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FACULTY OF ENGINEERING

The iHMMpredict algorithm

State sequence prediction in imprecise hidden Markov models

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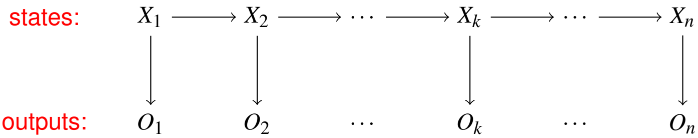
Imprecise Hidden Markov Model (iHMM)



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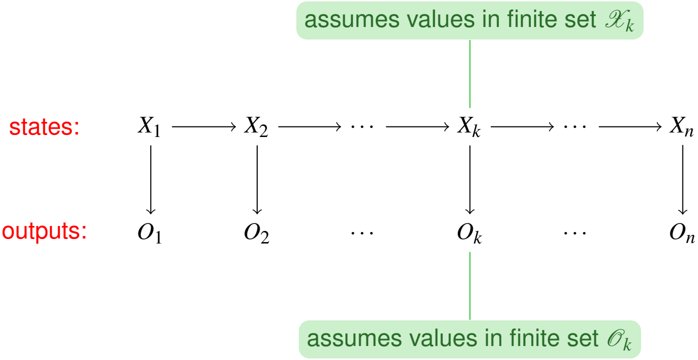

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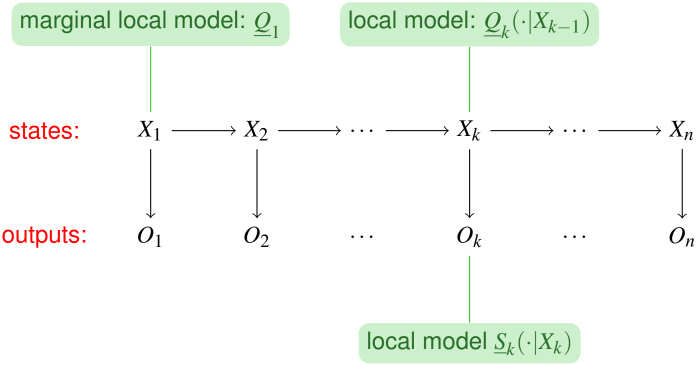
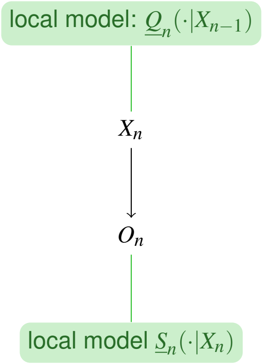


Recursive construction of the joint

First step



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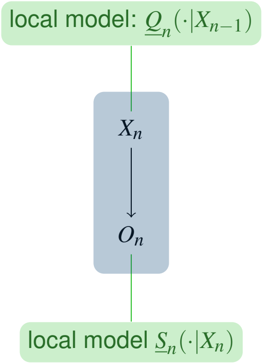

Joint model $\underline{P}_n(\cdot|X_{n-1}) := \underline{Q}_n(\underline{S}_n(\cdot|X_n)|X_{n-1})$

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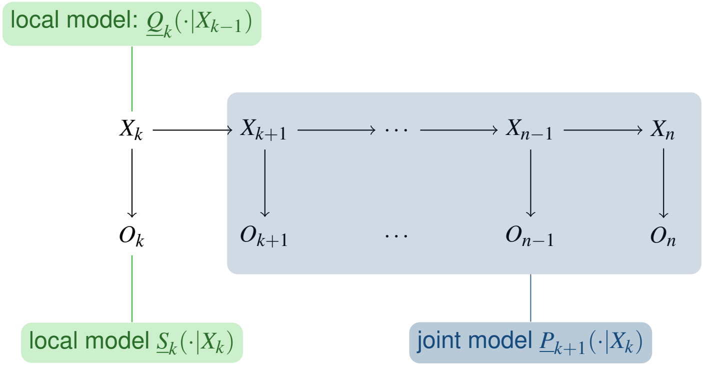
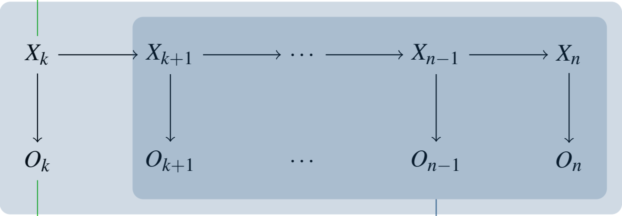
Recursive step



Recursive construction of the joint

Recursive step

local model: $\underline{Q}_k(\cdot|X_{k-1})$



local model $\underline{S}_k(\cdot|X_k)$

joint model $\underline{P}_{k+1}(\cdot|X_k)$

Independent natural extension $\underline{E}_k(\cdot|X_k) := \underline{S}_k(\cdot|X_k) \otimes \underline{P}_{k+1}(\cdot|X_k)$

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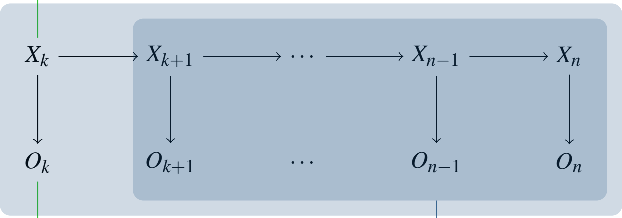
Recursive construction of the joint

Recursive step

local model: $\underline{Q}_k(\cdot|X_{k-1})$



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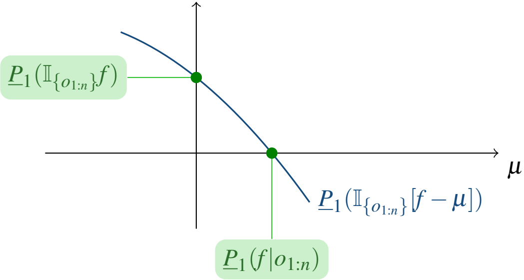
Joint model $\underline{P}_k(\cdot|X_{k-1}) := \underline{Q}_k(\underline{E}_k(\cdot|X_k)|X_{k-1})$

$\underline{E}_k(\cdot|X_k)$ is factorising $\Rightarrow$ very handy recursive expressions!

Updating the joint with the observations $o_{1:n}$

Since all local models are strictly positive: $\underline{P}_1(\{o_{1:n}\}) > 0$.

Generalised Bayes Rule yields uniquely coherent value of $\underline{P}_1(f|o_{1:n})$.



$$\underline{P}_1(f|o_{1:n}) \leq 0 \Leftrightarrow \underline{P}_1(\mathbb{I}_{\{o_{1:n}\}}f) \leq 0.$$

The notion of optimality we use: maximality

We define a partial order $\succ$ on state sequences:

$$\hat{x}_{1:n} \succ x_{1:n} \Leftrightarrow \underline{P}_1(\mathbb{I}_{\{\hat{x}_{1:n}\}} - \mathbb{I}_{\{x_{1:n}\}} | o_{1:n}) > 0$$

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A state sequence is optimal if it is undominated, or maximal:

$$\begin{aligned} \hat{x}_{1:n} \in \operatorname{opt}\left(\mathscr{X}_{1:n} | o_{1:n}\right) &\Leftrightarrow (\forall x_{1:n} \in \mathscr{X}_{1:n}) x_{1:n} \not\succ \hat{x}_{1:n} \\ &\Leftrightarrow (\forall x_{1:n} \in \mathscr{X}_{1:n}) \underline{P}_1\left(\mathbb{I}_{\{x_{1:n}\}} - \mathbb{I}_{\{\hat{x}_{1:n}\}} | o_{1:n}\right) \leq 0 \\ &\Leftrightarrow (\forall x_{1:n} \in \mathscr{X}_{1:n}) \underline{P}_1\left(\mathbb{I}_{\{o_{1:n}\}}[\mathbb{I}_{\{x_{1:n}\}} - \mathbb{I}_{\{\hat{x}_{1:n}\}}]\right) \leq 0, \end{aligned}$$

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&\Leftrightarrow (\forall x_{1:n} \in \mathscr{X}_{1:n}) \underline{P}_1\left(\mathbb{I}_{\{x_{1:n}\}} - \mathbb{I}_{\{\hat{x}_{1:n}\}} | o_{1:n}\right) \leq 0 \\
&\Leftrightarrow (\forall x_{1:n} \in \mathscr{X}_{1:n}) \underline{P}_1\left(\mathbb{I}_{\{o_{1:n}\}}[\mathbb{I}_{\{x_{1:n}\}} - \mathbb{I}_{\{\hat{x}_{1:n}\}}]\right) \leq 0,
\end{aligned}$$

$\underline{P}_1\left(\mathbb{I}_{\{o_{1:n}\}}[\mathbb{I}_{\{x_{1:n}\}} - \mathbb{I}_{\{\hat{x}_{1:n}\}}]\right)$ can be easily determined recursively!

The notion of optimality we use: maximality

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Aim of the algorithm:

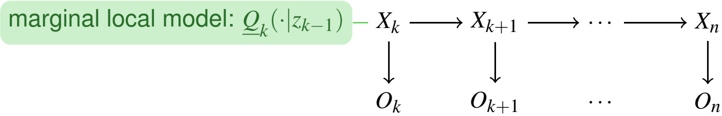
Determine $\operatorname{opt}\left(\mathscr{X}_{1:n} | o_{1:n}\right)$ efficiently.

More general optimality operators

$\mathrm{opt}\left(\mathscr{X}_{k-1}|z_{k-1},o_{k:n}\right)$

$$\hat{x}_{k:n} \in \mathrm{opt}\left(\mathscr{X}_{k-1}|z_{k-1},o_{k:n}\right)$$
$$\Leftrightarrow (\forall x_{k:n} \in \mathscr{X}_{k:n})\underline{P}_k(\mathbb{I}_{\{o_{k:n}\}}[\mathbb{I}_{\{x_{k:n}\}} - \mathbb{I}_{\{\hat{x}_{k:n}\}}]|z_{k-1}) \leq 0.$$

These are the optimal sequences for part of the original iHMM:



The corresponding joint lower prevision is precisely $\underline{P}_k(\cdot|z_{k-1})$.

The Principle of Optimality

$$\hat{x}_{k:n} \in \text{opt}\left(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n}\right) \Rightarrow \hat{x}_{k+1:n} \in \text{opt}\left(\mathscr{X}_{k+1:n} | \hat{x}_k, o_{k+1:n}\right).$$

And so we can say that:

$$\text{opt}\left(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n}\right) \subseteq S_k := \bigcup_{z_k \in \mathscr{X}_k} z_k \oplus \text{opt}\left(\mathscr{X}_{k+1:n} | z_k, o_{k+1:n}\right).$$

And therefore:

$$\text{opt}\left(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n}\right) = \text{opt}\left(S_k | z_{k-1}, o_{k:n}\right).$$

Checking if a final state $\hat{x}_n$ is optimal

$$\hat{x}_n \in \operatorname{opt}(\mathscr{X}_n|z_{k-1},o_k) \Leftrightarrow (\forall x_n \in \mathscr{X}_n)\underline{P}_n(\mathbb{I}_{\{o_n\}}[\mathbb{I}_{\{x_n\}} - \mathbb{I}_{\{\hat{x}_n\}}]|z_{n-1}) \leq 0.$$

This is OK if $x_n = \hat{x}_n$, so we can use the equivalent condition:

$$(\forall x_n \in \mathscr{X}_n \setminus \{\hat{x}_n\})\underline{Q}_n(\mathbb{I}_{\{x_n\}}\underline{S}_n(\{o_n\}|x_n) - \mathbb{I}_{\{\hat{x}_n\}}\overline{S}_n(\{o_n\}|\hat{x}_n)|z_{n-1}) \leq 0.$$

So for every z_{n-1} we can easily find $\operatorname{opt}(\mathscr{X}_n|z_{n-1},o_n)$.

Checking if a state sequence $\hat{x}_{k:n}$ is optimal

$$\hat{x}_{k:n} \in \operatorname{opt}(\mathscr{X}_{k:n}|z_{k-1}, o_{k:n})$$
$$\Leftrightarrow (\forall x_{k:n} \in \mathscr{X}_{k:n})\underline{P}_k(\mathbb{I}_{\{o_{k:n}\}}[\mathbb{I}_{\{x_{k:n}\}} - \mathbb{I}_{\{\hat{x}_{k:n}\}}]|z_{k-1}) \leq 0.$$

Since $\hat{x}_{k:n} \in S_k$, the inequality holds when $x_k = \hat{x}_k$.

This leads to the equivalent condition:

$$(\forall x_k \neq \hat{x}_k)(\forall x_{k+1:n} \in \mathscr{X}_{k+1:n})\underline{Q}_k(\mathbb{I}_{\{x_k\}}\beta(x_{k:n}) - \mathbb{I}_{\{\hat{x}_k\}}\alpha(\hat{x}_{k:n})|z_{k-1}) \leq 0.$$

where

$$\beta(z_{k:n}) := \underline{S}_k(\{o_k\}|z_k) \prod_{i=k+1}^{n} \underline{S}_i(\{o_i\}|z_i)\underline{Q}_i(\{z_i\}|z_{i-1})$$
$$\alpha(z_{k:n}) := \overline{S}_k(\{o_k\}|z_k) \prod_{i=k+1}^{n} \overline{S}_i(\{o_i\}|z_i)\overline{Q}_i(\{z_i\}|z_{i-1}).$$

Checking if a state sequence $\hat{x}_{k:n}$ is optimal

Equivalent formulations for $\hat{x}_{k:n} \in \operatorname{opt}(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n})$

$$(\forall x_k \neq \hat{x}_k)(\forall x_{k+1:n} \in \mathscr{X}_{k+1:n})\underline{Q}_k(\mathbb{I}_{\{x_k\}}\beta(x_{k:n}) - \mathbb{I}_{\{\hat{x}_k\}}\alpha(\hat{x}_{k:n})|z_{k-1}) \leq 0$$

is equivalent to

$$(\forall x_k \neq \hat{x}_k)\underline{Q}_k(\mathbb{I}_{\{x_k\}}\beta_k^{\max}(x_k) - \mathbb{I}_{\{\hat{x}_k\}}\alpha(\hat{x}_{k:n})|z_{k-1}) \leq 0,$$

where

$$\beta_k^{\max}(x_k) := \max_{\substack{z_{k:n} \in \mathscr{X}_{k:n} \\ z_k = x_k}} \beta(z_{k:n}).$$

Checking if a state sequence $\hat{x}_{k:n}$ is optimal

Equivalent formulations for $\hat{x}_{k:n} \in \operatorname{opt}(\mathscr{X}_{k:n}|z_{k-1}, o_{k:n})$

$$(\forall x_k \neq \hat{x}_k)\underline{Q}_k(\mathbb{I}_{\{x_k\}}\beta_k^{\max}(x_k) - \mathbb{I}_{\{\hat{x}_k\}}\alpha(\hat{x}_{k:n})|z_{k-1}) \leq 0$$

is equivalent to

$$(\forall x_k \neq \hat{x}_k)\alpha(\hat{x}_{k:n}) \geq \alpha_{threshold}(\hat{x}_k, x_k|z_{k-1}),$$

where

$$\alpha_{threshold}(\hat{x}_k, x_k|z_{k-1}) := \min\left\{\alpha \in \mathbb{R} \colon \underline{Q}_k(\mathbb{I}_{\{x_k\}}\beta_k^{\max}(x_k) - \alpha\mathbb{I}_{\{\hat{x}_k\}}|z_{k-1}) \leq 0\right\}$$

and finally, this is in turn equivalent to:

$$\alpha(\hat{x}_{k:n}) \geq \alpha^{\mathrm{opt}}(\hat{x}_k|z_{k-1}) := \max_{x_k \neq \hat{x}_k} \alpha_{threshold}(\hat{x}_k, x_k|z_{k-1}).$$

Determining the set of optimal $\hat{x}_{k:n}$

by backward-forward recursion

For $k = n \to 1$:

$$\text{opt}\left(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n}\right) = \{\hat{x}_{k:n} \in S_k \colon \alpha(\hat{x}_{k:n}) \geq \alpha^{\text{opt}}(\hat{x}_k | z_{k-1})\}$$

Now recurse forward:

First step: Fix any $\hat{x}_k \in \mathscr{X}_k$.
Then there is some optimal state sequence $z_{k:n} \in \text{opt}\left(\mathscr{X}_{k:n} | z_{k-1}, o_{k:n}\right)$ such that $z_k = \hat{x}_k$ if and only if

$$\max_{\substack{z_{k:n} \in \mathscr{X}_{k:n} \\ z_k = \hat{x}_k}} \alpha(z_{k:n}) =: \alpha_k^{\max}(\hat{x}_k) \geq \alpha^{\text{opt}}(\hat{x}_k | z_{k-1}).$$

Determining the set of optimal $\hat{x}_{k:n}$

by backward-forward recursion

Second step: Fix any $\hat{x}_k$ for which $\alpha_k^{\max}(\hat{x}_k) \geq \alpha^{\text{opt}}(\hat{x}_k|z_{k-1})$.
Consider any first state $\hat{x}_{k+1}$ of some element $\hat{x}_{k+1:n}$ of $\text{opt}\left(\mathscr{X}_{k+1:n}|\hat{x}_k, o_{k+1:n}\right)$ then:

$$\begin{aligned}
&\alpha(\hat{x}_{k:n}) \geq \alpha^{\text{opt}}(\hat{x}_k|z_{k-1}) \\
&\quad \Leftrightarrow \alpha(\hat{x}_{k+1:n})\overline{S}_{k+1}(\{o_{k+1}\}|\hat{x}_{k+1})\overline{Q}_{k+2}(\{\hat{x}_{k+2}\}|\hat{x}_{k+1}) \geq \alpha^{\text{opt}}(\hat{x}_k|z_{k-1}) \\
&\quad \Leftrightarrow \alpha(\hat{x}_{k+1:n}) \geq \alpha^{\text{opt}}(\hat{x}_{k:k+1}|z_{k-1}) := \frac{\alpha^{\text{opt}}(\hat{x}_k|z_{k-1})}{\overline{S}_k(\{o_k\}|\hat{x}_k)\overline{Q}_{k+1}(\{\hat{x}_{k+1}\}|\hat{x}_k)}
\end{aligned}$$

So we see that there is some $z_{k:n} \in \text{opt}\left(\mathscr{X}_{k:n}|z_{k-1}, o_{k:n}\right)$ such that $z_k = \hat{x}_k$ and $z_{k+1} = \hat{x}_{k+1}$ if and only if

$$\alpha_{k+1}^{\max}(\hat{x}_{k+1}) \geq \alpha^{\text{opt}}(\hat{x}_{k:k+1}|z_{k-1}).$$

Go on until you reach the end of the chain ...