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SOME CONTRIBUTIONS TO THE (GEOMETRIC) THEORY OF GENERALIZED POLYGONS

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Puzzling (with) Polygons

One of the most puzzling questions of non-mathematicians to a PhD-student in geometry is 'what exactly are you spending your days with?'. Well, I have been puzzling. My puzzles were - as most puzzles are - made of ordinary polygons, put together in a smart way and forming a puzzling object called a *'generalized polygon'*.

The official birth announcement of the generalized polygons was made in Tits' paper of 1959, *'Sur la trialité et certains groupes qui s'en déduisent'*. In this paper, Tits discovers the simple group ${}^3D_4$ by classifying certain maps, called *trialities* of D_4-geometries. The related geometries are what we call today 'generalized hexagons'. In a small appendix to the famous '59-paper, Tits introduces the notion of a *generalized n-gon.* Of course, these structures must have been in his head for some time then, and for example the projective planes - which are exactly the generalized 3-gons - had already been studied extensively at that moment. But anyway, from '59 on, the generalized n-gons come out of the shadow of the groups, and become geometries studied on their own. Some years later, Tits introduces - again for group-theoretical reasons - the notion of a *'building'*. The building bricks of buildings are the generalized polygons, which stresses again their importance in the study of incidence geometry.

The original paper of Tits already gives some examples of generalized polygons, arising from 'classical objects' such as quadrics and Hermitian varieties in projective space for example, which translates in nice geometric properties for these polygons. But one can do better: a free construction process provides examples of generalized n-gons, for every n. Recently, another method (using model theory) to construct infinitely many new examples of n-gons has been discovered. So at least there are polygons enough to investigate. Now what are the problems that one can look at?

From the point of view of discrete mathematics and combinatorics, one can be interested in *finite* examples of generalized polygons, i.e. polygons with a finite number of points. A famous theorem of Feit & Higman - proved with purely algebraic methods - states that such finite examples only exist for $n \in \{3, 4, 6, 8\}$. Surprisingly, these values of n turn up at various places in the theory - and this will also be the case in this thesis. Group-minded people are perhaps more interested in polygons with a very large automorphism group, the so-called *Moufang polygons*. The classification of all Moufang polygons (which only exist for $n \in \{3, 4, 6, 8\}$...) was announced by Tits in '76, is only recently completed by Tits & Weiss and is in the process of being published. This piece of the polygon-research is strongly related with algebraic objects as root systems and specific algebras. Within topology, the polygons are represented by the *topological n-gons*. The notion of a compact polygon for example has been very useful to prove topological counterparts of theorems about finite polygons. Last but not least, one can take the geometric point of view - which will also be the point of view of this thesis.

One thing that can happen when puzzling, is that some pieces disappear. Some puzzles then become worthless, but not our generalized polygons. Indeed, missing pieces will be the link between the different chapters of this work.

In the first part of this thesis, we are concentrating on our favourite puzzles, being the classical generalized hexagons. These hexagons have nice geometric properties. Now the question is: if we are only given some of the pieces of one of our favourite puzzles, can we recognize from which one they come? So in fact we look for pieces of our favourite puzzles (=geometric properties of the hexagons) that are typical for this puzzle, in this way obtaining *characterizations* of classical generalized hexagons. The characterizations we obtain are mainly based on regularity properties. In the second part, we leave the hexagons and consider really 'general' n-gons. Here we deal with puzzles missing so many pieces, that it is not clear any more that they actually arise from generalized polygons. These structures are called *'forgetful polygons'*, since their definition looks like the definition of a generalized polygon, where some lines seem to have been forgotten. It remains a question however whether a 'forgetful polygon' is necessary a forgetful generalized polygon. We investigate this more in detail for the case of the quadrangles (and so we are back at a small value of n). Next, two puzzles come into play, and we want to decide, given only partial information about them, whether they are the same or not. Let us be a bit more precise. A generalized polygon

can be seen as a graph, which allows us to define a distance function. If two generalized n-gons are given, and a bijective map between them preserving a specific distance, does this map extend to an isomorphism? The answer to this question is 'yes' in a lot of cases but, concerning counterexamples, again the small n-values turn up. In an appendix, we concentrate on the following problem. Suppose we are given some pieces (= some points) of one of our favourite puzzles (being in this case the finite dual split Cayley hexagon), and the following rule of play holds: whenever we have two pieces between which only one piece is missing (= two collinear points), we are allowed to plug in this missing piece (= add all the points on the joining line). With how many pieces do we have to start to end up with the complete puzzle (= to generate the whole point set)? We investigate this problem for some small cases with the help of the computer.
So far the rough sketch of the pieces of knowledge added by this thesis to the big puzzle of the theory of generalized polygons. A more extensive introduction to each problem can be found at the beginning of the relevant chapters.

One of the most interesting things about mathematical puzzles is that they are never solved 'completely': answering one question, related problems arise. This is the reason why you will meet the symbol  at various places in this thesis. Some of these questions only turned up when writing this thesis, others kept us puzzling for a while, but the pieces we collected were not enough to complete the whole thing.

At least three aspects distinguished me during these three years of research from a 'normal' puzzler. Indeed, many hobby puzzlers would envy the financial support of the FWO, the Fund for Scientific Research-Flanders. Also, puzzling was not at all a lonely business, being surrounded by the interest of the 'geometric part' of the maths department here. The puzzles I considered had no key included (since there was no key granted after all), but what I had was much better: a constant help desk, providing me puzzles, puzzle pieces and prepared to listen to all my puzzling attempts of solving. Thanks Hendrik.

Last but not least, I want to thank my parents and my sister for both listening to my puzzling doubts, and remembering me from time to time that puzzling is a game after all. My mother once taught me that, when making a puzzle, the corner pieces are the most important ones. Thanks for being these corners.

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Contents

Chapter 1

Introductory guide

The aim of the first chapter is to mention some highlights in the history of generalized polygons, and to introduce the notions that are needed to understand the pieces of knowledge we will add in the further chapters. This introduction is based on the monograph *'Generalized Polygons'* (Van Maldeghem [57]).

We opt to give three equivalent definitions of generalized polygons. The first one uses a lot of n-gons, and therefore explains why the polygons considered here can really be called 'generalized'. As Chapter 4 deals with distance-preserving maps, we give a definition in terms of the distance function. The third definition characterizes the incidence graph in a very compact way. The next sections contain the inevitable list of notions and properties that starts every introductory course on generalized polygons. We stress the notion of regularity, and introduce you to two classes of classical generalized hexagons, namely the split Cayley and the twisted triality hexagons. Since Chapter 2 concerns characterizations of these hexagons, we give an overview of the known geometric characterizations of the classical generalized hexagons. Finally, we explain how one can coordinatize generalized hexagons. Indeed, coordinatization is the main tool in appendix A. The calculations there are done by computer, but here we wanted to give the reader the same information about coordinatization as we told the computer.

1.1 Definitions

1.1.1 Incidence geometries

An **incidence geometry** (or **geometry** for short) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are two disjoint nonempty sets the elements of which are called **points** and **lines**, respectively, and where $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L} \cup \mathcal{L} \times \mathcal{P}$ is a symmetric relation, called the **incidence relation**. When a point p is incident with a line L, we also say that p lies on L or that L goes through p. Points and lines are called the **elements** of the geometry. A **flag** is a pair $\{p, L\}$, with p a point incident with the line L. The set of flags of the geometry is denoted by $\mathcal{F}$. **Adjacent** flags are distinct flags which have an element in common. An **antiflag** is a pair $\{p, L\}$ where p and L are not incident. The **dual** of the geometry Γ is the geometry Γ^D obtained by interchanging the roles of points and lines, i.e. $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathtt{I})$. We define the **double** 2Γ of Γ as the geometry with point set $\mathcal{F}$, line set $\mathcal{P} \cup \mathcal{L}$ and natural incidence relation (this definition is in fact the dual of the one given in [57]). A geometry is called **thick** if every element is incident with at least three other elements. A **thin** element is an element incident with exactly two other elements. If there exist constants s, t such that every line is incident with exactly $s + 1$ points, and every point is incident with exactly $t + 1$ lines, the pair (s, t) is called the **order** of Γ. If $s = t$, one says that Γ has order s. A geometry is said to be **finite** if both $\mathcal{P}$ and $\mathcal{L}$ are finite sets. A **subgeometry** of Γ is a geometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathtt{I}'$ the restriction of the relation $\mathtt{I}$ to $\mathcal{P}' \times \mathcal{L}' \cup \mathcal{L}' \times \mathcal{P}'$.

A **path** in Γ between the elements x and y is a sequence $(x = x_0, x_1, \ldots, x_k = y)$ of points and lines such that $x_{i-1} \mathtt{I} x_i$, for all $i \in \{1, 2, \ldots, k\}$. If $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$, the path is said to be **non-stammering**. The number k is called the **length** of the path. A non-stammering path of length k is called a **k-path**. If $x = y$, we talk about a **closed path**, and a non-stammering closed path is called a **circuit**. An **ordinary n-gon** is a closed path $(x_0, x_1, \ldots, x_{2n} = x_0)$ of length $2n > 2$ for which all x_i, $i \in \{0, 1, \ldots, 2n-1\}$ are distinct. The **distance** $\delta(x, y)$ between two elements x, y of Γ is the length of a shortest path joining x and y, if such a path exists. If not, then the distance between x and y is by definition ∞. Similarly, one can define a distance function on the set of flags.

The **incidence graph** of a geometry Γ is the graph with as vertex set $V = \mathcal{P} \cup \mathcal{L}$ and as edges the flags of Γ (hence adjacency in the incidence graph coincides with incidence in the geometry). The **girth** if this graph is the length of a minimal circuit.

1.1.2 Definition of a generalized n-gon

Let $n \geq 1$ be a natural number. A **weak generalized n-gon** is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ such that the following two axioms are satisfied:

(i) there are no ordinary k-gons in Γ, with $k < n$,

(ii) every two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in an ordinary n-gon.

If a weak generalized n-gon Γ is thick, we call it a **generalized n-gon**. As shown in [57] (Lemma 1.3.2), thickness is equivalent with the existence of at least one ordinary $(n+1)$-gon in Γ.

A generalized 2-gon (digon) is a rather trivial geometry in which every point is incident with every line. The generalized 3-gons are exactly the projective planes. Instead of 4-gons, 5-gons, 6-gons, 8-gons and n-gons, we shall also speak of quadrangles, pentagons, hexagons, octagons and polygons. A generalized n-gon consists of lots of ordinary n-gons, so-called **apartments**. This terminology is inherited from the theory of buildings. Indeed, generalized polygons are exactly the buildings of rank 2 (i.e. with two types of elements, namely points and lines), see for instance [57], section 1.3.7. Note that the dual of a generalized polygon is again a generalized polygon. Also, the double of a generalized n-gon is a weak generalized $2n$-gon with thin points and thick lines.

All weak non-thick generalized polygons arise either from ordinary polygons or from generalized polygons by inserting paths (see Structure Theorem of Tits, [57], section 1.6). So we will mainly be interested in (thick) generalized n-gons. In this case, one can prove that the generalized polygon Γ has an order (s, t) (s and t are also called the **parameters** of Γ). If n is odd, then necessarily $s = t$ (see [57], Lemma 1.5.3). There exist (thick) generalized n-gons for every n. Examples are provided by a free construction process due to Tits [52] (or see [57], section 1.3.13). We will give explicit examples of generalized quadrangles an hexagons below.

To conclude this section, we give two alternative definitions of generalized n-gons, the first in terms of the distance function, the second one using the incidence graph.

Lemma 1.1.1 (**[57], Lemma 1.3.5**) *A thick geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized n-gon if and only if the following axioms hold for the distance δ:*

(i) *If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\delta(x, y) = k < n$, then there is a unique path of length k joining x to y.*

(ii) For every $x \in \mathcal{P} \cup \mathcal{L}$, we have $n = max\{\delta(x,y) : y \in \mathcal{P} \cup \mathcal{L}\}$.

Lemma 1.1.2 (**[57], Lemma 1.3.6**) *A geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized n-gon if and only if the incidence graph of Γ is a connected graph of diameter n and girth $2n$, such that each vertex lies on at least three edges.*

1.1.3 A bunch of terminology

Let Γ be a generalized n-gon, $n \geq 3$. Two points p and q at distance 2 are called **collinear**, and we denote by pq the unique line joining p and q. Dually, two lines L and M at distance 2 are called **concurrent**, and the unique point $p = L \cap M$ incident with both is called the **intersection point** of L and M. For two elements x and y at distance 2, we also write $x \perp y$. Two elements x and y of Γ lying at maximal distance n are said to be **opposite**. If two elements x and y of Γ lie at distance $k < n$, the unique k-path between x and y is denoted by $[x,y]$. If k is even, $x \bowtie y$ denotes the unique element of $[x,y]$ at distance $k/2$ from both x and y. The unique element of $[x,y]$ incident with x is called the **projection of** y **onto** x, and denoted by $\mathrm{proj}_x y$. By definition, we put $\mathrm{proj}_x x = x$. When it suits us, we consider a path as a set so that we can take intersections of paths. For instance, if $[x,y] = (x = x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_k = y)$ and $[x,z] = (x = x_0, x_1, \ldots, x_i, x'_{i+1}, x'_\ell = z)$ with no x_j equal to any $x'_{j'}$, for some $0 < i \leq k, l$ and all j, $i < j \leq k$, and all j', $i < j' \leq l$, then we write $[x,y] \cap [x,z] = [x, x_i]$.

For any element x of Γ, and any integer $i \leq n$, we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x, and by $\Gamma_{\neq i}(x)$ the set of elements of Γ not at distance i from x. If κ is a set of natural numbers, then $\Gamma_\kappa(x)$ denotes the set of elements z of Γ for which $\delta(x,z) \in \kappa$. If p is a point, the set $\Gamma_1(p)$ of all lines through p is called a **line pencil**. Dually, if L is a line, the set $\Gamma_1(L)$ of all points on L is called a **point row**. For a point x, the set $x^\perp := \Gamma_2(x)$ is called the **perp** of x. The perp of a set X of points is the set of points collinear with every element of X. So $X^\perp = \bigcap_{x \in X} x^\perp$. For a point x of Γ, we denote by $x^{\perp\!\!\!\perp}$ the set of points not opposite x.

A (weak) **sub-n-gon** Γ' of a generalized n-gon is a subgeometry which is itself a (weak) generalized n-gon. If every line pencil of Γ' coincides with the corresponding line pencil of Γ, the subpolygon Γ' is called **ideal**. Dually, if every point row of Γ' coincides with the corresponding point row of Γ, the subpolygon Γ' is called **full**.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ be two generalized n-gons. An **isomorphism** or **collineation** of Γ onto Γ' is a bijection $\alpha : \mathcal{P} \to \mathcal{P}'$, inducing a bijection of $\mathcal{L}$ onto $\mathcal{L}'$ so that incidence and non-incidence is preserved, i.e. $p\mathtt{I}L \iff p^\alpha \mathtt{I}' L^\alpha$, for all $p \in \mathcal{P}$ and $L \in \mathcal{L}$. Lemma 1.3.14 of [57] states that, if $n \geq 4$, any bijection from $\mathcal{P}$ onto $\mathcal{P}'$ preserving collinearity and non-collinearity, extends to an isomorphism from Γ to Γ'. An **anti-isomorphism** of Γ onto Γ' is a collineation of Γ onto the dual Γ'^D of Γ'. An **automorphism** (**anti-automorphism**) of Γ is an isomorphism (anti-isomorphism) of Γ onto itself.

We say that a generalized polygon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is **embedded** in the projective space $\mathrm{PG}(d, \mathbb{K})$ if distinct points and lines of Γ are distinct points and lines of $\mathrm{PG}(d, \mathbb{K})$, with the natural incidence, and the point set of $\mathcal{P}$ generates $\mathrm{PG}(d, \mathbb{K})$.

1.2 Restrictions on the parameters

Theorem 1.2.1 (**Feit & Higman** [**21**]) *Finite generalized n-gons, $n \geq 3$, exist only for $n \in \{3, 4, 6, 8\}$.*

Lemma 1.2.2 ([**57**], **1.5.5**) *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a finite (weak) generalized n-gon of order (s, t), with $n \in \{3, 4, 6, 8\}$. Then we have*

$$|\mathcal{P}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1+s)(1+st) & \text{if } n = 4, \\ (1+s)(1+st+s^2t^2) & \text{if } n = 6, \\ (1+s)(1+st)(1+s^2t^2) & \text{if } n = 8. \end{cases}$$

Dually,

$$|\mathcal{L}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1+t)(1+st) & \text{if } n = 4, \\ (1+t)(1+st+s^2t^2) & \text{if } n = 6, \\ (1+t)(1+st)(1+s^2t^2) & \text{if } n = 8. \end{cases}$$

The following theorem is a combination of results of Feit & Higman [21], Higman [31] and Haemers & Roos [30].

Theorem 1.2.3 *Let Γ be a finite generalized n-gon of order (s, t).*

- *If $n = 4$, then $s \leq t^2$ and $t \leq s^2$.*

- *If* $n = 6$, *then* st *is a square;* $s \leq t^3$ *and* $t \leq s^3$.
- *If* $n = 8$, *then* $2st$ *is a square, in particular* $s \neq t$; $s \leq t^2$ *and* $t \leq s^2$.

Theorem 1.2.4 (**Thas [38],[40],[41],[42]**) *Let* Γ' *be an ideal weak sub-*n*-gon of order* (s', t) *of a finite generalized* n*-gon* Γ *of order* (s, t), *with* $s' \neq s$. *Then one of the following cases occurs.*

- $n = 4$ *and* $s \geq s't$ *and* $s \geq t \geq s'$;
- $n = 6$ *and* $s \geq s'^2 t$ *and* $s \geq t \geq s'$;
- $n = 8$ *and* $s \geq s'^2 t$.

A finite generalized hexagon of order (s, t) for which $s = t^3$ or $t = s^3$ is called an **extremal** hexagon.

The following two properties are used in Chapter 3. A **triad** of points in a generalized quadrangle is a triple of pairwise non-collinear points.

Proposition 1.2.5 (*i*) (**Bose & Shrikhande [6]**) *Let* Γ *be a finite generalized quadrangle of order* (s, t) *with* $t = s^2$. *Then for every triad* $\{x, y, z\}$ *of points, there are exactly* $s + 1$ *points collinear with* x, y *and* z.

(*ii*) (**Thas [40]**) *Let* Γ *be a finite generalized quadrangle of order* (s, t), *and* Γ' *an ideal subquadrangle of order* (s', t) *satisfying* $s = s't$. *Then every point of* Γ *not in* Γ' *lies on a unique line of* Γ'.

1.3 Some words on regularity

Let Γ be a generalized n-gon, and $2 \leq i \leq \frac{n}{2}$. If two elements x and y are opposite, the set $\Gamma_i(x) \cap \Gamma_{n-i}(y) = x^y_{[i]}$ is called the **distance-*i*-trace** of y with respect to x. For $i = 2$, it is convenient to call the distance-2-trace $x^y_{[2]}$ simply a **trace**, and denote it by x^y. The element x is **distance-*i*-regular** if distinct distance-i-traces with respect to x have at most 1 element in common (i.e. the distance i-traces with respect to x behave as lines, since they intersect in 0,1 or all elements). The element x is **regular** if it is distance-i-regular, for all $2 \leq i \leq \frac{n}{2}$. A generalized polygon is said to be **point-distance-*i*-regular** respectively **line-distance-*i*-regular** if all points respectively all lines are distance-i-regular. Instead of distance-i-regular, we sometimes use

i-**regular** for short (note that no confusion with the notion of 3-regularity in generalized quadrangles as defined in Payne & Thas [34] can occur, since distance-3-regularity is not defined for generalized quadrangles). A **regulus** is a distance-$\frac{n}{2}$-trace $x^y_{[\frac{n}{2}]}$, which we also denote by $\langle x, y\rangle$. If Γ is distance-$\frac{n}{2}$-regular, a regulus is determined by two of its elements u, v. In this case, we denote by $R(u,v)$ the unique regulus containing the elements u and v. If v and v are points (lines), we also talk about the **point regulus** (**line regulus**) through u and v.

Now let Γ be a point-distance-2-regular generalized hexagon, and p, q two opposite points of Γ. Define the following geometry $\Gamma(p,q)$. A point x of Γ belongs to $\Gamma(p,q)$ if $x \in y^z$ or $x \in z^y$, with $y \in p^q$, $z \in q^p$, and y, z opposite points. The lines of $\Gamma(p,q)$ are the lines of Γ containing at least two points of $\Gamma(p,q)$. Incidence is natural. Then $\Gamma(p,q)$ is the unique weak non-thick ideal subhexagon of Γ containing p and q (see [57], Lemma 1.9.10). Also, one proves that the point set of $\Gamma(p,q)$ is the union of the point sets of two projective planes $\Gamma^+(p,q)$ and $\Gamma^-(p,q)$. The points of $\Gamma^+(p,q)$ are the points of $\Gamma(p,q)$ at distance 0 or 4 from p; the lines of $\Gamma^+(p,q)$ are the traces x^y, where x is a point of $\Gamma^-(p,q)$, and y is a point of $\Gamma^+(p,q)$ opposite x. Similarly for $\Gamma^-(p,q) = \Gamma^+(q,p)$.

For later purposes, we mention the following result.

Theorem 1.3.1 (Van Maldeghem [56] *A generalized octagon cannot be point-distance-*2-*, nor point-distance-*3-*regular.*

1.4 Generalized quadrangles

1.4.1 Definition

We give an equivalent (more common) definition of generalized quadrangles.

A generalized quadrangle is a thick incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ such that the following axioms are satisfied:

(i) If p is a point of Γ not incident with the line L of Γ, then there exists a unique point incident with L and collinear with p.

(ii) The geometry Γ contains at least one antiflag.

We refer to axiom (i) as 'the main axiom' for a generalized quadrangle.

1.4.2 Classical generalized quadrangles

We mention some examples of classical[1] generalized quadrangles, including all finite ones.

- Let Q be a non-singular quadric in $\mathrm{PG}(d, \mathbb{K})$, with $\mathbb{K}$ a field, of Witt index 2 (i.e. the quadric contains lines but no subspaces of higher dimension). The points and lines of the quadric form a (weak) generalized quadrangle $\mathsf{Q}(d, \mathbb{K})$. In the finite case (putting $\mathbb{K} = \mathrm{GF}(q)$) only the dimensions $d = 3, 4, 5$ occur. We then obtain a (weak) generalized quadrangle of order $(q, 1)$ (sometimes also called a **grid**), (q, q) and (q, q^2), respectively. All lines of these quadrangles are regular.

- Let H be a non-singular Hermitian variety in $\mathrm{PG}(d, \mathbb{L})$, with $\mathbb{L}$ a skew field, of Witt index 2. The points and lines of the Hermitian variety form a generalized quadrangle $\mathsf{H}(d, \mathbb{L})$. In the finite case (putting $\mathbb{L} = \mathrm{GF}(q^2)$) only the dimensions $d = 3, 4$ occur. We then obtain a generalized quadrangle of order (q^2, q) and (q^2, q^3), respectively.

- Let τ be a symplectic polarity in $\mathrm{PG}(3, \mathbb{K})$, $\mathbb{K}$ a field. The points of $\mathrm{PG}(3, \mathbb{K})$ together with the absolute lines of τ define a generalized quadrangle $\mathsf{W}(\mathbb{K})$, called the symplectic quadrangle. In the finite case (putting $\mathbb{K} = \mathrm{GF}(q)$) we obtain a quadrangle of order (q, q) (denoted $\mathsf{W}(q)$).

One has the following isomorphisms between the finite classical generalized quadrangles (see Payne & Thas [34]):

- $\mathsf{Q}(4, q) \cong \mathsf{W}(q)^D$
- $\mathsf{W}(q) \cong \mathsf{W}(q)^D \iff q$ is even
- $\mathsf{Q}(5, q) \cong \mathsf{H}(3, q^2)^D$.

1.4.3 Generalized quadrangles of order $s = 2$.

For later purposes, we mention the following results.

- By a result of Cameron [10], a generalized quadrangle of order $(2, t)$ is necessarily finite, and hence has $t = 2$ or $t = 4$.

[1] For a motivation of the word ‘classical’ we refer to [57], Chapter 2.

- There is a unique generalized quadrangle of order 2, namely the symplectic quadrangle $\mathsf{W}(2)$. One has the following well-known construction of this generalized quadrangle. Let $S = \{1, 2, \dots, 6\}$. A duad is an unordered pair ij of distinct elements of S. A syntheme is a set $\{ij, kl, mn\}$ of three duads for which $\{i, j, k, l, m, n\} = S$. Now the geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with as points the duads of S, as lines the synthemes and symmetrized containment as the incidence relation, is the unique generalized quadrangle of order 2.

- There is a unique generalized quadrangle of order $(2, 4)$, namely the quadrangle $\mathsf{Q}(5, 2)$.

1.4.4 The construction of Payne

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a finite generalized quadrangle of order q admitting a regular line L. We define the following geometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$. The lines of Γ' are the lines of Γ different from L and not intersecting L. The points are of two types. Points of type (A) are the points of Γ not on the line L. Points of type (B) are the distance-2-traces containing L. Incidence is the incidence of Γ if defined, and symmetrized containment otherwise. Then the geometry Γ' is a generalized quadrangle of order $(q + 1, q - 1)$.

1.5 Classical generalized hexagons and octagons

1.5.1 Generalized hexagons

The examples of generalized hexagons given below are called ‘classical’, because they live on classical objects (namely quadrics) in projective space. They first appeared in Tits [49]. We sketch this construction, for proofs we refer to [57], section 2.4.

Let $\mathsf{Q}^+(7, \mathbb{K})$ be the non-singular hyperbolic quadric in $\mathrm{PG}(7, \mathbb{K})$ (with standard equation $X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0$). This quadric has Witt index 4, i.e. it contains 3-spaces (called the ‘generators’), but no subspaces of higher dimension. The generators can be divided into two families: two generators belong to the same family if they are disjoint or they intersect in a line. Any plane of $\mathsf{Q}^+(7, \mathbb{K})$ is contained in exactly two generators, one of each family. Now one can define the following geometry $\Omega(\mathbb{K})$. There are four types of elements. The 0-points are the points of $\mathsf{Q}(7, \mathbb{K})$, the lines

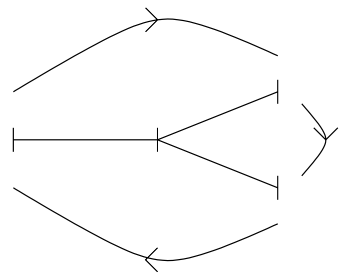


Figure 1.1: the D_4-diagram and a triality

are the lines of $\mathsf{Q}^+(7,\mathbb{K})$, the 1-points are the 3-spaces of the first family of generators and the 2-points are the 3-spaces of the second family. Denote by $\mathcal{P}^{(i)}$ the set of i-points. Incidence is containment if defined, and a 1-point and a 2-point are incident in $\Omega(\mathbb{K})$ if the corresponding 3-spaces intersect in a plane of the quadric. The diagram of this geometry $\Omega(\mathbb{K})$ is the so-called D_4-diagram (for an introduction to the theory of diagrams, see for instance Buekenhout [9]). One sees that the 0-,1- and 2-points play the same role in this geometry. In [57], section 2.4.6, it is explained how one can label the 1- and the 2-points in the same way as the 0-points, i.e. with an 8-tuple $(x_0, x_1, \ldots, x_7)$.

A **triality** of $\Omega(\mathbb{K})$ is a map

$$\theta : \mathcal{L} \to \mathcal{L},\ \mathcal{P}^{(0)} \to \mathcal{P}^{(1)},\ \mathcal{P}^{(1)} \to \mathcal{P}^{(2)},\ \mathcal{P}^{(2)} \to \mathcal{P}^{(0)},$$

preserving incidence and such that θ^3 is the identity.

An i-point p is called absolute if $p\mathtt{I}p^\theta$, a line is absolute if it is fixed by θ. Now one shows that if θ satisfies a weak additional assumption (basically saying that there are enough absolute points and lines), the geometry with as points the absolute i-points for a fixed i, and as lines the absolute lines, is a generalized hexagon.

Let σ be an automorphism of the field $\mathbb{K}$ of order 1 or 3, and consider the following map:

$$\tau_\sigma : \mathcal{P}^{(i)} \to \mathcal{P}^{(i+1)} : (x_j)_{j\in J} \to (x_j^\sigma)_{j\in J},\ i = 0,1,2 \bmod 3.$$

Then τ_σ is a triality, and the associated geometry of absolute i-points and lines is a generalized hexagon of order $(|\mathbb{K}|, |\mathbb{K}^{(\sigma)}|)$, with $\mathbb{K}^{(\sigma)}$ the subfield of $\mathbb{K}$ consisting of those elements of $\mathbb{K}$ that are fixed by σ. If $\sigma = 1$, the associated hexagon is called the **split Cayley hexagon** $\mathsf{H}(\mathbb{K})$. For $\mathbb{K} = \mathrm{GF}(q)$, this hexagon is denoted by $\mathsf{H}(q)$. Its order is (q,q). If $\sigma \neq 1$, the associated hexagon is called the **twisted triality hexagon** $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$. The dual of this hexagon is denoted by $\mathsf{T}(\mathbb{K}^{(\sigma)}, \mathbb{K}, \sigma)$. In the finite case, putting $\mathbb{K} = \mathrm{GF}(q^3)$, σ is necessarily the map $x \to x^q$, $\forall x \in \mathrm{GF}(q^3)$. The corresponding hexagon is denote by $\mathsf{T}(q^3, q)$, and has order (q^3, q). Its dual is denoted by $\mathsf{T}(q, q^3)$ and has order (q, q^3).

Note that Tits [49] classifies all trialities of the geometry $\Omega(\mathbb{K})$ having at least one absolute point. The only trialities producing thick generalized hexagons are the ones given above (the original definition of Tits of a generalized hexagon was weaker, such that it included also the geometries of absolute points and lines of the other trialities).

There are two other types of generalized hexagons that are also said to be classical, namely the mixed hexagons $\mathsf{H}(\mathbb{K}, \mathbb{K}')$ and the hexagons of type ${}^6D_4$ (and their duals). These hexagons are closely related to split Cayley hexagons over a field with characteristic 3 and to twisted triality hexagons respectively (they arise from them by a generalization in the choice of the coordinates, see [57], 3.5.3 and 3.5.9), but they do not exist in a finite version however. In the following, when talking about the **finite classical hexagons**, we will always mean the hexagons $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$. Their duals $\mathsf{H}(q)^D$ and $\mathsf{T}(q, q^3)$ are referred to as the **finite dual classical hexagons**. We now list some properties of the classical hexagons.

- The split Cayley hexagon $\mathsf{H}(\mathbb{K})$ is selfdual if and only if $\mathbb{K}$ is a perfect field of characteristic 3.
- All points of a split Cayley hexagon $\mathsf{H}(\mathbb{K})$ are regular. All lines are regular if and only if the field $\mathbb{K}$ has characteristic 3.
- All points and lines of a mixed hexagon are regular.
- All points of a twisted triality hexagon or a hexagon of type ${}^6D_4$ are regular. No line of such a hexagon is distance-2-regular.

Apart from the examples of generalized hexagons given above, other important examples are the ones related to the exceptional groups of type E_6 and E_8. Also these ones only exist in the infinite case.

Tits' description of the split Cayley hexagon
All the points of the split Cayley hexagon represented on the quadric $\mathsf{Q}^+(7, \mathbb{K})$ lie in fact in a certain hyperplane γ of $\mathrm{PG}(7, \mathbb{K})$. Conversely, all the points of $\mathsf{Q}^+(7, \mathbb{K})$ lying in this hyperplane γ are points of the hexagon. One now obtains the following description of the split Cayley hexagon $\mathsf{H}(\mathbb{K})$. The points

of the hexagon are all the points of a non-singular parabolic quadric $\mathsf{Q}(6,\mathbb{K})$ in $\mathrm{PG}(6,\mathbb{K})$ with standard equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2.$$

The lines of the hexagon are exactly the lines of $\mathsf{Q}(6,\mathbb{K})$ whose Grassmann coordinates satisfy the following six linear equations:

$$\begin{array}{lll} p_{12} = p_{34} & p_{54} = p_{32} & p_{20} = p_{35} \\ p_{65} = p_{30} & p_{01} = p_{36} & p_{46} = p_{31}. \end{array}$$

We now list some 'translations' of objects existing in the hexagon $\mathsf{H}(\mathbb{K})$ to this representation on the quadric $\mathsf{Q}(6,\mathbb{K})$.

- Two points in the hexagon are opposite if and only if they are non-collinear on the quadric (also the twisted triality hexagon has this property).

- A line of the quadric is either a line or a distance-2-trace of the hexagon.

- The points collinear in the hexagon with a point p are exactly the points of a fixed plane through p lying on the quadric. (The twisted triality hexagon has a similar property: the points collinear in the hexagon with a point p are contained in a plane through p lying on the quadric.)

- A line regulus in the hexagon is also a regulus (one set of generators of a hyperbolic quadric $\mathsf{Q}^+(3,\mathbb{K})$) on the quadric. A point regulus in the hexagon consists of the points of a plane intersecting the quadric in a non-singular conic.

- The points and lines of a thin ideal subhexagon $\Gamma(p,q)$ (see section 1.3) all lie in a hyperplane intersecting the quadric $\mathsf{Q}(6,q)$ in a non-singular hyperbolic quadric $\mathsf{Q}^+(5,q)$. The 'planes' $\Gamma^+(p,q)$ and $\Gamma^-(p,q)$ are indeed projective planes lying on the quadric $\mathsf{Q}(6,q)$.

The following result is due to Cohen & Tits [11].

Theorem 1.5.1 *A finite generalized hexagon of order (s,t) with $s = 2$ is isomorphic to one of the classical hexagons $\mathsf{H}(2)$, $\mathsf{H}(2)^D$ or $\mathsf{T}(2,8)$.*

1.5.2 Generalized octagons

The classical generalized octagons are generally called the **Ree-Tits octagons**. In the finite case, they only exist over fields GF(q) with $q = 2^{2e+1}$, and have - up to duality - order (q, q^2). There is no elementary description known of these classical octagons. In [57], section 2.5, a construction starting from a building of type F_4 is given. In Joswig & Van Maldeghem [32], one can find a description with coordinates.

1.6 Groups

Projectivities
For two opposite elements x and y of a generalized n-gon Γ, the projection map defines a bijection (denoted by $[x; y]$) from the set $\Gamma_1(x)$ to the set $\Gamma_1(y)$. For elements $x_0, \ldots, x_k$ with x_i opposite x_{i+1} for $0 \leq i < k$, the composition $[x_0; x_1] \ldots [x_{k-1}; x_k]$ is called a **projectivity** from x_0 to x_k. If $x_0 = x_k$, then we obtain a permutation of the set $\Gamma_1(x_0)$. The set of all such permutations of $\Gamma_1(x_0)$ is a group, called the **group of projectivities** of x_0 and denoted by $\Pi(x_0)$.

Elations and homologies
Let Γ be a generalized n-gon, and γ a fixed path of length $n-2$. A **γ-elation** (or **elation** for short) is a collineation of Γ fixing all elements incident with at least one element of γ. Let v, w be two opposite elements of Γ. A **$\{v, w\}$-homology** is a collineation fixing every element incident with v or w.

The Moufang property
Let $\gamma = (v_1, v_2, \ldots, v_{n-1})$ be a fixed $(n-2)$-path in a generalized n-gon Γ. Let v_0 be a fixed element incident with v_1, $v_0 \neq v_2$, and denote by $\mathcal{V}$ the set of all elements incident with v_0 different from v_1. Let v_n be a fixed element incident with v_{n-1}, $v_n \neq v_{n-2}$, and denote by $\mathcal{A}$ the set of all apartments containing v_0, γ and v_n. Then the set of all γ-elations forms a group G acting semi-regularly on both $\mathcal{V}$ and $\mathcal{A}$ (see [57], Proposition 4.4.3). If G acts transitively on the set $\mathcal{V}$ (or equivalently on the set $\mathcal{A}$), then the path γ is called a **Moufang path**. If all $(n-2)$-paths are Moufang paths, Γ is said to be a **Moufang polygon**. For a Moufang polygon, the collineation group generated by all elations is often called the **little projective group** of Γ.

A famous result of Tits [51], [53] and Weiss [58] states that Moufang n-gons exist only for $n \in \{3, 4, 6, 8\}$. All Moufang n-gons are classified, see Tits & Weiss [54]. In particular, all classical generalized polygons are Moufang polygons. The Moufang generalized hexagons are exactly the hexagons

$\mathsf{H}(\mathbb{K})$, $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$ and their duals, the mixed hexagons, the hexagons of type ${}^6D_4$ and the hexagons related to the exceptional groups E_6 and E_8.

The split Cayley hexagon
For later purposes, we mention that the little projective group of the hexagon $\mathsf{H}(\mathbb{K})$ is the group $G_2(\mathbb{K})$. It is exactly the group of automorphisms of this generalized hexagon that are induced by the group $\mathrm{PGL}(7, \mathbb{K})$ of linear transformations of $\mathrm{PG}(6, \mathbb{K})$ (the full automorphism group of $\mathsf{H}(\mathbb{K})$ is isomorphic to the semi-direct product $G_2(\mathbb{K}) : \mathrm{Aut}(\mathbb{K})$). In the finite case (putting $\mathbb{K} = \mathrm{GF}(q)$), we have

$$|G_2(q)| = q^6(q^6 - 1)(q^2 - 1).$$

Polygons arising from a BN-pair
Let Γ be a generalized polygon, and G a group of automorphisms of Γ. Suppose that G acts transitively on the ordered apartments of Γ. Then one says that Γ is a **Tits polygon** with respect to G. Fix an apartment Σ and a flag F in Σ. Denote by B the stabilizer in G of the flag F and by N the stabilizer in G of the (unordered) apartment Σ. Then one says that (B, N) is a **Tits system** in G for Γ. Conversely, from such a Tits system, one can reconstruct the polygon Γ (see Tits [50]). A polygon arising in this way is called a polygon **arising from a BN-pair**.

Weyl group
Consider an apartment Σ of a generalized n-gon Γ. The group of symmetries of Σ is the dihedral group D_{2n}. Let $\{p, L\}$ be a flag of Σ, and denote by s_p (s_L) the reflection about p (L). Then D_{2n} can be seen as the Coxeter group with generators s_p and s_L, i.e. $D_{2n} = \langle s_p, s_L || (s_p s_L)^n = (s_L s_p)^n = 1,\ s_p^2 = s_L^2 = 1\rangle$. Using the language of buildings, D_{2n} is the Weyl group of the apartment Σ. This allows us to define the Coxeter distance between flags of Γ. Let f_1 and f_2 be two flags of Γ, and Σ' an apartment containing f_1 and f_2. Then the **Coxeter distance** $\delta^*(f_1, f_2)$ between f_1 and f_2 is the unique element of D_{2n} mapping f_1 to f_2.

1.7 Ovoidal subspaces

An **ovoid** $\mathcal{O}$ of a generalized n-gon Γ, n even, is a set of mutually opposite points such that every element of Γ is at distance at most $n/2$ from at least one element of $\mathcal{O}$. A **spread** of a generalized polygon is the dual of an ovoid.

Let Γ be a generalized quadrangle. Then equivalently, an ovoid $\mathcal{O}$ of Γ is a set of points such that each line of Γ is incident with a unique point of $\mathcal{O}$. A

regular ovoid of a generalized quadrangle of order (s,t) is an ovoid $\mathcal{O}$ having the property that for any two points o_1 and o_2 of $\mathcal{O}$, $|\{o_1,o_2\}^{\perp\perp}| = t+1$ and $\{o_1,o_2\}^{\perp\perp} \subset \mathcal{O}$. A **geometric hyperplane** $\mathcal{H}$ of a generalized quadrangle is a proper subset of the point set such that for an arbitrary line L, either all the points of L belong to $\mathcal{H}$, or $|L \cap \mathcal{H}| = 1$.

Let Γ be a generalized hexagon. Reformulating the general definition, an ovoid $\mathcal{O}$ of Γ is a set of mutually opposite points such that each point of Γ not in $\mathcal{O}$ is collinear with a unique point of $\mathcal{O}$. The finite hexagon $\mathsf{T}(q^3,q)$ and its dual $\mathsf{T}(q,q^3)$ do not admit ovoids (by a counting argument). In fact, A. Offer recently proved that any finite generalized hexagon admitting an ovoid necessarily has order (q,q). An ovoid of a hexagon of order (q,q) contains q^3+1 points. Thas [44] gives the following construction of a spread of the hexagon $\mathsf{H}(q)$, which works for all possible values of q.

Let $\mathsf{H}(q)$ be defined on the quadric $\mathsf{Q} = \mathsf{Q}(6,q)$, and let γ be a hyperplane of $\mathrm{PG}(6,q)$ intersecting Q in a non-singular elliptic quadric $\mathsf{Q}^-(5,q)$. Then the lines of $\mathsf{H}(q)$ lying in γ constitute a spread of both the hexagon $\mathsf{H}(q)$ and the quadrangle $\mathsf{Q}(5,q)$. This spread is called the **Hermitian** or **classical spread** of the split Cayley hexagon. One has the following characterization of this spread.

Theorem 1.7.1 (Bloemen, Thas & Van Maldeghem [4], Theorem 9) *If $\mathcal{S}$ is a spread of $\mathsf{H}(q)$ for which holds that $\mathcal{S}$ is the union of q^2 line-reguli through L, for each $L \in \mathcal{S}$, then $\mathcal{S}$ is the Hermitian spread.*

For more constructions of ovoids of the classical hexagons, we refer the reader to Bloemen, Thas & Van Maldeghem [4].
An **ovoidal subspace** $\mathcal{O}$ of a generalized hexagon Γ is a proper subset of the point set of Γ such that each point of Γ not in $\mathcal{O}$ is collinear with a unique point of $\mathcal{O}$. By Brouns & Van Maldeghem [7], an ovoidal subspace is either an ovoid, (the point set of) a full subhexagon, or the set of points at distance 1 or 3 from a given line M. Dually, one defines a **dual ovoidal subspace**. A dual ovoidal subspace is either a spread (type S), (the line set of) an ideal subhexagon (type H), or the set of lines at distance 1 or 3 from a given point p (type P).

1.8 Characterizations of classical hexagons

Ronan's characterizations using regularity

Theorem 1.8.1 (Ronan [35]) *If Γ is a point-distance-2-regular generalized hexagon, then all points and lines are distance-3-regular and Γ is a Moufang* *hexagon. Conversely, up to duality, all points of any Moufang hexagon are regular.*

Every finite Moufang hexagon is classical or dual classical. This follows from a group-theoretical result of Fong & Seitz [22], [23], or alternatively from the classification of all Moufang polygons of Tits & Weiss [54]. Hence one has the following:

Theorem 1.8.2 *A finite generalized hexagon is point-distance-2-regular if and only if it is classical.*

Theorem 1.8.3 Ronan [36] *A finite extremal hexagon is classical if and only if it is distance-3-regular.*

Let Γ be a generalized hexagon, and x a point of Γ. Let y and z be two points opposite x and at distance 4 from each other, such that the point $y \bowtie z$ lies at distance 4 from x. Then the set $x^y \cap x^z$ is called an **intersection set**. By definition, an intersection set is never empty. (Note that our definition of intersection set is slighty different from the one in Ronan [37].) An intersection set of $\mathsf{H}(q)$ or $\mathsf{T}(q^3, q)$ contains 1 or $q+1$ points. An intersection set of $\mathsf{H}(q)^D$, q not a power of 3, or $\mathsf{T}(q, q^3)$ contains 2, respectively q^2+1 points.

Theorem 1.8.4 (Ronan [37]) *Let Γ be a distance-3-regular generalized hexagon.*

(*i*) *If for every intersection set $x^y \cap x^z$, one has $|x^y \cap x^z| = 1$ or $x^y = x^z$, then Γ is point-distance-2-regular and hence a Moufang hexagon.*

(*ii*) *If for every intersection set $x^y \cap x^z$, one has $|x^y \cap x^z| > 1$, then Γ is line-distance-2-regular and hence a Moufang hexagon.*

The characterizations above restricted to the finite case are summarized in Figure 1.2.

Hyperbolic and imaginary lines
Let Γ be a generalized n-gon, and x, y two non-collinear points of Γ at mutual distance $2j$. The **distance-j hyperbolic line** $H(x, y)$ is the set of points not opposite all elements not opposite x and y. If x and y are opposite, one also speaks of an **imaginary line**, notation $I(x, y)$. In fact, as shown in van Bon, Cuypers & Van Maldeghem [55], a distance-j hyperbolic line is exactly the intersection of all distance-j traces containing x and y. A distance-j

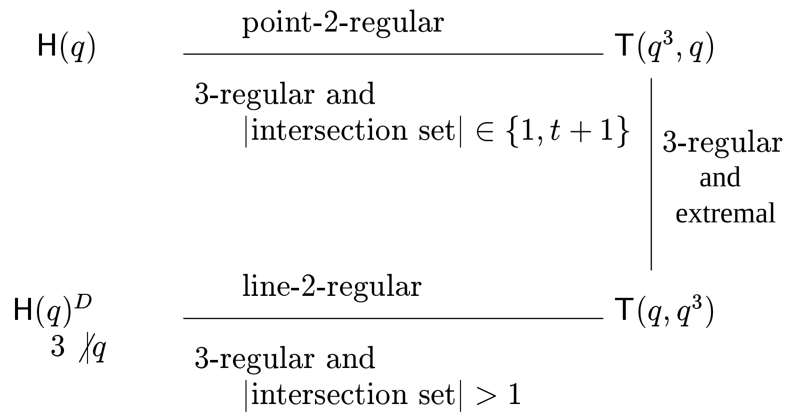


Figure 1.2: The finite Moufang hexagons

hyperbolic line $H(x, y)$ is called **long** if the projection of $H(x, y)$ onto any element L of Γ at distance $n - 1$ from all points of $H(x, y)$ is surjective onto $\Gamma_1(L)$ whenever it is injective. One proves that a long hyperbolic line $H(x, y)$ coincides with any distance-j-trace containing any two of its points. For a lot of results concerning hyperbolic lines in generalized n-gons, we refer to [55]. We now restrict to hexagons.

Theorem 1.8.5 (van Bon, Cuypers & Van Maldeghem [55])

(*i*) *All distance-2 hyperbolic lines of a generalized hexagon Γ are long if and only if Γ is isomorphic to $\mathsf{H}(\mathbb{K})$.*

(*ii*) *All imaginary lines of a generalized hexagon Γ are long if and only if Γ is isomorphic to $\mathsf{H}(\mathbb{K}')$, with $\mathbb{K}'$ a perfect field of characteristic 2.*

Characterizations using subpolygons

Theorem 1.8.6 (*i*) **(Van Maldeghem [57], Corollary 6.3.7)** *A generalized hexagon Γ is point-distance-2-regular (and hence a Moufang hexagon) if and only if every ordinary heptagon in Γ is contained in at least one ideal split Cayley hexagon.*

(*ii*) **(De Smet & Van Maldeghem [20])** *A finite generalized hexagon is isomorphic to $\mathsf{T}(q^3, q)$ if and only if every ordinary heptagon is contained in a proper ideal subhexagon.*

A characterization using span-regularity

A point p of a generalized hexagon is called **span-regular** if p is distance-2-regular, and if for every point x collinear with p, and every two points a and

b opposite x such that $p \in x^a \cap x^b$, the condition $|x^a \cap x^b| \geq 2$ implies $x^a = x^b$. The following characterization weakens the conditions of Theorem 1.8.1 in the finite case.

Theorem 1.8.7 (De Smet & Van Maldeghem [20], Brouns & Van Maldeghem [7]) *Let Γ be a finite hexagon containing a dual ovoidal subspace all the points of which are span-regular. If any two opposite points of Γ are contained in a thin ideal subhexagon, then Γ is classical.*

A characterization using intersections of traces

Theorem 1.8.8 (Thas [43]) *Let Γ be a finite generalized hexagon of order (s,t), $s \geq t$. Then Γ is isomorphic to $\mathsf{H}(q)$ if and only if any two distance-2-traces with respect to the same point meet in at least one point.*

For other characterizations, including some algebraic ones, we refer the reader to [57], Chapter 6.

1.9 Coordinatization

Generalized polygons can be coordinatized in a way similar to the projective planes. In this thesis, we will only need coordinatization of the classical generalized hexagons. The aim of this section is to give a rough idea how the labelling of the elements works (in the case of hexagons), and to show how one works with these coordinates. For a detailed description of the general coordinatization theory, we refer the reader to [57], Chapter 3.

1.9.1 Labelling of the elements

Let Γ be a generalized hexagon of order (s,t), and choose a fixed apartment Σ, called the **hat-rack** of the coordinatization. The elements of Σ are denoted as in Figure 1.3. Let R_1 and R_2 be two sets of cardinality s and t respectively, both containing a zero element, but no symbol ∞. The elements of the hat-rack are given coordinates as indicated on Figure 1.3 (coordinates of points will be written in parentheses, those of lines in square brackets). The points incident with L_0, different from x_0 or x_1 get a label (a), $a \in R_1 \setminus \{0\}$ in such a way that there is a bijection between R_1 and $\Gamma_1(L_0) \setminus \{x_0\}$. Similarly, the points on the line L_1 different from x_0 get a label $(0,b)$, and the points

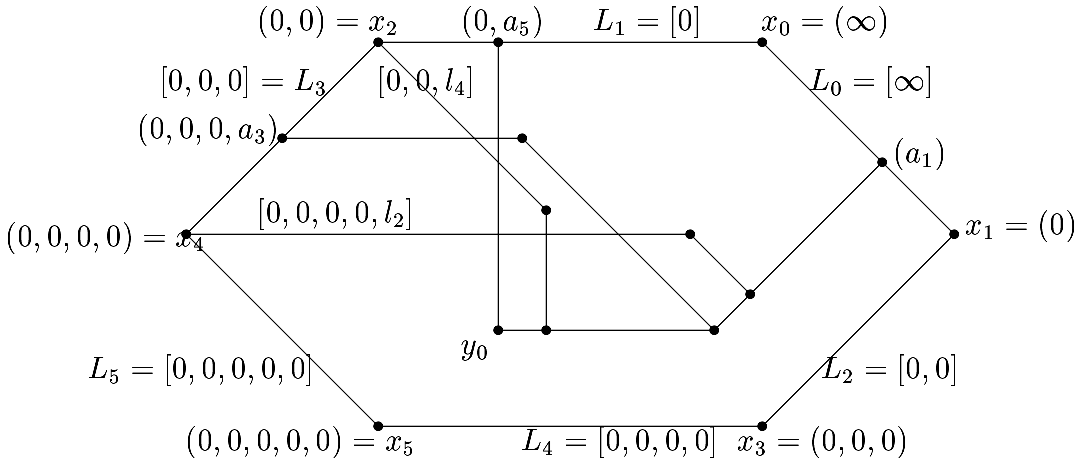


Figure 1.3: The hat-rack of the coordinatization

incident with L_2 different from x_1 are labelled $(0,0,a')$ [2]. Let $y = (a)$ be a point on L_0 different from x_0. Then the projection y' of y onto L_5 gets label $(a,0,0,0,0)$. Similarly, the points of L_3 different from x_4 and the points of L_4 different from x_3 are labelled $(0,0,0,a')$ and $(0,0,0,0,b)$ respectively. Dually (now using the set R_2) we give coordinates to the lines incident with the points of Σ. In this way, all elements incident with an element of the hat-rack Σ are already given coordinates. Now let y_0 be an arbitrary point opposite the point x_0, and put $(y_0 = z_5, M_4, z_3, M_2, z_1, L_0)$ the 5-path between y_0 and L_0. Suppose $z_1 = (a_1)$. Let a_3 (a_5) be the last coordinate of the projection of z_3 (z_5) onto the line L_3 (L_1), and l_2 (l_4) the last coordinate of the projection of M_2 (M_4) onto the point x_4 (x_2). Then we label the point z_i, $i = 3,5$ by $(a_1, l_2, \ldots, a_i)$ and the line M_i, $i = 2,4$ by $[a_1, \ldots, l_i]$ (see Figure 1.3).

In this way, all the points opposite x_0, and all the elements of Γ for which the projection onto x_0 is the line L_0 are given coordinates. Dually, all the lines opposite L_0 are labelled, hence also all other elements of Γ. For example, consider a point y collinear with x_0, not on the line L_0. The line x_0y has coordinate $[k]$. The projection of y onto the line L_4 has coordinate $(0,0,0,0,b)$, and y itself gets the label (k,b).

Now each i-tuple, $i \in \{0,1,\ldots,5\}$ (calling (∞) and $[\infty]$ 0-tuples) consisting alternately of elements of R_1 and R_2 corresponds to exactly one element of $\mathcal{P} \cup \mathcal{L}$ and conversely. Also, if the number of coordinates of two different

[2] In [57], it is explained how the labelling of the points on L_1 and L_2 can be related to the labelling of the points on L_0. We do not assume this 'normalization' here. This translates in the fact that we will have to make more choices - compared to [57] - to determine the coordinates of $\mathsf{H}(\mathbb{K})$ in the next paragraph.

elements differs by at least 2, these elements are not incident; if the number of coordinates differs by 1, the elements are incident if and only the coordinate-tuple of one of these elements is obtained from the coordinate-tuple of the other one by deleting the last coordinate. The only incident elements with the same number of coordinates different from 5 are (∞) and $[\infty]$. To obtain a complete description of the incidence in terms of coordinates, we only need a criterion to decide when two elements with 5 coordinates are incident. This is given by the operations Ψ_i and Φ_i, $i = 1,2,3,4$. Let p be a point with coordinates $(a, \ell, a', \ell', a'')$ and L a line with coordinates $[k, b, k', b', k'']$. Then $\Psi_i(k, a, \ell, a', \ell', a'')$ gives the $(n-i)$-th coordinate of the projection of the line $[k]$ onto the point p. Dually, $\Phi_i(a, k, b, k', b', k'')$ gives the $(n-i)$-th coordinate of the projection of the point (a) onto the line L. Using these operations, it can be expressed when the point p and the line L are incident (see Chapter 3 of [57]).

1.9.2 Coordinates of classical generalized hexagons

Consider the split Cayley hexagon $\mathsf{H}(\mathbb{K})$. This hexagon lies on the quadric with equation $X_0X_4+X_1X_5+X_2X_6 = X_3^2$ in $\mathrm{PG}(6,\mathbb{K})$. It is convenient to be able to go from the 6-dimensional space to the hexagon and back. Therefore, we choose coordinates in the following way (putting $R_1 = R_2 = \mathbb{K}$).

$$\begin{array}{rcl}
(1,0,0,0,0,0,0) & \rightarrow & (\infty), \\
(a,0,0,0,0,0,1) & \rightarrow & (a), \\
(0,0,0,0,0,1,0) & \rightarrow & (0,0), \\
(0,1,0,0,0,0,0) & \rightarrow & (0,0,0), \\
(0,0,1,0,0,0,0) & \rightarrow & (0,0,0,0), \\
(0,0,0,0,1,0,0) & \rightarrow & (0,0,0,0,0), \\
(b,0,0,0,0,1,0) & \rightarrow & (0,b), \\
(0,1,0,0,0,0,-a') & \rightarrow & (0,0,a'),
\end{array}$$

$$\begin{array}{rcl}
X_1 = X_2 = X_3 = X_4 = X_5 = 0 & \rightarrow & [\infty], \\
X_1 = X_2 = X_3 = X_4 = X_6 + kX_5 = 0 & \rightarrow & [k], \\
X_0 = X_2 = X_3 = X_4 = X_5 = 0 & \rightarrow & [0,0], \\
X_1 = X_3 = X_4 = X_6 = X_0 = 0 & \rightarrow & [0,0,0], \\
X_0 = X_2 = X_3 = X_5 = X_6 = 0 & \rightarrow & [0,0,0,0], \\
X_0 = X_1 = X_3 = X_5 = X_6 = 0 & \rightarrow & [0,0,0,0,0], \\
X_0 + lX_1 = X_2 = X_3 = X_4 = X_5 = 0 & \rightarrow & [0,l], \\
X_0 - k'X_2 = X_1 = X_3 = X_4 = X_6 = 0 & \rightarrow & [0,0,k'].
\end{array}$$

POINTS	
Coordinates in $\mathsf{H}(\mathbb{K})$	Coordinates in $\mathrm{PG}(6,\mathbb{K})$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$
LINES	
Coordinates in $\mathsf{H}(\mathbb{K})$	Coordinates in $\mathrm{PG}(6,\mathbb{K})$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1.1: Coordinatization of $\mathsf{H}(\mathbb{K})$.

This determines the coordinates of each point and line of $\mathsf{H}(\mathbb{K})$. The complete 'dictionary' for translation between the hexagon and the projective space is given in Table 1.1 (where $\langle p_1,p_2\rangle$ denotes the line through the points p_1 and p_2). The operations Ψ_i and Φ_i are the following:

$$\begin{cases} \Psi_1(k,a,l,a',l',a'') = a^3k+l-3a''a^2+3aa', \\ \Psi_2(k,a,l,a',l',a'') = a^2k+a'-2aa'', \\ \Psi_3(k,a,l,a',l',a'') = a^3k^2+l'-kl-3a^2a''k-3a'a''+3aa''^2, \\ \Psi_4(k,a,l,a',l',a'') = -ak+a'', \end{cases}$$

and

$$\begin{cases} \Phi_1(a,k,b,k',b',k'') = ak+b, \\ \Phi_2(a,k,b,k',b',k'') = a^3k^2+k'+kk''+3a^2kb+3bb'+3ab^2, \\ \Phi_3(a,k,b,k',b',k'') = a^2k+b'+2ab, \\ \Phi_4(a,k,b,k',b',k'') = -a^3k+k''-3ba^2-3ab'. \end{cases}$$

A similar table can be given for the coordinatization of the twisted triality hexagon. We will only need the coordinatization of $\mathsf{T}(q^3,q)$ once in this thesis, and refer to [57], Table 3.4 for the 'translations' used. Clearly, to coordinatize the duals of these hexagons, one only has to replace round and square brackets in the coordinatization above. Finally we remark that for a point p of the hexagon $\mathsf{H}(\mathbb{K})$ or $\mathsf{T}(\mathbb{K},\mathbb{K}^{(\sigma)},\sigma)$, the equations of the plane of the quadric containing $p^\perp$ can be calculated explicitely, see for instance Thas & Van Maldeghem [46].

Example
Let $\Gamma = \mathsf{H}(q)$, q even, $q \neq 2$, coordinatized as above. Suppose we are given the points p and p' with coordinates $(a,0,0,0)$ and $(c,0,d,0,0)$ respectively, $a,c,d \in \mathrm{GF}(q) \setminus \{0\}$. Then

$$\begin{array}{rcl}(a,0,0,0) & \rightarrow & (0,a,1,0,0,0,0),\\ (c,0,d,0,0) & \rightarrow & (d^2,0,c,d,1,0,0).\end{array}$$

The points not opposite the point p' in Γ (which are the points of the quadric collinear on the quadric with p') lie in the hyperplane γ with equation $X_0 + d^2X_4 + cX_6 = 0$, hence $\delta(p,p') \leq 4$. Since the point $(a,0)$ (corresponding to the point $(0,0,0,0,0,1,a)$ in $\mathrm{PG}(6,q)$) is collinear with p but opposite p', $\delta(p,p') = 4$. We now look for the projection L of p' onto p. The line L has coordinates $[a,0,0,0,x]$, and every point of L lies at distance ≤ 4 from p'. An arbitrary point on L different from $(a,0,0,0)$ has hexagon-coordinates $(t,at^3+x,at^2,a^2t^3+ax,at)$ (this is obtained by using the expressions for the Φ_i to calculate the coordinates of the projection of (t) onto the line L). Now expressing that such a point is contained in γ for all $t \in \mathrm{GF}(q)$, we obtain $x = \frac{d^2}{a}$.

1.10 More geometries

A **strongly regular graph** (notation $\mathrm{srg}(v,k,\lambda,\mu)$) is a graph with v vertices such that

(i) every vertex is adjacent to exactly k vertices;

(ii) for any two adjacent vertices x and y, there are exactly λ vertices adjacent to both x and y;

(iii) for any two non-adjacent vertices x and y, there are exactly μ vertices adjacent to both x and y.

The complement of a strongly regular graph G with parameters (v,k,λ,μ) is denoted by G^C, and is again a strongly regular graph with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.

An **association scheme** is a pair $(X, \mathcal{R})$ with X a finite set and $\mathcal{R} = (R_0, R_1, \ldots, R_d)$ a partition of $X \times X$ such that the following conditions are satisfied.

(i) $R_0 = \{(x,x) | x \in X\}$

(ii) The relations R_i are symmetric (i.e., $(x,y) \in R_i \Rightarrow (y,x) \in R_i$).

(iii) There exist integers p^i_{jk}, called the **intersection numbers**, having the following property: for all $(x,y) \in R_i$, there exist exactly p^i_{jk} elements $z \in X$ such that $(x,z) \in R_j$ and $(y,z) \in R_k$.

For an association scheme $(X, \mathcal{R})$, the matrices L_i with $(L_i)_{jk} = p^i_{jk}$ are called the **intersection matrices**. Define the matrix $\Delta = \mathrm{diag}(p^0_{00}, \ldots, p^0_{dd})$. Then in Brouwer, Cohen & Neumaier [8], section 2.2, it is shown that the matrices L_i have $d+1$ common eigenvectors. We denote these eigenvectors (normalized such that $(u_i)_0 = 1$) with $u_0, \ldots, u_d$. Then define $v_i = \Delta(u_i)$, $f_i = |X|/(u_i, v_i)$ and $q_{ijk} = \sum_l p^0_{ll}(u_i)_l(u_j)_l(u_k)_l$. The numbers f_i are called the **multiplicities** of the association scheme, and are necessarily integers (since they arise as multiplicities of eigenvalues of the adjacency matrices, see again [8], section 2.2).

Theorem 1.10.1 (Krein conditions [8], Theorem 2.3.2)
$q_{ijk} \geq 0$, *for* $0 \leq i, j, k \leq d$.

Chapter 2

Characterizations of (dual) classical generalized hexagons

2.1 Introduction

In the first section, we give a characterization of the finite classical hexagons defined over a field of characteristic two. This characterization is based on a property of these hexagons that was proved in Thas & Van Maldeghem [45], saying that no point of the hexagon can lie 'far away' from all the points of a point regulus. We formulate this property for an arbitrary finite polygon satisfying $s \geq t$, obtaining in this way a generalization of an existing characterization of the generalized quadrangle $\mathsf{W}(q)$. The obtained characterization excludes all finite octagons.

We then start from the following observation. Fix a point regulus R in a finite (dual) classical hexagon, and a line L lying at distance 3 from a point x of R and at distance 5 from all the points of $R \setminus \{x\}$. In the case of the split Cayley hexagon over a field of characteristic 2, all the points of $R \setminus \{x\}$ project onto the same point of L. By asking the property just mentioned for lines L in a particular position with respect to the regulus R, we obtain a characterization of the split Cayley hexagon over a finite field of even characteristic. In fact, this characterization weakens the condition of 'having long imaginary lines', as defined in van Bon, Cuypers & Van Maldeghem [55]. In the case of the dual classical hexagons, and the classical hexagons over a field of odd characteristic, the projection of the points of R onto L determines a bijection. We use this to obtain a characterization of all dual classical hexagons over a finite field of even characteristic, and a characterization of some extremal hexagons.

In a classical hexagon, all intersection sets containing at least two points, contain $t+1$ points. In a dual classical hexagon however, the size of an intersection set containing at least two points, is $1+t/q$ (with (q,t) the order of the hexagon). One could now ask whether a generalized hexagon for which all intersection sets have size $1+t/q$, is necessarily dual classical. This condition seems to be too loose to characterize these hexagons. Instead, we consider intersections of traces that are in a slightly more general position than intersection sets (called '(3,4)-position' later on), also containing $1+t/q$ points. In this way, we do obtain a characterization of all finite dual classical hexagons.

Payne & Thas [34] define the notion of anti-regular point in a generalized quadrangle. It is a conjecture that all finite anti-regular generalized quadrangles are isomorphic to the dual of $\mathsf{W}(q)$, q odd. We generalize this notion for hexagons, and prove that all finite anti-regular generalized hexagons are isomorphic to the dual of $\mathsf{H}(q)$, q not a power of three.

In Ronan [35] the point-distance-2-regular hexagons are characterized as follows: if a generalized hexagon is 3-regular, and all intersection sets behave as they should, then the points are 2-regular (see Theorem 1.8.4). We weaken the conditions of this characterization in two ways. First, we keep the assumption on the intersection sets, but ask the 3-regularity only for a certain subset of the points (namely for the points of an ovoidal subspace). Secondly, we start from a 3-regular hexagon, but only ask that all intersection sets with respect to a point in a certain subset (namely the points on lines of a dual ovoidal subspace) have the right size. The characterizations of the first three sections of this chapter are contained in Govaert [24] and Govaert & Van Maldeghem [25] and [26].

In the last section, we turn our attention to the Hermitian spread of the finite split Cayley hexagon. This spread arises by intersecting the underlying quadric Q with a hyperplane intersecting Q in an elliptic quadric $\mathsf{Q}^-(5,q)$. Since the points and lines of $\mathsf{Q}^-(5,q)$ form a generalized quadrangle, we obtain a generalized quadrangle $\Gamma_{\mathcal{S}}$ 'hidden' inside our hexagon. This quadrangle can be described using the spread (for example, the points of the quadrangle are exactly the points on spread lines). So the geometry $\Gamma_{\mathcal{S}}$ can be defined for an arbitrary spread $\mathcal{S}$ of the split Cayley hexagon. The fact that $\Gamma_{\mathcal{S}}$ is a generalized quadrangle is equivalent with a certain configuration of spread lines that is not allowed, and turns out to characterize the Hermitian spread.

For an arbitrary spread, one can consider the group of projectivities induced by the spread lines, i.e. considering only those projectivities that use spread lines. For the Hermitian spread, the associated group is a Singer group. We prove that this group characterizes the Hermitian spread as a spread of $\mathsf{H}(q)$. The results of this section will appear in Govaert & Van Maldeghem [27].

2.2 Characterizations of $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$, q even

Theorem 2.2.1 *Let Γ be a finite generalized hexagon. Then Γ is isomorphic to $\mathsf{H}(q)$ or to $\mathsf{T}(q^3, q)$, both with q even, if and only if $\Gamma_{\leq 4}(x) \cap \Gamma_{\leq 3}(L) \cap \Gamma_{\leq 3}(M)$ is nonempty for any point x and any pair of lines L, M of Γ.*

Proof. Let first Γ be isomorphic to $\mathsf{H}(q)$ or to $\mathsf{T}(q^3, q)$, both with q even, and L, M two lines of Γ. Let for any point x of Γ, $S_x = \Gamma_{\leq 4}(x) \cap \Gamma_{\leq 3}(L) \cap \Gamma_{\leq 3}(M)$. If $\delta(L, M) \leq 2$, then $\mathrm{proj}_L x \in S_x$. If $\delta(L, M) = 4$, then, with $N = L \bowtie M$, the point $\mathrm{proj}_N x \in S_x$. If finally L and M are opposite, then it is proved in Thas & Van Maldeghem [45], Lemma 5.2 that $S_x \neq \emptyset$ for any point x if and only if q is even.

So from now on we assume that Γ is a finite generalized hexagon with the property that $\Gamma_{\leq 4}(x) \cap \Gamma_{\leq 3}(L) \cap \Gamma_{\leq 3}(M)$ is nonempty for any point x, and for any two lines L, M. Let (s, t) be the order of Γ.

We first show that Γ is distance-3-regular. So let L and M be two opposite lines, and let $x, y, z \in \langle L, M \rangle$, $x \neq y \neq z \neq x$. Let $N \in \langle x, y \rangle$, $M \neq N \neq L$. Let $z' = \mathrm{proj}_M z$, $M' = \mathrm{proj}_{z'} N$, $z'' = \mathrm{proj}_N z'$ and $N' = \mathrm{proj}_{z''} z'$. We project the points of the regulus $\langle L, M \rangle$ onto the line N'. Since z'' is the image of at least two points (namely x and y), this projection is not injective. Because the regulus $\langle L, M \rangle$ and the line N' both contain $s + 1$ points, the projection of $\langle L, M \rangle$ onto N' cannot be surjective either, so there is a point w on N' which is not the projection of any point of $\langle L, M \rangle$. If N' would lie at distance 5 from every point of $\langle L, M \rangle$, then the point w on N' would be opposite every point of $\langle L, M \rangle$, a contradiction with the assumed property of Γ. Hence N' is at distance 3 from some point u of $\langle L, M \rangle$. If $u \neq z$, then we obtain an ordinary j-gon, $j < 6$, through the points $\mathrm{proj}_M u$, z' and $\mathrm{proj}_{N'} u$, a contradiction. Hence $u = z$, implying that $M' = zz'$. We have shown that

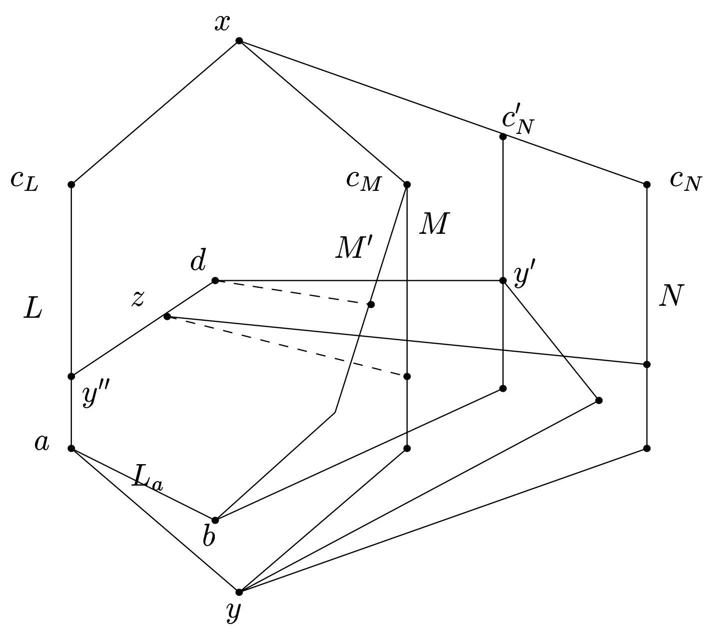


Figure 2.1: Proof of Theorem 2.2.1 (size of an intersection set is $t+1$).

the line $zz' = \mathrm{proj}_z M$ is at distance 4 from N. Similarly (by interchanging the roles of L and M), the line $\mathrm{proj}_z L$ is at distance 4 from N, implying that z is at distance 3 from N. Hence Γ is distance-3-regular.

Next we show that all intersection sets containing at least two points, have size $t+1$. Let x, y, L, M and N be as in the previous paragraph and put $a = \mathrm{proj}_L y$. Let L_a be any line through a, $ay \neq L_a \neq L$ (see Figure 2.1). Let c_N be the projection of x onto N and define c_L, c_M similarly. Let b be the projection of c_M on L_a. Then $x^y \cap x^b$ is an intersection set already containing the points c_L and c_M. We prove that $x^y = x^b$. Let c'_N be the projection of b onto xc_N. It suffices to show that $c_N = c'_N$. Suppose by way of contradiction these points are different. Let y' be the projection of y onto $\mathrm{proj}_{c'_N} b$. Note that $c'_N \neq y' \neq c'_N \bowtie b$. Hence the projection y'' of y' onto L is distinct from both a and c_L. Let $d = y' \bowtie y''$. We now project the points of the regulus $\langle L, N \rangle$ onto the line dy'. Since both x and y project onto the point y', this projection is not injective, so it cannot be surjective either. If the line dy' would lie at distance 5 from all points of $\langle L, N \rangle$, this would imply that the line dy' contains a point opposite all the points of $\langle L, N \rangle$, a contradiction. So we may assume that there is a point $z \in \langle L, N \rangle$ lying at distance 3 from dy'. If $\mathrm{proj}_L z \neq y''$, then the points z, $\mathrm{proj}_L z$, y'', d and $\mathrm{proj}_{dy'} z$ define a pentagon, a contradiction. Now clearly, the point z lies on the line $y''d$, $y'' \neq z \neq d$. Let M' be the line through c_M and $c_M \bowtie b$. The regulus

$\langle x,b\rangle$ contains the lines L, M' and $y'c'_N$. Since $\delta(d,L)=\delta(d,y'c'_N)=3$, the distance-3-regularity implies that also $\delta(d,M')=3$. Note that $\delta(z,M)=3$. But now the points z, d, $\mathrm{proj}_{M'}d$, c_M and $\mathrm{proj}_M z$ define a pentagon, the final contradiction, showing $x^y=x^b$. So all intersection sets containing at least two points, have size $t+1$. By Theorem 1.8.4 (i), all points are distance-2-regular, and by Theorem 1.8.2, Γ is a finite Moufang hexagon isomorphic to $\mathsf{H}(q)$ or to $\mathsf{T}(q^3,q)$. By Thas & Van Maldeghem [45], q is even. $\square$

In the proof of Theorem 2.2.1, we used the finiteness assumption in the following way: if one projects the points of a regulus $\langle L,M\rangle$ onto a line N at distance 5 from every point of $\langle L,M\rangle$, then the fact that this projection is not injective implies that it cannot be surjective either. This becomes exactly the additional assumption when generalizing Theorem 2.2.1 to the infinite case. We now obtain the following:

Theorem 2.2.2 *Let* Γ *be a generalized hexagon, not necessarily finite. Suppose that for any point* x *and any two lines* L,M *the set* $\Gamma_{\leq 4}(x)\cap\Gamma_{\leq 3}(L)\cap\Gamma_{\leq 3}(M)$ *is nonempty* (a). *Suppose moreover that if* L,M *are opposite, and* N *is a line at distance* 5 *from all elements of the regulus* $\langle L,M\rangle$, *then the projection of* $\langle L,M\rangle$ *onto* N *is injective whenever it is surjective* (b). *Then* Γ *is a point-distance-*2*-regular hexagon and hence a Moufang hexagon.*

In the infinite case however, the conditions stated in Theorem 2.2.2 do not completely characterize the characteristic 2 case. We investigate when the infinite split Cayley hexagon $\Gamma\cong\mathsf{H}(\mathbb{K})$ satisfies conditions (a) and (b) in Theorem 2.2.2. (For other types of Moufang hexagons, the condition on the underlying field becomes more involved.)
Let L and M be two opposite lines of Γ, x a point of Γ and $\mathsf{Q}=\mathsf{Q}(6,\mathbb{K})$ the quadric on which Γ is defined. The regulus $\langle L,M\rangle$ corresponds with the points of a conic C (lying in a plane π) on Q, and the set $\Gamma_{\leq 4}(x)$ corresponds with the set of points of Q lying in the tangent hyperplane ξ_x at Q in x. Let x be a point such that ξ_x and π intersect in a line R_x and put $S_x=\Gamma_{\leq 4}(x)\cap\langle L,M\rangle$.

- char $\mathbb{K}=2$. The line R_x is a line through the nucleus of C, hence R_x intersects C for all choices of x if and only if the field $\mathbb{K}$ is perfect. So let $\mathbb{K}$ be a perfect field of characteristic 2. Then (a) is satisfied. Property (b) follows from the fact that all imaginary lines of $\Gamma\cong\mathsf{H}(\mathbb{K})$, char $\mathbb{K}=2$, are long (see Theorem 1.8.5). Indeed, this fact implies that an imaginary line coincides with a regulus containing two of its points (see Lemma 2.4 in van Bon, Cuypers & Van Maldeghem [55]). The

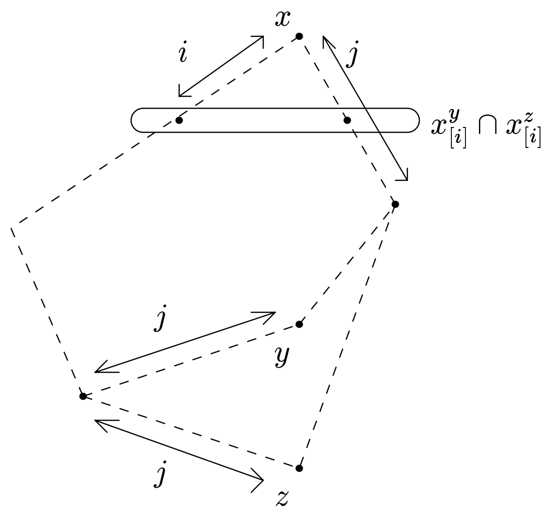


Figure 2.2: An (i, j)-intersection set.

projection of an imaginary line onto a line at distance 5 from all its points is either constant or injective (see [55], Corollary 2.3), hence (b).

- char $\mathbb{K} \neq 2$. The intersection of R_x with C is nonempty for all choices of x if and only if the field $\mathbb{K}$ is quadratically closed. So let $\mathbb{K}$ be a quadratically closed field of odd characteristic. Then (a) is satisfied, but (b) cannot be satisfied. Indeed, let x be a point for which S_x contains two points y and z. Let N be a line through x, $\mathrm{proj}_x y \neq N \neq \mathrm{proj}_x z$. Since x is opposite every point of $\langle L, M\rangle \setminus \{y, z\}$, the line N lies at distance 5 from every point of $\langle L, M\rangle$. The projection of $\langle L, M\rangle$ onto N is not injective (since both y and z project onto x). If (b) is satisfied, then there would be a point x' on N opposite every point of $\langle L, M\rangle$, contradicting (a).

We now want to generalize Theorem 2.2.1 to octagons. For this purpose, we first generalize the notion of an intersection set, and result 1.8.4 (i).

Let Γ be a generalized $2m$-gon and $2 \leq i \leq j \leq m$. Let x be an element of Γ. An (i, j)**-intersection set** S **with respect to** x is a set $x^y_{[i]} \cap x^z_{[i]}$, where y and z are opposite x, where $|x^y_{[j]} \cap x^z_{[j]}| \geq 2$ and where $|y^x_{[j]} \cap z^x_{[j]}| = 1$ if $j < m$. Note that, if (s, t) is the order of Γ, then $2 \leq |S| \leq t + 1$ if x is a point, and $2 \leq |S| \leq s + 1$ if x is a line.

A (2,2)-intersection set in a generalized hexagon now corresponds more or less with an intersection set as defined in section 1.8. The only difference is that a (2,2)-intersection set by definition already contains two elements. Using this new terminology, Theorem 1.8.4 (i) becomes:

> *If for every* $(2,j)$*-intersection set* $x^z_{[2]} \cap x^y_{[2]}$ *of a generalized hexagon* Γ*, one has* $x^z_{[2]} = x^y_{[2]}$*, then* Γ *is point-distance-2-regular.*

Indeed, we show that, if Γ is a generalized hexagon satisfying the conditions above, then Γ is distance-3-regular. Let x, y and z be three different points of a regulus $\langle L, M\rangle$, and N a line of $\langle x, y\rangle$, $L \neq N \neq M$. Since $x^y \cap x^z$ is a (2,3)-intersection set, $x^y = x^z$, hence $\delta(z, \mathrm{proj}_N x) = 4$. Interchanging the roles of x and y, we obtain $y^x = y^z$, hence $\delta(z, \mathrm{proj}_N y) = 4$. So $\delta(z, N) = 3$, showing the distance-3-regularity.

The following lemma is a generalization of Theorem 1.8.4 (i).

Lemma 2.2.3 *Let* Γ *be an arbitrary generalized* $2m$*-gon,* $m \geq 2$*, let* x *be a point of* Γ*, and let* $2 \leq i \leq m$*. Then* x *is distance-*i*-regular if and only* $x^y_{[i]} = x^z_{[i]}$*, for every* (i,j)*-intersection set* $x^y_{[i]} \cap x^z_{[i]}$ *with respect to* x*, for all* j*,* $i \leq j \leq m$*.*

Proof. Let x be a distance-i-regular point. Then every distance-i-trace in x is determined by two points. Since an (i,j)-intersection set $x^y_{[i]} \cap x^z_{[i]}$ with respect to x contains two elements by definition, $x^y_{[i]} = x^z_{[i]}$. So let now x be a point of Γ and suppose that $x^y_{[i]} = x^z_{[i]}$, for every (i,j)-intersection set $x^y_{[i]} \cap x^z_{[i]}$ with respect to x, for all j, $i \leq j \leq m$. Let y and z be two points opposite x such that $x^y_{[i]} \cap x^z_{[i]}$ contains the distinct points u_0 and w_0. Define the elements u_k and w_k, $0 \leq k \leq 2m-i$, as $u_k \mathtt{I} u_{k+1}$ and $w_k \mathtt{I} w_{k+1}$, $0 \leq k < 2m-i$, with $u_{2m-i} = w_{2m-i} = y$. Similarly, define the elements v_k and r_k, $0 \leq k \leq 2m-i$, as $v_k \mathtt{I} v_{k+1}$ and $r_k \mathtt{I} r_{k+1}$, $0 \leq k < 2m-i$, with $v_0 = u_0$, $r_0 = w_0$ and $v_{2m-i} = r_{2m-i} = z$. We claim that, in order to prove that $x^y_{[i]} = x^z_{[i]}$, we may assume that $w_{2m-2i} = r_{2m-2i}$. Indeed, suppose that $w_k = r_k$, for some k, $0 \leq k < 2m-2i$. We prove that we can assume $w_{k+1} = r_{k+1}$. Since w_k is opposite $u_{2m-2i-k}$, one has $\delta(r_{k+1}, u_{2m-2i-k}) = 2m-1$, so there is a unique chain

$$r_{k+1} \mathtt{I} r'_{k+2} \mathtt{I} \ldots \mathtt{I} r'_{2m-i} = y' = v'_{2m-i} \mathtt{I} v'_{2m-i-1} \mathtt{I} \ldots \mathtt{I} v'_{2m-2i-k} = u_{2m-2i-k}$$

(y' is the unique point of this chain opposite x). If $i+k \leq m$, then $S = x^y_{[i]} \cap x^{y'}_{[i]}$ is an $(i, i+k)$-intersection set with respect to x (indeed, the element $u_{2m-2i-k}$ lies at distance $i+k$ from both y and y', and the element w_k belongs to $x^y_{[i+k]} \cap x^{y'}_{[i+k]}$ and lies opposite $u_{2m-2i-k}$). If $i+k > m$, then $S = x^y_{[i]} \cap x^{y'}_{[i]}$

is an $(i, 2m-i-k)$- intersection set with respect to x (indeed, the element w_k lies at distance $2m-i-k$ from both y and y', and the element $u_{2m-2i-k}$ belongs to $x^y_{[2m-i-k]} \cap x^{y'}_{[2m-i-k]}$ and lies opposite w_k). Thus in both cases, $x^y_{[i]} = x^{y'}_{[i]}$. Since we were interested in $S = x^y_{[i]} \cap x^z_{[i]}$, we can as well consider $S' = x^{y'}_{[i]} \cap x^z_{[i]}$. So we may indeed assume $w_{k+1} = r_{k+1}$. Proceeding like this we can assume that $w_{2m-2i} = r_{2m-2i}$. But then $S' = x^y_{[i]} \cap x^z_{[i]}$ is an (i,i)-intersection set with respect to x (since S' contains the element u_0, and the element w_{2m-2i} lies at distance i from both y and z and is opposite u_0), hence $x^y_{[i]} = x^z_{[i]}$. This shows that x is distance-i-regular. □

Let Γ be a generalized $2m$-gon and $2 \leq i \leq m$. Let x be an element of Γ. A **half (i,i)-intersection set S with respect to x** is a set $x^y_{[i]} \cap x^z_{[i]}$, where y and z are opposite x, and $|y^x_{[i]} \cap z^x_{[i]}| = 1$ if $i < m$, $|y^x_{[i]} \cap z^x_{[i]}| \geq 1$ if $i = m$.

The following lemma is an immediate generalization of Theorem 1.8.4 (ii).

Lemma 2.2.4 *Let Γ be a generalized $2m$-gon, and $2 \leq k \leq m-1$.*

(i) *Suppose m is even. If Γ is line-distance-m-regular, and all half $(m+1-j, m+1-j)$-intersection sets with respect to any line X contain at least two elements, for $2 \leq j \leq k$, then Γ is line-distance-k-regular.*

(ii) *Suppose m is odd. If Γ is distance-m-regular, and all half $(m+1-j, m+1-j)$-intersection sets with respect to any point x contain at least two elements, for $2 \leq j \leq k$, then Γ is line-distance-k-regular.*

Proof. Suppose Γ is a generalized $2m$-gon satisfying the conditions of the lemma, for a fixed k, $2 \leq k \leq m-1$. Let L be a line of Γ and $L^{M_1}_{[k]} \cap L^{M_2}_{[k]}$ a (k,j)-intersection set with respect to L, $k \leq j$. We prove that $L^{M_1}_{[k]} = L^{M_2}_{[k]}$. Let v be an element of $L^{M_1}_{[j]} \cap L^{M_2}_{[j]}$, and w an element of ${M_1}^L_{[j]} \cap {M_2}^L_{[j]}$, v and w opposite. Let X be the element of the path $[L, w]$ at distance m from L. Let p be a point on L, $\mathrm{proj}_L v \neq p \neq \mathrm{proj}_L w$. Let $[p, M_1] = (p = y_0, y_1, \ldots, y_{2m-1} = M_1)$ and $[p, M_2] = (p = z_0, z_1, \ldots, z_{2m-1} = M_2)$. We prove that $y_i = z_i$, for $i \leq k$. So suppose by way of contradiction that $y_r = z_r$, but $y_{r+1} \neq z_{r+1}$, for a number r, $0 \leq r \leq k-1$. Denote by Y (Z) the unique element of the path $[p, M_1]$ $([p, M_2])$ at distance m from L and Y' (Z') the unique element of the path $[v, M_1]$ $([v, M_2])$ at distance m from L. Then $S = X^Y_{[m-r]} \cap X^Z_{[m-r]}$ is a half $(m-r-1, m-r-1)$-intersection set with respect to X. By assumption, S contains an element a, $\mathrm{proj}_X a \neq \mathrm{proj}_X L$. Since $\mathrm{proj}_w M_1 \neq \mathrm{proj}_w M_2$, also $\mathrm{proj}_X a \neq \mathrm{proj}_X w$. Let R_1 (R_2) be the element of the path $[Y, a]$ $([Z, a])$

at distance m from Y (Z). The regulus $\langle X,Y\rangle$ contains the elements L, M_1 and R_1. Since $\delta(Y',M_1)=\delta(Y',L)=m$, also $\delta(Y',R_1)=m$. Similarly, $\delta(Z',R_2)=m$. But now we obtain a circuit of length at most $4m+2(r-j+1)$ (determined by the paths $[a,R_1]$, $[R_1,Y']$, $[Y',v]$, $[v,Z']$, $[Z',R_2]$ and $[R_2,a]$), a contradiction. So $L^{M_1}_{[k]}=L^{M_2}_{[k]}$. By applying (the dual of) Lemma 2.2.3, we conclude that the line L is k-regular. $\square$

Theorem 2.2.5 *Let* Γ *be a finite generalized octagon of order* (s,t) *with* $s\geq t\geq 2$ *such that* $\Gamma_{\leq 6}(x)\cap\Gamma_{\leq 4}(y)\cap\Gamma_{\leq 4}(z)$ *is nonempty for all points* x,y,z *of* Γ. *Then* Γ *is point-distance-*3*-regular* (*and hence does not exist*).

Proof. Let Γ be a finite generalized octagon satisfying the conditions of the lemma. Note that $s>t$ by Theorem 1.2.3. Let x be any point of Γ. We first show that x is distance-4-regular. So let y and z be opposite x and $u,v\in x^y_{[4]}\cap x^z_{[4]}$, $u\neq v$. Let L be any line through x and let $y'=\mathrm{proj}_L y$, $z'=\mathrm{proj}_L z$. We show that $y'=z'$ and $\mathrm{proj}_{y'}y=\mathrm{proj}_{z'}z$. Suppose by way of contradiction that $y'\neq z'$. Let r be the unique point collinear with z and at distance 4 from z'. Suppose that all elements of $\langle x,y\rangle$ are at distance ≥ 6 from r. If we project all points of the regulus $\langle x,y\rangle$ which are at distance 6 from r onto r, then, since $|\langle x,y\rangle|=t+1=|\Gamma_1(r)|$, and since v and u are projected onto the same element, we see that there is at least one line N through r which lies at distance 7 from all those points, and hence from every element of $\langle x,y\rangle$. Since $t<s$, the projection of $\langle x,y\rangle$ onto N cannot be surjective, hence there is a point on N opposite every element of $\langle x,y\rangle$, a contradiction. Consequently there is a point w of $\langle x,y\rangle$ at distance ≤ 4 from r. If $\mathrm{proj}_x w\neq xy'$, then we obtain an ordinary j-gon, $j\leq 7$, through r,w,x,z', a contradiction. Hence w is the unique point at distance 2 from y' and at distance 4 from y. Now $\delta(r,w)\leq 4$ implies that $y'=z'$ and $\mathrm{proj}_{y'}y=\mathrm{proj}_{z'}z$. Interchanging the roles of x and y, we see that also $\delta(z,\mathrm{proj}_w y)=5$, implying that z lies at distance 4 from the point w. This shows that x is distance-4-regular. In particular, all (3,4)-intersection sets with respect to x have size $t+1$.

Next we show that all $(3,3)$-intersection sets with respect to x have size $t+1$. Let again y and z be opposite x such that $x^y_{[3]}\cap x^z_{[3]}$ is a (3,3)-intersection set. Let L be a line at distance 5 from x and 3 from both y and z and let M be a line of $x^y_{[3]}\cap x^z_{[3]}$, M opposite L. We prove that $x^y_{[3]}=x^z_{[3]}$. Let K be any line through x and put $y'=\mathrm{proj}_K y$ and $z'=\mathrm{proj}_K z$. Suppose by way of contradiction that $y'\neq z'$. Consider the line N at distance 3 from both z and z'. As before (using $t<s$), N cannot be at distance 7 from all elements of $\langle x,y\rangle$, hence N is at distance ≤ 5 from some element $w\in\langle x,y\rangle$.

If $\mathrm{proj}_x w \neq xy'$, then we obtain an ordinary j-gon, $j \leq 7$, containing the points w, $w \bowtie x$, x, z' and $\mathrm{proj}_N w$. So $\mathrm{proj}_x w = xy'$, which implies $y' = z'$. Put $N_z = \mathrm{proj}_{z'} z$ and $N_y = \mathrm{proj}_{y'} y$. We now show that $N_y = N_z$.

Suppose by way of contradiction that $N_y \neq N_z$. Let a be any point incident with N_z, $a \neq z'$, and let a' be the projection of a onto the line $\mathrm{proj}_y M$ (note that $a' \neq y$). Let L_a be the line at distance 3 from both a and a'. As before, L_a is not at distance 7 from every element of $\langle x, y\rangle$, so there exists a point $w' \in \langle x, y\rangle$ at distance ≤ 5 from L_a. If $w' \neq \mathrm{proj}_M y$, then we obtain an ordinary j-gon, $j \leq 7$, containing the points w', $w' \bowtie y$, y, a', $\mathrm{proj}_{L_a} a'$ and $\mathrm{proj}_{L_a} w'$. So $w' = \mathrm{proj}_M y$ and $\delta(w', L_a) \leq 5$ implies that $a' = w' \bowtie y$. So every point of the line N_z different from z' lies at distance 6 from the point $y \bowtie w'$, a contradiction. We conclude that $N_y = N_z$, showing that $x^y_{[3]} = x^z_{[3]}$. By Lemma 2.2.3, the point x is distance-3-regular. A point-distance-3-regular octagon does not exist (see Theorem 1.3.1), hence the result. $\square$

Combining the characterization of $\mathsf{W}(q)$ in Thas [39], with Theorems 2.2.1 and 2.2.5, we obtain the following.

Theorem 2.2.6 *Let Γ be a finite generalized $2m$-gon of order (s,t), with $s \geq t \geq 2$. Then Γ is isomorphic to $\mathsf{W}(q)$, to $\mathsf{H}(q')$ or to $\mathsf{T}(q'^3, q')$, with q' even, if and only if $\Gamma_{\leq 2m-2}(x) \cap \Gamma_{\leq m}(v) \cap \Gamma_{\leq m}(w)$ is nonempty for any point x, and for any pair of elements v, w, with v, w points if m is even, and v, w lines if m is odd.*

Remarks

1. In the case $m = 3$, the condition $s \geq t$ in the previous theorem is not needed.

2. It makes no sense to consider the cases v and w points if m is odd, or v and w lines if m is even. Indeed, let for example Γ be a generalized hexagon, and v, w opposite points in Γ. Let $z \in v^w$, and x a point at distance 3 from vz and 4 from v and z. The point x lies at distance 5 from every line of $\langle v, w\rangle$, hence $\langle v, w\rangle \cap \Gamma_{\leq 3}(x) = \emptyset$.

3. Theorem 2.2.6 characterizes the finite classical hexagons over a field of characteristic 2, but the assumptions kill every finite octagon, despite the fact that the Ree-Tits octagons too are defined over a field of characteristic 2. This shows once again that the Ree-Tits octagons play a special role in the theory of finite generalized polygons.

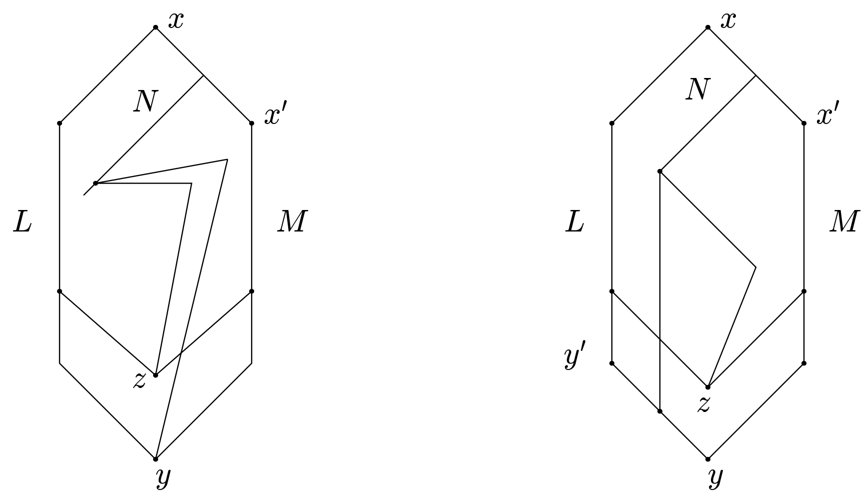
Generalize Theorem 2.2.6 *to the infinite case, i.e. find a 'nice' condition that forces* $2m \leq 6$.

Another attempt to generalize Theorem 2.2.1 to finite octagons is given in the following theorem.

Theorem 2.2.7 *Let* Γ *be a finite generalized octagon of order* (s,t)*. Suppose that for any point* x *and any two lines* L, M*, the set* $\Gamma_{\leq 6}(x) \cap \Gamma_{\leq 3}(L) \cap \Gamma_{\leq 5}(M)$ *is nonempty. Then* $s > t$*. If moreover, all* $(2,2)$*-intersection sets with respect to any line contain* $s+1$ *lines, then* Γ *is line-distance-*2*-regular* (*and hence does not exist*).

Proof. Let Γ be a finite generalized octagon of order (s,t) such that $\Gamma_{\leq 6}(x) \cap \Gamma_{\leq 3}(L) \cap \Gamma_{\leq 5}(M)$ is nonempty, for any point x and any two lines L, M. We first show that $t \leq s$. Suppose by way of contradiction that $s < t$. Let L and M be two opposite lines, and $x, y \in L^M_{[3]}$. Put $x' = \mathrm{proj}_M x$ and $v = x \bowtie x'$. Let N be an arbitrary line concurrent with vx', not through v or x', and $r = \mathrm{proj}_N y$. Let X be the set of points of $L^M_{[3]}$ at distance 6 from r, and Y the set of points of $L^M_{[3]}$ at distance 8 from r. Note that $X \cup Y = L^M_{[3]}$ and $x, y \in X$. We now project the points of X onto N; since $t > s$, at least one line N' through r is not the projection of any point of X, and hence N' lies at distance 7 from all points of $L^M_{[3]}$. The points of X lie opposite all points of $N' \setminus \{r\}$. We now project the points of Y onto the line N'. Since $|Y| \leq s-1$, at least one point w on N' lies opposite every point of $L^M_{[3]}$, a contradiction. This shows that $t > s$.

Now let L be a line of Γ. We show that all (2,3)-intersection sets with respect to L contain $s+1$ points. Let $L^{M_1}_{[3]} \cap L^{M_2}_{[3]}$ be a (2,3)-intersection set with respect to L. Let r be the point at distance 5 from L and at distance 3 from both M_1 and M_2, and r' a point in $L^{M_1}_{[3]} \cap L^{M_2}_{[3]}$, r' opposite r. Let v be an arbitrary point on L. We show that $R_1 := \mathrm{proj}_v M_1 = \mathrm{proj}_v M_2 =: R_2$. Suppose by way of contradiction that $R_1 \neq R_2$. Let r'' be the point collinear with r and at distance 3 from L. Let N be the line concurrent with M_2 at distance 4 from R_2. If all points of $L^{M_1}_{[3]}$ lie at distance 7 from N, then (since $|L^{M_1}_{[3]}| = |N|$ and both r'' and r' project onto the same element of N) there is a point on N opposite every point of $L^{M_1}_{[3]}$, a contradiction. Hence there is a point $x \in L^{M_1}_{[3]}$ at distance ≤ 5 from N. If x is not incident with R_1, we obtain an ordinary j-gon, $j \leq 7$, through x, v, $\mathrm{proj}_N v$ and $\mathrm{proj}_N x$. Hence $x \mathtt{I} R_1$, implying that $R_1 = R_2$, so $L^{M_1} = L^{M_2}$.



condition (I) condition (I')

Figure 2.3: A weakening of the condition for long imaginary lines

Completely similar, one shows that all $(2,4)$-intersection sets with respect to L contain $s+1$ points. Together with the assumption about the (2,2)-intersection sets, (the dual of) Lemma 2.2.3 implies that Γ is line-distance-2-regular, and hence cannot exist by Theorem 1.3.1. $\square$

Let x, y be opposite points of the hexagon $\mathsf{H}(q)$, q even, and L, M distinct lines in $\langle x, y\rangle$. Then the imaginary line $I(x,y)$ is long and hence coincides with the point regulus $\langle L, M\rangle$. It immediately follows from the definition of imaginary line that, if R is a line of Γ at distance 3 from the point x and at distance 5 from all points of $\langle L, M\rangle \setminus \{x\}$, then all points of $\langle L, M\rangle \setminus \{x\}$ project onto the same point of R. We now ask the above property for lines R in a particular position with respect to the regulus $\langle L, M\rangle$ and obtain a characterization of $\mathsf{H}(q)$, q even.

Consider the following property in a finite generalized hexagon Γ:

(I) Let L and M be two arbitrary opposite lines, x, y different points of $\langle L, M\rangle$ and $x' = \mathrm{proj}_M x$. Let N be an arbitrary line concurrent with xx', not through x or x'. Then $\mathrm{proj}_N y = \mathrm{proj}_N z$, for all $z \in \langle L, M\rangle \setminus \{x\}$.

Theorem 2.2.8 *A finite generalized hexagon Γ satisfies condition (I) if and only if Γ is isomorphic to $\mathsf{H}(q)$, q even.*

Proof. Suppose Γ is a finite generalized hexagon in which (I) holds. We first show that Γ is distance-3-regular. So let L and M be two opposite lines, x,y,z different points of $\langle L,M\rangle$ and $N \in \langle x,y\rangle$. We have to prove that $\delta(z,N)=3$. Put $p=\mathrm{proj}_L y$, $p'=\mathrm{proj}_M y$, $x'=\mathrm{proj}_N x$, $y'=\mathrm{proj}_N y$ and $z'=\mathrm{proj}_{xx'} z$. We show that $z'=x'$. Suppose by way of contradiction that $z'\neq x'$ and put $z''=\mathrm{proj}_{py} z'$. Suppose first $\mathrm{proj}_{z'} z \neq \mathrm{proj}_{z'} z''$. But this contradicts (I) since the projections of x and z onto the line through z'' and $z' \bowtie z''$ do not coincide, so $\mathrm{proj}_{z'} z = \mathrm{proj}_{z'} z''$. Note that $z' \bowtie z \neq z' \bowtie z''$ since otherwise, there would be an ordinary pentagon through the points z'', p, $\mathrm{proj}_L z$, z and $z \bowtie z'$. Let u be the projection of z' onto yp'. But now, noting that $\mathrm{proj}_{z'} u \neq \mathrm{proj}_{z'} z$, the projections of x and z onto the line through u and $u \bowtie z'$ do not coincide, again contradicting (I), so $x'=z'$. Interchanging the roles of x and y, we see that $y'=\mathrm{proj}_{yy'} z$. But this creates an ordinary pentagon containing x', y' and z, unless $\delta(z,N)=3$.

We next show that an imaginary line coincides with a regulus containing two of its points. Since Γ is finite, this will imply that all imaginary lines are long, which proves the result in view of Theorem 1.8.5. So let $I(x,y)$ be an imaginary line, and suppose by way of contradiction that there is a point $z \in R(x,y)$ not belonging to $I(x,y)$. This implies there exists a point a not opposite the points x and y, with $\delta(z,a)=6$. If $\delta(x,a)=2$ or $\delta(y,a)=2$, or if $\mathrm{proj}_a x = \mathrm{proj}_a y$, then the 3-regularity implies that $\delta(z,a)=4$, a contradiction. So suppose $\delta(x,a)=\delta(y,a)=4$ and $\mathrm{proj}_a x \neq \mathrm{proj}_a y$. Put $b = x \bowtie a$ and $c=\mathrm{proj}_{xb} y$. Again by the 3-regularity, the point z lies at distance 3 from the line through c and $c \bowtie y$. Now by (I), $\mathrm{proj}_{ab} z = \mathrm{proj}_{ab} y = a$, the final contradiction. $\square$

Consider the following weaker version of condition (I):

(I') Let L and M be two arbitrary opposite lines, x,y different points of $\langle L,M\rangle$ and $x'=\mathrm{proj}_M x$, $y'=\mathrm{proj}_L y$. Let N be an arbitrary line concurrent with xx', not through x or x' and at distance 4 from yy'. Then $\mathrm{proj}_N y = \mathrm{proj}_N z$, for all $z \in \langle L,M\rangle \setminus \{x\}$.

Corollary 2.2.9 *A finite generalized hexagon* Γ *of order* (s,t), $t \leq s$, *satisfies condition* (I') *if and only if* Γ *is isomorphic to* $\mathsf{H}(q)$, *q even.*

Proof. Suppose Γ satisfies (I'), and let L,M,x,x',y,y' be as in (I'). Let z be a point of $\langle L,M\rangle$, $x \neq z \neq y$, and put $z'=\mathrm{proj}_L z$. Let v be an arbitrary point on xx', $x\neq v\neq x'$ and $v'=\mathrm{proj}_{zz'} v$. If $\mathrm{proj}_v yy' = \mathrm{proj}_v zz'$, then condition (I') implies that $\delta(y, v' \bowtie v)=4$, which creates a pentagon containing z', y' and $v' \bowtie v$, so $\mathrm{proj}_v yy' \neq \mathrm{proj}_v zz'$. This shows that $t+1=$

$|\Gamma_1(v)| \geq |\langle x, y\rangle| = s+1$, so $t = s$. Now clearly, every line through v different from xx' lies at distance 4 from a line aa', with $a\mathtt{I}L$ and $a' \in \langle L, M\rangle$. Hence (I) is satisfied and the result follows. $\square$

2.3 Characterizations of $\mathsf{H}(q)^D$ and $\mathsf{T}(q, q^3)$

Consider the following property in a finite generalized hexagon Γ.

(C) If a point x is at distance 4 from an element y of the point regulus R, and if all elements of $R \setminus \{y\}$ are opposite x, then all elements of $R \setminus \{y\}$ are at distance 4 from $x \bowtie y$.

Lemma 2.3.1 *(i) The dual classical hexagons $\mathsf{H}(q)^D$ and $\mathsf{T}(q, q^3)$ satisfy property (C).*

(ii) The classical hexagons $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ satisfy property (C) if and only if q is odd.

Proof. Let Γ be a finite (dual) classical hexagon. Let x be a point of Γ at distance 4 from an element y of the point regulus $R = \langle M, N\rangle$, and suppose x is opposite every element of $R \setminus \{y\}$. Put $L = \mathrm{proj}_x y$. Note that, by the distance-3-regularity, M and N can be chosen arbitrarily in $\langle y, z\rangle$, for any $z \in R \setminus \{y\}$. In particular, we may choose M at distance ≤ 4 from L. If $\delta(L, M) = 2$, then $x \bowtie y$ is incident with M, hence lies at distance 4 from every element of $R \setminus \{y\}$. So in this case, property (C) is satisfied. Suppose now $\delta(L, M) = 4$. We show that this leads to a contradiction.

Let first Γ be dual classical. Note that the point $x \bowtie y$ does not belong to M, hence $x \bowtie y$ is opposite every element of $R \setminus \{y\}$. We project the points of the regulus R onto the line L. Since $|R| = q + 1$, and since also x is opposite every element of $R \setminus \{y\}$, there must be some point w incident with L which is at distance 4 from at least two points $z_1, z_2 \in R \setminus \{y\}$. Now by the distance-3-regularity we may choose N in such a way that it meets $\mathrm{proj}_{z_1} w$ (see Figure 2.4). Note that $\{\mathrm{proj}_y N, \mathrm{proj}_{z_1} N, \mathrm{proj}_{z_2} N\} \subseteq N^M$, and $\{\mathrm{proj}_y N, \mathrm{proj}_{z_1} N\} \subseteq N^L$. By the line-distance-2-regularity, we must have $\delta(L, \mathrm{proj}_{z_2} N) = 4$. But then, we obtain a pentagon containing $z_2 \bowtie w$, z_2, w and the line intersecting both L and $\mathrm{proj}_{z_2} N$, the final contradiction. Hence (C) is satisfied.

Let now Γ be a classical hexagon. Put $y' = \mathrm{proj}_M y$ and $v = \mathrm{proj}_{yy'} L$. We prove (using coordinates) that the projection θ of $\langle M, N\rangle \setminus \{y\}$ onto $L \setminus \{v\}$ is

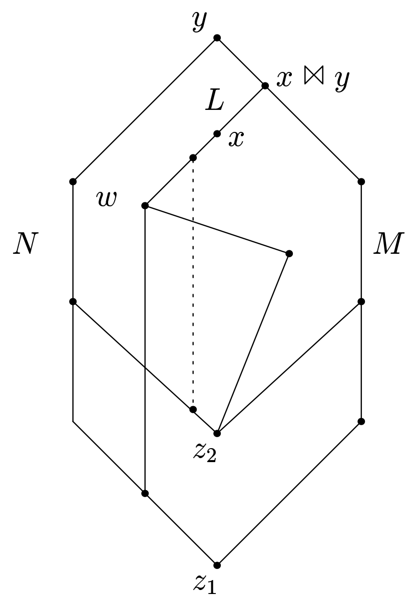


Figure 2.4: Proof of Lemma 2.3.1 if Γ is dual classical.

a bijection if and only if q is odd. Let first $\Gamma \cong \mathsf{T}(q^3,q)$. Choose coordinates in the following way: $y = (\infty)$, $yy' = [\infty]$, $M = [00]$, $N = [000]$, $v = (b)$, $b \in \mathrm{GF}(q^3) \setminus \{0\}$ and $L = [b,k]$, $k \in \mathrm{GF}(q)$. A point p of $\langle M, N\rangle \setminus \{y\}$ then has coordinates $(0,0,a,0,0)$, $a \in \mathrm{GF}(q^3)$. The projection of p onto L is the intersection of L with the tangent hyperplane ξ_p of $\mathsf{Q}(7,q^3)$ at p. We calculate this intersection in the projective space $\mathrm{PG}(7,q^3)$ (and therefore, we use Table 3.4 in [57]). Note that

$$p = (0,0,a,0,0) \leftrightarrow (a^{q+q^2},0,0,a^{q^2};1,0,0,-a^q),$$

$$L = [b,k] \leftrightarrow (-k,1,0,b^q;0,b^{q+q^2},0,-b^{q^2}) + t(b,0,0,0;0,0,1,0),$$

with $t \in (\mathrm{GF}(q^3) \cup \{\infty\})$, and

$$\xi_p \leftrightarrow X_0 - a^q X_3 + a^{q+q^2} X_4 + a^{q^2} X_7 = 0.$$

The point $\xi_p \cap L$ is then the point of L for which

$$t_p = \frac{k + a^q b^q + a^{q^2} b^{q^2}}{b}.$$

If θ is not bijective, then there exist two points $p(0,0,a,0,0)$ and $p'(0,0,a',0,0)$ belonging to $R \setminus \{y\}$ such that $t_p{=}t_{p'}$, or

$$b^q(a^q - a'^q) = b^{q^2}(a'^{q^2} - a^{q^2}).$$

Since $a \neq a'$ this becomes

$$b^{q^2-q}(a'-a)^{q^2-q} = -1.$$

If q is even, then one can choose p and p' such that $(a'-a) = b^{-1}$, hence θ is not bijective. Suppose now q is odd. Then -1 has to be a (q^2-q)th power, implying

$$d^{l(q^2-q)} = d^{\frac{q^3-1}{2}} (= -1),$$

with d a generating element of the multiplicative group $\mathrm{GF}(q^3) \setminus \{0\}$. This implies that

$$2l(q^2-q) \equiv 0 \bmod (q^3-1),$$

or

$$2lq \equiv 0 \bmod (q^2+q+1).$$

Since q^2+q+1 is odd, and not a multiple of q, l is necessarily a multiple of (q^2+q+1), or $l = l'(q^2+q+1)$. This implies

$$d^{l(q^2-q)} = d^{l'(q^2+q+1)(q^2-q)} = d^{l'q(q^3-1)} = 1,$$

hence $-1 = 1$, a contradiction, hence θ is a bijection.
If $\Gamma \cong \mathsf{H}(q)$, we obtain

$$t_p = \frac{k+2ab}{b}.$$

Now it is clear that also in this case, θ is a bijection if and only if q is odd. So the line L does not contain a point x opposite every point of $R \setminus \{y\}$ if and only if q is odd, showing (ii). $\square$

Lemma 2.3.2 *Let Γ be a finite generalized hexagon satisfying property (C). Then Γ is distance-3-regular.*

Proof. Let x, y, z be three distinct points of a regulus $\langle L, M\rangle$ and let $N \in \langle x, y\rangle$, $L \neq N \neq M$. We have to show that $\delta(z, N) = 3$. Let $v = \mathrm{proj}_N x$ and let $w = \mathrm{proj}_N y$.

Suppose first that there exists a line N' through v, $N' \neq vx$, at distance 5 from every element of $\langle L, M\rangle \setminus \{x\}$. Consider the projection of $\langle L, M\rangle$ onto N'. Since v is the image of at least two points of $\langle L, M\rangle$ (namely x and y), this projection is not injective, hence neither surjective. So there is a point incident with N' opposite every element of $\langle L, M\rangle \setminus \{x\}$. Property (C) implies now that every point of $\langle L, M\rangle \setminus \{x\}$ is at distance 4 from v. As a consequence, every point z' of $\langle L,M\rangle \setminus \{y\}$ is at distance 5 from every line N'' through w, $N'' \neq N$ (indeed, otherwise there arises a circuit of length $\delta(z',v)+\delta(v,N'')+\delta(N'',z') \leq 4+3+3$). Hence, interchanging the roles of v and w, we see that $\delta(z,v)=\delta(z,w)=4$, which is only possible if $\delta(z,N)=3$.

Now suppose that for every line $N' \neq vx$ through v there exists a point of $\langle L,M\rangle \setminus \{x\}$ at distance 3 from N'. By the previous paragraph, we can also assume that for every line $N'' \neq wy$ through w, there exists a point of $\langle L,M\rangle \setminus \{y\}$ at distance 3 from N''. Put $z' = \mathrm{proj}_L z$ and $z'' = \mathrm{proj}_M z$. Note that at least one of the lines zz' and zz'' is opposite N, otherwise $\delta(z,N)=3$ and we can go home. Suppose $\delta(zz'',N)=6$. Similarly, z is opposite at least one of v,w. Suppose $\delta(v,z)=6$. Let r be the projection of v onto zz'' (then $z \neq r \neq z''$). Put $R_1 = \mathrm{proj}_v r$ and $R_2 = \mathrm{proj}_r v$ (then $R_1 \neq N$). By assumption, there exists a point $y' \in \langle L,M\rangle \setminus \{x\}$ at distance 3 from R_1 (and note that $\mathrm{proj}_{R_1} y' \neq r \bowtie v$ because otherwise the points $y', \mathrm{proj}_M y', z'', r$ and $r \bowtie v$ define a pentagon). We consider the projection of $\langle L,M\rangle$ onto the line R_2. Since x and y' are both mapped onto $r \bowtie v$, this projection is, as above, not surjective, and hence, again using property (C) as before, r must be at distance 4 from every element of $\langle L,M\rangle \setminus \{z\}$, a final contradiction as, for example, $\delta(r,x)=6$. This shows the distance-3-regularity. □

Combining Lemma 2.3.2 and Theorem 1.8.3, we now obtain a characterization of some extremal classical hexagons.

Corollary 2.3.3 *If* Γ *is an extremal hexagon satisfying Property* (C)*, then it is isomorphic to* $\mathsf{T}(q,q^3)$ *or to* $\mathsf{T}(q'^3,q')$*,* q' *odd.*

Lemma 2.3.4 *Let* Γ *be a finite generalized hexagon satisfying property* (C)*. Then the following property holds in* Γ*: if a point* x *is at distance at most* 4 *from at least three points* y_1,y_2,y_3 *of a point regulus* R*, then* x *is at distance* 2 *from a unique element of* R *and at distance* 4 *from all other elements of* R*.*

Proof. Let Γ, x, R, y_1,y_2 and y_3 be as in the lemma. Note that Γ is distance-3-regular. If one of the points y_1,y_2,y_3 is collinear with x, or if at least two of these points have the same projection onto x, then the property above immediately follows from the distance-3-regularity. So suppose now that $\delta(x,y_i)=4$, $i=1,2,3$ and that the projections of the points y_1,y_2,y_3 onto x are all different. We look for a contradiction. Put $N=\mathrm{proj}_x y_1$. If N is at distance 3 from an element of $R\setminus\{y_1\}$ then, by the distance-3-regularity, $\mathrm{proj}_x y_2 = N$, a contradiction with the assumption. So we may assume that every point of $R \setminus \{y_1\}$ is at distance 5 from N. We then consider the projection of $R \setminus \{y_1\}$ onto N. Since $|R \setminus \{y_1\}| = |N| - 1$ and both y_2 and y_3 have the same image, there is some point w incident with N and distinct from $x \bowtie y_1$ opposite every element of $R \setminus \{y_1\}$. Property (C) implies that $\delta(y_2, x \bowtie y_1) = 4$, a contradiction. $\square$

If we want to use property (C) to characterize (some) dual classical hexagons, Lemma 2.3.1 (ii) shows that the condition q even is certainly necessary. It turns out that this condition is also sufficient.

Theorem 2.3.5 *Let Γ be a finite generalized hexagon of order (q,t), q even. Then Γ has property (C) if and only if Γ is isomorphic to $\mathsf{H}(q)^D$ or to $\mathsf{T}(q,q^3)$.*

Proof. Let Γ be a finite generalized hexagon of order (q,t), q even, satisfying property (C). Let x and y be two opposite points of Γ, and L an arbitrary line of Γ. We claim that there exists a line of the line regulus $\langle x, y \rangle$ at distance at most 4 from L. If L lies at distance ≤ 3 from a point of $R(x,y)$, then the claim follows because of the 3-regularity. So we may assume that L is at distance 5 from every point of $R(x,y)$. Hence, by Lemma 2.3.4, no point incident with L is the projection of at least 3 elements of $R(x,y)$. So every point on L is the projection of 0,1 or 2 points of $R(x,y)$. Since $|R(x,y)| = q+1$ is odd, there is a point v on L which is the projection of exactly one point z of $R(x,y)$. So v is opposite every point of $R(x,y) \setminus \{z\}$. Property (C) now implies that $v \bowtie z$ is at distance 4 from every point of $R(x,y) \setminus \{z\}$. By the distance-3-regularity, the projection of any element of $R(x,y) \setminus \{z\}$ onto $v \bowtie z$ is a line of $\langle x, y \rangle$ lying at distance 4 from L. This shows the claim. Now the property just shown is exactly the dual of the condition in Theorem 2.2.1, hence the result follows. $\square$

The property mentioned in Lemma 2.3.4*, together with the* 3*-regularity, is equivalent with property (C). But does this property on its own characterize the finite dual classical hexagons over a field of even characteristic ?*

Let x be a point of a generalized hexagon Γ, and y and z two points opposite x. We say that the points y and z are **in $(3,4)$-position with respect to x** if there exist points $v, w \in x^y \cap x^z$ such that $\mathrm{proj}_v y = \mathrm{proj}_v z$ but $\mathrm{proj}_w y \neq \mathrm{proj}_w z$ $(*)$. So there is a line at distance 3 from x, y and z and a point at distance 4 from both y, z and collinear with x (see the picture in Appendix B).

Lemma 2.3.6 *Let Γ be a finite generalized hexagon of order (q,t) isomorphic to $\mathsf{H}(q)^D$, q not divisible by 3, or to $\mathsf{T}(q,q^3)$. Then $|x^y \cap x^z| = 1 + t/q$ for all points x, y and z such that y and z are in $(3,4)$-position with respect to x.*

Proof. Let Γ be as above and v, w, x, y, z as in $(*)$. Let $L = \mathrm{proj}_v y = \mathrm{proj}_v z$, let $M_y = \mathrm{proj}_w y$, $M_z = \mathrm{proj}_w z$ and $u = v \bowtie y$. If z and u are collinear, then $x^y \cap x^z$ is an intersection set, hence $|x^y \cap x^z| = 1 + t/q$ (see section 1.8). Suppose $\delta(u,z) = 4$. Let a be the unique point of $\langle L, M_z \rangle$ at distance 4 from y. Because of the distance-3-regularity, we have $x^z = x^a$. Note that $x^y \cap x^a$ is an intersection set. Since we are working with the finite dual classical hexagons, we have $|x^a \cap x^y| = 1 + t/q$. This proves the lemma. $\square$

Theorem 2.3.7 *Let Γ be a finite generalized hexagon of order (q,t). Then Γ is isomorphic to $\mathsf{H}(q)^D$, q not divisible by 3, or to $\mathsf{T}(q,q^3)$ if and only if $|x^y \cap x^z| \leq 1 + t/q$ for all points x, y and z such that y and z are in $(3,4)$-position with respect to x.*

Proof. Let Γ be a finite generalized hexagon of order (q,t) satisfying the condition above on points in (3,4)-position. Note that this condition implies that every intersection set contains at most $1 + t/q$ points. We claim that each intersection set contains exactly $1 + t/q$ points. Indeed, let $p^{p'} \cap p^{p''}$ be an intersection set, put $r = p' \bowtie p''$ and $r' = r \bowtie p$. Now project the points of $p^{p'} \setminus \{r'\}$ onto the line $p''r$. Since every point of $p''r$ different from r (and there are q of these points) is the projection of at most t/q points of $p^{p'} \setminus \{r'\}$, every point of $p''r$ different from r is the projection of exactly t/q points, hence the claim.

Let x and y be opposite points, and $\{v, w\} \subseteq x^y$. Put $u = v \bowtie y$. Let L be an arbitrary line through x, $wx \neq L \neq vx$ and p a point on L, $p \neq x$. If there is a point $z \in u^w \setminus \{v\}$ at distance 4 from p, then $u^w \cap u^p$ is an intersection set, implying that p is the projection of exactly t/q points of $u^w \setminus \{v\}$ onto L. Hence each point on L different from x is the projection of either 0 or t/q elements of $u^w \setminus \{v\}$. Since $|L \setminus \{x\}| = q$ and $|u^w \setminus \{v\}| = t$, each point on L different from x is the projection of exactly t/q points of $u^w \setminus \{v\}$.

Now we prove that Γ is distance-3-regular. Put $w' = y \bowtie w$. Let z be a point of $\langle uv, ww' \rangle$ different from x or y. Consider the set

$$\mathcal{S} = \{(a,b) | a \in x^z \setminus \{v,w\}, b \in u^w \setminus \{v\} \text{ with } \delta(a,b) = 4\}.$$

We count the number of elements in $\mathcal{S}$. Fixing a, we obtain that $|\mathcal{S}| = (t-1) \cdot \frac{t}{q}$ by the previous paragraph. Now fix b. The points a then belong to $(x^b \cap x^z) \setminus \{v, w\}$. If $b \neq y$, then there at most $\frac{t}{q} - 1$ choices for a, since b and z lie in (3,4)-position with respect to x. Put $\ell = |x^y \cap x^z \setminus \{v, w\}|$. Now we obtain

$$(t-1) \cdot \frac{t}{q} \leq (t-1)(\frac{t}{q} - 1) + \ell,$$

implying $\ell = t - 1$, which means that $x^y = x^z$. Also, $\mathrm{proj}_r y = \mathrm{proj}_r z$ for all $r \in x^y \cap x^z$. Indeed, if not, then the points y and z would lie in (3,4)-position with respect to x, hence $|x^y \cap x^z| \leq \frac{t}{q} + 1 < t + 1$. This shows that Γ is distance-3-regular. Since every intersection set contains $1 + t/q$ points, and hence more than one point, Γ is line-distance-2-regular by Theorem 1.8.4 (ii). By Theorem 1.8.2, Γ is classical. $\square$

Remark. In the previous theorem, we did not assume that t is divisible by q, or that $q \leq t$.

Let x, y, z be points of a generalized hexagon Γ. If y and z are opposite x, then we denote

$$x^{\{y,z\}} = \{r \in x^y \cap x^z | \mathrm{proj}_r y \neq \mathrm{proj}_r z\}.$$

Let Γ be a finite generalized hexagon of order (q, q). We say that Γ is **anti-regular** if for any three points x, y, z, with y, z lying opposite x, the condition $|x^{\{y,z\}}| \geq 2$ implies that $|x^{\{y,z\}}| = 3$ and $x^{\{y,z\}} = x^y \cap x^z$.
We say that Γ is **weak anti-regular** if for any three points x, y, z, with y, z lying opposite x, the condition $|x^{\{y,z\}}| \geq 2$ implies that $|x^{\{y,z\}}| \geq 3$ and $x^{\{y,z\}} = x^y \cap x^z$.

Lemma 2.3.8 *The generalized hexagon $\mathsf{H}(q)^D$, with q not divisible by 3, is anti-regular (and hence also weak anti-regular).*

Proof. Let Γ be the hexagon $\mathsf{H}(q)^D$, q not a multiple of 3. Let y, z be points opposite a point x, and $v, w \in x^{\{y,z\}}$. Put $L_y = \mathrm{proj}_v y$, $L_z = \mathrm{proj}_v z$, $M_y = \mathrm{proj}_w y$, $M_z = \mathrm{proj}_w z$, $u' = w \bowtie y$ and $u'' = w \bowtie z$. We first claim that the pointwise stabilizer of $\{x, v, w, z, M_y\}$ in the automorphism group of Γ acts transitively on the set $\Gamma_1(v) \setminus \{vx, L_z\}$. Choose coordinates such that $v = (\infty)$, $L_z = [\infty]$ and the apartment through x, v, w and z is the hat-rack of the coordinatization. Consider the following homologies h_A:

$$\begin{array}{rrcl} h_A & : \ (a, l, a', l', a'') & \rightarrow & (A^{-1}a, l, Aa', Al', A^2a'') \\ & [k, b, k', b', k''] & \rightarrow & [Ak, A^2b, Ak', Ab', k''], \end{array}$$

with $A \in \mathrm{GF}(q) \setminus \{0\}$ (it suffices to give the action of h_A on the elements opposite (∞) and $[\infty]$, see 4.5.3 in [57]). Then each h_A fixes the apartment through x, v, z and w, and every line through the point w (since a line through w different from xw or M_z has coordinates $[0,0,0,0,k'']$). Also, these homologies h_A act transitively on the lines through u'' different from M_z or $u''z$ (since these lines have coordinates $[k,0,0,0,0]$), and hence on the lines through v different from vx or L_z. This shows the claim.

Now consider the following set

$$\mathcal{T} = \{x^{u_L} | u_L = (\mathrm{proj}_L u') \bowtie u', L \in \Gamma_1(v) \setminus \{vx\}\}.$$

Since the intersection of two traces belonging to $\mathcal{T}$ is an intersection set, two traces contained in $\mathcal{T}$ only have the points v and w in common. So, if R is an arbitrary line through x different from xv or xw, the q points u_L all project onto a different point of R, implying that the set $\mathcal{T}$ meets $x^z \setminus \{v,w\}$ in exactly $q-1$ points. Note that the traces x^z and $x^{u_{L_z}}$ only have the points v and w in common, since u_{L_z} and z are in (3,4)-position with respect to x. We claim that the $q-1$ traces of $\mathcal{T} \setminus \{x^{u_{L_z}}\}$ meet the trace x^z in a constant number of points. Indeed, we look for the image under h_A of $x^z \cap x^{u_L}$, $L \neq L_z$. The homology h_A leaves the set x^z invariant. The image under h_A of the trace x^{u_L} is the trace x^r, with r the unique point collinear with $u'' := h_A(u')$ (which is a point on M_y) and at distance 3 from $L' = h_A(L)$. Hence $|x^z \cap x^{u_L}| = |x^z \cap x^r|$. Because of the distance-3-regularity, we have $x^r = x^{u_{L'}}$, implying $|x^z \cap x^{u_L}| = |x^z \cap x^{u_{L'}}|$. Now by the transitivity of the homologies h_A mentioned in the first paragraph, it follows that each trace of $\mathcal{T} \setminus \{x^{u_{L_z}}\}$ meets x^z in a constant number (=3) of points. So in particular, $|x^y \cap x^z| = 3$.

Now let a be the unique point of $x^y \cap x^z$ different from v and w. If $\mathrm{proj}_a y = \mathrm{proj}_a z$, then the points y and z are in (3,4)-position with respect to x, implying $|x^y \cap x^z| = 2$, a contradiction. Hence $x^{\{y,z\}} = x^y \cap x^z$, which shows the lemma. $\square$

Remark. Lemma 2.3.8 can also easily be proved using coordinates instead of the more geometric arguments given above.

Theorem 2.3.9 *A finite generalized hexagon* Γ *of order* (q,q) *is weak anti-regular if and only if it is isomorphic to* $\mathsf{H}(q)^D$*, with* q *not divisible by* 3.

Proof. Let Γ be a finite weak anti-regular hexagon of order (q,q). Let y and z be points in (3,4)-position with respect to a point x; $v,w \in x^y \cap x^z$, with $\mathrm{proj}_v y = \mathrm{proj}_v z$ and $\mathrm{proj}_w y \neq \mathrm{proj}_w z$. We prove that $|x^y \cap x^z| = 2$. The result will then follow from Theorem 2.3.7. Put $u = v \bowtie y$ and $u' = w \bowtie y$. Suppose first that z and u are collinear, and let by way of contradiction

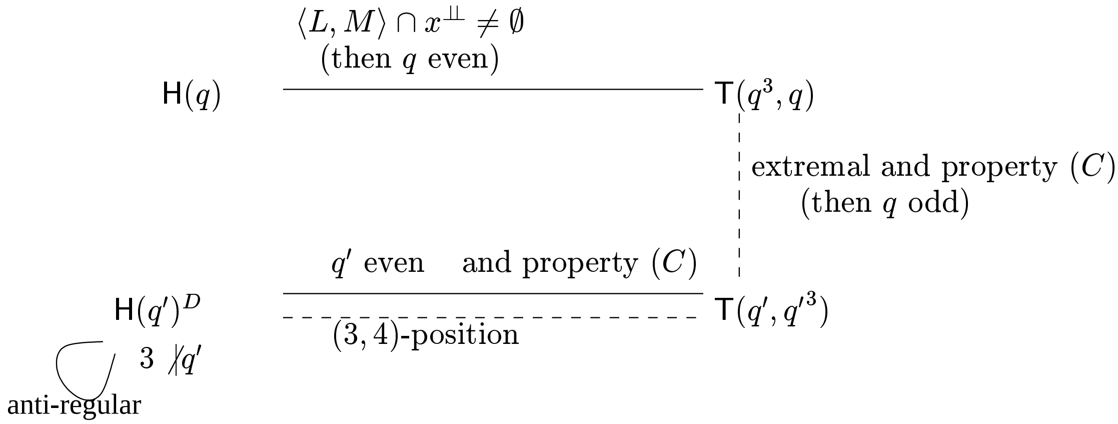


Figure 2.5: Characterizations obtained in sections 2.2 and 2.3.

$r \in x^y \cap x^z \setminus \{v, w\}$. Clearly, $\mathrm{proj}_r y \neq \mathrm{proj}_r z$. But now, the weak anti-regularity implies that also $\mathrm{proj}_v y \neq \mathrm{proj}_v z$, a contradiction.

Suppose now that $\delta(u, z) = 4$. Let $\mathcal{T} = \{x^r | r \in \Gamma_2(u') \cap \Gamma_4(v) \cap \Gamma_6(x)\}$. By the previous paragraph, every two elements of $\mathcal{T}$ have exactly the points v, w in common. By the weak anti-regularity, every set $T \cap x^z$, with $T \in \mathcal{T} \setminus \{x^y\}$, contains at least one element different from v, w. Since this gives rise to at least $q-1$ elements of $x^z \setminus \{v, w\}$, there is no room anymore in x^z for elements of $x^y \setminus \{v, w\}$, implying $|x^y \cap x^z| = 2$. $\square$

Remark. We call a finite generalized $2n$-gon Γ of order (q, q) **anti-regular** if for any three points x, y, z, with z, y both opposite x, such that $|\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)| \geq n-1$ and $|\Gamma_1(w) \cap \Gamma_{2n-3}(y) \cap \Gamma_{2n-3}(z)| = 0$, for at least $n-1$ elements w of $\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)$, we have that $|\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)| = n$ and $|\Gamma_1(w) \cap \Gamma_{2n-3}(y) \cap \Gamma_{2n-3}(z)| = 0$, for all elements w of $\Gamma_2(x) \cap \Gamma_{2n-2}(y) \cap \Gamma_{2n-2}(z)$.
In this way, one can see that the definition given earlier for an anti-regular hexagon generalizes the definition of anti-regularity given in Payne & Thas [34].

Proof Theorem 2.3.9 *under the weaker assumption that 'a certain set' of points of* Γ *is anti-regular.*

Some of the characterizations obtained in sections 2.2 and 2.3 are summarized in Figure 2.5.

2.4 A characterization of Moufang hexagons using ovoidal subspaces

Recall that for two opposite points x, y in a generalized hexagon, we say that the pair (x, y) is 3-regular if the set $\langle x, y\rangle$ is determined by any two of its lines.

Theorem 2.4.1 *Let Γ be a generalized hexagon in which all intersection sets $x^y \cap x^z$ either have size 1 or satisfy $x^y = x^z$. If Γ contains an ovoidal subspace $\mathcal{O}$ all the points of which are 3-regular, then Γ is 3-regular and hence a Moufang hexagon.*

Proof. The condition about the intersection sets is equivalent with the fact that every two opposite points x and y are contained in a (unique) thin ideal subhexagon, which we denote by $\mathcal{D}(x, y)$ (this follows for instance from Van Maldeghem [57], Lemma 1.9.10). Let x and y be two opposite points: we prove that the pair (x, y) is 3-regular. It is sufficient to find a point z at distance 3 from two lines of $\langle x, y\rangle$ such that one of the pairs (x, z) or (y, z) is 3-regular. We may assume that neither x nor y belongs to $\mathcal{O}$.

(a) Suppose there is a point $a \in \mathcal{D}(x, y)$ opposite x and at distance 4 from y such that the pair (a, x) is 3-regular. Let L_a be the line through $a \bowtie y$ at distance 3 from x and let z be an arbitrary point in $\langle L_a, M\rangle$, $L_a \neq M \in \langle x, y\rangle$, $z \neq x$. We show that $x^y = x^z$. Put $u = \mathrm{proj}_{L_a} z$, $u' = \mathrm{proj}_M a$, $X = \mathrm{proj}_{u'} a$ and z' the point of $\langle L_a, X\rangle$ collinear with u. Then $x^y = x^a$ and $x^{z'} = x^z$ because we work in the thin subhexagon $\mathcal{D}(x, y)$. Also $x^a = x^{z'}$ by the 3-regularity, hence $x^y = x^z$.

(b) Suppose in addition to (a) there is a point $b \in \mathcal{D}(x, y)$ opposite y and at distance 4 from x such that the pair (b, y) is 3-regular. Then the pair (x, y) is 3-regular. Indeed, let L_b be the line through $b \bowtie x$ at distance 3 from y. If $L_a \neq L_b$, put $M = L_a$ and $N = L_b$. If $L_a = L_b$, put $M = L_a$ and $N \in \langle x, y\rangle$, $N \neq L_a$. Applying (a), we see that for an arbitrary point $z \in \langle M, N\rangle$, $x \neq z \neq y$, $x^y = x^z$ and $y^x = y^z$, so z lies at distance 3 from every line of $\langle x, y\rangle$, and the pair (x, y) is 3-regular.

Let first $\mathcal{O}$ be an ovoid not containing x or y. Then $\mathcal{D}(x, y)$ contains 0, 1 or 2 points of $\mathcal{O}$. Suppose first $\mathcal{D}(x, y)$ contains two points a and b of $\mathcal{O}$. Up to interchanging x and y, one of the following situations occurs:

- $\delta(a,x) = 4 = \delta(b,y)$ and $\delta(a,y) = 6 = \delta(b,x)$.
  It immediately follows from (b) that the pair (x,y) is 3-regular.

- $\delta(a,x) = 2$, $\delta(b,x) = 4$ and $\delta(b,y) = 6$.
  Note that b lies at distance 4 from the point $a \bowtie y$. Let L_b be the line of $\langle x,y\rangle$ at distance 3 from b and L_a the line of $\langle x,y\rangle$ through a. Then (a) shows that

  (1) $y^x = y^{z'}$, for all points $z' \in \langle L_a, L_b\rangle$, $z' \neq y$.

  Consider the point v of y^x on L_b. Suppose first that the unique point o of $\mathcal{O}$ collinear with v does not lie on vy. Put $u = \mathrm{proj}_{L_a} o$. Note that $u \neq a$ since $o \notin \mathcal{D}(x,y)$. Let finally $z = u \bowtie (\mathrm{proj}_{L_b} u)$. Then applying (a), we obtain

  (2) $z^y = z^x = z^w$, for all $w \in \langle L_a, L_b\rangle$, $w \neq z$.

  Combining (1) and (2) as in (b), we see that the pair (x,y) is 3-regular. So we may now assume that o lies on vy. Consider an arbitrary point p of $\mathcal{D}(x,y)$ collinear with v, different from y or $v \bowtie x$. Since the line vp does not contain a point of $\mathcal{O}$, we can apply the previous argument (noting that $\mathcal{D}(x,y) = \mathcal{D}(x,p)$) to obtain that the pair (x,p) is 3-regular. But now again applying (b) shows that also the pair (x,y) is 3-regular.

- $\delta(a,x) = 2 = \delta(b,y)$.
  Let p be a point of $\mathcal{D}(x,y)$ collinear with $a \bowtie y$, different from a and y, and p' a point of $\mathcal{D}(x,y)$ collinear with $b \bowtie x$, different from b and x. Then the previous paragraph shows that both (x,p) and (y,p') are 3-regular pairs, so applying (b) gives that also the pair (x,y) is 3-regular.

Suppose now $\mathcal{D}(x,y)$ contains exactly 1 point a of $\mathcal{O}$. Then we have the following cases to consider:

- $\delta(a,x) = 2$.
  Let v be a point of x^y different from a and put $w = v \bowtie y$, $w' = a \bowtie y$. Denote by o the unique point of $\mathcal{O}$ collinear with v. If o lies on vw, then put $z = o \bowtie (\mathrm{proj}_{aw'} o)$. If o does not lie on vw, then put $u = \mathrm{proj}_{aw'} o$ and $z = u \bowtie (\mathrm{proj}_{vw} u)$. Now $\mathcal{D}(x,z)$ contains two points of $\mathcal{O}$, so the pair (x,z) (and hence the pair (x,y)) is 3-regular.

- $\delta(a,x) = 4$ and $\delta(a,y) = 6$.
  Let again L_a be the line of $\langle x,y\rangle$ at distance 3 from a. By (a), we already know that

(3) $y^x = y^z$, for all points z, $y \neq z \in \langle L_a, M\rangle$, $M \in \langle x, y\rangle$, $M \neq L_a$.

Choose a point $v \in y^x$, v not on L_a such that the unique point $o \in \mathcal{O}$ collinear with v does not lie on the line vy. Let L_v be the line of $\langle x, y\rangle$ through v. If o lies on L_v, then put $z = o \bowtie (\mathrm{proj}_{L_a} o)$. From the previous case, it is then clear that (x, z) is a 3-regular pair. If o does not lie on L_v, put $u = \mathrm{proj}_{L_a} o$ and $z = u \bowtie \mathrm{proj}_{L_v} u$. Then by (a),

(4) $z^y = z^x = z^w$, for all points w, $z \neq w \in \langle L_v, M\rangle$, $M \in \langle x, y\rangle$, $M \neq L_v$.

Combining (3) and (4), we again see that (x, y) is a 3-regular pair.

Suppose finally $\mathcal{D}(x, y)$ does not contain any point of $\mathcal{O}$. Similarly as before, we can find a point z at distance 3 from two lines of $\langle x, y\rangle$ for which the hexagon $\mathcal{D}(x, z)$ contains a point of $\mathcal{O}$, from which the result follows.

Suppose now $\mathcal{O} = \Gamma_1(M) \cup \Gamma_3(M)$, M a line of Γ at distance 5 from x and y. Then $\mathcal{D}(x, y)$ contains the line M, or $\mathcal{D}(x, y)$ intersects $\mathcal{O}$ in either 0 points or 2 collinear points. (Indeed, if $\mathcal{D}(x, y)$ contains a point of M, then M is a line of $\mathcal{D}(x, y)$. If $\mathcal{D}(x, y)$ contains a point $p \in \Gamma_3(M)$, but no point of M, then the line $L = \mathrm{proj}_p M$ of $\mathcal{D}(x, y)$ contains two points of $\mathcal{O}$. If $\mathcal{D}(x, y)$ would contain a point p' of $\mathcal{O}$, p' not on L, then also $\mathrm{proj}_L p' \in \mathcal{D}(x, y)$, a contradiction.) As before, the case that $\mathcal{D}(x, y)$ contains no point of $\mathcal{O}$ can be reduced to one of the other cases. If $\mathcal{D}(x, y)$ contains M, then the pair (x, y) is 3-regular because of (b). So we only have to consider the case that $\mathcal{D}(x, y)$ contains exactly two collinear points a and b of $\mathcal{O}$. We consider the following situations:

- $\delta(a, x) = 2 = \delta(b, y)$.
  Since M is concurrent with the line ab, we can find a point $z \in \mathcal{O}$ at distance 3 from ab and another line of $\langle x, y\rangle$, hence the result follows.

- $\delta(a, y) = 6 = \delta(b, x)$ (hence $\delta(a, x) = 4 = \delta(b, y)$).
  This is clear because of (b).

- $\delta(a, x) = 2$, $\delta(b, x) = 4$ and $\delta(b, y) = 6$.
  Note that the line ab is concurrent with M, and that a and b lie at distance 3 from M. Let L_a be the line of $\langle x, y\rangle$ through a and L' an arbitrary line of $\langle x, y\rangle$ different from L_a. Let u be the projection onto L' of the intersection of M and ab. Let finally z be the unique point of $\langle L_a, L'\rangle$ collinear with u. Clearly, $\mathcal{D}(x, z)$ contains M, hence the pair (x, z) is 3-regular, and so is (x, y).

Suppose finally that $\mathcal{O}$ is a full subhexagon. If $\mathcal{D}(x,y)$ contains at least one point p of $\mathcal{O}$, then it has at least an ordinary hexagon through p in common with $\mathcal{O}$. From this, it is easily seen that, if $\mathcal{D}(x,y)\cap\mathcal{O}$ is nonempty, either x or y lie in $\mathcal{O}$ (and then we are done), or $\mathcal{D}(x,y)$ contains points $o_1, o_2 \in \mathcal{O}$, $\delta(o_1,x)=\delta(o_2,y)=4$ and $\delta(o_2,x)=\delta(o_1,y)=6$. In the latter case, (x,y) is 3-regular because of (b). Again as before, the case that $\mathcal{D}(x,y)$ contains no point of $\mathcal{O}$ can be reduced to the previous one. $\square$

Let p be a point of a generalized hexagon Γ. We say that the point p is **intersection-regular**, if for every intersection set $p^x\cap p^y$, x and y opposite p, the condition $|p^x\cap p^y|>1$ implies $p^x=p^y$. [1] A subset $\mathcal{B}$ of the line set of a generalized hexagon Γ is called a **line blocking set** if every line of Γ not contained in $\mathcal{B}$ intersects at least one line of $\mathcal{B}$.

Theorem 2.4.2 *Let Γ be a distance-3-regular hexagon, and $\mathcal{B}$ a line blocking set in Γ. Suppose that all the points lying on any line of $\mathcal{B}$ are intersection-regular. Then Γ is point-distance-2-regular and hence a Moufang hexagon.*

Proof. Let Γ and $\mathcal{B}$ be as above, and denote by $\overline{\mathcal{B}}$ the points lying on any line of $\mathcal{B}$. Let p be an arbitrary point of Γ, and x,y points opposite p such that $p^x\cap p^y$ is an intersection set containing at least two points. We show that $p^x=p^y$. This will prove the result, in view of Theorem 1.8.4 (i). Put $z=x\bowtie y$, $v=p\bowtie z$ and $w\in p^x\cap p^y$, $w\neq v$. If $p\in\overline{\mathcal{B}}$, we are done, so assume $p\notin\overline{\mathcal{B}}$. Let o_1 be a point of $\overline{\mathcal{B}}$ on the line vp, o_2 a point of $\overline{\mathcal{B}}$ on the line wp and R a line of $\mathcal{B}$ containing o_1. Then we distinguish the following cases.

(i) $o_1\neq v$
Let L be a line through p, $vp\neq L\neq wp$. Suppose by way of contradiction that $y':=\mathrm{proj}_L y\neq\mathrm{proj}_L x$. Put $x'=\mathrm{proj}_{xz}y'$. Let a be the point of the regulus $R(z,w)$ collinear with o_1, and b the point of the regulus $R(y',z)$ collinear with o_1. Note that $\delta(a,\mathrm{proj}_y w)=\delta(a,\mathrm{proj}_x w)=3$ and $\delta(b,\mathrm{proj}_{x'}y')=\delta(b,\mathrm{proj}_y y')=3$. Also, $\delta(a,b)=4$ (indeed, if not, the points a,b and the distance-4-paths $[b,y]$ and $[a,y]$ would define a pentagon). Now ${o_1}^{x'}\cap {o_1}^y$ is an intersection set (containing the points v and b), hence ${o_1}^{x'}={o_1}^y$, contradicting $a\in {o_1}^y$ but $\delta(a,x')=6$.

(ii) $o_1=v$ and $R=vz$
Let w' be an arbitrary point of $p^x\setminus\{v,w\}$. Then $z^w\cap z^{w'}$ is an intersection set (containing the points v and x) hence, since $z\in\overline{\mathcal{B}}$, $z^w=z^{w'}$. Since $y\in z^w$, we have $\delta(y,w')=4$, showing $p^x=p^y$.

[1] Using the terminology of Ronan [37], a point is intersection-regular exactly if all intersection sets with respect to this point have size 1.

(*iii*) $o_1 = v$, $R \neq vz$ and $o_2 \neq w$.
Put $a = x \bowtie w$, $b = y \bowtie w$ and $c = \mathrm{proj}_R a$. Since $v^a = v^b$ (indeed, $v^a \cap v^b$ is an intersection set containing the points p and z), $\delta(b, c) = 4$. Put $x' = a \bowtie c$ and $y' = b \bowtie c$. Because $p^x \cap p^{x'}$ is an intersection set (containing the points v and w), and $o_2 \neq w$, case (*i*) above implies that $p^x = p^{x'}$. Similarly, $p^y = p^{y'}$. Since also $p^{x'} = p^{y'}$ by case (*ii*), we obtain $p^x = p^y$.

(*iv*) $o_1 = v$ and $o_2 = w$.
Let L be an arbitrary line through p, $vp \neq L \neq wp$. Suppose by way of contradiction that $y' := \mathrm{proj}_L y \neq \mathrm{proj}_L x =: x'$. We can assume that either both x' and y' belong to $\overline{\mathcal{B}}$ or, without loss of generality, $x' \notin \overline{\mathcal{B}}$. Put $x'' = \mathrm{proj}_{yz} x'$. Now $p^x \cap p^{x''}$ is an intersection set (containing v and x') and because of the assumptions, L contains a point of $\overline{\mathcal{B}}$ different from x'. Hence we can apply (*iii*) to obtain $p^x = p^{x''}$. Since $w \in p^x$, this implies $\delta(w, x'') = 4$, a contradiction.

$\square$

Remark. Let $\mathcal{O}$ be a dual ovoidal subspace of a hexagon Γ. Then $\mathcal{O}$ is a line blocking set, having the property that every line of Γ not contained in $\mathcal{O}$ intersects *exactly one* line of $\mathcal{O}$. If Γ is a finite generalized hexagon of order (s,t), then the set $\mathcal{O}$ has roughly size st^2, and $\overline{\mathcal{O}}$ (the set of points lying on any line of $\mathcal{O}$) has roughly size s^2t^2, while the point set of Γ itself has roughly size s^3t^2.

2.5 Two characterizations of the Hermitian spread in $\mathsf{H}(q)$

Let $\mathcal{S}$ be a spread of the generalized hexagon $\Gamma = \mathsf{H}(q)$, and define the following geometry $\Gamma_{\mathcal{S}}$. The points of $\Gamma_{\mathcal{S}}$ are the points of Γ on lines of the spread. For a point p not on any line of $\mathcal{S}$, we denote by $V_p^{\mathcal{S}}$ the set of $q+1$ points of $\Gamma_{\mathcal{S}}$ collinear with p (if there is only one spread $\mathcal{S}$ around, we use the notation V_p for short). Now the lines of $\Gamma_{\mathcal{S}}$ are the lines of the spread together with the sets $V_p^{\mathcal{S}}$, $p \in \Gamma \setminus \Gamma_{\mathcal{S}}$. Incidence is symmetrized containment. It is easy to check that $\Gamma_{\mathcal{S}}$ is a generalized quadrangle of order (q, q^2) if and only if the spread $\mathcal{S}$ satisfies the following property:

($\diamond$) Let L_1, L_2 and L_3 be three different lines of the spread $\mathcal{S}$, and x_1 a point on L_1. Put $x_2 = \mathrm{proj}_{L_2} x_1$ and $x_3 = \mathrm{proj}_{L_3} x_2$. If $\delta(x_1, x_3) = 4$, then necessarily $x_1 \bowtie x_2 = x_1 \bowtie x_3 = x_2 \bowtie x_3$.

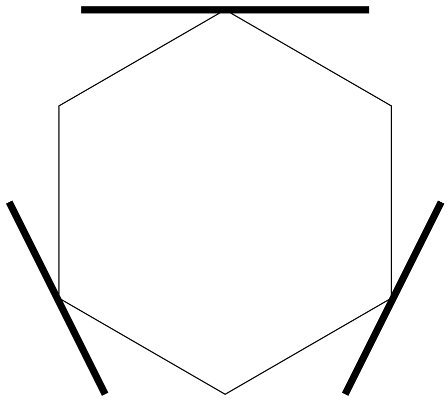


Figure 2.6: A forbidden configuration in $\mathsf{H}(q)$.

Property ($\diamond$) says that a configuration as in Figure 2.6 (where the bold lines are spread lines) is forbidden. Note that the lines L_1, L_2 and L_3 in ($\diamond$) necessarily belong to the same line regulus.

Theorem 2.5.1 *A spread $\mathcal{S}$ of the finite generalized hexagon $\mathsf{H}(q)$ is Hermitian if and only if the geometry $\Gamma_{\mathcal{S}}$ is a generalized quadrangle, if and only if $\mathcal{S}$ satisfies property ($\diamond$).*

Proof. If $\mathcal{S}$ is the Hermitian spread of $\mathsf{H}(q)$, then $\Gamma_{\mathcal{S}}$ is indeed a generalized quadrangle (namely the quadrangle $\mathsf{Q}(5,q)$), so assume now we have a spread $\mathcal{S}$ of $\mathsf{H}(q)$ satisfying property ($\diamond$). We prove that for any two lines of $\mathcal{S}$, the regulus defined by these lines is contained in $\mathcal{S}$. The result will then follow from Theorem 1.7.1.

Let p be a point of $\Gamma = \mathsf{H}(q)$ not on any line of the spread. We claim that V_p is in fact a distance-2-trace in Γ. Let a and a' be two different points of V_p and suppose by way of contradiction that the trace defined by a and a' contains a point b, $b \notin V_p$. Let L be the line of $\mathcal{S}$ through a, and let L' be an arbitrary line of Γ through b, different from bp. Let finally L'' be the unique spread line that is concurrent with L' and put $y = \mathrm{proj}_{L''}a$. Note that $\mathrm{proj}_a y \neq L$ (since spread lines are necessarily opposite). Because of the distance-2-regularity, the trace defined by a and b is equal to p^y, so $\delta(y,a') = 4$. If we denote by N the spread line through a', then $\mathrm{proj}_{a'}y \neq N$. But now we obtain a configuration forbidden by ($\diamond$), by considering the spread lines L, N and L'', together with the ordinary hexagon through a, p, a' and y, a contradiction. This shows our claim.

Let L_0 and L_1 be two different lines of $\mathcal{S}$, and $M \in R(L_0,L_1)$, $L_0 \neq M \neq L_1$. We show that $M \in \mathcal{S}$. Let p and p' be two different points belonging to the point regulus $\langle L_0, L_1\rangle$. By the previous paragraph, we know that $V_p = p^{p'}$ and $V_{p'} = p'^{p}$. But this implies that both $\mathrm{proj}_M p$ and $\mathrm{proj}_M p'$ have to lie on lines of $\mathcal{S}$. Because spread lines are opposite, this is only possible if $M \in \mathcal{S}$, which shows that $R(L_0, L_1)$ is contained in $\mathcal{S}$.

The theorem is proved. $\square$

Suppose Γ *is a finite generalized hexagon of order* (q,q) *having a spread* $\mathcal{S}$ *for which the geometry* $\Gamma_{\mathcal{S}}$ *is the classical quadrangle* $\mathsf{Q}(5,q)$. *Is* Γ *itself classical* ?

For an arbitrary generalized hexagon Γ, one could ask whether the existence in Γ of 'a lot of' spreads having property $(\diamond)$ is enough to conclude that Γ is classical. We now give a first result in this direction.

Let Γ be a generalized hexagon of order (q,q), and Σ a set of spreads of Γ such that the following conditions hold:

(1) Each element of Σ has property $(\diamond)$.

(2) For any two points a and b of Γ, with $\delta(a,b) = 4$, there exists a spread $\mathcal{S} \in \Sigma$ such that a and b lie on lines of $\mathcal{S}$.

(3) Let $\mathcal{S} \in \Sigma$, and a and b two points of Γ contained in a set $V_p^{\mathcal{S}}$, with $p = a \bowtie b$. For any point p' at distance 4 from a and opposite b and p, such that p' lies at distance 5 from the line of $\mathcal{S}$ through a, there exists a spread $\mathcal{S}' \in \Sigma$ for which p' lies on a line of $\mathcal{S}'$ and $V_p^{\mathcal{S}} = V_p^{\mathcal{S}'}$.

Proposition 2.5.2 *A finite generalized hexagon of order* (q,q) *contains a set of spreads* Σ *satisfying* $(1), (2), (3)$ *if and only if* Γ *is isomorphic with* $\mathsf{H}(q)$.

Proof. Let $\Gamma \cong \mathsf{H}(q)$. Let Σ be the set of Hermitian spreads of Γ. Then clearly properties (1) and (2) hold. Now let a, b and p' be as in (3). Then these three points define a plane intersecting the quadric $Q(6,q)$ in two concurrent lines. Through such a plane, there always exists a hyperplane intersecting $Q(6,q)$ in a non-singular elliptic quadric. Hence (3) holds. Now let Γ be a finite generalized hexagon of order (q,q) containing a set of spreads Σ satisfying (1), (2) and (3). By condition (2), it suffices to prove that, for any spread $\mathcal{S}$ of Σ, each $V_p^{\mathcal{S}}$ is a trace determined by two of its points. Fix a set $V_p^{\mathcal{S}}$, points a, a' of $V_p^{\mathcal{S}}$, and a point b on pa', b different from p and a'. Let p'

be a point opposite p, at distance 4 from both a and b. Now we claim that p' does not lie at distance 4 from a point of $V_p^{\mathcal{S}} \setminus \{a\}$. Indeed, suppose by way of contradiction there is a point $a'' \in V_p^{\mathcal{S}} \setminus \{a\}$ at distance 4 from p'. Suppose first p' lies at distance 5 from the line of $\mathcal{S}$ through a. By condition (3), there exists a spread $\mathcal{S}' \in \Sigma$ such that p' lies on a line of $\mathcal{S}'$, and $V_p^{\mathcal{S}} = V_p^{\mathcal{S}'}$. But now the forbidden configuration occurs in the ordinary hexagon through a, a'' and p', the contradiction. Next, consider the case that p' lies at distance 3 from the line of $\mathcal{S}$ through a. We can assume that the line $\mathrm{proj}_{a''}p'$ belongs to $\mathcal{S}$ (indeed, otherwise we obtain a contradiction with the previous case). Put p_1 the projection of a' onto the line $\mathrm{proj}_{p'}a$, and p_2 the projection of a' onto the line $\mathrm{proj}_{p'}a''$. Then as before, the line $\mathrm{proj}_{a'}p_1$ belongs to $\mathcal{S}$. But now the point p_2 lies at distance 4 from a', at distance 5 from the line of $\mathcal{S}$ through a' and opposite $a \in V_p$, hence (again by the previous case), p_2 lies opposite every element of $V_p \setminus \{a'\}$, contradicting $\delta(p_2, a'') = 4$.

This shows the claim. Hence an arbitrary point y opposite p and at distance 4 from both a and a', will be at distance 4 from every point of V_p, which shows that V_p is a trace determined by two of its points. □

Let Γ be a generalized hexagon admitting a spread $\mathcal{S}$, and let L_0 be a line of $\mathcal{S}$. Consider all projectivities $[L_0; L_1] \ldots [L_k, L_0]$ of the line L_0 for which the lines L_i, $0 \leq i \leq k$, belong to $\mathcal{S}$. These projectivities form a group, called the **group of projectivities of L_0 with respect to $\mathcal{S}$**, denoted by $\Pi_{\mathcal{S}}(L_0)$.

The groups $\Pi_{\mathcal{S}}(L)$, $L \in \mathcal{S}$ are all isomorphic, so we can define $\Pi_{\mathcal{S}} = \Pi_{\mathcal{S}}(L)$, for L an arbitrary line of the spread.

Lemma 2.5.3 *If $\mathcal{S}$ is the Hermitian spread of $\mathsf{H}(q)$, then $\Pi_{\mathcal{S}}$ is a Singer group.*

Proof. Let $\mathcal{S}$ be the Hermitian spread of $\mathsf{H}(q)$. Since $\mathcal{S}$ is also a spread of the quadrangle $\mathsf{Q}(5, q)$, the result follows from De Bruyn [15] (remembering that collinearity in $\mathsf{Q}(5, q)$ corresponds to non-opposition in $\mathsf{H}(q)$, so we are really talking about the same projectivities). We give an alternative proof using coordinates in the hexagon $\Gamma = \mathsf{H}(q)$. Let L_0, L_1 be two opposite lines of Γ, and choose coordinates such that $L_0 = [\infty]$ and $L_1 = [0, 0, 0, 0, 0]$. Let π be a hyperplane of $\mathrm{PG}(6, q)$ containing the lines L_0, L_1, and intersecting $\mathsf{Q}(6, q)$ in an elliptic quadric determining a Hermitian spread of Γ. Then for q odd, π has equation $X_1 = mX_5$, with m a non square in $\mathrm{GF}(q)$, or $mX_1 + X_3 + nX_5$, with $1 - 4mn$ a non square in $\mathrm{GF}(q)$, and for q even, π has equation $mX_1 + X_3 + nX_5$, with $\mathrm{Tr}(mn) = 1$. We proceed with the case that π has equation $X_1 = mX_5$ (the other cases are completely similar). A line N of

$\mathcal{S}$ not contained in $R(L_0, L_1)$ has coordinates $[mb', -mk'', k', b', k'']$, $k', b', k'' \in$ GF(q) (see for instance Bloemen, Thas & Van Maldeghem [4], section 3.1.2). Now it suffices to show that the projectivity $\sigma = [L_0; L_1][L_1; N][N; L_0]$ has exactly two 'imaginary' fixed points independent of the choice of N. Let $p = (x)$ be a point on L_0. Put $p_1 = \mathrm{proj}_{L_1} p$, $p_2 = \mathrm{proj}_N p_1$ and $p' = \mathrm{proj}_{L_0} p_2$. Then one easily calculates that the point p' has coordinate (x'), with $x = \frac{Ax'-B}{-mBx'+A}$, $A = k' - mb'k''$ and $B = b'^2 - mk''^2$. Hence $p = p^\sigma$ if and only if $x^2 = 1/m$, from which the result follows. $\square$

Theorem 2.5.4 *A spread $\mathcal{S}$ of the finite generalized hexagon $\mathsf{H}(q)$ is Hermitian if and only if $\Pi_{\mathcal{S}}$ is a Singer group.*

Proof. Let $\Gamma = \mathsf{H}(q)$ and $\mathsf{Q} = \mathsf{Q}(6, q)$ the quadric on which Γ is defined. Let L_0 and L_1 be two opposite lines of Γ. We first determine the number of Hermitian spreads containing the regulus $R(L_0, L_1)$. Note that $R(L_0, L_1)$ defines a 3-space $\mathcal{L}$ intersecting Q in a hyperbolic quadric $\mathsf{Q}^+(3, q)$. Let α be the polarity associated with the quadric Q. Then the plane $\mathcal{L}^\alpha$ intersects Q in a nondegenerate conic C which is the point regulus $\langle L_0, L_1\rangle$. Let γ' be a 5-space containing $\mathcal{L}$ and N the line in which γ' and the plane $\mathcal{L}^\sigma$ intersect. Then either $N \cap C = \{p_1, p_2\}$ (in this case, γ' intersects Q in a hyperbolic quadric $\mathsf{Q}^+(5, q)$ corresponding to the thin ideal subhexagon $\Gamma(p_1, p_2)$ of Γ), $N \cap C = \{p\}$ (in this case, γ' intersects Q in a cone $p\mathsf{Q}(4, q)$ corresponding to the set $p^{\perp\!\!\perp}$ in Γ) or $N \cap C = \emptyset$ (then γ' intersects Q in an elliptic quadric $Q^-(5, q)$ determining a Hermitian spread of Γ). So the number of Hermitian spreads through $R(L_0, L_1)$ is equal to the number of lines of $\mathcal{L}^\alpha$ not intersecting the conic C, which is $(q^2 - q)/2$.

Next, we show that for two opposite lines L_0 and L_1 of Γ, there is a bijective correspondence between the set of Hermitian spreads containing the regulus $R(L_0, L_1)$ and the set of Singer groups in $\mathrm{PGL}_2(q)$. Put $\mathrm{G} = G_2(q)$. Let $\mathcal{S}_1$ be a Hermitian spread containing $R(L_0, L_1)$, and put $H_1 = \Pi_{\mathcal{S}_1}$. Let H_2 be an arbitrary Singer group acting on L_0. Since every two Singer groups in $\mathrm{PGL}_2(q)$ are conjugate, there exists an element $\sigma' \in \mathrm{PGL}_2(q)$ for which $H_1^{\sigma'} = H_2$. Now choose σ in G_{L_0} such that $\sigma/\Gamma_1(L_0) = \sigma'$. Since the pointwise stabilizer of a line in $H(q)$ acts transitively on the lines opposite this line (this follows from the Moufang condition, see [57], Lemma 5.2.4 (ii)) we can choose an element β fixing L_0 pointwise such that $L_1^{\sigma\beta} = L_1$. Now $\sigma\beta$ maps $\mathcal{S}_1$ to a Hermitian spread $\mathcal{S}_2$ containing $R(L_0, L_1)$ for which $\Pi_{\mathcal{S}_2} = H_2$. This shows that every Singer group is the group belonging to a Hermitian spread containing $R(L_0, L_1)$. Furthermore, this spread is unique, since there are as many Hermitian spreads through a certain regulus as there are different Singer groups in $\mathrm{PGL}_2(q)$ (namely $(q^2 - q)/2$).

Let now $\mathcal{S} = \{L_0, L_1, \ldots, L_{q^3}\}$ be a spread of Γ such that $G = \Pi_{\mathcal{S}}$ is a Singer group. Because of the previous paragraph, we can define $\mathcal{S}_H$ to be the unique Hermitian spread containing $R(L_0, L_1)$ for which $G = \Pi_{\mathcal{S}_H}$. Let A be the set of lines M of the hexagon opposite L_0 and L_1 such that the projectivity $\beta_M := [L_0; M][M; L_1][L_1; L_0]$ belongs to $G \setminus \{e\}$. Let N be an arbitrary line of the hexagon opposite L_0 and L_1. If $N \in R(L_0, L_1)$, then $\beta_N = e$, so $N \notin A$. Suppose $N \notin R(L_0, L_1)$. Let γ be a 5-space containing L_0, L_1 and N. Then, as in the first paragraph of the proof, γ intersects the quadric $\mathsf{Q}(6, q)$ either in an elliptic quadric (case 1), a hyperbolic quadric (case 3) or in a cone $p\mathsf{Q}(4, q)$, with $p \in \langle L_0, L_1 \rangle$ (case 2).

(1) Note that all the lines of $\mathcal{S}_H$ not belonging to $R(L_0, L_1)$ are contained in A. In this way, we obtain $q^3 - q$ elements of A. If N does not belong to $\mathcal{S}_H$, then L_0, L_1 and N define a Hermitian spread corresponding to a group $G' \neq G$ (see the second paragraph of the proof), so $\beta_N \notin G \setminus \{e\}$ and $N \notin A$.

(2) Clearly, N lies at distance 3 from p. Let p' be the projection of p onto N. Then p' is a fixed point of β_N. Suppose there is a fixed point w on N different from p'. Put $w_0 = \mathrm{proj}_{L_0} w$ and $w_1 = \mathrm{proj}_{L_1} w$. Because w is a fixed point, we have $\delta(w_0, w_1) = 4$. Put $w' = w_0 \bowtie w_1$. Note that $\delta(w, w') = 6$ since we assumed $N \notin R(L_0, L_1)$. Because of the 2-regularity, $(w')^w = (w')^p$. This implies that w lies at distance 4 from the point of $(w')^p$ on the unique line L of $R(L_0, L_1)$ not opposite N, which is only possible if $p' \in L$. In this case every point on N is fixed. Indeed, let y be an arbitrary point on N, $y \neq p'$. Put $z_0 = \mathrm{proj}_{L_0} y$ and z the point of $\langle L_0, L_1 \rangle$ collinear with z_0. Let finally $z_1 = \mathrm{proj}_{L_1} z$ and $z_2 = \mathrm{proj}_L z$. Then $\{z_0, z_1, z_2\} \subseteq z^p$. Since $\delta(y, z_2) = \delta(y, z_0) = 4$, also $\delta(y, z_1) = 4$, showing that y is a fixed point of β_N. So if p' lies on a line of $R(L_0, L_1)$, $\beta_N = e$; if not, then β_N has exactly one fixed point.

(3) Let Γ' be the ideal subhexagon of order $(1, q)$ defined by the intersection of γ and Q, and let p, p' be the unique two points belonging to $\langle L_0, L_1 \rangle \cap \Gamma'$. Because L_0 and N are opposite lines, $\delta(p, N) \neq 1$. If $\delta(p, N) = 3$, then L_0, L_1 and N are contained in a hyperplane corresponding to case (2). Hence we may suppose that $\delta(p, N) = 5$. Put $w = \mathrm{proj}_N p$ and $v = p \bowtie w$. Note that v is a point of $p^{p'}$ not on the lines L_0 or L_1. If w lies on a line of $R(L_0, L_1)$, then $w \in (p')^p$ and we are back in case (2), so suppose vw is not a line of $R(L_0, L_1)$. Put $u_i = \mathrm{proj}_{L_i} p$ and $u'_i = \mathrm{proj}_{L_i} p'$, $i = 0, 1$. Then $\delta(w, u'_0) = \delta(w, u'_1) = 4$, so w is a fixed point of β_N. Also $w' := \mathrm{proj}_N u_0 = \mathrm{proj}_N u_1$, so w' is a second fixed

point of β_N. Suppose β_N has a fixed point f different from w and w'. Put $f_i = \text{proj}_{L_i} f$, $i = 0, 1$, and $f' = f_0 \bowtie f_1$ ($\delta(f_0, f_1) = 4$ since f is a fixed point). Note that $\delta(f, f') = 6$. Because of the 2-regularity, we have $(f')^p = (f')^f$. But this is a contradiction, since the point of $(f')^p$ on the line of $R(L_0, L_1)$ through v lies opposite f. So in this case, β_N has exactly two fixed points.

It follows that A contains exactly $q^3 - q$ lines (and these lines all belong to the spread $\mathcal{S}_H$). Also, β_N is the identity if and only if N is a line of $R(L_0, L_1)$ ($N \neq L_0, L_1$) or N is concurrent with a line of $R(L_0, L_1)$. Hence, since lines of a spread are mutually opposite, at most $q - 1$ lines of $\mathcal{S} \setminus \{L_0, L_1\}$ can give the identity. Consequently, at least (and hence exactly) $q^3 - q$ lines of $\mathcal{S}$ belong to A. Hence the spreads $\mathcal{S}$ and $\mathcal{S}_H$ can only differ in lines of the regulus $R(L_0, L_1)$. But now, applying the same argument to a regulus $R(L_0, M)$, M a line of A, shows that $\mathcal{S} = \mathcal{S}_H$. $\square$

Chapter 3

Forgetful Polygons

3.1 Introduction

For every non-incident point-line pair $\{p, L\}$ of an affine plane, there is *exactly one* line through p parallel with L. By replacing *exactly one* by *at most one*, the definition of a semi-affine plane is obtained. Since parallelism defines an equivalence relation, we can also say that in a semi-affine plane, any two lines are either concurrent or equivalent. Dually, a dual semi-affine plane is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathtt{I})$, together with an equivalence relation on the point set $\mathcal{P}$ such that every two lines meet, and every two points are either collinear or equivalent. In Dembowski [18], it is shown that every finite dual semi-affine plane Γ arises from a projective plane, in such a way that the points and lines of Γ are projective points and lines, and the equivalence classes are (pieces of) lines of the projective plane. In this chapter, we generalize the notion of a dual semi-affine plane to generalized polygons. The idea is that in the definition of generalized polygon, 'being collinear' is replaced by 'being collinear or equivalent'. We then ask the usual axioms about the distance between points and lines, but we do not ask anything about the distance between a point/line and a class, or between two classes. So the definition of these new structures looks pretty much the same as the one of a generalized polygon, except for the fact that some lines seem to have been 'forgotten'. This is the reason why we called these structures 'forgetful polygons'. In a way, the equivalence classes of points can be seen as 'the holes' in the memory of the forgetful polygon.

The main question now reads: does every forgetful polygon arise from a generalized polygon, by replacing (pieces of) lines by equivalence classes? Since one can construct infinite forgetful polygons using free constructions, we restrict ourselves to the finite case. For forgetful n-gons with n odd, the answer to the above question is 'yes', implying that there are no finite forgetful odd-gons, apart from the dual semi-affine planes. The case n even seems much harder. In this case, a positive answer to the question can be obtained if the forgetful polygon still remembers that at least one line and one equivalence class have the same number of points. If this condition is not satisfied, we prove that the forgetful polygon has 'nice' properties, for example the equivalence classes of points all have the same size. These classes can be much shorter than the lines however, and the memory of such a forgetful polygon seems to be too short to prove that they actually arise from generalized polygons. This is the reason why they got the name 'short forgetful polygons'.

We next turn our attention to the smallest short forgetful polygons, being the forgetful quadrangles. Here one can do a little better: if a forgetful quadrangle still remembers that there was a class and a line for which the number of points incident with it differs by at most one, then again the memory can be freshened up and the forgetful quadrangle indeed arises from a generalized quadrangle. We further investigate the structure of a short forgetful quadrangle, which gives rise to interesting objects such as strongly regular graphs. Furthermore, we give two families of examples of short forgetful quadrangles for which the classes are really short. Also these examples arise from generalized quadrangles, but not always in the expected way (this is, the classes are not always lines of the ambient generalized quadrangle). We then give some characterizations of the known examples of short forgetful quadrangles. We end this chapter with the following question: is it possible to classify all the forgetful quadrangles arising from a generalized quadrangle by 'forgetting lines'? At the moment, this question does not seem to be much easier than the original classification job...

3.2 Definitions and first examples

Let $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ be an incidence structure, and $\sim$ an equivalence relation on the point set $\mathcal{P}$. Denote by $\mathcal{C}$ the set of non-trivial equivalence classes of $\sim$. We say that a point x is incident with a class K of $\mathcal{C}$ if x belongs to K (and we also use the notation $x\mathtt{I}K$ for this). For two elements x and y of

$\mathcal{P} \cup \mathcal{L} \cup \mathcal{C}$, a **forgetful path** of length j between x and y is a sequence $(x = z_0, z_1, \ldots, z_{j-1}, z_j = y)$ of different elements of $\mathcal{P} \cup \mathcal{L} \cup \mathcal{C}$ such that $z_i \mathtt{I} z_{i+1}$, for $i = 0, \ldots, j-1$. If j is the length of a shortest forgetful path connecting x and y, we say that x and y are at distance j (notation $\delta(x, y) = j$).

Now $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$ is a **forgetful n-gon**, $n \geq 3$, if the following three axioms are satisfied:

(FP1) If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\delta(x, y) = k < n$, then there is a unique forgetful path of length k joining x to y.

(FP2) For every $x \in \mathcal{P} \cup \mathcal{L}$, we have $n = \max\{\delta(x, y) : y \in \mathcal{P} \cup \mathcal{L}\}$.

(FP3) Every line contains at least three points, every point is incident with at least three elements of $\mathcal{L} \cup \mathcal{C}$.

If $\mathcal{C} = \emptyset$, then clearly Γ is a generalized n-gon. So the generalized polygons give the first examples of forgetful polygons.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$ be a forgetful n-gon. The elements of $\mathcal{C}$ are called the **classes**. So except when mentioned differently, when talking about a class, we always mean a non-trivial class. The cardinality of the biggest class of $\mathcal{C}$ is denoted by g. A point which is only equivalent with itself is called an **isolated point**. Let x and y be two elements of $\mathcal{P} \cup \mathcal{L} \cup \mathcal{C}$ at distance $< n$. Then the element $\mathrm{proj}_x y$ is defined in the obvious way, and we use the notation $[x, y]$ for the shortest forgetful path between x and y. The set of lines through a point p (the set of points incident with a line L) is denoted by $\mathcal{L}_p$ ($\mathcal{P}_L$). The order of $\mathcal{L}_p$ ($\mathcal{P}_L$) is called the **degree** of p (L) and denoted by $|p|$ ($|L|$). In the following, if talking about a 'path' in Γ, we will always mean a 'forgetful path'.

Let $\Delta = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ be a finite generalized n-gon of order (s, t), $n \in \{4, 6, 8\}$. Then the following construction yields a forgetful n-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$.

(I) Let D be a set of disjoint lines of Δ. Put $\mathcal{P} = \mathcal{P}'$ and $\mathcal{L} = \mathcal{L}' \setminus D$. Two different points of $\mathcal{P}$ are said to be equivalent if and only if they both lie on the same line of D.

If Δ is a finite generalized quadrangle, then we also have the following examples.

(II) Let L be a line of Δ, X_1 a subset of the points of L, $1 \leq |X_1| \leq s+1$, and $X_2 = \Delta_1(L) \setminus X_1$. Denote by V the set of lines that intersect L

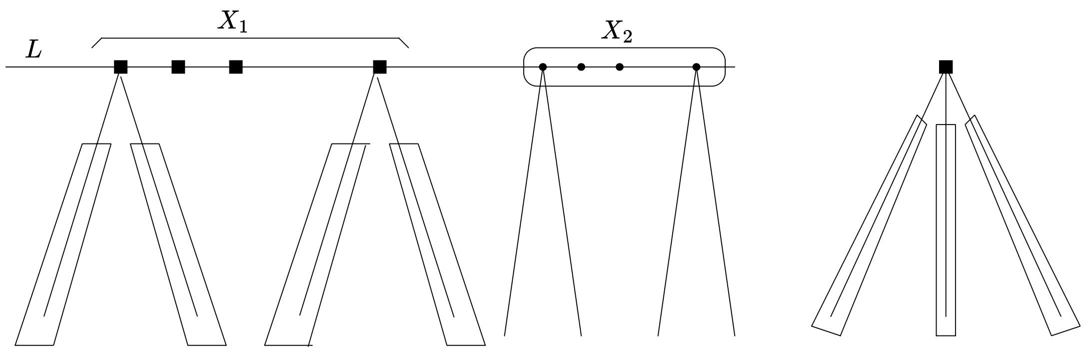


Figure 3.1: A forgetful quadrangle of type (II), and the special case $|X_1| = 1$.

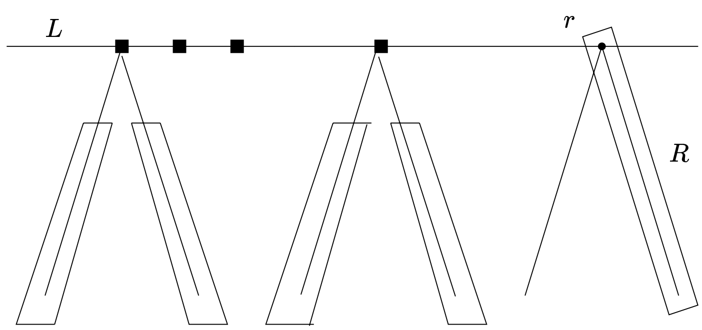


Figure 3.2: A forgetful quadrangle of type (III).

in a point of X_1. Then put $\mathcal{P} = \mathcal{P}' \setminus X_1$, $\mathcal{L} = \mathcal{L}' \setminus (V \cup \{L\})$. Two different points of $\mathcal{P}$ are said to be equivalent if and only if they both lie on the same line of $V \cup \{L\}$ (see Figure 3.1).

(III) Let R and L be two different lines of Δ through a point r. Let V be the set of lines concurrent with L, but not through the point r. Then put $\mathcal{P} = (\mathcal{P}' \setminus \Delta_1(L)) \cup \{r\}$, $\mathcal{L} = \mathcal{L}' \setminus (V \cup \{R, L\})$. Two different points of $\mathcal{P}$ are said to be equivalent if and only if they both lie on the same line of $V \cup \{R\}$ (see Figure 3.2).

Note that the above constructions also hold if Δ is not finite. However one can obtain more examples of infinite forgetful polygons by using free constructions, which shows that trying to classify forgetful polygons without the finiteness restriction is impossible.

3.3 Classification for n odd

Theorem 3.3.1 *There does not exist a finite forgetful generalized n-gon, n odd and $n \geq 5$.*

Proof.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$ be a finite forgetful n-gon, n odd and $n \geq 5$. The aim is to show that the geometry $(\mathcal{P}, \mathcal{L} \cup \mathcal{C}, \mathtt{I})$ is a finite generalized n-gon, hence Γ does not exist. Therefore it suffices to prove that, if K and K' are classes, and L an arbitrary line, then $\delta(K, L) \leq n-1$ (this is done in Step 1 and 2) and $\delta(K, K') \leq n-1$ (Step 3). In Step 0, we collect some general observations.

$\boxed{\text{Step 0}}$

Note that for two classes K and K', and an arbitrary line L, we certainly have that $\delta(K, L) \leq n+1$ and $\delta(K, K') \leq n+1$. Let K be an arbitrary class, $r \in K$, and N a line for which $\delta(N, K) = n-1$ and $\delta(N, r) = n$. Then $|N| = |r| + 1$. Indeed, the map

$$\begin{array}{rlll} \sigma: & \mathcal{L}_r & \rightarrow & \mathcal{P}_N \setminus \{\mathrm{proj}_N K\} \\ & L & \rightarrow & p, \qquad\qquad \text{with } \delta(p, L) = n-2, \end{array}$$

is a bijection. Suppose M is a line at distance $n+1$ from K. Since for any point p of K, the map

$$\begin{array}{rlll} \sigma: & \mathcal{L}_p & \rightarrow & \mathcal{P}_M \\ & L & \rightarrow & p', \quad \text{with } \delta(p', L) = n-2, \end{array}$$

is a bijection, $|M| = |p|$. Similarly, one shows that if z is an isolated point at distance n from a line N, then $|z| = |N|$.

Now fix a class K and suppose M is a line at distance $n+1$ from K, and $m\mathtt{I}M$. Let x be a point of K of degree k for which $N := \mathrm{proj}_m x$ is a line (note that such a point certainly exists). Then from the previous observations follows that $|M| = k$, every point of K has degree k and every line at distance $n-1$ from K contains $k+1$ points. Note also that $|M| = k$ implies $k \geq 3$, because of axiom (FP3). Let A be the element of $[x, m]$ at distance $\frac{n+1}{2}$ from x, $a = \mathrm{proj}_A x$ and $a' = \mathrm{proj}_A m$. The aim is now to show that the existence of the line M leads to a contradiction. In Step 1, we treat the case that, if $n \equiv 1 \bmod 4$, then A is not a class of size 2, and if $n \equiv 3 \bmod 4$, then a' is not a class of size 2. In Step 2, we get rid of the remaining cases.

$\boxed{\text{Step 1}}$

Suppose first that $n \neq 5$ and that in the case $n = 7$, $|A| \geq 3$, or a is a class of size at least 3. Let z be a point at distance $\frac{n-3}{2}$ from A such that $a \neq \mathrm{proj}_A z \neq a'$, and $N' := \mathrm{proj}_z A$ is a line. Let v be a point at distance $\frac{n-1}{2}$ from a' for which $A \neq \mathrm{proj}_{a'} v \neq \mathrm{proj}_{a'} m$ and $\mathrm{proj}_v a'$ is a line (note that it is always possible to choose z and v like this because of the assumptions on the degrees of a, a' and A). (The picture for $n = 9$ is given in Figure 3.3.) We claim that both z and v are isolated points with degree $k+1$.

Let z' be a point on N' different from z and from $\mathrm{proj}_{N'} A$. Let x' be a point of K different from x. Then $\delta(x', z') = n - 1$. We define the lines L and R as follows.

- Case $n \equiv 1 \bmod 4$. Note that A is a line. Let L be a line at distance $\frac{n-3}{2}$ from $l := x' \bowtie z'$ for which $\mathrm{proj}_l x' \neq \mathrm{proj}_l L \neq \mathrm{proj}_l z'$. Then $|L| = k+1$ since $\delta(L, K) = n - 1$. Let R be a line at distance $\frac{n-3}{2}$ from a, with $\mathrm{proj}_a x \neq \mathrm{proj}_a R \neq A$. Then $|R| = k+1$ since $\delta(R, K) = n - 1$.

- Case $n \equiv 3 \bmod 4$. If $n \neq 7$, let r be the point of $[x', z']$ at distance $\frac{n+1}{2}$ from x', and R a line at distance $\frac{n-5}{2}$ from r for which $\mathrm{proj}_r x' \neq \mathrm{proj}_r R \neq \mathrm{proj}_r z'$. If $n = 7$, let R be a line intersecting a, not through A or $\mathrm{proj}_a x$. Then in both cases, $|R| = k+1$ since $\delta(R, K) = n - 1$. Let L be a line at distance $\frac{n-1}{2}$ from $a'' := \mathrm{proj}_a x$, with $\mathrm{proj}_{a''} x \neq \mathrm{proj}_{a''} L \neq a$. Then $|L| = k+1$ since $\delta(L, K) = n - 1$.

Suppose z is not isolated, and let z'' be a point in the class Z containing z, $z'' \neq z$. Since $\delta(R, Z) = \delta(M, Z) = n - 1$ and $\delta(R, z'') = \delta(M, z'') = n$, we obtain $|R| = |z''| + 1 = |M|$ (see Step 0), a contradiction with $|R| = k+1$ and $|M| = k$. Hence z is isolated and $|z| = |L| = k+1$. The point z was arbitrarily chosen on N', so every point on N' different from $\mathrm{proj}_{N'} A$ is isolated and has degree $k+1$. From this also follows that each line T intersecting K, different from $\mathrm{proj}_x m$ contains $k+1$ points (indeed, one can always find a point of $N' \setminus \{\mathrm{proj}_{N'} A\}$ lying at distance n from T).

Suppose v is not isolated, and let v' be a point in the class V containing v, $v' \neq v$. Then $|M| = |v'| + 1 = |N'|$, a contradiction with $|N'| = k+1$ and $|M| = k$, hence v is isolated. If $n \equiv 1 \bmod 4$, then $|v| = |R| = k+1$. If $n \equiv 3 \bmod 4$, then consider a line X through x, $\mathrm{proj}_x m \neq X \neq \mathrm{proj}_x v$ (note that this is possible since $|x| = k \geq 3$). Since $\delta(v, X) = n$, we obtain $|v| = |X| = k+1$. This shows that both z and v are isolated points of degree $k+1$, as claimed.

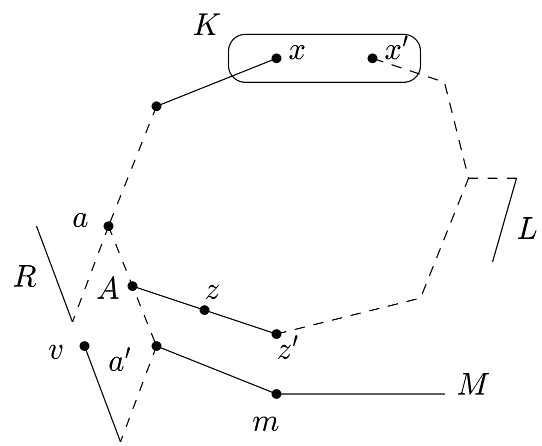


Figure 3.3: Proof of Step 1 for $n = 9$

Let R_z be a line through z, different from N'. Because $\delta(v, R_z) = n$, this line contains $k+1$ points, hence $\delta(R_z, K) = n-1$. The map

$$\begin{array}{rlcll} \sigma: & \mathcal{L}_z \setminus \{N'\} & \to & K \setminus \{x\} & \\ & R_z & \to & y, & \text{with } \delta(y, R_z) = n-2, \end{array}$$

is a bijection, from which follows that $|K| = |z| = k+1$.
Finally, we consider the point m. If m is isolated, then, with X a line through x different from $\text{proj}_x m$, $\delta(m, X) = n$ implies that $|m| = |X| = k+1$. Every point of K lies at distance $n-1$ from m, hence we need $k+1$ lines through m lying at distance $n-1$ from K, a contradiction since there are at most k such lines (indeed, $\delta(M, K) = n+1$). If m is not isolated, let m' be a point of the class K' containing m, $m' \neq m$. Then $\delta(m', x) = n-1$. Put $B = \text{proj}_x m'$. If $B \neq K$, then since $|B| = k+1$, $\delta(B, K') = n-1$ and $\delta(m, B) = n$, the point m has degree k. If $B = K$, put $y = \text{proj}_K m'$. Let Y be any line through y different from $\text{proj}_y m'$. Then, since $|Y| = k+1$, $\delta(Y, K') = n-1$ and $\delta(m, Y) = n$, the point m again has degree k. Since $|K| = k+1$, we need at least k lines through m lying at distance $n-1$ from K, a contradiction since there are at most $k-1$ such lines.

Case $n = 5$.
Let z be a point on the line A, $a \neq z \neq m$, and L a line intersecting ax not through a or x. If a would be equivalent with a point a'', $a'' \neq a$, then $|M| - 1 = |a''| = |L| - 1$ (by Step 0), a contradiction with $|L| = k+1$ and $|M| = k$. Hence the point a is isolated. Let N'' be a line through a, $ax \neq N'' \neq am$ (such a line exists, since the degree of the isolated point a is at least 3). Note that $|N''| = k+1$. Similarly as above for the point a,

we obtain that z is isolated (now using N'' instead of L). From $\delta(z, L) = 5$ and $|L| = k+1$ follows $|z| = k+1$. By considering a line at distance 5 from both a and z, we conclude that $|a| = k+1$. Suppose first that m has degree at least 3. Then let v be a point collinear with m, v not on the lines M or A. Again by using the same argument as above for the point a, we obtain that v is isolated and has degree $k+1$. From this follows that every line R_z through z, $R_z \neq A$, contains $k+1$ points. Suppose now m has degree 2, and let r be a point equivalent with m, $r \neq m$. Then $|r| = k$ since $|N''| = k+1$, implying that also in this case every line R_z through z, $R_z \neq A$, contains $k+1$ points. We now proceed similarly as in the case $n > 5$.

Case $n = 7$, $|A| = 2$ and a, a' are lines.
We show that this case cannot occur. Let Z be the class containing A, and $z \in Z$, $z \neq A$. Put $a'' = \mathrm{proj}_a x$. Let Y be a line at distance 4 from $a''x$, and at distance 5 from both x and a''. Then $|Y| = k+1$. Let y be a point on a, $a'' \neq y \neq A$ and Y' a line at distance 3 from a'', and at distance 4 from $a''x$ and a. Then $|Y'| = k+1$. If $y \sim y'$, $y' \neq y$, then $|M| - 1 = |y'| = |Y'| - 1$, a contradiction, hence y is isolated. Since $\delta(y, Y) = 7$, $|y| = k+1$. Let finally x' be a point of K, $x' \neq x$. Then $\delta(x', z) = 6$. The line $\mathrm{proj}_{x'} z$ contains $k+1$ points (because it lies at distance 7 from y), implying that $|A| = k \geq 3$. This is a contradiction with the assumption.

Case $n = 7$, a a class of size 2.
Define z and v as in the general case. We have to give another argument to conclude that the points z and v are isolated (since the line R defined in the general case cannot be found). Let $x' \in K$, $x' \neq x$. Then $\delta(x', m) = 6$. Put $R = \mathrm{proj}_m x'$.

- Suppose first R is a line. Then $|R| = k+1$. If $z \sim z'$, $z' \neq z$, then $|M| - 1 = |z'| = |R| - 1$, a contradiction, hence the point z is isolated. Also, $|z| = |R'| = k+1$, with R' a line at distance 3 from $a'' = \mathrm{proj}_a x$ for which $a \neq \mathrm{proj}_{a''} R' \neq a''x$.

- Suppose now R is a class and let w be the point of $[x', R]$ at distance 2 from x'. Using the same argument as for the point z above (with a' in the role of R), we obtain that w is isolated. Also, $|w| = k+1$ (since $|w| = |Az|$). Hence an arbitrary line through $a'' = \mathrm{proj}_a x$, different from $a''x$, contains $k+1$ points. Again using the same argument as above, it now follows easily that also in this case the point z is isolated. By considering a line intersecting $a''x$ not through x or a'', we obtain $|z| = k+1$.

If $v \sim v'$, $v \neq v'$, then $|Az| - 1 = |v'| = |M| - 1$, a contradiction, hence v is isolated. For a line X intersecting M, not through m, holds that $|z| = |X| = |v|$, hence $|v| = k + 1$. The rest of the proof is similar as in the general case.

This finishes the case $n \equiv 1 \bmod 4$ and A not a class of size 2, or $n \equiv 3 \bmod 4$, and a' not a class of size 2. Note that for n =5 (7), the element A (a') can be chosen to be a line, so Step 2 does not concern these cases.

Step 2

$n \equiv 1 \bmod 4$, $n > 5$ and A a class of size 2.

In the following, we will construct a point p' such that $\delta(p', A) = n$, $\delta(p', a) = n - 1$ for which both $\mathrm{proj}_{p'}a$ and the element of $[a, p']$ at distance $\frac{n+1}{2}$ from a are lines, and such that $|p'| = k$ if p' is not isolated, and $|p'| = k + 1$ if p' is isolated. Step 1 then implies that every line through p' lies at distance $n - 2$ from a point of A. Hence $|A| \geq k \geq 3$, a contradiction.

Case $n = 9$.
Let p' be a point at distance 5 from the line $X = \mathrm{proj}_x m$, with $x \neq \mathrm{proj}_X p' \neq \mathrm{proj}_X m$ and such that the path $[X, p']$ only consists of points and lines. If p' is isolated, then let y be a point at distance 3 from K and 4 from x, and y' the point of $[y, p']$ at distance 2 from y. By considering a line R at distance 3 from y', $\mathrm{proj}_{y'}y \neq \mathrm{proj}_{y'}R \neq \mathrm{proj}_{y'}p'$, we see that $|p'| = |R| = k + 1$. If p' is contained in a non-trivial class Z, then let $p'' \in Z$, $p'' \neq p'$, and $x' \in K$, $x' \neq x$. Let L be a line at distance 3 from the point $z := x' \bowtie p''$, $\mathrm{proj}_z p'' \neq \mathrm{proj}_z L \neq \mathrm{proj}_z x'$. Since $|L| = k + 1$, p' has degree k. So we constructed a point p' as claimed.

Case $n \equiv 1 \bmod 8$, $n > 9$
Let p be the point of $[x, a]$ at distance $\frac{\delta(x,a)}{2} - 2 = \frac{n-9}{4}$ from x, and p' a point at distance $\frac{3n-11}{4}$ from p such that $\mathrm{proj}_p a \neq \mathrm{proj}_p p' \neq \mathrm{proj}_p x$ and such that the path $[p, p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. We show that $|p'| = k + 1$. Consider a point y at distance 6 from x such that $\mathrm{proj}_x p \neq \mathrm{proj}_x y \neq K$. Then $\delta(p', y) = n - 1$. Let γ be the union of the paths $[p', y]$ and $[y, x]$. Let z be the element of γ at distance $\frac{n+3}{2}$ from x, and z' a point at distance $\frac{n-9}{2}$ from z such that $\mathrm{proj}_z p' \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z y$. Then any line through z' different from the projection of K onto z' contains $k+1$ points (since it lies at distance $n - 1$ from K), and lies at distance n from p', hence $|p'| = k + 1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. We show that $|p'| = k$. Consider a point y at distance 4 from x such that $\mathrm{proj}_x p \neq \mathrm{proj}_x y \neq K$. Then $\delta(p'', y) = n - 1$. Let c be the point of $[y, p'']$ at distance $\frac{n-5}{2}$ from y and let c' be a point at distance $\frac{n-9}{2}$ from c such that $\mathrm{proj}_c p'' \neq \mathrm{proj}_c c' \neq \mathrm{proj}_c y$. Then any line $R_{c'}$ through c' different from the projection of K onto c' contains $k+1$ points (because it lies at distance $n-1$

from K). Since $\delta(R_{c'},K')=n-1$ and $\delta(R_{c'},p')=n$, Step 0 implies that $|p'|=|R_{c'}|-1=k$. Now the point p' satisfies all the conditions above.

Case $n\equiv 5 \bmod 8$, $n>5$
Let p be the point of $[x,a]$ at distance $\frac{n-5}{4}$ from x, and p' a point at distance $\frac{3n-7}{4}$ from p such that $\mathrm{proj}_p a \neq \mathrm{proj}_p p' \neq \mathrm{proj}_p x$ and such that the path $[p,p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. Let z be the point of $[x,p']$ at distance $\frac{n-5}{2}$ from x, and z' a line at distance $\frac{n+1}{2}$ from z such that $\mathrm{proj}_z x \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z p'$. Then $|p'|=|z'|=k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. Let y be a point of K, $y\neq x$. Then $\delta(p'',y)=n-1$. Consider a line z' at distance $\frac{n-3}{2}$ from the point $z=y\bowtie p''$ for which $\mathrm{proj}_z p'' \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z y$. Then $|p'|=|z'|-1=k$.

<u>$n\equiv 3 \bmod 4$, $n>7$ and a' a class of size 2.</u>

Similarly as in the case $n\equiv 1 \bmod 4$, we construct a point p' at distance n from a' and at distance $n-1$ from A, for which both $\mathrm{proj}_{p'}A$ and the element of $[A,p']$ at distance $\frac{n+3}{2}$ from A are lines, and such that either $|p'|=k+1$ if p' is isolated, or $|p'|=k$ if p' is not isolated. The result will then follow.

Case $n\equiv 3 \bmod 8$
Let p be the point of $[x,A]$ at distance $\frac{n-3}{4}$ from x, and p' a point at distance $\frac{3n-9}{4}$ from p such that $\mathrm{proj}_p A \neq \mathrm{proj}_p p' \neq \mathrm{proj}_p x$ and such that the path $[p,p']$ only consists of points and lines except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. Let y be a point at distance 4 from x such that $\mathrm{proj}_x A \neq \mathrm{proj}_x y \neq K$. Then $\delta(p',y)=n-1$. Let γ be the union of the paths $[x,y]$ and $[y,p']$, and z the point of γ at distance $\frac{n+1}{2}$ from x. Let finally z' be a line at distance $\frac{n-5}{2}$ from z such that $\mathrm{proj}_z y \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z p'$. Then $|p'|=|z'|=k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p. Consider a point y at distance 2 from x such that $\mathrm{proj}_x p \neq \mathrm{proj}_x y \neq K$. Let z be the point of $[y,p'']$ at distance $\frac{n-3}{2}$ from y, and z' a line at distance $\frac{n-5}{2}$ from z such that $\mathrm{proj}_z y \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z p''$. Then $|p'|=|z'|-1=k$.

Case $n\equiv 7 \bmod 8$, $n>7$
Let p be the point on $[x,A]$ at distance $\frac{n-7}{4}$ from x, and p' a point at distance $\frac{3n-13}{4}$ from p such that $\mathrm{proj}_p A \neq \mathrm{proj}_p p' \neq \mathrm{proj}_p x$ and such that the path $[p,p']$ only consists of points and lines, except possibly for the element $\mathrm{proj}_p p'$. Suppose first that p' is isolated. Let z be the point of $[x,p']$ at distance $\frac{n-7}{2}$ from x, and z' a line at distance $\frac{n+3}{2}$ from z such that $\mathrm{proj}_z x \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z p'$. Then $|p'|=|z'|=k+1$. Suppose now that p' is contained in a class K', and p'' is a point of K' different from p'. Let y be a point at distance 3 from K and at distance 4 from x. Then $\delta(p'',y)=n-1$. Let z be the point

of $[y,p'']$ at distance $\frac{n-3}{2}$ from y, and z' a line at distance $\frac{n-5}{2}$ from z such that $\mathrm{proj}_z y \neq \mathrm{proj}_z z' \neq \mathrm{proj}_z p''$. Then $|p'| = |z'| - 1 = k$.

Hence we have shown that for any class K and an arbitrary line M, $\delta(M,K) \leq n-1$.

$\boxed{\text{Step 3}}$

Suppose there exist two classes K_1 and K_2 at distance $n+1$ from each other. We look for a contradiction. Let $x \in K_1$ be arbitrary. Since by the results of Step 1 and 2, any line through x lies at distance $n-1$ from K_2, the map

$$\begin{array}{rlll} \sigma: & \mathcal{L}_x & \to & K_2 \\ & L & \to & y, \quad \text{with } \delta(L,y) = n-2, \end{array}$$

is a bijection, hence $|K_2| = |x| =: k$ and all points in K_1 have the same degree k. Fix points $x \in K_1$ and $y \in K_2$. Note that $\delta(x,y) = n-1$. Let z be the element of $[x,y]$ at distance $\frac{n-3}{2}$ from x. If $n \equiv 1 \bmod 4$, then we can assume without loss of generality that z is a line (indeed, if z is a class, then interchange the roles of K_1 and K_2). If $n=5$, let w be a point on z, different from x or $x \bowtie y$. If $n > 5$, let w be a point at distance $\frac{n-3}{2}$ from z such that $\mathrm{proj}_z x \neq \mathrm{proj}_z w \neq \mathrm{proj}_z y$ (this is possible because z was assumed not to be a class) and $\mathrm{proj}_w z$ is a line (this is not possible if $n=7$, z has degree 2 and $x \bowtie y$ is a line, see case (1) below). Let finally W be a line through w, $W \neq z$ if $n=5$, $W \neq \mathrm{proj}_w z$ if $n>5$. Then $|W| = k+1$, since $\delta(K_1, W) = n-1$. But also $\delta(K_2, W) = n-1$ by Step 1 and 2, hence there is a point in K_2 which has degree k. By repeating the argument at the beginning of this step, we obtain that also $|K_1| = k$, and every point of K_2 has degree k. Note that this implies that every line intersecting K_1 or K_2 contains $k+1$ points.
Let the point w be as above. If w is isolated then, by considering a line intersecting K_1, but not through x, we obtain that $|w| = k+1$. Since every line through w lies at distance $n-1$ from K_2, we need at least $k+1$ points in K_2, a contradiction. Suppose now w is contained in a non-trivial class K, and let w' be a point of K, $w' \neq w$. Then $|w'| = k$. Indeed, $|w'| = |R| - 1$, with R an arbitrary line through x different from $\mathrm{proj}_x y$. But then the map

$$\begin{array}{rlll} \sigma: & \mathcal{L}_{w'} & \to & K_1 \setminus \{x\} \\ & L & \to & v, \qquad \text{with } \delta(L,v) = n-2, \end{array}$$

is a bijection, hence $|K_1| = k+1$, the final contradiction since $|K_1| = k$.

(1) $n=7$, $x \bowtie y$ a line and z non-isolated.
Let L be a line intersecting $x \bowtie y$, $\delta(L,K_1) = \delta(L,K_2) = 6$. Then $|L| = k+1$, hence every point in $K_2 \setminus \{y\}$ has degree k. By repeating the argument at

the beginning of this step, we obtain that also $|K_1| = k$, and every point of K_2 has degree k. Let w be a point in the class containing z, $w \neq z$. Let Y be a line intersecting K_1, Y not through x. Then $|w| = |Y| - 1 = k$. Since every line through w lies at distance 6 from a point of $K_2 \setminus \{y\}$, $|K_2| = k+1$, a contradiction.

This shows that for two classes K_1 and K_2, $\delta(K_1, K_2) \leq n-1$. Now the theorem follows. $\square$

3.4 Square forgetful pentagons

A semi-plane is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ together with an equivalence relation on the point set and the line set respectively, such that every two lines (points) are either concurrent (collinear) or equivalent. These structures were introduced in Dembowski [19] (appendix 7.4). The aim of this section is to generalize the notion of a semi-plane to n-gons. So in fact, we want to make the notion of a forgetful polygon selfdual.

Let $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ be an incidence structure, $\sim_{\mathcal{P}}$ an equivalence relation on the point set $\mathcal{P}$ and $\sim_{\mathcal{L}}$ an equivalence relation on the line set $\mathcal{L}$. Denote by $\mathcal{C}_{\mathcal{P}}$, $\mathcal{C}_{\mathcal{L}}$, the set of non-trivial classes of $\sim_{\mathcal{P}}$ respectively $\sim_{\mathcal{L}}$. Completely similar as in section 3 we define square forgetful paths and distances between elements of $\mathcal{P} \cup \mathcal{L} \cup \mathcal{C}_{\mathcal{P}} \cup \mathcal{C}_{\mathcal{L}}$, isolated points/lines and the degree of a point/line.

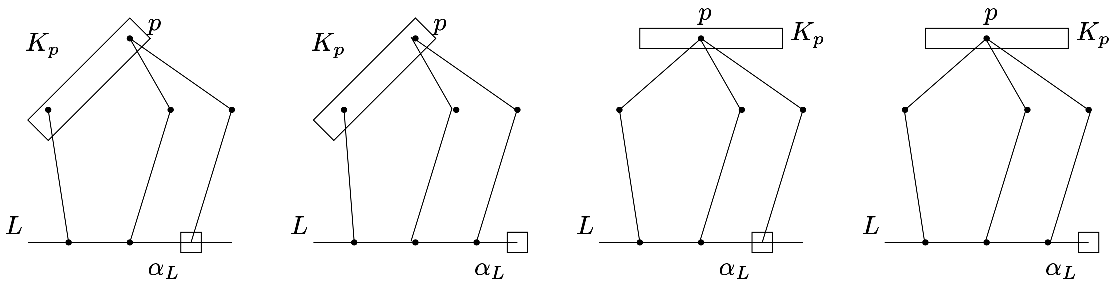
Now $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim_{\mathcal{P}}, \sim_{\mathcal{L}})$ is a **square forgetful n-gon**, n odd, $n \geq 3$, if the following axioms are satisfied:

(DFP1) If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\delta(x,y) = k < n$, then there is a unique square forgetful path of length k joining x to y.

(DFP2) For every $x \in \mathcal{P} \cup \mathcal{L}$, we have $n = \max\{\delta(x,y) : y \in \mathcal{P} \cup \mathcal{L}\}$.

(DFP3) Every line and every point class is incident with at least three points, every point and every line class is incident with at least three lines.

Note that a square forgetful 3-gon is a semi-plane (but not conversely, since in the definition of semi-plane, nothing is required for the size of the equivalence classes). The classification of all finite semi-planes (see Dembowski [19]) is not completed. In fact, there exists an example of a semi-plane that does not arise from a projective plane by ‘forgetting’ points and lines, see Baker [2]. We now give some partial results for the classification of finite square forgetful



$(a) : |L| = k$ $(b) : |L| = k+1$ $(c) : |L| = k-1$ $(d) : |L| = k$

Figure 3.4: possibilities if $|p| = k$.

pentagons. Moreover, we show that if one adds the assumption that at least one point class and one line class have 'the right size' (this is, one more than the degree of a non-isolated element), then this structure arises from a finite generalized pentagon, and hence cannot exist. The generalization of this result for $n \geq 7$, as well as the full classification of all finite square forgetful n-gons seems to be out of reach at this moment.

Let Γ be a finite square forgetful pentagon admitting non-isolated points and lines. Throughout this section, we denote by K_p the point class containing the non-isolated point p, and by α_L the line class containing the non-isolated line L.

We start with some observations similar to the ones in Step 0 of the proof of Theorem 3.3.1. Let p be a point at distance 5 from the line L, and put $|p| = k$. Then the following cases can occur (see Figure 3.4).

- $\delta(K_p, L) = 4$.
  (a) If there exists a line at distance 3 from p and equivalent with L, then $|L| = |p| = k$.
  (b) If no such line exists (for example when the line L is isolated), then $|L| = |p| + 1 = k + 1$.

- $\delta(K_p, L) = 6$ or the point p is isolated.
  (c) If there exists a line at distance 3 from p and equivalent with L, then $|L| = |p| - 1 = k - 1$.
  (d) If no such line exists (for example when the line L is isolated), then $|L| = |p| = k$.

These observations will be used throughout, without further reference to them.

Lemma 3.4.1 *If a point class K and a line class α lie at distance 5 from each other, then every line of α lies at distance 4 from K, and dually every point of K lies at distance 4 from α.*

Proof. Let K be a point class, $K = \{r_1, r_2, \ldots, r_{t+1}\}$, and α a line class at distance 5 from K and 4 from r_1. Suppose by way of contradiction that α contains a line L at distance 6 from K. Put $k = |r_1|$. Then $|L| = k - 1$ and $|z| \in \{k-1, k\}$, for $z \in K \setminus \{r_1\}$.

$\boxed{\text{Claim 1}}$ *All points of K have degree k.*
Suppose first that K contains at least two points r_2 and r_3 of degree $k-1$. Let x be a point on L. Without loss of generality, we can assume $N := \mathrm{proj}_x r_2$ is a line. If $r_2 \bowtie x$ would be a line class, then $|r_1| = |N| = |r_3|$, a contradiction. If N would be isolated, then $|r_1| + 1 = |N| = |r_3| + 1$, again a contradiction. So the line N is non-isolated, and the line class α_N lies at distance 5 from K. Since $|r_1| = k$, $|N| \in \{k, k+1\}$, and since $|r_3| = k-1$, $|N| \in \{k, k-1\}$. Hence $|N| = k$, there exists a line $N' \in \alpha_N$ at distance 3 from r_1, and all lines of α_N lie at distance 5 from r_3. Since $|r_2| = k - 1$ and $\delta(r_2, N) = 3$, $|N'| = k - 1$. But since the point r_3 has degree $k - 1$ and does not lie at distance 3 from a line equivalent with N', we obtain $|N'| = k$, a contradiction.

Suppose now that K contains a unique point r_3 of degree $k - 1$. Then all the points of K different from r_3 lie at distance 4 from α. Put $N_i = \mathrm{proj}_\alpha r_i$, $i = 1, \ldots, t+1$, $i \neq 3$. Note that $|N_i| = k$. The lines incident with a point r_i, $i \neq 3$, are isolated. Indeed, suppose that a line M through r_1 is not isolated, and let $M' \in \alpha_M \setminus \{M\}$. Then $|r_2| = |M'| = |r_3|$, a contradiction. With a dual argument, one sees that the points incident with the lines N_i are isolated. Let now R be a line at distance 4 from K and at distance 3 from r_3 for which $\mathrm{proj}_R r_3$ is a point. We claim that R is isolated $(*)$. Suppose by way of contradiction that R' is a line equivalent with R, $R' \neq R$. If $\delta(R', K) = 6$, then $|r_3| - 1 = |R'| \in \{|r_2|, |r_2| - 1\}$, a contradiction. So $\delta(R', K) = 4$. Without loss of generality, we can assume that $\delta(r_2, R') = 3$. But now we obtain $|r_3| = |R'| \in \{|r_1|, |r_1| + 1\}$, again a contradiction, which shows the claim. Note also that $|R| = |r_1| + 1 = k + 1$. Now by $(*)$ and the fact that every point is incident with at least 3 lines, it is clear that we can always find an isolated line of degree $k + 1$ at distance 5 from the isolated point $x_i := r_i \bowtie \alpha$, $i = 1, 2$, hence $|x_i| = k + 1$. From this follows that also the lines through r_1 or r_2, and the points on the lines N_1 and N_2 have degree $k + 1$.

We now show that the class K contains exactly $k + 1$ points. Let y be a point on the line N_2 different from x_2, and A a line through y, $A \neq N_2$. Then $|A| \in \{|x_1|, |x_1| - 1\} = \{k + 1, k\}$. If $\delta(A, K) = 6$, then $|A| \leq |r_3| = k - 1$, a contradiction. Hence each line through y lies at distance 4 from K, implying that $|K| \geq |y| = k+1$. Since also every point of K lies at distance 3 from a unique line through y, $|K| = k+1$.

Consider an arbitrary point p on the line L. Since p lies at distance 4 from every point of K, we need at least $k+1$ elements of $\mathcal{L} \cup \mathcal{C}_{\mathcal{P}}$ incident with p and at distance 4 from K. Since the line L lies at distance 6 from K, $|p| \geq k+1$ if p is non-isolated, and $|p| \geq k+2$ if p is isolated. If p is isolated, then $|p| = |r_1 x_1| = k+1$, a contradiction, so we can assume that p is non-isolated. Without loss of generality, we can assume that $\mathrm{proj}_p r_2 \neq K_p$. Note also that $\mathrm{proj}_{r_2} p \neq K$, since L and K lie at distance 6. If $\delta(K_p, x_2 r_2) = 4$, then $|p| = |x_2 r_2| - 1 = k$, which is too small. So $\delta(K_p, x_2 r_2) = 6$, implying that $|p| = |x_2 r_2| = k+1$. Since $|K| = k+1$ and $\delta(L, K) = 6$, this implies that the class K_p lies at distance 4 from K. Then necessarily, the projection of K_p onto K is the point r_3 (indeed, if $\mathrm{proj}_K K_p = r_i$, $i \neq 3$, then any line through r_i different from $\mathrm{proj}_{r_i} p$ would contain $|p|+1 = k+2$ points, a contradiction). Now let B be the projection of p onto r_3, and B' a line intersecting B in a point not belonging to K or K_p. Note that by $(*)$, the line B' is isolated and has degree $k+1$. Then $|p| = |B'| - 1 = k$, the final contradiction. Hence we have shown that all points of K have the same degree k. $\diamond$

Now it follows that every point of K lies at distance 4 from the class α, and hence at distance 3 from a line of degree k belonging to α. Put $x_i = r_i \bowtie x_i$ and $N_i = \mathrm{proj}_\alpha r_i$, $i = 1, \ldots, t+1$. As before, the points on the lines N_i are isolated.

$\boxed{\text{Claim 2}}$ *The points x_i have degree $k+1$.*
We claim that $|x_i| \in \{k, k+1\}$. Indeed, let A be a line at distance 3 from r_1 such that $K \neq \mathrm{proj}_{r_1} A \neq \mathrm{proj}_{r_1} x_1$ and such that the projection of r_1 onto A is a point. Then $|A| \in \{|r_2|, |r_2|+1\} = \{k, k+1\}$, from which follows that $|x_1| \in \{k, k+1, k+2\}$. If $|x_1| = k+2$, then $|A| = k+1$, there exists a line A' equivalent with A at distance 3 from x_1 but every line of α_A different from A lies at distance 5 from all points of K. So $\delta(K, A') = 6$. But now $|r_1| - 1 = |A'| = |r_2|$, a contradiction. Hence the claim.

Suppose $|x_1| = k$. We look for a contradiction. Let M be an arbitrary line intersecting $x_1 r_1$ in a point z different from x_1 and r_1. Suppose M is not isolated, and let M' be a line equivalent with M, different from M. Then $|M'| = |x_1| - 1 = k - 1$. This implies that the line M' lies at distance 6 from K, and (since $\delta(r_2, M') = 5$ and $|M'| = |r_2| - 1$) that there is a line M'' of α_M at distance 3 from r_2. Then $|x_1| - 1 = |M''| = |r_1|$, a contradiction. Hence the line M is isolated and has degree $k+1$ $(**)$. From this easily follows that every point on one of the lines N_i, different from x_1 has degree $k+1$.

We next show that K contains exactly $k+1$ points. Let R be a line intersecting N_1 in a point z' different from x_1. Let y be a point on N_1 different from x_1 and not on R. Note that $|y| = k+1$. If R is non-isolated, then $|y|-1 = |R'| = |x_1|-1$, with R' a line belonging to $\alpha_R \setminus \{R\}$, a contradiction. Hence R is isolated and has degree $k+1$, so R lies at distance 4 from K. It now follows that $|K| = |z'| = k+1$.

Let p be an arbitrary point on the line L. As before, $|K| = k+1$ implies that $|p| \geq k+2$ if p is isolated, and $|p| \geq k+1$ if p is non-isolated. If p would be isolated, then $|p| = k+1$ by $(**)$, so p is non-isolated. Again by $(**)$, $|p| \in \{k, k+1\}$. Hence $|p| = k+1$. This implies that $\delta(K_p, K) = 4$. Since the line N_1 has degree k, every point of K_p different from p has degree k. Suppose first $\delta(K_p, r_i) = 3$, for some i, $i \neq 1$. Consider the line $A = r_i x_i$. Let y be an arbitrary point on N_1 different from x_1. If A would be isolated, then $|x_1| = |A| = |y|$, a contradiction, so A is non-isolated. Then $|A| \in \{|x_1|, |x_1|-1\}$, and $|A| \in \{|y|, |y|-1\}$, implying $|A| = k$. But also $|A| \in \{|p|, |p|+1\}$, a contradiction. So necessarily $\delta(K_p, r_1) = 3$. Let p' be a point of K_p different from p and $\mathrm{proj}_{K_p} r_1$. The line $x_1 r_1$ is not isolated (indeed, otherwise $|p| = |x_1 r_1| = |p'|$). Now clearly (by comparing with the degrees of p and p'), $|x_1 r_1| = k+1$. This implies that there is a line B equivalent with $x_1 r_1$ at distance 3 from p, and no line of $\alpha_{x_1 r_1}$ lies at distance 3 from p'. Now $|B| = |r_2| = k$, but also $|B| = |p'|+1 = k+1$, a contradiction. This shows that $|x_1| = k+1$. $\diamond$

$\boxed{\text{Claim 3}}$ *Every point on a line N_i has degree $k+1$.*
Let y be a point on the line N_1, y different from x_1. Let R be a line through the point r_1 different from $x_1 r_1$. If R is isolated, then $|y| = |R| = |x_2| = k+1$. Suppose R is not isolated. Since $|\alpha_R| \geq 3$, there exists a line $R' \in \alpha_R$ at distance 5 from both y and r_1. Hence $|R'| = |r_2|$ and $|y| \in \{|R'|, |R'|+1\} = \{k, k+1\}$. Suppose by way of contradiction that $|y| = k$. Let z be a point on N_1 different from x_1 and y, and A a line through z different from N_1. Since the degrees of x_1 and y are different, it is easy to see that the line A is necessarily isolated, and has degree $|x_2| = k+1$. From this follows that every point on one of the lines N_i, $i \neq 1$ has degree $k+1$, and that every line through z lies at distance 4 from K. We now show that $|z| = k+1$, which will imply that the class K contains exactly $k+1$ points. If $|z| \neq k+1$, then as before, $|z| = k$. Consider a line B at distance 3 from y and at distance 4 from N_1 for which the projection of y onto B is a point. Because the degrees of x_1 and z are different, the line B is non-isolated. Choose a line B' equivalent with B at distance 5 from x_1 and different from B. Then $|y|-1 = |B'| \in \{|x_1|, |x_1|-1\}$, a contradiction. So $|z| = |K| = k+1$. Completely similar as in the previous claim, one shows that this contradicts

the degree of a point p on the line L. $\diamond$

$\boxed{\text{Claim 4}}$ *The class K contains $k+1$ points*
Let y be a point on the line N_1 different from x_1. Then $|K| \leq |y| = k+1$. Suppose by way of contradiction that $|K| < k+1$. Then there exists a line Y through y at distance 6 from K. Since $|Y| \in \{|x_2|, |x_2|-1\}$ and also $|Y| \in \{|r_1|, |r_1|-1\}$, we obtain $|Y| = k$. This implies that Y is non-isolated, all the lines of α_Y lie at distance 5 from any point of K, and for every point z on the line N_2, there exists a line Y_z equivalent with Y and at distance 3 from z (since z is an isolated point with degree $k+1$). Note that $Y_z \neq x_2r_2$ (indeed, $\delta(r_2, Y_z) = 5$), so all these lines Y_z are different. Hence we need at least k ($=|N_2|$) lines in α_Y different from Y. Now every line of $\alpha_Y \setminus \{Y\}$ lies at distance 4 from a unique point on L. Hence $|L| \geq k$, a contradiction. So $|K| = k+1$. $\diamond$

We can now finish the proof of the lemma. Similarly as in the proof of Claim 1, the fact $|K| = k+1$ implies that the degree of a point p on the line L is at least $k+1$ if p is non-isolated, and at least $k+2$ if p is isolated. Let R be a line concurrent with N_1, not through the point x_1. Note that if R is isolated, then $|R| = |x_2| = k+1$, and if R is non-isolated, then $|R'| = |x_1| - 1 = k$, for any line R' of $\alpha_R \setminus \{R\}$. Suppose p is isolated. If R is isolated, then $|p| = |R| = k+1$, a contradiction. If R is non-isolated, then let R' be a line of α_R different from R at distance 5 from p. This implies $|p| \leq |R'|+1$, a contradiction. So the point p is non-isolated. If R is isolated, then $|p| \leq |R| = k+1$. If R is non-isolated, then $|p| \leq |R'|+1 = k+1$, with R' a line of $\alpha_R \setminus \{R\}$ at distance 5 from p. So in both cases, $|p| = k+1$, implying that $\delta(K, K_p) = 4$. Without loss of generality, we can assume $\delta(r_1, K_p) = 3$. Let M be the line intersecting both K and K_p, and p' an arbitrary point of K_p different from p and not on M. Put p'' the point of K_p on M. Let N be a line intersecting M in a point and at distance 4 from K and K_p. Note that $|p'| = |N_1| = k$. The line N is non-isolated since otherwise $|p|+1 = |N| = |p'|+1$, a contradiction. By comparing the degrees of p and p', we see that $|N| = k+1$, implying that there exists a line N' equivalent with N at distance 3 from p, and that every line of α_N lies at distance 5 from p'. Let $N'' \in \alpha_N \setminus \{N, N'\}$. Then $\delta(p, N'') = \delta(p'', N'') = 5$, but since the point p' above was chosen arbitrarily in $K_p \setminus \{p, p''\}$, N'' also lies at distance 5 from every point of $K_p \setminus \{p, p''\}$. Hence $|p| - 1 = |N''| = |p'|$, the final contradiction. Now the lemma is proved. $\square$

Corollary 3.4.2 *Let K be a point class containing a point p with degree k.*

(*i*) *If there exists a line class α at distance 5 from K, then $|K| = |\alpha|$ and every point of K and every line of α have the same degree k.*

(*ii*) *All points in the same point class have the same degree. Dually, all lines in the same line class have the same degree.*

Proof. Observation (*i*) follows immediately from Lemma 3.4.1. Now let K be a point class, and $\{r_1, r_2, r_3\} \subseteq K$. Let L be a line at distance 4 from K and 3 from r_3. If L is isolated, then $|r_1|+1 = |L| = |r_2|+1$, hence $|r_1| = |r_2|$. If L is non-isolated, then $|r_1| = |L| = |r_2|$ by (*i*). So any two points of K have the same degree, showing (*ii*). $\square$

Proposition 3.4.3 (*i*) *All non-isolated points and lines have the same degree k.*

(*ii*) *An isolated element has degree k or $k+1$.*

(*iii*) *Any point class or line class has size at most $k+1$.*

Proof. Let K be an arbitrary point class, and k the degree of the points of K. Let L be an arbitrary non-isolated line, and $L' \in \alpha_L \setminus \{L\}$. If L intersects K, then $|L'| = k$, hence by Corollary 3.4.2 (*ii*) also $|L| = k$. If L lies at distance 4 from K, then by Lemma 3.4.1, $|L| = k$. If finally L lies at distance 6 from K, then also α_L lies at distance 6 from K, and $|L| = |p| = k$, with p an arbitrary point of K. So all non-isolated lines have the same degree k, which is equal to the degree of the points of K. Dually all non-isolated points have the same degree k. This shows (*i*). Now let p be an isolated point and suppose $|p| \notin \{k, k+1\}$. Let L be a line at distance 5 from p. If L is non-isolated (and thus has degree k) then $p \in \{|L|, |L|+1\}$, a contradiction. Hence every line at distance 5 from p is isolated and has degree $|p|$. Dually, every point at distance 5 from a line at distance 5 from p is isolated. It is now easy to deduce that all elements are isolated, so Γ is not forgetful at all. Hence (*ii*). Now let K be an arbitrary point class, and r a point at distance 5 from K. Since each point of K lies at distance 4 from r, $|K|$ is at most the number of elements of $\mathcal{L} \cup \mathcal{C}_{\mathcal{P}}$ incident with r, showing (*iii*). $\square$

Theorem 3.4.4 *If Γ contains at least one point class and at least one line class of size $k+1$, then Γ arises from a finite generalized pentagon, and hence cannot exist.*

Proof. Let $\mathcal{K}$ be a point class of size $k+1$, and $\mathcal{A}$ a line class of size $k+1$. Let p be a point at distance 5 from $\mathcal{K}$. Since p lies at distance 4 from every point of $\mathcal{K}$, and since p is incident with at most $k+1$ elements of $\mathcal{L} \cup \mathcal{C}_{\mathcal{P}}$, p

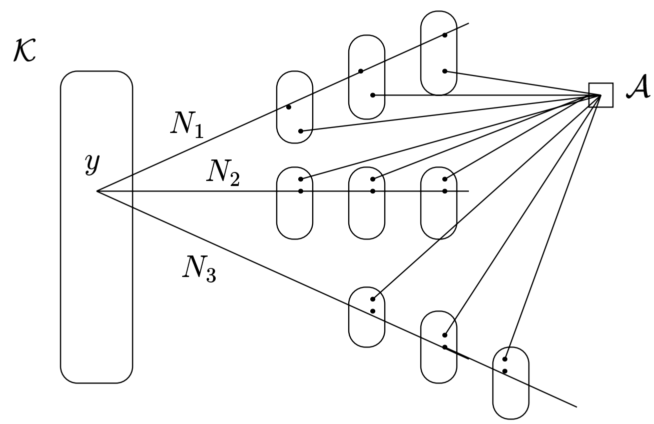


Figure 3.5: Proof of Theorem 3.4.4.

is incident with exactly $k+1$ elements of $\mathcal{L} \cup \mathcal{C}_{\mathcal{P}}$ and every element incident with p lies at distance 4 from $\mathcal{K}$. This shows that each line and each point class lies at distance at most 4 from $\mathcal{K}$. Also, every isolated point at distance 5 from $\mathcal{K}$ necessarily has degree $k+1$. Dual results hold for the line class $\mathcal{A}$. Now we show that any point class K' of size $\leq k$ lies at distance 3 from $\mathcal{A}$. By Corollary 3.4.2 (i), $\delta(K', \mathcal{A}) \neq 5$. Suppose $\delta(K', \mathcal{A}) = 7$. Let L be a line belonging to $\mathcal{A}$, and p a point of K'. Then $\delta(p, L) = 5$. Let R be a line through p at distance 4 from L. Since every line of $\mathcal{A}$ lies at distance 4 from R, and since R is incident with at most $k+1$ elements of $\mathcal{P} \cup \mathcal{C}_{\mathcal{L}}$, the point p lies at distance 3 from a line of $\mathcal{A}$, a contradiction. So $\delta(K', \mathcal{A}) = 3$, showing that every class of size $\leq k$ is intersected by a line of $\mathcal{A}$.

Now suppose by way of contradiction that there exists a point class K with $|K| \leq k$. Let x be the point of K at distance 3 from $\mathcal{K}$, and y the point of $\mathcal{K}$ collinear with x. Put $N_1 = xy$, and let N_2 be an arbitrary line through y different from N_1. Let N' be a line intersecting N_1 in a point different from x or y. Then N' is isolated. Indeed, if not, then $|K| = |\alpha_{N'}| = |\mathcal{K}|$ by Corollary 3.4.2 (i), a contradiction. Note that $|N'| = k+1$ $(*)$.

Let r be an arbitrary point on N_2 different from y. We claim that every line L through r different from N_2 lies at distance 4 from K. Indeed, if L is isolated, then $|L| = k+1$ (because $\delta(L, \mathcal{K}) = 4$) and hence, noting that $|x| = k$, $\delta(K, L) = 4$. Suppose L is non-isolated. Then $|\alpha_L| = |\mathcal{K}| = k+1$ (by Corollary 3.4.2 (i)). It then follows as in the first paragraph of the proof that K is intersected by a line of α_L. Hence the claim. If r were isolated, then $|r| = k+1$ by $(*)$, hence $|K| = k+1$, a contradiction. So the point r is non-isolated and has degree k. Since $|K| \geq |r|$, we conclude that K contains exactly k points, and that the classes K and K_r mutually lie at distance

6. But now the class K_r also contains at most k points (since otherwise $\delta(K, K_r) = 4$ by the first paragraph of the proof). By symmetry, we now obtained the following situation. Let $N_1, \ldots, N_k$ be the lines through the point y. Then each point r on one of the lines N_i, $r \neq y$, belongs to a class K_r^i of size k, and K_r^i lies at distance 6 from every class $K_{r'}^j$ intersecting a line N_j, $i \neq j$, with $K_{r'}^j \neq \mathcal{K}$ (see Figure 3.5). Now consider the classes K_r^i of size k intersecting the line N_i. Note that $N_i \notin \mathcal{A}$, since otherwise any class intersecting N_j, $j \neq i$, not through y would have size $|\mathcal{A}| = k+1$. So we need at least $k-1$ lines in $\mathcal{A}$ intersecting one of the classes K_r^i (keeping in mind that every point class of size $\leq k$ is intersected by a line of $\mathcal{A}$). Note that these $k-1$ lines do not intersect the classes $K_{r'}^j$, for $j \neq i$, since the classes K_r^i and $K_{r'}^j$ lie at distance 6 from each other. Because $|y| \geq 3$, we need at least $3(k-1)$ lines in $\mathcal{A}$, the final contradiction.

Hence each point class (and dually also each line class) has size $k+1$, implying that the distance between two point classes and between a line and a point class is at most 4 (and dually for the line classes). Now it immediately follows that the geometry $(\mathcal{P} \cup \mathcal{C}_{\mathcal{L}}, \mathcal{L} \cup \mathcal{C}_{\mathcal{P}}, \mathtt{I})$ is a finite generalized pentagon, and hence does not exist. $\square$

Remarks

- All non-classified semi-planes have the property that the size of any equivalence class is at most the largest occuring degree. Hence the analogon of the extra assumption in Theorem 3.4.4 for square forgetful 3-gons would kill all non-classified semi-planes.

- The major problem for generalizing the results above to square forgetful n-gons with $n \geq 7$ is that the proof of Claim 4 in Lemma 3.4.1 does not go through.

3.5 Classification results for n even

Let for the rest of this section, Γ be a forgetful n-gon, n even, admitting non-isolated points.

Lemma 3.5.1 *Every line contains the same number l of points, and $l \geq g$.*

Proof. Let L and L' be two arbitrary lines. Note that $\delta(L, L') \leq n$ by axiom (FP2). If $\delta(L, L') = n$, then the projection map defines a bijection between $\mathcal{P}_L$ and $\mathcal{P}_{L'}$ (by axiom (FP1)), hence $|L| = |L'|$. If L and L' meet in a point

p, then consider a line M at distance $n-1$ from p for which $L \neq \mathrm{proj}_p M \neq L'$ (such a line exists by axiom (FP3)). Then $\delta(L,M) = \delta(L',M) = n$, hence $|L| = |M| = |L'|$. If finally $2 < \delta(L,L') < n$, then let N be a line at distance $n-\delta(L,L')$ from L such that $\mathrm{proj}_L N \neq \mathrm{proj}_L L'$, for which the path between L and N only contains points and lines. Then $|L| = |N|$; and because $\delta(N,L') = n$ we also have $|N| = |L'|$. So all lines contain the same number of points. Now let G be a class of size g, and L a line at distance n from G. Since for every point x of G, there is a unique point x' on L for which $\delta(x,x') = n-2$, and since all these points are different, we obtain $l \geq g$. $\square$

Lemma 3.5.2 *Let K be a class of size at least* 3*. Then all the points in K have the same degree, or K is a forgetful quadrangle of type* (III).

Proof. Let K be a class of size at least 3, and suppose that there are two points $p_1, p_2 \in K$ for which $|p_1| \neq |p_2|$. Suppose K' is a class different from K and x a point of K'. Since every point of K lies at distance $< n$ from any line through x, $\delta(K,K') \leq n+2$.
Suppose $\delta(K,K') = n+2$. Since the map

$$\begin{array}{llll} \sigma: & \mathcal{L}_x & \rightarrow & \mathcal{L}_{p_i} \\ & L & \rightarrow & L', \quad \text{with } \delta(L,L') = n-2, \end{array}$$

is a bijection, we have $|x| = |p_i|$, $i = 1,2$, a contradiction.
Suppose $\delta(K,K') = n$, and $\delta(p_1,K') = \delta(p_2,K') = n+1$. Let y be a point of K' at distance $n-1$ from K. Then the map

$$\begin{array}{llll} \sigma': & \mathcal{L}_y \setminus \{\mathrm{proj}_y K\} & \rightarrow & \mathcal{L}_{p_i} \\ & L & \rightarrow & L', \quad \text{with } \delta(L,L') = n-2, \end{array}$$

is a bijection, hence $|y| = |p_i| + 1$, $i = 1,2$, a contradiction.
Suppose $\delta(K,K') = n$, and $\delta(p_1,K') = \delta(p_2,K') = n-1$. Let $r_i = \mathrm{proj}_{K'} p_i$, $i = 1,2$, and let $p_3 \in K \setminus \{p_1,p_2\}$. If $\delta(p_3,K') = n+1$, then $|r_1| - 1 = |p_3| = |r_2| - 1$. If $\delta(p_3,K') = n-1$, then $|r_1| = |p_3| = |r_2|$. Since $|p_1| = |r_2|$ and $|p_2| = |r_1|$, we obtain a contradiction in both cases.
We conclude that $\delta(K,K') < n$, or $\delta(K,K') = n$ but then exactly one of p_1 and p_2 lies at distance $n-1$ from K'. Completely similar, one shows that any isolated point w lies at distance at most $n-1$ from K, and that if $\delta(w,K) = n-1$, then either $\delta(w,p_1) = n-2$ or $\delta(w,p_2) = n-2$.

Suppose first that no point at distance $n-1$ from K is isolated. Let $p_3 \in K \setminus \{p_1,p_2\}$. Without loss of generality, we can assume $|p_2| \neq |p_3|$. Let K' be a class at distance n from K for which $\delta(p_1,K') = n-1$ and put

$r_1 = \mathrm{proj}_{K'}p_1$. Then $\delta(p_2, K') = n+1$. If $\delta(p_3, K') = n+1$, then $|p_3| = |r_1| - 1 = |p_2|$, a contradiction, hence $\delta(p_3, K') = n-1$. Put $r_3 = \mathrm{proj}_{K'}p_3$. Now $|p_1| = |r_3| = |p_2| + 1$. Let K'' be a non-trivial class at distance n from K for which $\delta(p_2, K'') = n-1$, and put $r_2 = \mathrm{proj}_{K''}p_2$. Let finally r be a point of K'' different from r_2. If $\delta(r, K) = n-1$, then $|p_2| = |r| = |p_1| + 1$. If $\delta(r, K) = n+1$, then $|p_1| = |r| = |p_2| - 1$. So in both cases, we obtain a contradiction with $|p_2| + 1 = |p_1|$.

So we may assume that there exists an isolated point w at distance $n-1$ from K for which $\delta(p_1, w) = n-2$. Then $|w| = |z| + 1$, for all points z of K different from p_1. Let v be a point at distance $n-2$ from p_2 and $n-1$ from K for which $\mathrm{proj}_v p_2$ is a line. Let p_3 be an arbitrary point of K different from p_1 and p_2. If v would be isolated, then we obtain $|p_1| + 1 = |v| = |p_3| + 1$, hence (since $|p_2| = |p_3|$) also $|p_1| = |p_2|$, a contradiction. So v is contained in a class K'. By the first paragraph of the proof, $\delta(p_1, K') = n+1$ and hence $|v| = |p_1| + 1$. Since the degree of an arbitrary point z of K different from p_1 is $|p_2|$, such a point z lies at distance $n-1$ from K' (if not, the degree of z would be $|v| - 1 = |p_1|$). Also, it is now clear that $|v| = |p_3|$, hence $|p_3| = |p_1| + 1$. Note that any isolated point at distance $n-1$ from K necessarily lies at distance $n-2$ from p_1. Now let w' be an arbitrary point at distance $n-1$ from K and at distance $n-2$ from p_1 such that $\mathrm{proj}_{w'}p_1$ is a line. We show that w' is isolated. Suppose by way of contradiction that w' is equivalent with a point w'', $w'' \neq w'$. Since w'' does not lie at distance $n-2$ from any point of K, we see that $|p_2| = |w''| = |p_1| - 1$, a contradiction. Finally, we claim that every point u of K' (with K' as above) lies at distance $n-2$ from a point of $K \setminus \{p_1\}$, and $|u| = |p_2|$. Suppose by way of contradiction that the class K' contains a point v' at distance $n+1$ from K. Then $|v'| = |p_2| - 1 = |p_1|$. Since K' contains at least two points of degree $|p_2|$ (namely the projections of p_2 and p_3 onto K'), every isolated point at distance $n-1$ from K' has to lie at distance $n-2$ from v'. Let γ be a fixed n-path between p_1 and v', and x the element of γ at distance $n/2+1$ from v'. If x is not a point of degree two or a class containing only two points, then consider a point y at distance $n/2-1$ from x for which $\mathrm{proj}_y p_1$ is a line and $\mathrm{proj}_x p_1 \neq \mathrm{proj}_x y \neq \mathrm{proj}_x v'$. The point y is isolated (since $\delta(y, K) = n-1$ and $\delta(y, p_1) = n-2$) and lies at distance $n-1$ from K' (if $\delta(y, K') = n+1$, all the points of K' would have equal degree) but at distance n from v', a contradiction. If x is a class or a point of degree two, then let R be a line at distance $n/2-1$ from $x' = \mathrm{proj}_x p_1$ for which $\mathrm{proj}_{x'}p_1 \neq \mathrm{proj}_{x'}R \neq \mathrm{proj}_{x'}v'$. Then the line R contains at least two isolated points, hence at least one isolated point at distance n from v', a contradiction. This shows the claim.

So we obtained the following situation $(\diamond)$: if K' is an arbitrary non-trivial class at distance n from K, then $|K'| = |K| - 1$ and each point of K' lies at distance $n-2$ from a unique point of $K \setminus \{p_1\}$. The degree of a point in K' is equal to $|z| = |p_1| + 1$, with z an arbitrary point of $K \setminus \{p_1\}$. A point w' at distance $n-1$ from K for which $\mathrm{proj}_{w'} K$ is a line is isolated if and only if $\delta(w', p_1) = n-2$. Moreover, $|K| = g = l$. Indeed, consider a line L at distance n from K. Suppose $l > |K|$. Then L contains a point y which lies at distance n from all the points of K. This contradicts the observations at the beginning of the proof. Consequently, $l = |K| = g$.

Suppose $n = 4$. We show that Γ is of type (III). Note that every non-trivial class different from K lies at distance 4 from K and hence has size $g-1$. It is also easy to see that, if two classes K' and K'' lie at distance 4, $K' \neq K \neq K''$, then every point of K' lies at distance 3 from a point of K'' and conversely. Indeed, let $z \in K'$, $\delta(z, K'') = 3$, and suppose y is a point of K'' at distance 5 from K'. Then $|y| = |z| - 1$, a contradiction with the fact that all points in K' and K'' have degree $|p_1| + 1$.
We define the following equivalence relation $\sim_C$ on the classes of size $g-1$:

$$K_1 \sim_C K_2 \Leftrightarrow \delta(K_1, K_2) = 6.$$

The transitivity of $\sim_C$ is shown as follows: suppose $K_1 \sim_C K_2$, $K_1 \sim_C K_3$, but $\delta(K_2, K_3) = 4$. Let L be a line intersecting both K_2 and K_3. Every point of K_1 has to lie at distance 2 from a unique point of L, not belonging to K_2 or K_3, hence $|L| \geq |K_1| + 2 = g + 1$, a contradiction. We associate a symbol ∞_i, $i = 1, \ldots, s$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, s$. A line of Δ is either a line of Γ; the set K (with K the unique class of size g); the set of points of a class of size $g-1$ together with the symbol of its equivalence class, or the set of points $\{\infty_1, \ldots, \infty_s, p_1\}$. Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s, k), by checking the main axiom. We illustrate this for two cases. First, let p be a point of Γ, $p \notin K$, and L a line of Δ containing the points of a class K' and the symbol ∞_j, $p \notin K'$. Suppose p is not collinear in Γ with any point of K'. If p is not isolated, then it is contained in a class K'' of size $g-1$ which necessarily lies at distance 6 from K', hence $K' \sim_C K''$. So p is collinear in Δ with the point ∞_j. If p is isolated, then p is collinear in Γ with p_1, see ($\diamond$). Since no point of K' is collinear with p_1, and $|K'| = |pp_1| - 1$, p has to be collinear in Γ with a point of K'. Secondly, let p be a symbol ∞_j, and L a line of Γ. Let K' be a class such that $K' \cup \{\infty_j\}$ is a line of Δ. Suppose K' does not intersect L. By projecting the points of K' onto L, we see that there is a unique point y on L not collinear with any of the points of K'. If y would be isolated, then $y \perp p_1$ and we obtain a contradiction by projecting the points of K' onto the line yp_1, hence y is contained in a non-trivial class K''. If $|K''| = g$, then $y = p_1$ and the result follows. If $|K''| = g - 1$, then $K' \sim_C K''$, hence the result. Now clearly, Γ is a forgetful quadrangle of type (III).

Suppose $n \geq 6$. We look for a contradiction. Let K' be a class at distance n from K (such a class exists by ($\diamond$)), and $z \in K'$. We construct a line M for which $\delta(M, K) = n - 2$, $\delta(M, K') = n$, $\delta(M, p_1) = n - 3$ and such that there exists a point $r \in K'$ for which $\delta(\mathrm{proj}_M p_1, r) = n - 2$. Fix a line L through p_1 and put $z' = \mathrm{proj}_L z$. If $n \equiv 2 \bmod 4$, let m be the point of $[z, z']$ at distance $n/2 - 3$ from z' and let M be a line at distance $n/2 - 2$ from m for which $\mathrm{proj}_m p_1 \neq \mathrm{proj}_m M \neq \mathrm{proj}_m z$. (Note that, for $n = 6$, $z' = m$, but since $|z'| \geq |z''| = |p_1| + 1 \geq 3$, with $z'' \in K' \setminus \{z\}$, the line M exists.) If $n \equiv 0 \bmod 4$, consider the element N of $[z, z']$ at distance $n/2 - 3$ from z'. If N is a line or a class containing at least three points, then let M be a line at distance $n/2 - 2$ from N such that $\mathrm{proj}_N p_1 \neq \mathrm{proj}_N M \neq \mathrm{proj}_N z$. If N is a class of size 2, then let N' be a line at distance $n/2 - 3$ from $x = \mathrm{proj}_N z'$ such that $\mathrm{proj}_x p_1 \neq \mathrm{proj}_x N' \neq \mathrm{proj}_x z$. Fix a point $z'' \in K' \setminus \{z\}$ and put $y = \mathrm{proj}_{N'} z''$. Note that if $\mathrm{proj}_y z''$ is a line, then $|y| \geq |z| = |p_1| + 1 \geq 3$. Hence it is possible to choose a line M through y different from N' or $\mathrm{proj}_y z''$. So in each case, we constructed a line M as claimed. Now all the points on M different from $\mathrm{proj}_M p_1$ are isolated (see $\diamond$). Since $|K'| = |M| - 1$, there exists a point a on M which lies at distance n from all the points of K'. Hence $|a| = |r| = |p_2|$, the final contradiction since a is isolated (which implies $|a| = |p_2| + 1$). $\square$

From now on, we assume that two points belonging to a class of size at least 3, have the same degree.

Lemma 3.5.3 *The points of a class K of size 2 have the same degree.*

Proof. Let $K = \{p_1, p_2\}$ and suppose by way of contradiction that $|p_1| \neq |p_2|$. For an arbitrary class K', one shows similarly as in the proof of Lemma 3.5.2 that $\delta(K, K') \leq n$, that $\delta(K, K') = n$ implies that $|K'| = 2$ and that the two points of K' have different degrees. Also, any isolated point lies at distance at most $n - 1$ from K. We first claim that the degrees of p_1 and p_2 differ by one. Let L be a line at distance n from K. Since $l \geq 3$, there is at least one point x on L at distance n from p_1 and p_2. Since x is not isolated, this point is contained in a class K' of size 2 at distance n from K. Without loss of generality, we can assume that the point y of K' different from x lies at distance $n - 2$ from p_2. From this follows that $|p_2| - 1 = |x| = |p_1| = |y| - 1$,

hence the claim. From now on, we assume that $|p_2| = |p_1| + 1$.
Now the following observation ($\diamondsuit$) can easily been shown. If $K_1 = \{q_1, q_2\}$ and $|q_1| \neq |q_2|$, then (up to interchanging q_1 and q_2) $|q_1| = |q_2| - 1$, and for any class $K_2 = \{r_1, r_2\}$ at distance n from K_1, we have (up to interchanging r_1 and r_2) either $\delta(q_1, r_1) = \delta(q_2, r_2) = n - 2$ with $|q_1| = |r_2|$ and $|q_2| = |r_1|$ or $\delta(q_2, r_2) = n - 2$, $\delta(q_1, K_2) = \delta(r_1, K_1) = n + 1$ and $|q_i| = |r_i|$, $i = 1, 2$.

First consider the case $n = 4$. We start by showing that there are no isolated points. Let L be an arbitrary line not intersecting K, and put $r_i = \mathrm{proj}_L p_i$, $i = 1, 2$. Let z be a point on L, $r_1 \neq z \neq r_2$. Then z is not isolated (because of the observations at the beginning of the proof), hence z is contained in a class $K_z = \{z, z'\}$ at distance 4 from K and by ($\diamondsuit$), $|z| = |p_1|$ and $|z'| = |p_2|$. If r_2 would be isolated, then $|p_1| + 1 = |r_2| = |z'| + 1$, a contradiction since $|z'| = |p_1| + 1$, hence r_2 is contained in a class $K_{r_2} = \{r_2, r_2'\}$, and $|r_2| = |p_2|$ (indeed, if $|r_2| = |p_1|$, then the two points r_2 and z of degree $|p_1|$ would be collinear, contradicting ($\diamondsuit$)). If r_1 would be isolated, then $|r_2'| + 1 = |r_1| = |z'| + 1$, again a contradiction. Now it is clear that no point of Γ is isolated. Hence every point is contained in a class of size 2, and has degree $|p_1|$ or $|p_2|$. Put $h = \frac{|\mathcal{P}|}{2}$. We count the number of pairs (p, L), L a line of Γ through the point p, with p a point of degree $|p_i|$. Since by ($\diamondsuit$) every line contains at most 1 point of degree $|p_1|$, we obtain

$$h|p_1| \leq |\mathcal{L}| \ (i = 1),$$

$$h(|p_1| + 1) \geq 2|\mathcal{L}| \ (i = 2).$$

Hence $|p_1| \leq 1$, the final contradiction.

Now consider the case $n \geq 6$ (in fact, the argument below also works for $n = 4$ except when $l = 3$). Choose an element x at distance $n/2 + 1$ from p_1 and at distance $n/2 + 2$ from K such that x is a line if $n \equiv 0 \mod 4$. Suppose first that, if $n = 6$, the point x can be chosen such that either x is isolated (implying $|x| \geq 3$) or $\mathrm{proj}_x p_1$ is a class. Let M_1 and M_2 be two lines at distance n from K and at distance $n/2 - 2$ from x such that $\mathrm{proj}_x M_1 \neq \mathrm{proj}_x M_2$ (note that such lines exist because of the assumptions just made). On each of the lines M_i, $i = 1, 2$, there is at least one point m_i that lies at distance $n + 1$ from the class K. Because of the first paragraph of the proof, the point m_i is contained in a class K_i of size 2, and by ($\diamondsuit$), $|m_1| = |p_1| = |m_2|$ (recall that $|p_1| = |p_2| - 1$). But since $\delta(m_1, m_2) = n - 2$, this is a contradiction with ($\diamondsuit$). Suppose now that $n = 6$ and that we cannot choose a point x as above. This implies in particular that every point collinear with p_1 and at distance 3 from K is isolated. Then let again x be a point at distance 5 from K for which $\delta(x, p_1) = 4$, M_1 a line through x,

$M_1 \neq \mathrm{proj}_x K$ and m_1 a point on M_1 at distance 6 from both p_1 and p_2. Again, the point m_1 is not isolated, and the class K_1 containing m_1 lies at distance 6 from K. Hence $K_1 = \{m_1, m_1'\}$ and by ($\diamondsuit$), also $|m_1| = |m_1'| - 1$. Now the point $x \bowtie p_1$ lies at distance 5 from K_1 and 4 from m_1, and has degree at least 3. Hence we can apply the argument of the general case above (with K_1 in the role of K and $x \bowtie p_1$ in the role of x) to obtain a contradiction with $|m_1| \neq |m_1'|$ □

Lemma 3.5.4 (*i*) *If two classes K and K' lie at distance n, then every point of K lies at distance $n-1$ from K' and vice versa, hence $|K| = |K'|$.*

(*ii*) *All non-isolated points have the same degree k.*

Proof. Let K and K' be two classes for which the points have degree k and k' respectively. Suppose first that $\delta(K, K') = n+2$. Let $x \in K$ and $x' \in K'$. Then $k = |x| = |x'| = k'$. Suppose now that $\delta(K, K') = n$. Then there are points $x \in K$ and $x' \in K'$ such that $\delta(x, x') = n-2$. If there exist points $y \in K$, $y \neq x$ and $y' \in K'$ such that $\delta(y, y') = n-2$, then $|x'| = |y|$, hence $k = k'$. But if no such points exist, then, for an arbitrary point $y \in K$, $y \neq x$, and a point $y' \in K'$, $y' \neq x'$, we have $|y'| = |y| = |x'| - 1$, contradicting $|x'| = |y'|$. Hence $k = k'$. Note that if K would contain a point z at distance $n+1$ from K', then $k = |z| = |x'| - 1 = k - 1$, a contradiction. This shows (*i*). Now choose a class K' at minimal distance from K (if such a class does not exist, (*ii*) is proved). We show that the points in K and the points in K' have the same degree. We can assume $\delta(K, K') \leq n-2$. Let X be the element at distance $\frac{\delta(K,K')}{2}$ from both K and K' (note that X cannot be a class because of the minimality of $\delta(K, K')$). Consider a point x at distance $n - 1 - \frac{\delta(K,K')}{2}$ from X such that $\mathrm{proj}_X K' \neq \mathrm{proj}_X x \neq \mathrm{proj}_X K$ (such a point exists since again by the minimality of $\delta(K, K')$, X is not a class of size 2 or a point of degree 2) and such that $\mathrm{proj}_x X$ is not a class. If x is isolated, then it is easy to see that $k = k'$ (indeed, then $|x| = |z| + 1$, for z an arbitrary point of K or K' different from the projection of X onto K or K'). If x is contained in a non-trivial class K'', then K'' lies at distance n from both K and K', hence the result. Now (*ii*) easily follows. □

Lemma 3.5.5 *One of the following situations occurs:*

(*i*) *There is a unique isolated point of degree k. In this case, Γ is a generalized quadrangle of type* (II)*, with $|X_1| = s$.*

(*ii*) *Any isolated point has degree $k+1$, and lies at distance at most $n-1$ from any class.*

Proof. Let w be an isolated point. We first prove that $|w| \in \{k, k+1\}$. For an arbitrary class K, we have $\delta(w, K) \leq n+1$. If there is a class K at distance $n+1$ from w, then $|w| = k$, and if there is a class K at distance $n-1$ from w, then $|w| = k+1$. Now choose a class K at minimal distance from w. We can assume $\delta(w, K) \leq n-3$. Put $v = \mathrm{proj}_K w$. Let X be the element on the shortest path between v and w at distance $\frac{\delta(v,w)}{2}$ from w. Then X is not a class, hence it is possible to choose a point x at distance $n - \frac{\delta(v,w)}{2}$ from X such that $\mathrm{proj}_X w \neq \mathrm{proj}_X x \neq \mathrm{proj}_X v$. If x is isolated, then (since $\delta(x, K) = n \pm 1$), $|x| = k$ or $|x| = k+1$. Since opposite isolated points have the same degree, also $|w| \in \{k, k+1\}$. If x is not isolated, then, with K' the class containing x, $\delta(w, K') = n \pm 1$, hence also $|w| \in \{k, k+1\}$. So we conclude that every isolated point has degree k or $k+1$; if it has degree $k+1$, then it cannot lie at distance $n+1$ from any class; if it has degree k, then it cannot lie at distance $n-1$ from any class.

<u>$n = 4$</u>

Suppose there exists an isolated point w of degree k. We show that all other isolated points have degree $k+1$. Note that all points collinear with w are isolated. Let K be an arbitrary class, and $x, y \in K$. Let L_1 and L_2 be two different lines through w and put $x_i = \mathrm{proj}_{L_i} x$, $y_i = \mathrm{proj}_{L_i} y$, $i = 1, 2$. Then x_i and y_i are isolated points of degree $k+1$ (since they lie at distance 3 from K). If w' is a second isolated point of degree k, then w' is collinear with the points x_i and y_i, $i = 1, 2$ (since isolated points at distance $n = 4$ have the same degree), hence $w = w'$. So w is the unique isolated point of degree k. From this it immediately follows that a point is isolated if and only if it is collinear with w. Let G be a class of size g. Since every point of a line L through w, different from w, is collinear with a unique point of G, it follows that $l = g+1$. Note that this implies that all classes have size g. Indeed, for an arbitrary non-trivial class K, every point of L different from w has to be collinear with a unique point of K and vice versa, hence $|K| = g$. Now we define the following equivalence relation $\sim_C$ on the classes of size g:

$$K_1 \sim_C K_2 \Leftrightarrow \delta(K_1, K_2) = 6.$$

The transitivity of $\sim_C$ is shown as follows: suppose $K_1 \sim_C K_2$, $K_1 \sim_C K_3$, but $\delta(K_2, K_3) = 4$. Let L be a line intersecting both K_2 and K_3. Then each point of K_1 has to be collinear with a unique point of the line L, not belonging to K_2 or K_3, hence $l \geq g+2$, a contradiction. We associate a symbol ∞_i, $i = 1, \ldots, s$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, s$. A line of Δ is either a line of Γ, the set of points of a class of $\mathcal{C}$ together with the symbol ∞_j of its equivalence class, or the set of points $\{\infty_1, \ldots, \infty_s, w\}$. Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s, k). Hence Γ is a forgetful quadrangle of type (II), with $|X_1| = s$.

$\underline{n \geq 6}$

We show that an isolated point of degree k cannot exist. So let by way of contradiction, w be an isolated point of degree k. Let S be the set of points x at distance $n-2$ from w for which $\mathrm{proj}_x w$ is a line. Clearly, S is nonempty and consists of isolated points. We first show that all the points of S have degree $k+1$. Suppose by way of contradiction that S contains a point x of degree k. Since opposite isolated points have the same degree, it is easy to see that all the points of S then have the same degree k. We can now always find a point y of S at distance $n-1$ from a certain class, which is a contradiction. Indeed, let K be an arbitrary non-trivial class at minimal distance from w. If $\delta(w, K) = n+1$, let v be a point of K and γ a fixed n-path between v and w. If $\delta(w, K) < n+1$, let $v = \mathrm{proj}_K w$ and $\gamma = [v, w]$. Let X be the element of γ at distance $\frac{\delta(v,w)}{2}$ from both v and w. Since X cannot be a class or a non-isolated point, it is possible to choose a point y at distance $n - 2 - \frac{\delta(v,w)}{2}$ from X such that $\mathrm{proj}_X w \neq \mathrm{proj}_X y \neq \mathrm{proj}_X v$ and $\mathrm{proj}_y X$ is not a class. Now the point y belongs to S and lies at distance $n-1$ from K, the contradiction. We conclude that all the points of S have degree $k+1$.

Now let x be a point at distance n from w. Then x cannot be isolated. Indeed, if x is isolated, then $|x| = |w| = k$, but it is easy to see that there exists a point y of S opposite x, hence $|x| = |y| = k+1$, a contradiction. Also, the class K_x containing x cannot lie at distance $n-1$ from w. Let L be a line through w. Since $\delta(w, K_x) = n+1$, projecting the points of K_x onto L shows that $l \geq |K_x| + 1$. Let γ be a fixed n-path between x and w, and X the element of γ at distance $n/2+1$ from x. If $n > 6$ and X is not a class of size two, then consider a line M at distance $n/2 - 2$ from X for which $\mathrm{proj}_X x \neq \mathrm{proj}_X M \neq \mathrm{proj}_X w$. Since the points of M, different from $\mathrm{proj}_M w$, are contained in S, they all have to lie at distance $n-1$ from K_x, hence $|K_x| = |M| = l$, a contradiction. If $n = 6$, then the same argument can be applied except if X is a point of degree 2 for which both $\mathrm{proj}_X x$ and $\mathrm{proj}_X w$ are lines. In this case, consider a line M at distance 3 from w and at distance 6 from the class K' containing X. Then similarly as above, we obtain $|K'| = |M| = l$, but this is a contradiction since by Lemma 3.5.4(i),

$|K'| = |K_x| \leq l-1$. Finally, if X is a class containing exactly two points, we proceed as follows. Let Y be the element of γ at distance $\frac{\delta(w,X)-3}{2}$ from w and, if Y is not a class of size 2, M a line at distance $n - \frac{\delta(w,X)+3}{2}$ from Y such that $\mathrm{proj}_Y w \neq \mathrm{proj}_Y M \neq \mathrm{proj}_Y X$. Note that $\delta(w,M) = n-3$ and $\delta(M,X) = n$. The points of M different from $\mathrm{proj}_M w$ (and there are at least 2 of them) are contained in S, hence lie at distance $n-2$ from a point of X (different from $\mathrm{proj}_X w$). This is a contradiction, since $|X| = 2$. If Y is a class of size 2, then we repeat the argument above with Y in the role of X. In this way, we obtain that there are no isolated points of degree k, and the lemma is proved. □

From now on, we assume that any isolated point has degree $k+1$.

Lemma 3.5.6 *If there exists a class X, with $1 < |X| < g$, then Γ is a forgetful quadrangle of type* (II)*, with $X = X_2$ and $1 < |X_1| < s$.*

Proof.

<u>$n = 4$</u>
Let X be a class of size $< g$, and G a class of size g. Note that $\delta(X,G) = 6$, because of Lemma 3.5.4(i). By projecting the points of G onto a line L intersecting X, we see that $|L| \geq g+1$. Suppose there is a second class X' with $|X'| < g$. If $\delta(X,X') = 4$, then let (X,x,M,x',X') be a 4-path between X and X'. Every point of G is collinear with a point of M different from x or x', and all these points are isolated (indeed, if $\mathrm{proj}_M p$, with $p \in G$, is not isolated, then the class K' containing $\mathrm{proj}_M p$ would satisfy $|G| = |K'| = |X|$ because of Lemma 3.5.4(i)). Hence there are at least g isolated points on M. Now let L be a line through x different from M, and assume there is a class K intersecting L, but not containing x. Because of Lemma 3.5.5(ii), every isolated point of M is collinear with a point of K, different from $\mathrm{proj}_K x$. Hence $|K| \geq g+1$, a contradiction. So all the points on L different from x are isolated (this makes at least g isolated points on L). But since, again by Lemma 3.5.5(ii), every isolated point of L is collinear with a point of X', $|X'| \geq g+1$, a contradiction. So all points at distance 3 from X are isolated, and $\delta(X,X') = 6$. Let N be a line intersecting X. Since N contains at least g isolated points, $|X'| \geq g$, a contradiction. Hence X is the unique class of size $< g$, and a point is isolated if and only if it lies at distance 3 from X. By projecting the points of a class G of size g onto a line L intersecting X, we see that $l = g+1$. As in the proof of Lemma 3.5.5, it is now possible to define the following equivalence relation on the classes of size g:

$$K_1 \sim_C K_2 \Leftrightarrow \delta(K_1,K_2) = 6.$$

Note that it is possible to find two classes of size g at distance 4 (indeed, consider the points on a line at distance 4 from X), hence $\sim_C$ defines at least two equivalence classes. We associate a symbol ∞_i, $i = 1, \dots, r$ to each equivalence class C_i of $\sim_C$. Now define the following geometry $\Delta = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \dots, r$. A line of Δ is either a line of Γ, the set of points of a class of size g together with the symbol ∞_j of its equivalence class, or the set of points $\{\infty_1, \dots, \infty_r\} \cup X$. Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (s, k) with $s = r + |X| - 1$, hence Γ is a forgetful quadrangle of type (II), with $1 < |X_1| < s$. We check for example the main axiom for a 'point' ∞_i and a line L of Γ, L not intersecting X. Suppose ∞_i is not collinear in Δ with any point of L. Let K be a class belonging to the equivalence class of $\sim_C$ with symbol ∞_i. Then K does not intersect L. Every point x of L is collinear with a point of K. (Indeed, this is clear if x is isolated. If x belongs to a non-trivial class K', then $\delta(K, K') = 4$, hence x is collinear with a point of K'.) Hence $|K| \geq |L| = g + 1$, a contradiction.

<u>$n = 6$</u>
We treat this case separately, because here the reasoning is slightly different from the general case.
Let by way of contradiction X be a class for which $|X| < g$. Let G be a class of size g. Then $\delta(X, G) \in \{4, 8\}$. If $\delta(X, G) = 8$, then $l \geq g + 1$. Indeed, choose $x \in X$ and L a line through x. By projecting the points of G onto the line L, we see that $|L| \geq g + 1$.

We next show that, if $\delta(X, G) = 4$, then $l = g$. Let (X, x, M, y, G) be the 4-path between X and G. Consider a line N concurrent with M not through x or y. Then the points on N different from $\mathrm{proj}_M N$ are isolated (indeed, a class K intersecting M but not containing $\mathrm{proj}_M N$ would satisfy $|X| = |K| = |G|$ by Lemma 3.5.4(i)). So N contains at least $g - 1$ isolated points. Now let L be a line through y different from M, and suppose there is a class K intersecting L, different from G. Since $\delta(X, K) = 6$, $|K| = |X| < g$. By Lemma 3.5.5(ii), there is a bijection between the points of K different from $\mathrm{proj}_L y$ and the points on N different from $\mathrm{proj}_M N$. This implies that $|K| = l$, hence $|K| = g = l$, a contradiction. So all the points on L are isolated. Now let y' be a point of G different from y, and L' a line through y'. Not all points on L' different from y' can be isolated (since otherwise Lemma 3.5.5(ii) would imply $|X| = |L'| \geq g$), so there is a class K' intersecting L', $K' \neq G$. Then $|K'| = |L| \geq g$, and hence $|K'| = |L| = g$.

First, suppose $\delta(X, G) = 8$ (so $l \geq g + 1$ in this case), and let $\gamma = (x, \dots, y)$ be a fixed 6-path between points $x \in X$ and $y \in G$. Let M be the element of γ at distance 3 from both x and y. Suppose first M is a class. If $|M| = g$, then $\delta(X, M) = 4$ implies $l = g$. If $|M| < g$, then $\delta(M, G) = 4$ implies $l = g$. Hence we obtain a contradiction in both cases, so M is necessarily a line. Clearly, the points on M different from the projection of x or y onto M are isolated. Also the point $z = \mathrm{proj}_M x$ is isolated. Indeed, this point cannot be contained in a class of size $< g$ (since such a class would lie at distance 6 from G). But z cannot lie in a class of size g either, since such a class would lie at distance 4 from X, implying $l = g$. Hence we have at least g isolated points on M. Now let N be a line through y different from $\mathrm{proj}_y M$, and suppose there is a class K intersecting N in a point different from y. Since there is a bijection between the points of M and the points of K, we obtain $|K| = g + 1$, a contradiction. Hence N contains at least g isolated points, but this implies (using Lemma 3.5.5(ii)) that $|X| \geq g$, a contradiction. This shows that $\delta(X, G) \neq 8$.

Next, suppose $\delta(X, G) = 4$ (so $l = g$ in this case). Let again (X, x, M, y, G) be the 4-path between X and G. Let L'' be a line at distance 3 from y such that $\mathrm{proj}_y L''$ is a line different from M. Suppose there is a class K intersecting L'', but not containing the point $\mathrm{proj}_{L''} y$. Since $\delta(G, K) = 6$, $|K| = g$, but this contradicts $\delta(K, X) = 8$ and the previous paragraph. Hence all $g - 1$ points of L'' different from $\mathrm{proj}_{L''} y$ are isolated. Consequently $|X| = g$ (again by Lemma 3.5.5(ii)), a contradiction. We conclude that all classes have size g. Hence X cannot exist.

<u>$n > 6$</u>
Let by way of contradiction X be a class for which $|X| < g$. Let G be a class of size g. Then $\delta(X, G) \leq n + 2$ and $\delta(X, G) \neq n$.

We first claim that if $\delta(X, G) = n + 2$, $\gamma = (x, \ldots, y)$ is an arbitrary n-path between points $x \in X$ and $y \in G$, and M is the element of γ at distance $n/2$ from both x and y, then either $n \equiv 2 \bmod 4$ and M is a class of size two, or $n = 8$ and there does not exist a line through M different from $\mathrm{proj}_M x$ and $\mathrm{proj}_M y$. $(*)$ So assume $\delta(X, G) = n + 2$, and suppose by way of contradiction that there is a path $\gamma = (x, \ldots, y)$ between points $x \in X$ and $y \in G$ such that the element M (with M as above) is not a class of size two if $n \equiv 2 \bmod 4$, or such that there does exist a line through M different from $\mathrm{proj}_M x$ and $\mathrm{proj}_M y$ if $n = 8$. By projecting the points of G onto the line $\mathrm{proj}_x M$, we obtain $l \geq g + 1$. Let L be a line at distance $n/2 - 3$ from M such that $\mathrm{proj}_M x \neq \mathrm{proj}_M L \neq \mathrm{proj}_M y$ (L exists because of the assumptions on M). Then the points on L different from $\mathrm{proj}_L M$ are isolated (indeed, a class K intersecting L but not containing $\mathrm{proj}_L M$ would satisfy $\delta(X, K) = \delta(G, K) = n$, so $|X| = |K| = |G|$, a contradiction). Now let N be a line through y different from $\mathrm{proj}_y M$. Since a class K' intersecting

N, $K' \neq G$, would satisfy $|K'| = |L| \geq g+1$ (using Lemma 3.5.5(ii)), every point on N is isolated. But (again using Lemma 3.5.5(ii)) this implies $|X| \geq |N|-1 \geq g$, a contradiction. This shows the claim.

We next show that $\delta(X,G) \neq j$, with $j \leq n/2+1$.

Suppose that $\delta(X,G) = 4$, and let (X, x, xy, y, G) be a 4-path between X and G. Let L be a line at distance $n-3$ from y such that $\mathrm{proj}_y L \neq xy$, and such that the element M of the path $[x, L]$ at distance $n/2$ from x is not a class of size 2 if $n \equiv 2 \bmod 4$, or such that there can be chosen a line through M not belonging to $[x, L]$ if $n = 8$. Let r be a point on L, $r \neq \mathrm{proj}_L y$. Then r is isolated. Indeed, suppose by way of contradiction that r is contained in a class K. Since $\delta(G,K) = n$, $|K| = |G| = g$. This implies that $\delta(X,K) = n+2$. But now, by considering the n-path between $x \in X$ and $r \in K$ containing L, we see that K cannot exist because of $(*)$. So the line L contains at least $g-1$ isolated points. By Lemma 3.5.5(ii), there is a bijection between the points of $X \setminus \{x\}$ and the points on L different from $\mathrm{proj}_L y$, a contradiction with $|X| < g$.

We proceed by induction on $\delta(X,G)$. Let $4 \leq k < n/2+1$ and suppose that $\delta(X,G') > k$, for any class G' of size g. Then we first claim that there does not exist a class G'' of size g at distance $n+2-k$ from X such that the element of the path $[X, G'']$ at distance $n/2-k+1$ from G'' is not a class of size 2 if $n \equiv 2 \bmod 4$ $(**)$. Suppose by way of contradiction a class G'' as above does exist. Let z be a point of G'' not belonging to $[X, G'']$, and L a line at distance $k-3$ from z such that $\mathrm{proj}_z L \neq G''$. All the points on L different from $\mathrm{proj}_L z$ are isolated. Indeed, if K' would be a class intersecting L, $\mathrm{proj}_L z \notin K'$, then $|K'| < g$ contradicts $\delta(K', G'') \leq k$ and the previous paragraph, but $|K'| = g$ contradicts $\delta(K', X) = n+2$ and $(*)$. So there are at least $g-1$ isolated points on L. By Lemma 3.5.5(ii), there is a bijection between the points of $L \setminus \{\mathrm{proj}_L z\}$ and the points of $X \setminus \mathrm{proj}_X G''$. Hence $|X| = g$, a contradiction. This shows $(**)$. Now we show that $\delta(X, G'') \neq k+2$, $k < n/2$, for any class G'' of size g. So suppose by way of contradiction that $\delta(X, G'') = k+2$, and let y be the element of G'' at distance $k+1$ from X. Let L be a line at distance $n-1-k$ from y, and $n-k$ from G'' such that $\mathrm{proj}_y L \neq \mathrm{proj}_y X$ and such that the element of the path $[G'', L]$ at distance $n/2-k+1$ from G'' is not a class of size 2 if $n \equiv 2 \bmod 4$. Then every point r on L different from $\mathrm{proj}_L y$ is isolated. Indeed, suppose by way of contradiction that r is contained in a class K. Since $\delta(K, G'') = n+2-k$, $(**)$ implies that $|K| = g$. But if $|K| = g$, then $\delta(X,K) = n+2$, which contradicts $(*)$. Hence r is isolated. Now again by Lemma 3.5.5(ii), there is a bijection between the points of L and X, implying $|X| \geq g$, a contradiction.

So we obtained the following: if $\delta(X,G) = j$, then $j > n/2+1$ and the element of a j-path between X and G at distance $j-(n/2+1)$ from G is a class K of size two. But now, applying this result on the classes G and K (note that $|K| < |G|$ and $\delta(K,G) < n/2-1$ if $j \neq n+2$) leads to the final contradiction. This shows that X cannot exist.

□

We summarize the situation reached so far. If Γ is a finite forgetful n-gon, n even, then Γ is either a generalized n-gon, a forgetful quadrangle of type (II) (with $1 < |X_1| < s+1$) or type (III), or there exist parameters g, k, d such that the following axioms are satisfied:

(S1) Every isolated point is incident with exactly $k+1$ lines, every non-isolated point is incident with exactly k lines ($k \geq 2$).

(S2) Every class has the same size g, $g > 1$.

(S3) Every line contains $g+d$ points, $d \geq 0$.

Let now Γ be a finite forgetful n-gon, n even, satisfying axioms (S1), (S2) and (S3). The parameter d is called the **deficiency** of Γ.

Lemma 3.5.7 *If $d = 0$, then Γ is a forgetful n-gon of type* (I).

Proof. Suppose $d = 0$. Define the geometry $\Delta = (\mathcal{P}, \mathcal{L} \cup \mathcal{C}, \mathtt{I})$. Then Δ is a generalized n-gon. Indeed, we only have to check that a point p and a class K lie at distance at most $n-1$ from each other. Suppose by way of contradiction that $\delta(p,K) = n+1$. But then projecting the points of K onto an arbitrary line L through p shows that $|L| \geq g+1$, a contradiction. □

Lemma 3.5.8 *If $n = 4$ and $d = 1$, then Γ is a forgetful quadrangle of type* (II)*, with $|X_1| \in \{1, s+1\}$.*

Proof. Suppose $d = 1$, so $l = g+1$. As in Lemma 3.5.6, we define an equivalence relation $\sim_C$ on the classes (which are all of size g), and associate a symbol ∞_i, $i = 1, \ldots, r$ to each equivalence class C_i of $\sim_C$.

Suppose first there are no isolated points. Then define the following geometry $\Delta = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$. A point of Δ is either a point of Γ or a symbol ∞_i, $i = 1, \ldots, r$. A line of Δ is a line of Γ, the set of points of a class K together with the symbol of its equivalence class, or the set of points $\{\infty_1, \ldots, \infty_r\}$.

Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Similarly as in Lemma 3.5.6, one shows that Δ is a finite generalized quadrangle of order (g, k), hence Γ is a forgetful quadrangle of type (II), with $|X_1| = s + 1$.

Suppose now there is an isolated point w. We show that any two classes of size g lie at distance 6 (hence $\sim_C$ has only one equivalence class). Note first that no line through w only contains isolated points. Indeed, if L_0 would be a line through w full of isolated points, then for a class G of size g (which necessarily lies at distance 3 from w), Lemma 3.5.5(ii) implies that $|G| = |L_0| = g + 1$, a contradiction. Now let $L_0, \ldots, L_k$ be the lines through w and X_0 a class of size g intersecting L_0. Since $l = g + 1$, there is a unique point x_i on L_i, $i = 1, \ldots, k$ that is not collinear with any point of X_0. Hence by lemmas 3.5.5(ii) and 3.5.4(i), x_i is contained in a class X_i of size g for which $\delta(X_0, X_i) = 6$. Since $\sim_C$ is an equivalence relation, also $\delta(X_j, X_{j'}) = 6$, for $j, j' \in \{1, \ldots, k\}$, $j \neq j'$. Suppose now there exists a class K of size g, $K \neq X_i$, $i = 0, \ldots, k$. Then without loss of generality, we can assume that K intersects L_0 in the point y. Because of the construction of the classes X_i, the point y lies at distance 3 from every class X_i, $i = 0, \ldots, k$. Hence there exists a line N_i through y, $i = 0, \ldots, k$ such that $\delta(N_i, X_i) = 2$ (note that all these lines N_i are different because the classes X_i mutually lie at distance 6). But since y is not isolated, it has degree k, a contradiction. So $\sim_C$ has a unique equivalence class with associated symbol ∞. Now define the following geometry $\Delta = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$. A point of Δ is either a point of Γ or the symbol ∞. A line of Δ is either a line of Γ or the set of points of a class K together with the symbol ∞. Incidence is the incidence of Γ if defined, or symmetrized containment otherwise. Then it is easy to see that Δ is a finite generalized quadrangle of order (g, k), hence Γ is a forgetful quadrangle of type (II), with $|X_1| = 1$. $\square$

A forgetful n-gon, n even, satisfying (S1), (S2) and (S3) with $d \geq 2$ if $n = 4$ and $d \geq 1$ if $n \geq 6$, is called a **short forgetful n-gon**. This name refers to both the short classes and the short memory of these objects. (Indeed, their memory seems to be too short to prove that they arise from generalized polygons.)

We now obtained the following theorem:

Theorem 3.5.9 *A finite forgetful n-gon, n even, is either a generalized n-gon, a forgetful n-gon of type* (I), (II) *or* (III)*, or a short forgetful n-gon.*

Proposition 3.5.10 *Let Γ be short forgetful polygon.*

(*i*) *Γ contains at least two classes.*

(*ii*) *Γ contains either 0 or at least two isolated points.*

Proof. Let Γ be a short forgetful polygon. Suppose there is a unique non-trivial class K $(K \neq \mathcal{P})$. Then by Lemma 3.5.5(ii), any line at distance n from K contains g points, so $d = 0$, a contradiction. This shows (i). Suppose now Γ contains a unique isolated point w. Let K be a class at distance $n-1$ from w, L_1 the line intersecting K at distance $n-3$ from w, and L_2 an arbitrary line through $\mathrm{proj}_K w$, different from L_1. Let $\mathcal{S}$ be the set of points at distance $n+1$ from K. Since $d \neq 0$, $S \neq \emptyset$. Each line at distance $n-2$ from L_1 (L_2) and at distance n from K contains d points of $\mathcal{S}$. Conversely, every point of $\mathcal{S}$ is on a unique line lying at distance n from K and at distance $n-2$ from L_1 (L_2). Hence, denoting by $\mathcal{R}_i$ the set of lines at distance n from K and $n-2$ from L_i, $i = 1, 2$, $|\mathcal{S}| = d|\mathcal{R}_i|$. But $|\mathcal{R}_1| - |\mathcal{R}_2| = 1$, since the point w is incident with $k+1$ lines, but every point at distance $n-3$ from L_2 is incident with k lines. This is a contradiction, showing (ii) . $\square$

3.6 Short forgetful quadrangles

3.6.1 General properties

Let Γ be a finite short forgetful quadrangle, with parameters (g, k, d). Recall that an isolated point of Γ lies at distance 3 from any class, and that, if two classes K_1 and K_2 lie at distance 4, then any point of K_1 (K_2) lies at distance 3 from K_2 (K_1).

Lemma 3.6.1 *Every line of Γ contains a constant number ρ of isolated points.*

Proof. Suppose there is a line L of Γ only containing isolated points. Let K be any class. Then the map

$$\begin{array}{rccl} \sigma: & \mathcal{P}_L & \to & K \\ & x & \to & y, \quad \text{with } \delta(x,y) = 2, \end{array}$$

is a bijection between the points on L and the points of K, so $l = g$, a contradiction with $l \geq g+2$. Hence every line intersects at least one class.

Let K be a fixed class of size g, and L_1 (L_2) a line intersecting K and containing ρ_1 (ρ_2) isolated points. Each point of Γ at distance 5 from K lies at distance 3 from L_1 and L_2, and each line at distance 4 from K contains exactly d points at distance 5 from K. So counting the number of points of Γ at distance 5 from K, we obtain

$$\rho_1 kd + (g+d-\rho_1-1)(k-1)d = \rho_2 kd + (g+d-\rho_2-1)(k-1)d,$$

hence $\rho_1 = \rho_2$. Now let K_1 (K_2) be a class of size g such that every line intersecting K_1 (K_2) contains exactly ρ_1 (ρ_2) isolated points. We count the number of points of Γ:

$$\begin{aligned} |\mathcal{P}| &= g + gk(g+d-1) + \rho_1 kd + (g+d-\rho_1-1)(k-1)d \\ &= g + gk(g+d-1) + \rho_2 kd + (g+d-\rho_2-1)(k-1)d, \end{aligned}$$

implying $\rho_1 = \rho_2$. This shows the lemma. □

Lemma 3.6.2 *(i) Either $\rho = 0$ or $\rho = g-(d-1)(k-1)$.*

(ii) If $\rho \neq 0$, then $|\mathcal{L}| = gk(k+1)$. If $\rho = 0$, then $|\mathcal{L}| = k((d-1)(k-1)+gk)$.

Proof. Let $\mathcal{I}$ be the set of isolated points. If K is a class of size g, then every isolated point lies at distance 3 from K, hence $|\mathcal{I}| = gk\rho$. Also, every line of Γ intersects K, or lies at distance 3 from a fixed point of K, hence

$$|\mathcal{L}| = gk + k(\rho k + (g+d-\rho-1)(k-1)).$$

Counting the number of pairs (i, L), $i \in \mathcal{I}$, $L \in \mathcal{L}$, $i \mathrm{I} L$, we obtain:

$$gk\rho(k+1) = (gk + k(\rho k + (g+d-\rho-1)(k-1))\rho.$$

If $\rho \neq 0$, this simplifies to $\rho = g-(d-1)(k-1)$, showing (i). Now by using this in the expression for $|\mathcal{L}|$ above, we obtain (ii). □

Define the following graph G_Γ. The vertices of G_Γ are the classes of Γ. Two vertices are adjacent if and only if the corresponding classes lie at distance 6.

Lemma 3.6.3 *(i) If $\rho \neq 0$, then G_Γ is a*

$$\mathrm{srg}((k+1)(kd+1-k), kd, k-1, d))$$

and G_Γ^C is a

$$\mathrm{srg}((k+1)(kd+1-k), k^2(d-1), (d-1)(k^2+1)-dk, k(k-1)(d-1)).$$

(ii) If $\rho = 0$, then, with $f = \frac{d(d-1)(k-1)}{g}$, G_Γ is a $\mathrm{srg}(1 + k(g+d-1) + (k-1)d + f, (k-1)d + f, k - d - 1 + f, f)$ *and G_Γ^C is a*

$$\mathrm{srg}(1 + k(g+d-1) + (k-1)d + f, k(g+d-1), d - 1 + k(g-1), kg).$$

Proof. We determine the parameters of G_Γ^C. The number of classes follows from $|\mathcal{P}| = g + gk(g+d-1) + (g+d-1-\rho)(k-1)d + \rho kd$ and $|\mathcal{I}| = gk\rho$, with $\mathcal{I}$ the set of isolated points. Now let K be a fixed class, and $r \in K$. A class K' lies at distance 4 from K if and only if K' contains a point collinear with r, hence there are $k(g+d-1-\rho)$ classes lying at distance 4 from K. Let K' be a fixed class, with $\delta(K, K') = 4$, and (K, r, R, r', K') a 4-path between K and K'. A class K'' lies at distance 4 from both K and K' if and only if K'' intersects R (not in r or r' of course), or K'' intersects a line L through r, $L \neq R$, in a point v ($v \neq r$) that lies at distance 3 from K'. Note that every isolated point on such a line L necessarily lies at distance 3 from K'. Hence there are $g + d - 2 - \rho + (k-1)(g-1-\rho)$ classes at distance 4 from K and K'. Let finally $\overline{K}$ be a class at distance 6 from K. A class K'' lies at distance 4 from both K and $\overline{K}$ if and only if there exists a line N through r such that K'' intersects N in a point at distance 3 from $\overline{K}$. Since any line N through r contains exactly $g - \rho$ non-isolated points at distance 3 from $\overline{K}$, we have in total $k(g-\rho)$ classes at distance 4 from K and $\overline{K}$. $\square$

If $\rho \neq 0$, then $g | d(d-1)k$ is a necessary condition for the existence of Γ. Also, the fact that the multiplicities of the eigenvalues of G_Γ are integers, gives necessary conditions on (g, k, d) for the existence of G_Γ, and hence for Γ itself.

Is there is similar structure (a distance-regular graph for example) on the classes of a short forgetful hexagon without isolated points ?

Let Γ be a short forgetful quadrangle without isolated points. For a point x of Γ, we denote by K_x the class containing x. We define the following relations $\mathcal{R} = (R_0, R_1, R_2, R_3, R_4)$ on $\mathcal{P}$.

$$\begin{aligned}
R_0 &= \{(x,x) | x \in \mathcal{P}\}\\
R_1 &= \{(x,y) \in \mathcal{P}^2 | x \perp y\}\\
R_2 &= \{(x,y) \in \mathcal{P}^2 | \delta(x,y) = 4 \text{ and } \delta(K_x, K_y) = 4\}\\
R_3 &= \{(x,y) \in \mathcal{P}^2 | \delta(x,y) = 4 \text{ and } \delta(K_x, K_y) = 6\}
\end{aligned}$$

Lemma 3.6.4 *The pair $(\mathcal{P}, \mathcal{R})$ is an association scheme.*

Proof. We have to prove that the intersection numbers p^i_{jk} are defined. These numbers are easily determined using Lemma 3.6.3. (We only mention the non-zero intersection numbers for which $j \leq k$).

$p^0_{00} = 1,\ p^0_{11} = k(g+d-1),\ p^0_{22} = g-1,\ p^0_{33} = k(g-1)(g+d-1),\ p^0_{44} = (k-1)d(g+d-1),\ p^1_{01} = 1,\ p^1_{11} = g+d-2,\ p^1_{13} = (k-1)(g-1),\ p^1_{14} = d(k-1),\ p^1_{23} = g-1,\ p^1_{33} = (g-1)(gk-2k+d),\ p^1_{34} = d(g-1)(k-1),\ p^1_{44} = d(d-1)(k-1),\ p^2_{02} = 1,\ p^2_{13} = k(g+d-1),\ p^2_{22} = g-2,\ p^2_{33} = k(g-2)(g+d-1),\ p^2_{44} = (k-1)(d^2+dg-d),\ p^3_{03} = 1,\ p^3_{11} = k-1,\ p^3_{12} = 1,\ p^3_{13} = (k-1)(g-2)+g+d-2,\ p^3_{14} = d(k-1),\ p^3_{23} = g-2,\ p^3_{33} = (g-2)(gk-2k+d)+(g-1)(k-1),\ p^3_{34} = d(g-1)(k-1),\ p^3_{44} = d(d-1)(k-1),\ p^4_{04} = 1,\ p^4_{11} = k,\ p^4_{13} = k(g-1),\ p^4_{14} = k(d-1),\ p^4_{24} = g-1,\ p^4_{33} = k(g-1)^2,\ p^4_{34} = k(d-1)(g-1),\ p^4_{44} = g(k-d-1)+d(d-1)(k-1).$ □

The intersection matrix L_3 of the association scheme $(\mathcal{P},\mathcal{R})$ has five distinct eigenvalues. The corresponding normalized eigenvectors $u_0, u_1, \ldots, u_4$ are:

$$\begin{aligned}
u_0 &= (1,1,1,1,1)\\
u_1 &= (1,\frac{1}{k},\frac{-1}{g-1},\frac{-1}{k(g-1)},0)\\
u_2 &= (1,\frac{-1}{g+d-1},1,\frac{-1}{g+d-1},\frac{g}{d(g+d-1)})\\
u_3 &= (1,\frac{-1}{g+d-1},\frac{-1}{g-1},\frac{1}{(g-1)(g+d-1)},0)\\
u_4 &= (1,\frac{d-1}{k(g+d-1)},1,\frac{d-1}{k(g+d-1)},\frac{-g}{(k-1)(g+d-1)}).
\end{aligned}$$

Theorem 3.6.5 *If Γ is a short forgetful quadrangle without isolated points, then $k \leq (l-1)^2$.*

Proof. This follows from the Krein condition $q_{332} \geq 0$, with $q_{332} = \sum_l p^0_{ll}(u_2)_l(u_3)^2_l$. □

The multiplicities of the association scheme $(X,\mathcal{R})$ are the following:

$$
\begin{aligned}
f_0 &= 1\\
f_1 &= \frac{(g+d)(kg+(d-1)(k-1))k(g-1)}{g(k+g+d-1)}\\
f_2 &= \frac{d(g+d-1)(kg+(d-1)(k-1))}{g(d+k-1)}\\
f_3 &= \frac{(g-1)(g+d)(g+d-1)(kg+(d-1)(k-1))}{g(g+d+k-1)}\\
f_4 &= \frac{(g+d)k(k-1)(g+d-1)}{g(k+d-1)}.
\end{aligned}
$$

The fact that these multiplicities are integers, gives additional necessary conditions on (g,k,d) for the existence of Γ.

3.6.2 Examples of short forgetful quadrangles

Subquadrangle type

Let Δ be a finite generalized quadrangle of order (s,t), having an ideal (possibly thin) subquadrangle Δ' of order (s',t), $s' \geq 1$. Then we define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$. The points of Γ are the points of Δ not contained in the subquadrangle Δ'. The lines of Γ are the lines of Δ that do not intersect Δ'. Incidence is the incidence of Γ. Two points of Γ are equivalent if and only if they are on a line of Δ'. So the equivalence classes correspond to the sets $\Delta_1(L) \setminus \Delta'_1(L)$, with L a line of Δ that is also a line of the subquadrangle Δ'. It is now easy to see that Γ is a short forgetful quadrangle, with parameters $g = s - s'$, $k = t$, $d = s'+1$ and $\rho = s - s't$ (the value of ρ follows from the proof of Theorem 1.2.5(ii)). Note that in this example, G_Γ corresponds with the line graph of Δ'. If a short forgetful quadrangle Γ arises from this construction, we say that Γ is of **subquadrangle type**.

One has the following examples of this construction in the classical case.

Δ	Δ'	$\Gamma(g,k,d)$
$\mathsf{W}(q)$	dual grid	$(q-1,q,2)$, $\rho=0$
$\mathsf{H}(3,q^2)$	dual grid	$(q^2-1,q,2)$, $\rho=q^2-q$
$\mathsf{H}(3,q^2)$	$\mathsf{W}(q)$	$(q^2-q,q,q+1)$, $\rho=0$
$\mathsf{H}(4,q^2)^D$	$\mathsf{H}(3,q^2)^D$	$(q^3-q,q^2,q+1)$, $\rho=0$

Ovoid type

Let Δ be a finite generalized quadrangle of order (s,t), admitting a regular ovoid $\mathcal{O}$. Then we define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$. The points of Γ are the points of Δ not belonging to $\mathcal{O}$. The lines of Γ are the lines of Δ. Two points of Γ are equivalent if and only if they are contained in a set $\{o_1, o_2\}^\perp$, for $o_1, o_2 \in \mathcal{O}$. Incidence is the incidence of Δ. Then Γ is a forgetful quadrangle with parameters $g = k = t + 1$, $d = s - t - 1$ and $\rho = 0$. Indeed, we check that for a non-incident point-line pair (p, L) of Γ, there exists a unique point $r \mathtt{I} L$, r collinear or equivalent with p. Note that there exists a path $[p, L] = (p, L', p', L)$ in the generalized quadrangle Δ. If $p' \in \mathcal{O}$, then p' is not collinear in Δ with any point of L. Let in this case o be a point of $\mathcal{O}$ collinear with p, $o \neq p'$. The point $r = \mathrm{proj}_L o$ is then the unique point on L equivalent with p, since $\{p, r\} \subseteq \{o, p'\}^\perp$. If $p' \notin \mathcal{O}$, then p' is the unique point on L collinear in Δ with p. If p would be equivalent with a point $p'' \mathtt{I} L$, the point p' would be a point of the ovoid $\mathcal{O}$, a contradiction. This shows that Γ is a forgetful quadrangle. If a short forgetful quadrangle Γ arises from this construction, we say that Γ is of **ovoid type**.

Note that a forgetful quadrangle of ovoid type has the following property:

(O) If there is a line intersecting three different (possibly trivial) classes K, K' and K'', then any line intersecting two of these classes, also intersects the third one.

The known regular ovoids giving rise to a short forgetful quadrangle (thus with $d \geq 2$) are the following.

- Let Δ be the $(q+1) \times (q+1)$-grid (so a thin generalized quadrangle of order $(q, 1)$), $q \geq 4$, and $\mathcal{O}$ the points on one of the diagonals. Then the associated short forgetful quadrangle has parameters $g = k = 2$ and $l = q$. The graph G_Γ^C is the triangular graph $\mathsf{T}(q+1)$.

- Let Δ be the generalized quadrangle $\mathsf{H}(3, q^2)$, and $\mathcal{O}$ the points of a Hermitian curve $\mathcal{H}$ lying on $\mathsf{H}(3, q^2)$. Then the associated short forgetful quadrangle has parameters $g = k = q + 1$ and $l = q^2$. The vertices of the graph G_Γ correspond to the lines intersecting $\mathcal{H}$ in $q + 1$ points, and two vertices of G_Γ are adjacent if the corresponding lines intersect in a point not belonging to $\mathcal{H}$.

Application. The other known examples of regular ovoids all occur in generalized quadrangles of order $(q + 1, q - 1)$ (see for instance De Bruyn [16] section 2.6.2). By applying the above construction on a generalized quadrangle Δ of order $(q + 1, q - 1)$ admitting a regular ovoid $\mathcal{O}$, one obtains a forgetful quadrangle Γ with $g = k = q$ and $d = 1$. By Lemma 3.5.8, Γ then arises from a generalized quadrangle Δ' by applying construction (II), with $|X_1| = s + 1$. Clearly, Δ' has order (q, q). Since property (O) holds, the line L of Δ' corresponding to the set X_1 has to be regular. One can now easily see that the quadrangle Δ arises by applying the construction of Payne on the generalized quadrangle Δ' with L as regular line. The regular ovoid $\mathcal{O}$ corresponds with the points of type (B) (using the notation of section 1.4.4). So any generalized quadrangle of order $(q+1, q-1)$ having a regular ovoid $\mathcal{O}$, arises from the construction of Payne. This result can be found in Payne [33], section 3.

3.6.3 Characterization results

Lemma 3.6.6 *Let Γ be a short forgetful quadrangle with $k \leq g$ and satisfying property (O). Then Γ is of ovoid type.*

Proof. Let Γ be a short forgetful quadrangle satisfying the conditions of the lemma. We first claim that Γ does not contain isolated points. Indeed, suppose first $0 < \rho < l - 1$ and let w be an isolated point. Let L be a line through w. Then there are at least two non-trivial classes K and K' intersecting L. Since K and K' lie at distance 4, there is a line N different from L intersecting K and K', hence, by property (O), N is incident with w. Now a 'triangle' arises, the contradiction. Suppose $\rho \geq l-1$. Let L be a line at distance 4 from a non-trivial class K. Note that $l - 1 \geq g + 1$. Now by Lemma 3.5.5, every isolated point of L lies at distance 3 from K, implying $|K| \geq g + 1$, again a contradiction. Hence $\rho = 0$.

We now prove that $g = k$. Let L_1 be a line of Γ and $K_1, \ldots, K_l$ the classes intersecting L_1. Then by Lemma 3.5.4(i) there exists a set of lines $\mathcal{S} = \{L_1, \ldots, L_g\}$ such that every line of $\mathcal{S}$ intersects $K_1, \ldots, K_l$. Since every line of Γ intersects at most one line of $\mathcal{S}$ (otherwise a 'triangle' would arise), there are $(g+d)g(k-1)$ lines intersecting a line of $\mathcal{S}$. Hence $g + (g+d)g(k-1) \leq |\mathcal{L}| = k((d-1)(k-1) + gk)$. Using $k \neq 1$, this simplifies to

$$(g - k)(g + d - 1) \leq 0,$$

hence $g \leq k$. Since also $k \leq g$, we obtain $g = k$. Note that in particular, this implies that every line of Γ intersects exactly one line of $\mathcal{S}$.

Define the following equivalence relation on the set of lines of Γ. Two lines L_1 and L_2 of Γ are equivalent if and only if there exist at least two classes intersecting both L_1 and L_2 (the fact that this is an equivalence relation immediately follows from property (O)). To each equivalence class C_i of lines of Γ, we associate a symbol ∞_i, $i = 1, \ldots, r$. Then define the following geometry $\Delta = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. A point of Δ is either a point of Γ or a symbol ∞_i. The lines of Δ are the lines of Γ. Incidence is the incidence of Γ if defined, and symmetrized containment otherwise. Then Δ is a generalized quadrangle of order $(l, k-1)$. Indeed, we check the main axiom for a non-incident point-line pair (p, L) of Δ.

(i) p a point of Γ.
Either p is collinear in Γ (and hence in Δ) with a point p' of L, or p is equivalent with a point p' of L. In the former case, no line through p can be equivalent with L (otherwise p would be equivalent with a point of L), so p' is the unique point of Δ on L that is collinear in Δ with p. In the latter case, let p'' be a point on L, different from p'. The point p lies at distance 3 from the class K containing p'', hence the line R through p intersecting K is equivalent with the line L. Now the symbol ∞_i corresponding to the equivalence class containing R and L is the unique point of Δ collinear with p and incident in Δ with L.

(ii) $p = \infty_i$.
Let C_i be the equivalence class of lines corresponding to ∞_i. By the second paragraph of the proof, every line not belonging to C_i intersects a unique line of C_i. This implies that the line L is concurrent in Δ with a unique line of Δ through the point ∞_i.

The points ∞_i form an ovoid $\mathcal{O}$ of Δ. Moreover, $\mathcal{O}$ is a regular ovoid. Indeed, let ∞_i and ∞_j be two different equivalence classes of lines. Let L and L' be two lines of the class ∞_i. Then by the second paragraph of the proof, L (L') meets a unique line M (M') of the class ∞_j in a point r (r'). Note that $M \neq M'$, since otherwise a 'triangle' would arise. Suppose that r and r' are not equivalent. Let K be the class containing r. Since $L \sim L'$, K intersects the line L' in a point a, $a \neq r$, and since $M \sim M'$, K intersects M' in a point b, $r \neq b \neq r'$. But now a 'triangle' arises through the points a, b and r', a contradiction. Hence the lines of ∞_i and ∞_j meet in the set of points of a class K. From $\{\infty_i, \infty_j\}^{\perp} = K$ follows easily that the ovoid $\mathcal{O}$ is regular. Now clearly, Γ is of ovoid type. $\square$

Theorem 3.6.7 *Let Γ be a short forgetful quadrangle without isolated points, such that $g = k$ and $l \geq (g-1)^2$. Then Γ is of ovoid type.*

Proof. Let Γ be a short forgetful quadrangle without isolated points, such that $g = k =: q + 1$, and $l = q^2 + r$, $r \geq 0$. We show that Γ has property (O).

Suppose first $g = 2$. Let L be a line of Γ, and $K_i = \{r_i, r'_i\}$, $i = 1, 2, 3$, three different classes intersecting the line L in the point r_i. Since $g = 2$, $r'_1r'_2$, $r'_1r'_3$ and $r'_2r'_3$ are lines. This gives rise to a triangle, unless $r'_1r'_2r'_3$ is a line. So in this case, property (O) is satisfied. From now on, we assume $g \geq 3$. Let K be a fixed class, and put $K = \{a_0, \ldots, a_q\}$. Let L be a fixed line through a_0 and $C = \{K_1, \ldots, K_{q^2+r-1}\}$ the set of classes different from K intersecting L. Note that, since $\delta(K, K_i) = 4$, $i = 1, \ldots, q^2 + r - 1$, every point of K is collinear with exactly one point of each class K_i, $i = 1, \ldots, q^2+r-1$. Put b_i, $i = 0, \ldots, q$ the point of K_1 collinear with the point a_i of K. We claim that if at least one class of C intersects a line concurrent with K, then at least $q+r-1$ classes of C intersect this line. Indeed, consider the class K_1 and the line a_1b_1. Let V be the set of the q^2+r-2 points of $K_2, \ldots, K_{q^2+r-1}$ that are collinear with b_1 (these points exist, since $\delta(K_1, K_i) = 4$, $i = 2, \ldots, q^2+r-1$). No point of V lies on a line a_ib_i, $i = 0, 2, \ldots, q$, since these lines already contain a point that is equivalent with b_1 (otherwise a 'triangle' would arise). Hence every point of V either lies on a_1b_1, or on a line through a_i, different from a_ib_i, with $2 \leq i \leq q$. Since a line through a_i, $2 \leq i \leq q$, different from a_ib_i can contain at most one point of V, the line a_1b_1 contains at least $q^2 + r - 2 - q(q - 1) = q + r - 2$ points of V. So at least $q + r - 1$ classes of C intersect the line a_1b_1. This shows the claim. Note also that, if the line a_1b_1 contains exactly $q + r - 2$ points of V, then each line through a_i, $2 \leq i \leq q$, different from a_ib_i, contains exactly one point of V $(*)$.

Suppose first that there is a line M concurrent with K that is intersected by exactly $q + r - 1$ classes of C. Without loss of generality, we can assume that $M = a_1b_1$. By $(*)$, this means that every line through a_i, $i = 2, \ldots, q$, intersects at least one, and hence at least $q + r - 1$ classes of C. So in total, a point a_i, $2 \leq i \leq q$, lies at distance 3 from at least $(q + 1)(q + r - 1)$ classes of C. Hence $q^2 + r - 1 \geq (q + r - 1)(q + 1)$, implying that $r = 0$ and that every line through a_i, $2 \leq i \leq q$, intersects exactly $q - 1$ classes of C. By symmetry, also every line through a_1 intersects exactly $q - 1$ classes of C. Let, without loss of generality, $C' = \{K_1, \ldots, K_{q-1}\}$ be the set of $q - 1$ classes of C intersecting the line a_1b_1. The point b_1 has to be collinear with a point $p_i \in K_i$, for $q \leq i \leq q^2 - 1$ (note that these K_i are the classes of $C \setminus C'$). By $(*)$ each line through a_i, $2 \leq i \leq q$, different from a_ib_i contains a point collinear with b_1 and belonging to a class of $C \setminus C'$. Let C_1 be the set of $q - 1$ classes of C intersecting the line a_2b_2, and C_2 the set of $q^2 - q$ classes intersecting a_2b_2, different from K and not belonging to C_1. Since exactly $q - 1$ classes intersecting a_2b_2 different from K also intersect the line a_0b_0 (namely the classes of C_1), every line concurrent with K but not through a_2 is intersected by exactly $q - 1$ classes of $C_1 \cup C_2$ (this follows from the argument above, applied on $C_1 \cup C_2$ instead of C). Now since K_1 lies at distance 4 from every class of C_2, there is a set $V' = \{v_1, \ldots, v_{q^2-q}\}$ of $q^2 - q$ points collinear with b_1 and belonging to a class of C_2. These points cannot be collinear with a_i, $i \geq 2$ (since the points collinear with b_1 on lines through a_i, $i \geq 2$, belong to classes of C, and $C \cap C_2 = \emptyset$). So all the points of V' are collinear with a_0 (and there are at most q such points), or lie on the line $a_1 b_1$. So at least $q^2 - 2q + 1$ classes of $C_1 \cup C_2$ intersect the line $a_1 b_1$. This implies $q^2 - 2q + 1 \leq q - 1$, hence $q = 1$ (then $g = 2$, a contradiction) or $q = 2$. But if $q = 2$, then $l = 4$, hence $d = l - g = 1$, a contradiction.

We may now assume that if at least one class of C intersects a line concurrent with K, then at least $q + r$ classes of C intersect this line. For each point a_i, $2 \leq i \leq q$, let A_i be the number of lines through a_i that do not intersect any class of C. If $A_i = 0$ for some i, then every line through a_i contains at least $q + r$ points belonging to one of the $q^2 + r - 1$ classes of C. This would imply that $(q+1)(q+r) \leq q^2 + r - 1$, a contradiction. Hence $M := \min_{2 \leq i \leq q} A_i \neq 0$. We next claim that every line through a_i, $2 \leq i \leq q$, that intersects at least one class of C, intersects at least $(q-1)(M+1) + r$ classes of C. Let V again be the set of the $q^2 + r - 2$ points of $K_2, \ldots, K_{q^2+r-1}$ that are collinear with b_1. Now the number of points of V that do not lie on the line $a_1 b_1$ is at most $(q-1)(q-M)$, hence there are at least $q^2 + r - 1 - (q-1)(q-M) = (q-1)(M+1) + r$ classes of C that intersect the line $a_1 b_1$. This shows the claim. Now consider a point a_j of K, $2 \leq j \leq q$, for which $A_j = M$. Then there are $q + 1 - M$ lines through a_j such that each of these lines intersects at least $(q-1)(M+1) + r$ classes of C, hence

$$(q + 1 - M)(qM + q + r - M - 1) \leq q^2 + r - 1,$$

implying

$$M \leq \frac{-r}{q-1} \text{ or } M \geq q.$$

Since $0 < M \leq q$, we have $M = q$, meaning that if a line concurrent with K intersects a class of C, it intersects every class of C. This is exactly property (O), so the conditions of Lemma 3.6.6 are satisfied, and Γ is of ovoid type. $\square$

Corollary 3.6.8 *If Γ is a short forgetful quadrangle without isolated points, satisfying $k = g \neq 2$, then $l \leq (g-1)^2$.*

Proof. Suppose by way of contradiction that Γ is a short forgetful quadrangle without isolated points, with $k = g =: q + 1$ and $l = (g-1)^2 + r = q^2 + r$,

$r \geq 1$. Then because of Theorem 3.6.7, Γ is of ovoid type, and there exists a generalized quadrangle of order (l, q), with $l > q^2$, a contradiction since $q \neq 1$. $\square$

Lemma 3.6.9 *Let Γ be a short forgetful quadrangle for which G_Γ is the line graph of a generalized quadrangle. Then Γ is of subquadrangle type.*

Proof. Let Γ be a short forgetful quadrangle for which G_Γ is the line graph of a generalized quadrangle Δ' of order (s, t). Then one calculates (using Lemma 3.6.3) that $s = d - 1$, $t = k$ and that, if $\rho = 0$, $g = (k-1)(d-1)$. Each point of Δ' corresponds to a (maximal) clique of size $k+1$ in the graph G_Γ, so to $k+1$ classes lying at distance 6 from each other. Also, every two classes at distance 6 from each other are contained in a unique clique of size $k+1$, and every class belongs to exactly d $(k+1)$-cliques. We denote the points of Δ' by ∞_i, $i = 1, \ldots, v$. Now define the following geometry $\Delta = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. The points of Δ are the points of Γ and the symbols ∞_i. There are two types of lines of Δ. The lines of type (A) are the lines of Γ. A line of type (B) consists of the points of a class K of Γ, together with the symbols $\infty_1, \ldots, \infty_d$ of the $(k+1)$-cliques containing K. Incidence is the incidence of Γ if defined, and symmetrized containment otherwise. Then Δ is a generalized quadrangle of order $(l-1, k)$. Indeed, we check the main axiom for a non-incident point-line pair (p, L) of Δ.

- (i) p a point of Γ, L type (A). Immediate.
- (ii) p a point of Γ, L type (B). Let K be the class of Γ such that all its points are incident with L. If p is isolated, then p is collinear with a unique point of K, and p is not incident with any line of type (B). Hence the path $[p, L]$ exists and is unique in Δ. If p is non-isolated and is collinear in Γ with a (necessarily unique) point of K, then the class K' containing p lies at distance 4 from K, hence K' is not contained in any $(k+1)$-clique containing K, so $[p, L]$ is unique in Δ. If finally p is non-isolated and lies at distance 5 from K, then the class K' containing p belongs to a unique $(k+1)$-clique through K, so p is collinear in Δ with exactly one of the symbols ∞_i on L.
- (iii) $p = \infty_j$, L type (A). There are exactly $gk(k+1)$ lines intersecting one of the $k+1$ classes of the clique corresponding to ∞_j. Since this is equal to the number of lines of Γ, the path $[p, L]$ exists and is unique in Δ.

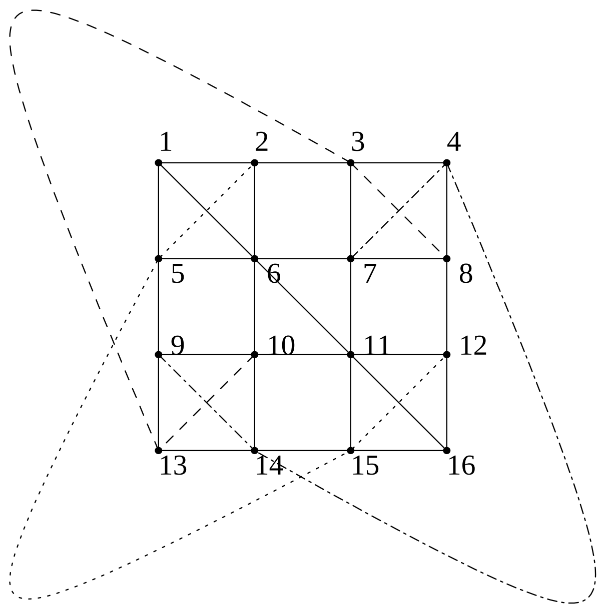


Figure 3.6: The complement of the Shrikhande graph.

(*iv*) $p = \infty_j$, L type (B). This follows immediately from the fact that Δ' is a generalized quadrangle.

Since Δ' is an ideal subquadrangle of Δ, Γ is of subquadrangle type. $\square$

Remark. Let Γ be a short forgetful quadrangle with parameters (g, k, d) such that, if $\rho = 0$, $g = (k-1)(d-1)$. If every two adjacent vertices of G_Γ are contained in a clique of size $k+1$, then G_Γ is the line graph of a generalized quadrangle (see Brouwer, Cohen & Neumaier [8] Lemma 1.15.1), hence Γ is of subquadrangle type by Lemma 3.6.9.

Theorem 3.6.10 *Let Γ be a short forgetful generalized quadrangle with parameters $(g, k, 2)$.*

(*i*) *If Γ contains isolated points, then Γ is of subquadrangle type.*

(*ii*) *If $g = k-1$, then Γ is of subquadrangle type.*

Proof. Let Γ be a short forgetful quadrangle with $d = 2$, and suppose that either Γ contains isolated points, or the parameters of Γ satisfy $g = k-1$ (which implies that there are no isolated points because of Lemma 3.6.2(i)). Put $l = s+1$, $g = s-1$ and $k = t$. Then in both cases, G_Γ is a srg $((t+1)^2, 2t, t-1, 2)$.

Suppose first $t \neq 3$. Then G_Γ is the $(t+1) \times (t+1)$-grid (or equivalently, the line graph of a thin generalized quadrangle of order $(1, t)$) (see Bose [5], or De Clerck & Van Maldeghem [17], Theorem 4). By Lemma 3.6.9, Γ is of subquadrangle type, which proves the theorem in this case. Suppose now $t = 3$. Then G_Γ is a srg (16,6,2,2). Any strongly regular graph with these parameters is either the 4×4-grid or the Shrikhande graph (see for example Brouwer, Cohen & Neumaier [8], Theorem 3.12.4). In the former case, the theorem again follows as before, so assume that G_Γ is the Shrikhande graph. We show that this leads to a contradiction. For convenience, we work with the complementary graph G_Γ^C. Label the classes of G_Γ^C with $K_1, \ldots, K_{16}$ as in Figure 3.6. Remember that Γ is a short forgetful quadrangle with $g = s-1$, $l = s+1$, $k = 3$ and $\rho = s-3$, so every line contains exactly 4 non-isolated points. Also, two vertices are adjacent in G_Γ^C if and only if the corresponding classes lie at distance 4 from each other. We now make the following observations.

(a) Every vertex v of G_Γ^C is contained in exactly 3 maximal cliques of size 4 (which only intersect in the vertex v). This implies the following property for Γ: if there exists a line of Γ intersecting the four classes K_i, K_j, K_m and K_n, then there exist exactly $g = s-1$ lines intersecting the classes K_i, K_j, K_m and K_n.

(b) If p is an isolated point of Γ, then p lies at distance 3 from every non-trivial class, so the point p will determine four cliques in G_Γ^C, each of size 4 (corresponding to the four lines through p). Hence p will determine a partition of the vertices of G_Γ^C into four disjoint maximal cliques.

We will call a line intersecting the four classes K_i, K_j, K_m and K_n, an (i, j, m, n)-line.

Case $g = 2$[1].
Put $K_i = \{a_i, b_i\}$. Without loss of generality, we can assume $a_1 \perp a_2 \perp a_6 \perp a_5$ Because $\delta(K_2, K_5) = 4$, the point a_2 is collinear with b_5. This implies that $b_5 \perp b_1$ and $a_5 \perp a_1$. We can argue similarly for the 'squares' A_{13}, A_{22}, A_{31} and A_{33} (labelling the 'squares' in figure 3.6 as the elements of a 3×3-matrix). Hence we can choose the notation such that we obtain the lines $a_5a_6a_7a_8$, $a_9a_{10}a_{11}a_{12}$, $a_2a_6a_{10}a_{14}$ and $a_3a_7a_{11}a_{15}$. Also, $a_3 \perp a_4$, $a_{13} \perp a_{14}$, $a_{15} \perp a_{16}$, $a_4 \perp a_8$, $a_9 \perp a_{13}$ and $a_{12} \perp a_{16}$. Since $\delta(K_3, K_{10}) = 4$, the point a_3 is collinear with b_{10}. This implies that $a_2 \perp a_3$. But now there is no room

[1] We advise the patient reader to make a large picture similar to Figure 3.6, but with the vertices replaced by classes containing two points.

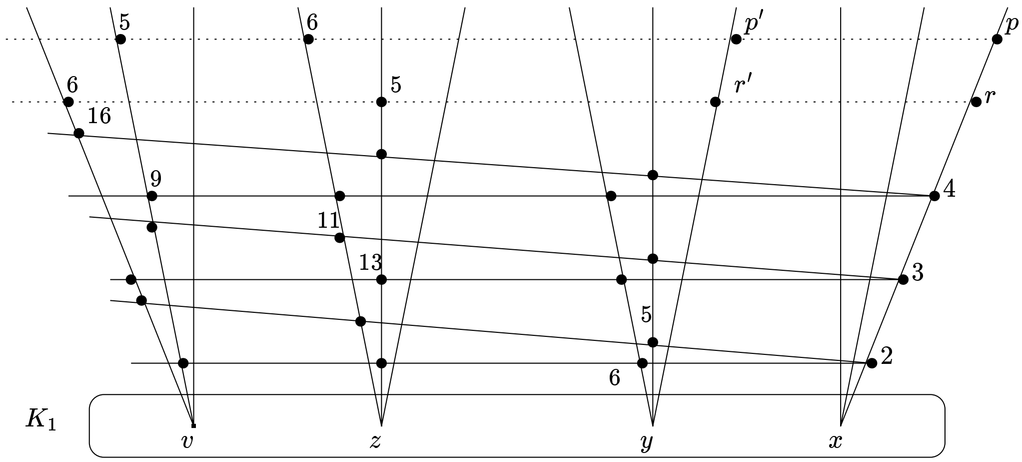


Figure 3.7: Case $g = 4$ in the proof of Theorem 3.6.10

any more for the (1,6,11,16)-line through a_1 (indeed, $a_1 \perp b_6$, $a_1 \perp b_{11}$, hence $b_6 \perp b_{11}$, a contradiction since $\delta(a_6, a_{11}) = 4$).

Case $g = 4$.
Note that through every point of K_1, there is a line of type $(1, 2, 3, 4)$, $(1, 5, 9, 13)$ and $(1, 6, 11, 16)$. Put $K_1 = \{x, y, z, v\}$. For $w \in \{x, y, z, v\}$, let L_H^w (L_V^w, L_D^w) be the (1,2,3,4)-line ((1,5,9,13)-line, (1,6,11,16)-line respectively) through w, and let b_i^w be the point of the class K_i that is collinear with w, for $i \in \{2, 3, 4, 5, 9, 13, 6, 11, 16\}$. The point b_2^x has to be collinear with a point b_5^w of K_5 and a point $b_6^{w'}$ of K_6, $w, w' \in \{y, z, v\}$. Without loss of generality, we can assume $w = y$. We first show that also $w' = y$. Suppose by way of contradiction that $w' \neq y$. The line $b_2^x b_6^{w'}$ has to contain a point r collinear with y. But $r \not\mathrel{I} L_H^y$ since $b_2^x \sim b_2^y$, $r \not\mathrel{I} L_D^y$ since $b_6^{w'} \sim b_6^y$ and $r \not\mathrel{I} L_V^y$ since $b_2^x \perp b_5^y$. Since $k = 3$, there is no room for the point r, a contradiction. Hence we can assume $b_2^x \perp b_6^y$. Note that the projections of z and v on the line $b_2^x b_5^y$ are both isolated, and are incident with the lines L_D^z and L_D^v respectively. Similarly the projections of z and v on the line $b_2^x b_6^y$ are both isolated, and are incident with the lines L_V^z and L_V^v respectively. Now the point b_3^x is collinear with a point b_{11}^w of K_{11} and a point b_{13}^w of K_{13}, $w \in \{y, z, v\}$. We show that $y \neq w$. Suppose by way of contradiction that $w = y$. Then as before, the projections of z (v) onto the lines $b_3^x b_{11}^y$ and $b_3^x b_{13}^y$ are isolated and are incident with L_V^z and L_D^z (L_V^v and L_D^v) respectively. The point b_4^x is collinear with a point b_9^u of K_9, and a point b_{16}^u of K_{16}, $u \in \{y, z, v\}$. The line $b_4^x b_9^u$ has to contain an isolated point incident with one of the lines L_D^z or L_D^v, a contradiction since the four isolated points on these lines are already collinear with b_2^x or b_3^x. Hence we can assume without loss of generality that

$w = z$ (so $b_3^x \perp b_{11}^z$ and $b_3^x \perp b_{13}^z$) and that $b_4^x \perp b_9^v$ and $b_4^x \perp b_{16}^v$. (This is the situation drawn in Figure 3.7. When a point has the label i in this figure, it belongs to the class K_i.) Let p and r (p' and r') be the two isolated points on the line L_H^x (L_H^y). Since through both p and r, there is a line intersecting K_5 and K_6 (by observation (b)), and since p and r lie at distance 4 from the points b_5^y and b_6^y, we can assume that $rr'b_5^z b_6^v$ and $pp'b_5^v b_6^z$ are lines. But now one easily shows that there is no room any more for the line through r intersecting the classes K_9 and K_{11}.

Case $g \neq 2, 4$.
Note that through every point of K_1, there is a line of type $(1,2,3,4)$, $(1,5,9,13)$ and $(1,6,11,16)$. Since $g \neq 2$, the geometry Γ contains isolated points. Let $a_1 \in K_1$ and let r be an isolated point on the (1,2,3,4)-line L through a_1. Each line through r contains $s-3$ isolated points, which necessarily lie at distance 3 from K_1, and hence each line through r different from L contains exactly 2 non-isolated points at distance 3 from K_1. So in total there are 6 non-isolated points collinear with r, at distance 3 from K_1 and not incident with L. These 6 points belong to the classes K_5, K_6, K_9, K_{11}, K_{13} and K_{16} (since according to observation (b), the point r determines the partition $(1,2,3,4), (5,6,7,8)$, $(9,10,11,12)$, $(13,14,15,16)$). Hence we need at least 6 lines concurrent with K_1, not through a_1 and not of type (1,2,3,4), implying that $g \geq 4$. Now we claim that every line through r different from L intersects $g-3$ lines of type $(1,2,3,4)$ different from L. Indeed, consider for example the (5,6,7,8)-line R through r. The points b_5 and b_6 of K_5 and K_6 on R lie at distance 3 from K_1, hence R has to intersect a (1,5,9,13)-line and a (1,6,11,16)-line concurrent with K_1. But R cannot intersect two (1,5,9,13)-lines concurrent with K_1, since this would give rise to a 'triangle'. Similarly R cannot intersect two (1,6,11,16)-lines concurrent with R. Hence every line concurrent with both R and K_1 and not through b_5 or b_6 is necessarily a (1,2,3,4)-line. This shows the claim. So each of the three lines through r different from L intersects $g-3$ lines of type $(1,2,3,4)$ different from L. Since all these lines of type (1,2,3,4) at distance 3 from r are different, $3(g-3) \leq g-1$, implying that $g = 4$. This case was excluded before. $\square$

Application 3.6.11 *Let Γ be a short forgetful quadrangle with $k = 2$. If Γ contains isolated points, then Γ is of subquadrangle type. If Γ does not contain isolated points, then Γ is of ovoid type. In both cases, the corresponding generalized quadrangles are uniquely determined.*

Proof. Let Γ be a short forgetful quadrangle with $k = 2$. Suppose first that Γ contains isolated points. Since every two adjacent vertices of G_Γ

are contained in a clique of size $k+1=3$ (see Lemma 3.6.3(i)), Γ is of subquadrangle type by Lemma 3.6.9. Using the notation of section 3.6.2, the associated generalized quadrangle Δ has order $t=2$. Hence Δ is isomorphic with $\mathsf{H}(3,4)$. Suppose now Γ does not contain isolated points. If $g=k=2$, then Γ is of ovoid type, by Lemma 3.6.6. Again using the notation of section 3.6.2, we see that the associated generalized quadrangle Δ is an $(l+1)\times(l+1)$-grid. We now show that the case $g>2$ leads to a contradiction.

$\boxed{g=3}$

Let G be a fixed class of size 3, put $G=\{r_1,r_2,r_3\}$, and let R_i and R'_i be the two lines through the point r_i, $i=1,2,3$. Let a_1 be a point on R_1 different from r_1, and L the line through a_1 different from R_1. Let A be the class containing a_1. Without loss of generality, we can assume that L intersects the lines R_2 and R_3 in a point of the classes B and C respectively. Put $A=\{a_1,a_2,a_3\}$, $B=\{b_1,b_2,b_3\}$ and $C=\{c_1,c_2,c_3\}$, with $x_i\perp r_i$, $i=1,2,3$ and $x\in\{a,b,c\}$. Note that $b_1,c_1\in R'_1$, $a_2,c_2\in R'_2$ and $b_3,a_3\in R'_3$. Also, since $\delta(A,B)=\delta(A,C)=\delta(B,C)=4$, we obtain $b_1\perp a_2$, $c_1\perp a_3$ and $c_2\perp b_3$. Let d_1 be a point on R_1 different from a_1 and r_1. Then the class $D=\{d_1,d_2,d_3\}$ cannot intersect R'_2. Indeed, if $d_2\in R'_2$, then $d_3\in R'_3$ (using $\delta(A,D)=4$), but then $\delta(B,D)=4$, so $b_2\perp d_1$ or $b_2\perp d_2$, a contradiction. Hence $d_2\in R_2$ and $d_3\in R_3$. Since $\delta(B,D)=\delta(C,D)=4$, we have $d_1\mathtt{I}c_2b_3$, hence d_1 is determined on R_1, implying $|R_1|=3$, a contradiction with $l=g+d\geq 5$.

$\boxed{g=4}$

Similarly to the previous case.

$\boxed{g\geq 5}$

Let $G=\{r_1,r_2,\ldots,r_g\}$ be a class and L a line at distance 4 from G. For $i=1,\ldots,g$, put $a_i=\mathrm{proj}_L r_i$, K_i the class containing a_i, $R_i=a_ir_i$ and R'_i the line through r_i different from R_i. Clearly, K_2 intersects the lines R'_i, $i\neq 2$, K_3 intersects the lines R'_i, $i\neq 3$, and K_4 intersects the lines R'_i, $i\neq 4$. Since K_3 and K_4 both have a point on R'_1, $\delta(K_3,K_4)=4$. Let a'_3 be the point of K_3 on the line R'_4, and a'_4 the point of K_4 on the line R'_3. The line L and the lines R'_i, $i\neq 3,4$, intersect both K_3 and K_4. So we already have $g-1$ lines intersecting K_3 and K_4. Since none of these lines contains the points a'_3 or a'_4, a'_3 and a'_4 have to be collinear. Put $L'=a'_3a'_4$. Clearly, the line L' intersects R_1, say in a point of a class B. Note that $B\neq K_i$, $i=1,\ldots,g$, and that B does not intersect the line L. Since L' intersects the lines R_2 and R_i, $i=5,\ldots,g$, the class B intersects the lines R'_2, R_3, R_4 and R'_i, $i=5,\ldots,g$. Now $\delta(B,K_2)=4$, since these classes have collinear points on the line R'_5. This is a contradiction, since the point a_2 is not collinear with any point of B (indeed, the line L through a_2 does not intersect R'_i, $i=1,\ldots,g$). □

3.6.4 Short forgetful quadrangles ‘arising’ from generalized quadrangles.

We say that a forgetful quadrangle Γ **arises from a generalized quadrangle** Δ if the points of Γ are points of Δ, the lines of Γ are (parts of) lines of Δ, and each class of Γ is a subset of the point set of a line of Δ. Note that all examples of forgetful quadrangles, except possibly from the short forgetful quadrangles of ovoid type, arise in this way.

Let Δ be a finite generalized quadrangle of order (s,t), and W a set of points in Δ satisfying the following conditions.

(W1) There exist constants g and l, $g \geq 2$, $g+2 \leq l \leq s+1$, such that every line of Δ intersects W in either $s+1$, s, $s+1-g$ or $s+1-l$ points (these lines are called respectively W-, T-, C-, and F-lines).

(W2) If a point $w \in W$ lies on an F-line L, then every line through w different from L is a W-line.

(W3) There exists a constant k, $2 \leq k \leq t$ such that any point p not belonging to W either lies on exactly k F-lines and one C-line, or on exactly $(k+1)$ F-lines and no C-line (so each point not contained in W lies on exactly $(t-k)$ T-lines).

Then we construct a short forgetful quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I}, \sim)$ in the following way. The points of Γ are the points of Δ not belonging to W. The lines of Γ are the F-lines. Incidence is the incidence of Δ. Two points of Γ are equivalent if they lie on a C-line. Clearly, Γ is a short forgetful quadrangle with parameters $(g,k,l-g)$. Also, every short forgetful quadrangle arising from a generalized quadrangle can be constructed in this way. We give two examples of this construction.

1. Let Δ be the unique finite generalized quadrangle of order (4,2). Let L_1, L_2 be two opposite lines of Δ, and $S = \{L'_0, \ldots, L'_4\}$ the five lines concurrent with both L_1 and L_2. Let W be the set of points lying on $L_1 \cup L_2 \cup L'_0 \cup \ldots \cup L'_4$. Then every line of Δ not contained in W intersects W in 1 or 3 points. Indeed, a line concurrent with L_1 or L_2 different from L'_i, $i = 0, \ldots, 4$ intersects W in 1 point. A line opposite both L_1 and L_2 and intersecting at least 2 lines of S, intersects exactly three lines of S, since every triad of lines in Δ has exactly 3 centers (see Theorem 1.2.5(i)). From this, it easily follows that every line opposite both L_1 and L_2 intersects W in exactly 3 points. Now it is readily

seen that the set W satisfies properties (W1), (W2) and (W3). The corresponding short forgetful quadrangle Γ has $k = 2$ and does not contain isolated points, hence is of ovoid type and already classified in application 3.6.11 (Γ arises from the 5×5-grid).

2. Let Δ be a generalized quadrangle of order (q, q), and Δ' a subquadrangle of Δ of order $(1, q)$. Then $\Delta \setminus \Delta'$ defines a short forgetful quadrangle Γ of subquadrangle type. Let Δ_1 be a generalized quadrangle of order (q^2, q) containing Δ, and Δ_2 a generalized quadrangle of order (q^2, q^3) containing Δ_1. Then the set $W = (\Delta_2 \setminus \Delta) \cup \Delta'$ in Δ_2 satisfies conditions (W1), (W2) and (W3): the C-lines are the lines of Δ intersecting Δ', the F-lines are the lines of Δ not intersecting Δ', the T-lines are the lines of Δ_2 intersecting $\Delta \setminus \Delta'$ in a unique point. Note that also the set $W' = (\Delta_1 \setminus \Delta) \cup \Delta'$ satisfies properties (W1), (W2) and (W3) (and gives rise to the same forgetful quadrangle as the set W).

Classifying all short forgetful quadrangles arising from a generalized quadrangle boils down to classifying all sets W satisfying the conditions (W1), (W2) and (W3). The following results can easily be obtained:

- If $l = s + 1$ and no T-lines exist, then Γ is of subquadrangle type.
- If $l = s$, then $k = 2$ hence Γ is classified in 3.6.11.

Classifying all sets W seems to be quite difficult. Note that example 2 above shows that the set W can be 'large' compared with the short forgetful quadrangle obtained from it. Also, different sets W can give rise to the same short forgetful quadrangle. So restricting the classification of all short forgetful quadrangles to the ones arising from a generalized quadrangle does not seem to make the question easier.

Do the short forgetful quadrangles of ovoid type (or at least the 'classical examples') arise from generalized quadrangles ?

Chapter 4

Distance-preserving maps

4.1 Introduction

Any isomorphism between two generalized n-gons preserves all distances and conversely, every bijective map between two generalized n-gons preserving *all* distances, defines an isomorphism. The aim of the present chapter is to weaken that condition. The inspiration for this problem came from the the theorem of Beckman and Quarles (see for instance [3]) stating that a permutation of the point set of a Euclidean real space preserves distance i between points (for some positive real number i) if and only if it preserves all distances.

Let us be a bit more precise. In fact, we consider two versions of the problem. In the first version, we look at surjective maps between the point sets of two generalized n-gons if i is even, and between the point sets and the line sets of two generalized n-gons if i is odd, preserving distance i. With 'preserving distance i', we mean that two elements are at distance i *if and only if* their respective images are at distance i. The question now reads: does this map extend to an isomorphism? We show that the answer to the question is 'yes' if i is not equal to the maximal distance. In case i is the maximal distance, a counterexample arises for the split Cayley hexagon. Two natural problems turn up: do only hexagons give rise to counterexamples - and is the split

Cayley hexagon the only hexagon giving rise to a counterexample? If one restricts to the finite case, we prove that counterexamples can only occur for hexagons of order (s, s). In the infinite case, if there is enough transitivity around, the only counterexample-hexagon is the one described for the classical hexagon. We use the arguments of these proofs in two applications. First, we determine the intersection of the line sets of two classical generalized hexagons living on the same quadric in 6-space. Secondly, we prove the (well-known) maximality of the group $G_2(q)$ in $O_7(q)$ in an entirely geometric way.

In the second version of the problem, we consider maps between the flags of two generalized n-gons, and ask the same question. For this problem, the answer is 'yes' up to one counterexample arising in the smallest generalized quadrangle. A variation on this problem can be obtained by asking that a certain Coxeter distance between the flags is preserved. Here, the counterexample has to give up and we obtain a 'yes' in all cases.

We conclude with some words about the proofs of the main theorems of this chapter. We are given a map preserving a certain distance i. In fact, this means that we look at our polygon with a pair of glasses that only allows us to see whether two elements lie at distance i or not. The aim is to find a property that distinguishes the collinear points among all the other pairs of points, and that one can see with such a pair of glasses. However, the quote 'Mathematics is as love, the idea is easy but it can become difficult' really applies to this chapter. Indeed, the proofs are quite technical, since they are for general n. We therefore advise the reader to keep an appropriate n in mind when going through them (and to make a lot of pictures...). To make some arguments a bit more explicit, a few specific examples are included before the proofs.

The results of this chapter are contained in Govaert & Van Maldeghem [28] and [29].

4.2 Main theorems and some words about the proof

Theorem 4.2.1 (Point-Line Theorem)

- *Let Γ and Γ' be two generalized n-gons, $n \geq 4$, let i be an even integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ'. If for every two points a, b of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends (in a unique way) to an isomorphism from Γ to Γ'.*

- *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an odd integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ', and from the line set of Γ onto the line set of Γ'. If for every point-line pair $\{a, b\}$ of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α defines an isomorphism from Γ to Γ'.*

Theorem 4.2.2 (Flag Theorem) *Let Δ and Δ' be two generalized m-gons, $m \geq 2$, let r be an integer satisfying $1 \leq r \leq m$, and let α be a surjective map from the set of flags of Δ onto the set of flags of Δ'. If for every two flags f, g of Δ, we have $\delta(f, g) = r$ if and only if $\delta(f^\alpha, g^\alpha) = r$, then α extends to an (anti)isomorphism from Δ to Δ', except possibly when Δ and Δ' are both isomorphic to the unique generalized quadrangle of order $(2, 2)$ and $r = 3$.*

Theorem 4.2.3 (Special Flag Theorem) *Let (W, S) be the Coxeter system associated with the dihedral group $W = D_{2m}$ of order $2m$. Let Δ and Δ' be two generalized m-gons, $m \geq 2$, let w be a non-trivial element of $W \setminus S$, and let α be a surjective map from the set of flags of Δ onto the set of flags of Δ'. Denote by δ^* the Coxeter distance between flags in both Δ and Δ'. If for every two flags f, g of Δ, we have $\delta^*(f, g) = w$ if and only if $\delta^*(f^\alpha, g^\alpha) = w$, then α extends to an (anti)isomorphism from Δ to Δ'. If moreover, the length of w is not maximal in W, then α extends to an isomorphism from Δ to Δ'.*

If $i = 1$ in Theorem 4.2.1, then the result is obvious from the definition of isomorphism. The case $i = 2$ is exactly Lemma 1.3.14 in Van Maldeghem [57].

The case $r = 1$ in Theorem 4.2.2 can be found in Tits [50], Theorem 3.21, and the case $r = m$ was proved in Abramenko & Van Maldeghem [1], Corollary 5.2 (in the latter paper, the case $r = m$ is proved for spherical and twin buildings). Also, Abramenko & Van Maldeghem recently proved the analogue of Theorem 4.2.3 for buildings.

Point-Line Theorem

We first show that the map α from Γ to Γ' is necessarily a bijection. Let a and b be two points of Γ for which $a^\alpha = b^\alpha$, and suppose α preserves distance i (note that we can assume $i \geq 3$). Since any element of Γ' at distance i from a^α is also at distance i from b^α, the set of elements of Γ at distance i from a coincides with the set of elements at distance i from b. This is easily seen to be a contradiction. Indeed, let γ be a fixed minimal path joining a and b. If $\delta(a, b) \neq 2i$, let c be an element at distance i from a such that c belongs to γ if $i \leq \delta(a, b)$, or such that the path $[a, c]$ contains γ if $i > \delta(a, b)$.

Then $\delta(b,c) \neq i$. If $k = 2i$, let m be the element of γ at distance i from a, $m' = \mathrm{proj}_m a$, and c an element incident with m', $m \neq c \neq \mathrm{proj}_{m'} a$. Then also $\delta(a,c) = i$ but $\delta(b,c) \neq i$.

Now clearly, it is enough to prove that α preserves collinearity. The result will then follow from Van Maldeghem [57] Lemma 1.3.14.

This will be done as follows. If α preserves a certain distance i, then we look for a property P that characterizes distance 2 in terms of distance i and 'not distance i'. So two points a and b of Γ lie at distance 2 if and only if $P(a,b)$ is satisfied. If the same characterization of collinearity holds in the polygon Γ', then we are sure that also the images a^α and b^α lie at distance 2. Hence α preserves collinearity, and we are done.

<u>Flag Theorem</u>

Let Γ and Γ' be the doubles of Δ and Δ', respectively. Put $n = 2m$. Then Γ and Γ' are generalized n-gons, $n \geq 6$, with thin points (corresponding to the flags of Γ and Γ') and with thick lines (corresponding to the points an lines of Γ and Γ'). Put $2r = i$. The map α induces a surjective map (which we may also denote by α) from the point set of Γ to the point set of Γ' preserving distance i. As in the previous case, one shows that α is a bijection. So in section 4.4.6, we in fact prove the following:

If α is a bijection from the point set of Γ to the point set of Γ' such that for every two points a, b of Γ, we have $\delta(a,b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α preserves collinearity, except possibly when Γ and Γ' are isomorphic to the unique generalized octagon of order $(2,1)$ and $i = 6$.

The result will then follow from Theorem 3.21 in Tits [50].

In this way, we reduced Theorem 4.2.2 to a particular case of Theorem 4.2.1 for weak polygons with thin points and thick lines. We will not gain so much by doing that, because a separate proof remains necessary. But the intuition is easier.

<u>Special Flag Theorem</u>

As above, we again consider the doubles Γ, Γ' and the associated map α from Γ to Γ'. Let i' be the length of the element w of W, and $i = 2i'$ (note that $i' \neq 1$ since $w \notin S$). If i' is even, then two flags of Δ lie at Coxeter distance w if and only if they lie at distance i', hence in this case, we are back to the situation of Theorem 4.2.2. Also if $i' = m$, it is clear that Theorem 4.2.3 adds nothing new. Suppose i' is odd, $i' \neq m$. Then, for two points a and b lying at distance i in Γ, either both $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$ correspond to points of Δ, or both $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$ correspond to lines of Δ. In the former case,

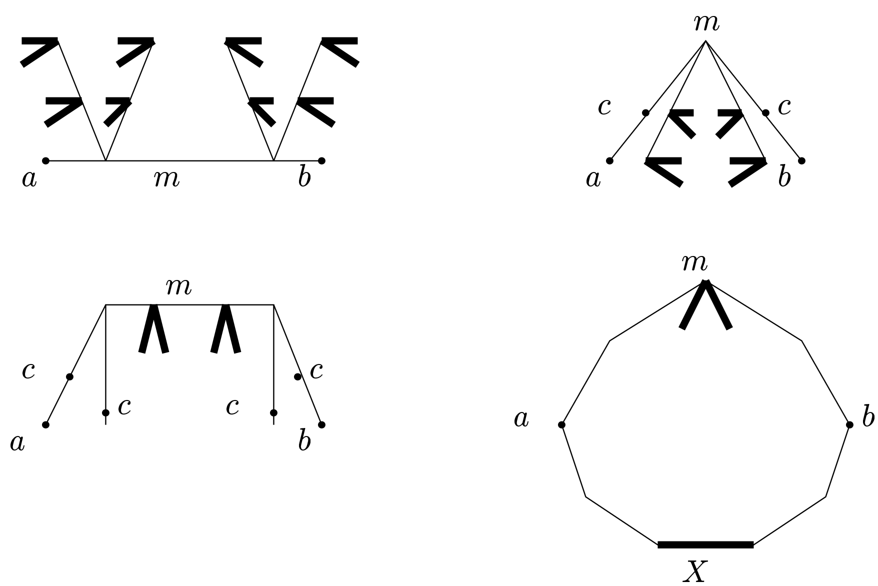


Figure 4.1: Example 1

we say that $\delta(a,b) = i_p$, in the latter that $\delta(a,b) = i_L$. So we know that α preserves either distance i_p or distance i_L. Without loss of generality, we assume that α preserves distance i_p. So in section 4.7.4, we will in fact prove the following result:

If α is a bijection from the point set of Γ to the point set of Γ' such that for every two points a, b of Γ, we have $\delta(a,b) = i_p$ if and only if $\delta(a^\alpha, b^\alpha) = i_p$, $2 < i < n$, $i \equiv 2 \bmod 4$, then α preserves collinearity.

From Theorem 3.21 in Tits [50] then follows that Δ and Δ' are (anti)isomorphic. But for $2 < i < n$, $i \equiv 2 \bmod 4$, the fact that distance i_p is preserved, contradicts Δ and Δ' being anti-isomorphic. Hence in this case, Δ and Δ' are necessarily isomorphic.

4.3 Some examples by pictures

Example 1: Point-Line Theorem, $n = 9$, $i = 5$

For two points a and b of Γ, we first consider the set of lines $T_{a,b} = \Gamma_5(a) \cap \Gamma_5(b)$. Put $k = \delta(a,b)$ and $m = a \bowtie b$. From Figure 4.1, one sees that for any line L of $T_{a,b}$, the paths $[a, L]$ and $[b, L]$ both contain m, except if $k = 8$.

Now let S be the set of pairs of points (a,b) for which there exists a point c different from a and b satisfying $T_{a,b} \subseteq \Gamma_5(c)$. If $\delta(a,b) = 2$, then clearly such a point c cannot exist. If $\delta(a,b) \in \{4,6\}$, the possibilities for the point c are indicated in Figure 4.1. Note that these are the only possible positions for the point c. For example, if $\delta(a,b)=4$, it is easy to see that the point c has to lie at distance 3 from any line through m different from am or bm. Hence c has to be collinear with m and lies on am or bm. Similarly if $\delta(a,b)=6$. Finally if $\delta(a,b)=8$, the point c cannot exist. Indeed, c would have to lie at distance 5 every line through m different from the projections of a and b onto m. Since there are at least two such lines (noting that both s and t are infinite in this case), c lies at distance 4 from m and either $\mathrm{proj}_m a = \mathrm{proj}_m c$ or $\mathrm{proj}_m b = \mathrm{proj}_m c$. But now it is easy to see that c cannot lie at distance 5 from the line X (see picture). So the set S contains exactly the pairs of points at mutual distance 4 or 6.

Next, let S' be the set of pairs of points (a,c) for which there exists a point b satisfying $(a,b) \in S$ and $T_{a,b} \subseteq \Gamma_5(c)$ (so in fact, we collect all the pairs (a,c) and (b,c) indicated in Figure 4.1). Clearly, the set S' contains exactly the pairs of points at mutual distance 2, 4 or 6. Now $S' \setminus S$ is exactly the set of pairs of collinear points. This example corresponds with Case 4.4.3 in the proof of Theorem 4.2.1.

Example 2: Point-Line Theorem, $n=9$, $i=8$

Let a and b be two points at distance $k \neq 8$. We again consider the set $T_{a,b} = \Gamma_8(a) \cap \Gamma_8(b)$. For the cases $k=2,4$, all possible positions of a point belonging to $T_{a,b}$ are indicated in Figure 4.2. For $k=6$, a subset of $T_{a,b}$ is indicated. Now we look for the pairs of points (c,c'), c and c' distinct from a and b such that

$$T_{v,v'} \subseteq \Gamma_8(w) \cup \Gamma_8(w'), \text{ whenever } \{v,v',w,w'\} = \{a,b,c,c'\}.$$

It is easy to see that if $k=2,4$, the possibilities (c_1,c_1') and (c_2,c_2') indicated on the picture indeed satisfy the condition above. In the proof of Theorem 4.2.1, Case 4.4.5, we will show that these are the only possibilities for the pair (c,c') if $k=2,4$, and that if $k=6$ and a pair (c,c') with the above properties exists, the points c and c' are necessarily as in the picture.

Now for two points a and b at distance 2,4 or 6, we collect all the points c and c' as above in a set $C_{a,b}$. If $k=2$ or 4, it is impossible to find a point x at distance 8 from all the points of $C_{a,b} \cup \{a,b\}$. If $k=6$, one can find such a point (see for instance the point x on the picture). So we can characterize 'being at distance 2 or 4'. Now if $\delta(a,b)=4$, all points of $C_{a,b}$ lie at mutual distance 2 or 4, which is not true for $\delta(a,b)=2$. Hence we can distinguish distance 2. This example corresponds with Case 4.4.5 in the proof of Theorem 4.2.1.

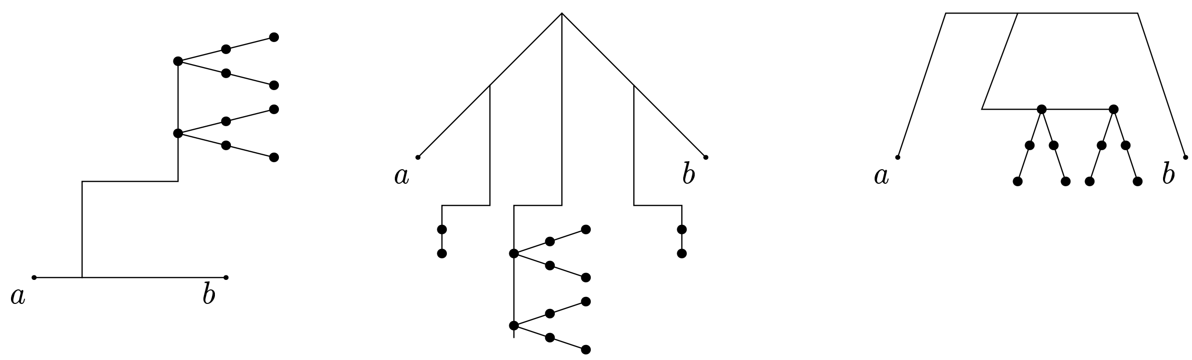


Figure 4.2: Example 2: the set $T_{a,b}$

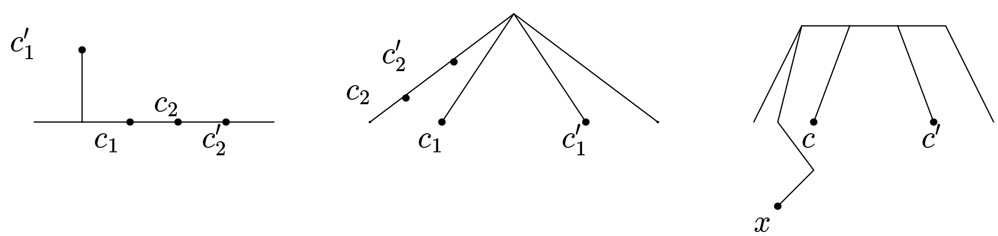


Figure 4.3: Example 2: the set $C_{a,b}$

$\boxed{\text{Example 3: Flag Theorem, } n = 24,\ i = 16}$

We first look for the pairs (a, b), a, b points of Γ, for which the set $T_{a,b} := \Gamma_i(a) \cap \Gamma_i(b)$ is empty. By Figure 4.4, it is intuitively clear that this will be the case if and only if $\delta(a, b) \in \{4, 8, 12\}$. Indeed, if $\delta(a, b) = k \equiv\ 2 \bmod 4$, one can find a point $c \in T_{a,b}$ at distance $16 - k/2$ from a line at distance $k/2$ from both a and b. If $\delta(a, b) = k \equiv\ 0 \bmod 4$ and $c \in T_{a,b}$, a circuit of length at most $k + 2i$ (≥ 48) arises (noting that the paths $[a, c]$ and $[b, c]$ cannot meet in an element 'in the middle' of a and b, since points are thin), hence $k \geq 16$. So we can characterize the set of distances $\kappa = \{4, 8, 12\}$.

Next, for two points a and b lying at a distance contained in κ, we consider the set $R_{a,b} = \Gamma_\kappa(a) \cap \Gamma_\kappa(b)$. For two points at distance 12, which is the biggest distance in κ, a point of $R_{a,b}$ will necessarily lie on the path $[a, b]$, while this is not the case for smaller κ-distances (see Figure 4.5). We will then characterize distance 12 as the κ-distance for which $|R_{a,b}|$ is minimal.

Finally, we show that two points a and b are opposite if and only if the set $\Gamma_{12}(a) \cap \Gamma_{12}(b)$ contains exactly two points c, d, and moreover, $\Gamma_{12}(c) \cap \Gamma_{12}(d) = \{a, b\}$. Hence we recovered opposition, and the theorem follows from Abramenko & Van Maldeghem [1]. This example corresponds with

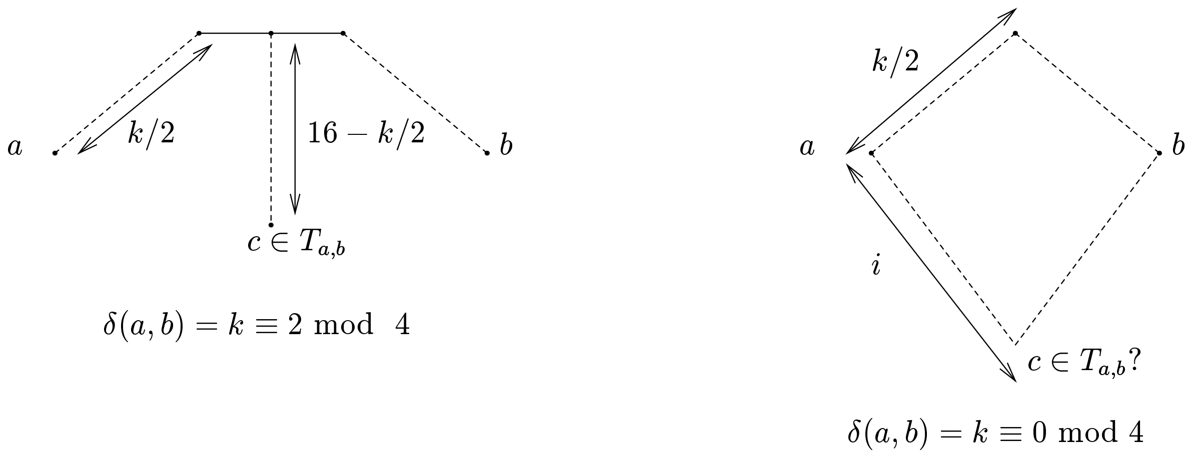


Figure 4.4: Example 3: the set $T_{a,b}$

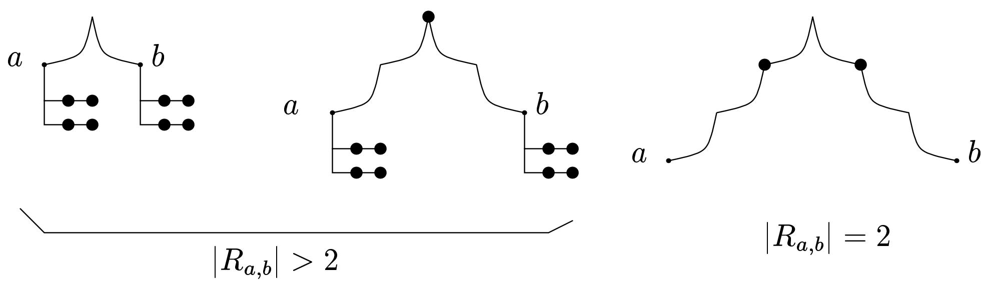


Figure 4.5: Example 3: the set $R_{a,b}$

cases 4.7.2 and 4.7.3 in the proof of Theorem 4.2.2.

4.4 Proof of the Point-Line Theorem

Throughout, we put $T_{a,b} := \Gamma_i(a) \cap \Gamma_i(b)$, for points a, b of Γ.

4.4.1 Case $i < \frac{n}{2}$

Let a and b be two points at distance k, and γ a fixed k-path joining a and b. Denote by m the element of γ at distance $\frac{k}{2}$ from both a and b.

Claim 1. *The set $T_{a,b}$ is empty if and only if $k > 2i$.*
Proof. Suppose first $\delta(a,b) > 2i$ and let by way of contradiction x be an element of $T_{a,b}$. Note that x cannot lie on γ, nor do the paths $[a,x]$ and $[b,x]$ meet on γ. Hence there arises a circuit of length at most $k + 2i < 2n$ (determined by γ and the paths $[a,x]$ and $[x,b]$), a contradiction. Suppose

now $\delta(a,b) \leq 2i$. Then any element x at distance $i - \frac{k}{2}$ from m for which $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$ belongs to $T_{a,b}$ (note that such an element exists because Γ is thick). $\diamond$

Claim 2. *Suppose $k = 2$. Then there does not exist a point c, $c \neq a, b$ at distance i from every element of $T_{a,b}$.*
Proof. Suppose c is a point for which $T_{a,b} \subseteq \Gamma_i(c)$. Let $x \in T_{a,b}$, and put $[a,x] = (a = x_0, x_1, x_2, \ldots, x_i = x)$. Note that necessarily $x_1 = ab$ and $x_2 \neq b$. Let y be an element incident with x_{i-1} different from x_{i-2} and x. Then $y \in T_{a,b}$. Since $\delta(c,x) = i = \delta(c,y)$, necessarily $\delta(c, x_{i-1}) = i-1$. Proceeding like this we obtain $\delta(c, x_j) = j$, for all j, hence $c \in \{a,b\}$. $\diamond$

Claim 3. *Suppose $2 < k \leq 2i$. Then there exists a point c, $c \neq a, b$ at distance i from every element of $T_{a,b}$.*
Proof. Let $x \in T_{a,b}$ and define v as $[a,v] = [a,b] \cap [a,x]$. Let $j = \delta(a,v)$. Combining $[b,v]$ and $[v,x]$, we obtain a path of length $\ell = k + i - 2j$ between b and x. If $\ell > n$, then by combining this path with the i-path joining b and x, a circuit of length at most $k + 2i - 2j < 2n$ arises, a contradiction. Hence $\ell \leq n$, so $\ell = \delta(b,x) = i$. This implies $j = k/2$, hence every element x of $T_{a,b}$ lies at distance $i - \frac{k}{2}$ from m, with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. Now clearly, a point c on the line $N := \mathrm{proj}_a b$ different from a and from the projection of b onto N satisfies $T_{a,b} \subseteq \Gamma_i(c)$. $\diamond$

Let S be the set of pairs of points (a,b) for which $T_{a,b} \neq \emptyset$, and such that there does not exist a point c different from a and b satisfying $T_{a,b} \subseteq \Gamma_i(c)$. Then by the claims above, S is the set of pairs of collinear points. This ends the proof of the case $i < \frac{n}{2}$.

4.4.2 Case $i = \frac{n}{2} + 1$, i even

Since the case $n = 6$ and $i = 4$ is considered in 4.4.4, we can assume $n > 6$. Let S be the set of pairs of distinct points (a,b) such that $\delta(a,b) \neq i$ and the set $T_{a,b}$ contains at least two points at distance i from each other. We claim that a pair (a,b) belongs to S if and only if $\delta(a,b) < i$. Suppose first that $0 \neq \delta(a,b) = k$, $k < i$ and put $m = a \bowtie b$. Consider a point c at distance $i - k/2$ from m such that $\mathrm{proj}_m a \neq w := \mathrm{proj}_m c \neq \mathrm{proj}_m b$ (note that $\delta(c,w) = i - k/2 - 1 > 0$). Let v be the element of $[c,w]$ at distance $i/2$ from c (such an element exists since $i/2 \leq i - k/2 - 1$). Consider a point c' at distance $i/2$ from v such that $\mathrm{proj}_v m \neq \mathrm{proj}_v c' \neq \mathrm{proj}_v c$. The points c and c' are both points of $T_{a,b}$ and lie at distance i from each other.

Now let $\delta(a,b) = k > i$ and suppose by way of contradiction that $c, c' \in T_{a,b}$ with $\delta(c,c') = i$. If $\mathrm{proj}_a c \neq \mathrm{proj}_a c'$, then we have a path of length $2i$ between

c and c' containing a. This implies that $\delta(c,c') \geq 2n-2i > i$, a contradiction. Suppose now that $\mathrm{proj}_a c = \mathrm{proj}_a c'$. Define v as $[a,c] \cap [a,c'] = [a,v]$. If we put $\delta(a,v) = j$, then there is a path of length $\ell = 2i - 2j \leq n$ between c and c', hence $\ell = \delta(c,c')$. Now $\ell = i$ implies $j = i/2$, hence $v = a \bowtie c = a \bowtie c'$. Similarly, $b \bowtie c = b \bowtie c' =: v'$. Now there arises a circuit of length at most $2i < 2n$ (determined by the paths $[v,c]$, $[c,v']$, $[v',c']$ and $[c',v]$) unless $v = v'$. But this implies there arises a path between a and b of length at most i, the final contradiction. Our claim is proved.

Put $\kappa = \{2, \ldots, i-2\}$. Then $(a,b) \in S$ if and only if $\delta(a,b) \in \kappa$.

We claim that two distinct points a and b are collinear if and only if $\delta(a,b) \in \kappa$ and $\Gamma_\kappa(a) \subseteq \Gamma_\kappa(b) \cup \Gamma_i(b)$. Indeed, if $\delta(a,b) = 2$ and x is a point at distance $j \in \kappa$ from a, then $\delta(b,x) \in \{j-2, j, j+2\} \subseteq \kappa \cup \{i\}$. Now suppose $\delta(a,b) = k$, $2 < k < i$. Then $k = i - j$, $0 < j < i-2$. Consider a point c at distance $j+2$ from a such that $\mathrm{proj}_a c \neq \mathrm{proj}_a b$. Then $\delta(a,c) \in \kappa$, but $\delta(b,c) = i+2 \notin \kappa \cup \{i\}$. This shows the claim. Hence we can distinguish distance 2 and so α preserves collinearity of points.

4.4.3 Case $\frac{n}{2} < i < n-2$

Let a and b be points of Γ at distance k, and m an element at distance $k/2$ from both a and b. Note that $T_{a,b}$ is never empty. For the case $i = \frac{n}{2}+1$, i even and $k = n$, we assume that $|\Gamma_1(m)| \geq 4$ (so $s \geq 3$). This can be done, since this case was in fact already handled without assumptions on the order in the previous paragraph[1]. We first prove the following claims. For later purposes, we remark that these claims also hold for the case $i \in \{n-2, n-1, n\}$, with a similar proof[2].

Claim 1. *Every element y of $T_{a,b}$ lies at distance $i - k/2$ from m with* $\mathrm{proj}_m a \neq \mathrm{proj}_m y \neq \mathrm{proj}_m b$ *if and only if $k < 2(n-i)$.*

Proof. Suppose first that $2(n-i) \leq k$. Consider an apartment Σ through a and b containing m, and let m' be the element of Σ opposite m. Then any element y at distance $i - (n - \frac{k}{2})$ from m' with $\mathrm{proj}_{m'} a \neq \mathrm{proj}_{m'} y \neq \mathrm{proj}_{m'} b$ belongs to $T_{a,b}$ and $\mathrm{proj}_a y \neq \mathrm{proj}_a m$ (hence y is not as in the claim above). Now suppose $k < 2(n-i)$ and let $y \in T_{a,b}$. Let j be the length of the path $[a,b] \cap [a,y]$. Then there is a path of length $\ell = k - 2j + i$ between b and

[1]The reason we want to include this case in the proof for $\frac{n}{2} < i < n-2$ is that in this way, we can characterize distance 2 by the same property for all i, $\frac{n}{2} < i < n-2$ if $s, t \geq 3$. This will be useful in the proof of Application 4.4.1

[2]The only difference is that in the proof of Claim 2 for the case $i = n-1$, one considers the set $T' \cup \{x \in T_{a,b} \mid \delta(x,m) = i - k/2 + 2$ and $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b\}$.

y. If $\ell \leq n$, then $\ell = i$ and so necessarily $j = k/2$. Hence y is an element as claimed. If $\ell > n$, then $\delta(b, y) \geq 2n - \ell > i$, a contradiction. $\diamond$

Claim 2. Suppose there exists a point c at distance i from every element of $T_{a,b}$. Then c lies at distance $k/2$ from m, and $\mathrm{proj}_m a = \mathrm{proj}_m c$ *or* $\mathrm{proj}_m b = \mathrm{proj}_m c$.

Proof. Suppose c is a point at distance i from every element of $T_{a,b}$. Consider the set

$$T' = \{x \in T_{a,b} \mid \delta(x, m) = i - k/2 \text{ and } \mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b\}.$$

Then we may assume that T' contains at least two elements y and y' at distance 2 from each other. Indeed, this is clear if $i - \frac{k}{2} > 1$ or if $|\Gamma_1(m)| \geq 4$. If $i = \frac{k}{2} + 1$, then either $i = \frac{n+1}{2}$ (then necessarily n odd, hence $s, t \geq 3$), or $i = \frac{n}{2} + 1$ (thus $k = n$). In the latter case, if $|\Gamma_1(m)| = 3$, i is odd by assumption (thus $t = 2$), and we can apply the dual reasoning. (This is, we consider exactly the same arguments as given here, but with a and b lines instead of points. Note that this is allowed since by the fact that i is odd, α is also defined on the line set - see also the last paragraph of 4.4.3.)

Put $w = y \bowtie y'$. Then $\delta(c, w) = i - 1$ (noting that $i \neq n - 1$). Put $\gamma = [c, w]$. We show that γ contains m. Suppose by way of contradiction that this is not true. Define the element z as $[w, c] \cap [w, m] = [w, z]$. Put $\gamma' = [z, c]$. An element y'' of T' either lying on γ' (if $\delta(c, m) \geq i - k/2$) or such that the path $[y'', m]$ contains γ' (otherwise), clearly does not lie at distance i from c, a contradiction. So the point c has to lie at distance $k/2$ from m. But if $\mathrm{proj}_m a \neq \mathrm{proj}_m c \neq \mathrm{proj}_m b$, then similarly we can find an element of T' not at distance i from c. The assertion follows. $\diamond$

Claim 3. Suppose $k = 2$ or $k \geq 2(n - i)$. Then there does not exist a point c, $c \neq a, b$ at distance i from every element of $T_{a,b}$.

Proof. If $k = 2$, then this follows immediately from Claim 2. So suppose $k \geq 2(n - i)$. We first show the following property for an element v of the path $[a, \mathrm{proj}_m a]$.

(*) There exists $y \in T_{a,b}$ such that $[a, y] \cap [a, m] = [a, v]$ if and only if $\delta(a, v) \leq j := i - n + k/2$.

Indeed, let v be an element of the path $[a, \mathrm{proj}_m a]$, put $j' = \delta(a, v)$. If there is an element $y \in T_{a,b}$ such that $[a, y] \cap [a, m] = [a, v]$, then there arises a circuit of length at most $k + 2i - 2j'$, so necessarily, $j' \leq j$. Suppose now $j' \leq j$. Let Σ be an apartment through v and b containing m, but not the element $\mathrm{proj}_v a$, and let m' be the element of Σ at distance $n + j' - k/2$ from b for which $\mathrm{proj}_b m' \neq \mathrm{proj}_b m$. Let finally y be an element at distance $i - (n + j' - k/2)$ from m' such that $\mathrm{proj}_{m'} b \neq \mathrm{proj}_{m'} y \neq \mathrm{proj}_{m'} v$. Then y is as in (*).

Now suppose c is a point at distance i from every element of $T_{a,b}$. We may assume that, if $\delta(m, c) \neq n$, then $\mathrm{proj}_m a = \mathrm{proj}_m c$. If $\delta(a, c) \neq n$, then we define the element z as $[m, c] \cap [m, a] = [m, z]$; otherwise we define z as $[\mathrm{proj}_m a, c] \cap [\mathrm{proj}_m a, a] = [\mathrm{proj}_m a, z]$. Note that by Claim 2, $\delta(c, z) = \delta(a, z) =: \ell$.

Suppose first $\ell > j$, and consider an element y of $T_{a,b}$ such that $[a, y] \cap [a, m] = [a, v]$, with $\delta(a, v) = j$. Then we obtain a path of length $d = i + 2l - 2j$ between y and c. Clearly, $d \neq i$. But if $d > n$, $\delta(y, c) = i$ implies there arises a circuit of length at most $2(i + l - j) < 2n$, a contradiction. Similarly, if $\ell \leq j$, then an element $y \in T_{a,b}$ such that $\mathrm{proj}_z c \in [a, y]$, does not lie at distance i from c, the final contradiction. $\diamond$

Now let S be the set of pairs of points (a, b) for which there exists a point c, $a \neq c \neq b$, such that $T_{a,b} \subseteq \Gamma_i(c)$. Then a pair (a, b) with $\delta(a, b) = k$ belongs to S if and only if $2 < k < 2(n - i)$ (note that there are always even numbers k satisfying these inequalities because $i < n - 2$). Indeed, if $k = 2$ or $k \geq 2(n - i)$, this is Claim 3. If $2 < k < 2(n - i)$ then by Claim 1 it suffices to consider a point c at distance $k/2$ from m with $\mathrm{proj}_m c \in \{\mathrm{proj}_m a, \mathrm{proj}_m b\}$.

Define $S' = \{(a, c) \mid \exists b \in \mathcal{P}$ such that $T_{a,b} \subseteq \Gamma_i(c)\}$. Suppose $(a, c) \in S'$, and b is such that $T_{a,b} \subseteq \Gamma_i(c)$. Then $(a, b) \in S$ and by Claim 2, $\delta(a, c) \leq \delta(a, b)$. Hence S' is exactly the set of pairs of points at distance $< 2(n - i)$ from each other. Now clearly $S' \setminus S$ is precisely the set of all pairs of collinear points.

So we found a property characterizing distance 2 between points, which is independent of the order of Γ except for the case $i = \frac{n}{2} + 1$ and i odd. Consider this case. There, if $t \geq 3$, there was no problem, but if $t = 2$, the property we found was exactly the dual of the original one, and hence is valid for all orders of Γ except for $s = 2$. Now we still have to distinguish the cases $s = 2$ and $t = 2$ (indeed, it might be possible that the order of Γ is $(2, t)$, while the order of Γ' is $(s', 2)$). This is done as follows. Define the set S' as above. If $s = 2$, $t > 2$, hence S' is exactly the set of pairs of collinear points. If $t = 2$, it is easy to see that all the pairs of collinear points are included in S'. Now consider the set S'' of pairs (a, b) of S' for which there exist at least two points x, x' such that (a, x), (b, x), (a, x') and (b, x') belong to S'. If $s = 2$, then S'' is empty. If $t = 2$, then $s \geq 3$, hence S'' is nonempty (considering points a, b, x, x' on the same line). This distinguishes the cases $s = 2$ and $t = 2$. The theorem is now proved for this case.

4.4.4 Case $i = n - 2$

Case $n = 6$

Let C be the set of pairs of points (a, b), $\delta(a, b) \neq 4$, such that for every point y in $T_{a,b}$, there exists a point y' in $T_{a,b}$, $y' \neq y$ and $\delta(y, y') \neq 4$, with the property that $\Gamma_4(y) \cap T_{a,b} = \Gamma_4(y') \cap T_{a,b}$ (**). We show that C is the set of pairs of collinear points.

If a and b are collinear points, a point y of $T_{a,b}$ lies at distance 3 from ab and $b \neq \mathrm{proj}_{ab} y := x \neq a$. Then any point y' on yx, $y' \neq x, y$ satisfies $\Gamma_4(y) \cap T_{a,b} = \Gamma_4(y') \cap T_{a,b}$, hence $(a, b) \in C$.

Suppose now that $(a, b) \in C$ with $\delta(a, b) = 6$. We look for a contradiction. Put $\mathcal{M} = \Gamma_3(a) \cap \Gamma_3(b)$. If x is a point of $T_{a,b}$, then either x lies on a line of $\mathcal{M}$, or x is a point at distance 3 from a line A through a and from a line B through b, with A opposite B. Let y be a point of $T_{a,b}$ on a line M of $\mathcal{M}$, and y' a point such that (**) is satisfied. Clearly, y' cannot be incident with an element of $\mathcal{M}$. Let $A = \mathrm{proj}_a y'$ and $B = \mathrm{proj}_b y'$. If $A \neq \mathrm{proj}_a y$ and $B \neq \mathrm{proj}_b y$, then the point $\mathrm{proj}_M y'$ lies at distance 4 from y' but not from y, a contradiction. Suppose $A = \mathrm{proj}_a y$. Let M' be the line of $\mathcal{M}$ concurrent with B. Then the point $\mathrm{proj}_{M'} y$ lies at distance 4 from y but is opposite y', a contradiction. Hence $(a, b) \notin C$.

So we distinguished distance 2 and the theorem follows.

Case $n \neq 6$

$\boxed{\text{Step 1: the set } S_{a,b}}$

For two points a and b, we define

$$S_{a,b} = \{x \in \mathcal{P} \mid \Gamma_{n-2}(x) \cap T_{a,b} = \emptyset\}.$$

Note that by symmetry, $x \in S_{a,b}$ implies $a \in S_{b,x}$ and $b \in S_{a,x}$. We claim the following:

(*i*) If $\delta(a, b) = 2$ and $s \geq 3$, then $S_{a,b} = (\Gamma_2(a) \cup \Gamma_2(b)) \setminus \Gamma_1(ab)$. If $\delta(a, b) = 2$ and $s = 2$, then $S_{a,b} = (\Gamma_2(a) \cup \Gamma_2(b)) \setminus \{a, b\}$.

(*ii*) If $\delta(a, b) = 4$, then $\{a \bowtie b\} \subseteq S_{a,b} \subseteq \{a \bowtie b\} \cup [\Gamma_2(a \bowtie b) \cap \Gamma_4(a) \cap \Gamma_4(b)]$. If $t \geq 3$, then $S_{a,b} = \{a \bowtie b\}$.

(*iii*) If $k := \delta(a,b) \notin \{2,4,n\}$, then every $x \in S_{a,b}$ lies at distance $k/2$ from $a \bowtie b =: m$ with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. If moreover $s > 2$ and $k \equiv 2 \bmod 4$, or $t > 2$ and $k \equiv 0 \bmod 4$, then $S_{a,b} = \emptyset$.

(*iv*) If $\delta(a,b) = n$, then let γ be an arbitrary path of length n joining a and b, let m be the middle element of γ and put $v_a = \mathrm{proj}_m a$, $v_b = \mathrm{proj}_m b$. Then

$$\begin{aligned} S_{a,b} \subseteq \; & (\Gamma_{n/2}(m) \cap \Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+1}(v_b)) \\ & \bigcup (\Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(a)) \\ & \bigcup (\Gamma_{n/2+1}(v_b) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(b)). \end{aligned}$$

If moreover $s > 2$ and $n \equiv 2 \bmod 4$, or $t > 2$ and $n \equiv 0 \bmod 4$, then

$$\begin{aligned} S_{a,b} \subseteq \; & (\Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(a)) \\ & \bigcup (\Gamma_{n/2+1}(v_b) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(b)). \end{aligned}$$

We prove these claims.

(*i*) Suppose $\delta(a,b) = 2$. Since any point y of $T_{a,b}$ lies at distance $n-3$ from the line ab, with $\mathrm{proj}_{ab} y \notin \{a,b\}$, it follows that every point collinear with a or b, not on the line ab, belongs to $S_{a,b}$. Also, if $s = 2$, then the unique point of ab different from a and b is an element of $S_{a,b}$. Let x be an arbitrary point in $S_{a,b}$. Put $j = \delta(x,a)$. If $j = s = 2$, then there is nothing to prove, so we may assume $(j,s) \neq (2,2)$. Suppose first there exists a j-path γ between a and x containing ab, but not the point b. Let v be the element on γ at distance $j/2$ from a, and consider an element y at distance $n-2-j/2$ from v such that $\mathrm{proj}_v y \notin \{\mathrm{proj}_v a, \mathrm{proj}_v b, \mathrm{proj}_v x\}$. Note that such an element v exists because $(j,s) \neq (2,2)$. Then y lies at distance $n-2$ from a, b and x, a contradiction. So we can assume that $\mathrm{proj}_{ab} x = a$. If $j = 2$, then again, there is nothing to prove. So we may assume $2 < j < n$ (the case $j = n$ is contained in the previous case, or can be obtained from the present case by interchanging the roles of a and b). Let v be an element at distance $n-j-1$ from the line ab such that $a \neq \mathrm{proj}_{ab} v \neq b$. Note that v and x are opposite and $\delta(a,v) = n-j$. Consider an element v' incident with v, different from $\mathrm{proj}_v a$, and let v'' be the projection of x onto v'. Let w be the element of $[x, v'']$ at distance $j/2-2$ from v''. An element y at distance $j/2-2$ from w such that $\mathrm{proj}_w x \neq \mathrm{proj}_w y \neq \mathrm{proj}_w v''$ lies at distance $n-2$ from a, b and x, a contradiction. Claim (*i*) is proved.

$(ii-iv)$ We proceed by induction on the distance k between a and b, the case $k=2$ being Claim (i) above. Suppose $\delta(a,b)=k>2$ and let m be an element at distance $k/2$ from both a and b. Note that, if $\delta(a,b)=4$, the point $a \bowtie b$ indeed belongs to $S_{a,b}$ (indeed, in this case, every element of $T_{a,b}$ either lies at distance $n-4$ from $a \bowtie b$ or lies opposite $a \bowtie b$ in an apartment containing a,b). Suppose x is an arbitrary element of $S_{a,b}$ and put $\ell=\delta(x,m)$.

Suppose first that, if $\ell \neq n$, $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. Then we have the following possibilities:

1. Suppose $\ell < k/2$. Then $\delta(a,x) < k$ and we apply the induction hypothesis on $S_{a,x}$. Since $b \in S_{a,x} \neq \emptyset$, $\delta(a,x) < n$ and $m \neq a \bowtie x$, we have $\delta(a,x) \in \{2,4\}$. Hence either $\delta(a,b)=4$ and $x=a \bowtie b$ (which is a possibility mentioned in (ii)), or $\delta(a,b)=6$ and x lies on m, or $\delta(a,b)=8$ and $x=m$. But in these last two cases, the "position" of b contradicts the induction hypothesis applied on $S_{a,x}$.
2. Suppose $\ell \geq k/2$. Let γ'' be an ℓ-path between m and x containing neither $\mathrm{proj}_m a$ nor $\mathrm{proj}_m b$. Put $\gamma' = [a,m] \cup \gamma''$. Let w be the element on γ' at distance $(\ell + k/2)/2$ from both x and a. Note that w belongs to the path γ''. If $\ell = k/2$ (and hence $w=m$) and either $k \equiv 2 \bmod 4$ and $s=2$, or $k \equiv 0 \bmod 4$ and $t=2$, then there is nothing to prove. Otherwise, there exists an element y of Γ at distance $n-2-(k/2+\ell)/2$ from w such that $\mathrm{proj}_w a \neq \mathrm{proj}_w y \neq \mathrm{proj}_w x$ and $\mathrm{proj}_w b \neq \mathrm{proj}_w y$. Now y lies at distance $n-2$ from a, b and x, a contradiction.

Suppose now x is a point of $S_{a,b}$ at distance ℓ from m, $0<\ell<n$, for which $\mathrm{proj}_m x = \mathrm{proj}_m a$. Let $[a,m] \cap [x,m] = [v,m]$, and put $i' = \delta(v,a)$. (Note that ℓ and $k/2$ have the same parity.) We have the following possibilities:

1. Suppose $\ell = k/2+2$ and $i' < k/2-1$ or $\ell \leq k/2$. Again $\delta(a,x)<k$ and applying the induction hypothesis on $S_{a,x}$, we obtain a contradiction as in Case 1 above.
2. Suppose $n > \ell > k/2+2$. Let z be an element at distance $n-\ell$ from m with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$, and Z an element incident with z and different from $\mathrm{proj}_z m$. Put $h = \mathrm{proj}_Z x$. Then $\delta(h,x) = n-2$ and $\delta(m,h) = n-\ell+2$. Let $j = n-2-\delta(h,m)-k/2$. Let h' be the element on the $(n-2)$-path between x and h at distance $j/2$ from h. An element y at distance $j/2$ from h' such

that $\mathrm{proj}_{h'}x \neq \mathrm{proj}_{h'}y \neq \mathrm{proj}_{h'}h$ lies at distance $n-2$ from a, b and x, a contradiction.

3. Suppose $\ell = k/2+2$, $i' = k/2-1$ and $k < n-1$. Then $\delta(b,x) = k+2$ and v lies at distance $k/2+1$ from both b and x. Let Σ be an apartment containing x, b and v, and let v' be the element in Σ opposite v. Let $w = \mathrm{proj}_v a$, $w' = \mathrm{proj}_{v'} w$ (note that $\delta(w,v') = n-1$ and $w' \neq \mathrm{proj}_{v'}x, \mathrm{proj}_{v'}b$) and d the length of the path $[w,a] \cap [w,w']$. Note that $d \leq k/2-2$. For an element y not opposite w', let w''_y be the element such that $[w,w'] \cap [y,w'] = [w''_y, w']$. Consider now an element y such that $\delta(w''_y, w') = k/2-d-2$ and $\delta(w''_y, y) = d$. Then y lies at distance $n-2$ from a, b and x, a contradiction.
4. If $\ell = k/2+2$, $i' = n/2-1$ and $k = n$, there is nothing to prove.
5. Suppose finally $\ell = k/2+2$, $i' = k/2-1$ and $k = n-1$. Then $\delta(b,x) = n-1$. Let b' and x' be the elements of the path $[b,x]$ at distance $(n-1)/2-1$ from b and x, respectively. Since $a \in S_{b,x}$, either $\delta(a,b') = (n+1)/2$ or $\delta(a,x') = (n+1)/2$ (this is what we proved up to now for the "position" of a point of $S_{b,x}$). But since we obtain a path between a and b' (x') of length $d = (3n-5)/2$ (passing through $\mathrm{proj}_m a$), the triangle inequality implies $\delta(a,b'), \delta(a,x') \geq 2n-d > (n+1)/2$, a contradiction.

This completes the proof of our claims.

Step 2: the sets O and $\overline{O}$.

For a point $c \in S_{a,b}$, we define the set

$$C_{a,b;c} = \{c' \in S_{a,b} | S_{c,c'} \cap \{a,b\} \neq \emptyset\}.$$

Now let O be the set of pairs of points (a,b), $\delta(a,b) \neq n-2$ for which $|S_{a,b}| > 1$ and $|C_{a,b;c}| > 1$, $\forall c \in S_{a,b}$. We claim that O contains only pairs of collinear points and pairs of opposite points, and all pairs of collinear points are included in O.

Suppose first $s \geq 3$ and $t \geq 3$. Then by Step 1, only pairs of collinear points and pairs of opposite points can satisfy $|S_{a,b}| > 1$. If a and b are collinear points then, for a point $c \in S_{a,b} \cap \Gamma_2(a)$, every point y collinear with a not on the lines ab or ac belongs to $C_{a,b;c}$ (since $a \in S_{c,y}$), hence the claim. Note that if n is odd, there are no pairs of opposite points. So in this case, O is the set of pairs of collinear points, which proves the theorem for the case n odd.

Let now $s = 2$ or $t = 2$ (so n is even).

Suppose first $\delta(a, b) = 2$ and let c be a point of $S_{a,b}$. If c is collinear with a, then any point y' collinear with a not on the lines ac or ab belongs to $C_{a,b;c}$ (since $a \in S_{c,y'}$), hence $(a, b) \in O$.

Suppose $\delta(a, b) = 4$. If $(a, b) \in O$, then (because of the condition $|S_{a,b}| > 1$ and (ii) in Step 1) necessarily $t = 2$. But for a point $c \in S_{a,b}$ different from $a \bowtie b$, we obtain $C_{a,b;c} = \{a \bowtie b\}$, a contradiction.

Suppose finally $\delta(a, b) = k$, $4 < k < n-2$ and suppose by way of contradiction that (a, b) belongs to O. Put $m = a \bowtie b$ and let x be a fixed point of $S_{a,b}$. Note that by (iii) in Step 1, $|\Gamma_1(m)| = 3$. Let x' be an element of $S_{a,b}$ different from x. Because $|\Gamma_1(m)| = 3$, we have $\mathrm{proj}_m x = \mathrm{proj}_m x'$, so $\delta(x, x') \leq k - 2 < n$. But now $\delta(a, x \bowtie x') = \delta(b, x \bowtie x') \geq k/2 + 1 > \delta(x, x')/2$, so neither a nor b belongs to $S_{x,x'}$, the final contradiction. Hence the claim.

Let $\overline{O}$ be the set of pairs of points (a, b) satisfying $\delta(a, b) \neq n - 2$, $(a, b) \notin O$ and such that there exists a point $c \in S_{a,b}$ for which (a, c) and (b, c) both belong to O. Then by considering the results of Step 1, one easily obtains that $\overline{O}$ contains only pairs (a, b) of points at mutual distance 4 or n, and that all distance-4-pairs are contained in $\overline{O}$ (indeed, consider the point $c = a \bowtie b$).

Step 3: the set of pairs of collinear points

Case $n = 8$ and $s = 2$
Note that $t \geq 4$. For a point $c \in S_{a,b}$, we define the set

$$R_c := \{x \in S_{a,b} | x \neq c \text{ and } a \in S_{x,c}\}.$$

Let O' be the subset of O of pairs of points (a, b) for which there exists a point $c \in S_{a,b}$ with the following property:

$(*)$ every point r for which $R_c \subseteq \Gamma_6(r)$ satisfies $(r, a) \in \overline{O}$.

We claim that O' is exactly the set of pairs of collinear points. Let first $(a, b) \in O$ with $\delta(a, b) = 2$. Let c be a point collinear with a not on the line ab. Now one easily checks that R_c contains the set of points collinear with a not on the lines ab or ac. Hence clearly, a point r lying at distance 6 from all the points of R_c lies at distance 4 from a, so $(a, r) \in \overline{O}$. Let now $(a, b) \in O$ with $\delta(a, b) = 8$, m a point at distance 4 from both a and b, $a' = \mathrm{proj}_m a$ and $b' = \mathrm{proj}_m b$. Then a point c of $S_{a,b}$ lies at distance 6 from m, 5 from a' and 8 from a, or at distance 6 from m, 5 from b' and 8 from b. Suppose first $\delta(a', c) = 5$. Suppose c' is a point of R_c. If $\delta(c', a') = 5$ then, since $s = 2$, the projections of c and c' onto a' coincide. From this, it is easily seen (using the results of Step 1) that $a \notin S_{c,c'}$. Hence all points of R_c lie at distance 5 from b'. But now the point $y := \mathrm{proj}_{b'} b$ lies at distance 6 from all points of R_c and from a, hence $(a, y) \notin \overline{O}$ and property $(*)$ is not satisfied for the point c. Suppose now $\delta(b', c) = 5$. Then as above, one easily sees that a point of R_c lies at distance 5 from a'. The point $y' := \mathrm{proj}_{a'} a$ lies at distance 6 from all points of R_c, but is collinaer with a, hence $(y', a) \notin \overline{O}$ and property $(*)$ is not satisfied for the point c. This shows that $(a, b) \notin O'$, hence the claim. Next we distinguish this case from the case $s > 2$. For $s > 2$, one can also consider the set O' defined as above. It is easy to see that all pairs of collinear points are included in O'. Now let O'' be the subset of O' consisting of these pairs (a, b) for which there exist at least two points x, x' different from a and b such that (a, x), (a, x'), (b, x) and (b, x') belong to O'. If $s = 2$, O'' is empty (since x and x' should be two different points collinear with the collinear points a and b), but if $s \neq 2$, every pair of collinear points is included in O''. This distinguishes the cases $s = 2$ and $s > 2$. Hence from now on, we can assume that, if $n = 8$, then $s > 2$.

The general case

Let (a, b) be a pair of points of O, and c, c' points of Γ. Consider the following conditions:

(T1) $T_{a,b} \subseteq \Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$

(T2) $(c, y), (c', y) \notin O$, $\forall y \in T_{a,b}$,
if $n \geq 12$, $(c, y), (c', y) \notin \overline{O}$, $\forall y \in T_{a,b}$

(T3) $(x, y) \in O$, for any two distinct points $x, y \in \{a, b, c, c'\}$

(T3′) $(x, y) \in \overline{O}$, for any two distinct points $x, y \in \{a, c, c'\}$ or $x, y \in \{b, c, c'\}$

(T4) $T_{c,c'} \subseteq \Gamma_{n-2}(a) \cup \Gamma_{n-2}(b)$

(T5) $\exists! x_1 \in S_{a,b} : \forall x' \in S_{a,b}, |S_{x_1,x'} \cap \{a, b\}| = 1$,
$(x_1, v) \in O$ for any point $v \in \{a, b, c, c'\}$,
$(x_1, y) \notin O$, $\forall y \in T_{a,b}$,
$x_1 = S_{a,c} \cap S_{a,c'} \cap S_{b,c} \cap S_{b,c'} \cap S_{c,c'}$.

Let $\mathcal{T}$ be the set of conditions T1, T2, T3 and T4. Let $\mathcal{T}'$ be the set of conditions T1, T2, T3, T3′ and T5.

Claim 1. *Let $\delta(a, b) = 2$. If $s \geq 3$, there exist points c, c' such that conditions $\mathcal{T}$ are satisfied, but there do not exist points c, c' for which conditions $\mathcal{T}'$ hold. If $s = 2$ and $n > 8$, there exist points c, c' such that conditions $\mathcal{T}'$ hold.*

Proof. Suppose first $s \geq 3$. Choose c and c' on the line ab, different from a and b. Then it is easy to see that the conditions $\mathcal{T}$ hold (note that (T2) is satisfied because $n \neq 6$). Clearly, there does not exist a point y in $S_{a,b}$ for which both (a, y) and (b, y) belong to O, hence (T5) cannot be satisfied. Suppose now $s = 2$. Let x_1 be the unique point on ab, different from a and b, and c and c' points on two different lines (different from ab) through the point x_1. Then it is easy to see that the conditions $\mathcal{T}'$ hold (the condition $(c, y) \notin O$, $\forall y \in T_{a,b}$ does not hold if $n = 8$, which is the reason we treated the octagons separately). $\diamond$

Claim 2. Let $\delta(a, b) = n$. If $s \geq 3$, there do not exist points c, c' such that conditions $\mathcal{T}$ are satisfied, and if moreover $n > 8$, there do not exist points c, c' such that conditions $\mathcal{T}'$ hold. If $s = 2$ and $n > 8$, there do not exist points c, c' such that conditions $\mathcal{T}'$ are satisfied.

Proof. Suppose by way of contradiction a, b, c and c' are such that $s \geq 3$ and conditions $\mathcal{T}$ hold or $n > 8$ and conditions $\mathcal{T}'$ hold. Let $m \in \Gamma_{n/2}(a) \cap \Gamma_{n/2}(b)$, $a' = \mathrm{proj}_m a$ and $b' = \mathrm{proj}_m b$. For an element x at distance j from m, $0 \leq j \leq n/2 - 3$, such that $a' \neq \mathrm{proj}_m x \neq b'$, define the following set:

$$T_x = \{y \in T_{a,b} | \delta(x, y) = n/2 - 2 - j, \mathrm{proj}_x a \neq \mathrm{proj}_x y \neq \mathrm{proj}_x b\}.$$

Note that T_x is the subset of $T_{a,b}$ of elements y for which the path $[a, y]$ contains x. We first prove that for any set T_x,

($\diamond$) there does not exist a point $v \in \{c, c'\}$ such that $T_x \subseteq \Gamma_{n-2}(v)$.

Put $\mathcal{M} = \Gamma_{n/2}(a) \cap \Gamma_{n/2}(b)$. Suppose $T_m \subseteq \Gamma_{n-2}(v)$, for a point $v \in \{c, c'\}$. It is easy to see that $\delta(v, m) = n/2$ and $\mathrm{proj}_m a = \mathrm{proj}_m v$ or $\mathrm{proj}_m b = \mathrm{proj}_m v$. Without loss of generality, we assume $\mathrm{proj}_m a = \mathrm{proj}_m v$, hence $\delta(a, v) \leq n - 2$. Suppose first $s \geq 3$ and conditions $\mathcal{T}$ hold. But then by condition (T3), $\delta(a, v) \leq n - 2$ implies $\delta(a, v) = 2$ and v is a point at distance $n/2$ from m lying on the line $L = \mathrm{proj}_a m$. This implies that for an arbitrary element m' of $\mathcal{M}$, $m' \neq m$, $T_{m'} \cap \Gamma_{n-2}(v) = \emptyset$ (note that $T_m \cap T_{m'} = \emptyset$), so $T_{m'} \subseteq \Gamma_{n-2}(v')$, with $\{v, v'\} = \{c, c'\}$. We obtain a contradiction by considering a third element of $\mathcal{M}$. Suppose now conditions $\mathcal{T}'$ hold. Note that $n > 8$ by assumption. In this case, $\delta(a, v) \leq n - 2$ implies $\delta(a, v) = 4$ (see Condition (T3$'$)). Then since $x_1 \in S_{a,v}$, x_1 is either the point $a \bowtie v$ or lies on the projection of b onto $a \bowtie v$ (and the latter can only occur if $t = 2$). In any of these cases, $\delta(x_1, b) \notin \{2, n\}$, contradicting condition (T5). This shows ($\diamond$) for the case $j = 0$.

Let x be an element at distance $j = 1$ from m such that $\mathrm{proj}_m a \neq x \neq \mathrm{proj}_m b$. Suppose $T_x \subseteq \Gamma_{n-2}(v)$, with $v \in \{c, c'\}$. Then again it is easy to show that

$\delta(v,x) = \delta(x,a) = n/2+1$ and $\mathrm{proj}_x v = \mathrm{proj}_x a = m$. If $\mathrm{proj}_m a = \mathrm{proj}_m v$ or $\mathrm{proj}_m b = \mathrm{proj}_m v$, then we are back in the previous case, which led to a contradiction, so suppose $\mathrm{proj}_m a \neq \mathrm{proj}_m v \neq \mathrm{proj}_m b$. Consider the n-path between a and v that contains m. Then we can find a point y of $T_{a,b}$ on this path that is collinear with v, in contradiction with condition (T2). This shows ($\diamond$) for the case $j=1$. Note that thus no element of $\{c,c'\}$ lies at distance $n/2$ from m.

We now proceed the proof of ($\diamond$) by induction on the distance j between x and m. Let $j>1$. Consider an element x at distance j from m such that $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. Suppose by way of contradiction that $T_x \subseteq \Gamma_{n-2}(v)$, with $v \in \{c,c'\}$. Let $x' = \mathrm{proj}_x m$. Then it is again easy to show that $\delta(v,x) = \delta(a,x) = n/2+j$ and $\mathrm{proj}_x v = x'$. Remark that $\mathrm{proj}_{x'} a \neq \mathrm{proj}_{x'} v$, since otherwise $T_{x'} \subseteq \Gamma_{n-2}(v)$ (since $\delta(v,x') = n/2+j-1$ and $\delta(x',w) = n/2-j-1$, with $w \in T_{x'}$), in contradiction with the induction hypothesis. Suppose first that if $j=2$ we do not have the case $t=2$ and $n \equiv 0 \bmod 4$ or $s=2$ and $n \equiv 2 \bmod 4$. Consider now an element z incident with the element $w = \mathrm{proj}_{x'} a$, but different from $\mathrm{proj}_w a$, from $\mathrm{proj}_w b$ and from x' (such an element exists, because of the restrictions above). But then we have $\delta(v,w') = n$, for every element w' of T_z, so $T_z \subseteq \Gamma_{n-2}(v')$, with $\{c,c'\} = \{v,v'\}$, a contradiction with the induction hypothesis.

Let $j=2$, $t=2$ and $n \equiv 0 \bmod 4$ (note that $n \geq 12$ since $j \leq \frac{n}{2}-3$). Let $L = mx$ and let γ' be the path of length $n+2$ between a and v consisting of the paths $[a,L]$ and $[L,v]$. Then the element of γ' at distance 4 from v is contained in $T_{a,b}$, contradicting (T2).

Let $j=2$, $s=2$ and $n \equiv 2 \bmod 4$. Note that m and x are lines, and we assumed conditions $\mathcal{T}'$. The arguments given above for $t=2$ and $n \equiv 0 \bmod 4$ also work for this case, except when $n=10$. We give a general argument. Put $a' = \mathrm{proj}_m a$ and $b' = \mathrm{proj}_m b$. By Claim (iv) of Step 1, there are essentially two possibilities for x_1. First, suppose the point x_1 lies at distance $n/2+2$ from m and at distance $n/2+1$ from a'. Then there arises an n-path γ' between a and x_1 sharing the path $[a,a']$ with γ. Let a'' be the projection of x_1 onto a'. Since a'' is a line at distance $n/2$ from both a and x_1, and $v \in S_{x_1,a}$ (by Condition (T5)), either the distance between v and a'' is $n/2$ (which is not true), or the distance between v and a'' is $n/2+2$, which is again impossible. Secondly, x_1 cannot lie at distance $n/2$ from m, since this would contradict condition (T5) (x_1 would be collinear with a point of $T_{a,b}$).

This completes the proof of ($\diamond$).

Suppose $s \geq 3$.
Consider now a line L at distance $j = n/2-3$ from m, such that $\mathrm{proj}_m a \neq$

$\mathrm{proj}_m L \neq \mathrm{proj}_m b$. The points on L different from the projection of m onto L are points of T_L. By ($\diamond$), we know that $T_L \not\subseteq \Gamma_{n-2}(v)$, for $v \in \{c, c'\}$. Since $s \geq 3$, T_L contains at least 3 points, so we may suppose that at least two points of them are contained in $\Gamma_{n-2}(v)$, with $v \in \{c, c'\}$. This implies that v is at distance $n-3$ from L, so at distance $n-4$ from a unique point x of L. If $x = \mathrm{proj}_L a$, then $T_L \subseteq \Gamma_{n-2}(v)$, a contradiction, so we can assume that $x \neq \mathrm{proj}_L a$. Let first $n \neq 8$ or $t \geq 3$. Then consider a line L' incident with $\mathrm{proj}_L a$, $L' \neq L$, at distance $n-3$ from both a and b (such a line always exists because of our assumptions). Now $T_{L'} \cap \Gamma_{n-2}(v) = \emptyset$ (because all points of $T_{L'}$ lie opposite v), so $T_{L'}$ is contained in $\Gamma_{n-2}(v')$, with $\{v, v'\} = \{c, c'\}$, contradicting ($\diamond$).

Let now $n = 8$ and $t = 2$ (hence we assume conditions $\mathcal{T}$ hold). Then $\delta(v', x) = 6$, with $\{v, v'\} = \{c, c'\}$, $T_L \not\subseteq \Gamma_6(v')$ and $\delta(v, v') \in \{2, 8\}$. Now for each potential v', it is possible to construct a point of $T_{v,v'}$ not at distance $n-2$ from a nor from b, a contradiction with condition (T4). For example, let us do the case $\delta(v, v') = 2$ in detail. Since v does not lie at distance 6 from a or b, we obtain $\delta(v, a) = \delta(v, b) = 8$, hence $\delta(a, vv') = \delta(b, vv') = 7$, $v' \neq \mathrm{proj}_{vv'} a \neq v$ and $v' \neq \mathrm{proj}_{vv'} b \neq v$. Also $\mathrm{proj}_{vv'} a \neq \mathrm{proj}_{vv'} b$, since otherwise we would obtain a point of $T_{a,b}$ not at distance 6 from v nor from v'. Now let N be the line at distance 3 from b and at distance 4 from vv'. Then the points of N different from $\mathrm{proj}_N v$ are points of $T_{v,v'} \setminus \Gamma_6(b)$, but not all these points lie at distance 6 from a, a contradiction.

This ends the proof of Claim 2 for the case $s \geq 3$.

Suppose $s = 2$.
Note that we assume $n > 8$ and conditions $\mathcal{T}'$ hold. We keep the same notation as in the previous paragraph. Now the only possibility (to rule out) that we have not considered yet (because it does not occur in the previous case) is the case that c and c' both lie at distance $n-2$ from different points u and u' on L, $\delta(c, L) = \delta(c', L) = n-1$ and u and u' different from the projection w of a onto L.

Suppose $n > 10$ (otherwise some of the notations introduced below don't make sense). Put $L' = \mathrm{proj}_w a$ and $l' = \mathrm{proj}_{L'} a$. Suppose first the unique point z on L' at distance $n-2$ from c is not l'. Then consider a line K through z, different from L' and from $\mathrm{proj}_z c$. Because c is at distance n from all the points of K different from z (which are elements of $T_{a,b}$) we conclude that $T_K \subseteq \Gamma_{n-2}(c')$, a contradiction to ($\diamond$). So $[c, L']$ contains l'. Define the element p as $[l', m] \cap [l', c] = [l', p]$. Suppose $p \neq m$ and let $j = \delta(l', p)$. Consider the element z' on $[c, p]$ at distance $j+3$ from p. Note that z' is a line at distance $n-3$ from both a and b. Since c is not at distance $n-2$ from any of the points of $T_{z'}$, we conclude that $T_{z'} \subseteq \Gamma_{n-2}(c')$, a contradiction to ($\diamond$). If $p = m$, but $a' \neq \mathrm{proj}_m c \neq b'$, we obtain a similar contradiction considering the line z' at distance $n/2 - 3$ from m on the path $[c, m]$ (note that this path does not contain a' or b'). So the path $[c, l']$ contains a' or b' (hence $\delta(c, m) = n/2 + 4$). Suppose without loss of generality $[c, l']$ contains a'. Consider now the element q defined by $[m, c] \cap [m, a] = [m, q]$. Then we first show that q coincides either with a' (Case 2 below), or with the element $a'' = \mathrm{proj}_{a'} a$ (Case 1 below). Indeed, if not, then $\delta(a, c) < n$, which implies that $\delta(a, c) = 4$ (by Condition (T3$'$)) and $x_1 = a \bowtie c$. Since $(b, x_1) \in O$, $\delta(b, x_1)$ is then equal to n. Now it is easy to see that there exists an element of $T_{a,b}$ for which the projection onto ax_1 is different from a and from x_1, a contradiction (such an element would be at distance $n - 2$ from x_1, which would imply that $x_1 \notin S_{a,b}$). One checks that in the case $n = 10$, we end up with the same possibilities.

Case 1 Consider the element $m' \in [a'', c]$ that is at distance 2 from a''. By Step 1, a point of $S_{a,c}$ lies at distance $n/2$ or $n/2+2$ from m'. Because of the conditions, $x_1 \in S_{a,c}$. We now check the different positions of x_1 (using $x_1 \in S_{a,b}$ and Step 1). If x_1 lies at distance $n/2+1$ from a', then $\delta(x_1, m') = n/2+4$, a contradiction. If x_1 lies at distance $n/2+1$ from b', there arises a path of length $n/2+6$ between x_1 and m', which is again a contradiction, since $n > 8$. Note that x_1 cannot lie at distance $n/2$ from m because $(x_1, y) \notin O$ for $y \in T_{a,b}$.

Case 2 Suppose first x_1 lies at distance $n/2 + 1$ from b'. Note that as in Case 1, $\delta(x_1, m) \neq n/2$. Let b_0 be the projection of x_1 onto b'. Then a point of $S_{x_1,b}$ lies at distance $n/2$ or $n/2 + 2$ from b_0. Because of the conditions, $c \in S_{x_1,b}$. But we have a path of length $n/2 + 6$ between c and b_0 (containing $[c, a']$), a contradiction since $n \neq 8$. So we know that x_1 lies at distance $n/2 + 1$ from a'. Let $a_0 = \mathrm{proj}_{a'} x_1$. Suppose the projections of c and x_1 onto a' are not equal (which only occurs if $n \equiv 2 \bmod 4$, since $s = 2$). Since $c \in S_{x_1,a}$, the distance between c and a_0 is either $n/2$ or $n/2+2$, a contradiction ($\delta(c, a_0) = n/2+4$). So the projection of c onto a' is the element a_0. Suppose $\mathrm{proj}_{a_0} c \neq \mathrm{proj}_{a_0} x_1$. Since the distance between c and a_0 is $n/2+2$, and $c \in S_{a,x_1}$, the point c has to lie at distance $n/2+1$ from either a' or $\mathrm{proj}_{a_0} x_1$, which is not true. So $\mathrm{proj}_{a_0} c = \mathrm{proj}_{a_0} x_1 := h$. Note that the projections of c and x_1 onto h are certainly different, since we know that the distance between c and x_1 is either n or 2, and the last choice would contradict the fact that $a \in S_{x_1,c}$. Now consider the projection m' of c onto h. This is an element at distance $n/2$ from both c and x_1. Now $\delta(b, m') = n/2 + 4$, which contradicts the fact that $b \in S_{c,x_1}$.

This ends the case $s = 2$ and the proof of Claim 2. $\diamond$

By Claims 1 and 2 above, no pairs $(a, b) \in O$ satisfy conditions $\mathcal{T}'$ if $n > 8$ and $s > 2$, while if $n > 8$ and $s = 2$, all pairs of collinear points belong to O and satisfy conditions $\mathcal{T}'$. This distinguishes the cases $s = 2$ and $s > 2$. If $s = 2$, let O' be the subset of O of pairs (a, b) for which there exist points c and c' such that conditions $\mathcal{T}'$ hold. If $s \geq 3$, let O' be the subset of O of pairs (a, b) for which there exist points c and c' such that conditions $\mathcal{T}$ hold. Now it is clear that O' is the set of pairs of collinear points. Hence α preserves collinearity.

4.4.5 Case $i = n - 1$

We can obviously assume $n \geq 6$. If $n = 6$ and $s = t = 2$, then an easy counting argument yields the result, so we exclude this case in the following.

$\boxed{\text{Step 1: the set } O_{a,b}}$

For two points a and b with $\delta(a, b) \neq n - 1$, let $O_{a,b}$ be the set of pairs of points $\{c, c'\}$, c and c' different from a and from b, for which

$$T_{v,v'} \subseteq \Gamma_{n-1}(w) \cup \Gamma_{n-1}(w'),$$

whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$. For a pair $\{c, c'\} \in O_{a,b}$, we claim the following:

(*i*) If $\delta(a, b) = 2$, then either c and c' are different points on the line ab (distinct from a and b), or, without loss of generality, c is a point on ab and $c' \in \Gamma_3(ab)$ with $\mathrm{proj}_{ab} c' \notin \{a, b, c\}$. Moreover, all the pairs $\{c, c'\}$ obtained in this way are elements of $O_{a,b}$.

(*ii*) If $\delta(a, b) = 4$, then either c and c' are collinear points on the lines am or bm (where $m = a \bowtie b$) different from m, or c and c' are points collinear with m, at distance 4 from both a and b, and at distance 4 from each other. Again, all the pairs $\{c, c'\}$ obtained in this way, are elements of $O_{a,b}$.

(*iii*) Let $4 < \delta(a, b) = k \neq n - 1$ and m an element at distance $k/2$ from both a and b. Then c and c' are points at distance $k/2$ from m, at distance k from both a and b, and at distance k from each other (but such pairs $\{c, c'\}$ do not necessarily belong to $O_{a,b}$).

If $\delta(a,b) = 2$, then an element x of $T_{a,b}$ is either opposite the line ab, or lies at distance $n-3$ from a unique point on ab, different from a and from b. If $\delta(a,b) = 4$, then an element x of $T_{a,b}$ either lies at distance $n-1$ or $n-3$ from $m = a \bowtie b$ with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$ or lies at distance $n-3$ from a point x' on am or bm, $x' \notin \{a,b,m\}$ with $am \neq \mathrm{proj}_{x'} x \neq bm$. It is now easy to see that the given possibilities for c and c' in (i) and (ii) indeed satisfy the claim for $\delta(a,b) = 2$ and $\delta(a,b) = 4$, respectively. Note that if $s = 2$, only the second possibility of (ii) remains.

Let $\delta(a,b) = k \neq n-1$ and let m be a fixed element at distance $k/2$ from both a and b. Suppose $\{c,c'\} \in O_{a,b}$. For an element y with $\delta(m,y) = j \leq n - k/2 - 2$ and $\mathrm{proj}_m a \neq \mathrm{proj}_m y \neq \mathrm{proj}_m b$, we define the following set:

$$T_y = \{x \in T_{a,b} \mid \delta(x,y) = (n \pm 1) - j - k/2 \text{ and } \mathrm{proj}_y x \neq \mathrm{proj}_y m \text{ if } \delta(x,y) \neq n\}.$$

For an element y with $\delta(m,y) = n - k/2 - 1$ and $\mathrm{proj}_m a \neq \mathrm{proj}_m y \neq \mathrm{proj}_m b$, we define T_y as the set of elements at distance 2 from y, not incident with $\mathrm{proj}_y m$. For an element y with $\delta(m,y) = n - k/2$ and $\mathrm{proj}_m a \neq \mathrm{proj}_m y \neq \mathrm{proj}_m b$, we define T_y as the set of elements incident with y, different from $\mathrm{proj}_y m$. Note that $T_y \subseteq T_{a,b}$.

First we make the following observation. Let y be an element for which the set T_y is defined, and for which $\delta(m,y) \leq n - k/2 - 2$. Then there exists an element $v \in \{c,c'\}$ such that $T_y \subseteq \Gamma_{n-1}(v)$ if and only if $\delta(v,y) = \delta(a,y)$ and $\mathrm{proj}_y v = \mathrm{proj}_y a$ or $\mathrm{proj}_y v = \mathrm{proj}_y b$.

Now we prove claims (i), (ii) and (iii) above by induction on the distance k between a and b. Let $k \geq 2$. In the sequel, we include the proof for the case $k = 2$ in the general case.

Suppose first there exists an element $v \in \{c,c'\}$ such that $T_m \subseteq \Gamma_{n-1}(v)$. Then, by the previous observation, $\delta(v,m) = \delta(m,a) = k/2$ and we may assume that $\mathrm{proj}_m v = \mathrm{proj}_m a$. This implies that $\delta(a,v) \leq k-2$, so we can apply the induction hypothesis on $T_{a,v}$. Put $\{c,c'\} = \{v,v'\}$. If $k = 2$, we obtain $a = v$, a contradiction. If $k = 4$, then v is a point on the line am, $v \neq m$, and the only remaining possibility, considering the induction hypothesis and the condition $T_{a,v} \subseteq \Gamma_{n-1}(b) \cup \Gamma_{n-1}(v')$ is that v' is also a point on am, different from m. This is indeed a possibility mentioned in (ii). If $k > 4$, the position of b contradicts again the fact that $T_{a,v} \subseteq \Gamma_{n-1}(b) \cup \Gamma_{n-1}(v')$ and the induction hypothesis. Indeed, the element at distance $\delta(a,v)/2$ from both a and v belongs to the path $[a, \mathrm{proj}_m a]$ and hence does not lie at distance $\delta(a,v)/2$ from b. In this way, we described all the possibilities for the points c and c' in case there is a point $v \in \{c,c'\}$ for which $T_m \subseteq \Gamma_{n-1}(c)$. So from now on, we assume that there does not exist an element $v \in \{c, c'\}$ such that $T_m \subseteq \Gamma_{n-1}(v)$.

Let l be any element incident with m, different from the projection of a or b onto m. Suppose there exists a point $v \in \{c, c'\}$ such that $T_l \subseteq \Gamma_{n-1}(v)$. Then $\delta(v, l) = \delta(l, a) = k/2+1$ and we can assume that $\mathrm{proj}_l v = \mathrm{proj}_l a = m$. Since $T_m \not\subseteq \Gamma_{n-1}(v)$, we also know that $\mathrm{proj}_m a \neq \mathrm{proj}_m v =: w \neq \mathrm{proj}_m b$. Put $\{v, v'\} = \{c, c'\}$.

Suppose first $k = 2$. Then v is a point on the line ab. We now show that the point v' lies at distance 2 or 4 from v such that $\mathrm{proj}_v v' = m$. Indeed, suppose $\mathrm{proj}_v v' \neq m$ or $\delta(v, v') = n$. If $\delta(v, v') \neq n$, put $\gamma' = [v, v']$. If $\delta(v, v') = n$, let γ' be an arbitrary n-path between v and v' not containing m. Let x be an element of $T_{a,b}$ at distance $n-3$ from v such that either x lies on γ', or $[v, x]$ contains γ'. Then x is an element of $T_{a,b}$ not at distance $n-1$ from v or v', a contradiction. So we can assume that $\delta(v, v') < n$ and $\mathrm{proj}_v v' = m$. Suppose first $4 < \delta(v, v')$. Let Σ be an arbitrary apartment through v and v'. Then the unique element of Σ at distance $n-3$ from v and belonging to $T_{a,b}$, does not lie at distance $n-1$ from v', a contradiction, so the distance between v and v' is 2 or 4. Suppose now $\delta(v, v') = 4$ and $\mathrm{proj}_{ab} v' = b$. Then we obtain a contradiction interchanging the roles of b and v (noting that $T_{a,v} \subseteq \Gamma_{n-2}(b) \cup \Gamma_{n-2}(v')$). So v' is a point on ab, or v' is a point at distance 3 from ab for which the projection onto ab is different from a, b or v, as claimed in (i).

Suppose now $k \neq 2$. Let $w' = \mathrm{proj}_w v$. Since the distance between v and any element of $T_{w'}$ is less than or equal to $n-3$, we have that $T_{w'} \subseteq \Gamma_{n-1}(v')$, from which follows that $\delta(v', w') = \delta(w', a) = k/2+2$ and $\mathrm{proj}_{w'} v' = \mathrm{proj}_{w'} a = w$. Since $T_m \not\subseteq \Gamma_{n-1}(v')$, we either have that v' is a point at distance $k/2$ from m for which the projection onto m is different from w and $\mathrm{proj}_m a \neq \mathrm{proj}_m v' \neq \mathrm{proj}_m b$ (as required in (ii) and (iii)), or v' is a point at distance $k/2+2$ from m for which the projection onto m is w. In the latter case, let z be the projection of v' onto w (then $\delta(v, z) = \delta(v', z) = k/2$) and consider an element x at distance $n-1-k/2-2$ from z such that $\mathrm{proj}_z v \neq \mathrm{proj}_z x \neq \mathrm{proj}_z v'$. Then x is an element of $T_{a,b}$ at distance $n-3$ from both v and v', a contradiction. In this way, we described all the possibilities for the points c and c' in case there is a point $v \in \{c, c'\}$ and an element l as above for which $T_l \subseteq \Gamma_{n-1}(c)$. So from now on, we assume that there does not exist an element $v \in \{c, c'\}$ such that $T_l \subseteq \Gamma_{n-1}(v)$, for any l as above.

We now prove that (under the assumption just made)

($\diamond$) if y is an element for which the set T_y is defined, with $\delta(m, y) > 1$, then there does not exist a point $v \in \{c, c'\}$ such that $T_y \subseteq \Gamma_{n-1}(v)$.

This is done by induction on the distance j between y and m.

So let by way of contradiction l be an element at distance j from m, $j > 1$, for which the set T_l is defined and such that there exists an element $v \in \{c, c'\}$ with $T_l \subseteq \Gamma_{n-1}(v)$. Put $\{v, v'\} = \{c, c'\}$. Let first $j < n - k/2 - 1$. Then $\delta(v, l) = \delta(l, a) = k/2 + j$ and $w := \text{proj}_l v = \text{proj}_l a$ but by the induction hypothesis, $u := \text{proj}_w v \neq \text{proj}_w a$. Let $w' = \text{proj}_u v$. Note that the distance between w' and an element of $T_{w'}$ is $(n \pm 1) - k/2 - (j + 1)$, so an element of $T_{w'}$ lies at distance at most $n - 3$ from v. We conclude that $T_{w'} \subseteq \Gamma_{n-1}(v')$, from which follows that $\delta(v', w') = \delta(a, w') = k/2 + j + 1$ or $\delta(v', w') = n - 3$ (the latter is possible only if $j = n - k/2 - 2$), and $\text{proj}_{w'} v' = \text{proj}_{w'} a = u$. Let $\text{proj}_w a = u'$. First suppose $\delta(v', w') \neq n - 3$. From the assumptions, it follows that $\text{proj}_w v' \neq u'$. Depending on whether the projection of v' onto w is u or not, the distance between v' and u' is $k/2 + j + 2$ or $k/2 + j$. Note that $\delta(v, u') = k/2 + j$. Now consider an element x at distance $(n - 1) - (k/2 + j)$ from u' such that $\text{proj}_{u'} x \neq w$, and such that x either lies on $[u', b]$, or $[u', x]$ contains $[u', b]$. Then x is an element of $T_{v,v'}$ not contained in $\Gamma_{n-1}(a) \cup \Gamma_{n-1}(b)$, a contradiction. If $\delta(v', w') = n - 3$, then we similarly obtain a contradiction. So $T_y \not\subseteq \Gamma_{n-1}(v)$, for any element y at distance j from m.

Now let $j = n - k/2 - 1$. Note that T_l consists of all elements at distance 2 from l, not incident with $l' = \text{proj}_l m$. Then $\delta(v, l) = n - 1$ or $\delta(v, l) = n - 3$, and in both cases, $\text{proj}_l v = \text{proj}_l a$. If $\delta(v, l) = n - 1 (= \delta(a, l))$, we proceed as in the previous paragraph and end up with a contradiction. So let $\delta(v, l) = n - 3$. Suppose $n = k = 6$ and $s = 2$. Then l' is the unique point on m different from $\text{proj}_m a$ and $\text{proj}_m b$. Since no line through l' lies at distance 5 from v, every line through l' distinct from m has to lie at distance 5 from v', hence $\delta(v', l') \in \{4, 6\}$. Now for each of the positions of v and v', it is easy to construct a line at distance 5 from exactly two points of $\{a, b, v, v'\}$, contradicting the initial conditions. So we can assume $(n, s) \neq (6, 2)$. First suppose that $\text{proj}_{l'} v \neq \text{proj}_{l'} a = w$. Now consider an element w' incident with w, $l' \neq w' \neq \text{proj}_w a$ and $w' \neq \text{proj}_w b$ (such an element always exists by the restrictions made above). Then $T_{w'} \subseteq \Gamma_{n-1}(v)$, a contradiction since $\delta(m, w') = j - 1$. So $\text{proj}_{l'} v = w$. Let $[u, m] = [v, m] \cap [w, m]$ and put $u' = \text{proj}_u v$. Suppose first that $\text{proj}_m a \neq u' \neq \text{proj}_m b$ and $v \neq m$ ($v = m$ can occur only if $k = 4$). Then $T_{u'} \subseteq \Gamma_{n-1}(v')$. Indeed, if we put $i = \delta(u, l)$, then $\delta(v, u') = n - 4 - i$ and $\delta(m, u') = n - k/2 - i$. So the distance between u' and an element of $T_{u'}$ is $i \pm 1$, and the distance between v and an element of $T_{u'}$ is at most $n - 3$. So $T_{u'}$ is contained in $\Gamma_{n-1}(v')$, which is a contradiction since $\delta(m, u') < j$ (indeed, $i \geq 2$). Suppose finally $u' = \text{proj}_m a$ or $v = m$. If $k = 2$, we end up with a point v lying on ab (namely $v = \text{proj}_m l$). But then, if $s \neq 2$, for an arbitrary point x on m, different from a, b and v, we have that $T_x \subseteq \Gamma_{n-1}(v)$, in contradiction with our assumptions. If $s = 2$, we end up with a point v' at distance 3 from m and collinear with a or b, which contradicts $T_{a,v} \subseteq \Gamma_{n-1}(b) \cup \Gamma_{n-1}(v')$. If $k = 4$, we end up with $v = m$, but then the position of b contradicts the fact that $T_{a,v} \subseteq \Gamma_{n-1}(b) \cup \Gamma_{n-1}(v')$ and the (general) induction hypothesis. Finally, if $k > 4$, then $\delta(v, a) \leq \delta(v, \mathrm{proj}_m a) + \delta(a, \mathrm{proj}_m a) = k - 4$. Now the position of b contradicts again the fact that $T_{a,v} \subseteq \Gamma_{n-1}(b) \cup \Gamma_{n-1}(v')$ and the (general) induction hypothesis.

Let finally $j = n - k/2$. Note that T_l consists of all elements incident with l, different from the projection l' of m onto l. Then $\delta(v, l') = n - 1$ or $\delta(v, l') = n - 3$. Note that, in both cases, $\mathrm{proj}_{l'} v \neq \mathrm{proj}_{l'} a$. Indeed, $\mathrm{proj}_{l'} v = \mathrm{proj}_{l'} a$ would imply that $T_{l'} \subseteq \Gamma_{n-1}(v)$, a contradiction with our assumptions. Suppose first $\delta(v, l') = n - 3$. Let $l'' = \mathrm{proj}_{l'} v$. Since no element incident with l'' is at distance $n-1$ from v, we have $T_{l''} \subseteq \Gamma_{n-1}(v')$, which implies that $\delta(v', l')$ is either $n-3$ or $n-1$ and $\mathrm{proj}_{l'} v' \neq \mathrm{proj}_{l'} a$. Consider now the element on $[a, l']$ at distance 2 from l'. This is an element of $T_{v,v'}$ which is at distance $n - 3$ from both a and b, a contradiction. Suppose now $\delta(v, l') = n - 1$. Let x be the element on $[v, l']$ at distance 2 from l'. Since x is the only element of $T_{l'}$ not at distance $n - 1$ from v, this element x lies at distance $n - 1$ from v'. But then $\delta(v', l')$ is either $n - 1$ or $n - 3$. If $\mathrm{proj}_{l'} v' = \mathrm{proj}_{l'} a$, then $T_{l'} \subseteq \Gamma_{n-1}(v')$, a contradiction with our assumptions. If $\mathrm{proj}_{l'} v' \neq \mathrm{proj}_{l'} a$, then again the element on $[a, l']$ at distance 2 from l' is an element of $T_{v,v'}$ at distance $n - 3$ from both a and b, the final contradiction. This proves $(\diamond)$.

So we can now assume $T_y \not\subseteq \Gamma_{n-1}(v)$ for all $v \in \{c, c'\}$ and for any appropriate element y. Consider an element l at distance $n - k/2$ from m such that the projection of l onto m is different from the projections of a and b onto m. Let u be the projection of m onto l. Since $T_l \not\subseteq \Gamma_{n-1}(c)$ and $T_l \not\subseteq \Gamma_{n-1}(c')$, there is an element x incident with l, different from u, at distance $n - 1$ from c but not from c', and an element y incident with l, different from u, at distance $n - 1$ from c' but not from c. So $\delta(x, c') = n - 3 = \delta(y, c)$ and $\mathrm{proj}_x c' \neq l \neq \mathrm{proj}_y c$. But from this follows that, for an arbitrary element l' incident with u, $l \neq l' \neq \mathrm{proj}_u a$, we have $T_{l'} \subseteq \Gamma_{n-1}(c)$, a contradiction. This proves the claims (i), (ii) and (iii).

Step 2: the set $C_{a,b}$

For two points a, b, let $C_{a,b}$ be the set containing a, b, and all points c for which there exists a point c' such that $\{c, c'\} \in O_{a,b}$. Now let S be the set of pairs of points (a, b), $\delta(a, b) \neq n - 1$, for which there does not exist an element at distance $n - 1$ from all the points of $C_{a,b}$. We claim that, if $s > 2$, S contains exactly the pairs of points (a, b) for which $\delta(a, b) = 2$ or $\delta(a, b) = 4$

and if $s = 2$, $S = \emptyset$.

First assume $\delta(a,b) = 2$. If $s = 2$, then clearly $(a,b) \notin S$, so suppose $s > 2$. Let by way of contradiction w be an element at distance $n-1$ from all points of $C_{a,b}$. Since all the points of the line ab are contained in $C_{a,b}$, w lies opposite ab. If v is an arbitrary point on ab, different from a and from b, then the element on $[v,w]$ that is collinear with v, is contained in $C_{a,b}$, but lies at distance $n-3$ from w, a contradiction. Suppose now $\delta(a,b) = 4$. If $s = 2$, consider an element w at distance $n-2$ from the line am, with $a \neq \mathrm{proj}_{am} w \neq m$. This element lies at distance $n-1$ from all points of $C_{a,b;c}$ hence $(a,b) \notin S$. Suppose $s > 2$. Let by way of contradiction w be an element at distance $n-1$ from all points of $C_{a,b}$. Then w lies at distance $n-1$ from all the points collinear with $m = a \bowtie b$, which is not possible. Finally suppose $4 < \delta(a,b) = k \neq n-1$. Let a' be the element on a fixed k-path joining a and b at distance $k/2 - 1$ from a, and x an element at distance $(n-1) - (k/2 - 1)$ from w with $\mathrm{proj}_{a'} a \neq \mathrm{proj}_{a'} w \neq \mathrm{proj}_{a'} b$. Then w lies at distance $n-1$ from all points of $C_{a,b}$. Our claim is proved.

Step 3: the set S' of pairs of collinear points

Suppose $s > 2$. Let S' be the subset of S containing all the pairs (a,b) with the property that there exist points x and x' belonging to $C_{a,b}$ such that $(x,x') \notin S$. Then S' contains exactly the pairs of collinear points. Indeed, if $\delta(a,b) = 2$, we can find points x and x' in $C_{a,b}$ at distance 6 from each other, while if $\delta(a,b) = 4$, then $C_{a,b} \subseteq \Gamma_2(m)$. If $s = 2$, then, since both s and t are infinite for n odd, n is even and hence $t > 2$. In this case, we dualize the arguments given above, i.e. we consider a and b lines instead of points (this is allowed since i is odd). Now we only have to distinguish between the case $s = 2$ and $s > 2$. By Step 2, if $s = 2$, the set S is empty while if $s \neq 2$, S contains all pairs of collinear points.

This completes the proof of the case $i = n-1$.

4.4.6 Case $i = n/2$

Let a and b be two elements at distance k, k even, and m an element at distance $k/2$ from both a and b. Then it is easy to see that, if $k \neq n$, an arbitrary element $x \in T_{a,b}$ lies at distance $n/2 - k/2$ from m such that $\mathrm{proj}_m a \neq \mathrm{proj}_m c \neq \mathrm{proj}_m b$. Now we define the set $S_{a,b}$ as the set of elements c, $a \neq c \neq b$, for which $T_{a,b} \subseteq \Gamma_{n/2}(c)$.

Suppose first i is odd, and $s > 2$. Let S be the following set:

$$S = \{(a,b) \in \mathcal{P}^2 \cup \mathcal{L}^2 : |S_{a,b}| \geq 2 \text{ and } \exists c,d \in S_{a,b} : T_{a,b} \neq T_{c,d}\}.$$

Then a pair (a,b) of points or lines belongs to S if and only if $2 < \delta(a,b) < n$. Indeed, suppose to fix the ideas that a and b are points. If $2 \neq \delta(a,b) \neq n$, then consider two points c and d on the line $L = \mathrm{proj}_a b$, different from a or $\mathrm{proj}_L b$ (this is possible since $s > 2$). If $\delta(a,b) = 2$, then $S_{a,b} = \emptyset$. Let finally $\delta(a,b) = n$. Then a point c for which $T_{a,b} \subseteq \Gamma_{n/2}(c)$ is necessarily contained in any set $\Gamma_{n/2}(x) \cap \Gamma_{n/2}(y)$, with $x, y \in T_{a,b}$. We now show that for points c, d satisfying $T_{a,b} \subseteq T_{c,d}$, necessarily $T_{a,b} = T_{c,d}$ (and hence $(a,b) \notin S$). Suppose by way of contradiction c and d are such that $T_{a,b} \subsetneq T_{c,d}$, but R is an element belonging to $T_{c,d}$ for which $\delta(a,R) \neq n/2$. Define v as $[\mathrm{proj}_c R, a] \cap [\mathrm{proj}_c R, d] = [\mathrm{proj}_c R, v]$ and put $\delta(c,v) = j$. Note that $j < n/2$. Define w as $[\mathrm{proj}_a v, b] \cap [\mathrm{proj}_a v, c] = [\mathrm{proj}_a v, w]$ and put $\delta(a,w) = j'$. Let R' be the element of $T_{a,b}$ on the n-path between a and b containing $\mathrm{proj}_a v$. Suppose first R' lies on the path $[a,w]$. Then, joining the paths $[d,v]$ and $[v,R']$, one obtains a path of length $\ell = \frac{3n}{2} - 2j$ between d and R'. Note that $\delta(d,R') = n/2$. If $\ell \leq n$, this implies that $\ell = n/2$, hence $j = n/2$, a contradiction. If $\ell > n$, there arises a circuit of length at most $2n - 2j < 2n$, again a contradiction. So R' does not belong to the path $[a,w]$, implying $j' < n/2$. Now similarly, one obtains a contradiction with $\delta(c,R') = n/2$. Hence $T_{a,b} = T_{c,d}$ for this case. So we obtained that $(a,b) \in S$ if and only if $\delta(a,b) \in \kappa = \{4,\ldots,n-2\}$[3]. If $n = 6$, then S is the set of all pairs of elements of Γ at distance 4 from each other, which ends the proof in this case (because of Paragraph 4.4.4). So assume $n \geq 10$.

Define the following sets S' and S'':

$$S' = \{(p,L) \in \mathcal{P} \times \mathcal{L} | \Gamma_{n/2}(p) \subseteq \Gamma_\kappa(L)\},$$
$$S'' = \{(a,b) \in \mathcal{P}^2 | \exists L \in \mathcal{L} : (a,L),(b,L) \in S'\}.$$

We claim that $(p,L) \in S'$ if and only if $\delta(p,L) \leq n/2 - 4$. Suppose $\delta(p,L) = k \leq n/2 - 4$. A line X at distance $n/2$ from p lies at distance at most $k + n/2 \leq n - 4$ from L. A line L' concurrent with L lies at distance $k+2, k$ or $k-2$ from p, hence $\delta(L',p) \neq n/2$. This shows that $\Gamma_{n/2}(p) \subseteq \Gamma_\kappa(L)$. If $\delta(p,L) = n/2 - 2$, a line L' concurrent with L for which $\mathrm{proj}_L p \neq \mathrm{proj}_L L'$ lies at distance $n/2$ from p, but $\delta(L,L') \notin \kappa$, hence $(p,L) \notin S'$. If finally $k \geq n/2$, then one can easily find a line at distance $n/2$ from p and opposite L. This shows the claim. Now it immediately follows that a pair $(a,b) \in S''$ if and only if $\delta(a,b) \leq n-8$. Then $S'' \setminus (S \cap S'')$ is the set of pairs of collinear points, which concludes the proof.

Let now $s = 2$. The case $n = 6$ and $t = 2$ follows from an easy counting, so we can assume $t > 2$. Then for two points a and b, $\delta(a,b) = 4$ if and only

[3] Alternatively, one can now argue as in Case 4.4.3.

if $|S_{a,b}| = 2$. Indeed, if $\delta(a, b) = 4$, then the points on am and bm, different from a, b or m, are exactly the elements of $S_{a,b}$. If $\delta(a, b) = 2$, then $S_{a,b} = \emptyset$. If $4 < \delta(a, b) = k < n$, then any point x at distance $k/2$ from m for which $\mathrm{proj}_m a = \mathrm{proj}_m x$ or $\mathrm{proj}_m b = \mathrm{proj}_m x$ belongs to $S_{a,b}$. If finally $\delta(a, b) = n$, then the set $\Gamma_{n/2}(x) \cap \Gamma_{n/2}(y)$, with $x, y \in T_{a,b}$ contains only one element different from a and b (since i is odd and $s = 2$), hence $|S_{a,b}| \leq 1$. So we recovered distance 4 and by Subsections 4.4.3 and 4.4.4, the result follows in this case.

If i is even and $s \neq 2$, the proof is similar to the case i odd (the only difference is that for the sets S and S', we consider pairs of points). Let finally i be even and $s = 2$. Then one easily shows that for two points a and b, $\delta(a, b) = n-2$ if and only if $|T_{a,b}| = 1$, which again ends the proof (see case Subsection 4.4.4).

Now we still have to distinguish the cases $s = 2$ and $s > 2$. Let R be the set of pairs of points (a, b) for which $|S_{a,b}| = 2$ and, putting $S_{a,b} = \{x, y\}$, either $|S_{a,x}| \neq 2$ or $|S_{a,y}| \neq 2$. Suppose for two points a and b, $|S_{a,b}| = 2$. Then either $\delta(a, b) = 4$, $s = 2$ and x, y are the unique points on the lines am and bm different from a, m and b, or (possibly) $\delta(a, b) = n$. In the former case, a is collinear with x or y, hence $S_{a,x} = \emptyset$ or $S_{a,y} = \emptyset$, implying $(a, b) \in R$. In the latter case, putting $S_{a,b} = \{x, y\}$, it is easy to see that $S_{a,b} = S_{a,v}$ for $v \in \{x, y\}$, (since we already showed that $T_{a,b} = T_{a,v} = T_{x,y}$ for $v \in \{x, y\}$), hence $(a, b) \notin R$. So if $s = 2$, then $R \neq \emptyset$, while if $s > 2$, $R = \emptyset$. This distinguishes these cases and ends the proof. $\square$

Application 4.4.1 • *Let Γ, Γ' be a generalized n-, respectively a generalized m-gon, $n \neq m$ and $n, m \geq 4$. Let i be an even integer satisfying $1 \leq i \leq n$, $1 \leq i \leq m$. Furthermore, suppose that the orders of Γ and Γ' do not contain 2. Then there does not exist a surjective map α from the point set of Γ onto the point set of Γ' preserving [4] distance i.*

- *Let Γ, Γ' be a generalized n-, respectively a generalized m-gon, $n \neq m$ and $n, m \geq 4$. Let i be an odd integer satisfying $1 \leq i \leq n$, $1 \leq i \leq m$. Furthermore, suppose that the orders of Γ and Γ' do not contain 2. Then there does not exist a surjective map from the point set of Γ onto the point set of Γ', and from the line set of Γ onto the line set of Γ' preserving distance i.*

Proof. This application follows by combining parts of the proof of Theorem 4.2.1. As before, one shows that α is bijective. If in the proof of

[4] 'preserving' means as before that for any two points a, b of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$

Theorem 4.2.1 collinearity of points is characterized by the same property for Γ and Γ', then it follows that α would preserve distance 2. This is a contradiction with $n \neq m$. Indeed, without loss of generality, we can assume $n < m$. Let Σ be an ordinary n-gon in Γ. Then, if α preserves collinearity, the points of Σ are mapped onto the points of a (stammering) closed path of length at most $2n$ in Γ', hence all these points must be sent to $\Gamma'_1(R)$, for some line R of Γ'. Since any two points of Γ are contained in an apartment, this implies that all the points of Γ are mapped onto points of R, contradicting the bijectivity of α.

So we only have to consider the cases where the characterization of distance 2 is not the same for Γ and Γ' (and this roughly corresponds with the subsections in the proof of Theorem 4.2.1). Therefore we start with some observations. Let Γ be a generalized n-gon, $n \geq 4$, for which the order (s,t) satisfies $s,t \geq 3$ and i an integer, $2 \leq i \leq n$.

(1) *There exist points a,b of Γ for which $T_{a,b} = \emptyset$ if and only if $i < \frac{n-1}{2}$.*

(2) *There exist points a,b,c for which $T_{a,b} \subseteq \Gamma_i(c)$ if and only if $i < n-2$.*
Proof. Suppose first $i < n-2$ and let a,b be points at distance 4. Then each element x of $T_{a,b}$ lies at distance $i-2$ from $w := a \bowtie b$, with $\mathrm{proj}_w a \neq \mathrm{proj}_w x \neq \mathrm{proj}_w b$. A point c on the line aw, $a \neq c \neq w$ satisfies $T_{a,b} \subseteq \Gamma_i(c)$. Suppose now $i \geq n-2$ and let a,b be two points at distance k. Let w be a fixed element at distance $k/2$ from both a and b. Since $|\Gamma_1(w)| \geq 4$, we can argue as in subsection 4.4.3, Claims 2 and 3 to obtain that a point c as above cannot exist. $\diamond$

Define $S_{a,b} = \{x \in \mathcal{P} \,|\, \Gamma_i(x) \cap T_{a,b} = \emptyset\}$.

(3) *If $i = n-2$, there exist points a,b for which $S_{a,b} \neq \emptyset$, if $i \in \{n-1, n\}$, $S_{a,b} = \emptyset$ for any two points a and b.*
Proof. The claim for $i = n-2$ was shown in Step 1 of Case 4.4.4. Suppose now $i \in \{n-1, n\}$. Let a,b be points at distance k, and w an element at distance $k/2$ from both a and b. Let, by way of contradiction, c be a point of $S_{a,b}$. Suppose first there exists a shortest path between c and w not containing $\mathrm{proj}_w a$ or $\mathrm{proj}_w b$. Put $j = \delta(c,w)$. If $j = k/2$, then any point x of $T_{a,b}$ at distance $i - \frac{k}{2}$ from w for which $\mathrm{proj}_w x \notin \{\mathrm{proj}_w a, \mathrm{proj}_w b, \mathrm{proj}_w c\}$ lies at distance i from c, a contradiction (note that x exists since both $s,t \geq 3$). If $j < k/2$, let X be the element of the path $[a,c]$ at distance $j/2 + k/4$ from both a and c. Note that X belongs to the path $[a,w]$. Now for an arbitrary element x at distance $i-(j/2+k/4)$ from X with $\mathrm{proj}_X a \neq \mathrm{proj}_X x \neq \mathrm{proj}_X c$ (note that such a point x belongs to $T_{a,c}$) there arises a path of length $\ell = k + i - 2(j/2 + k/4)$ between b and x (consisting of the paths $[b,X]$ and

$[X, x]$). Since $\ell \geq n-1$, we can choose the element x to lie at distance i from b, a contradiction with $c \in S_{a,b}$. If $j > k/2$, let X be the element of the path $[c, w]$ at distance $j/2 + k/4$ from c. Since $j/2 + k/4 \leq n-1$, it is possible to find a point x at distance $i - (j/2 + k/4)$ from X for which $\mathrm{proj}_X a \neq \mathrm{proj}_X x \neq \mathrm{proj}_X c$. Such a point x lies at distance i from a, b and c, a contradiction. So we can assume that a shortest path between w and c contains $\mathrm{proj}_w a$. Define the element v as $[w, a] \cap [w, c] = [w, v]$. Let $j = \delta(v, c)$ and $\ell = \delta(v, a)$. Let γ' be the path between a and c obtained by joining $[a, v]$ and $[v, c]$, and X the element of γ' at distance $\frac{\ell+j}{2}$ from both a and c. Let x be an element at distance $i - \frac{\ell+j}{2}$ from X such that $\mathrm{proj}_X x \notin \{\mathrm{proj}_X a, \mathrm{proj}_X b, \mathrm{proj}_X c\}$ (note that such a point x belongs to $T_{a,c}$). There arises a path of length $\geq n-1$ between b and x (consisting of the paths $[b, X]$ and $[X, x]$). Hence we can choose the point x to lie at distance i from b, the final contradiction. $\diamond$

(4) *If $i = n = m-1$, then α cannot exist.*
Proof. Suppose $i = n = m-1$. Let for two points a, b of Γ or Γ', $O_{a,b}$ be the set of pairs of points $\{c, c'\}$, c and c' different from a and b, for which $T_{v,v'} \subseteq \Gamma_i(w) \cup \Gamma_i(w')$ whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$, and $C_{a,b}$ the set containing a, b and all points c for which there exists a point c' such that $\{c, c'\} \in O_{a,b}$. Let S^Γ be the set of pairs of points (a, b) of Γ with $\delta(a, b) \neq i$, for which there does not exist an element at distance i from all the points of $C_{a,b}$. Similarly, we define the set $S^{\Gamma'}$ of pairs of points of Γ'. In the proof of Theorem 4.2.1, Case 4.4.5, it was shown that $S^{\Gamma'}$ contains exactly the pairs of points at mutual distance 2 or 4. We claim that S^Γ contains the set of pairs of collinear points, and that for two collinear points a and b of Γ, one has $C_{a,b} = \Gamma_1(ab)$ (and to prove this, similar arguments as in the proof of Case 4.4.5 are used).

So let a and b be two collinear points of Γ, $i = n$ and suppose $\{c, c'\} \in O_{a,b}$. For an element y at distance $0 \leq j \leq n-2$ from $m := ab$, with $\mathrm{proj}_m y \notin \{a, b\}$, define the set $T_y = \{x \in T_{a,b} | \delta(x, y) = n-j-1\}$. It is easy to see that there does not exist a point x distinct from a and b such that $T_m \subseteq \Gamma_n(x)$. Let y be a point incident with m, $y \neq a, b$ and suppose there is an element $v \in \{c, c'\}$ for which $T_y \subseteq \Gamma_n(v)$. Then it is easy to see that v is necessarily a point on m different from a, b and y. Since such a point v is not opposite any point of T_v, $T_v \subseteq \Gamma_n(v')$, with $\{v, v'\} = \{c, c'\}$. This implies that also the point v' is incident with m, and v' is different from a, b and v. Noting that every point of $T_{a,b}$ lies at distance $n-2$ from a unique point of m, it is now easy to see that such a pair $\{c, c'\}$ indeed belongs to $O_{a,b}$. From now on, we assume that no point of $\{v, v'\}$ satisfies $T_y \subseteq \Gamma_n(v)$, for a point y incident with m. We proof by induction on $\delta(m, y)$ that no point of $\{v, v'\}$ satisfies

$T_y \subseteq \Gamma_n(v)$, for any element y for which the set T_y is defined.

Consider an element y for which the set T_y is defined, with $\delta(y,m) = j > 1$. Suppose there is an element $v \in \{c,c'\}$ for which $T_y \subseteq \Gamma_n(v)$. Then $\delta(y,v) = \delta(y,a)$ and $\mathrm{proj}_y a = \mathrm{proj}_y v = y'$ but, using the induction hypothesis, $\mathrm{proj}_{y'} a \neq \mathrm{proj}_{y'} v =: y''$. Note that $T_{y''} \cap \Gamma_n(v) = \emptyset$, hence $T_{y''} \subseteq \Gamma_n(v')$, with $\{v,v'\} = \{c,c'\}$. This implies that $\delta(y'',v') = \delta(y'',a)$ and $\mathrm{proj}_{y''} a = \mathrm{proj}_{y''} v' = y'$ but $\mathrm{proj}_{y'} a \neq \mathrm{proj}_{y'} v'$. Now let z be an element at distance $n - \delta(v,y')$ $(= n - \delta(v',y'))$ from y' such that either the path $[y',z]$ contains the path $[y',a]$, or z belongs to the path $[y',a]$. Then z belongs to $T_{v,v'}$, but does not lie at distance n from a or b, a contradiction. So we have now shown that $T_y \not\subseteq T_v$, for any element y and $v \in \{c,c'\}$. Let L be a line at distance $n-2$ from ab such that $\mathrm{proj}_{ab} L \notin \{a,b\}$. Since neither c nor c' lies opposite all the points of T_L, we can assume that $\delta(c,L) = \delta(c',L) = n-3$ and the projections of c and c' onto L are different from $\mathrm{proj}_L ab$, and different from each other. Now the element on the path $[L,ab]$ at distance 3 from L belongs to $T_{c,c'}$ but is not opposite a or b, the final contradiction. This shows the claim concerning S^Γ.

Let (a,b) be a pair of points of Γ belonging to S^Γ for which $\delta(a,b) = 2$. Then by the above, for any two distinct points c,c' belonging to $C_{a,b} \setminus \{a,b\}$, one has $T_{a,b} \subseteq \Gamma_n(c) \cup \Gamma_n(c')$, hence $(c,c') \in O_{a,b}$. However, no two points in $S^{\Gamma'}$ have this propery. Indeed, if $(a,b) \in S^\Gamma$ with $\delta(a,b) = 2$, then consider a point c on ab, and a point c' collinear with c not incident with ab; if $\delta(a,b) = 4$, consider a point c on the line am (with $m = a \bowtie b$) distinct from m, and a point c' collinear with m not on the lines am or bm. This shows that α cannot exist. $\diamond$

Now let Γ and Γ' be as above. Without loss of generality, we can assume $n < m$. If $i \leq \frac{n-1}{2}$ (hence also $i \leq \frac{m-1}{2}$), we can characterize distance 2 with the same property for Γ and Γ' (because of subsection 4.4.1). The case $i \geq \frac{n-1}{2}$ but $i < \frac{m-1}{2}$ would contradict (1). Indeed, for a pair (a,b) of points of Γ' for which $T_{a,b} = \emptyset$, one would obtain, with $(a',b') = (a^{\alpha^{-1}}, b^{\alpha^{-1}})$, $T_{a',b'} \neq \emptyset$, contradicting the fact that α preserves the cardinality of $T_{a,b}$. Hence we can assume from now on that $i \geq \frac{n-1}{2}$ and $i \geq \frac{m-1}{2}$.

From (2), we deduce that either $\frac{n-1}{2} \leq i < n-2$ and $\frac{m-1}{2} \leq i < m-2$ or $i \geq n-2$ and $i \geq m-2$. The case $i = m-2$ and $i \in \{n-1,n\}$ contradicts (3), so we can assume $i = n = m-1$ or $\frac{n-1}{2} \leq i < n-2$ and $\frac{m-1}{2} \leq i < m-2$. The first case is exluded by (4), so consider the latter case. If $i \geq \frac{n+1}{2}$ and $i \geq \frac{m+1}{2}$, we can apply the result of Case 4.4.3. So the remaining cases are $i \in \{\frac{m-1}{2}, \frac{m}{2}\}$ and either $i \geq \frac{n+1}{2}$ or $i \in \{\frac{n-1}{2}, \frac{n}{2}\}$.

- $i = \frac{m-1}{2}$
  In this case $|T_{a,b}| = 1$ for two points a and b of Γ' at distance $m-1$ from each other. If $i \geq \frac{n+1}{2}$ no points a', b' of Γ for which $|T_{a',b'}| = 1$ can be found. If $i = \frac{n}{2}$, then points a', b' of Γ for which $|T_{a,b}| = 1$ only exist if $s = 2$ or $t = 2$, contradicting the assumption on the order of Γ.

- $i = \frac{m}{2}$
  Note that this implies $i \geq \frac{n+1}{2}$. Let S_Γ ($S_{\Gamma'}$) be the set of pairs of points (a, b) of Γ (Γ') for which there exist two elements c, d satisfying $T_{a,b} \subseteq \Gamma_i(c) \cap \Gamma_i(d)$ and $T_{a,b} \neq T_{c,d}$. We claim that $(a, b) \in S_\Gamma$ ($S_{\Gamma'}$) if and only if $2 < \delta(a, b) < 2(n-i)$ ($2 < \delta(a, b) < m$). If a and b are points of Γ', this was shown in Case 4.4.6. If a and b belong to Γ, this follows from Case 4.4.3. Now one can proceed similarly as in the proof of Case 4.4.3 (defining the set S') to characterize distance 2 in the same way for Γ and Γ', from which the result.

This proves the application. $\square$

Is the same result true if $n > m$ and we do not require that the map α is surjective ?

4.5 Some exceptions and applications to the Point-Line Theorem

Counterexample for the case $i = n$

Let $\mathsf{H}(\mathbb{K})$ be the split Cayley hexagon defined on the quadric $\mathsf{Q}(6, \mathbb{K})$. Now we choose an automorphism α of $\mathsf{Q}(6, \mathbb{K})$ which does not preserve the line set of $\mathsf{H}(\mathbb{K})$. Such an automorphism α induces a permutation of the points of $\mathsf{H}(\mathbb{K})$. Because opposition and non-opposition in the hexagon corresponds with non-collinearity and collinearity respectively on the quadric, α has the property that $\delta(x, y) = 6$ if and only if $\delta(x^\alpha, y^\alpha) = 6$, for any two points x, y of $\mathsf{H}(\mathbb{K})$. But clearly, α does not preserve collinearity. Hence we have produced a counterexample to Theorem 4.2.1 for $i = n = 6$.

Note that the previous class of counterexamples contains finite hexagons (putting $\mathbb{K}$ equal to any finite field). We now show that, for the finite case, the only counterexamples must be hexagons of order (s, s). If there is enough transitivity around, then these are the only counterexamples (see below for a precise statement).

Theorem 4.5.1 *Let Γ and Γ' be two finite generalized n-gons of order (s,t) and (s',t'), respectively, let α be a bijection between the points of Γ and Γ', and fix an even number i, $2 < i \leq n$. If for every two points x and y of Γ, $\delta(x,y) = i \iff \delta(x^\alpha, y^\alpha) = i$, then either α extends to an isomorphism between Γ and Γ', or else we have $n = i = 6$ and $s = t = s' = t'$.*

Proof. By Theorem 4.2.1 we may assume that $i = n$. First consider the case $n = i = 6$. We may assume $s \neq t$. Then clearly, also $s' \neq t'$. Let a, b be two points of Γ. If $\delta(a,b) = 2$, then $|\Gamma_6(a) \cap \Gamma_6(b)| = s^2t^2(s-1)$. If $\delta(a,b) = 4$, then $|\Gamma_6(a) \cap \Gamma_6(b)| = st(t-s+st(s-1))$. These two numbers are different since $s \neq t$. Hence either two points at distance 4 are always mapped onto collinear points, or two points at distance 4 are always mapped onto points at distance 4. In the latter case, the theorem is proved. In the former case, we obtain by counting the number of points collinear with a fixed point in Γ - and this should be equal to the number of points at distance 4 from a fixed point in Γ' - that $s(t+1) = (t'+1)s'^2t'$ and similarly $s'(t'+1) = (t+1)s^2t$. Combining these, we obtain the contradiction $sts't' = 1$.

Next consider the case $n = i = 8$. We first prove that $(s,t) = (s',t')$. Indeed, we already have $(1+s)(1+st)(1+s^2t^2) = (1+s')(1+s't')(1+s'^2t'^2)$, and also, looking at the number of points opposite a given point, $s^4t^3 = s'^4t'^3$. Suppose $st \neq s't'$. Putting $X = st$ and $X' = s't'$ in the first equation (thus eliminating t and t') and then substituting $s' = sX^3/X'^3$ in the equation obtained, we get, after dividing by $X - X'$ the following quadratic equation in X:

$$(X'^3 - sX'^2 - sX' - s)X^2 + (X'^4 + X'^3 - sX'^2 - sX')X + (X'^5 + X'^4 + X'^3 - sX'^2) = 0.$$

Note that the expression $X'^3 - sX'^2 - sX' - s$ is always positive for $X' = s't' \geq 8$ (this can been shown using $s \leq t^2$ and hence $s^4 = \frac{s'^4t'^3}{t^3} \leq \frac{s'^4t'^3}{s^{3/2}}$, implying $s \leq s'^{8/11}t'^{6/11}$). From this follows that the quadratic equation above has no positive solutions. This proves $st = s't'$ and hence (combined with $s^4t^3 = s'^4t'^3$) $s = s'$ and $t = t'$. Now let a, b be two points of Γ. If $\delta(a,b) = 2$, then $\ell_2 := |\Gamma_8(a) \cap \Gamma_8(b)| = (s-1)s^3t^3$. If $\delta(a,b) = 4$, then $\ell_4 := |\Gamma_8(a) \cap \Gamma_8(b)| = s^2t^2(st(s-1)+t-s)$. These two numbers are different because $s \neq t$. Let $\ell_6 = |\Gamma_8(a) \cap \Gamma_8(b)|$ with $\delta(a,b) = 6$. Notice that ℓ_6 is a constant, independent of a, b. If $\ell_6 \neq \ell_2$, then clearly α must preserve collinearity. Likewise, if $\ell_6 \neq \ell_4$, then α must preserve distance 4. The result now follows from Theorem 4.2.1. $\square$

Theorem 4.5.2 *Let Γ and Γ' be two generalized n-gons, $n \in \{6, 8\}$, and suppose that Γ' has an automorphism group acting transitively on the set of* *pairs of points at mutual distance* $n-2$ *(this is in particular satisfied if* Γ' *is a Moufang* n*-gon, or if* Γ' *arises from a* BN*-pair). Suppose there exists a bijection* α *from the point set of* Γ *to the point set of* Γ' *such that, for any pair of points* a, b *of* Γ*, we have that* a *is opposite* b *if and only if* a^α *is opposite* b^α*. If* α *is not an isomorphism, then* $\Gamma \cong \Gamma' \cong \mathsf{H}(\mathbb{K})$ *and for any isomorphism* $\beta : \Gamma \to \Gamma'$*, the permutation of the points of* Γ *defined by* $\alpha\beta^{-1}$ *arises as in the counterexample above.*

Proof. Let first $n = 6$. Let x and y be two collinear points for which $x' := x^\alpha$ and $y' := y^\alpha$ lie at distance 4 (these exist since otherwise α is an isomorphism by Lemma 1.3.14 of [57]). We look for the image of the line $L := xy$. Note that a point z, $x \neq z \neq y$, lies on L if and only if there is no point of Γ opposite exactly one point of the set $\{x, y, z\}$ (see for instance [1]). Since this property is preserved by α, it is easy to check that a point z of the line L has to be mapped onto a point of the distance-2 hyperbolic line $H := H(x', y')$. Now we claim that H is a long distance-2 hyperbolic line. Indeed, let K be a line of Γ' at distance 5 from all the points of H, and suppose that the projection of H onto K is not surjective. This would imply that there is a point opposite all the points of H, so in particularly opposite all the points of L^α. Applying α^{-1}, we see that there would be a point opposite all the points of L, a contradiction. Our claim follows. (In fact, the very same argument shows that $L^\alpha = H$.) So Γ' contains a long hyperbolic line. The transitivity condition on the group of automorphisms of Γ' now easily implies that all distance-2 hyperbolic lines are long. From Theorem 1.8.5 (i) then follows that $\Gamma' \cong \mathsf{H}(\mathbb{K})$, and we may actually put $\Gamma' = \mathsf{H}(\mathbb{K})$.

Moreover, since the map α preserves distance 6, we obtain a representation of Γ on $\mathsf{Q}(6, \mathbb{K})$ with the property that opposition in Γ coincides with opposition in $\mathsf{Q}(6, \mathbb{K})$ (the latter viewed as a polar space: opposite points are just non-collinear points). Now, it is easy to see that, if x is any point of Γ (whose point set is identified with the point set of $\mathsf{Q}(6, \mathbb{K})$), then the set $\Gamma_2(x)$ is contained in a plane π_x of $\mathsf{Q}(6, \mathbb{K})$ (indeed, the space generated by $\Gamma_2(x)$ in $\mathrm{PG}(6, \mathbb{K})$ is a singular subspace of $\mathsf{Q}(6, q)$). If a point y of π_x would be at distance 4 from x in Γ, then y would be at distance ≤ 4 from all points in $\Gamma_1(x)$, a contradiction. Hence we can apply Theorem 1.2 of Cuypers & Steinbach [14] to obtain $\Gamma \cong \mathsf{H}(\mathbb{K})$. It is clear that, for a given isomorphism $\beta : \Gamma \to \mathsf{H}(\mathbb{K}) = \Gamma'$, the map $\alpha\beta^{-1}$ can be seen as a permutation of the point set of $\mathsf{Q}(6, \mathbb{K})$ preserving opposition and collinearity, hence it is an isomorphism of $\mathsf{Q}(6, \mathbb{K})$. The result follows.

Let now $n = 8$. Let x and y be two collinear points in Γ for which $x' = x^\alpha$ and

$y' = y^{\alpha}$ lie at distance 4 or 6. Completely similar as above, one shows that the image of $L = xy$ is the long distance-2 hyperbolic line or the long distance-3 hyperbolic line defined by x' and y'. The transitivity condition now implies that either all distance-2 hyperbolic lines or all distance-3 hyperbolic lines are long. This contradicts Theorem 1.3 resp. Theorem 2.6 of van Bon, Cuypers & Van Maldeghem [55].

The theorem is proved. $\square$

Remark. The previous theorem means in fact that, for hexagons and octagons with a fairly big automorphism group, Theorem 4.2.1 remains true if we rephrase the conclusion as: " ... then Γ and Γ' are isomorphic", and if we do not insist on the fact that α defines that isomorphism. Also, we have only considered the important values $n = 6, 8$. Using the results of van Bon, Cuypers & Van Maldeghem [55], we can allow for more (though all odd) values, such as $n = 5, 7, 9$. Indeed, for example for $n = 9$ (under the same transitivity conditions as in the theorem above) the existence of α would imply that either all distance-2, distance-3 or distance-4 hyperbolic lines are long. But each of these values contradicts Theorem 2.6 in [55].

Given two finite generalized hexagons Γ and Γ' of order (q, q) and a bijection from the points of Γ to the points of Γ' preserving opposition but not collinearity. Is $\Gamma \cong \Gamma'$?

Lemma 4.5.3 *Let Γ be a generalized hexagon, and let α be a permutation of the point set of Γ preserving the opposition relation. Then the set S of lines L of Γ such that $\Gamma_1(L)^{\alpha} = \Gamma_1(M)$, for some line M of Γ, is a dual ovoidal subspace in Γ.*

Proof. We have to show that every line of Γ not in S is concurrent with a unique line of S. We first claim that

(a) if L and L' are two lines of S at distance 4, then also the line $L \bowtie L'$ belongs to S,

(b) if L and L' are two concurrent lines of S, then all the lines concurrent with both L and L' belong to S.

Indeed, let first $\delta(L, L') = 4$, with $L, L' \in S$. Let M and M' be the lines of Γ incident with all the images (under α) of L and L', respectively. All points of L except for $\mathrm{proj}_L L'$ are opposite all points of L' except for $\mathrm{proj}_{L'} L$;

hence all elements of $\Gamma_1(M) \setminus (\mathrm{proj}_L L')^\alpha$ are opposite all elements of $\Gamma_1(M') \setminus (\mathrm{proj}_{L'} L)^\alpha$. Hence M and M' must be at distance 4 from each other, and $x := (\mathrm{proj}_L L')^\alpha$ must be collinear with $x' := (\mathrm{proj}_{L'} L)^\alpha$. Consequently the points of the line $L \bowtie L'$ are mapped onto the points of the line xx'. This proves (a). A similar argument shows (b).

Now suppose L is a line of Γ not belonging to S. We know that, by the proof of Theorem 4.5.2, the image under α of $\Gamma_1(L)$ is a certain distance-2 hyperbolic line $H(x, y)$. Put $a := x \bowtie y$. The point $a' := a^{\alpha^{-1}}$ is not opposite any element of $\Gamma_1(L)$, hence it is collinear with a unique point $b \in \Gamma_1(L)$. It now easily follows that the line $a'b$ belongs to S. This shows that L is concurrent with at least one line L' belonging to S. By (a) and (b), this line is unique. $\square$

Application 1. *The intersection of the line sets of two generalized hexagons* $\Gamma \cong \mathsf{H}(\mathbb{K})$ *and* $\Gamma' \cong \mathsf{H}(\mathbb{K})$ *on the same quadric* $\mathsf{Q}(6, \mathbb{K})$ *is a dual ovoidal subspace in both these hexagons.*

Proof. Denote by S the intersection of the line set of the two hexagons Γ and Γ' living on the same quadric $\mathsf{Q}(6, q)$. By a simple change of coordinates, one easily verifies that for both Γ and Γ', coordinates can be chosen as in section 1.9.2. Hence there exists an automorphism θ of the quadric $\mathsf{Q}(6, \mathbb{K})$ mapping Γ to Γ'. (This also follows directly from Tits' classification of trialities in [49].) Now θ preserves the opposition relation in the hexagons. Applying Lemma 4.5.3, we obtain that $\theta^{-1}(S)$ is a dual ovoidal subspace in Γ, so S is a dual ovoidal subspace in Γ'. Applying θ^{-1}, we conclude that S is also a dual ovoidal subspace in Γ. $\square$

Remark. A similar result is true for the symplectic quadrangle $\mathsf{W}(\mathbb{K})$ over some field $\mathbb{K}$. But there, the proof is rather easy, because the intersection of the line sets of two symplectic quadrangles naturally represented in $\mathrm{PG}(3, \mathbb{K})$ boils down (dually using the Klein correspondence) to the intersection of a non-singular quadric $\mathsf{Q}(4, \mathbb{K})$ in $\mathrm{PG}(4, \mathbb{K})$ with a hyperplane. Hence this intersection is always a dual geometric hyperplane (of classical type).

Determine the intersection of the set of reguli of two generalized hexagons $\Gamma \cong \mathsf{H}(\mathbb{K})$ *and* $\Gamma' \cong \mathsf{H}(\mathbb{K})$ *lying on the same quadric* $\mathsf{Q}(6, \mathbb{K})$.

The aim of the second application is to prove the maximality of the group $G_2(q)$ in $O_7(q)$ in an entirely geometric way. We first prove some general results about dual ovoidal subspaces in the hexagon $\mathsf{H}(q)$. Note that, if Γ and Γ' are two hexagons isomorphic to $\mathsf{H}(q)$ such that the line sets of Γ and Γ'

intersect in a dual ovoidal subspace which is a spread $\mathcal{S}$, then $\mathcal{S}$ is necessarily Hermitian. Indeed, if two lines belong to $\mathcal{S}$, then clearly so do all lines of the regulus defined by those two lines on $\mathsf{Q}(6,q)$. By Theorem 1.7.1, the spread $\mathcal{S}$ is Hermitian. In the following, we say that a dual ovoidal subspace of $\mathsf{H}(q)$ is of type S′ if it is a Hermitian spread.

Lemma 4.5.4 *Let N_{X} be the number of dual ovoidal subspaces of type X in $\mathsf{H}(q)$, then we have*

$$\begin{cases} N_{\mathrm{P}} = q^5+q^4+q^3+q^2+q+1, \\ N_{\mathrm{H}} = \frac{q^3(q^3+1)}{2}, \\ N_{\mathrm{S}'} = \frac{q^3(q^3-1)}{2}. \end{cases}$$

The automorphism group $G_2(q)$ of $\mathsf{H}(q)$ acts transitively on the dual ovoidal subspaces of type P, H *and* S′ *respectively.*

Proof. A dual ovoidal subspace of type P (H) is determined by one point (a pair of opposite points), hence the result follows for these types of subspaces. In Theorem 2.5.4, it was shown that there are exactly $\frac{q^2-q}{2}$ Hermitian spreads containing a fixed line regulus. Noting that there are $q^4(q^4+q^2+1)$ line reguli, and that each Hermitian spread contains $\frac{q^2(q^3+1)}{q+1}$ line reguli, $N_{S'}$ follows. The stabilizer in $G_2(q)$ of a Hermitian spread $\mathcal{S}$ is the group $U_3(q)$, hence the length of the orbit of $\mathcal{S}$ is $\frac{|G_2(q)|}{|U_3(q)|}$, which is equal to $N_{S'}$. We conclude that $G_2(q)$ acts transitively on the set of Hermitian spreads. □

Lemma 4.5.5 *Let $\Gamma = \mathsf{H}(q)$ be defined on $\mathsf{Q}(6,q)$. Then there are exactly $q+1$ copies of Γ on $\mathsf{Q}(6,q)$ containing a given dual ovoidal subspace of Γ of type* S′*, exactly q containing one of type* P*, and exactly $q-1$ containing one of type* H.

Proof. Let $\mathcal{O} = \Gamma_1(p) \cup \Gamma_3(p)$ be a dual ovoidal subspace of type P in Γ, and suppose Γ' is a copy of $\mathsf{H}(q)$ also containing $\mathcal{O}$. Let M be a line at distance 3 from p. Suppose there exists a point x on M at distance 4 from p for which $\Gamma_1(x) = \Gamma_1'(x)$. We show that this implies $\Gamma = \Gamma'$. Therefore, it suffices to prove that Γ and Γ' share at least one apartment, and all lines concurrent with one of three consecutive concurrent lines L_1, L_2, L_3 of that apartment (see Van Maldeghem [57], proof of Theorem 6.3.1). Let Σ be an apartment of Γ containing x and p and denote by z the point of Σ opposite x. Let p' be the point of Σ opposite p, let L be the line of Σ through z different from pz and put $y = \mathrm{proj}_L x$ (everything in the hexagon Γ). Since y is the intersection of the line L (which also belongs to the hexagon Γ') with the tangent hyperplane of Q in x, the point y is also the unique point on L at distance 4 from x in the hexagon Γ'. Let N be the projection of y onto x in Γ. Since N is the unique line through x containing no points opposite y (opposition seen on the quadric), N is also the projection of y onto x in Γ'. Now the point p' is the unique point on the line N not opposite z. Hence $p'y$ is a line of Γ', so Γ and Γ' share the apartment Σ. Completely similar, one shows that if Γ and Γ' share two opposite lines X, X', and all lines through a certain point v of X, then they also share all lines through the unique point on X' not opposite v. Let a be a point on the line $\mathrm{proj}_p M$, different from p and $p \bowtie x$, and $a' = \mathrm{proj}_{p'y} a$ in Γ. Note that Γ and Γ' share the path $[a, a']$ and all lines concurrent with the line through a' and $a \bowtie a'$ (because this line is opposite pz). Hence Γ and Γ' share the lines concurrent with the line M. We conclude that $\Gamma = \Gamma'$. This shows that a hexagon containing $\mathcal{O}$ is completely determined by the choice of a plane $x^\perp$ on Q through the line M, different from the plane containing p. Hence there are at most q such hexagons.

Let $\mathcal{O}$ be a dual ovoidal subspace of type S$'$ in Γ and suppose Γ' is a copy of $\mathsf{H}(q)$ also containing $\mathcal{O}$. Let L_0 be a fixed line of the spread $\mathcal{O}$ and suppose Γ and Γ' share the lines through a certain point x_0 on L_0. We show that $\Gamma = \Gamma'$. Let $L_0, L_1, \ldots, L_q$ be the lines of a regulus contained in $\mathcal{O}$ through L_0. Then as in the first paragraph of the proof, it follows that on each line L_i, $i > 0$, there is a point x_i not opposite x_0 such that $\Gamma_1(x_i) = \Gamma'_1(x_i)$ (and these points x_i mutually lie at distance 4 in both Γ and Γ'). Now let N be a line of $\mathcal{O}$ opposite every line L_i. Because the spread has property $(\diamond)$ (see section 2.5), the projections of the points x_i on the line N are all different. Hence each line concurrent with N belongs to both Γ and Γ'. Since every line of $\mathcal{O} \setminus \{N\}$ is opposite N, it follows that every line concurrent with a spread line belongs to both Γ and Γ'. We conclude that $\Gamma = \Gamma'$. Since there are $q+1$ choices for the plane ${x_0}^\perp$ through L_0, this shows that there are at most $q+1$ hexagons containing $\mathcal{O}$.

Let $\mathcal{O}$ be a dual ovoidal subspace of type H in Γ, and suppose Γ' is a copy of $\mathsf{H}(q)$ also containing $\mathcal{O}$. Let Σ be a fixed apartment contained in $\mathcal{O}$, and L a line of Σ. Then there are at least two points x, y on L for which the planes $x^\perp$ and $y^\perp$ are the same for Γ and Γ'. If Γ and Γ' also share a plane $z^\perp$, $z \mathtt{I} L$, $z \neq x, y$ (and there are $q-1$ choices for such a plane), then Γ and Γ' coincide. Indeed, if we denote by Π the set of planes of Q containing the line L, then the map

$$\begin{array}{rlcl} \sigma: & \Gamma_1(L) & \to & \Pi \\ & p & \to & \pi_p, \end{array}$$

with π_p the plane corresponding to $p^\perp$ in Γ, defines a projectivity of the line L. So σ is completely determined by the choice of the planes in x, y and z. Hence the two hexagons share all lines concurrent with any line of Σ, implying $\Gamma = \Gamma'$. So there are at most $q-1$ copies of Γ containing $\mathcal{O}$ in this case.

In total, we obtain at most $N := (q-1)N_P + (q-2)N_H + qN_{S'}$ hexagons on Q, different from Γ and intersecting Γ in a dual ovoidal subspace. By application 1, the number $N+1$ has to be equal to $\frac{|\mathrm{PGO}_7(q)|}{|G_2(q)|}$, from which the result. $\square$

Application 2. *The group $G_2(q)$ is maximal in $O_7(q)$.*

Proof. Recall that $O_7(q)$ is the derived group of $\mathbf{PSO}_7(q)$ and is simple. It coincides with $\mathbf{PSO}_7(q)$ if q is even, and has order $|\mathbf{PSO}_7(q)|/2$ if q is odd. Let g be any element of $O_7(q)$ not belonging to the automorphism group $G_2(q)$ of $\mathsf{H}(q)$. Let G be the group generated by $G_2(q)$ and g. We show that $G = O_7(q)$. Clearly it suffices to show that $|G| = |O_7(q)|$. To that end, we look at the orbit O of $\mathsf{H}(q)$ under G. This orbit contains images of $\mathsf{H}(q)$ the line set of which intersect $\mathsf{H}(q)$ in dual ovoidal subspaces. By the transitivity of $G_2(q)$ on the three types of dual ovoidal subspaces of $\mathsf{H}(q)$, there are a constant number of elements of O meeting $\mathsf{H}(q)$ in each of the three types of dual ovoidal subspaces. Hence we may assume that there are exactly k elements of O whose line set contains a given dual ovoidal subspace of type P of $\mathsf{H}(q)$. Similarly we define the numbers ℓ and m for type H and type S$'$, respectively. Hence in total, we have

$$N := 1 + k(q^5 + q^4 + q^3 + q^2 + q + 1) + \ell\frac{q^3(q^3+1)}{2} + m\frac{q^3(q^3-1)}{2}$$

elements in O, with $k \leq q-1$, with $\ell \leq q-2$ and with $m \leq q$. We know that $N|G_2(q)|$ $(= |G|)$ divides the order of $O_7(q)$, in particular, it divides the order of $\mathbf{PSO}_7(q)$, which is $q^3(q^4-1)|G_2(q)|$. Hence N divides $q^3(q^4-1)$. Since $(k, \ell, m) \neq (0, 0, 0)$, we see that $N > q^5$. Hence q must divide N, implying q divides $1+k$. Since $0 \leq k \leq q-1$, this means that $k = q-1$. Hence

$$N = q^6 + \ell\frac{q^3(q^3+1)}{2} + m\frac{q^3(q^3-1)}{2}$$

divides $q^3(q^4-1)$. We may write $N = abcd$, where a divides q^3, where b divides q^2+1, where c divides $q+1$ and where d divides $q-1$. If q is even, then a, b, c, d are unique, since every two of the numbers q^3, q^2+1, $q+1$ and $q-1$ are relatively prime. For q odd, there may be different possibilities, and we will make advantage of that below.

First suppose that q is even. Then both c and d are odd, and hence one can divide by 2 modulo c or d. We have $0 \equiv N \bmod c \equiv 1+m \bmod c$ and $0 \equiv N \bmod d \equiv 1+\ell \bmod d$. Hence $m \geq c-1$ and $\ell \geq d-1$. Since $ab \leq q^3(q^2+1)$, we also have

$$(q^2+1)cd - c\frac{q^3-1}{2} - d\frac{q^3+1}{2} \geq 0.$$

This implies

$$d(c(q^2+1) - \frac{q^3+1}{2}) \geq c\frac{q^3-1}{2},$$

which on its turn implies that $c(q^2+1) - \frac{q^3+1}{2} \geq 0$. Hence $c > \frac{q-1}{2}$. Similarly $d > \frac{q-1}{2}$. Since d divides $q-1$, we necessarily have $d = q-1 = \ell+1$. Also, $c \in \{\frac{q+1}{2}, q+1\}$ (or $c = \frac{q}{2} = 1$, hence $m = 2$ and $N = q^3(q^4-1)$, so we are done). If $c = \frac{q+1}{2}$, then $m \in \{\frac{q-1}{2}, q\}$. But clearly, $m = \frac{q-1}{2}$ leads to a contradiction (the N derived from that value does not divide $q^3(q^4-1)$, because it is bigger than half of that number, and not equal to it). Hence $m = q$ and therefore $c = q+1$. We obtain $N = q^3(q^4-1)$ and so $|G| = |O_7(q)|$. This completes the case q even.

Now suppose that q is odd. We essentially try to give a similar proof as for q even, but the arguments need a little more elementary computations. Note that for q odd, $|O_7(q)| = \frac{q^3(q^4-1)}{2}|G_2(q)|$. Hence, we may choose c in such a way that it divides $\frac{q+1}{2}$. We easily compute $N \equiv 1+m \bmod c$. Similarly, we obtain $N \equiv 1+\ell \bmod d/i$, where $i \in \{1,2\}$, depending on the fact whether d divides $\frac{q-1}{2}$ ($i=1$) or not ($i=2$). In any case, estimating cd as for q even, we obtain $d > \frac{q-1}{2}$ and $c > \frac{q-1}{2i}$. For $i=1$, this is a contradiction (because d cannot exist!). Hence $i=2$ and $d = q-1$. Consequently $\ell \in \{q-2, \frac{q-3}{2}\}$. Also, $c \in \{\frac{q+1}{4}, \frac{q+1}{2}\}$ and hence $m \in \{\frac{q-3}{4}, \frac{q-1}{2}\}$. Clearly $\ell = q-2$ leads to an order of G which is bigger than $|O_7(q)|$. And $m = \frac{q-3}{4}$ leads to an order of G that is bigger than half the order of $O_7(q)$. Hence $(\ell, m) = (\frac{q-3}{2}, \frac{q-1}{2})$ and this implies that $|G| = |O_7(q)|$. The application is proved. □

4.6 The exception in the Flag Theorem

Let $\mathsf{W}(2)$ be the symplectic quadrangle of order $(2,2)$. Its automorphism group is isomorphic to the symmetric group $\mathbf{S}_6$, which is isomorphic to the

$\mathsf{W}(2)$	$\{1,\ldots,6\}$	$\mathrm{PG}(1,9)$
point	(ij)	Baer subline of one orbit under $\mathrm{PSL}_2(9)$
line	$(ij)(kl)(mn)$	Baer subline of the other orbit under $\mathrm{PSL}_2(9)$
incidence	containment	disjoint sublines

Table 4.1: Representations of $\mathsf{W}(2)$

linear group $\mathbf{P\Sigma L}_2(9)$. It is well known that the duads of a 6-set correspond to one orbit under $\mathrm{PSL}_2(9)$ of the set of Baer sublines of $\mathrm{PG}(1,9)$, and that the synthemes of a 6-set correspond to the other orbit (see [12], page 4). Since the duads and the synthemes of a 6-set are the points and lines of $\mathsf{W}(2)$, one obtains a representation of $\mathsf{W}(2)$ on the projective line $\mathrm{PG}(1,9)$ (see Table 4.1). We now investigate how one can recognize the flags of $\mathsf{W}(2)$ in this representation. In the following, we use the notation $[a,b,c,d]$ for the Baer subline through the points of $\mathrm{PG}(1,9)$ with affine coordinates (a), (b), (c) and (d). Also, we denote the Baer subline corresponding to the point p of $\mathsf{W}(2)$ with B_p (and refer to this subline as 'the point B_p'). Similarly for the lines of $\mathsf{W}(2)$.

Now fix two points of $\mathrm{PG}(1,9)$ for which we choose coordinates (∞) and (0). Assume the Baer subline $[\infty,0,1,-1]$ corresponds with a point of $\mathsf{W}(2)$. Then also the Baer subline $[\infty,0,i,-i]$ is a point. The other Baer sublines containing (∞) and (0) (namely $[\infty,0,1+i,-1-i]$ and $[\infty,0,1-i,i-1]$) are lines. So fixing the subline $B_p=[\infty,0,1,-1]$, there are 12 Baer sublines corresponding to lines and intersecting B_p in exactly two points of $\mathrm{PG}(1,9)$. The other three Baer sublines corresponding to lines are disjoint from B_p (these are the sublines $B_{L_1}=[i,-i,1+i,1-i]$, $B_{L_2}=[1+i,1-i,i-1,-i-1]$ and $B_{L_3}=[i-1,-i-1,i,-i]$). So there are 180 pairs of Baer sublines (B_p,B_L) for which B_p is a point, B_L is a line and $|B_p\cap B_L|=2$, and 45 pairs of sublines (B_p,B_L) for which B_p is a point, B_L is a line and B_p and B_L (as subsets of the point set of $\mathrm{PG}(1,9)$) are disjoint. Since the automorphism group of $\mathsf{W}(2)$ acts transitively on both the set of flags and the set of antiflags (and there are respectively 45 and 180 of them), we deduce that a point p and a line L of $\mathsf{W}(2)$ are incident precisely when the Baer sublines B_p and B_L are disjoint. Hence we may identify a flag of $\mathsf{W}(2)$ with the pair of points of $\mathrm{PG}(1,9)$ not contained in either of the two disjoint Baer sublines. This identification is bijective since there are 45 flags and 45 pairs of points, and every pair of points occurs by the 2-transitivity of $\mathrm{PSL}_2(9)$.

Let p,L_1,L_2,L_3 be as above. It is now clear that the flags (p,L_i) and (p,L_j) correspond with disjoint pairs whose union forms the Baer subline L_k, $(i,j,k)=(1,2,3)$. Now let (p,L) and (p'',L'') be two flags at distance

4 from each other, and $(p,L),(L,p'),(p',L'),(L',p''),(p'',L'')$ a 4-path between these two flags. Choose $B_p=[\infty,0,1,-1]$, $B_L=[i,-i,1+i,1-i]$, $B_{p'}=[\infty,-1,i-1,-i-1]$ and $B_{L'}=[0,1,i,1+i]$. The cross-ratio of the pairs corresponding to the flags (p,L) and (p',L') is a square in $\mathrm{GF}(9)\setminus\mathrm{GF}(3)$. Now $B_{p''}$ has to be disjoint from $B_{L'}$ and intersects B_L in two points. Since the pair of the flag (p'',L') is different from the pair corresponding to the flag (p,L) (and using the fact that through two points of $\mathrm{PG}(1,9)$, there are exactly two Baer sublines of point-type), these two pairs necessarily meet in exactly one point. From the observations above, we deduce that

(1) flags at distance 1 correspond to disjoint point pairs whose union forms a Baer subline (the latter corresponds to the unique element of $\mathsf{W}(2)$ which, together with the intersection of the two flags, forms again a flag distinct from both original flags);

(2) flags at distance 2 correspond to disjoint point pairs $\{a,b\}$ and $\{c,d\}$ such that the cross-ratio $(a,b;c,d)$ is a square in $\mathrm{GF}(9)\setminus\mathrm{GF}(3)$;

(3) flags at distance 3 correspond to non-disjoint pairs of points;

(4) flags at distance 4 correspond to disjoint point pairs $\{a,b\}$ and $\{c,d\}$ such that the cross-ratio $(a,b;c,d)$ is a non-square in $\mathrm{GF}(9)$.

It is now clear that an arbitrary permutation of the points of $\mathrm{PG}(1,9)$, which does not belong to $\mathbf{P\Gamma L}_2(9)$, preserves the set of flags of $\mathsf{W}(2)$, even preserves the distance 3, but does not extend to an (anti)automorphism of $\mathsf{W}(2)$. Note that $\mathsf{W}(2)$ does not provide a counterexample to Theorem 4.2.3.

Remark. Our description makes it obvious that the graph on the flags of $\mathsf{W}(2)$ where adjacency is being at distance 3, is the strongly regular graph with parameters $(v,k,\lambda,\mu)=(45,16,8,4)$ obtained from a 10-set by taking as vertices the pairs of points and adjacency being non-disjoint.

We give an explicit example of a bijection of the flags of $\mathsf{W}(2)$ preserving distance 3, but not preserving distance 1. Let F be a fixed flag of $\mathsf{W}(2)$ and G a flag at distance 3 from F. Then we define the flags G_F and G'_F as follows. Suppose to fix the ideas that $F=(p,L)$ and $G=(p',L')$ with L and L' concurrent lines (see Figure 4.6). Let $m=\mathrm{proj}_L p'$. Let L'' be the unique line through m, different from L and L', and x the point on L'' of the trace containing p and p'. Let x' be the unique point on L'' different from x and m. Then we define $G_F=(x,L'')$ and $G'_F=(x',L'')$. Dually if p and p' are collinear points. Now we define the following map θ_F between the flags of $\mathsf{W}(2)$.

$$\theta_F(G)=\begin{cases} G_F & \text{if } \delta(F,G)=3\\ G & \text{if } \delta(F,G)\neq 3.\end{cases}$$

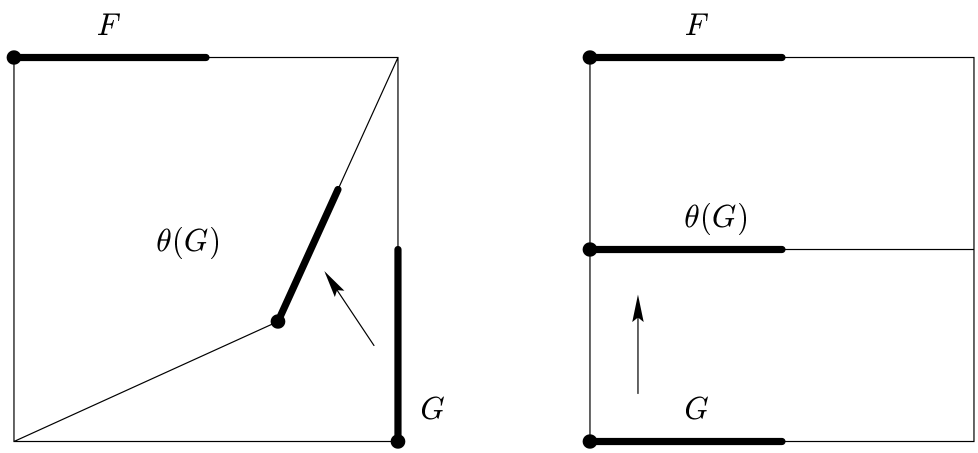


Figure 4.6: A counterexample to Theorem 4.2.2.

Then it is easy to check that θ_F preserves distance 3 between the flags of $\mathsf{W}(2)$, but not distance 1. (In fact, for a flag G not at distance 3 from F, it suffices to show that the set $T_{F,G}$ is preserved.) By choosing G'_F instead of G_F, one obtains another map with this property.

Exceptions to the Special Flag Theorem

We explain the restriction $r \notin S$ in Theorem 4.2.3. Suppose Γ is a generalized n-gon, $n \geq 3$ and $S = \{s_p, s_L\}$. Suppose, to fix the ideas, that n is even and that two adjacent flags lie at Coxeter distance s_p if and only they have a point in common. Now the following relation is easily seen to be an equivalence relation on the set $\mathcal{F}$ of flags of Γ:

$$F \sim F' \Longleftrightarrow \delta^*(F, F') = s_p, \text{ with } F, F' \in \mathcal{F}.$$

Let p and p' be two opposite points of Γ, and denote by $\mathcal{F}_p$ ($\mathcal{F}_{p'}$) the set of flags containing p (p'). For a flag F in $\mathcal{F}_p$, define F^α to be the unique flag of $\mathcal{F}_{p'}$ at distance $n-1$ from F. Similarly, one defines the map α on $\mathcal{F}_{p'}$. For a flag F not contained in $\mathcal{F}_p \cup \mathcal{F}_{p'}$, we define $F^\alpha = F$. Now α is a bijection on the set of flags of Γ preserving Coxeter distance s_p, but clearly not s_L, hence α does not extend to an (anti)automorphism of Γ.

4.7 Proof of the Flag Theorem

4.7.1 Case $i < n/2$

Let S be the set of pairs of points (a, b) of Γ satisfying $\delta(a, b) \neq i$ and $T_{a,b} = \emptyset$. We claim that a pair (a, b) belongs to S if and only if $\delta(a, b) > 2i$ or $\delta(a, b) = k < 2i$, with $k \equiv 0 \bmod 4$ and $0 \neq k \neq i$. Indeed, let (a, b) be an arbitrary pair of points of Γ. We distinguish the following possibilities.

(i) $\delta(a, b) = k > 2i$.
Suppose by way of contradiction that $c \in T_{a,b}$. If, for $k < n$, $\text{proj}_a c \neq \text{proj}_a b$, then there arises a circuit of length at most $k + 2i < 2n$, a contradiction. So we can assume that, for $k < n$, $\text{proj}_a c = \text{proj}_a b$ and symmetrically, $\text{proj}_b c = \text{proj}_b a$. But then again, a circuit of length $< 2n$ arises, unless the paths $[a, c]$ and $[b, c]$ meet on $[a, b]$ (with $[a, b]$ any n-path between a and b if $k = n$). Clearly, this contradicts $i < k/2$, so $T_{a,b} = \emptyset$.

(ii) $\delta(a, b) = k < 2i$, with $k \equiv 0 \bmod 4$ and $0 \neq k \neq i$.
Note that $a \bowtie b$ is a point. Suppose by way of contradiction that $c \in T_{a,b}$. If $\text{proj}_a c \neq \text{proj}_a b$, then we obtain a path of length $k + i$ between b and c (consisting of the paths $[b, a]$ and $[a, c]$). Since this cannot be the i-path between b and c, there arises a circuit of length $< 2n$, a contradiction. Hence we may assume $\text{proj}_a c = \text{proj}_a b$ and $\text{proj}_b c = \text{proj}_b a$. In this case, since there are no circuits of length $< 2n$, the paths $[a, c]$ and $[b, c]$ must meet on $[a, b]$, necessarily in $a \bowtie b$ (and $c \neq a \bowtie b$ since $i > k/2$). This is impossible since $a \bowtie b$ is a thin point.

(iii) $\delta(a, b) = k < 2i$, with $k \equiv 2 \bmod 4$ and $k \neq i$.
Any point c at distance $i - \frac{k}{2}$ from $M := a \bowtie b$ with $\text{proj}_M a \neq \text{proj}_M c \neq \text{proj}_M b$ belongs to $T_{a,b}$ (since M is thick, such a point c can be found). So $(a, b) \notin S$.

(iv) The cases $\delta(a, b) = 0, i, 2i$ are trivial.

This shows the claim. We put $\kappa = \{\delta(a, b) \,|\, (a, b) \in S\}$ (hence $\kappa = \{k \in \mathbb{N} \,|\, 2i < k \leq n \text{ or } k < 2i, k \equiv 0 \bmod 4 \text{ and } 0 \neq k \neq i\}$).

Case $i \equiv 0$ mod 4

Let S' be the set of pairs (a, b) of distinct points of Γ such that $i \neq \delta(a, b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_\kappa(b) = \emptyset$. We claim that S' is exactly the set of pairs of collinear points of Γ. Indeed, let (a,b) be an arbitrary pair of distinct points of Γ, $i \neq \delta(a,b) \notin \kappa$. There are three possibilities.

(*i*) $\delta(a,b) = 2$.
Every point at distance i from a but not at distance i from b lies at distance $i \pm 2 \equiv 2 \bmod 4$ from b, which is not a distance belonging to κ. Hence $(a,b) \in S'$.

(*ii*) $\delta(a,b) = k \equiv 2 \bmod 4$, $2 < k < 2i$.
Let L be the line of $[a,b]$ at distance $k/2 - 2$ from a, and let c be any point at distance $i - (k/2 - 2)$ from L such that $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = i+4$. The latter is a multiple of 4. So, if $i \neq 4$, then $4 + i < 2i$ and $\delta(b,c) \in \kappa$. If, on the other hand, $i = 4$, then necessarily $k = 6$. In this case, re-choose the point c at distance 4 from a with $\mathrm{proj}_a c \neq \mathrm{proj}_a b$. Then $\delta(b,c) = 10 \in \kappa$. Hence in both cases $(a,b) \notin S'$.

(*iii*) $\delta(a,b) = 2i$.
Let L be the unique line of $[a,b]$ at distance $i/2 - 1$ from a and let c be any point at distance $i/2 + 1$ from L such that $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = 2i+2$, hence $\delta(b,c) \in \kappa$. Consequently $c \in \Gamma_i(a) \cap \Gamma_\kappa(b)$, implying $(a,b) \notin S'$.

Our claim is proved.

Case $i \equiv 2$ **mod** 4

We proceed similarly as above. Now S' is the set of pairs (a,b) of distinct points of Γ such that $i \neq \delta(a,b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_{\neq i}(b) \subseteq \Gamma_\kappa(b)$ and we again claim that S' is exactly the set of pairs of collinear points of Γ. So let (a,b) be an arbitrary pair of distinct points of Γ, $i \neq \delta(a,b) \notin \kappa$. There are three possibilities.

(*i*) $\delta(a,b) = 2$.
Every point at distance i from a but not at distance i from b lies at distance $i \pm 2$ from b, which is a distance belonging to κ. Hence $(a,b) \in S'$.

(*ii*) $\delta(a,b) = k \equiv 2 \bmod 4$, $2 < k < 2i$.
We consider a point c as in 4.7.1(*ii*). Then $\delta(b,c) = i + 4$ implies $\delta(b,c) \notin \kappa$.

(*iii*) $\delta(a,b) = 2i$.
Let L be the unique line of $[a,b]$ at distance $i/2$ from a and let c be any point at distance $i/2$ from L such that $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = 2i$, hence $\delta(b,c) \notin \kappa$. Consequently c is in $\Gamma_i(a) \cap \Gamma_{\neq i}(b)$, but not in $\Gamma_\kappa(b)$, implying $(a,b) \notin S'$.

This shows the claim and completes the proof of Case $i < n/2$.

4.7.2 Case $i = n/2$

$n = 8$ **and** Γ **has order** $(2,1)$

Note that also Γ' has order $(2,1)$ by the bijectivity of α.

We first distinguish distance 6. Let a and b be two different points of Γ. If $\delta(a,b) = 2$, then a point x belongs to $T_{a,b}$ if and only if x lies at distance 3 from the line ab, hence $|T_{a,b}| = 2$. If $\delta(a,b) = 6$, then $T_{a,b} = \{c\}$, with c the unique point on the line $a \bowtie b$ not collinear with a or b. If finally $\delta(a,b) = 8$, the points of $T_{a,b}$ necessarily lie on one of the 8-paths between a and b, hence $|T_{a,b}| = 2$. So $\delta(a,b) = 6$ if and only if $\delta(a,b) \neq i$ and $|T_{a,b}| = 1$. Unfortunately, all straightforward counting arguments do not lead to a distinction between points at distance 2 or 8. Hence we give a more sophisticated reasoning.

Let a, b be points of Γ at distance 2 or 8 from each other. Put $T_{a,b} = \{c,d\}$ and $S = \{a,b,c,d\}$. We claim that there is a unique point x such that

(*) $\Gamma_4(x) \cap S = \emptyset$ and $\Gamma_6(x) \cap S = \emptyset$.

Indeed, if a and b are collinear, then c and d are collinear points such that the line cd meets the line ab in a point $x \notin S$. One can easily check that x is the only point of Γ that satisfies (*). If $\delta(a,b) = 8$, then S is contained in the unique apartment through a and b. Note that Γ is the double of the unique generalized quadrangle $\mathsf{W}(2)$ of order 2. In $\mathsf{W}(2)$ the points a,b,c,d correspond to flags whose union is an apartment Σ in $\mathsf{W}(2)$ (see Figure 4.7). There is a unique point u (respectively a unique line U) in $\mathsf{W}(2)$ opposite every point (respectively line) of Σ and u is incident with U. The flag $\{u,U\}$ corresponds in Γ with the unique point x satisfying (*). This proves our claim.

Now if $\delta(a,b) = 8$, then there exists a point y of $T_{a,x}$ at distance 6 from b (see Figure 4.7, in fact, every point of $T_{a,x}$ has this property) while if $\delta(a,b) = 2$, every point of $T_{a,x}$ is collinear with b. Hence we can distinguish distance 2 and the theorem follows.

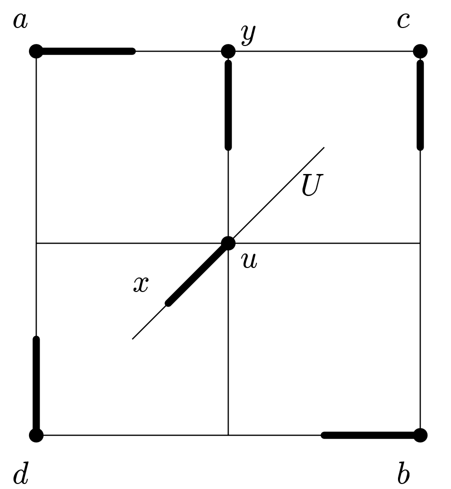


Figure 4.7: Case $i = n/2$ and Γ the double of $\mathsf{W}(2)$.

The general case

Here we assume that, if $n = 8$, then Γ contains lines with more than 3 points. Note also that, since i is even, necessarily $n \equiv 0 \bmod 4$.

In this case, we show that we can recover opposition. Let a, b be points of Γ. We claim that $\delta(a, b) = n$ if and only if

(**) $|T_{a,b}| = 2$ and, putting $T_{a,b} = \{c, d\}$, $T_{c,d} = \{a, b\}$.

Obviously, if a and b are opposite, then they satisfy (**). So we may assume that $\delta(a, b) =: k < n$. We distinguish three cases.

(i) $k \equiv 0 \bmod 4$, $k \neq n$.
We show that $T_{a,b} = \emptyset$. Suppose by way of contradiction that $c \in T_{a,b}$. Assume first that $\mathrm{proj}_a b = \mathrm{proj}_a c$. Note that the path $[a, c]$ does not contain $[a, b]$, and since $k \neq 2i$, the point c does not belong to $[a, b]$. Hence we can define the line L as $[a, b] \cap [a, c] = [a, L]$. Let $j = \delta(a, L)$. There is a path of length $k + i - 2j$ between b and c, consisting of the paths $[b, L]$ and $[L, c]$. If $k + i - 2j \leq n$, then $\delta(b, c) = i$ implies $j = k/2$, hence $L = a \bowtie b$, contradicting the fact that L is a line. Hence $k + i - 2j > n$. But now $\delta(b, c) = i$ implies there is a circuit of length at most $(k + i - 2j) + i = n + k - 2j < 2n$, a contradiction.

The case $\mathrm{proj}_a b \neq \mathrm{proj}_a c$ corresponds with $j = 0$ in the previous argument.

(ii) $k = n-2$.
Let c be an arbitrary element of $T_{a,b}$ ($T_{a,b}$ is easily seen to be nonempty; this will also follow from our next argument). Similarly as in (i) above, one shows that $[a,b] \cap [a,c] \cap [b,c] = a \bowtie b =: L$. But then $c\mathtt{I}L$ and $\mathrm{proj}_L a \neq c \neq \mathrm{proj}_L b$. So if (a,b) satisfies (**), then L contains 4 points $c, d, \mathrm{proj}_L a, \mathrm{proj}_L b$. But every point on $M := \mathrm{proj}_a b$ distinct from $\mathrm{proj}_M b$ belongs to $T_{c,d}$. Similarly for $M' := \mathrm{proj}_b a$. Note that $M \neq L$ and $M' \neq L$ since $n-2 \neq 2$. Hence, since also $M' \neq M$, we conclude by thickness of those lines that $|T_{c,d}| \geq 4$. So (a,b) does not satisfy (**).

(iii) $k \equiv 2 \bmod 4$ and $k \neq n-2$.
Every point c at distance $\frac{n-k}{2}$ from the line $L := a \bowtie b$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$ belongs to $T_{a,b}$. So if $|T_{a,b}| = 2$, then necessarily $\frac{n-k}{2} = 3$ and both L and $\mathrm{proj}_c L$ are incident with exactly 3 points (note that $\frac{n-k}{2} = 1$ corresponds with case (ii) above). We put $T_{a,b} = \{c,d\}$. As in (ii) above, $|T_{c,d}| \geq 4$ whenever $\mathrm{proj}_a b \neq \mathrm{proj}_b a$. Hence we may assume that a and b are incident with L and that $k = 2$ and $i = 4$. But this is Case 4.7.2.

So we obtained that α preserves opposition. By Abramenko & Van Maldeghem [1], Corollary 5.2, this completes the proof of Case $i = n/2$.

4.7.3 Case $n/2 < i < n-2$

Let S be the set of pairs of points (a,b) of Γ such that $T_{a,b} = \emptyset$. Put $\kappa = \{k \in \mathbb{N} \mid 0 < k \leq 2n-2i-4 \text{ and } k \equiv 0 \bmod 4\}$. We claim that $(a,b) \in S$ if and only if $\delta(a,b) \in \kappa$. Indeed, let a,b be points of Γ. Put $k = \delta(a,b)$.

(i) $0 < k \leq 2n-2i-4$ and $k \equiv 0 \bmod 4$.
Similarly as in 4.7.1(ii), one shows that $T_{a,b} = \emptyset$ in this case.

(ii) $k \leq 2n-2i-2$ and $k \equiv 2 \bmod 4$.
Here, a point $c \in T_{a,b}$ can be found similarly as in 4.7.1(iii).

(iii) $k \geq 2n-2i$.
Let c' be a point opposite b and at distance $n-k$ from a (c' lies in some apartment containing a,b). Let X be a line incident with c', distinct from $\mathrm{proj}_{c'} a$ if $k \neq n$. Clearly, there is a point $x\mathtt{I}X$, $x \neq c'$, with x opposite b. Then $\delta(c',x) = 2$, and an inductive argument shows that

there is a point c'' opposite b with $\delta(c',c'') = k - 2n + 2i$ and with $\mathrm{proj}_{c'}a \neq \mathrm{proj}_{c'}c''$ if $k \neq n$. Note that $\delta(a,c'') = 2i - n \neq 0$. Let $c \in \Gamma_i(b) \cap \Gamma_{n-i}(c'')$ be such that $\mathrm{proj}_{c''}c \neq \mathrm{proj}_{c''}a$ (c is the point at distance i from b lying on the n-path between b and c'' not containing $\mathrm{proj}_{c''}a$). Clearly, c belongs to $T_{a,b}$.

This shows our claim.

Case $i \equiv 0$ mod 4 and $i \leq 2n - 2i - 4$

In this case i precisely belongs to κ. We claim that two distinct points a and b are collinear in Γ if and only if $\delta(a,b) \notin \kappa$ and $R := \Gamma_i(a) \cap \Gamma_{\neq i}(b) \cap \Gamma_\kappa(b)$ is empty. Indeed, let a, b be two arbitrary distinct points of Γ, $\delta(a,b) \notin \kappa$.

(i) $\delta(a,b) = 2$.
This is similar to 4.7.1(i).

(ii) $\delta(a,b) \equiv 0$ mod 4.
Note that $i < k := \delta(a,b) < 2i$. Let $c \in \Gamma_i(a) \cap \Gamma_{k-i}(b)$ (choose c on a k-path between a and b). Then $c \in R$ because $\delta(b,c) = k - i$ is distinct from i, it is a multiple of 4 and it is at most $2n - 2i - 4$ (for $i \leq 2n - 2i - 4 < 2n - k - 4$).

(iii) $2 \neq \delta(a,b) \equiv 2$ mod 4.
First let $i < k := \delta(a,b) < 2i - 2$. Let $L \in \Gamma_{i-1}(a) \cap \Gamma_{k-i+1}(b)$ and let $c\mathrm{I}L$ with $\mathrm{proj}_L a \neq c \neq \mathrm{proj}_L b$. Then we show that $c \in R$. Indeed, $\delta(b,c) = k - i + 2$, so $\delta(b,c) = i$ implies $k/2 + 1 = i$, a contradiction. Also, $\delta(b,c) \equiv 0$ mod 4 and the inequalities $i \leq 2n - 2i - 4$ and $k \leq 2i - 6$ imply $\delta(b,c) \leq 2n - 2i - 4$. Consequently $\delta(b,c) \in \kappa$.

Now suppose $k = 2i - 2$. This implies, since $2i \geq n + 2$, that $k \geq n$, hence $k = n$ and $n = 2i - 2$. Let $L \in \Gamma_{i-3}(a) \cap \Gamma_{n-i+3}(b)$ and let $c \in \Gamma_3(L)$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. Then $c \in \Gamma_i(a) \cap \Gamma_{\neq i}(b)$ (because $\delta(b,c) = n - i + 6 = (2i - 2) - i + 6 \neq i$). Also, $\delta(b,c)$ is a multiple of 4. If $n \geq 22$, then one verifies that $\delta(b,c) = n - i + 6 \leq 2n - 2i - 4$, hence $c \in R$. If $n < 22$, then, since i is a multiple of 4, the only possibility is $(n,k,i) = (14,14,8)$. But then $\kappa = \{4,8\}$ and we can distinguish distance 4 in Γ; hence also distance 2 by Subsection 4.7.1.

Finally let $k := \delta(a,b) < i$. Put $L = \mathrm{proj}_b a$. Let $c \in \Gamma_{i-k+1}(L)$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. As above, one checks that $c \in R$.

This shows our claim.

Case $i \equiv 2$ mod 4 and $i \leq 2n - 2i - 4$

Here, we claim that two distinct points a, b of Γ are collinear if and only if $i \neq \delta(a,b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_{\neq i}(b) \subseteq \Gamma_\kappa(b)$. Indeed, let a, b be two arbitrary distinct points of Γ, $i \neq \delta(a,b) \notin \kappa$. We have the following cases.

(i) $\delta(a,b) = 2$.
This is similar to 4.7.1(i).

(ii) $\delta(a,b) = k \equiv 0 \bmod 4$, $k > 2n - 2i - 4$.
Note that $i < k < 2i$. Let c be the point of a fixed k-path between a and b at distance i from a. Then $i \neq \delta(b,c) = k - i \equiv 2 \bmod 4$, hence $\delta(b,c) \notin \kappa$.

(iii) $2 \neq \delta(a,b) = k \equiv 2 \bmod 4$.
(a) Suppose first $k > i$. Let L be the line of a fixed k-path between a and b at distance $i-1$ from a. If $i \neq k/2+1$, let c be a point incident with L, $\mathrm{proj}_L a \neq c \neq \mathrm{proj}_L b$. Then $i \neq \delta(b,c) = k - i + 2 \notin \kappa$. If $i = k/2+1$, let L' be the line concurrent with L and closest to a, and c a point at distance 3 from L' for which $\mathrm{proj}_{L'} a \neq \mathrm{proj}_{L'} c \neq \mathrm{proj}_{L'} b$. Then $\delta(a,c) = i$ and $\delta(b,c) = k - i + 6 \equiv 2 \bmod 4$ (note that $i \geq 6$), hence $i \neq \delta(b,c) \notin \kappa$.

(b) Suppose now $k < i$. In this case, a point c at distance $i - k + 1$ from the line $R := \mathrm{proj}_b a$ for which $\mathrm{proj}_R a \neq \mathrm{proj}_R c \neq b$ will do the job.

This shows the claim.

Case $i \geq 2n - 2i - 2$

We claim that two points a, b of Γ are at distance $2n - 2i - 4$ from each other if and only if $\delta(a,b) \in \kappa$ and $T^\kappa_{a,b} := \Gamma_\kappa(a) \cap \Gamma_\kappa(b)$ contains exactly $(2n-2i-8)/4 =: \ell$ elements. Indeed, let a, b be two distinct points of Γ such that $\delta(a,b) \in \kappa$. We distinguish the following cases.

(i) $\delta(a,b) = 2n - 2i - 4$.
Note that there are exactly ℓ elements of $T^\kappa_{a,b}$ contained in $[a,b]$ (indeed, every point x of $[a,b]$ different from a, b for which $\delta(a,x) \equiv 0 \bmod 4$ belongs to $T^\kappa_{a,b}$.) Conversely, we show that every element of $T^\kappa_{a,b}$ is contained in $[a,b]$. Suppose $c \in T^\kappa_{a,b}$ and c not on the path $[a,b]$. If

$\mathrm{proj}_a c \neq \mathrm{proj}_a b$, we obtain a circuit of length $\leq 3(2n-2i-4) < 2n$ (indeed, $3i \geq 2n-2$), a contradiction. So we can assume $\mathrm{proj}_a c = \mathrm{proj}_a b$ and $\mathrm{proj}_b c = \mathrm{proj}_b a$. Let in this case, $[a,c] \cap [a,b] = [a,L]$ and $r = \delta(a,L)$. Since $c \in T^{\kappa}_{a,b}$, $\delta(a,c) \equiv 0 \bmod 4$. We obtain a path of length $d = \delta(a,b) + \delta(a,c) - 2r$ between b and c (joining the paths $[b,L]$ and $[L,c]$). If $d > n$, $\delta(b,c) \notin \kappa$ since otherwise we obtain a circuit of length $< 2n$. If $d \leq n$, then $\delta(b,c) = d \equiv 2 \bmod 4$, hence $\delta(b,c) \notin \kappa$.

(ii) $\delta(a,b) := k \in \kappa \setminus \{2n-2i-4\}$.
On the path $[a,b]$, we already find $k/4 - 1$ members of $T^{\kappa}_{a,b}$. Now let $h \in \kappa$ with $h > k$. Then every point $x \in \Gamma_h(a) \cap \Gamma_{h-k}(b)$ belongs to $T^{\kappa}_{a,b}$. Now for each such h, we find at least two such points. Indeed, consider points x at distance $h-k$ from b for which $\mathrm{proj}_b a \neq \mathrm{proj}_b x$. The number of choices for h is $\frac{2n-2i-4}{4} - \frac{k}{4} = \ell + 1 - \frac{k}{4}$, hence together with the points of $T^{\kappa}_{a,b}$ on $[a,b]$ we obtain at least $2\ell + 1 - k/4$ elements of $T^{\kappa}_{a,b}$. This number is bigger than ℓ, since $\ell > k/4 - 1$.

So we can recover distance $2n-2i-4$. By the previous cases, this is enough to recover collinearity. This completes the proof of Case $n/2 < i < n-2$.

4.7.4 Case $i = n-2$

It is convenient to treat the cases $n = 6, 8$ separately.

Case $n = 6$

Here, $i = 4$, so we only have to distinguish distance 2 from 6. But for opposite points a, b, the set $T_{a,b}$ contains points at mutual distance $i = 4$, while this is not the case for collinear points a, b. Hence in this case α preserves collinearity.

Case $n = 8$

First suppose that Γ is the double of a quadrangle Δ of order $(2,t)$ (with t automatically finite). Then $t = 2, 4$ and Δ is unique. Notice that by the bijectivity of α, in this case Δ and Δ' have the same order. If $t = 2$, then there is nothing to prove (this was the exception). Suppose $t = 4$. For two points a, b at distance k, put $l_k = |T_{a,b}|$. Then it is easily verified that $l_2 = 8$ if ab contains 3 points, $l_2 = 24$ if ab contains 5 points, $l_4 = 8$ and $l_8 = 16$. Hence α preserves opposition and we are done.

So from now on we may assume that all lines of Γ have at least 4 points. We claim that two distinct points a, b of Γ are collinear if and only if $\delta(a,b) \neq 6$ and there are no distinct points $c, c' \in \Gamma_6(b) \cap \Gamma_{\neq 6}(a)$ satisfying $T_{a,b} \subseteq \Gamma_6(c) \cup \Gamma_6(c')$. Indeed, if $\delta(a,b) = 4$, then we take two different points c, c' (unequal a) on the unique line through a at distance 5 from b; if $\delta(a,b) = 8$, then we take $\{c, c'\} = \Gamma_2(a) \cap \Gamma_6(b)$. In these cases one easily checks that $T_{a,b} \subseteq \Gamma_6(c) \cup \Gamma_6(c')$. Now let $\delta(a,b) = 2$. Suppose by way of contradiction that there do exist two points c, c' as above. Let L be an arbitrary but fixed line meeting the line ab but not through a or b. Then the set of points $R = \Gamma_3(L) \setminus \Gamma_1(ab)$ is contained in $T_{a,b}$ and hence is a subset of $\Gamma_6(c) \cup \Gamma_6(c')$. Either $\delta(a,c) = 4$ or $\delta(a,c) = 8$ (and similarly for c'). First suppose $\delta(a,c) = 4$. Clearly, for any line $M \neq ab$ meeting L, there is exactly one point $x \mathtt{I} M$ at distance 6 from c. Hence there are at least 2 points of $T_{a,b}$ on M and opposite c, implying that the line M must be at distance 5 from c'. Since there are at least 3 such lines M, we similarly have that $\delta(c', L) = 3$, and $\delta(c', ab) = 1$ (because $\mathrm{proj}_L c'$ cannot be on a line M, so must be incident with ab), contradicting $\delta(b, c') = 6$.

So we showed that $\delta(a,c) = 8$ and symmetrically, also $\delta(a,c') = 8$. So $\delta(c, L) = \delta(c', L) = 7$, and hence, since $\delta(c, ab) = 7$, there must be a unique line $M_c \neq ab$ meeting L having distance 5 to c. Similarly, there is such a line $M_{c'}$ at distance 5 from c'. Now let $M \in \Gamma_2(L) \setminus \{M_c, M_{c'}, ab\}$. Since $\delta(c, M) = \delta(c', M) = 7$, at most two points on M are covered by $\Gamma_6(c) \cup \Gamma_6(c')$, a contradiction with the fact that the line M contains at least 3 points of $T_{a,b}$. This proves our claim. So α preserves collinearity and the theorem follows.

Case $n > 8$

Suppose first that, up to duality, Δ (or Δ') has order $(2,t)$ with t finite (hence $n \in \{12, 16\}$), or has order $(3,3)$ (and then $n = 12$). Then the same holds for Δ' (or Δ). We now give a similar counting argument as in 4.7.4. Let first $n = 12$ and Δ a hexagon of order (s,t). Then, with as before $l_k := |T_{a,b}|$ for $\delta(a,b) = k$, a rather easy counting in Γ shows the following:

$$
\begin{aligned}
&l_2 = (s-1)s^2t^2 \text{ if } |ab| = s+1 \\
&l_2 = (t-1)s^2t^2 \text{ if } |ab| = t+1 \\
&l_4 = s^2t^2 \\
&l_6 = st(2s(t-1) + (s-1)(s+t)) \text{ if } |ab| = s+1 \\
&l_6 = st(2t(s-1) + (t-1)(s+t)) \text{ if } |ab| = t+1 \\
&l_8 = st(s+t-2)^2 \\
&l_{12} = (s+t-2)((s-1)(t-1)^2 + (t-1)(s-1)^2 + 2s(t-1) + 2t(s-1)).
\end{aligned}
$$

Now $(s,t) \in \{(2,2),(2,8),(3,3)\}$. In any of these cases, one checks that $l_{12} \notin \{l_2, l_4, l_6, l_8\}$, hence we can distinguish opposition. Similarly if $n = 16$ and Δ has order $(2,4)$.

So from now on we may assume that Δ has order $(s,t) \neq (3,3)$ with $s,t \geq 3$, or $\{s,t\} = \{2,\infty\}$. We divide the proof in several steps.

$\boxed{\text{Step 1: the set } S_{a,b}}$

For any three points a,b,c of Γ, define $T_{a,b,c} := \Gamma_{n-2}(a) \cap \Gamma_{n-2}(b) \cap \Gamma_{n-2}(c)$. Let a,b be two arbitrary points of Γ not at distance $n-2$, then we define

$$S_{a,b} = \{c \in \Gamma_{\neq(n-2)}(a) \cap \Gamma_{\neq(n-2)}(b) \mid T_{a,b,c} = \emptyset\}.$$

Note that, by symmetry, $c \in S_{a,b}$ implies $b \in S_{a,c}$ and $a \in S_{b,c}$.

We will prove the following claims (where $w = a \bowtie b$ whenever defined).

Claim 1. $\delta(a,b) = 2$.
If the line ab contains at least 4 points, then $S_{a,b} = \Gamma_3(ab)$. Otherwise, $S_{a,b} = \Gamma_{\{1,3,7\}}(ab) \setminus (\{a,b\} \cup \Gamma_6(a) \cup \Gamma_6(b))$.

Claim 2. $\delta(a,b) = 4$.
Here, $S_{a,b} = \Gamma_1(aw) \cup \Gamma_1(bw) \cup \Gamma_4(a) \cup \Gamma_4(b) \setminus (\{a,b\} \cup \Gamma_4(w))$.

Claim 3. $2 \neq \delta(a,b) = k \equiv 2 \bmod 4$, $k \leq n-4$.
Here $S_{a,b} \subseteq \{x \in \Gamma_{\leq k/2+2}(w) \mid \mathrm{proj}_w a \neq \mathrm{proj}_w x \neq \mathrm{proj}_w b\}$. If $k = 6$, then no point incident with w belongs to $S_{a,b}$. Also, if w contains at least 4 points, then no point of $\Gamma_{k/2}(w)$ belongs to $S_{a,b}$.

Claim 4. $4 \neq \delta(a,b) = k \equiv 0 \bmod 4$, $k \leq n-4$, $n \neq 12$.
Put $A = \mathrm{proj}_w a$ and $B = \mathrm{proj}_w b$. Also, define

$$\begin{aligned} S'_{a,b} &= \{x \in \Gamma_{\{k/2-1,k/2+1\}}(A) \mid \mathrm{proj}_A a \neq \mathrm{proj}_A x \neq w\} \\ &\quad \cup \{x \in \Gamma_{\{k/2-1,k/2+1\}}(B) \mid \mathrm{proj}_B b \neq \mathrm{proj}_B x \neq w\}. \end{aligned}$$

If $k \neq 8$, then $S_{a,b} \subseteq S'_{a,b}$. If $k = 8$ and if both $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$ contain at least 4 points, then $\{w\} \subseteq S_{a,b} \subseteq S'_{a,b} \cup \{w\}$. If $k = 8$ and either $\mathrm{proj}_a b$ or $\mathrm{proj}_b a$ contains exactly three points (and suppose without loss of generality that $\mathrm{proj}_b a$ has size 3), then $\{w,e\} \subseteq S_{a,b} \subseteq S'_{a,b} \cup \{w,e\}$, where e is incident with $\mathrm{proj}_b a$ and distinct from both b and $b \bowtie w$.

Claim 5. $\delta(a,b) = k = 8$ and $n = 12$.
Here, with the notation of Claim 4, we have, if $s,t \geq 3$, then $w \in S_{a,b} \subset S'_{a,b} \cup (\Gamma_8(a) \cap \Gamma_8(b)) \cup \{w\}$. If $\{s,t\} = \{2,\infty\}$ (and we may

assume without loss of generality that $A' := \mathrm{proj}_a b$ is incident with infinitely many points), then $\{w, e\} \subseteq S_{a,b} \subset S'_{a,b} \cup (\Gamma_8(a) \cap \Gamma_8(b)) \cup S''_{a,b} \cup \{w, e\}$, where $S''_{a,b} = \{x \in \Gamma_{11}(B) \mid \mathrm{proj}_B b \neq \mathrm{proj}_B x \neq w\} \cup \{x \in \Gamma_7(A') \mid \mathrm{proj}_{A'} w \neq \mathrm{proj}_{A'} x \neq a\}$.

We will prove these claims by induction on $\delta(a, b)$.

Claim 1.
Let c be an arbitrary point of Γ, $a \neq c \neq b$. Note that the points of $T_{a,b}$ all lie at distance $n-3$ from the line ab. First assume that $\mathrm{proj}_{ab} c = a$. Put $j = \delta(c, a)$. If $j = 2$, then $\delta(c, x) = n$, for all $x \in T_{a,b}$, hence $c \in S_{a,b}$. Suppose $j > 2$. Let y be a point at distance $n - j - 1$ from ab for which $b \neq \mathrm{proj}_{ab} y \neq a$. Then $\delta(c, y) = n$. Let Y be the line incident with y and different from $\mathrm{proj}_y a$. On the line Y, there is at least one point opposite c and at distance $n - j + 2$ from a, b. Proceeding like this, one constructs a point c' opposite c with $\delta(a, c') = \delta(b, c') = n-4$. Now let L be a line incident with c' and different from $\mathrm{proj}_{c'} a$. Then $\mathrm{proj}_L c \in T_{a,b,c}$, hence $c \notin S_{a,b}$.

So we may assume $\mathrm{proj}_{ab} c \notin \{a, b\}$. Put $j + 1 = \delta(c, ab)$. If $j = 0$, then clearly $c \in S_{a,b}$ if and only if ab is incident exactly three points. If $j = 2$, then clearly c always belongs to $S_{a,b}$. Suppose $j > 2$. Let $L \in \Gamma_{j-1}(c) \cap \Gamma_2(ab)$. If $j = 4$, then clearly there are points at distance $n - 5$ from L which belong to $T_{a,b,c}$. If $j = 6$ and $|\Gamma_1(ab)| = 3$, then one verifies $c \in S_{a,b}$. If $|\Gamma_1(ab)| > 3$, then similarly as in the previous paragraph, we find a point $x \in T_{a,b,c}$ with $\mathrm{proj}_{ab} x \notin \{\mathrm{proj}_{ab} c, a, b\}$. Finally if $j > 6$, then, as before, we find a point x in $T_{a,b,c}$ with $\delta(x, L) = n - 5$, and with $\mathrm{proj}_L x \notin \{\mathrm{proj}_L c, \mathrm{proj}_L ab\}$.

Claim 2.
Note that all points of $T_{a,b}$ lie opposite w in an apartment through a and b. Let c be an arbitrary point of Γ distinct from a, b and put $j = \delta(w, c)$. Without loss of generality, we may assume that a minimal path from c to w contains aw, except if $c = w$. But in the latter case, clearly $c \in S_{a,b}$. So from now on $c \neq w$. If $j = 2$, then clearly c is opposite every point of $T_{a,b}$, hence $c \in S_{a,b}$. Now suppose $j > 2$. Let Σ be an apartment containing b, c. Suppose first $j \equiv 0 \mod 4$ and let M be the line of Σ at distance $n - 1 - j/2$ from both b, c and at distance $n + 1 - j/2$ from w. If $j = 4$ and $\delta(a, c) = 4$, then $\mathrm{proj}_M a \in T_{a,b,c}$. If $j = 4$ and $\delta(a, c) = 2$, then $c \in S_{a,b}$ would imply $b \in S_{a,c}$, contradicting Claim 1. So we can assume $j \geq 8$. Note that $\delta(a, M) = n + 3 - j/2$. We construct a point of $T_{a,b,c}$ as follows. Let x' be a point at distance $j/2 - 3$ from M for which $\mathrm{proj}_M x'$ does not belong to Σ. Then a and x' are opposite. Let X' be a line through x', $X' \neq \mathrm{proj}_{x'} M$ and $x = \mathrm{proj}_{X'} a$. Then $x \in T_{a,b,c}$, showing that $c \notin S_{a,b}$. Suppose now $j \equiv 2 \mod 4$. If $j \neq n$, then we consider an apartment Σ'

containing $[b, \mathrm{proj}_c w]$, but not containing c. If $j = n$, then we consider an apartment Σ' containing $[b, L]$, with L the line of $[aw, c]$ at distance 1 from c, and containing the projection of b onto L (note that we can assume $\mathrm{proj}_L b \neq c$ since $\delta(b, c) \neq n - 2$ for a point $c \in S_{a,b}$). In this way we obtain a path of length $h = 2n - j \equiv 2 \bmod 4$ between b and c (combining the path $[c, L]$ with the path between b and L contained in Σ', but not containing w). We now argue similarly as before. Let M be the line of Σ' at distance $n - j/2$ from both b and c, and at distance $n - j/2 + 2$ from w. Suppose $j \geq 10$. Note that $\delta(a, M) = n - j/2 + 4$. Let x' be a point at distance $j/2 - 4$ from M for which $\mathrm{proj}_M x'$ does not belong to Σ'. Then x' and a are opposite. As before, the projection of a onto a line X through x' different from $\mathrm{proj}_{x'} M$ belongs to $T_{a,b,c}$, showing $c \notin S_{a,b}$. Now let $j = 6 = \delta(a, c)$. In this case, the projection of c onto M belongs to $T_{a,b,c}$, hence also $c \notin S_{a,b}$. If finally $j = 6$ and $\delta(a, c) = 4$ then $c \in S_{a,b}$. Indeed, let (a, L, p, L', c) be the 4-path between a and c, and x an arbitrary point of $T_{a,b}$. Then either $\mathrm{proj}_L x \neq p$, implying $\delta(x, c) = n$, or $\mathrm{proj}_L x = p$, implying $\delta(x, c) \leq n - 4$. This shows Claim 2.

We now proceed by induction on $\delta(a, b)$.

Claim 3.
Let c be any point of Γ. Suppose $\mathrm{proj}_w c = \mathrm{proj}_w a$. As before, put $\delta(w, c) = j$ (note that j is odd since w is a line). If $j > k/2 + 2$, then we can find a point in $T_{a,b,c}$ at distance $n - 2 - k/2$ from w. Indeed, let x be a point at distance $n - j$ from w for which $\mathrm{proj}_w a \neq \mathrm{proj}_w x \neq \mathrm{proj}_w b$. Then c and x are opposite, but $\delta(a, x) = \delta(b, x) < n - 2$. Put $h = n - 2 - \delta(a, x)$. It is easy to see that one can find a point x' at distance $h - 2$ from x, $\mathrm{proj}_x x' \neq \mathrm{proj}_x w$, and x' opposite c. Now the projection of c onto any line X incident with x' and different from $\mathrm{proj}_{x'} w$ is a point of $T_{a,b,c}$. Hence $c \notin S_{a,b}$. If $j \leq k/2 + 2$, then one calculates $\delta(a, c) \leq k/2 - 2 + j - 2 < k$. Now if c would be in $S_{a,b}$, then $b \in S_{a,c}$. We check that this contradicts the induction hypothesis. So suppose $b \in S_{a,c}$. It is easy to verify that $\delta(a, c) \notin \{2, 4\}$. Hence, for any element x of $S_{a,c}$ (so also for b), we have $\delta(x, a \bowtie c) \leq \frac{\delta(a,c)}{2} + 2 < \frac{k}{2} + 2$ by the induction hypothesis. Clearly, c cannot lie on the path $[w, a]$. If the path $[w, c]$ contains the path $[w, a]$, then we obtain a path of length $\frac{3k}{4} + \frac{j}{2}$ between b and $a \bowtie c$, a contradiction. So suppose this is not the case. Put $[w, a] \cap [w, c] = [w, L]$ and $r = \delta(w, L)$. Then we obtain a path of length $k/2 + j - 2r < n$ between a and c. Since $\frac{k}{4} + \frac{j}{2} - r \leq \frac{k}{2} - r + 1 = \delta(a, L) + 1$, the element $a \bowtie c$ belongs to $[a, b]$, or is incident with an element of $[a, b]$. Hence $\delta(b, a \bowtie c) \geq k/2 + 2$, a contradiction.

Now suppose $\mathrm{proj}_w a \neq \mathrm{proj}_w c \neq \mathrm{proj}_w b$ and $j \geq k/2 + 4$. If $j \neq k/2 + 6$, then similarly as before, we can find a point $c' \in T_{a,b,c}$ with $|[w, c'] \cap [w, c]| = 3$. Suppose $j = k/2 + 6$. Let w' be the element of $[a, b]$ at distance $k/2 - 2$

from a. Suppose first $n \geq 12$. Let Z be a line at distance $n - k/2 - 7$ from w' for which $\mathrm{proj}_{w'}a \neq \mathrm{proj}_{w'}Z \neq \mathrm{proj}_{w'}w$. Note that $\delta(a, Z) = n - 9$ and $\delta(c, \mathrm{proj}_Z a) = n$. Now consider the path γ' consisting of the union of the paths $[a, Z]$ and $[Z, c]$ (which has length $2n - 10$). Let M be the line of γ' at distance $n - 5$ from both a and c, and M' a line intersecting M, $\delta(M', a) = \delta(M', c) = n - 3$. Now the point $\mathrm{proj}_{M'}b$ belongs to $T_{a,b,c}$; showing $c \notin S_{a,b}$. If $n = 10$, then necessarily $k = 6$ and $\delta(a, c) = 8$ (but in this case, $c \notin S_{a,b}$) or $\delta(a, c) = 10$. In the latter case, we argue similarly as above, choosing for γ' the 10-path between a and c containing $\mathrm{proj}_a b$. The assertions for $k = 6$ and $|\Gamma_1(m)| \geq 4$ are easy and left for the reader.

Claim 4-5.

Let c again be an arbitrary point of Γ. If $c = w$, then $c \in S_{a,b}$ implies $b \in S_{a,c}$, and by the induction hypothesis this only happens if $k = 8$ (and in this case one easily verifies that indeed $w \in S_{a,b}$). So we may assume that $c \neq w$ and, without loss of generality, that there is a minimal path from c to w containing A. Put $j = \delta(c, w)$ and let ℓ be the distance from w to the unique element X of $[a, w]$ closest to c.

$\underline{j < k/2 + 2\ell}$

In this case, we can apply the induction hypothesis. Indeed, the condition above implies that the path between a and c consisting of $[a, X]$ and $[X, c]$ (which has length $k/2 + j - 2\ell$) is a path of length less than k, so we can apply the induction hypothesis on $S_{a,c}$. Suppose first the path $[w, c]$ contains the path $[w, a]$. Then similarly as in the proof of Claim 3, one shows that $b \in S_{a,c}$ would contradict the induction hypothesis on $S_{a,c}$. So we can assume $\ell < k/2$. Suppose now $j \leq k/2$. This condition implies that the element $a \bowtie c$ belongs to $[a, b]$ (since $\delta(a, a \bowtie c) = k/4 + j/2 - \ell \leq k/2 - \ell$), so $\delta(b, a \bowtie c) \geq k/2 + \ell$. Suppose $\delta(a, c) \neq 2, 4$. Then the fact that $b \in S_{a,c}$ implies $\delta(b, a \bowtie c) \leq \delta(a, c)/2 + 2 = k/4 + j/2 - \ell + 2$. This can only be satisfied if $j \geq k/2 + 4\ell - 4$, which is again only satisfied if $\ell < 2$. Hence $\ell = 1$ and necessarily $j = k/2$, which is one of the cases mentioned in Claim 4. Clearly, $\delta(a, c) \neq 4$ unless $k = 8$ and $c = w$. If $\delta(a, c) = 2$, then $\delta(b, ac) = k - 1$, which is only possible if $(s, t) = (2, \infty)$, $k = 8$, $\mathrm{proj}_a b$ is a line containing exactly three points and c is the unique point on the line $\mathrm{proj}_a b$ at distance 8 from b (this gives the exception mentioned in Claim 4). Suppose now $k/2 < j < k/2 + 2\ell$ (which implies that $a \bowtie c$ is either a line of $[X, c]$ or a point on a line of this path). Note that $\delta(a, c) \neq 2, 4$. We obtain a path of length $d = k/4 + j/2 + l$ between b and $a \bowtie c$. If $d \leq n$ and $l > 1$, then $d > \delta(a, c)/2 + 2$, a contradiction. But $d \leq n$ and $\ell = 1$ implies $k/2 < j < k/2 + 2$, a contradiction. If finally $d > n$, then $\delta(b, a \bowtie c) \geq 2n - d > \delta(a, c)/2 + 2$, again a contradiction.

So we may assume that $j \geq k/2 + 2l$. Let L be the line of $[w, b]$ at distance 3 from w, and x a point on L different from the projections of w and b onto L.

$\underline{k/2 + 2l \leq j \leq n - 4}$

We use the same method as in the proof of Claim 2. Let y be a point at distance $n - j - 3$ from L such that the projection onto L is x, and Y the line through y different from the projection of x onto y. Note that $\delta(c, Y) = n - 1$. We obtain a path γ' between b and c (consisting of the paths $[b, Y]$ and $[Y, c]$) of length $d = 2n + k/2 - j - 6$. Suppose $d \equiv 2 \bmod 4$. Consider the line M of γ' at distance $d/2$ from both b and c. Let ℓ' be the length of the path between a and $\mathrm{proj}_M b$ consisting of $[a, L]$ and $[L, \mathrm{proj}_M b]$. Then $\ell' = n + k/4 - j/2 + 2$. If $j \geq k/2 + 4$ (which is certainly satisfied if $\ell > 1$), then $d/2 \leq n - 5$ and $\ell' \leq n$ (and $d/2 = n - 5$ if and only if $\ell' = n$). Suppose first $\ell' < n$. Then it is possible to find a point z at distance $n - 4 - d/2$ from M with $\mathrm{proj}_M a \neq \mathrm{proj}_M z \neq \mathrm{proj}_M b$ that is opposite a. If Z is the line through z different from $\mathrm{proj}_z b$, then there is a unique point on Z (and thus contained in $T_{b,c}$) at distance $n - 2$ from a, hence $c \notin S_{a,b}$. Suppose now $\ell' = n$ (then $d/2 = n - 5$ and $j = k/2 + 4$). If we find a point z' on M, $\mathrm{proj}_M a \neq z' \neq \mathrm{proj}_M b$, that lies opposite a, then the projection of a onto the line through z' different from M is a point of $T_{b,c}$ at distance $n - 2$ from a, implying $c \notin S_{a,b}$. If we cannot find such a point z', then M is a line containing three points and $\mathrm{proj}_M a \neq \mathrm{proj}_M c \neq \mathrm{proj}_M b$. A point v on the line W through $\mathrm{proj}_M a$, different from M, $\mathrm{proj}_W c \neq v \neq \mathrm{proj}_W a$, then is an element of $T_{b,c}$ at distance $n - 2$ from a, hence $c \notin S_{a,b}$. The case $\ell = 1$ and $j = k/2 + 2$ is the second remaining case mentioned in Claim 4. Suppose $d \equiv 0 \bmod 4$. Consider a point on Y different from y or $\mathrm{proj}_Y c$, and the line Y' through this point different from Y. Then joining the paths $[b, Y']$ and $[Y', c]$ gives a path γ' between b and c of length $\equiv 2 \bmod 4$. Then we proceed similarly as in the previous case. We again obtain the possibility $\ell = 1$ and $j = k/2 + 2$ mentioned in Claim 4.

$\underline{j = n - 2}$

If it is possible to choose the point x on L such that $\delta(c, x) = n$, we proceed as in the previous paragraph. If it is not possible to choose x as above, L contains exactly 3 points, and $\mathrm{proj}_L b \neq \mathrm{proj}_L c = x \neq \mathrm{proj}_L a$. In this case, let γ' be the union of the paths $[b, x]$ and $[x, c]$. If this gives a path of length $\equiv 0 \bmod 4$, then let y' be a point at distance 3 from the line $L' = \mathrm{proj}_x c$, $x \neq \mathrm{proj}_{L'} y' \neq \mathrm{proj}_{L'} c$, and Y'' the line through y' different from $\mathrm{proj}_{y'} b$. Put γ'' the path (of length $\equiv 2 \bmod 4$) joining $[b, Y'']$ and $[Y'', c]$. Now proceeding similarly as before with the path γ' or γ'', this gives one of the possibilities mentioned in Claim 5 for the case $(s, t) = (2, \infty)$.

$\underline{j = n}$
A similar reasoning as before gives the other exceptions mentioned in Claim 5.

The claims are proved.

In order to make future arguments uniform, we redefine the set $S_{a,b}$ for two points a, b of Γ in the case $n = 12$ as follows. Put

$$\widetilde{S}_{a,b} = S_{a,b} \setminus \{x \in S_{a,b} \,|\, \Gamma_{10}(x) \cap S_{a,b} \neq \emptyset\}.$$

If $\delta(a,b) = 2$ with $|ab| = \infty$ or $\delta(a,b) = 4$, then $\widetilde{S}_{a,b} = S_{a,b}$. If $\delta(a,b) = 2$ with $|ab| = 3$, then $\widetilde{S}_{a,b} = \Gamma_1(L)$, with L the unique line concurrent with ab not through a, b. For the cases $n = 6$ or $n = 8$ and $\{s,t\} \neq \{2,\infty\}$, we content ourselves with the observation $\widetilde{S}_{a,b} \subseteq S_{a,b}$. Suppose finally $\delta(a,b) = 8$ and $\{s,t\} = \{2,\infty\}$. Now with the notation of Claim 5, if $x \in S_{a,b} \cap \Gamma_8(a) \cap \Gamma_8(b)$, then $\delta(x,e) = 10$, hence $x \notin \widetilde{S}_{a,b}$. If $x \in S_{a,b} \cap \Gamma_7(A')$, $\delta(x,w) = 10$, hence $x \notin \widetilde{S}_{a,b}$. If finally $x \in S_{a,b} \cap \Gamma_{11}(B)$ then since $\delta(x,b) \neq n-2$, $\delta(x,e) = 10$, so $x \notin \widetilde{S}_{a,b}$. We conclude that $\widetilde{S}_{a,b} \subseteq S'_{a,b} \cup \{w, e\}$. We write $S_{a,b}$ for $\widetilde{S}_{a,b}$ from now on.

$\boxed{\text{Step 2: the set } C_{a,b;c}}$
Let $c \in S_{a,b}$. We keep the same notation as in Step 1. Then we define $C_{a,b;c} = \{c' \in S_{a,b} \,|\, S_{c,c'} \cap \{a,b\} \neq \emptyset\}$.

For $\delta(a,b) = k \equiv 2 \bmod 4$ and $k \notin \{2, n-2, n\}$, we will prove that $C_{a,b;c}$ is always empty, except possibly in the following cases (with $w := a \bowtie b$):

(1) $\delta(c,w) = k/2 - 2$.
Here, a point $c' \in C_{a,b;c}$ lies at distance $k/2 - 2$ from w, with $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$.

(2) $\delta(c,w) = k/2 + 2$.
Here, a point $c' \in C_{a,b;c}$ lies at distance $k/2 + 2$ from w and either $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$ or $\mathrm{proj}_w c = \mathrm{proj}_w c' =: z$ (and let $\{w, Z\} = \Gamma_1(z)$) but $\mathrm{proj}_Z c \neq \mathrm{proj}_Z c'$; if $\{s,t\} = \{2,\infty\}$ and $k = 6$, then there is an extra possibility (*) for c' described below.

Indeed, let $\delta(c,w) = j$ and suppose $c' \in C_{a,b;c}$, $\delta(c',w) = j'$.

Suppose first $\mathrm{proj}_w c = \mathrm{proj}_w c'$. Then $\delta(c,c') \leq j + j' - 4 \leq k$ (because $j, j' \leq k/2 + 2$ by Claim 3 above). Without loss of generality we may assume $a \in S_{c,c'}$. Then, if $\delta(c,c') \notin \{2,4\}$,

$$\delta(a, c \bowtie c') \leq \frac{\delta(c,c')}{2} + 2 \leq \frac{k}{2} + 2.$$

Since clearly $\delta(a, c\bowtie c') \geq k/2+2$ ($c\bowtie c'$ lies on $[c,c']$!), this implies $j = j' = k/2+2$. Using Claim 2 and 3 above, one checks that $\delta(c,c') \neq 4$ (indeed, the only possibility would be $k=6$ and c or c' a point on $a\bowtie b$, but this is excluded by Claim 3). If $\delta(c,c') = 2$, then it is easy to see that we necessarily have $\{s,t\} = \{2,\infty\}$, $k=6$ and

(*) c, c' are collinear points on a line incident with exactly 3 points and both c, c' are at distance 5 from w.

These are some of the possibilities mentioned in (2).

Suppose now $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$. Here, $\delta(c,c') = j + j' \leq k+4$. We may again assume $a \in S_{c,c'}$. Then, if $\delta(c,c') = 2$, we must have $k=6$ by Claim 1 above (noting that the line w contains at least 4 points in this case). But this contradicts $c \in S_{a,b}$ and Claim 3. Also, it is easily verified that $\delta(c,c') \neq 4$. Now for $\delta(c,c') \notin \{2,4\}$, we obtain the following possibilities.

(a) $j + j' \equiv 2 \bmod 4$.
By Claim 3 above, $w = c\bowtie c'$ and $j = j'$. Since $a \in S_{c,c'}$ and $c, c' \in S_{a,b}$, we have $k/2-2 \leq j \leq k/2+2$. The case $j = j' = k/2+2$ corresponds to the remaining part of possibility (2). The case $j = j' = k/2$ contradicts Claim 3 above (noting w contains at least 4 points here). Finally, the case $j = j' = k/2-2$ corresponds to possibility (1).

(b) $j + j' \equiv 0 \bmod 4$.
Without loss of generality we may assume $j > j'$. By $a \in S_{c,c'}$ and Claim 4, $c\bowtie c' = \mathrm{proj}_w c$ and hence $j = j'+2$. Furthermore, $k/2 = (j+j')/2 \pm 1$. This implies that either j or j' is equal to $k/2$, contradicting $c, c' \in S_{a,b}$ and Claim 3.

This proves (1) and (2).

Step 3: the sets D_2 and D_4 if $s,t \geq 3$ for both Δ and Δ'

The aim of Step 3 is to construct sets D_2 and D_4 consisting of all pairs of points of Γ at mutual distance 2 and 4, respectively, possibly containing some pairs of opposite points as well. Therefore, we first define the sets D_2' and D_4', as follows.

A pair (a,b) of points of Γ belongs to D_2' if

(1) $|S_{a,b}| > 1$ and $\delta(a,b) \neq n-2$;

(2) $|C_{a,b;c}| > 1$, for all $c \in S_{a,b}$;

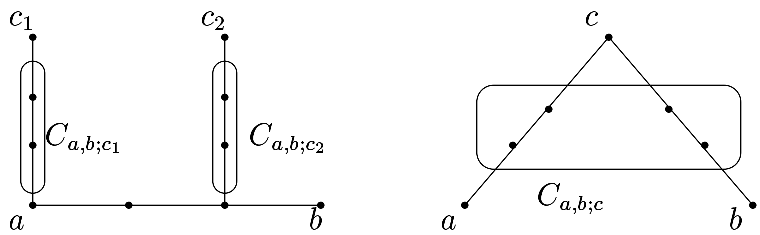


Figure 4.8: Proof of Step 3.

(3) there exists a point $c \in S_{a,b}$ such that c itself and all points $c' \in C_{a,b;c}$ satisfy Property P(c) and P(c') respectively, with

P(z) If $y \in C_{a,b;z}$ and $x \in T_{y,z}$, then x is at distance $n-2$ from all points of $C_{a,b;z} \cup \{a\}$ but exactly one;

(4) for all $c \in S_{a,b}$ and all $c', c'' \in C_{a,b;c}$ we have $S_{c,c'} \cap \{a,b\} = S_{c,c''} \cap \{a,b\}$ and $C_{a,b;c} \setminus \{c'\} = C_{a,b;c'} \setminus \{c\}$.

A pair (a,b) of points of Γ belongs to D'_4 if

(1′) $|S_{a,b}| > 1$ and $\delta(a,b) \neq n-2$;

(2′) there exists a point $c \in S_{a,b}$ such that $C_{a,b;c} \neq \emptyset$ and such that no point of Γ is at distance $n-2$ from all the points of $C_{a,b;c}$.

We show the following assertions.

If $\delta(a,b) = 2$, *then* $(a,b) \in D'_2 \setminus D'_4$.

Proof. Clearly, (1) holds. For $c \in S_{a,b}$, one easily sees $C_{a,b;c} = \Gamma_1(\mathrm{proj}_c ab) \setminus \{c, \mathrm{proj}_{ab} c\}$. Now (2) and (4) are clear, while (2′) cannot be satisfied. Every point $c \in S_{a,b}$ collinear with a (such c exists) satisfies P(c), whence (3). $\diamond$

If $\delta(a,b) = 4$, *then* $(a,b) \in D'_4 \setminus D'_2$.

Proof. Clearly, (1′) holds. Now we put $c = a \bowtie b$. Then $C_{a,b;c} = \Gamma_1(ac) \cup \Gamma_1(bc) \setminus \{a,b,c\}$. So it is clear that (2′) is satisfied, but (4) does not hold. Indeed, let $c' \in \Gamma_1(ac)$ and $c'' \in \Gamma_1(bc)$, $c', c'' \notin \{a,b,c\}$. Then $S_{c,c'} \cap \{a,b\} = \{b\}$ and $S_{c,c''} \cap \{a,b\} = \{a\}$. $\diamond$

If $\delta(a,b) \equiv 2 \bmod 4$ *with* $2 \neq \delta(a,b) < n-2$, *then* $(a,b) \notin D'_2 \cup D'_4$.

Proof. Put $w = a \bowtie b$. Suppose by way of contradiction that $(a,b) \in D'_2$. Let c be a point for which (3) holds and c', c'' two distinct arbitrary elements of $C_{a,b;c}$. By Step 2 the paths $[w,c']$ and $[w,c'']$ have at most 3 elements in common with $[w,c]$ and $\delta(w,c') = \delta(w,c) = \delta(w,c'')$. Since, by the last part of (4), $c'' \in C_{a,b;c} \setminus \{c'\}$ implies $c'' \in C_{a,b;c'} \setminus \{c\}$, it follows from Step 2 that also the paths $[w,c']$ and $[w,c'']$ have at most 3 elements in common. Put $\{c,c',c''\} = \{c_1,c_2,c_3\}$. In the following, we construct a point x for which $\delta(a,x) \neq n-2$, and such that x lies at distance $n-2$ from exactly 2 points of $\{c_1,c_2,c_3\}$. Put $j = \delta(w,c_1)$. Suppose first $\mathrm{proj}_w c_3 \notin \{\mathrm{proj}_w c_1, \mathrm{proj}_w c_2\}$. Let x be a point at distance $n-2-j$ from w such that the path $[w,c_3]$ contains the path $[w,x]$ or vice versa. Then $\delta(a,x) \neq n-2$ and $\Gamma_{n-2}(x) \cap \{c_1,c_2,c_3\} = \{c_1,c_2\}$. Assume now $\mathrm{proj}_w c_1 = \mathrm{proj}_w c_2 = \mathrm{proj}_w c_3$. Note that $j = k/2+2$. Let x be a point at distance $n-k/2$ from w such that the path $[w,x]$ contains $[w,c_3]$. Then again $\delta(a,x) \neq n-2$ and $\Gamma_{n-2}(x) \cap \{c_1,c_2,c_3\} = \{c_1,c_2\}$. So the point x is as claimed. Now the existence of x contradicts (3). Indeed, suppose $\Gamma_{n-2}(x) \cap \{c_1,c_2,c_3\} = \{c_1,c_2\}$. Then property $\mathrm{P}(c_1)$ (with $y = c_2$) is not satisfied, since x does not lie at distance $n-2$ from the two points a and c_3 of C_{a,b,c_1}. So $(a,b) \notin D'_2$.

Now suppose by way of contradiction that $(a,b) \in D'_4$. Let $c \in S_{a,b}$ be as in (2′) and Σ any apartment through a,b. By Step 2, the points of $C_{a,b;c}$ all lie at the same distance from w. Hence a point x of Σ at distance $n-2-\delta(w,c)$ from w lies at distance $n-2$ from all elements of $C_{a,b;c}$, contradicting (2′) $\diamond$.

For a pair (a,b) of points of Γ, we define

$$\overline{S}_{a,b} = \{x \in S_{a,b} \mid (a,x),(b,x) \in D'_2 \cup D'_4\}.$$

Now a pair (a,b) of points of Γ belongs to D_2 (respectively D_4) if

(1″) $(a,b) \in D'_2$ ($(a,b) \in D'_4$ respectively);

(2″) $|\overline{S}_{a,b}| > 1$;

(3″) for any point x of Γ, there are at least 2 points of $\overline{S}_{a,b}$ not lying at distance $n-2$ from x.

We show the following assertions.

If $\delta(a,b) = 2$, then $(a,b) \in D_2$; if $\delta(a,b) = 4$, then $(a,b) \in D_4$.
Proof. If $\delta(a,b) = 2$, then clearly $S_{a,b} = \overline{S}_{a,b}$; if $\delta(a,b) = 4$, then (putting $w = a \bowtie b$) $\Gamma_1(aw) \cup \Gamma_1(bw) \subseteq \overline{S}_{a,b} \cup \{a,b\}$. Hence (2″) and (3″) are satisfied. $\diamond$

If $\delta(a,b) \equiv 0 \bmod 4$ with $4 \neq \delta(a,b) < n-2$, then $(a,b) \notin D_2 \cup D_4$.
Proof. If $\delta(a,b) > 8$, then Claim 4 of Step 1 implies that for any $c \in S_{a,b}$

either $\delta(a,c) \equiv 2 \bmod 4$ or $\delta(b,c) \equiv 2 \bmod 4$ (and also, these distances are not equal to 2,4 or n); hence $\overline{S}_{a,b} = \emptyset$ (and $(2'')$ is not satisfied). If $\delta(a,b) = 8$ and $n \neq 12$, then similarly $\overline{S}_{a,b} = \{a \bowtie b\}$ (and again $(2'')$ is not satisfied). If $\delta(a,b) = 8$ and $n = 12$, then $\overline{S}_{a,b} \subseteq (\Gamma_8(a) \cap \Gamma_8(b)) \cup \{a \bowtie b\}$. But then, if $(2'')$ holds, then $(3'')$ cannot be satisfied by considering the point $a \bowtie (a \bowtie b)$. $\diamond$

Hence we have shown that D_2 consists of all pairs of collinear points of Γ and some (or possibly no) pairs of opposite points; likewise D_4 consists of all pairs of points of Γ at mutual distance 4 and some (or possibly no) pairs of opposite points.

Step 4: the set Ω of pairs of collinear points if $s, t \geq 3$ for both Δ and Δ'

We define the set Ω of pairs of points of Γ as follows. A pair (a, b) belongs to Ω if it belongs to D_2 and if there exists a pair of points $(c, c') \in D_2$, with $\{a, b\} \cap \{c, c'\} = \emptyset$, satisfying

(1) whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$, then $T_{v,v'} \subseteq \Gamma_{n-2}(w) \cup \Gamma_{n-2}(w')$;

(2) for any two distinct points $x, y \in \{a, b, c, c'\}$, we have $(x, y) \in D_2$;

(3) whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$, then for all $z \in T_{v,v'}$, we have $(w, z), (w', z) \notin D_2 \cup D_4$.

We claim that Ω is precisely the set of pairs of collinear points of Γ. Indeed, let $(a, b) \in D_2$ be arbitrary.

First suppose $\delta(a, b) = 2$. Then we can choose two distinct points c, c' on the line ab (with $\{a, b\} \cap \{c, c'\} = \emptyset$). It is easy to check that (c, c'), which obviously belongs to D_2, satisfies (1), (2) and (3) above. We now show for later purposes that, if $(c, c') \in D_2$ satisfies (1), (2) and (3), then both c and c' are incident with the line ab. First assume $c \in \Gamma_2(a)$. If c is not incident with the line ab, then $\delta(b, c) = 4$ and so $(b, c) \notin D_2$. Hence $c \mathtt{I} ab$. If c' is not incident with ab, then it must be opposite a, b and c, and hence $\mathrm{proj}_{ab} c' \notin \{a, b, c\}$. But then the point y collinear with c' on the path $[c', ab]$ belongs to $T_{a,b}$ and contradicts (3) since $(c', y) \in D_2$. So we may assume that both c, c' are opposite a, b. But then again the point y collinear with c' on the path $[c', ab]$ contradicts (3) since $(c', y) \in D_2$.

Hence we have shown that

$(*)$ if $(a, b) \in \Omega$ and $\delta(a, b) \neq 2$, then for any pair of distinct points $c, c' \in D_2$ satisfying (1), (2) and (3), we must have $\delta(x, y) = n$, for any two distinct points x, y in $\{a, b, c, c'\}$.

Indeed, if two elements of $\{a, b, c, c'\}$ would be collinear, then we can let them play the roles of a and b in the previous paragraph and obtain a contradiction (by remarking that all conditions (1) up to (3) are symmetric in a, b, c, c').

Now suppose $\delta(a, b) = n$. We must show $(a, b) \notin \Omega$. Suppose by way of contradiction that there exists a pair of points $(c, c') \in D_2$, with $\{a, b\} \cap \{c, c'\} = \emptyset$, and satisfying conditions (1), (2) and (3). If $n \equiv 2 \bmod 4$, we choose a fixed line M of Γ at distance $n/2$ from both a and b. If $n \equiv 0 \bmod 4$, we choose a fixed line M at distance $n/2 + 1$ from both a and b (such a line can be obtained as follows: fix a line A through a and let B be the line through b opposite A; let a' be a point on A, $a \neq a' \neq \mathrm{proj}_A b$, and put $b' = \mathrm{proj}_B a'$; let then M be the line of $[a', b']$ at distance $n/2 - 1$ from both a' and b'). In both cases (by possibly interchanging the roles of the two lines through a, and hence also of those through b), we may assume that M contains more than four points (this follows from our assumption that at most one of the parameters s, t is equal to 3). Let Y be a line at distance j from M, $0 \leq j \leq n - 3 - \delta(a, M)$, with $\mathrm{proj}_M b \neq \mathrm{proj}_M Y \neq \mathrm{proj}_M a$ (note that $\delta(a, M) < n - 3$ since $n > 8$). Define the following sets T_Y:

$$T_Y := \{x \in \mathcal{P} \mid \delta(x, Y) = (n-2) - \delta(a, M) - j \text{ and } \mathrm{proj}_Y a \neq \mathrm{proj}_Y x \neq \mathrm{proj}_Y b\}.$$

Note that $T_Y \subseteq T_{a,b}$, hence by (1), $T_Y \subseteq \Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$. We first prove, by induction on $j = \delta(Y, M)$, that $T_Y \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, for all lines Y for which the set T_Y is defined.

First let $j = 0$. Then $Y = M$. Suppose $T_M \subseteq \Gamma_{n-2}(c)$. Then it is easy to see that $\delta(a, M) = \delta(c, M)$ and $\mathrm{proj}_M a = \mathrm{proj}_M c$ or $\mathrm{proj}_M b = \mathrm{proj}_M c$. Assume $\mathrm{proj}_M a = \mathrm{proj}_M c$. This implies that $\delta(a, c) \leq n - 2$, so (since $(a, c) \in D_2$ by (2)), $\delta(a, c) = 2$, contradicting $(*)$. Hence $T_M \not\subseteq \Gamma_{n-2}(v)$ for any $v \in \{c, c'\}$. Now let $j = 2$. So let N be a line concurrent with M, not through the projection of a or b onto M. Suppose $T_N \subseteq \Gamma_{n-2}(c)$. Then $\delta(c, N) = \delta(a, N) = \delta(a, M) + 2$ and $\mathrm{proj}_N c = \mathrm{proj}_N a$ but $\mathrm{proj}_M a \neq \mathrm{proj}_M c \neq \mathrm{proj}_M b$ (because otherwise $T_M \subseteq \Gamma_{n-2}(c)$). Now $\delta(a, M) = \delta(c, M)$. If $\delta(a, M) = n/2$, then the point y on $[M, c]$ collinear with c belongs to $T_{a,b}$. If $\delta(a, M) = n/2 + 1$, then the point y on $[M, c]$ at distance 4 from c belongs to $T_{a,b}$. In both cases, $(c, y) \in D_2 \cup D_4$, contradicting (3). Hence $T_N \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$ for all lines N concurrent with M, not through the projection of a or b onto M.

Now let $j \geq 4$ be arbitrary, $j \leq n - 3 - \delta(a, M)$ and let Y be a line at distance j from M with $\mathrm{proj}_M b \neq \mathrm{proj}_M Y \neq \mathrm{proj}_M a$. Suppose $T_Y \subseteq \Gamma_{n-2}(c)$. Let $[Y, M] =: (Y, p, Y', p', Z, \ldots, M)$ (with possibly $Z = M$). Then $\delta(c, Y) = \delta(a, Y) = \delta(a, M) + j$ and $\mathrm{proj}_p c = \mathrm{proj}_p a = Y'$ but $\mathrm{proj}_{p'} a \neq \mathrm{proj}_{p'} c$

(otherwise $T_{Y'} \subseteq \Gamma_{n-2}(c)$, contradicting the induction hypothesis). Let Y'' be the line through $\mathrm{proj}_{Y'}c = p''$, different from Y'. Now it is readily checked that $T_{Y''} \cap \Gamma_{n-2}(c) = \emptyset$, so (1) implies $T_{Y''} \subseteq \Gamma_{n-2}(c')$. Since also $\delta(Y'', M) = j$, we have that $\delta(c', Y'') = \delta(a, Y'') = \delta(a, M) + j$, $\mathrm{proj}_{p''}c' = \mathrm{proj}_{p''}a = Y'$ but $\mathrm{proj}_{p'}a \neq \mathrm{proj}_{p'}c'$. Let X be a line concurrent with Z, not through p' or the projection of a or b onto Z. Consider a line L at distance $n-1-\delta(a,M)-j$ from X with $\mathrm{proj}_X M \neq \mathrm{proj}_X L$ (then all the points of L except from $\mathrm{proj}_L a$ are points of $T_{a,b}$). Since $\delta(c, L) = \delta(c', L) = n-1$, there is exactly one point of L at distance $n-2$ from c, and the same for c'. This is a contradiction with (1), since L contains at least 3 points of $T_{a,b}$. Hence $T_Y \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, for all lines Y for which the set T_Y is defined.

Now consider a line K at distance $n-5-\delta(a,M)$ from M for which the set T_K is defined. Let R, R' and R'' be three different lines concurrent with K at distance $n-3-\delta(a,M)$ from a (such lines exist because $s, t \geq 3$ and since, if $K = M$, which occurs if $n = 10$ or $n = 12$, then M contains at least three points different from $\mathrm{proj}_M a$ and $\mathrm{proj}_M b$ by assumption). We already know that $T_R \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, so the only remaining possibility for the points c and c' is that (since T_R contains at least 3 points and necessarily $T_R \subseteq \Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$) up to interchanging c and c', the point c lies at distance $n-4$ from a point r on R, r not on K, with $\mathrm{proj}_r c \neq R$. Because then c is opposite all but one point of $T_{R'}$, we must have that the point c' lies at distance $n-4$ from a point r' on R', r' not on K, with $\mathrm{proj}_{r'}c' \neq R'$. But now at most two points of $T_{R''}$ will be contained in $\Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$, a contradiction with (1) and the fact that $T_{R''}$ contains at least 3 points. This shows that the points c, c' cannot exist, so $(a, b) \notin \Omega$.

This shows that α preserves collinearity if both the orders of Δ and Δ' do not contain a 2, and completes the proof in this case.

Step 5: the sets D_2, D_2' and D_4 if $\{s,t\} = \{2, \infty\}$ for both Δ and Δ'

The aim of Step 5 is to construct sets D_2, D_2' and D_4 (for the case $\{s,t\} = \{2, \infty\}$) consisting of all pairs of points of Γ at mutual distance 2 (and the joining lines have infinitely many points or exactly three points, for D_2 and D_2' respectively) and 4, respectively, possibly containing some pairs of opposite points as well. Therefore, we first define the sets E_2 and E_4, as follows.

A pair (a, b) of points of Γ belongs to E_4 if

(1) $|S_{a,b}| > 1$ and $\delta(a, b) \neq n-2$;

(2) there is a point $c \in S_{a,b}$ such that $|C_{a,b;c}| = \infty$ and such that no point x of Γ satisfies $\{a, c\} \cup C_{a,b;c} \subseteq \Gamma_{n-2}(x)$.

A pair (a, b) of points of Γ belongs to E_2 if

(1′) $|S_{a,b}| > 1$ and $\delta(a, b) \neq n - 2$;

(2′) no point lies at distance $n - 2$ from all elements of $S_{a,b}$;

(3′) for every point $c \in S_{a,b}$ we have $|C_{a,b;c}| = 1$, and, putting $C_{a,b;c} = \{c'\}$, we must have $(c, c') \in E_4$.

Note that the sets E_2 and E_4 are disjoint, because of properties (2) and (3′). We show the following assertions.

If two points a, b are collinear in Γ and the line ab contains exactly three points, then $(a, b) \in E_4$ and $(a, b) \notin E_2$.

Proof. Note that $\Gamma_3(ab) \subseteq S_{a,b}$, hence (1) holds. Let e be the unique point on ab different from a and b, and c a point of $\Gamma_3(ab)$ collinear with e. Then $\Gamma_1(ec) \subseteq C_{a,b;c} \cup \{c\}$; showing that (2) holds for this point c. So $(a, b) \in E_4 \setminus E_2$. $\diamond$

If two points a, b are collinear in Γ and the line ab contains infinitely many points, then $(a, b) \in E_2$ and $(a, b) \notin E_4$.

Proof. In this case, $S_{a,b} = \Gamma_3(ab)$, hence (1′) and (2′) hold. For any point $c \in S_{a,b}$, the set $C_{a,b;c}$ contains exactly one point, namely the unique point on the line $L = \mathrm{proj}_c ab$ different from c and $\mathrm{proj}_L a$. This shows that (3′) holds, hence $(a, b) \in E_2 \setminus E_4$. $\diamond$

If two points a, b are at mutual distance 4 in Γ, then $(a, b) \in E_4$ and $(a, b) \notin E_2$.

Proof. Put $c = a \bowtie b$. Then $C_{a,b;c}$ contains the set $\Gamma_1(ac) \cup \Gamma_1(bc) \setminus \{a, b, c\}$. This shows that (2) holds for the point c, hence $(a, b) \in E_4 \setminus E_2$. $\diamond$

If $\delta(a, b) \equiv 2 \bmod 4$ with $2 \neq \delta(a, b) < n - 2$, then $(a, b) \notin E_2 \cup E_4$.

Proof. Suppose by way of contradiction that $(a, b) \in E_4$, and let c be a point in $S_{a,b}$ satisfying (2). But since any element of $C_{a,b;c}$ lies at distance $\delta(a \bowtie b, c)$ from $a \bowtie b$, one can easily find a point at distance $n - 2$ from all points of $C_{a,b;c} \cup \{a, c\}$; a contradiction, hence $(a, b) \notin E_4$. Suppose now $(a, b) \in E_2$, and let $c \in S_{a,b}$ with $C_{a,b;c} = \{c'\}$. Put $w := a \bowtie b$. If $\delta(w, c) = k/2 - 2 = \delta(w, c')$, then $\delta(c, c') = k - 4 \equiv 2 \bmod 4$. Hence $(c, c') \in E_4$ implies $\delta(c, c') = 2$, so $k = 6$ and c, c' are incident with m. This contradicts Claim 3 in Step 1. If $\delta(w, c) = k/2 + 2 = \delta(w, c')$, then either $\delta(c, c') = k + 4 \equiv 2 \bmod 4$ or $\delta(c, c') = k \equiv 2 \bmod 4$ (and in both cases, we obtain a contradiction with $(c, c') \in E_4$) or $k = 6$ and c, c' are collinear points on a line with exactly 3 points (this is the exception mentioned in (*)). But in the latter case, all points of $S_{a,b}$ lie at distance $k/2+2$ from $a \bowtie b$, and we easily find a point at distance $n-2$ from every point of $S_{a,b}$, contradicting (2′). Hence $(a,b) \notin E_2$. $\diamond$

Now we define

$$\overline{S}_{a,b} = \{x \in S_{a,b} \,|\, (a,x),(b,x) \in E_2 \cup E_4\}.$$

Completely similar as in Step 3, one shows that if two points a,b satisfy $4 < \delta(a,b) = k \equiv 0 \bmod 4$, $k < n-2$, then $|\overline{S}_{a,b}| \neq \infty$. Also, for two points a,b with $\delta(a,b) \in \{2,4\}$, $|\overline{S}_{a,b}| = \infty$.

By definition, a pair (a,b) of points of Γ belongs to D_2 if $(a,b) \in E_2$ and $|\overline{S}_{a,b}| = \infty$. Also, a pair (a,b) of points of Γ belongs to D_2' if $(a,b) \in E_4$, $|\overline{S}_{a,b}| = \infty$ and there are some $c,c' \in S_{a,b}$ such that $(a,c),(b,c') \in D_2$. Finally, D_4 consists precisely of those pairs (a,b) of points of $E_4 \setminus D_2'$ that satisfy $|\overline{S}_{a,b}| = \infty$. We conclude that D_2 consists of all pairs (a,b) of collinear points with $|\Gamma_1(ab)| = \infty$, possibly together with some pairs of opposite points; D_2' consists of all pairs (a,b) of collinear points with $|\Gamma_1(ab)| = 3$, possibly together with some pairs of opposite points; D_4 consists of all pairs (a,b) of points at mutual distance 4, possibly together with some pairs of opposite points.

Step 6: the set Ω of pairs of collinear points if $\{s,t\} = \{2,\infty\}$ for both Δ and Δ'

Note that $n \equiv 0 \bmod 4$ (indeed, remember that (s,t) was the order of the corresponding $n/2$-gon Δ). We first pin down the set $\overline{\Omega}$ of pairs (a,b) of collinear points with $|\Gamma_1(ab)| = \infty$. Therefore we define $V_{a,b}$, for two arbitrary points a,b of Γ, as $V_{a,b} = \Gamma_{n-2}(a) \setminus \Gamma_{n-2}(b)$. Now let $\overline{\Omega}$ be the set of pairs (a,b) of D_2 such that there exist points c,c',c'' in Γ, all distinct from a and from b, with the following properties.

(1) $V_{a,b}$ is the disjoint union of the sets $V_{a,b} \cap \Gamma_{n-2}(c)$, $V_{a,b} \cap \Gamma_{n-2}(c')$ and $V_{a,b} \cap \Gamma_{n-2}(c'')$;

(2) $(a,c'),(a,c'') \in D_2'$; $(b,c'),(b,c'') \in D_4$ and $(a,c),(b,c) \in D_2$;

(3) no point x in $\Gamma_{n-2}(a) \cap \Gamma_{n-2}(c)$ satisfies $(b,x) \in D_2 \cup D_2'$; likewise no point x in $\Gamma_{n-2}(b) \cap \Gamma_{n-2}(c)$ satisfies $(a,x) \in D_2 \cup D_2'$;

(4) $a \in S_{b,c'} \cap S_{b,c''}$;

(5) if $\{u,v,w\} = \{c,c',c''\}$, then $v,w \in S_{a,u} \cap S_{b,u}$.

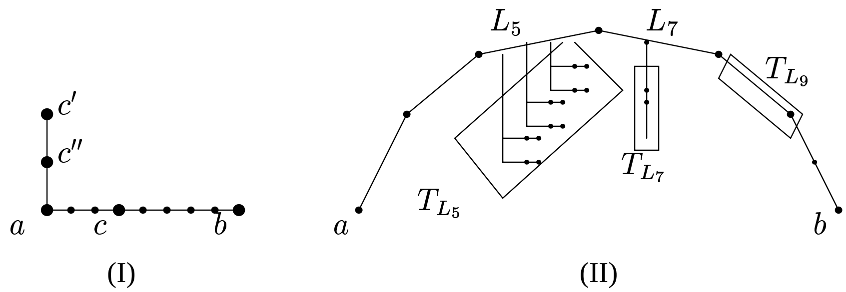


Figure 4.9: (I) : $(a, b) \in \overline{\Omega}$ (II) : The sets T_{L_j} for $n = 12$.

We now show that $\overline{\Omega}$ is the set of pairs (a, b) of collinear points with $|\Gamma_1(ab)| = \infty$. Clearly, if $\delta(a, b) = 2$ and $|\Gamma_1(ab)| = \infty$, then choosing $c \in \Gamma_1(ab) \setminus \{a, b\}$ arbitrarily, and putting $\{c', c''\} = \Gamma_2(a) \setminus \Gamma_1(ab)$, we see that $(a, b) \in \overline{\Omega}$ (see Figure 4.9 (I)).

So there remains to show that no pair of opposite points belongs to $\overline{\Omega}$. By way of contradiction, let (a, b) be a pair of opposite points of Γ belonging to $\overline{\Omega}$. Let c, c', c'' be as in (1) up to (5) above.

We claim that $\delta(a, x) = n = \delta(b, x)$, for $x \in \{c, c', c''\}$. Indeed, by (2), we already know that $\delta(a, c')$ and $\delta(a, c'')$ are either 2 or n. Suppose $\delta(a, c') = 2$. Then $\delta(b, ac') = n - 1$. But $b \in S_{a,c'}$ (which is Condition (4)) implies $\delta(b, ac') \leq 7$ (by Claim 1 of Step 1), a contradiction. Similarly for (a, c''). Also, $\delta(b, c')$ and (b, c'') are either 4 or n. Suppose $\delta(b, c') = 4$ and put $w = b \bowtie c'$. Then $\delta(a, bw) = n - 1$. Condition (4) states that $a \in S_{b,c'}$, hence by Claim 2 of Step 1, $\delta(a, bw) \leq 7$, a contradiction. Similarly for (b, c''). By (2), we also know that $\delta(a, c)$ and $\delta(b, c)$ are either 2 or n. If $\delta(a, c) = 2$ (and so $\delta(b, c) = n$), then the point collinear with b at distance $n - 3$ from the line ac lies at distance $n - 2$ from both a and c, contradicting Condition (3). Similarly for $\delta(b, c)$. This show the claim.

Form now on until the end of the proof, we assume that n is "large enough" (the generic case) in certain arguments. When n is too small, then either the given argument can be skipped or a separate but easier argument can be given (and we do not do that explicitly).

Let γ be the path of length n between a and b for which the line of γ through b contains exactly three points. Denote by L_j the line of γ at distance j (j is odd!) from a and define for $n/2 - 1 \leq j \leq n - 3$,

$$T_{L_j} = \{x \in \mathcal{P} | \delta(x, L_j) = n - 2 - j, \mathrm{proj}_{L_j} a \neq \mathrm{proj}_{L_j} x\}.$$

Note that the sets T_{L_j} are subsets of $V_{a,b}$, and that these sets consist of unions of certain sets $\Gamma_1(L) \setminus \{\mathrm{proj}_L a\}$, with $|\Gamma_1(L)| = \infty$ (see Figure 4.9 (II)).

For an element z at distance $\leq n-2-j$ from L_j for which $\mathrm{proj}_{L_j}a \neq \mathrm{proj}_{L_j}z \neq \mathrm{proj}_{L_j}b$, we define the set

$$T_z = \{x \in \mathcal{P} | \delta(x,z) = n-2-j-\delta(z,L_j), \mathrm{proj}_z a \neq \mathrm{proj}_z x\}.$$

Note that T_z is the subset of T_{L_j} containing the points x for which $[x, L_j]$ contains z.

Let Z be a line for which the set T_Z is defined. We first show by induction on $i_Z := n - \delta(a, Z)$ that

($\diamond$) for such a line Z there exist points $v, v' \in \{c, c', c''\}$ such that $T_Z \subset \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$. Moreover, for any two points $z', z'' \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a, \mathrm{proj}_Z b\}$, we have that $T_{z'} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$, with $T_{z'} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{z'} \cap \Gamma_{n-2}(v')$, implies $T_{z''} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$, with $T_{z''} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{z''} \cap \Gamma_{n-2}(v')$.

Suppose first $i_Z = 3$. Then Z is a line at distance $n-3$ from a containing an infinite number of points, and $T_Z = \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$. Condition (1) implies that there exists a point $x \in \{c, c', c''\}$ at distance $n-2$ from at least two points of T_Z, hence at distance $n-2$ from all but at most one point of T_Z. This shows ($\diamond$) for the line Z.

Now we assume $i_Z = 5$. Then necessarily $|\Gamma_1(Z)| = 3$. If $Z = L_{n-5}$, then $T_Z = T_N$, with N the unique line concurrent with Z and different from $\mathrm{proj}_Z a$ and $\mathrm{proj}_Z b$, and ($\diamond$) follows. So suppose $Z \neq L_{n-5}$. Let r and r' be the two points on Z different from $\mathrm{proj}_Z a$, and let R, R' be the lines through r respectively r' different from Z. Put Z' the line through $\mathrm{proj}_Z a$, different from Z. We claim that

(*) no point $v \in \{c, c', c''\}$ is at distance $n-2$ from exactly one point of $\Gamma_1(R) \setminus \{\mathrm{proj}_R a\}$ and from exactly one point of $\Gamma_1(R') \setminus \{\mathrm{proj}_{R'} a\}$.

Indeed, suppose some $v \in \{c, c', c''\}$ is at distance $n-2$ from exactly one point of T_R and from exactly one point of $T_{R'}$. Since by Condition (1) every point of $V_{a,b}$ lies at distance $n-2$ from exactly one point of $\{c, c', c''\}$, there exists a point $v' \in \{c, c', c''\} \setminus \{v\}$ such that $\delta(v', R) = n-3$ and $\mathrm{proj}_R v = \mathrm{proj}_R v'$. But then $\mathrm{proj}_{R'} v \neq \mathrm{proj}_{R'} v'$ (because $\delta(v', \mathrm{proj}_{R'} v') = n-2$ and $\mathrm{proj}_{R'} v \in V_{a,b}$) and so the unique point of $\{c, c', c''\} \setminus \{v, v'\}$ lies at distance $n-2$ from all but two or three points of R', which is impossible. Our claim is proved.

If R is not contained in $\Gamma_{n-2}(v)$ for a point $v \in \{c, c', c''\}$, then some point $w \in \{c, c', c''\}$ is at distance $n-4$ from exactly one point of T_R, and at distance $n-2$ from all the other points of T_R. But then there is exactly one point of $T_{R'}$ at distance $n-2$ from w. So the only possibility to satisfy Condition (1) is that a point $w' \in \{c, c', c''\} \setminus \{w\}$ is at distance $n-4$ from exactly one point of $T_{R'}$ (namely $\mathrm{proj}_{R'}w$) and at distance $n-2$ from all the other points of this set and at distance $n-2$ from $\mathrm{proj}_R w$. Whence ($\diamond$).

If $T_R \subset \Gamma_{n-2}(v)$ for some $v \in \{c, c', c''\}$, then $\delta(v, r) = n-4$ and $\mathrm{proj}_r c = Z$. If $\mathrm{proj}_Z v = r'$, then we consider a line Z'' concurrent with Z', different from Z and not through $\mathrm{proj}_{Z'}a$ or $\mathrm{proj}_{Z'}b$ (this is possible since $|Z'| = \infty$). But now $\delta(v, Z'') = n-1$ and $\delta(v, \mathrm{proj}_{Z''}Z) = n-2$, so v is at distance $n-2$ from exactly two non-collinear points of $T_{Z''}$, contradicting (*). So $\mathrm{proj}_Z v = \mathrm{proj}_Z a$ and $T_Z \subseteq \Gamma_{n-2}(v)$.

This shows ($\diamond$) for the line Z.

Now suppose $i_Z > 5$. Put $j = n - i_Z = \delta(a, Z)$. Suppose first that $|\Gamma_1(Z)| = 3$. If $Z = L_j$ (i.e., if Z belongs to γ), then $T_{L_j} = T_L$, with L the unique line concurrent with L_j and not contained in γ, and with $i_L = i_Z - 2$. So the result follows from the induction hypothesis. Hence we may assume that Z does not belong to γ. Put $\Gamma_1(Z) = \{x, x_1, x_2\}$ with $x = \mathrm{proj}_Z a$, put $L = \mathrm{proj}_x a$ and let X_i be the line through x_i distinct from Z, $i = 1, 2$. By the induction hypothesis, there are two cases to consider.

(i) There exists $v \in \{c, c', c''\}$ such that $T_{X_1} \subseteq \Gamma_{n-2}(v)$. We show that $T_{X_2} \subseteq \Gamma_{n-2}(v)$. Indeed, $\delta(v, x_1) = j+1$ and $\mathrm{proj}_{x_1} v = Z$. If $\mathrm{proj}_Z v \neq x_2$, then clearly $T_{X_2} \subset \Gamma_{n-2}(v)$. If $\mathrm{proj}_Z v = x_2$, then consider an arbitrary point p at distance $n-4-j$ from L for which $\mathrm{proj}_L p \notin \{x, \mathrm{proj}_L a, \mathrm{proj}_L b\}$. The point p lies at distance $n-2$ from v and a contradiction to (*) arises in the set T_p (indeed, we find exactly two non-collinear points of T_p lying at distance $n-2$ from v).

(ii) Suppose now that we are not in case (i) and there exist $v, v' \in \{c, c', c''\}$, $v \neq v'$, such that $T_{X_1} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$, with $T_{X_1} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{X_1} \cap \Gamma_{n-2}(v')$. From the proof of the case $i_Z = 5$ now follows that $\delta(v, x_1) = \delta(v', x_1) = \delta(a, x_1) + 6 = j + 7$. (Indeed, let Y be a line at distance $n-5$ from a for which the path $[a, Y]$ contains the path $[a, Z]$ and let Y_1, Y_2 be the two lines concurrent with Y and at distance $n-3$ from a. Then the proof of the case $i_Z = 5$ shows that we can assume $\delta(v, Y_1) = n-3$, $\delta(v', Y_2) = n-3$, $\delta(v, Y) = \delta(v', Y) = n-1$ and, with Y' the line concurrent with Y at distance $n-7$ from a, $\mathrm{proj}_{Y'}v = \mathrm{proj}_{Y'}a = \mathrm{proj}_{Y'}v'$.) If $\delta(v, x_2) = \delta(v', x_2) = j+7$, then $T_{X_2} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$ with $T_{X_2} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{X_2} \cap \Gamma_{n-2}(v')$. Suppose now by way of contradiction that $\delta(v, x_2) = j+5$. Then we consider a point p at distance $n-(j+8)$ from L such that $\mathrm{proj}_L a \neq \mathrm{proj}_L p \neq x$ and $\mathrm{proj}_L p \neq \mathrm{proj}_L b$. Put $\Gamma_1(p) = \{\mathrm{proj}_p a, R\}$. Note that v and p are opposite points of Γ. Put $[R,v] = (R, p', R', p'', \ldots, v)$, and let r be any point incident with R', $p' \neq r \neq p''$. Then considering T_r and v, we obtain a contradiction to (*).

This shows ($\diamond$) for the case $|\Gamma_1(Z)| = 3$.

Suppose now $|\Gamma_1(Z)| = \infty$. Suppose first $Z \notin \gamma$. Let x be any point on Z different from $\mathrm{proj}_Z a$. By the induction hypothesis, there are two cases.

(i) There exists $v \in \{c, c', c''\}$ such that $T_x \subseteq \Gamma_{n-2}(v)$. Similarly as above, one shows that in this case $T_y \subseteq \Gamma_{n-2}(v)$, for all $y \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, except possibly for one point $x^* \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, in which case $T_{x^*} \cap \Gamma_{n-2}(v) = \emptyset$.

(ii) There exists $v \in \{c, c', c''\}$ such that $T_x \neq T_x \cap \Gamma_{n-2}(v) \neq \emptyset$. Again similarly as above, one shows that in this case $T_y \neq T_y \cap \Gamma_{n-2}(v) \neq \emptyset$ for all $y \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, except possibly for one point $x^* \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, in which case $T_{x^*} \cap \Gamma_{n-2}(v) = \emptyset$.

Combining (i), (ii) and $|\Gamma_1(Z)| = \infty$ (and using the fact that the three sets $T_Z \cap \Gamma_{n-2}(v)$, $v \in \{c, c', c''\}$ are disjoint), we readily deduce ($\diamond$). If $Z \in \gamma$, then a similar reasoning shows the result.

So we have shown ($\diamond$) for all appropriate lines Z.

Suppose now that there exists $v \in \{c, c', c''\}$ with $T_{L_{n-3}} \subseteq \Gamma_{n-2}(v)$. We look for a contradiction. Note that $\delta(v, L_{n-3}) = n-3$ and $\mathrm{proj}_{L_{n-3}} a = \mathrm{proj}_{L_{n-3}} v$. Define $j \in \mathbb{N}$ as $[a, L_{n-3}] \cap [v, L_{n-3}] = [L_j, L_{n-3}]$. Suppose first $n/2 < j \leq n-5$. Then v lies at distance j from L_j. Consider a point p at distance $n-j-4$ from L_{j-2} satisfying $\mathrm{proj}_{L_{j-2}} a \neq \mathrm{proj}_{L_{j-2}} p \neq \mathrm{proj}_{L_{j-2}} b$. Then p lies at distance $n-2$ from v and at distance $n-6$ from a, and we obtain a contradiction to (*) by considering T_p and v. Suppose now $j \leq n/2 - 1$. Then $\delta(a,v) \leq \delta(a, L_j) + \delta(L_j, v) \leq n-2$, the final contradiction (since $\delta(a,v) = n$, for $v \in \{c, c', c''\}$).

Hence, since at least one element of $\{c, c', c''\}$ must be at distance $n-2$ from infinitely many points of L_{n-3}, there exists $v \in \{c, c', c''\}$ satisfying $\delta(v, L_{n-3}) = n-3$ and $\mathrm{proj}_{L_{n-3}} v \notin \gamma$ (remembering v is opposite b). Now v lies at distance $n-2$ from exactly one point of $T_{L_{n-5}}$, so there is a $v' \in \{c, c', c''\} \setminus \{v\}$ such that $T_{L_{n-5}} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$. Hence v' lies at distance $n-1$ from L_{n-5} and $\mathrm{proj}_{L_{n-5}} v' \notin \gamma$. Note that both v and v' are opposite the point $w := \mathrm{proj}_{L_{n-7}} b$, hence $\delta(v, L_{n-7}) = n-1 = \delta(v', L_{n-7})$. Let j be defined as $[L_{n-7}, a] \cap [L_{n-7}, v] = [L_{n-7}, L_j]$ and let j' be defined as $[L_{n-7}, a] \cap [L_{n-7}, v'] = [L_{n-7}, L_{j'}]$ (these are well-defined since $a \notin [L_{n-7}, v] \cup [L_{n-7}, v']$). Then $\delta(v, L_j) = j+6$ and $\delta(v', L_{j'}) = j'+6$, with $\mathrm{proj}_{Lj} v, \mathrm{proj}_{Lj'} v' \notin \gamma$.

Suppose first $n/2 - 2 < j$ and, if $n \equiv 0 \bmod 8$, $j \neq n/2 - 1$.

(*i*) If $|\Gamma_1(L_j)| = 3$, then, because of the conditions on j, the line L_{j-2} has infinitely many points and the set $T_{L_{j-2}}$ is defined. We proceed similarly as in (*ii*) of the proof of ($\diamond$), case $i_Z > 5$ and $|\Gamma_1(Z)| = 3$ (see above) to obtain a contradiction with (*).

(*ii*) If $|\Gamma_1(L_j)| = \infty$, then let $x = \mathrm{proj}_{L_j} v$. Calculating distances, it is easy to check that $T_x \cap \Gamma_{n-2}(v) = \emptyset$ and $T_{x'} \cap \Gamma_{n-2}(v) \neq \emptyset$, for all points $x' \in \Gamma_1(L_j) \setminus \{x, \mathrm{proj}_{L_j} a, \mathrm{proj}_{L_j} b\}$. This contradicts ($\diamond$).

We now treat the remaining cases. Note that in the foregoing, we may interchange the roles of j and j'.

(*iii*) If $n \equiv 0 \bmod 8$ and $j \leq n/2 - 1$, then $j' \leq n/2 - 1$ and $|\Gamma_1(L_{n/2-1})| = 3$. Note that $\{j, j'\} \subseteq \{n/2-1, n/2-3\}$ (since both v and v' are opposite a). If $j = j' = n/2-1$, then $T_{L_{n/2-1}} \cap \Gamma_{n-2}(v) = \emptyset = T_{L_{n/2-1}} \cap \Gamma_{n-2}(v')$, so $T_{L_{n/2-1}} \subset \Gamma_{n-2}(v'')$, with $\{v, v', v''\} = \{c, c', c''\}$. But this implies that $\delta(v'', L_{n/2-1}) = n/2 - 1$ and $\mathrm{proj}_{L_{n/2-1}} v'' \in \gamma$. So, calculating distances, we see that either $\delta(a, v'') < n$ or $\delta(b, v'') < n$, a contradiction.

Suppose $j = n/2 - 1$ and $j' = n/2 - 3$, or $j = j' = n/2 - 3$. Let a' and a'' be the two points on the line L_3 at distance $n-2$ from b. Then at least one of these two points lies at distance $n-2$ from the points v, v' and b, contradicting Condition (5) (namely $v' \in S_{b,v}$). This concludes the case $n \equiv 0 \bmod 8$.

(*iv*) If $n \equiv 4 \bmod 8$ and $j \leq n/2 - 3$, then again the case $j < n/2 - 3$ can not occur. So $j = n/2 - 3$, and by symmetry, also $j' = n/2 - 3$. We proceed similarly as in the last part of (*iii*) above to obtain a contradiction with Condition (5).

This shows that a pair of opposite points never belongs to $\overline{\Omega}$. Hence $\overline{\Omega}$ consists precisely of all pairs (a, b) of collinear points with $\Gamma_1(ab) = \infty$.

Finally, we define

$$\overline{\overline{\Omega}} = \{(a, b) \in D_2' | (\forall z \in \Gamma_{n-2}(b))((a, z) \notin \overline{\Omega})\}.$$

Clearly, if a and b are collinear points with $|ab| = 3$, then $(a,b) \in \overline{\overline{\Omega}}$. But if $\delta(a,b) = n$ then, with L the line through a containing infinitely many points, the point $\mathrm{proj}_L b$ lies at distance $n-2$ from b, and $(a, \mathrm{proj}_L b) \in \overline{\Omega}$. Hence $(a,b) \notin \overline{\overline{\Omega}}$, and $\overline{\overline{\Omega}}$ consists precisely of all pairs (a,b) of collinear points with $\Gamma_1(ab) = 3$.

Now $\Omega := \overline{\Omega} \cup \overline{\overline{\Omega}}$ is the set of all pairs of collinear points of Γ. This shows that α preserves collinearity in case both the orders of Δ and Δ' contain a 2.

Step 7: Distinction between the orders

By the results of the previous steps, we know that the given bijection extends to an isomorphism between Δ and Δ' if the orders of Δ and Δ' both contain a 2, or if they both do not contain a 2. We now want to exclude the remaining case. So suppose by way of contradiction that every line of Γ contains at least 4 points, and that Γ' has lines containing exactly 3 points. Note that we can assume that both Γ and Γ' are infinite, and $n \equiv 0 \bmod 4$. In the case of Γ', the set $\overline{\Omega}$ defined in Step 6 is non-empty (since it contains all pairs of collinear points a, b for which $|\Gamma_1'(ab)| = \infty$). We now show that the set $\overline{\Omega}$ (with $\overline{\Omega}$ as in Step 6) defined for the polygon Γ is empty. Since the size of the set $\overline{\Omega}$ is preserved by α, this is a contradiction. First, one has to determine the sets D_2, D_2' and D_4 as defined in Step 5 for the polygon Γ. It is easy to check that D_2 and D_2' can only contain pairs of points (a,b) for which $\delta(a,b) = n$. The set D_4 consists of all pairs of points (a,b) for which $\delta(a,b) = 4$, or for which $\delta(a,b) = 2$ such that the line ab is concurrent with a line containing infinitely many points, possibly together with some pairs of opposite points. Now suppose $(a,b) \in \overline{\Omega}$. Then we have $\delta(a,b) = n$, and there exist points c, c', c'' in Γ such that the conditions (1) up to (5) listed in Step 6 are satisfied. Similarly as in Step 6, one proves that $\delta(a,v) = \delta(b,v) = n$, for $v \in \{c, c', c''\}$.

Let γ be a fixed n-path between a and b such that the line of γ through a contains infinitely many points. Let L_0 be the line of γ at distance $\frac{n}{2} - 1$ from a. Let L_j be a line at distance j from L_0, $0 \leq j \leq \frac{n}{2} - 2$, for which $\mathrm{proj}_{L_0} a \neq \mathrm{proj}_{L_0} L_j \neq \mathrm{proj}_{L_0} b$. For such a line L_j, we define the set T_{L_j} as follows:

$$T_{L_j} = \{x \in \mathcal{P} | \delta(x, L_j) = n - 2 - j - \delta(a, L_j), \mathrm{proj}_{L_j} a \neq \mathrm{proj}_{L_j} x \neq \mathrm{proj}_{L_j} b\}.$$

The sets T_{L_j} are subsets of the set $V_{a,b}$ and consist of unions of sets $\Gamma_1(L) \setminus \{\mathrm{proj}_L a\}$. Note that by the choice of γ, these sets $\Gamma_1(L)$ contain infinitely many points.

We prove by induction on j that there does not exist a point $v \in \{c, c', c''\}$ and a line L_j for which the set T_{L_j} is defined such that $T_{L_j} \subseteq \Gamma_{n-2}(v)$. This is done in three parts. In (A), we handle the case $j = 0$ or $j = 2$, in (B) the case $2 < j \le \frac{n}{2} - 4$ and in (C) the case $j = \frac{n}{2} - 2$.

(A) Let L_2 be a line concurrent with L_0, $\mathrm{proj}_{L_0} a \neq \mathrm{proj}_{L_0} L_2 \neq \mathrm{proj}_{L_0} b$. Suppose there exists a point $v \in \{c, c', c''\}$ such that $T_{L_2} \subseteq \Gamma_{n-2}(v)$. Then it is easy to see that $\delta(v, L_0) = \delta(a, L_0)$, implying $\delta(a, v) < n$, a contradiction.

(B) Suppose there exists a line L_j, $2 < j \le \frac{n}{2} - 4$ for which the set T_{L_j} is defined, and a point $v \in \{c, c', c''\}$ such that $T_{L_j} \subseteq \Gamma_{n-2}(v)$. Denote by $(L_0, a_1, L_2, \ldots, a_{j-1}, L_j)$ the j-path between L_0 and L_j. Then $\delta(a, L_j) = \delta(v, L_j)$, $\mathrm{proj}_{L_j} v = a_{j-1}$ but $a_{j-3} \neq \mathrm{proj}_{L_{j-2}} v =: z$ (since otherwise, we obtain a contradiction with the induction hypothesis). Let Z be the projection of v onto z (note that $\delta(Z, L_0) = j$). Then it is easily verified that $T_Z \cap \Gamma_{n-2}(v) = \emptyset$ hence, with $\{v, v', v''\} = \{c, c', c''\}$, $T_Z \subseteq \Gamma_{n-2}(v') \cup \Gamma_{n-2}(v'')$. Since for any line Z' concurrent with L_{j-2} and different from Z and L_{j-4}, $T_{Z'} \subseteq \Gamma_{n-2}(v)$, one also has $T_{Z'} \cap (\Gamma_{n-2}(v') \cup \Gamma_{n-2}(v'')) = \emptyset$. Let N be a line at distance ℓ, $\delta(a, Z) \le \ell \le n-3$, from a for which the path $[a, N]$ contains the path $[a, Z]$ (implying that the set T_N is defined and contained in T_Z). We claim that there does not exist a point $v' \in \{c, c', c''\} \setminus \{v\}$ for which $T_N \subseteq \Gamma_{n-2}(v')$. This is shown by induction on ℓ. In (B1), we consider the case $\ell = \delta(a, Z)$ and in (B2) the case $\delta(a, Z) < \ell \le n - 3$.

(B1) Suppose $N = Z$ and $T_Z \subseteq \Gamma_{n-2}(v')$. Then as before, $\delta(a, Z) = \delta(v', Z)$, $\mathrm{proj}_Z v' = \mathrm{proj}_Z a = z$ but $\mathrm{proj}_{L_{j-2}} a \neq \mathrm{proj}_{L_{j-2}} v' =: z'$. Now for a line Z'' concurrent with L_{j-2} and not through the points z, z' or a_{j-3}, the set $T_{Z''}$ is contained in both $\Gamma_{n-2}(v)$ and $\Gamma_{n-2}(v')$, contradicting condition (1) (note that such a line Z'' exists since we are in the case $s, t \ge 3$).

(B2) Suppose $\delta(a, Z) < \ell \le n - 3$ and $T_N \subseteq \Gamma_{n-2}(v')$. Let N' be the line of the path $[N, a]$ concurrent with N, and d the intersection point of N and N'. Then $\delta(a, N) = \delta(v', N)$, $\mathrm{proj}_N v' = \mathrm{proj}_N a = d$ but $\mathrm{proj}_{N'} a \neq \mathrm{proj}_{N'} v' =: d'$. Put $B = \mathrm{proj}_{d'} v'$. It is easy to check that $T_B \cap \Gamma_{n-2}(v') = \emptyset$, implying $T_B \subseteq \Gamma_{n-2}(v'')$ (with $\{v, v', v''\} = \{c, c', c''\}$). But now we obtain as before a contradiction with condition (1), by considering the set $T_{B'}$ for a line B' concurrent with N' not through the points d, d' or $\mathrm{proj}_{N'} a$. This shows that there does not exist a point $v' \in \{c, c', c''\} \setminus \{v\}$ for which $T_N \subseteq \Gamma_{n-2}(v')$.

Now let M be a line at distance $n-5$ from a for which the path $[a, M]$ contains the path $[a, Z]$. Let M_1, M_2 and M_3 be three distinct lines concurrent with M and at distance $n-3$ from a. Since T_{M_1} is not contained in one of the sets $\Gamma_{n-2}(v')$ or $\Gamma_{n-2}(v'')$, we can assume without loss of generality that v' lies at distance $n - 3$ from the line M_1, and that $y = \mathrm{proj}_{M_1} v'$ is a point of T_{M_1}. Then v' lies at distance $n-2$ from exactly one point of T_{M_2}, hence the point v'' lies at distance $n - 3$ from the line M_2, and $y' = \mathrm{proj}_{M_2} v''$ is contained in T_{M_2}. But now the set T_{M_3} cannot be covered by $\Gamma_{n-2}(v') \cup \Gamma_{n-2}(v'')$, the final contradiction. So we have shown that there does not exist a line L_j with $j \leq \frac{n}{2} - 4$ and a point $v \in \{c, c', c''\}$ such that $T_{L_j} \subseteq \Gamma_{n-2}(v)$.

(C) Suppose $j = \frac{n}{2} - 2$ and $T_N \subseteq \Gamma_{n-2}(v)$. Hence N lies at distance $n-3$ from a. Let M be the line of $[a, N]$ concurrent with N and put a_M the projection of a onto M. For an arbitrary point x_i on M different from a_M, we denote by M_i the line through x_i different from M. Without loss of generality, we choose $N = M_1$. Since $\mathrm{proj}_{M_1} v = x_1$ and $\mathrm{proj}_M v = x_2$ (with $x_2 \neq a_M$ by the induction hypothesis), $\Gamma_{n-2}(v) \cap T_{M_2} = \emptyset$. Hence there exists a point $v' \in \{c, c', c''\}$, $v' \neq v$ lying at distance $n-3$ from the line M_2. Since v covers every set T_{M_i}, $i \neq 2$, it is clear that $\mathrm{proj}_{M_2} v' \neq x_2$. Hence v' lies at distance $n-2$ from a unique point z of T_{M_1}, a contradiction.

Let M and the lines M_i be defined as in the previous paragraph. We now have the following situation. Since the set T_{M_1} contains infinitely many points, there is a point $v \in \{c, c', c''\}$ at distance $n-2$ from all but one point of T_{M_1}. Note that this point v lies at distance $n-2$ from exactly one point of the sets T_{M_i}, $i \neq 1$. We deduce that the line M contains exactly 4 points, and that there are points v', v'' of $\{c, c', c''\}$, with $\{v, v', v''\} = \{c, c', c''\}$ lying at distance $n-2$ from all but one point of T_{M_2}, T_{M_3} respectively. But now the projections of v and v' onto the line M_3 have to coincide, so we obtain a point at distance $n-2$ from a, b, v and v', contradicting Condition (5). This ends the proof of Step 7 and hence the case $i = n-2$.

The theorem is now proved. $\square$

4.8 Proof of the Special Flag Theorem

We first introduce some notation. A line in the thin polygon Γ (Γ') corresponding with a point in Δ (Δ') will be called a p-line, a line in Γ (Γ') corresponding with a line in Δ (Δ') will be called an L-line. Two points of the thin polygon Γ (Γ') at distance k, $k \neq n$ and $k \equiv 2 \bmod 4$ are said to be at distance k_p (k_L) if both $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$ are p-lines (L-lines). For the rest of this section, δ will always refer to this 'extended' distance function. Recall that we only have to consider the case $i \equiv 2 \bmod 4$, $2 < i < n$. Furthermore, put $T_{a,b} := \Gamma_{i_p}(a) \cap \Gamma_{i_p}(b)$

4.8.1 Case $i_p \leq n/2$

Let S be the set of pairs of points (a, b) of Γ satisfying $\delta(a, b) \neq i_p$ and $T_{a,b} = \emptyset$. We claim that

$$(a, b) \in S \Longleftrightarrow \begin{cases} \delta(a, b) \geq 2i \\ \delta(a, b) = k < 2i, & k \equiv 0 \bmod 4 \\ \delta(a, b) = k_L < 2i, & k \equiv 2 \bmod 4. \end{cases}$$

Indeed, let (a, b) be an arbitrary pair of points of Γ. We distinguish the following possibilities.

(i) $\delta(a, b) > 2i$ or $\delta(a, b) = k < 2i$ with $k \equiv 0 \bmod 4$.
If $i \neq n/2$ then in Case 4.7.1 *(i)* and *(ii)* in the proof of Theorem 4.2.2 it was shown that $\Gamma_i(a) \cap \Gamma_i(b) = \emptyset$, hence also $T_{a,b} = \emptyset$. If $i = n/2$, then it is easy to see that $\Gamma_i(a) \cap \Gamma_i(b) \neq \emptyset$ if and only if $\delta(a, b) = n$, which is case *(ii)* below.

(ii) $\delta(a, b) = 2i$.
Suppose first $i = n/2$. In this case, $\Gamma_i(a) \cap \Gamma_i(b)$ contains exactly two points x and y, namely lying on one of the two distance-n-paths between a and b. But since $\delta(a, b) \equiv 0 \bmod 4$, the lines $\mathrm{proj}_a x$ and $\mathrm{proj}_b x$ cannot be of the same type. Similarly for y. Hence neither x nor y is contained in $T_{a,b}$. Suppose now $i < n/2$. Without loss of generality, we can assume $\mathrm{proj}_a b$ is a p-line and $\mathrm{proj}_b a$ is an L-line. Suppose by way of contradiction $T_{a,b}$ contains a point x. Clearly, $x \neq a \bowtie b$ and $\mathrm{proj}_a b = \mathrm{proj}_a x$. Put $[a, R] = [a, b] \cap [a, x]$ and $j = \delta(a, R)$. Since $\mathrm{proj}_b x \neq \mathrm{proj}_b a$ (indeed, the line $\mathrm{proj}_b x$ is necessarily a p-line), there arises a circuit of length at most $2(2i - j) < 2n$, a contradiction.

(iii) $\delta(a, b) = k_L < 2i$ with $k \equiv 2 \bmod 4$.
Suppose $x \in T_{a,b}$. Clearly, $\mathrm{proj}_a b \neq \mathrm{proj}_a x$ and $\mathrm{proj}_b a \neq \mathrm{proj}_b x$. So there arises a circuit of length at most $k + 2i < 4i \leq 2n$, the contradiction.

(iv) $\delta(a, b) = k_p < 2i$ with $k \equiv 2 \bmod 4$.
In this case, any point x at distance $i - k/2$ from $M := a \bowtie b$ for which $\mathrm{proj}_M a \neq \mathrm{proj}_M x \neq \mathrm{proj}_M b$ belongs to $T_{a,b}$.

This shows the claim. Put $\kappa = \{\delta(a, b) \mid (a, b) \in S\}$.

Now let S' be the set of pairs (a, b) of distinct points of Γ such that $i_p \neq \delta(a, b) \notin \kappa$ and $\Gamma_{i_p}(a) \cap \Gamma_{\neq i_p}(b) \subseteq \Gamma_\kappa(b)$. We claim that $(a, b) \in S'$ if and only if $\delta(a,b) = 2_p$. Let (a,b) be an arbitrary pair of distinct points of Γ satisfying $i_p \neq \delta(a,b) \notin \kappa$. Hence $i_p \neq \delta(a,b) = k_p$, $k < 2i$. There are two possibilities.

(a) $\delta(a,b) = 2_p$.
Any point x of $\Gamma_{i_p}(a) \cap \Gamma_{\neq i_p}(b)$ lies at distance $i-2 \equiv 0 \bmod 4$ from b, hence $\delta(b,x) \in \kappa$.

(b) $i_p \neq \delta(a,b) = k_p$, $2 < k < 2i$.
Put $M := a \bowtie b$ and M' the line concurrent with M at distance $k/2-2$ from a. Let x be a point at distance $i-(k/2-2)$ from M' for which $\mathrm{proj}_{M'}b \neq \mathrm{proj}_{M'}x \neq \mathrm{proj}_{M'}b$. Then $\delta(a,x) = i_p$ and the length of the path consisting of $[b,M']$ and $[M',x]$ is $i+4 \leq n$, hence $\delta(b,x) = i+4$. Since $i+4 \equiv 2 \bmod 4$ and $i+4 < 2i$, $\delta(b,x) \notin \kappa$.

This shows the claim.

Let finally S'' be the set of pairs (a,b) of distinct points of Γ with $\delta(a,b) \neq i_p$ and for which there exists a point c such that either $\delta(a,c) = 2_p$ and $\delta(b,c) = i_p$ or $\delta(a,c) = i_p$ and $\delta(b,c) = 2_p$. Then S'' is the set of pairs of points at distance $i-2$ from each other. Indeed, let $(a,b) \in S''$ and put $k = \delta(a,b)$. Clearly, $k \in \{i-1, i_L, i+2\}$. Without loss of generality we can assume there is a point c such that $\delta(a,c) = 2_p$ and $\delta(b,c) = i_p$. This is easily seen to be a contradiction unless $k = i-2$.

By Theorem 4.2.2, α preserves collinearity. This ends the case $i_p < n/2$.

4.8.2 Case $n/2 < i_p < n-2$

Let S be the set of pairs of points (a,b) of Γ satisfying $\delta(a,b) \neq i_p$ and $T_{a,b} = \emptyset$. We claim that

$$(a,b) \in S \Longleftrightarrow \begin{cases} \delta(a,b) = k \leq 2n-2i, & k \equiv 0 \bmod 4 \\ \delta(a,b) = k_L \leq 2n-2i-2, & k \equiv 2 \bmod 4. \end{cases}$$

Indeed, let (a,b) be an arbitrary pair of points of Γ. We distinguish the following possibilities.

(i) $\delta(a,b) = k_p$, $k \equiv 2 \bmod 4$.
Any point x at distance $i-k/2$ from $M := a \bowtie b$ for which $\mathrm{proj}_M a \neq \mathrm{proj}_M x \neq \mathrm{proj}_M b$ is contained in $T_{a,b}$.

(ii) $\delta(a,b) = k_L$, $k \equiv 2 \bmod 4$, $k \leq 2n-2i-2$.
Suppose $x \in T_{a,b}$. Then $\mathrm{proj}_a b \neq \mathrm{proj}_a x$ and $\mathrm{proj}_b a \neq \mathrm{proj}_b x$. Hence there arises a circuit of length at most $k+2i < 2n$, a contradiction.

(*iii*) $\delta(a,b) = k_L$, $k \equiv 2 \bmod 4$, $k > 2n-2i-2$.
Let Σ be an apartment containing a and b, and X the element of Σ at distance $n-k/2$ from both a and b for which $\mathrm{proj}_a X$ and $\mathrm{proj}_b X$ are p-lines. Any point x at distance $i-(n-k/2)$ from X for which $\mathrm{proj}_X a \neq \mathrm{proj}_X x \neq \mathrm{proj}_X b$ belongs to $T_{a,b}$.

(*iv*) $\delta(a,b) = k$, $k \equiv 0 \bmod 4$, $k \leq 2n-2i$.
It is easy to see that $\Gamma_i(a) \cap \Gamma_i(b) \neq \emptyset$ if and only if $k = 2n-2i$. But if $k = 2n-2i$, the only points of $\Gamma_i(a) \cap \Gamma_i(b)$ lie in an apartment containing a and b and opposite the element $a \bowtie b$. Hence either $\mathrm{proj}_a x$ or $\mathrm{proj}_b x$ is an L-line, showing that $T_{a,b} = \emptyset$.

(*v*) $\delta(a,b) = k$, $k \equiv 0 \bmod 4$, $k > 2n-2i$.
Fix a k-path γ between a and b. Without loss of generality, we can assume that the element X of γ incident with a is a p-line. Let x be a point on X different from a and from the projection of b onto X. Let Σ be an apartment containing x and b, and Y the line of Σ at distance $\frac{2n-k}{2}+1$ from b for which $\mathrm{proj}_b Y$ is a p-line. Note that also $\delta(a,Y) = \frac{2n-k}{2}+1$ and $\mathrm{proj}_a Y$ is a p-line. Now any point y at distance $i - (\frac{2n-k}{2}+1)$ from Y for which $\mathrm{proj}_Y a \neq \mathrm{proj}_Y y \neq \mathrm{proj}_Y b$ belongs to $T_{a,b}$.

This shows the claim. Put $\kappa = \{\delta(a,b) \mid (a,b) \in S\}$.

Case $i \leq \frac{2n+2}{3}$

Let S' be the set of distinct points of Γ such that $\delta(a,b) \notin \kappa$ and $\Gamma_{i_p}(a) \cap \Gamma_{\neq i_p}(b) \subseteq \Gamma_\kappa(b)$ and symmetrically $\Gamma_{i_p}(b) \cap \Gamma_{\neq i_p}(a) \subseteq \Gamma_\kappa(a)$.

We claim that $(a,b) \in S'$ if and only if $\delta(a,b) = 2_p$. Let (a,b) be an arbitrary pair of distinct points of Γ for which $\delta(a,b) \notin \kappa$. Then the following cases can occur.

(*i*) $\delta(a,b) = k_p$, $k \equiv 2 \bmod 4$.
Similarly as in Case 4.8.1, (*a*)-(*b*), one shows that $(a,b) \in S'$ if and only if $\delta(a,b) = 2_p$.

(*ii*) $\delta(a,b) = k_L$, $k \equiv 2 \bmod 4$, $k \geq 2n-2i+2$.
Let Σ be an apartment through a and b, and X the line of Σ at distance $\frac{2n-k}{2}-2$ from a for which $\mathrm{proj}_a X$ is a p-line. Let x be a point at distance $i-\delta(a,X)$ from X for which $\mathrm{proj}_X a \neq \mathrm{proj}_X x \neq \mathrm{proj}_X b$. Then $\delta(b,x) = j_p$ with $j = i+4$. Hence $\delta(b,x) \notin \kappa$.

(*iii*) $\delta(a,b) = k$, $k \equiv 0 \bmod 4$, $k \geq 2n - 2i + 4$.
Fix a k-path γ between a and b. Without loss of generality, we can assume that the element of γ incident with a is a p-line. Consider a point x at distance i_p from b and opposite a. (One can construct such a point x as follows. Let x' be a point at distance $n - k$ from b for which $\mathrm{proj}_b x$ is a p-line. Let X be a line incident with x' and different from $\mathrm{proj}_{x'} b$ if $k \neq n$, and X the p-line through b if $k = n$. Since $\delta(a, X) = n-1$, it is possible to choose a point $x'' \mathtt{I} X$ at distance $n-k+2$ from b and opposite a. Proceeding like this, we obtain a point x as claimed). Now $\delta(a,x) \notin \kappa$.

This shows the claim. Similarly as in Case 4.8.1 it now follows that α preserves collinearity, which ends the proof.

Case $\frac{2n+2}{3} < i < \frac{3n}{4}$

Noting that $i \equiv 2 \bmod 4$, the condition above implies $n \geq 30$. Let S' be the set of distinct points of Γ such that $\delta(a,b) \in \kappa$ and $\Gamma_{i_p}(a) \cap \Gamma_\kappa(b) \neq \emptyset$ or $\Gamma_{i_p}(b) \cap \Gamma_\kappa(a) \neq \emptyset$.

We claim that $(a,b) \in S'$ if and only if $\delta(a,b) = k \equiv 0 \bmod 4$, $3i - 2n + 2 \leq k \leq 2n - 2i$ (note that this interval is nonempty, since $n \geq 8$). Let (a,b) be an arbitrary pair of distinct points of Γ for which $\delta(a,b) \in \kappa$. Then the following cases can occur.

(*i*) $\delta(a,b) = k_L$, $k \equiv 2 \bmod 4$, $k \leq 2n - 2i - 2$.
Let x be a point at distance i_p from a. If $i + k \leq n$, then $\delta(x,b) = i + k \notin \kappa$ (indeed, $k + i \leq 2n - 2i$ implies $3i \leq 2n - k$, but we assumed $3i > 2n + 2$). If $i + k > n$ and $\delta(b,x) \in \kappa$, we obtain a circuit of length at most $i + k + 2n - 2i < 2n$ (noting that $k < i$), a contradiction. Hence $\Gamma_{i_p}(a) \cap \Gamma_\kappa(b) = \emptyset$. Symmetrically, also $\Gamma_{i_p}(b) \cap \Gamma_\kappa(a) = \emptyset$. So $(a,b) \notin S'$.

(*ii*) $\delta(a,b) = k$, $k \equiv 0 \bmod 4$, $k \leq 2n - 2i$.
Without loss of generality, we can assume $\mathrm{proj}_a b$ is a p-line and $R := \mathrm{proj}_b a$ is an L-line. As in (*i*) above, it follows that $\Gamma_{i_p}(b) \cap \Gamma_\kappa(a) = \emptyset$. Note that $i > k$. Suppose first $k \geq 3i - 2n + 2$. Let x be a point at distance $i - k + 1$ from R for which $\mathrm{proj}_R a \neq \mathrm{proj}_R x \neq b$. Then $\delta(a,x) = i_p$ and $\delta(b,x) = i - k + 2 \equiv 0 \bmod 4$. Since $i - k + 2 \leq 2n - 2i$, $\delta(b,x) \in \kappa$, hence $(a,b) \in S'$. Suppose now $k < 3i - 2n + 2$ and let x be a point at distance i_p from a. Suppose by way of contradiction that

$\delta(b,x) \in \kappa$. If $[a,x]$ contains $[a,b]$, then $\delta(b,x) = j_p$ with $j = i-k$, hence $\delta(b,x) \notin \kappa$. Put $[a,x] \cap [a,b] = [a,R']$ and $r = \delta(a,R')$. Note that we obtain a path of length $k+i-2r$ between b and x (consisting of the paths $[b,R']$ and $[R',x]$). If $k+i-2r \le n$, then $\delta(b,x) \in \kappa$ implies $k+i-2r \le 2n-2i$. But $k+i-2r > i-k > 2n-2i$, a contradiction. If $k+i-2r > n$, then we obtain a circuit of length at most $k+i-2r+2n-2i < 2n$, a contradiction. So $(a,b) \notin S'$.

This shows the claim. Put $\lambda = \{\delta(a,b) \,|\, (a,b) \in S'\}$. Hence $\lambda = \{k \in \mathbb{N} \,|\, k \equiv 0 \bmod 4 \text{ and } 3i-2n+2 \le k \le 2n-2i\}$.

Let S'' be the set of distinct points of Γ such that $\delta(a,b) \in \lambda$ and $|\Gamma_\lambda(a) \cap \Gamma_\lambda(b)|$ is finite. Define $T^\lambda_{a,b} := \Gamma_\lambda(a) \cap \Gamma_\lambda(b)$.

We claim that $(a,b) \in S''$ if and only if $\delta(a,b) = k \equiv 0 \bmod 4$, $4n-5i+2 \le k \le 2n-2i$ (note that this interval is nonempty, since $i \ge \frac{2n+2}{3}$). Let (a,b) be an arbitrary pair of distinct points of Γ for which $\delta(a,b) \in \lambda$. Then we distinguish the following cases.

(*i*) $\delta(a,b) = k \le 4n-5i-2$
Let x be a point a distance $2n-2i$ from a for which the path $[a,x]$ contains the path $[a,b]$. Then $\delta(b,x) = 2n-2i-k \ge 3i-2n+2$, so clearly, $\delta(b,x) \in \lambda$. Since $2n-2i-k \ge 4$ and not both the p-lines and the L-lines are finite (indeed, $n > 16$), we obtain that $T^\lambda_{a,b}$ is infinite.

(*ii*) $\delta(a,b) = k \ge 4n-5i+2$
We show that the points of $T^\lambda_{a,b}$ all belong to the path $[a,b]$, which implies that this set is finite. Let x be a point for which $\delta(a,x) \in \lambda$. Suppose first $\mathrm{proj}_a b \neq \mathrm{proj}_a x$. Then we obtain a path of length $k + \delta(a,x)$ between b and x. If $k+\delta(a,x) \le n$, then $\delta(b,x) = k+\delta(a,x) \ge (4n-5i+2)+(3i-2n+2) = 2n-2i+4$, showing that $\delta(b,x) \notin \lambda$. If $k+\delta(a,x) > n$, then $\delta(b,x) \in \lambda$ implies there is a circuit of length at most $k+\delta(a,x)+2n-2i \le 6n-6i < 2n$, a contradiction. Suppose now that $\mathrm{proj}_a b = \mathrm{proj}_a x$. If the path $[a,x]$ contains the path $[a,b]$, then $\delta(b,x) = \delta(a,x)-k \le (2n-2i)-(4n-5i+2) = 3i-2n-2$, showing $\delta(b,x) \notin \lambda$. Now suppose x does not lie on the path $[a,b]$ and put $[a,x] \cap [a,b] = [a,R']$ and $r = \delta(a,R')$. If $k+\delta(a,x)-2r \le n$, then $\delta(b,x) = k+\delta(a,x)-2r \equiv 2 \bmod 4$, hence $\delta(b,x) \notin \lambda$. If $k+\delta(a,x)-2r > n$, then $\delta(b,x) \in \lambda$ implies there is a circuit of length at most $k+\delta(a,x)-2r+2n-2i \le 6n-6i-2r < 2n$, the final contradiction.

This shows the claim.

Now put $A := \max\{|T^\lambda_{a,b}| \mid (a,b) \in S''\}$. Since for a pair $(a,b) \in S''$, the points of $T^\lambda_{a,b}$ all lie on $[a,b]$, the number of points in $T^\lambda_{a,b}$ only depends from the distance between a and b. Note that $A \neq 0$. We claim that for a pair $(a,b) \in S''$, $|T^\lambda_{a,b}| = A$ if and only if $\delta(a,b) = 2n-2i$. Indeed, let (a,b) be a pair of points contained in S'' with $\delta(a,b) = k \neq 4n-5i+2$ (this is possible since $4n-5i+2 \neq 2n-2i$) and for which $|T^\lambda_{a,b}| \neq 0$. We show that $|T^\lambda_{a,b}| > |T^\lambda_{a,b'}|$, with b' the point of $[a,b]$ at distance 4 from b. Clearly, we may assume $|T^\lambda_{a,b'}| \neq 0$. Let x be a point of $T^\lambda_{a,b'}$. Then x also belongs to $T^\lambda_{a,b}$ because $\delta(x,b) = \delta(x,b') + 4$, $\delta(x,b') \in \lambda$ and $\delta(x,b') < 2n-2i$ (indeed, $\delta(x,b') = 2n-2i$ would imply $\delta(a,x) = 0$). This shows that $|T^\lambda_{a,b}| \geq |T^\lambda_{a,b'}|$. Let y be the point of $T^\lambda_{a,b'}$ at minimal distance from a. Now let y' be the point of $[a,b]$ at distance $\delta(a,y)$ from b. Then $\delta(b,y') \in \lambda$ and $\delta(a,y') = \delta(b',y) + 4 \in \lambda$. So $y' \in T^\lambda_{a,b}$ but $y' \notin T^\lambda_{a,b'}$ since $\delta(b',y') = \delta(b,y') - 4 = \delta(a,y) - 4 \notin \lambda$ since $\delta(a,y)$ was chosen to be minimal in λ. We thus showed that $|T^\lambda_{a,b}| > |T^\lambda_{a,b'}|$. Now the claim immediately follows.

We thus recovered distance $2n-2i$. By Theorem 4.2.2, this ends the proof in this case.

Case $\frac{3n}{4} \leq i < n-2$

Note that $4(n-i) \leq n$. If $4(n-i) < n$, let S' be the set of pairs (a,b) of distinct points of Γ such that $\delta(a,b) \notin \kappa$ and $|\Gamma_\kappa(a) \cap \Gamma_\kappa(b)| = 1$. Let S'' be the set of pairs of points (a,c) for which there exists a point b such that $(a,b) \in S'$ and $\Gamma_\kappa(a) \cap \Gamma_\kappa(b) = \{c\}$. If $4(n-i) = n$, let S' be the set of pairs (a,b) of distinct points of Γ such that $\delta(a,b) \notin \kappa$ and $\Gamma_\kappa(a) \cap \Gamma_\kappa(b) = \{c,d\}$, with $\Gamma_\kappa(c) \cap \Gamma_\kappa(d) = \{a,b\}$. Let S'' be the set of pairs of points (a,c) for which there exists a point b such that $(a,b) \in S'$ and $c \in \Gamma_\kappa(a) \cap \Gamma_\kappa(b)$. We claim that in both cases S'' is exactly the set of pairs of points at distance $2n-2i$ from each other. We have the following possibilities to consider.

(i) $\delta(a,b) = k_p$, $k < 2n-2i$.
Every point x at distance $2n-2i-k/2$ from the line $M := a \bowtie b$ for which $\mathrm{proj}_M a \neq \mathrm{proj}_M x \neq \mathrm{proj}_M b$ belongs to $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$. Moreover, since $2n-2i-k/2 \geq 3$, there are at least two collinear points c, d contained in $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$. So for these two points c, d, $\Gamma_\kappa(c) \cap \Gamma_\kappa(d) \neq \{a,b\}$. We conclude that $(a,b) \notin S'$.

(ii) $\delta(a,b) = k$, $k > 4(n-i)$.
In this case, it is easy to see that $\Gamma_\kappa(a) \cap \Gamma_\kappa(b) = \emptyset$ (indeed, if not, then necessarily $4(n-i) = n$, contradicting $k > 4(n-i)$). So $(a,b) \notin S'$.

(*iii*) $\delta(a,b) = k \equiv 0 \bmod 4$, $2(n-i) < k \leq 4(n-i)$.
Suppose first $k < 4(n-i)$. Then the point c (d) of $[a,b]$ at distance $2n-2i$ from a (b) belongs to $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$. If $a \in \Gamma_\kappa(c) \cap \Gamma_\kappa(d)$, then clearly, every point on the line $R' := \mathrm{proj}_a b$ different from $\mathrm{proj}_{R'} b$ also belongs to $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$, hence $\Gamma_\kappa(c) \cap \Gamma_\kappa(d) \neq \{a,b\}$. So if $k < 4(n-i)$, $(a,b) \notin S'$. Now consider the case $k = 4(n-i)$. If $4(n-i) < n$, then it is easy to see that $\Gamma_\kappa(a) \cap \Gamma_\kappa(b) = \{c\}$, with $c = a \bowtie b$, so $(a,b) \in S'$ and c lies at distance $2n-2i$ from both a and b. If $4(n-i) = n$, then it is easy to see that $\Gamma_\kappa(a) \cap \Gamma_\kappa(b) = \{c,d\}$, with c and d the unique points on the two n-paths joining a and b. Hence also $\Gamma_\kappa(c) \cap \Gamma_\kappa(d) = \{a,b\}$ (so $(a,b) \in S'$), and both c,d lie at distance $2n-2i$ from a,b.

(*iv*) $\delta(a,b) = k \equiv 2 \bmod 4$, $2(n-i) < k < 4(n-i)$.
Let R_a (R_b) be the line of $[a,b]$ at distance $2n-2i-1$ from a (b), and c (d) a point on R_a (R_b) different from both $\mathrm{proj}_{R_a} a$ and $\mathrm{proj}_{R_a} b$ ($\mathrm{proj}_{R_b} a$ and $\mathrm{proj}_{R_b} b$). In this way, we obtain two distinct points c,d of $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$ for which $\Gamma_\kappa(c) \cap \Gamma_\kappa(d) \neq \{a,b\}$, except if $k = 4(n-i)-2$ (then $R_a = R_b = a \bowtie b$) and $a \bowtie b$ contains exactly 3 points. In the latter case, it is easily seen that the unique point c on $M := a \bowtie b$ distinct from $\mathrm{proj}_M a$ and $\mathrm{proj}_M b$ is the unique point of $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$ (hence $(a,b) \in S'$), and c lies at distance $2n-2i$ from both a and b.

This shows the claim. We thus recovered distance $2n-2i$, and the result follows by Theorem 4.2.2.

4.8.3 Case $i_p = n-2$

Note that $n \equiv 0 \bmod 4$. Let S be the set of pairs of points (a,b) of Γ satisfying $\delta(a,b) \neq i_p$ and $T_{a,b} = \emptyset$. We claim that $(a,b) \in S$ if and only if $\delta(a,b) \in \{2_L, 4\}$. Indeed, we distinguish the following cases.

(*i*) $\delta(a,b) = k_p$, $k \equiv 2 \bmod 4$
Any point x at distance $i - k/2$ from the line $M := a \bowtie b$ for which $\mathrm{proj}_M a \neq \mathrm{proj}_M x \neq \mathrm{proj}_M b$ belongs to $T_{a,b}$.

(*ii*) $\delta(a,b) = k_L$, $k \equiv 2 \bmod 4$
If $k = 2_L$, then any point x of $\Gamma_i(a) \cap \Gamma_i(b)$ lies at distance $n-3$ from the line ab, hence $x \notin T_{a,b}$. If $k \geq 6$, let Σ be an apartment through a and b, and M' the line of Σ opposite $a \bowtie b$. Then any point x at distance $n-2-(\frac{2n-k}{2})$ from M' for which $\mathrm{proj}_{M'} a \neq \mathrm{proj}_{M'} x \neq \mathrm{proj}_{M'} b$ belongs to $T_{a,b}$ (and such points exist because $k \geq 6$).

(*iii*) $\delta(a,b) = k$, $k \equiv 0 \bmod 4$

If $k = 4$, then it is easy to see that any point x of $\Gamma_i(a) \cap \Gamma_i(b)$ lies opposite $a \bowtie b$ in an apartment containing a and b. But then either $\mathrm{proj}_a x$ or $\mathrm{proj}_b x$ is an L-line, hence $x \notin T_{a,b}$. If $k \geq 8$, then let R be the p-line through a. Without loss of generality, we can assume that $R = \mathrm{proj}_a b$ if $k \neq n$. Let a' be a point on R, $a \neq a' \neq \mathrm{proj}_R b$. Let Σ be an apartment containing b and a', and R' the line of Σ at distance $\frac{2n-k+2}{2}$ from b for which $\mathrm{proj}_b R'$ is a p-line (note that R' is indeed a line since $2n - k + 2 \equiv 2 \bmod 4$). Then any point x at distance $n - 2 - (\frac{2n-k+2}{2})$ from R' for which $\mathrm{proj}_{R'} a \neq \mathrm{proj}_{R'} x \neq \mathrm{proj}_{R'} b$ belongs to $T_{a,b}$ (and such points exist because $k \geq 8$).

This shows the claim. Put $\kappa = \{2_L, 4\}$. Now it is easy to verify that two points a and b lie at distance 2_L if and only if $\delta(a,b) \in \kappa$, and there exist points $c, c' \in \Gamma_\kappa(a) \cap \Gamma_\kappa(b)$, $c \neq c'$, for which $\delta(c,c') \notin \kappa$. (Indeed, if $\delta(a,b) = 2_L$, one can consider points c, c' lying on a line M intersecting ab in a point x different from a or b, $c \neq x \neq c'$. If $\delta(a,b) = 4$, then assume that $\mathrm{proj}_b a$ is an L-line. Since $n > 6$, the only points in $\Gamma_\kappa(a) \cap \Gamma_\kappa(b)$ lie on $\mathrm{proj}_b a$, hence lie at distance 2_L from each other.) So we can distinguish distance 4. By Theorem 4.2.2 this ends the proof in this case. $\square$

Appendix A

Minimal generating sets in $\mathsf{H}(q)^D$

A.1 Introduction

In this chapter, the following question keeps us puzzling: how many points do you need to generate the whole point set of one of the classical hexagons (in the sense that two distinct collinear points generate the points on the joining line)? We tackle this question for the finite dual split Cayley hexagon, since in this case, some strong results are available in Thas & Van Maldeghem [48]. Indeed, they show that our hexagon admits an embedding in 13-dimensional projective space, implying that we will need at least 14 points to generate the whole point set. They also prove that the point set of this hexagon is generated by the point sets of three thin full subhexagons lying 'close' to each other. In the first section, we show how one can select 15 points to generate these three subhexagons. Hence 15 points are always enough. If the underlying field is of prime order, then it follows that one can do with 14 points. Now the question becomes: is it possible in the general case to delete one point of this generating set of 15 points, and still to generate the whole point set? We let the computer work for us, and obtain that this is indeed possible for $q = 4, 8, 9, 16$. Now it is up to us to be smarter and faster than the computer, and beat it by proving the result for all values of q. But

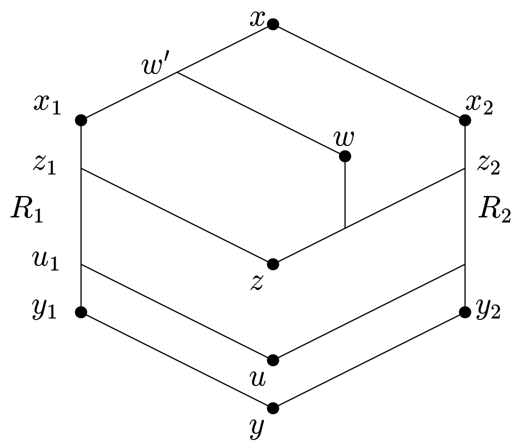


Figure A.1: A generating set of 9 points for $\Gamma(R_1, R_2)$

for the moment, it seems that we are losing this competition. So the partial result reads : $14 \Rightarrow (1+q)(1+q^2+q^4)$ if q is prime or $q \in \{4, 8, 9, 16\}$...

Throughout this chapter, we put $\Gamma \cong \mathsf{H}(q)^D$. Denote by m the size of a minimal set of points $\mathcal{M}$ of Γ such that the points of $\mathcal{M}$ generate Γ (in the sense that two distinct collinear points generate all points on the joining line).

A.2 $14 \leq m \leq 15$

The generalized hexagon $\mathsf{H}(q)^D$ admits an embedding in $\mathrm{PG}(13, q)$ (for an explicit description, see Thas & Van Maldeghem [48]). This implies that the point set of Γ generates $\mathrm{PG}(13, q)$, hence $m \geq 14$. For the hexagon $\mathsf{H}(2)^D$, there exists a generating set of 14 points (see Thas & Van Maldeghem [48] or Cooperstein [13]). So from now on, we assume that $q > 2$.

In Thas & Van Maldeghem [48], the following useful property is proved.

Theorem A.2.1 *Let L_1 and L_1' be two arbitrary opposite lines of Γ. Let L_2 and L_2' be two distinct lines of the regulus $R(L_1, L_1')$ both distinct from L_1 and L_1'. Then the union of the point sets of the three thin full subhexagons $\Gamma(L_1, L_1')$, $\Gamma(L_2, L_2')$ and $\Gamma(L_1, L_2)$ generates the point set of Γ.*

Consider a thin full subhexagon $\Gamma(R_1, R_2)$ of Γ. Note that $\Gamma(R_1, R_2)$ is the double of a desarguesian projective plane $\Delta \cong \mathrm{PG}(2, q)$ (where the lines of $\Gamma(R_1, R_2)$ correspond with the points and lines of Δ, and the points of $\Gamma(R_1, R_2)$ correspond with the flags of Δ). Let x, y, z be three different points of the regulus $\langle R_1, R_2\rangle$, and define $v_i = \mathrm{proj}_{R_i} v$, with $v \in \{x, y, z\}$. Let finally

w' be a point on the line xx_1 different from x or x_1, and $w = w' \bowtie \mathrm{proj}_{zz_2} w'$. Now we consider the subset Γ' of the point set of $\Gamma(R_1, R_2)$ generated by the set $\mathcal{N} = \{x, x_1, x_2, y, y_1, y_2, z, w\}$ (see Figure A.1). Denote by Δ' the subset of the plane Δ corresponding with the set Γ'. Then Δ' is a subplane of Δ. Indeed, if p_1 and p_2 are two points of Δ', then these two points correspond with two lines M_1 and M_2 of $\Gamma(R_1, R_2)$ that are generated by $\mathcal{N}$. Since M_1 and M_2 necessarily lie at distance 4, the line $M_1 \bowtie M_2$ of $\Gamma(R_1, R_2)$ is also generated, and corresponds with the line p_1p_2. So the line p_1p_2 is contained in Δ'. Similarly, every two lines of Δ' meet. The ordinary octagon in Γ' through the points $w', w, \mathrm{proj}_{zz_2} w, z_2, y_2, y, y_1, x_1$ corresponds with a quadrangle in Δ', so Δ' is a projective plane isomorphic to $\mathrm{PG}(2, q')$. Now we distinguish two cases.

- q is prime.
  In this case, $\Delta = \Delta'$, so $\Gamma(R_1, R_2)$ is generated by the 8 points of $\mathcal{N}$. Note that $\mathcal{N}$ is a minimal generating set for $\Gamma(R_1, R_2)$, since there actually exists an embedding of $\Gamma(R_1, R_2)$ in $\mathrm{PG}(7, q)$ (for a description, see for instance Thas & Van Maldeghem [47]).

- q is not prime.
  Choose a point u of the regulus $\langle R_1, R_2 \rangle$, different from x, y, z and such that, with $u_1 = \mathrm{proj}_{R_1} u$, the cross-ratio $(x_1, y_1; z_1, u_1)$ generates the field $\mathrm{GF}(q)$. Again as before, the set generated by $\mathcal{N} \cup \{u\}$ corresponds with a subplane Δ'' of Δ, but because of the choice of u, $\Delta = \Delta''$ (indeed, $\Delta'' \cong \mathrm{PG}(2, q'')$, but $\mathrm{GF}(q'')$ contains a generating element of the field $\mathrm{GF}(q)$). So $\Gamma(R_1, R_2)$ is generated by the 9 points of $\mathcal{N} \cup \{u\}$. As in the previous case, this is a generating set of minimal size, because there exists an embedding of $\Gamma(R_1, R_2)$ in $\mathrm{PG}(8, q)$ (for a description, see again Thas & Van Maldeghem [47]).

Now choose a set $\mathcal{M}'$ of 15 points in Γ as indicated in Figure A.2, with $(p_0, p_1; p_2, p_3)$ a generating element of the field $\mathrm{GF}(q)$. Then by the previous observations, these points generate the thin full subhexagons $\Gamma(p_0p_3, r_0r_3)$, $\Gamma(r_0r_3, s_0s_3)$ and $\Gamma(s_0s_3, t_0t_3)$. Theorem A.2.1 implies that $\mathcal{M}'$ generates Γ. Note that if q is prime, the set $\mathcal{M}' \setminus \{u\}$ still generates Γ. So we obtained the following theorem.

Theorem A.2.2 *If q is prime, then $m = 14$. If q is not prime, then $14 \leq m \leq 15$.*

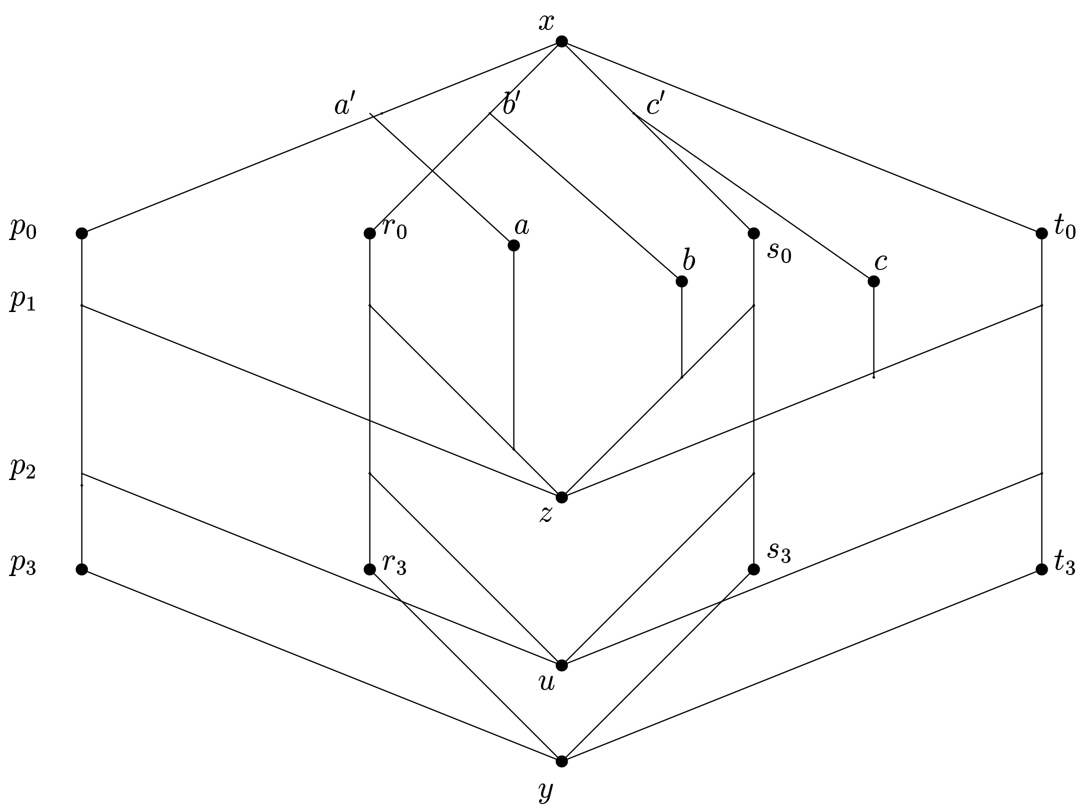


Figure A.2: A generating set of 15 points for $\mathsf{H}(q)^D$

A.3 $m = 14$ if $q \in \{4, 8, 9, 16\}$

Put $q = p^h$, p prime. Let $\mathcal{M}'$ be a generating set as in figure A.2. We want to delete one point of $\mathcal{M}'$ to obtain a minimal generating set. More precisely, we delete the point u. Put $\mathcal{M} = \mathcal{M}' \setminus \{u\}$. Note that $\mathcal{M}$ certainly generates three subhexagons $\Gamma'(p_0p_3, r_0r_3)$, $\Gamma'(r_0r_3, s_0s_3)$ and $\Gamma'(s_0s_3, t_0t_3)$ of order $(p, 1)$. We now investigate whether it is possible to select the points a, b, c and t_0 such that the set $\mathcal{M}$ still generates Γ. Our tools are coordinates and the computer.

choice of coordinates

Let the apartment through x, y, p_0 and r_0 be the hat-rack of the coordinatization, with $x = (\infty)$, $xp_0 = [\infty]$, $y = (00000)$, $xr_0 = [0]$. Without loss of generality, we can choose $xs_0 = [1]$. Further we choose $xt_0 = [k]$, $z = (00100)$, $a' = (a)$, $b' = (0b)$ and $c' = (1c)$.

idea of the computer program

The first idea is of course to give the computer the coordinates of the 14 points we selected, and let it do the generating work. This is, whenever two collinear points are already in the set of generated points, then all the points on the joining line are added to this set. If at the end, the program tells us that $(q+1)(q^4+q^2+1)$ distinct points were generated, we are done. Alternatively, we give the computer the coordinates of the 9 lines in the set $\overline{\mathcal{M}} = \{p_0x, p_3y, p_1z, r_0r_3, s_0s_3, t_0t_3, aa', bb', cc'\}$ (note that these lines are certainly generated by $\mathcal{M}$). The job then becomes: whenever two lines L and L' at distance 4 from each other are contained in $\overline{\mathcal{M}}$, add the line $L \bowtie L'$ to $\overline{\mathcal{M}}$. If at the end, the set $\overline{\mathcal{M}}$ contains $(q+1)(q^4+q^2+1)$ distinct lines, then the set $\mathcal{M}$ of 14 points we started from, is indeed a generating set for Γ. We give a method to make the program a bit faster. Denote by Γ'_1, Γ'_2 and Γ'_3 the three thin subhexagons $\Gamma'(p_0p_3, r_0r_3)$, $\Gamma'(r_0r_3, s_0s_3)$ and $\Gamma'(s_0s_3, t_0t_3)$. Then each Γ'_i is contained in a full subhexagon Γ_i of order $(q, 1)$. Dually, in $\mathsf{H}(q)$, these Γ_i correspond to ideal subhexagons lying in a hyperplane γ_i of $\mathrm{PG}(6, q)$. Suppose that at a certain moment, a line L is generated for which the corresponding point of Γ^D lies in a hyperplanes γ_j of $\mathrm{PG}(6, q)$, $j \in \{1, 2, 3\}$, and such that L lies at distance 4 from a line L' contained in the thin subhexagon Γ'_j. Then in the following step, the line $L \bowtie L'$ of Γ_j will be generated. Now let d_0, d_1, d_2 be the three points on L belonging to Γ'_j. If $\mathrm{proj}_{L'} L \notin \{d_0, d_1, d_2\}$, and the cross-ratio $(d_0, d_1; d_2, \mathrm{proj}_{L'} L)$ generates $\mathrm{GF}(q)$, then we know by the previous section that the thin full subhexagon Γ_j can be generated. This implies that it is possible to generate the whole point regulus $R(x, y)$, hence the generated set contains the set $\mathcal{M}'$ we started from, and we are done.

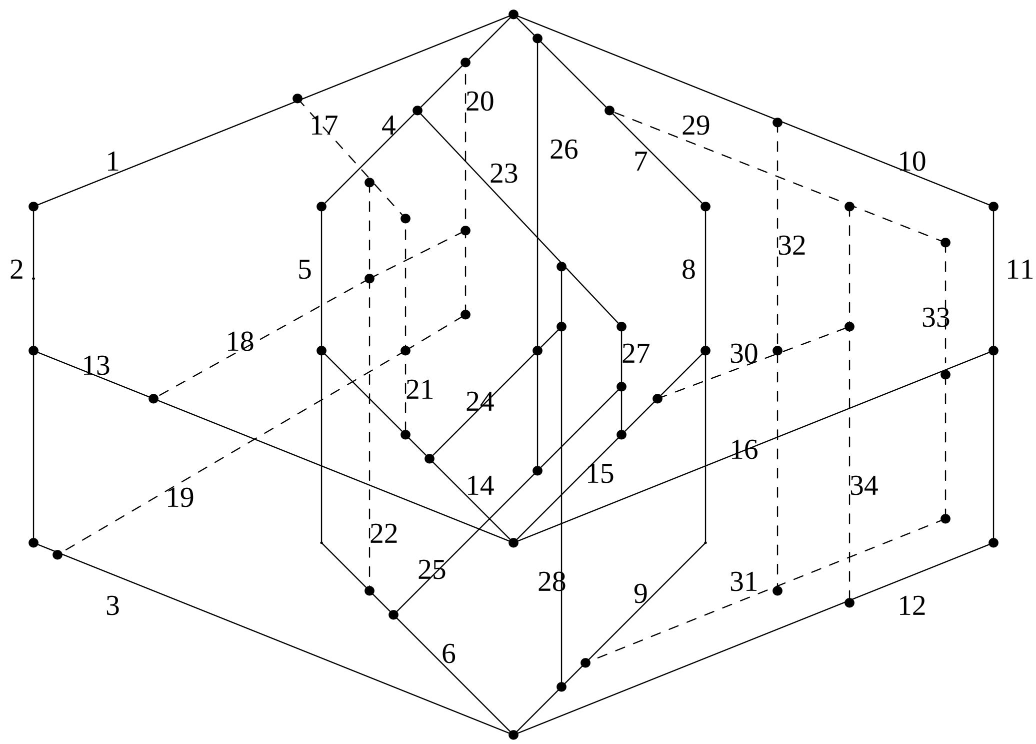


Figure A.3: numbering of the lines M_i

In the following, we give a generating set obtained by this method for the cases $q = 4, 8, 16, 9$ respectively. Put $M_1 = xp_0$, $M_2 = p_0p_3$, $M_3 = yp_3$, $M_5 = r_0r_3$, $M_8 = s_0s_3$ and $M_{11} = t_0t_3$. We further label the lines of the three subhexagons $\Gamma'(M_2, M_5)$ $(= \Gamma'_1)$, $\Gamma'(M_5, M_8)$ $(= \Gamma'_2)$ and $\Gamma'(M_8, M_{11})$ $(= \Gamma'_3)$ with $M_1, \ldots, M_k$, $k = 34$ for $p = 2$ and $k = 68$ for $p = 3$. The lines M_i, $i = 1, \ldots, 34$ are numbered as in Figure A.3. The subhexagons Γ_i of order $(q, 1)$ containing Γ'_i correspond with the following hyperplanes γ_i of $\mathrm{PG}(6, q)$:

$$\begin{aligned}
\gamma_1 &: \quad X_3 = 0 \\
\gamma_2 &: \quad X_3 - X_5 = 0 \\
\gamma_3 &: \quad kX_1 + (1 + k)X_3 + X_5 = 0.
\end{aligned}$$

The computer results are given by the diagrams, where the numbers refer to the lines M_i, and where, if a line N splits into two lines N_1 and N_2, this has

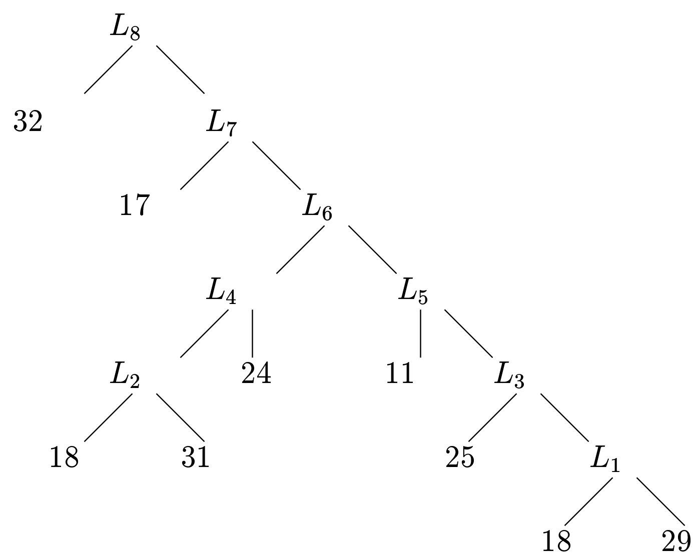


Figure A.4: a minimal generating set for $\mathsf{H}(4)^D$

to be read as $N = N_1 \bowtie N_2$.

Case $\mathsf{H}(4)^D$

Let α be a generating element of GF(4) (satisfying $\alpha^2 = 1 + \alpha$). We choose

$$\begin{cases} a = \alpha^2 \\ b = 1 \\ c = \alpha^2 \\ k = \alpha. \end{cases}$$

$$\begin{array}{ll}
M_{11} = [\alpha, 0, 0] & L_1 = [1, \alpha^2, 1, 0, 1] \\
M_{17} = [\alpha^2, 0] & L_2 = [\alpha^2, 0, \alpha^2, \alpha, 1] \\
M_{18} = [0, \alpha, 0, 1, 0] & L_3 = [\alpha^2, \alpha, \alpha^2, \alpha] \\
M_{24} = [1, 1, 1, 0, 1] & L_4 = [1, 1, 1, 0, 0] \\
M_{25} = [1, 1, 1, 1, 1] & L_5 = [\alpha, 0, 0, \alpha^2, \alpha^2] \\
M_{29} = [1, \alpha^2, 1] & L_6 = [0, 0, 1, 0, 0] \\
M_{31} = [\alpha, \alpha, \alpha^2, \alpha^2, 1] & L_7 = [\alpha^2, 0, 0, 1] \\
M_{32} = [\alpha, \alpha, \alpha^2] & L_8 = [\alpha, \alpha, \alpha^2, 0, 1]
\end{array}$$

The line L_8 is contained in Γ_2 and lies at distance 4 from the line M_8 of Γ_2'. Also, L_8 lies opposite the lines $M_7 = [1]$, $M_9 = [1, 0, 0, 0, 0]$ and $M_{15} = [1, 0, 0, 1, 0]$, which are the lines of Γ_2' intersecting M_8. This means that a line of Γ_2 not contained in Γ_2' will be generated. Since GF(4) contains no subfield different from GF(2), we are done.

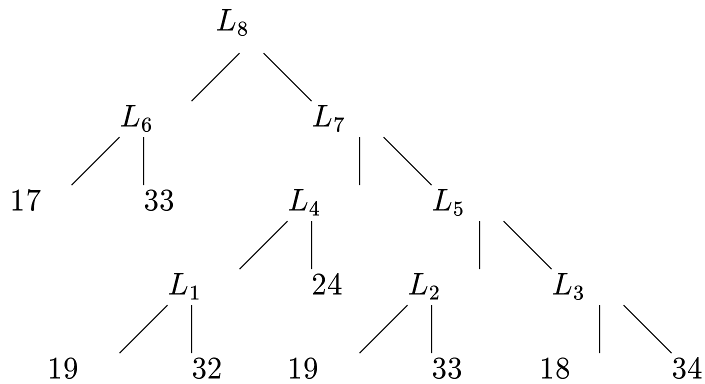


Figure A.5: a minimal generating set for $\mathsf{H}(8)^D$

Case $\mathsf{H}(8)^D$

Let α be the generating element of GF(8) (satisfying $\alpha^3 = 1+\alpha$). We choose

$$\begin{cases} a = \alpha^2 \\ b = 1 \\ c = \alpha \\ k = \alpha. \end{cases}$$

$$\begin{array}{ll}
M_{17} = [\alpha^2, 0] & L_1 = [\alpha, \alpha, \alpha^5, \alpha^2, \alpha^4] \\
M_{18} = [0, \alpha^5, 0, 1, 0] & L_2 = [\alpha^4, 0, \alpha, \alpha^5, \alpha^4] \\
M_{19} = [0, \alpha^5, 0, 0, 0] & L_3 = [\alpha^4, 0, \alpha, \alpha^4, \alpha^4] \\
M_{24} = [1, 1, 1, 0, 1] & L_4 = [\alpha^3, \alpha, \alpha^4, 1, \alpha^5] \\
M_{32} = [\alpha, \alpha, \alpha^5] & L_5 = [\alpha^4, 0, \alpha] \\
M_{33} = [1, \alpha, \alpha^5, 0, \alpha^2] & L_6 = [\alpha^2, 0, \alpha^2, \alpha^3] \\
M_{34} = [1, \alpha, \alpha^5, 1, \alpha^2] & L_7 = [\alpha^4, 0, \alpha, \alpha^3, \alpha^2] \\
 & L_8 = [\alpha^2, 1, \alpha^4, 1, \alpha^4]
\end{array}$$

The line L_8 is contained in Γ_2, lies at distance 4 from the line $M_9 = [1,0,0,0,0]$ and lies opposite the lines $M_6 = [0,0,0,0,0]$, $M_8 = [1,0,0]$ and $M_{28} = [0,1,1,1,1]$. Similarly as in the previous case, we are done.

Case $\mathsf{H}(16)^D$

Let α be the generating element of GF(16) (satisfying $\alpha^4 = 1+\alpha$). We choose

$$\begin{cases} a = \alpha^{14} \\ b = \alpha^8 \\ c = \alpha^8 \\ k = \alpha. \end{cases}$$

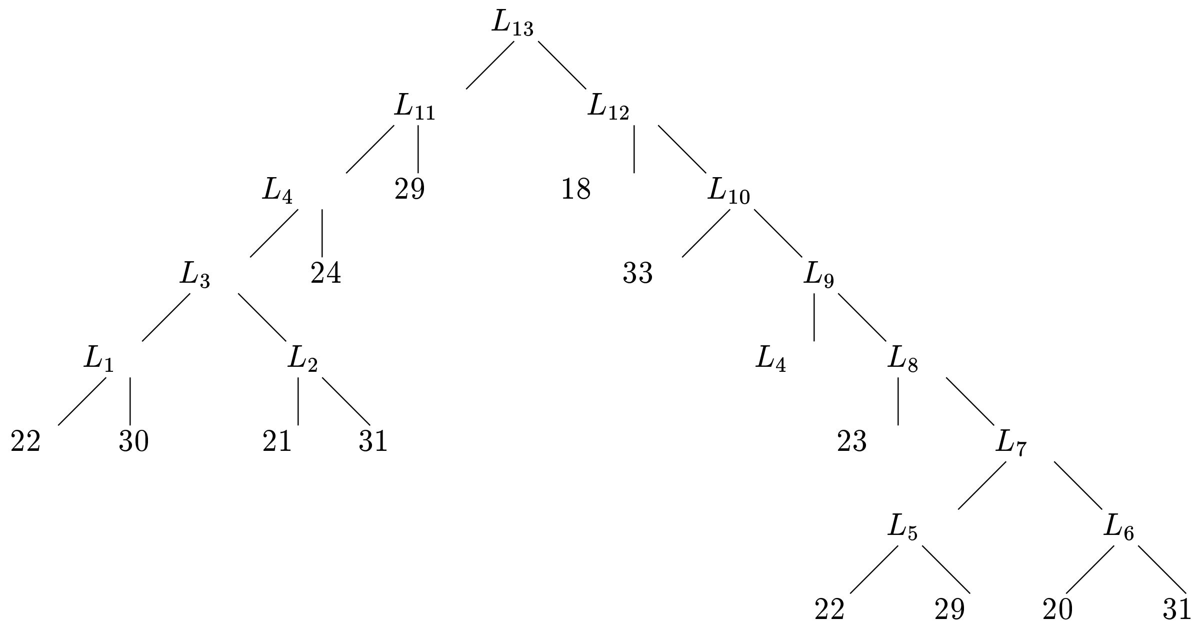


Figure A.6: a minimal generating set for $\mathsf{H}(16)^D$

$$
\begin{array}{ll}
M_{18} = [0, \alpha, 0, 1, 0] & L_1 = [\alpha^5, \alpha^{10}, \alpha^9, \alpha^9, \alpha^4] \\
M_{20} = [0, \alpha, 0] & L_2 = [\alpha^5, \alpha^{10}, \alpha^9, \alpha^7, \alpha^4] \\
M_{21} = [\alpha^{14}, 0, 1, 0] & L_3 = [\alpha^5, \alpha^{10}, \alpha^9] \\
M_{22} = [\alpha^{14}, 0, 0, 0] & L_4 = [\alpha^5, \alpha^{10}, \alpha^9, \alpha^{12}, \alpha^{13}] \\
M_{23} = [0, \alpha^8, \alpha^8] & L_5 = [1, \alpha^8, \alpha^{14}, \alpha^7, \alpha^{14}] \\
M_{24} = [1, \alpha^7, \alpha^7, \alpha^3, \alpha^7] & L_6 = [0, \alpha, 0, \alpha^{10}, \alpha^9] \\
M_{29} = [1, \alpha^8, \alpha^{14}] & L_7 = [\alpha^2, \alpha^8, \alpha^7, \alpha^6, \alpha^6] \\
M_{30} = [\alpha^7, \alpha^4, \alpha^{10}, \alpha^{12}, \alpha] & L_8 = [0, \alpha^8, \alpha^8, \alpha^{10}, \alpha^5] \\
M_{31} = [\alpha^7, \alpha^4, \alpha^{10}, \alpha^{11}, \alpha] & L_9 = [\alpha^2, \alpha^5, \alpha^{13}, \alpha^8] \\
M_{33} = [1, \alpha^8, \alpha^{14}, \alpha, \alpha^5] & L_{10} = [\alpha^4, \alpha^9, \alpha, \alpha^{11}, \alpha^9] \\
 & L_{11} = [1, \alpha^8, \alpha^{14}, \alpha^9, \alpha^{10}] \\
 & L_{12} = [\alpha^9, \alpha^3, 1, \alpha^{13}, \alpha^6] \\
 & L_{13} = [\alpha^5, \alpha, \alpha^{10}, \alpha^3, \alpha^8]
\end{array}
$$

The line L_{13} is contained in Γ_2 and lies at distance 4 from the line $M_{26} = [1, \alpha^7, \alpha^7]$ of Γ_2' and opposite the lines $M_7 = [1]$, $M_{24} = [1, \alpha^7, \alpha^7, \alpha^3, \alpha^7]$ and $M_{25} = [1, \alpha^7, \alpha^7, \alpha^{14}, \alpha^7]$ (which are the lines of Γ_2' intersecting M_{26}).

Denote by p_0, p_1, p_2, p_3 the projections of respectively M_7, M_{24}, M_{25} and L_{13} on the line M_{26}. Then these points have coordinates:

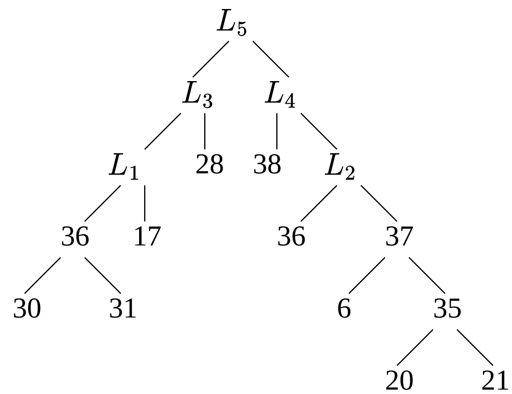


Figure A.7: a minimal generating set for $\mathsf{H}(9)^D$

$$\begin{cases} p_0 = (1, \alpha^7) \\ p_1 = (1, \alpha^7, \alpha^7, \alpha^3) \\ p_2 = (1, \alpha^7, \alpha^7, \alpha^{14}) \\ p_3 = (1, \alpha^7, \alpha^7, \alpha^7). \end{cases}$$

In PG$(6, q)$, the point p_i, $i = 0, \ldots, 3$, corresponds with the line pr_i, with $p = (0, 1, 0, 1, 0, 1, \alpha^7)$ and

$$\begin{cases} r_0 = (1, 0, 0, 0, 0, 0, 1) \\ r_1 = (1, 0, 1, \alpha^7, 1, \alpha^7, \alpha^3) \\ r_2 = (0, 0, 1, \alpha^7, 1, \alpha^7, \alpha^{14}) \\ r_3 = (\alpha, 0, 1, \alpha^7, 1, \alpha^7, \alpha^7). \end{cases}$$

The points r_0, r_1, r_2 and r_3 are collinear. It is now easy to calculate that $(r_0, r_1; r_2, r_3)$ equals α, hence we are done.

Case $\mathsf{H}(9)^D$
Let α be the generating element of GF(9) (satisfying $\alpha^2 = 1 - \alpha$). We choose

$$\begin{cases} a = \alpha \\ b = \alpha \\ c = \alpha^5 \\ k = \alpha \end{cases}$$

In this case the subhexagons Γ'_i, $i = 1, 2, 3$, are of order $(3, 1)$, the union of their line sets contains 68 lines M_i, $i = 1, \ldots, 68$. We again choose the lines

$M_1, \ldots, M_{34}$ as in Figure A.3.

$$\begin{array}{lll}
M_6 = [0,0,0,0,0] & M_{30} = [\alpha,\alpha^5,\alpha^3,\alpha^4,\alpha^5] & L_1 = [\alpha,0,1,\alpha^6] \\
M_7 = [1] & M_{31} = [\alpha,\alpha,\alpha^7,1,\alpha] & L_2 = [\alpha^5,\alpha^5,\alpha^7,\alpha^7,\alpha^2] \\
M_{17} = [\alpha,0] & M_{35} = [0,\alpha^7,0,\alpha^4,0] & L_3 = [\alpha^7,\alpha^5,\alpha^5,\alpha^4,1] \\
M_{20} = [0,\alpha^7,0] & M_{36} = [1,\alpha,\alpha^3,1,\alpha] & L_4 = [1,\alpha^3,\alpha^7,0,\alpha^6] \\
M_{21} = [\alpha,0,1,0] & M_{37} = [\alpha^5,0,0,0] & L_5 = [\alpha^3,1,\alpha^7,\alpha^6,\alpha^2] \\
M_{25} = [1,\alpha^3,\alpha^7,\alpha^6,\alpha^7] & M_{38} = [1,\alpha^3,\alpha^7] & \\
M_{28} = [0,\alpha,\alpha,\alpha^6,\alpha^5] & &
\end{array}$$

The line L_5 belongs to the subhexagon Γ_2, and lies at distance 4 from the line M_{28} of this Γ_2'. The lines of Γ_2' intersecting M_{28} are the lines $M_{23} = [0,\alpha,\alpha]$, $M_9 = [1,0,0,0,0]$, $M_{24} = [1,\alpha^7,\alpha^3,\alpha,\alpha^3]$ and $M_{39} = M_{28} \bowtie M_{38} = M_{28} \bowtie (M_7 \bowtie M_{25}) = [1,\alpha^3,\alpha^7,\alpha^7,\alpha^7]$. Since L_5 lies opposite all the lines of Γ_2' intersecting M_{28}, we are done.

A.4 Comments

Our initial aim was to use the computer results to see how one can construct a generating set for $\mathsf{H}(q)^D$, for all values of q. One thing we did was checking whether for example diagram A.5 also goes through in other fields of characteristic 2. So we imitated the calculations of the computer, but now with general coordinates a, b, c and k. The conclusion was that this 'path' only works for the field GF(8). Another attempt was to concentrate on the values of a, b, c and k instead of on the diagrams, and see similarities between the results for GF(4) and GF(8). For these fields, the computer program does not need much time, so we could really try all possible values of a, b, c and k. We then tried to see 'symmetries' in the good choices, and made a guess for what a good choice in the field GF(16) could be. We seem not to be very good clairvoyants, since our predictions turned out to be wrong... So looking at the computer data, no answers, but all the more questions turned up. For example, sometimes the generated set of lines was not the whole line set of the hexagon, and not the line set of the three subhexagons over the prime field, but something in between. Moreover, the size of such a set was often the same for distinct choices of a, b, c and k. We would of course like to explain these mysteries in a geometric way. But perhaps this puzzle was not designed for geometers at all, and came into our hands only by accident?

Appendix B

Veelhoeken in Veelvoud

B.1 Van veelhoek naar veralgemeende veelhoek

Veralgemeende veelhoeken zijn meetkundige structuren die werden ingevoerd door de (van oorsprong Belgische) wiskundige Jacques Tits in 1959, in een appendix van het artikel *'Sur la trialité et certains groupes qui s'en déduisent'*. Oorspronkelijk stonden de veralgemeende veelhoeken ten dienste van de groepentheorie, maar al gauw begon men zich te interesseren voor deze structuren op zich. Zoals de naam al laat vermoeden, zijn veralgemeende veelhoeken opgebouwd uit veel gewone veelhoeken. We geven nu een precieze definitie.

Een *veralgemeende n-hoek*, $n \geq 2$, is een meetkunde Γ bestaande uit een verzameling punten $\mathcal{P}$, een verzameling rechten $\mathcal{L}$, en een relatie $\mathtt{I}$, de zogenaamde 'incidentierelatie' die beschrijft wanneer een punt op een rechte ligt (of een rechte door een punt gaat), zodat aan de volgende axioma's voldaan is:

(i) in de meetkunde Γ zijn geen gewone k-hoeken, met $k < n$ te vinden,

(ii) door elke twee punten, twee rechten, of een punt en een rechte is steeds een gewone n-hoek te vinden,

(iii) er is ergens in de meetkunde een gewone $(n+1)$-hoek te vinden.

Axioma (iii) kan men ook vervangen door het volgende axioma:

$(iii)'$ Elke rechte bevat minstens 3 punten, en door elk punt gaan minstens 3 rechten.

Gebruik makende van deze axioma's kan men aantonen dat elke rechte eenzelfde aantal ($= s+1$) punten bevat, en dat door een punt eenzelfde aantal ($= t+1$) rechten gaan. We zeggen dan dat Γ *orde* (s,t) heeft. Zijn er slechts een eindig aantal punten en rechten, dan wordt Γ een *eindige veralgemeende veelhoek* genoemd. De veralgemeende 3-hoeken zijn juist de projectieve vlakken, en werden al uitgebreid bestudeerd vóór de andere leden van de veralgemeende veelhoeken-familie het levenslicht zagen.

Er bestaan voorbeelden van veralgemeende n-hoeken voor elk natuurlijk getal n. Merkwaardig genoeg bestaan eindige veralgemeende n-hoeken enkel voor $n \in \{3,4,6,8\}$. Voor deze waarden van n zijn er wat we noemen *klassieke voorbeelden*, d.w.z. voorbeelden die nauw verwant zijn met favoriete objecten van meetkundigen, zoals kwadrieken in projectieve ruimtes.

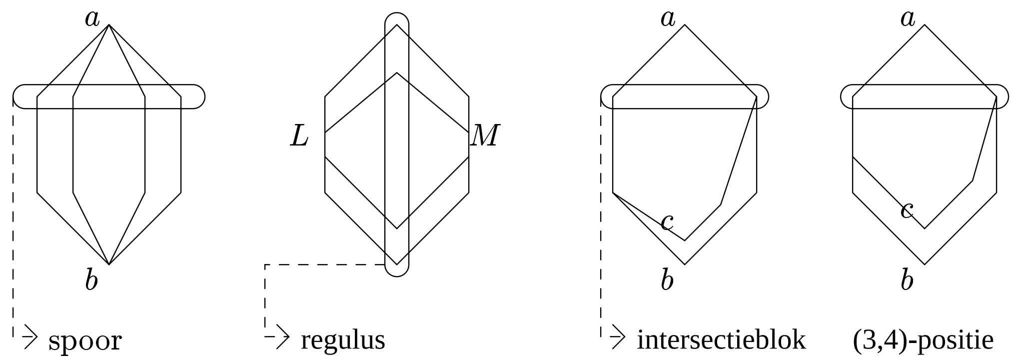
Een belangrijk begrip in een veralgemeende veelhoek is de *afstand*. Een pad in een veralgemeende veelhoek bestaat uit een opeenvolging van punten en rechten die incident zijn. De lengte van een pad wordt gedefinieerd als het aantal stappen dat je moet zetten om van het begin naar het einde te wandelen. Een pad (p, L, p', L', p'') tussen de punten p en p' heeft dus lengte 4. Omdat elke twee elementen van een veralgemeende n-hoek bevat zijn in een gewone n-hoek, is het duidelijk dat, om van een element in een ander element te geraken, hoogstens n stappen nodig zijn. De afstand tussen twee elementen is dus hooguit n. Elementen op afstand n worden *tegenvoeters* genoemd.

Veralgemeende veelhoeken zijn niet enkel het speelgoed van meetkundigen, maar duiken ook op in eerder algebraïsch of groep-theoretisch onderzoek. In deze thesis gaan we echter de meetkundige toer op. De bedoeling is enkele (los van elkaar staande) problemen te bekijken, en zo 'onze veelhoekjes' bij te dragen aan de grote veelhoek-puzzel.

Hoofdstuk 1 bundelt de gebruikte definities en stellingen. Voor een uitgebreidere kennismaking met de theorie van de veralgemeende veelhoeken verwijzen we naar het standaardwerk *Generalized Polygons*, Van Maldeghem [57].

B.2 Ken uw klassiekers

Voor de klassieke voorbeelden van veralgemeende veelhoeken vertaalt het verband met meetkundige objecten als kwadrieken zich in 'mooie' eigenschappen van deze veelhoeken. Nog mooier is het als die eigenschappen het klassieke voorbeeld in kwestie *karakteriseren*, d.w.z. van zodra een willekeurige veralgemeende veelhoek die eigenschap heeft, is hij klassiek. In hoofdstuk 2



worden een aantal karakteriseringen van klassieke veralgemeende zeshoeken gegeven. Hierbij zijn de volgende begrippen in een veralgemeende zeshoek essentieel (zie ook de figuur):

- Neem twee tegenvoetse punten a en b. Op elke rechte door a ligt een uniek punt op afstand 4 van b. Deze verzameling punten wordt het *spoor* a^b met basispunt a genoemd. Als alle sporen van een veralgemeende zeshoek Γ (dus met willekeurig basispunt) zich gedragen zoals rechten (d.w.z. twee verschillende sporen snijden in ten hoogste 1 punt), dan wordt Γ *punt*-2-*regulier* genoemd.

- Neem twee tegenvoetse rechten L en M. De verzameling punten op afstand 3 van L en M wordt de *regulus* bepaald door L en M genoemd. Als de reguli van een veralgemeende zeshoek Γ zich gedragen zoals rechten (d.w.z. twee verschillende reguli snijden in ten hoogste 1 rechte), dan wordt Γ 3-*regulier* genoemd.

Het begrip regulariteit ligt aan de grondslag van heel wat meetkundige karakteriseringen van veralgemeende zeshoeken. Een belangrijke karakterisering is deze van Ronan [35], die zegt dat, van zodra een veralgemeende zeshoek punt-2-regulier is, de zeshoek noodzakelijk klassiek is.

In een 2-reguliere zeshoek wordt dus een voorwaarde opgelegd op de doorsnede van elke twee sporen met zelfde basispunt. Een aantal karakteriseringen pogen die voorwaarde te verzwakken. Een *intersectieblok* is een doorsnede van twee sporen a^b en a^c, maar waarbij de punten b en c zodanig gekozen zijn dat er een punt bestaat collineair met b en c, en op afstand 4 van het punt a (zie figuur). Dit begrip werd ingevoerd door Ronan (zie [37]). In een klassieke zeshoek van de orde (q, t) die niet punt-2-regulier is, bevat zo'n intersectieblok juist $t/q + 1$ punten. De vierde tekening in de figuur geeft aan hoe het begrip intersectieblok iets kan veralgemeend worden: we eisen nu dat er een rechte bestaat op afstand 3 van a, b en c. In dit geval zeggen we dat b en c in $(3,4)$-positie liggen ten opzicht van a. Dit geeft de volgende karakterisering.

Stelling *Een eindige veralgemeende zeshoek van de orde* (q,t) *is duaal klassiek als en slechts als* $|a^b \cap a^c| \leq t/q + 1$*, voor elk drietal punten* a, b *en* c *zodat* b *en* c *in* $(3,4)$*-positie liggen ten opzichte van* a*.*

We geven nu nog enkele voorbeelden van karakteriseringen uit hoofdstuk 2.

Stelling *Zij* Γ *een eindige veralgemeende zeshoek. Dan is* Γ *isomorf met de klassieke zeshoek* $\mathsf{H}(q)$ *of* $\mathsf{T}(q^3, q)$*, beide met* q *even, als en slechts als voor elk punt* x*, en elke twee tegenvoetse rechten* L *en* M*, er steeds een punt in de regulus bepaald door* L *en* M *bestaat dat niet tegenvoets* x *ligt.*

(C) Veronderstel dat een punt a op afstand 4 ligt van juist één punt r van een regulus R, en tegenvoets alle andere punten van die regulus, dan liggen alle punten van $R \setminus \{r\}$ op afstand 4 van het unieke punt dat collineair is met a en r.

Stelling *Zij* Γ *een eindige veralgemeende zeshoek van de orde* (s, s^3) *of* (s'^3, s') *die aan voorwaarde* (C) *voldoet. Dan is* Γ *isomorf met één van de klassieke zeshoeken* $\mathsf{T}(s, s^3)$ *of* $\mathsf{T}(s'^3, s')$*, met* s' *oneven.*

Stelling *Zij* Γ *een eindige veralgemeende zeshoek van de orde* (q,t)*,* q *even. Dan voldoet* Γ *aan eigenschap* (C) *als en slechts als* Γ *isomorf is met één van de duaal klassieke zeshoeken* $\mathsf{H}(q)^D$*,* q *niet deelbaar door* 3 *of* $\mathsf{T}(q, q^3)$*.*

B.3 Vergeethoeken

De bedoeling van hoofdstuk 3 is een soort veralgemening van het begrip veralgemeende veelhoek te definiëren. Daarvoor inspireerden we ons op de definitie van een (duaal) semi-affien vlak, een structuur ingevoerd door Dembowski. Bij een affien vlak is er voor elk niet-incident punt-rechte paar $\{p, L\}$ juist één rechte door p 'parallel' met L. Door 'juist één' te vervangen door 'hoogstens één', bekomt men de definitie van een semi-affien vlak. Aangezien parallellisme een equivalentierelatie definieert, kunnen we ook zeggen dat elke twee rechten snijdend of parallel zijn. We bekijken nu de duale structuur, d.w.z. met een equivalentierelatie op de punten in plaats van op de rechten. In de bekomen structuur zullen dus elke twee rechten snijden, en elke twee punten collineair of equivalent zijn. Sommige rechten zijn dus 'vergeten', en vervangen door equivalentieklassen van punten (de ‘gaten’ in het geheugen van de veelhoek).

Om dit te veralgemenen naar n-hoeken starten we met een meetkunde $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, en een equivalentierelatie op de puntenverzameling. Een *gewone n-vergeethoek* is dan een gewone n-hoek in deze meetkunde, maar waarbij sommige zijden vervangen kunnen zijn door equivalentieklassen van punten. De meetkunde Γ wordt een *n-vergeethoek* genoemd als de volgende axioma's voldaan zijn:

(i) in de meetkunde Γ zijn geen gewone k-vergeethoeken, met $k < n$ te vinden,

(ii) door elke twee punten, twee rechten, of een punt en een rechte is steeds een gewone n-vergeethoek te vinden,

(iii) elke rechte bevat minstens drie punten, door elk punt gaan minstens twee rechten, door een punt dat enkel equivalent is met zichzelf gaan minstens drie rechten.

Duidelijkerwijs voldoet een veralgemeende n-hoek aan bovenstaande axioma's, door elke equivalentieklasse van grootte 1 te nemen. De eindige duale semi-affiene vlakken (dus de 3-vergeethoeken) zijn geclassificeerd; ze ontstaan uit een projectief vlak (dus een veralgemeende 3-hoek) door (stukken van) rechten te vervangen door equivalentieklassen. Dit brengt ons op ideeën om n-vergeethoeken te constueren. Zo kunnen we bijvoorbeeld starten met een veralgemeende n-hoek Δ, en een verzameling niet-snijdende rechten in Δ. Door elk van deze rechten te vervangen door een equivalentieklasse, bekomen we een n-vergeethoek. Een ander voorbeeld verkrijg je door te starten van een veralgemeende vierhoek Δ, en hieruit één punt p weg te laten. Elke rechte door p wordt een equivalentieklasse, het resultaat is een 4-vergeethoek waarbij elke equivalentieklasse juist 1 punt minder bevat dan een willekeurige rechte.

De vraag is nu of elke eindige vergeethoek eigenlijk niets anders is dan een vergeetachtig geworden veralgemeende veelhoek. In hoofdstuk 3 wordt de volgende stelling aangetoond:

Stelling *Een eindige n-vergeethoek, met n oneven, wordt steeds bekomen uit een eindige veralgemeende n-hoek, en bestaat dus enkel voor $n = 3$.*

Een vergeethoek wordt *kort van geheugen* genoemd als er parameters (g, k, d) bestaan zodat:

- door elk punt juist k rechten en één niet-triviale klasse, of $k+1$ rechten en geen niet-triviale klasse gaan,

- elke niet-triviale klasse juist g punten bevat,

- elke rechte $g+d$ punten bevat, $d \geq 1$.

Bij een vergeethoek die kort van geheugen is zijn de rechten dus 'langer' dan de klassen.

Stelling *Een eindige n-vergeethoek, met n even, wordt bekomen uit een eindige veralgemeende n-hoek, of is kort van geheugen.*

In hoofdstuk 3 concentreren we ons dan verder op de 4-vergeethoeken die kort van geheugen zijn. We tonen aan dat, als $d = 1$, deze ook bekomen kunnen worden uit een veralgemeende 4-hoek, en geven voorbeelden en karakteriseringen van 4-vergeethoeken die werkelijk zeer kort van geheugen zijn. Hierbij blijft het wel een vraagteken of deze voorbeelden ook bekomen kunnen worden uit een veralgemeende 4-hoek op de gebruikelijke manier, d.w.z. door het vergeten van rechten.

B.4 Veelhoeken door een speciale bril

Zoals reeds gezegd is in een veralgemeende veelhoek een afstand gedefinieerd. Als tussen twee veralgemeende veelhoeken een afbeelding bestaat die alle afstanden bewaart, dan is die afbeelding een isomorfisme. De vraag is nu of we deze voorwaarde kunnen verzwakken. Veronderstel dus dat tussen twee veralgemeende n-hoeken een afbeelding bestaat die een bepaalde afstand i bewaart: is deze afbeelding een isomorfisme? In hoofdstuk 4 tonen we het volgende aan:

Stelling

- *Zij Γ en Γ' twee veralgemeende n-hoeken, $n \geq 4$, i een even getal, $2 \leq i \leq n-1$ en α een surjectieve afbeelding van de puntenverzameling van Γ naar de puntenverzameling van Γ'. Als voor elke twee punten a en b van Γ, $\delta(a,b) = i$ als en slechts als $\delta(a^\alpha, b^\alpha) = i$, dan is α uit te breiden tot een isomorfisme tussen Γ en Γ'.*

- *Zij Γ en Γ' twee veralgemeende n-hoeken, $n \geq 2$, i een oneven getal, $1 \leq i \leq n-1$ en α een surjectieve afbeelding van de puntenverzameling van Γ naar de puntenverzameling van Γ' en van de rechtenverzameling van Γ naar de rechtenverzameling van Γ'. Als voor elk punt-rechte paar $\{a,b\}$ van Γ, $\delta(a,b) = i$ als en slechts als $\delta(a^\alpha, b^\alpha) = i$, dan definieert α een isomorfisme tussen Γ en Γ'.*

De idee van het bewijs van deze stelling is de volgende. De bedoeling is aan te tonen dat α collineariteit bewaart (dan volgt het gestelde uit een reeds bestaande karakterisering van isomorfismen). We kijken nu naar de gegeven veelhoek met een speciale bril, die ons enkel toelaat te zien of twee elementen al dan niet op afstand i gelegen zijn. Door deze bril zien Γ en Γ' er hetzelfde uit, wegens de afbeelding α. Als we nu een manier kunnen vinden om met deze bril toch collineariteit te ontdekken, dan weten we dus dat α ook collineariteit bewaart. Het bewijs reduceert zich dus tot het zoeken naar een eigenschap waarin twee collineaire punten zich onderscheiden van twee niet-collineaire punten, en die geformuleerd kan worden door enkel gebruik te maken van 'afstand i' en 'niet afstand i'.

Voor $i = n$ zal het niet mogelijk zijn de stelling te bewijzen, aangezien er een tegenvoorbeeld bestaat voor één van de klassieke zeshoeken. We kunnen natuurlijk hopen dat er niet al te veel tegenvoorbeelden zijn. Zo vragen we ons in hoofdstuk 4 af of misschien alleen zeshoeken aanleiding geven tot tegenvoorbeelden. Voor eindige veralgemeende veelhoeken kunnen we inderdaad bewijzen dat alleen zeshoeken van de orde (q, q) problemen geven. Een andere vraag is of onder de zeshoeken alleen die éne klassieke zeshoek een tegenvoorbeeld is. Hier krijgen we een positief antwoord op voorwaarde dat de automorfismengroep van de zeshoek in kwestie voldoende groot is.

Een variant op bovenstaande stelling verkrijgen we door te gaan kijken naar afbeeldingen gedefinieerd op de vlaggenverzameling van een veralgemeende n-hoek. In dit geval kunnen we bewijzen dat er maar één tegenvoorbeeld is, namelijk voor de kleinste veralgemeende vierhoek.

B.5 Strategische verzamelingen in een klassieke zeshoek

In appendix A concentreren we ons op de klassieke zeshoek $H(q)^D$. Veronderstel dat een bericht moet doorgegeven worden naar alle punten van deze zeshoek, en dat, van zodra twee collineaire punten van het bericht op de hoogte zijn, ook alle punten op de rechte door deze twee punten kunnen geïnformeerd worden. Hoeveel startpunten zijn minimaal nodig om op die manier alle $(q + 1)(q^4 + q^2 + 1)$ punten op de hoogte te brengen? Zo'n verzameling startpunten die de hele zeshoek voortbrengen, noemen we een *strategische verzameling*. De uitdaging is nu een strategische verzameling met een minimaal aantal punten te vinden. Een stelling bewezen door Thas & Van Maldeghem (zie [48]) leert dat zeker een strategische verzameling van

15 punten bestaat, en dat je altijd minstens 14 punten zal nodig hebben. Als q een priemgetal is, dan kan het steeds met 14 punten. In appendix A proberen we nu uit zo'n strategische verzameling van 15 punten, er één te selecteren, zodat de verzameling zonder dit punt nog steeds strategisch blijft. Om een idee te hebben van het te bewijzene (14 is mogelijk, of 14 is zeker niet mogelijk) pakten we de kleinste gevallen aan met de computer. In deze gevallen bleek er inderdaad een 'overbodig' punt te zitten in de verzameling van 15 punten. Of en hoe dit te veralgemenen is voor een willekeurige waarde van q, blijft voorlopig een goed bewaard 'militair' geheim van deze zeshoek, dus het gedeeltelijk resultaat dat we bekomen is: $14 \Rightarrow (q+1)(q^4+q^2+1)$ als q priem is, of $q \in \{4, 8, 9, 16\}$...

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Index of Notations

$\Gamma_i(x)$	the set of elements of Γ at distance i from x
$x^\perp$	the set of points collinear with the point x
$x^{\perp\!\!\!\perp}$	the set of points not opposite the point x
$x^y_{[i]}$	$\Gamma_i(x) \cap \Gamma_{n-i}(y)$, with $\delta(x,y) = n$
x^y	$x^y_{[2]}$, with $\delta(x,y) = n$
$\langle x, y\rangle$	$x^y_{[\frac{n}{2}]}$, with $\delta(x,y) = n$
$R(x,y)$	(in 3-regular hexagon) the regulus containing x and y
$a \bowtie b$	the unique element at distance $\frac{\delta(a,b)}{2}$ from a and b (if defined)
$[x,y]$	the shortest path between x and y (if defined)
$[x;y]$	projectivity from $\Gamma_1(x)$ to $\Gamma_1(y)$

$\mathsf{H}(q)$	finite split Cayley hexagon of order q
$\mathsf{H}(q)^D$	finite dual split Cayley hexagon of order q
$\mathsf{T}(q^3,q)$	finite twisted triality hexagon of order (q^3,q)
$\mathsf{T}(q,q^3)$	finite dual twisted triality hexagon of order (q,q^3)

Index