \alpha_{2n}(E)$, for which the recurrence relation reads

$$E\,\tilde{\alpha}_{2n} = \sqrt{n(n+a-1)}\,\tilde{\alpha}_{2n-2} + \sqrt{(n+1)(n+a)}\,\tilde{\alpha}_{2n+2}.$$

If we compare this with equation (3.26), we see that both formulas are almost the same. The unknown objects $\tilde{\alpha}_{2n}(E)$ can therefore be identified with the normalized Meixner-Pollaczek functions. We have

$$\tilde{\alpha}_{2n}(E) = \tilde{P}_n(\frac{E}{2}).$$

We have found the coefficients $\alpha_{2n}(E)$ which determine the formal eigenvector (6.4). They can be written as

$$\alpha_{2n}(E) = (-1)^n \sqrt{\frac{(a)_n}{n!}}\, \mathcal{A}_0(E)\, {}_2F_1\left(\left. \begin{matrix} -n, \frac{a+iE}{2} \\ a \end{matrix} \right| 2 \right), \tag{6.5}$$

with

$$\mathcal{A}_0(E) = \frac{\left|\Gamma\left(\frac{a+iE}{2}\right)\right|}{\sqrt{2^{2-a}\,\pi\,\Gamma(a)}}.$$

So far, we have used the language of formal eigenvectors, but the result can also be formulated in a different mathematical way. Herein, $\hat{H}_b$ acts as a Jacobi operator on

the basis $\{e_{2n}|n \in \mathbb{Z}_+\}$ of $V_0 \cong \ell^2(\mathbb{Z}_+)$. Next, one defines a map Λ from $\ell^2(\mathbb{Z}_+)$ to square integrable functions $L^2(\mathbb{R}, w(E)dE)$, where $w(E) = |\Gamma(\frac{a}{2} + iE)/2|$ is the weight function corresponding to Meixner-Pollaczek polynomials, by

$$(\Lambda e_{2n})(E) = (-i)^n \sqrt{\frac{2^a\, n!}{\pi\, \Gamma(n+a)}}\, P_n(\frac{E}{2}).$$

Then $\Lambda \circ \hat{H}_b = M_E \circ \Lambda$, i.e. Λ intertwines $\hat{H}_b$ acting in $\ell^2(\mathbb{Z}_+)$ with the multiplication operator M_E on $L^2(\mathbb{R}, w(E)dE)$, see [52, Prop. 3.1]. Since we are really dealing with aspects of Wigner quantization here, we shall not overload it with stricter terminology and just use the for physicists more familiar language of formal eigenvectors, delta-functions, etc.

We can now rely on the spectral theorem to find the spectrum of $\hat{H}_b$ in V_0: it is equal to the support of the weight function of the Meixner-Pollaczek polynomials. This weight function, accompanying the polynomials under the integral in equation (3.24), has the real axis as its support. As a result, the spectrum of $\hat{H}_b$ in V_0 is $\mathbb{R}$.

The same technique will give us the spectrum of $\hat{H}_b$ in V_1. We start by defining a formal eigenvector $u_1(E)$, determined by the coefficients $\alpha_{2n+1}(E)$. These are calculated in the same way as the coefficients $\alpha_{2n}(E)$. The analysis and results can be copied exactly, but the parameter a has to be changed into $a+1$. Thus we have

$$\alpha_{2n+1}(E) = (-1)^n \sqrt{\frac{(a+1)_n}{n!}}\, \mathcal{A}_1(E)\; {}_2F_1\left(\begin{matrix} -n, \frac{a+1+iE}{2} \\ a+1 \end{matrix} \middle| 2 \right), \qquad (6.6)$$

with

$$\mathcal{A}_1(E) = \frac{\left|\Gamma\left(\frac{a+1+iE}{2}\right)\right|}{\sqrt{2^{1-a}\,\pi\,\Gamma(a+1)}}.$$

The reason for this is that in V_0 the lowest weight vector is e_0 and $h\, e_0 = a\, e_0$ characterizes this representation. In V_1 on the other hand, e_1 is the lowest weight vector, and the corresponding lowest weight is $a+1$. The conclusion in this case is similar: the spectrum of $\hat{H}_b$ in V_1 is $\mathbb{R}$.

Combining these results, we have:

Theorem 6.2 *In the $\mathfrak{osp}(1|2)$ representation ρ_a with representation space V, the Hamiltonian $\hat{H}_b$ has formal eigenvectors*

$$u_0(E) = \sum_{n=0}^{\infty} \alpha_{2n}(E) e_{2n},$$

with coefficients determined by equation (6.5), and

$$u_1(E) = \sum_{n=0}^{\infty} \alpha_{2n+1}(E) e_{2n+1},$$

with coefficients (6.6). The spectrum of $\hat{H}_b$ in V is $\mathbb{R}$ with multiplicity 2.

It is important to note that the coefficients $\alpha_n(E)$ have been chosen in such a way that the vectors $u_0(E)$ and $u_1(E)$ are delta function normalized vectors. To support this statement, we first observe that

$$\langle \alpha_{2m}(E), \alpha_{2n}(E) \rangle = \int_{-\infty}^{+\infty} \alpha_{2m}^*(E)\, \alpha_{2n}(E)\, dE = \delta_{mn}, \tag{6.7}$$

which is a direct consequence of equation (3.25) and the fact that the Meixner-Pollaczek polynomials are real. We then multiply both sides of equation (6.4) by $\alpha_{2m}^*(E)$ and integrate. With the help of equation (6.7) this yields

$$\int_{-\infty}^{+\infty} u_0(E)\, \alpha_{2m}^*(E)\, dE = \sum_{n=0}^{\infty} e_{2n} \int_{-\infty}^{+\infty} \alpha_{2n}(E)\, \alpha_{2m}^*(E)\, dE = e_{2m}.$$

Now we have an integral expression for e_{2m}, which is used to obtain

$$\begin{aligned} u_0(E') &= \sum_{m=0}^{\infty} a_{2m}(E') \int_{-\infty}^{+\infty} u_0(E)\, \alpha_{2m}^*(E)\, dE \\ &= \int_{-\infty}^{+\infty} \left(\sum_{m=0}^{\infty} \alpha_{2m}(E')\, \alpha_{2m}^*(E) \right) u_0(E)\, dE. \end{aligned}$$

The defining property (3.6) of the Dirac delta function implies that

$$\sum_{m=0}^{\infty} \alpha_{2m}^*(E)\, \alpha_{2m}(E') = \delta(E - E').$$

From this, we can draw the immediate conclusion that $u_0(E)$ is a delta function normalized vector. Indeed:

$$\langle u_0(E), u_0(E') \rangle = \sum_{n=0}^{\infty} \alpha_{2n}^*(E)\, \alpha_{2n}(E') = \delta(E - E').$$

In an analogous way one can prove the same property for the vector $u_1(E)$, since equation (6.7) is also valid for odd indices. More generally, when a vector is decomposed in a certain orthonormal basis (in this case the $\mathfrak{osp}(1|2)$ representation space basis e_n), and when the coordinates (the coefficients $\alpha_n(E)$) are also orthonormal, then the vector is normalized with respect to the delta function. We will encounter this scenario for the eigenvectors of $\hat{x}$, $\hat{p}$ and the Hamiltonian of the free particle $\hat{H}_f$.

Spectrum of $\hat{x} = \frac{(b^+ + b^-)}{\sqrt{2}}$

The spectrum of the operator $\hat{x}$ will be related to generalized Hermite polynomials, and the method of reaching this relation is quite similar as before. The biggest difference is that we now have to take the entire representation space V into account. Thus, the formal eigenvector $v(x)$ of $\hat{x}$ is defined by

$$v(x) = \sum_{n=0}^{\infty} \beta_n(x) e_n, \tag{6.8}$$

satisfying the relation $\hat{x}\, v(x) = x\, v(x)$. The left hand side of this eigenvalue equation can be written as

$$\sum_{n=0}^{\infty} \beta_{2n}(\sqrt{n}\, e_{2n-1} + \sqrt{n+a}\, e_{2n+1}) + \sum_{n=0}^{\infty} \beta_{2n+1}(\sqrt{n+a}\, e_{2n} + \sqrt{n+1}\, e_{2n+2}),$$

using $\hat{x} = (b^+ + b^-)/\sqrt{2}$ together with equations (6.3). We can then compare the coefficients of e_n on both sides of the equation, which will result in a pair of recurrence relations. We have

$$\begin{aligned} x\, \beta_{2n} &= \sqrt{n}\, \beta_{2n-1} + \sqrt{n+a}\, \beta_{2n+1}, \\ x\, \beta_{2n+1} &= \sqrt{n+a}\, \beta_{2n} + \sqrt{n+1}\, \beta_{2n+2}. \end{aligned}$$

It should come as no surprise that we recognize the pair of recurrence relations (3.19) of the normalized generalized Hermite functions. Therefore, we can identify the unknown coefficients $\beta_n(x)$ with those functions:

$$\beta_n(x) = \tilde{T}_n^{(a)}(x).$$

The explicit expression for the $\beta_{2n}(x)$ is given by

$$\beta_{2n}(x) = (-1)^n \sqrt{\frac{(a)_n}{n!}}\, \mathcal{B}_0(x)\, {}_1F_1\left(\begin{matrix} -n \\ a \end{matrix} \middle| x^2 \right), \tag{6.9}$$

with

$$\mathcal{B}_0(x) = \frac{|x|^{a-\frac{1}{2}}\, e^{-\frac{x^2}{2}}}{\sqrt{\Gamma(a)}}.$$

A similar expression describes the β's with an odd index:

$$\beta_{2n+1}(x) = (-1)^n \sqrt{\frac{(a+1)_n}{n!}}\, \mathcal{B}_1(x)\, {}_1F_1\left(\begin{matrix} -n \\ a+1 \end{matrix} \middle| x^2 \right), \tag{6.10}$$

with

$$\mathcal{B}_1(x) = \frac{x\,|x|^{a-\frac{1}{2}}\, e^{-\frac{x^2}{2}}}{\sqrt{\Gamma(a+1)}}.$$

The same arguments as for the spectrum of $\hat{H}_b$ allow us to determine the spectrum of $\hat{x}$. We have

Theorem 6.3 *In the $\mathfrak{osp}(1|2)$ representation ρ_a with representation space V, the position operator $\hat{x}$ has formal eigenvectors*

$$v(x) = \sum_{n=0}^{\infty} \beta_n(x) e_n,$$

with coefficients determined by equations (6.9) and (6.10). The spectrum of $\hat{x}$ in V is equal to $\mathbb{R}$, which is the support of the weight function of the generalized Hermite polynomials.

Note that the major difference with Theorem 6.2 lies in the fact that the spectrum of the position operator does not have a double multiplicity. Each eigenvalue x belongs to exactly one eigenvector $v(x)$.

Spectrum of $\hat{p} = \frac{i(b^+ - b^-)}{\sqrt{2}}$

Determining the spectrum of $\hat{p}$ is now just a formality because of the similar expressions for $\hat{x}$ and $\hat{p}$. In fact, if we denote the formal eigenvector of $\hat{p}$ for the eigenvalue p by

$$w(p) = \sum_{n=0}^{\infty} \gamma_n(p) e_n,$$

then it is not hard to see that the coefficients $\gamma_n(p)$ can be written as

$$\gamma_n(p) = i^n \beta_n(p),$$

where the $\beta_n(p)$ are given by equations (6.9) and (6.10). Thus, these coefficients are again generalized Hermite functions and we have

Theorem 6.4 *In the $\mathfrak{osp}(1|2)$ representation ρ_a with representation space V, the momentum operator $\hat{p}$ has formal eigenvectors*

$$w(p) = \sum_{n=0}^{\infty} i^n \beta_n(p) e_n,$$

with coefficients determined by equations (6.9) and (6.10). Just like $\hat{x}$, $\hat{p}$ has a spectrum in V that is equal to $\mathbb{R}$.

As pointed out earlier in this section, the eigenvectors of $\hat{x}$ and $\hat{p}$ are normalized with respect to the delta function. Thus we have

$$\langle v(x), v(x') \rangle = \delta(x - x')$$

and

$$\langle w(p), w(p') \rangle = \delta(p - p'),$$

ensuing from the fact that $\beta_n(x)$ and $\gamma_n(p)$ are orthonormal functions.

6.1.3 Generalized wave functions

In the previous section we have constructed formal eigenvectors for all relevant operators $\hat{H}_b$, $\hat{x}$ and $\hat{p}$. Finding the wave functions corresponding to the physical states $u_0(E)$, $u_1(E)$, $v(x)$ and $w(p)$ requires the mutual inner products between these vectors. Therefore, we will compute these inner products before analyzing the wave functions explicitly.

Mutual inner products

This section is dedicated to the purely mathematical calculation of all inner products between the vectors $u_0(E)$, $u_1(E)$, $v(x)$ and $w(p)$. Two existing formulas will be

crucial in these calculations. One of them is found in [101, Proposition 2] and states that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, b\\ a\end{matrix}\middle| y\right) {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| x\right) t^n \\ = \quad (1-t)^{b-a}(1-t+yt)^{-b}\, e^{\frac{xt}{t-1}}\, {}_1F_1\left(\begin{matrix}b\\ a\end{matrix}\middle| \frac{xyt}{(1-t)(1-t+yt)}\right),\end{aligned} \tag{6.11}$$

where $|t| < 1$. The other formula is the following approximation [99, equation (7.9)]:

$$\lim_{z\to\infty} {}_1F_1\left(\begin{matrix}c\\ a\end{matrix}\middle| z\right) = \lim_{z\to\infty} \frac{\Gamma(a)}{\Gamma(c)}\, e^z\, z^{c-a} \sum_{n=0}^{\infty} \frac{(a-c)_n\,(1-c)_n}{n!}\, z^{-n}. \tag{6.12}$$

These two equations will appear useful in the following.

Let us first examine the inner products between the eigenvectors of $\hat{x}$ and $\hat{H}_b$. We will only calculate

$$\langle v(x), u_0(E)\rangle = \sum_{n=0}^{\infty} \alpha_{2n}(E)\beta_{2n}^*(x)$$

explicitly, since $\langle v(x), u_1(E)\rangle$ is found in a highly similar way. The functions $\beta_n(x)$ are real, so the complex conjugation can be dropped. Using equations (6.5) and (6.9), we can write $\langle v(x), u_0(E)\rangle$ as

$$\mathcal{A}_0(E)\,\mathcal{B}_0(x) \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, \frac{a+iE}{2}\\ a\end{matrix}\middle| 2\right) {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| x^2\right).$$

In order to determine this summation, we need to add a factor t^n to the nth summand and then take the limit for $t \to 1$, with $t < 1$. Using equation (6.11), we rewrite $\sum_{n=0}^{\infty} \alpha_{2n}(E)\beta_{2n}(x)t^n$ as

$$\mathcal{A}_0(E)\,\mathcal{B}_0(x)(1-t)^{\frac{-a+iE}{2}}\,(1+t)^{\frac{-a-iE}{2}}\, e^{\frac{x^2t}{t-1}}\, {}_1F_1\left(\begin{matrix}\frac{a+iE}{2}\\ a\end{matrix}\middle| \frac{2x^2t}{(1-t)(1+t)}\right). \tag{6.13}$$

The limit for $t \to 1$ is found with the help of equation (6.12). After simplification,

we have

$$\begin{aligned}\langle v(x), u_0(E)\rangle &= \mathcal{A}_0(E)\,\mathcal{B}_0(x)\,\frac{\Gamma(a)}{\Gamma(\frac{a+iE}{2})}\,\lim_{t\to 1} e^{\frac{x^2 t}{1+t}}\,(|x|\sqrt{2t})^{-a+iE}(1+t)^{-iE}\\ &= \frac{\left|\Gamma(\frac{a+iE}{2})\right|}{\Gamma(\frac{a+iE}{2})}\,\frac{|x|^{iE-\frac{1}{2}}}{\sqrt{\pi}\,2^{iE+2}}\\ &= \mathcal{C}_0(E)\,\frac{|x|^{iE-\frac{1}{2}}}{2\sqrt{\pi}},\end{aligned} \tag{6.14}$$

where $|\mathcal{C}_0(E)|^2 = 1$. A similar derivation provides us with the inner product $\langle v(x), u_1(E)\rangle$. It is

$$\begin{aligned}\langle v(x), u_1(E)\rangle &= \frac{\left|\Gamma(\frac{a+1+iE}{2})\right|}{\Gamma(\frac{a+1+iE}{2})}\,\frac{x\,|x|^{iE-\frac{3}{2}}}{\sqrt{\pi}\,2^{iE+2}}\\ &= \mathcal{C}_1(E)\,\frac{x|x|^{iE-\frac{3}{2}}}{2\sqrt{\pi}},\end{aligned} \tag{6.15}$$

where $|\mathcal{C}_1(E)|^2 = 1$.

We can deduce the inner products between the formal eigenvectors of $\hat{p}$ and $\hat{H}_b$ using (6.14) and (6.15). First observe that

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\, z\right) = (1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\middle|\, \frac{z}{z-1}\right),$$

together with $\mathcal{A}_0(E) = \mathcal{A}_0(-E)$, implies that

$$(-1)^n \alpha_{2n}(E) = \alpha_{2n}(-E).$$

A similar identity holds for odd indices of the coefficients $\alpha_n(E)$: $(-1)^n\alpha_{2n+1}(E) = \alpha_{2n+1}(-E)$. We have used a standard hypergeometric identity that can be found in every book with an introduction to hypergeometric functions, such as [50] or [97]. It is then possible to see that

$$\langle w(p), u_0(E)\rangle = \langle v(p), u_0(-E)\rangle \tag{6.16}$$

and

$$\langle w(p), u_1(E)\rangle = -i\,\langle v(p), u_1(-E)\rangle\,, \tag{6.17}$$

where the inner products with a vector v can be found in equations (6.14) and (6.15).

What remains to be calculated is the inner product between the eigenvectors of the position and momentum operators $\hat{x}$ and $\hat{p}$. This inner product can be written as:

$$\langle v(x), w(p)\rangle = \sum_{n=0}^{\infty} i^n \beta_n(x)\beta_n(p),$$

which splits up as a sum over even indices and a sum over odd indices. Both series can be computed in an analogous fashion. For the even part we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\beta_{2n}(x)\beta_{2n}(p)& \\ = \quad \mathcal{B}_0(x)\mathcal{B}_0(p)\sum_{n=0}^{\infty}(-1)^n\frac{(a)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| x^2\right){}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| p^2\right).&\end{aligned} \tag{6.18}$$

If we manage to write one of the ${}_1F_1$-series as a ${}_2F_1$ Gauss hypergeometric function, we can rely on equation (6.11) to simplify this equation. We shall make use of the following identity (see e.g. [50]):

$$\lim_{b\to\infty}{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle|\frac{z}{b}\right) = {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle| z\right). \tag{6.19}$$

Applying this to ${}_1F_1\left(\begin{smallmatrix}-n\\ a\end{smallmatrix}\middle| x^2\right)$, we see that the summation in the right-hand side of equation (6.18) can be calculated with the help of equation (6.11). After some simplifications, this summation translates to

$$2^{-a}e^{\frac{p^2}{2}}\lim_{b\to\infty}\left(\frac{b}{b-\frac{x^2}{2}}\right)^b {}_1F_1\left(\begin{matrix}b\\ a\end{matrix}\middle|\frac{x^2p^2}{2(x^2-2b)}\right).$$

Now the limit for $b\to\infty$ can be taken for both factors. The first becomes $e^{x^2/2}$, while the second is ${}_0F_1\left(\begin{smallmatrix}-\\ a\end{smallmatrix}\middle| -x^2p^2/4\right)$. Putting all this together, we find that the even part of the inner product $\langle v(x), w(p)\rangle$ is

$$\sum_{n=0}^{\infty}(-1)^n\beta_{2n}(x)\beta_{2n}(p) = \frac{|xp|^{a-\frac{1}{2}}}{2^a\,\Gamma(a)}\,{}_0F_1\left(\begin{matrix}-\\ a\end{matrix}\middle| -\frac{x^2p^2}{4}\right).$$

The odd part of the inner product is found in the same way, so we have that $\langle v(x), w(p)\rangle$ equals

$$\frac{|xp|^{a-\frac{1}{2}}}{2^a\,\Gamma(a)}\left({}_0F_1\left(\begin{matrix}-\\ a\end{matrix}\middle| -\frac{x^2p^2}{4}\right)+\frac{ixp}{2a}\,{}_0F_1\left(\begin{matrix}-\\ a+1\end{matrix}\middle| -\frac{x^2p^2}{4}\right)\right). \tag{6.20}$$

In the canonical case, many of these expressions simplify significantly. We discuss these simplifications for $a = 1/2$ in Section 6.1.3.

Generalized wave functions and the canonical case

Consider an arbitrary state of the system $|\psi\rangle$, written in Dirac's bra-ket notation. Assume that the eigenstates of the position operator are denoted by $|x\rangle$. Then equation (3.5) tells us that the spatial wave function of the system is found by

$$\psi(x) = \langle x|\psi\rangle\,.$$

Similarly, an inner product describes the wave function in the momentum space:

$$\psi(p) = \langle p|\psi\rangle\,,$$

where $|p\rangle$ represents the momentum eigenstates.

We have written the position and momentum eigenvectors as $v(x)$ and $w(p)$ respectively. Their inner product, given by equation (6.20), represents the wave function of the particle being located at position x, when the system is in the momentum eigenstate p. This result is compatible with the canonical case where $a = 1/2$. In canonical quantization, $\hat{x}$ and $\hat{p}$ are known. The operator $\hat{x}$ is simply multiplication with x and $\hat{p} = -i\partial_x$. For $a = 1/2$ equation (6.20) reduces to

$$\langle v(x), w(p)\rangle = \frac{1}{\sqrt{2}\,\Gamma(\frac{1}{2})}\big(\cos(xp) + i\sin(xp)\big) = \frac{1}{\sqrt{2\pi}}\,e^{ixp},$$

which is an eigenfunction of the canonical interpretation of the operator $\hat{p}$ with eigenvalue p.

The case where the eigenstate $|\psi\rangle$ corresponds to the energy E needs to be handled with a little more care, for there are two energy eigenstates corresponding to E. Both $u_0(E)$ and $u_1(E)$ belong to the same energy eigenvalue, thus inducing two independent wave functions $\psi_E^0(x)$ and $\psi_E^1(x)$. The previous results (6.14) and (6.15) allow us to write

$$\psi_E^0(x) = \mathcal{C}_0(E)\,\frac{|x|^{iE-\frac{1}{2}}}{2\sqrt{\pi}} \tag{6.21}$$

and

$$\psi_E^1(x) = \mathcal{C}_1(E)\,\frac{x|x|^{iE-\frac{3}{2}}}{2\sqrt{\pi}}, \tag{6.22}$$

with $|\mathcal{C}_0|^2 = |\mathcal{C}_1|^2 = 1$. Therefore the general wave function of the particle when the system's energy equals E must be of the form

$$\psi_E^{(a)}(x) = A\psi_E^0(x) + B\psi_E^1(x), \tag{6.23}$$

with A and B complex coefficients satisfying $|A|^2 + |B|^2 = 1$. This result is compatible with the canonical case as well, which is no surprise since $\psi_E^{(a)}(x)$ is practically independent of a. In fact, in equations (6.21) and (6.22) a only appears explicitly in the phase factors $\mathcal{C}_0(E)$ and $\mathcal{C}_1(E)$. For $\hat{x} = x$ and $\hat{p} = -i\partial_x$, the Hamiltonian $\hat{H}_b$ converts into

$$\hat{H}_b = -i(x\partial_x + \frac{1}{2}),$$

which, for $a = 1/2$, indeed has $\psi_E^0(x)$ and $\psi_E^1(x)$ as eigenfunctions with eigenvalue E. Moreover, the general wave function (6.23) is normalized. We have (omitting the superscript (a) for clarity)

$$\langle \psi_{E'}(x), \psi_E(x) \rangle = \int_{-\infty}^{+\infty} \psi_{E'}^*(x)\psi_E(x)dx = \delta(E' - E),$$

which follows from equation (3.7)

In summary, we have

Theorem 6.5 *In the Wigner quantization of the Hamiltonian $\hat{H}_b = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$, the wave function of the particle with position coordinate x, when the total energy of the system equals E is given by*

$$\psi_E^{(a)}(x) = A\,\mathcal{C}_0(E)\,\frac{|x|^{iE-\frac{1}{2}}}{2\sqrt{\pi}} + B\,\mathcal{C}_1(E)\,\frac{x|x|^{iE-\frac{3}{2}}}{2\sqrt{\pi}},$$

with $|\mathcal{C}_0|^2 = |\mathcal{C}_1|^2 = 1$ and $|A|^2 + |B|^2 = 1$.

The wave function of the particle with position coordinate x, when the system's momentum is equal to p is given by

$$\phi_p^{(a)}(x) = \frac{|xp|^{a-\frac{1}{2}}}{2^a\,\Gamma(a)}\left({}_0F_1\left(\begin{matrix}-\\ a\end{matrix}\middle|-\frac{x^2p^2}{4}\right) + \frac{ixp}{2a}\,{}_0F_1\left(\begin{matrix}-\\ a+1\end{matrix}\middle|-\frac{x^2p^2}{4}\right)\right).$$

Both results are compatible with the known expressions for these wave functions in canonical quantization, which occurs when $a = 1/2$.

The wave function in the momentum basis is a superposition of two independent wave functions determined by (6.16) and (6.17).

6.2 The Hamiltonian of the free particle

Curiously, although many one-dimensional Hamiltonians have been studied before in the context of Wigner quantization, the simplest of them all had been forgotten until now. We choose to fill this lacuna next to the Berry-Keating-Connes Hamiltonian because, concerning Wigner quantization, there are a lot of similarities between $\hat{H}_b$ and the Hamiltonian of the free particle $\hat{H}_f$. Since most of the proofs and methods are very much alike, we will not mention any calculations unless they differ significantly from analogous computations in the previous section.

6.2.1 Relation with the $\mathfrak{osp}(1|2)$ Lie superalgebra

The system of a particle having no potential energy, is described by the Hamiltonian

$$\hat{H}_f = \frac{1}{2}\hat{p}^2, \tag{6.24}$$

where $\hat{p}$ is the momentum operator of the particle. We note that the Hamiltonian is independent of the position operator $\hat{x}$. Performing the Wigner quantization for the free particle starts with writing down Hamilton's equations and the equations of Heisenberg for this system. Hamilton's equations give

$$\dot{\hat{p}} = -\mathrm{op}\left(\frac{\partial H_f}{\partial x}\right) = 0, \qquad \dot{\hat{x}} = \mathrm{op}\left(\frac{\partial H_f}{\partial p}\right) = \hat{p}.$$

Together with the equations of Heisenberg (for $\hbar = 1$)

$$[\hat{H}_f, \hat{p}] = -i\dot{\hat{p}}, \qquad [\hat{H}_f, \hat{x}] = -i\dot{\hat{x}}$$

we obtain a set of compatibility conditions

$$[\hat{H}_f, \hat{p}] = 0, \qquad [\hat{H}_f, \hat{x}] = -i\hat{p},$$

which is equivalent to

$$[\hat{p}^2, \hat{x}] = -2i\hat{p}. \tag{6.25}$$

Just as for the Berry-Keating-Connes Hamiltonian, the operators $\hat{x}$ and $\hat{p}$ subject to the current compatibility conditions (6.25) generate the Lie superalgebra $\mathfrak{osp}(1|2)$. This is not so straightforward to prove as before.

Theorem 6.6 *If the operators $\hat{x}$ and $\hat{p}$, subject to the relation $[\hat{p}^2, \hat{x}] = -2i\hat{p}$, are considered to be odd elements of some superalgebra, then they must generate the Lie superalgebra $\mathfrak{osp}(1|2)$.*

Proof. We introduce the operators b^+ and b^- as before, see equation (6.2), by

$$b^{\pm} = \frac{\hat{x} \mp i\hat{p}}{\sqrt{2}}.$$

The aim of this proof is to show that b^+ and b^- satisfy the defining relations (2.16) of $\mathfrak{osp}(1|2)$:

$$\big[\{b^-, b^+\}, b^{\pm}\big] = \pm 2b^{\pm}.$$

First we want to rewrite the compatibility conditions (6.25) in terms of b^+ and b^-. For this purpose, we start by noticing that

$$[\hat{p}^2, b^{\pm}] = b^+ - b^-. \tag{6.26}$$

Either of these two relations can be fully rewritten in terms of b^+ and b^-. The result is the same, so let us focus on the commutator of $\hat{p}^2$ and b^+. Substituting

$$\hat{p}^2 = -\frac{1}{2}\big((b^+)^2 + (b^-)^2 - \{b^+, b^-\}\big)$$

in this relation, we obtain

$$[\{b^+, b^-\}, b^+] - [(b^-)^2, b^+] = 2(b^+ - b^-).$$

Writing down the commutators and anticommutators explicitly, it is straightforward to see that

$$[(b^-)^2, b^+] = -[\{b^+, b^-\}, b^-],$$

so that finally both equations in (6.26) are equivalent to

$$[\{b^+, b^-\}, b^+ + b^-] = 2(b^+ - b^-). \tag{6.27}$$

Next, assume that b^+ and b^- are odd elements of a Lie superalgebra L, that is $b^{\pm} \in L_{\bar{1}}$, where $L_{\bar{1}}$ is the odd part of the Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$. Anticommutators of the odd elements are situated in the even part $L_{\bar{0}}$ of L. By the definition of the

Lie superbracket, it is then necessary that the commutator of an odd and an even element must be an odd element again. Thus we can put

$$[\{b^+, b^-\}, b^+] = -[(b^+)^2, b^-] = Ab^+ + Bb^- \tag{6.28}$$

for some constants A and B. The first equality again follows from explicit computation of the commutators and anticommutators. With the help of equation (6.27) we then find:

$$[\{b^+, b^-\}, b^-] = -[(b^-)^2, b^+] = (-A+2)b^+ - (B+2)b^-. \tag{6.29}$$

We are able to determine the constants A and B by calculating $[(b^+)^2, (b^-)^2]$ in two different ways. Indeed, from

$$\begin{aligned}[(b^+)^2, (b^-)^2] &= b^+[b^+, (b^-)^2] + [b^+, (b^-)^2]b^+ \\ &= -2(A-2)(b^+)^2 - (B+2)\{b^+, b^-\}\end{aligned}$$

and

$$\begin{aligned}[(b^+)^2, (b^-)^2] &= b^-[(b^+)^2, b^-] + [(b^+)^2, b^-]b^- \\ &= -2B(b^-)^2 - A\{b^+, b^-\}\end{aligned}$$

we can conclude that $A = 2$ and $B = 0$. Substituting this into (6.28) and (6.29), we obtain the $\mathfrak{osp}(1|2)$ defining relations (2.16). □

Thus, the operators $\hat{x}$ and $\hat{p}$ subject to the compatibility conditions (6.25), have solutions in terms of $\mathfrak{osp}(1|2)$ generators b^+ and b^-. Clearly $\hat{x}$ and $\hat{p}$ have the same expression as before, and the Hamiltonian $\hat{H}_f$ can be written as

$$\hat{H}_f = -\frac{1}{4}((b^+)^2 + (b^-)^2 - \{b^+, b^-\})$$

Moreover, since $\hat{x}$ and $\hat{p}$ together with the compatibility conditions generate $\mathfrak{osp}(1|2)$, no other Lie superalgebra solutions exist.

6.2.2 Energy spectrum of the free particle

In this section, we will show that the spectrum of the Hamiltonian (6.24) is equal to $\mathbb{R}^+$ with double multiplicity. The Hamiltonian of the free particle written in terms of the $\mathfrak{osp}(1|2)$ generators $b^\pm$ shows that $\hat{H}_f$ is an even operator in $\mathfrak{osp}(1|2)$:

$$\hat{H}_f = -\frac{1}{2}(e - f - h),$$

with h, e and f defined in equation (2.17). Together with the $*$-condition $(b^{\pm})^{\dagger} = b^{\mp}$ this implies that the $*$-representations of $\mathfrak{osp}(1|2)$ described in Section 2.4 come into the picture again. These representations ρ_a have a representation space $V = V_0 \oplus V_1$ which can be written as a direct sum of two subspaces. As we have already argued, an even operator can be regarded in both $\mathfrak{su}(1,1)$ subspaces V_0 and V_1 separately, which explains how we will end up with a double multiplicity in the energy spectrum.

Let $z_0(E)$ be a formal eigenvector of $\hat{H}_f$ with eigenvalue E in the subspace V_0. We write this eigenvector as

$$z_0(E) = \sum_{n=0}^{\infty} \epsilon_{2n}(E) e_{2n},$$

where the $\epsilon_{2n}(E)$ are coefficients we wish to determine. We can let $\hat{H}_f$ operate on this eigenvector, which gives

$$\hat{H}_f z_0(E) = -\frac{1}{2} \sum_{n=0}^{\infty} \epsilon_{2n}(E) \Big(\sqrt{n(n+a-1)}\, e_{2n-2} - (2n+a)\, e_{2n} + \sqrt{(n+1)(n+a)}\, e_{2n+2} \Big).$$

Since $z_0(E)$ is postulated to be an eigenvector of $\hat{H}_f$ with eigenvalue E, we also have

$$\hat{H}_f z_0(E) = \sum_{n=0}^{\infty} E\, \epsilon_{2n}(E) e_{2n}.$$

Collecting the coefficients of e_{2n} in both expressions for $\hat{H}_f z_0(E)$, delivers us a three term recurrence relation for the unknown functions $\epsilon_{2n}(E)$:

$$(2n - 2E + a)\, \epsilon_{2n} = \sqrt{n(n+a-1)}\, \epsilon_{2n-2} + \sqrt{(n+1)(n+a)}\, \epsilon_{2n+2}.$$

This is recognized to be the recurrence relation (3.22) of the normalized Laguerre functions with $\alpha = a - 1$ and $x = 2E$. So we can identify the $\epsilon_{2n}(E)$ with these functions:

$$\epsilon_{2n}(E) = \tilde{L}_n^{(a-1)}(2E) = \sqrt{\frac{(a)_n}{n!}}\, \mathcal{E}_0(E)\, {}_1F_1\left(\begin{matrix} -n \\ a \end{matrix} \middle| 2E \right),$$

with

$$\mathcal{E}_0(E) = \frac{E^{\frac{a-1}{2}}\, e^{-E}\, \sqrt{2^a}}{\sqrt{\Gamma(a)}}.$$

Moreover, due to the chosen normalization we have

$$\int_0^{+\infty} \epsilon_{2m}(E)\epsilon_{2n}(E)dE = \delta_{mn}.$$

The Laguerre polynomials have a weight function with a positive support, so the spectrum of $\hat{H}_f$ in V_0 is $\mathbb{R}^+$. In a similar way, one can define a formal eigenvector of $\hat{H}_f$ with eigenvalue E in V_1:

$$z_1(E) = \sum_{n=0}^{\infty} \epsilon_{2n+1}(E)e_{2n+1}.$$

The same procedure as before reveals that

$$\epsilon_{2n+1}(E) = \tilde{L}_n^{(a)}(2E) = \sqrt{\frac{(a+1)_n}{n!}}\,\mathcal{E}_1(E)\;{}_1F_1\left(\begin{matrix}-n\\ a+1\end{matrix}\middle|\, 2t\right),$$

with

$$\mathcal{E}_1(E) = \frac{E^{\frac{a}{2}}\, e^{-E}\,\sqrt{2^{a+1}}}{\sqrt{\Gamma(a+1)}}.$$

Thus, we can conclude that the spectrum of $\hat{H}_f$ in V_1 is also equal to $\mathbb{R}^+$. Combining both results, we obtain

Theorem 6.7 *The spectrum of $\hat{H}_f = -(e-f-h)/2$ in the representation space V equals $\mathbb{R}^+$ with multiplicity two.*

Now that we have found the energy eigenstates of $\hat{H}_f$, the corresponding wave function remains to be sought. Crucial for this is the determination of some inner products.

6.2.3 Remaining inner products

Since $\langle v(x), w(p)\rangle$ is already known, the remaining inner products to be found are

$$\langle v(x), z_0(E)\rangle \qquad \text{and} \qquad \langle v(x), z_1(E)\rangle$$

and

$$\langle w(p), z_0(E)\rangle \qquad \text{and} \qquad \langle w(p), z_1(E)\rangle\,.$$

The first two are rather straightforward, once the right identity is traced. In [81, equation (5.11.3.7)] we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| x\right) {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| y\right) t^n \\ = \quad (1-t)^{-a}\, e^{\frac{t(x+y)}{t-1}}\, {}_0F_1\left(\begin{matrix}-\\ a\end{matrix}\middle| \frac{txy}{(1-t)^2}\right),\end{aligned} \tag{6.30}$$

for $|t| < 1$. The eigenvectors of $\hat{x}$ and $\hat{H}_f$ have an inner product that can be found by means of (6.30). We have

$$\begin{aligned}\langle v(x), z_0(E)\rangle &= \sum_{n=0}^{\infty} \beta_{2n}(x)\, \epsilon_{2n}(E) \\ &= \mathcal{B}_0(x)\mathcal{E}_0(E) \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n}{n!}\, {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| x^2\right) {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| 2E\right).\end{aligned}$$

Using (6.30) this simplifies to

$$\langle v(x), z_0(E)\rangle = \frac{1}{\sqrt{2^a}\,\Gamma(a)}\, |x|^{a-\frac{1}{2}}\, E^{\frac{a-1}{2}}\, {}_0F_1\left(\begin{matrix}-\\ a\end{matrix}\middle| -\frac{2E\,x^2}{4}\right). \tag{6.31}$$

In a similar way, one obtains

$$\langle v(x), z_1(E)\rangle = \frac{1}{\sqrt{2^{a+1}}\,\Gamma(a+1)}\, x|x|^{a-\frac{1}{2}}\, E^{\frac{a}{2}}\, {}_0F_1\left(\begin{matrix}-\\ a+1\end{matrix}\middle| -\frac{2E\,x^2}{4}\right). \tag{6.32}$$

The other pair of inner products is a harder nut to crack. We present the highlights of the computation. Omitting a factor of $\mathcal{B}_0(p)\mathcal{E}_0(E)$, the calculation of the inner product $\langle w(p), z_0(E)\rangle$ boils down to finding the summation

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| p^2\right) {}_1F_1\left(\begin{matrix}-n\\ a\end{matrix}\middle| 2E\right). \tag{6.33}$$

We apply equation (6.19) on the hypergeometric function with argument $2E$, before using equations (6.11) and (6.12) to simplify the result. After a while, we obtain

$$\Gamma(a)\, p^{-2a} \lim_{b\to\infty} \frac{1}{\Gamma(b)} \left(\frac{p^2 b}{2E}\right)^b \exp\left(\frac{p^2(2E-b)}{2E}\right).$$

For very large b, the Gamma function can be written as $\sqrt{2\pi}\, b^{b-1/2} \exp(-b)$, so we can simplify further to reach

$$\frac{\Gamma(a)}{\sqrt{2}}\, p^{-2a} \exp(p^2) \lim_{b\to\infty} \sqrt{\frac{b}{\pi}} \exp\left(b \left[1 + \ln(\frac{p^2}{2E}) - \frac{p^2}{2E} \right] \right).$$

At this point, we can introduce the delta function by one of its limit definitions (see for instance [16])

$$\delta(x) = \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{\pi\epsilon}} \exp(-x^2/\epsilon).$$

Moreover, properties of the delta function allow us to write

$$\delta\left(\sqrt{\frac{p^2}{2E} - \ln(\frac{p^2}{2E}) - 1} \right) = 2E\sqrt{2}\,\delta(p^2 - 2E),$$

so that the summation (6.33) can be written as

$$\Gamma(a)\, e^{2E} (2E)^{1-a}\, \delta(p^2 - 2E).$$

Finding the desired inner product $\langle w(p), z_0(E)\rangle$ requires adding a factor $\mathcal{B}_0(p)\mathcal{E}_0(E)$ to the summation (6.33). Simplifying once again finally gives

$$\langle w(p), z_0(E)\rangle = \sqrt{2}\sqrt{|p|}\,\delta(p^2 - 2E). \tag{6.34}$$

Analogously, one finds

$$\langle w(p), z_1(E)\rangle = \sqrt{2}\frac{p}{\sqrt{|p|}}\,\delta(p^2 - 2E). \tag{6.35}$$

At last we are ready to determine the generalized wave functions for the free particle and to compare our results with the canonical case.

6.2.4 Generalized wave function and the canonical case

The free particle has been thoroughly investigated in the past, which makes it easy to check if our results are compatible with what is known. Luckily, they are. We have two wave functions belonging to the same energy eigenvalue E. In the position basis, we shall call them $\psi_E^0(x)$ and $\psi_E^1(x)$ and they are defined as the inner products

(6.31) and (6.32) respectively. The general wave function of the particle when the system's energy equals E, written in the position basis must then be

$$\psi_E^{(a)}(x) = A\psi_E^0(x) + B\psi_E^1(x), \tag{6.36}$$

with A and B complex coefficients satisfying $|A|^2 + |B|^2 = 1$. This wave function is normalized with respect to the delta function and it is compatible with the canonical case when $a = 1/2$. Indeed, for $a = 1/2$ equations (6.31) and (6.32) become

$$\psi_E^0(x) = \frac{(2E)^{-\frac{1}{4}}}{\sqrt{\pi}} \cos(x\sqrt{2E})$$

and

$$\psi_E^1(x) = \frac{(2E)^{-\frac{1}{4}}}{\sqrt{\pi}} \sin(x\sqrt{2E}).$$

These are both eigenfunctions with eigenvalue E of the canonical interpretation of the Hamiltonian $\hat{H}_f = -\partial_x^2/2$. Note that, in the canonical case, the solution (6.36) is equivalent to

$$\left(\frac{A - iB}{\sqrt{2}}\right) \frac{(2E)^{-\frac{1}{4}}}{\sqrt{2\pi}} e^{ix\sqrt{2E}} + \left(\frac{A + iB}{\sqrt{2}}\right) \frac{(2E)^{-\frac{1}{4}}}{\sqrt{2\pi}} e^{-ix\sqrt{2E}},$$

with $|(A - iB)/\sqrt{2}|^2 + |(A + iB)/\sqrt{2}|^2 = 1$. This is the more traditional way of writing the normalized wave function of the free particle, as a superposition of a plane wave moving to the right, and a plane wave going to the left. Note that this canonical notation for the wave function is in accordance with the fact that the energy can be written as $E = p^2/2$ in this case.

Theorem 6.8 *In the Wigner quantization of the Hamiltonian $\hat{H}_f = \frac{\hat{p}^2}{2}$, the wave function of the particle with position coordinate x, when the total energy of the system is equal to E is given by*

$$\begin{aligned}\psi_E^{(a)}(x) = &\frac{A}{\sqrt{2^a}\,\Gamma(a)} |x|^{a-\frac{1}{2}} E^{\frac{a-1}{2}} {}_0F_1\left(\begin{matrix} - \\ a \end{matrix}\middle| -\frac{2E\,x^2}{4}\right) \\ &+ \frac{B}{\sqrt{2^{a+1}}\,\Gamma(a+1)} x|x|^{a-\frac{1}{2}} E^{\frac{a}{2}} {}_0F_1\left(\begin{matrix} - \\ a+1 \end{matrix}\middle| -\frac{2E\,x^2}{4}\right),\end{aligned}$$

with $|A|^2 + |B|^2 = 1$. This result is compatible with the known expression for this wave function in canonical quantization, which occurs when $a = 1/2$.

In the momentum basis, the wave function is a superposition of the independent wave functions determined by equations (6.34) and (6.35).

6.3 Conclusions

We have looked at the Wigner quantization of two different one-dimensional systems in this chapter. The first system is described by the Berry-Keating-Connes Hamiltonian $\hat{H} = \hat{x}\hat{p}$ and in the second part the Hamiltonian of the free particle $\hat{H}_f = \hat{p}^2/2$ is investigated.

Although both systems are entirely different, there are a lot of similarities in their Wigner quantization. Each time solutions for the compatibility conditions were found in terms of generators of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. Moreover, the self-adjointness of the position and momentum operators defined a $*$-structure on the Lie superalgebra. For the odd elements the $*$-structure can be written as $(b^{\pm})^{\dagger} = b^{\mp}$, so by the results of Section 2.4 the actions of the operators $\hat{x}$ and $\hat{p}$ are found by looking at the positive discrete series representations of $\mathfrak{osp}(1|2)$, which are characterized by a parameter a. We find that the representation space on which all operators act, can be written as a direct sum $V = V_0 \oplus V_1$. Both Hamiltonians also act on this Hilbert space, but their action is confined to one of the two subspaces. The fact that these Hamiltonians have a spectrum with double multiplicity must be attributed to this observation, with the two subspaces V_0 and V_1 having the same structure.

Likewise, there are some differences to be found when looking at the two Hamiltonians. The spectrum of the Berry-Keating-Connes Hamiltonian covers the entire real axis, while the spectrum of $\hat{H}_f$ only contains positive energy values. The reason for this is that the orthogonal polynomials describing the energy eigenstates have a different support. The support of the weight function of the Meixner-Pollaczek polynomials, appearing for $\hat{H} = \hat{x}\hat{p}$ equals $\mathbb{R}$, while the weight function of the Laguerre polynomials, related to the free particle system, has the positive real axis as its support.

The framework of Wigner quantization is less restrictive than canonical quantization. Therefore, generalized wave functions of the systems can be constructed. These wave functions, two for each Hamiltonian, are expected to be dependent on the representation parameter a. Remarkably, for $\hat{H} = \hat{x}\hat{p}$ this is not very much the case. Only the phase factors of the two independent wave functions contain a. In contrast with this, we find that the essential structure of the wave functions for the

free particle is altered when a changes. Here, we have an actual generalization of the canonical solution.

The latter remark suggests that one is able to retrieve the canonical case in some way. Indeed, one specific representation of $\mathfrak{osp}(1|2)$ corresponds to the canonical picture. For each Hamiltonian we find back the known canonical results for $a = 1/2$.

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