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Vakgroep Zuivere Wiskunde
en Computeralgebra

Ovoids and Spreads of Finite Classical Generalized Hexagons and Applications

An De Wispelaere

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Proefschrift voorgelegd aan
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Preface

Most of my friends and family know that I am a PhD student at Ghent University. However, this is mostly as far as it goes and I only have myself to blame for that. This is because, right in the beginning, whenever anyone would ask me to describe my job I would say, enthusiastic as I was, that I was starting research on various topics based on generalized hexagons. And what is a generalized hexagon? Well ... by then I realized that – apart from saying that it is a structure that contains points and lines and is filled with ordinary hexagons, but no pentagons, quadrangles and triangles – this is an object that is very hard to grasp for a non-mathematician. So after seeing a dozen puzzled faces when answering this question, I decided that it might in fact be easier to say that

> "I am an assistant at the university and give exercises to mathematics courses ... oh yes, and I also do *some* research".

And quite frankly things could not be further from the truth as I mainly focus on research and give *some* exercise courses and the ratio of these two is about 90 to 10. In my defense: it is very hard to dispute the fact that

> no, not everything is known in mathematics;
> no, finite geometry is not like the geometry that is taught in high school;
> and no, there is no *real world* application for what I am doing!

In many aspects finite geometry is like modern art. There is no greater purpose in the creation of something new; and there is only a small population of people throughout the world who like it, but those who do, LOVE IT! So let me guide you through my PhD thesis and show you the art work that I have created during the past three years.

Chapters 1 and 2 are dedicated to countless elementary definitions, every one of which is known. The main purpose of the first chapter is to familiarize

the reader with the concept of generalized polygons. In Chapter 2, special attention is given to generalized hexagons and all of their particularities.

Chapter 3 is concerned with the common point reguli of two split Cayley hexagons represented on the same parabolic quadric $\mathsf{Q}(6, q)$. As a first application we construct a 2-design that is – according to specific values of q – a subdesign of or isomorphic to the Hölz design. To conclude this chapter, we give a computer free proof of a result by V.D. Tonchev, i.e. the only unitals contained in the Hölz design on 28 points are Hermitian and Ree unitals.

The next two chapters deal with the construction of new substructures of the known generalized hexagons, where Chapter 4 tackles the order 3 and Chapter 5 treats the order 4 case.

In Chapter 4 we construct and classify all distance-j ovoids and distance-j spreads of the known generalized hexagon of order 3 and hereby provide a geometric interpretation of all maximal subgroups of the exceptional group $\mathsf{G}_2(3)$.

In Chapter 5 we construct two non-isomorphic distance-2 ovoids of the known generalized hexagon of order 4. With these distance-2 ovoids arise new types of two-weight codes, strongly regular graphs and two-character sets. Moreover, we construct a new infinite class of two-character sets in $\mathsf{PG}(5, q^2)$ for all values of q.

Chapter 6 provides a geometrical interpretation of the eigenvalues of the point graph of a generalized hexagon of order q, as well as a list of a number of codes that arise from generalized hexagons with small parameters.

And finally, Chapter 7 stipulates a characterization of the one-point extension of $\mathsf{H}(2)$.

A lot of these results were initiated by computer searches. GAP and some of its packages, like *pg*, *grape* and *guava* helped me get familiar and acquainted with the split Cayley hexagons and made these objects a bit more transparent to work with. The main challenge was then to geometrically explain every new results and hence eliminate all use of the computer. Nevertheless, I truly acknowledge the help that programs like Gap offered me.

Writing a PhD thesis of course leaves many people to thank. First of all, I would like to thank the *Fund for Scientific Research - Flanders (Belgium)* for granting me the opportunity to become a full-time artist and for allowing me to attend conferences all over the world.

But above all, I wish to express my gratitude to my supervisor Prof. Dr. H. Van Maldeghem for it was he who opened my eyes to the art of geometry, encouraged and helped me whenever needed and became a good friend along the way. Also, I consider myself very fortunate to be a member of a research group many fellow geometers are jealous of. Not only are we surrounded by some of the greatest geometers of our time, but the collegiality that I have felt during the past few years has also been incredibly strong.

On this note, I would like to say that I consider many colleagues to have become personal friends. Tom and Stefaan, for instance, have been great friends right from the beginning and every step of the way. And even though she has only been with us for a short period of time, also Nele became a very close friend. I wish her all the best in her new challenge at the Artevelde Hogeschool. Pieter, Stefaan and David had been my badminton-buddies for over a year, when we were sad to see Pieter leave. Trying to fill the void we had to replace him by Jan, Koen, Nathalie and occasionally a substituting Patrick – not one, but four new badminton friends.

On a more personal level, I would like to thank my parents for their unconditional love and support, and David, for always believing in me, for always being – or pretending to be, which is just as good – interested in hearing all about both my failures and creations, and most importantly for being the loving husband he is!

Contents

1 Introductory guide

This first chapter will essentially be occupied with definitions, some elementary examples and a few basic properties that are fundamental to the rest of this work. I thought long and hard on a way to start this chapter with the definition of a generalized hexagon, which is the central notion of my thesis. However, I realized I would have to let this idea slide seeing I need to clarify some basic concepts, recall definitions that will be used and establish notations. Hence I re-evaluated my goals for this chapter and made the purpose of these first few sections to arrive at the definition of generalized polygons in general. A whole new chapter will then be devoted to generalized hexagons and all of their particularities.

The first two sections give a brief introduction to graphs and codes. In Section 1.3 we turn our attention to projective spaces over finite fields and elaborate on polarities, quadrics and Hermitian varieties, m-systems, unitals, ovals and hyperovals, Grassmann coordinates and, finally, two-character sets. In Section 1.4 and 1.5 we introduce geometries and designs. Finally, the last section turns to the main subject of this chapter, namely generalized polygons.

We emphasize that this chapter is only intended as an introduction to certain aspects of finite incidence geometry. Many text books exist which treat the topic in much more detail than we have room for here. We refer the interested reader to F. Buekenhout [7], R. Hill [35], D.R. Hughes and F.C. Piper [39], S.E. Payne and J.A. Thas [45], H. Van Maldeghem [67], and other books in the bibliography.

1.1 Graphs

We will assume that the basic definitions of graph theory are already familiar to the reader. In this section we give a brief review on those which will be useful to us. At the same time, we introduce notations and terminology.

> A *graph* Γ is a pair (V, E), consisting of a set V and a set E of unordered pairs of V. The elements of V are called the *vertices* of the graph Γ, while the elements of E are called the *edges*.

Two elements, x and y, of V are called *adjacent* if $\{x, y\} \in E$ and we write $x \sim y$. Whenever two vertices are not adjacent we denote this by $x \not\sim y$ (note that $x \not\sim x$ for every $x \in V$).

A *clique* of a graph Γ is a set of vertices of Γ such that any two of them are adjacent. A *coclique*, on the contrary, is a set of vertices such that no two of them are adjacent.

The set of all vertices that are adjacent to some fixed x is called the *neighborhood* of x and will be denoted by Γ_x.

The *complement* $\overline{\Gamma}$ of a graph Γ is the graph with the same vertex set V, but different edge set $\overline{E}$, consisting of all unordered pairs of vertices that are not in E. A graph (W, F) is an *induced subgraph* of the graph $\Gamma = (V, E)$ provided that W is a subset of V and F consists of all edges of Γ contained in W.

A graph is called *bipartite* if the vertex set can be partitioned into two disjoint sets, such that no vertices of the same set are adjacent.

In a graph, a *simple path* of *length* n from x to y is a sequence $x = x_0, x_1, \ldots, x_n = y$ of vertices in which the vertices of each successive pair, x_i, x_{i+1}, $0 \leq i < n$, are adjacent and in which no vertex is repeated, except that possibly $x_0 = x_n$. In that case, such a path of length at least 3 will be called a *circuit*. Two vertices x and y of the graph Γ are at *distance* $d(x, y)$, provided there exists a path of length $d(x, y)$ joining x and y, but no shorter one. Clearly a vertex has distance 0 to itself and distance 1 to all vertices in its neighborhood. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x.

A graph Γ is said to be *connected* if there exists a path between any two of its vertices, otherwise it is called *disconnected*. The *diameter* of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is defined as the maximal value of the distance function $d(x, y)$ (note that this definition can only be applied to connected graphs). The *girth* of Γ is the length of its shortest circuit. If no circuits exist, then

the girth is considered to be infinity. A *tree* is a connected graph with girth infinity, and so every graph with girth infinity is the disjoint union of trees (i.e. each connected component is a tree).

A graph is *regular* of *valency* $k > 0$, or *k-regular*, if each vertex is adjacent to k vertices.

> A *distance regular graph* Γ with diameter d, is a regular and connected graph of valency k with the following property. There are natural numbers
>
> $$b_0 = k, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d$$
>
> such that for each pair of vertices, (x, y), at distance j, we have
>
> 1. $|\Gamma_{j-1}(y) \cap \Gamma_1(x)| = c_j$, $(1 \leq j \leq d)$;
> 2. $|\Gamma_{j+1}(y) \cap \Gamma_1(x)| = b_j$, $(0 \leq j \leq d-1)$.

The *intersection array* of Γ is defined by $i(\Gamma) = \{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$.

> A distance regular graph of diameter 2 is better known as a *strongly regular graph*. In this use, we denote Γ by $srg(v, k, \lambda, \mu)$ where
>
> 1. Γ contains v vertices;
> 2. every vertex has valency k;
> 3. any two mutually adjacent vertices are adjacent to a constant number λ of vertices; and any two mutually non-adjacent vertices are adjacent to a constant number μ of vertices.

It is easy to check that an $srg(v, k, \lambda, \mu)$ is equivalent with a distance regular graph with intersection array $\{k, k-1-\lambda; 1, \mu\}$.

1.2 Codes

For an extensive introduction to this subject we refer to *A first course in Coding Theory* by R. Hill [35]. Here we merely recall some basic definitions and standard notations that are used later in this work.

A *q-ary code* is a given set of sequences of symbols where each symbol is chosen from a set F_q (the *alphabet*) of q distinct elements.

In a code of *length* n each *codeword* is a sequence of n symbols. From now on we assume that the alphabet F_q is the Galois field $\mathsf{GF}(q)$, with q a prime power, and we regard $(F_q)^n$ as the vector space $\mathsf{V}(n,q)$.

A *linear code* over $\mathsf{GF}(q)$ is just a subspace of $\mathsf{V}(n,q)$.

The *weight* of a vector x in $\mathsf{V}(n,q)$ is defined to be the number of non-zero entries of x. One of the most useful properties of a linear code is that its *minimum distance* is equal to the smallest of the weights of the non-zero codewords (see [35]).

A *two-weight code* is a code in which all codewords – as the name already indicates – have one out of two weights.

If C is a k-dimensional subspace of $\mathsf{V}(n,q)$, then the linear code C is called an $[n,k]$-*code*, or sometimes, if we wish to specify the minimum distance d of C, an $[n,k,d]$-*code*.

One of the advantages of a linear code is that it can be specified by simply giving a basis of k codewords (whereas in a non-linear code we may have to list all codewords). A $k \times n$ matrix whose rows form a basis of a linear $[n,k]$-code is called a *generator matrix* of the code.

Let C be a linear $[n,k,d]$-code over F_q. The *packing radius* of C is defined to be the value $\rho(C) = \frac{d-1}{2}$.

The *covering radius* of C is $r(C) = \max_x \min_c \delta(x,c)$ with $x \in F_q^n$ and $c \in C$, and where δ denotes the *Hamming distance* (i.e. the number of entries in which the symbols of two codewords disagree) on F_q^n.

The code C is said to be *perfect* if $r(C) = \rho(C)$.

1.3 Projective spaces over finite fields

The subject of projective and affine spaces is so large and so fundamental to all of those who work in geometry that we will assume that the reader is familiar with the basic ideas of projective geometry. Thorough references on finite projective spaces are *Projective geometries over finite fields* by J.W.P. Hirschfeld and *General Galois geometries* by J.W.P. Hirschfeld and J.A. Thas, respectively.

Projective spaces can be defined over arbitrary fields. We restrict ourselves to the finite fields as these are the only type of fields we will encounter.

> We will use the classical notation $\mathsf{PG}(n, q)$ for a Desarguesian projective space of dimension n over the Galois field $\mathsf{GF}(q)$.

In the next section we specialize to the definition of different sorts of polarities in $\mathsf{PG}(n, q)$. In Section 1.3.2 we introduce quadrics and Hermitian varieties in an algebraic manner, state some properties and define substructures that will be useful later on. In Section 1.3.3 we turn our attention to m-systems on quadrics with particular emphasis on those on the quadric $\mathsf{Q}(6, q)$. In Section 1.3.4 we define unitals, ovals and hyperovals and in Section 1.3.5 Grassmann coordinates are introduced. Finally, in Section 1.3.6 we provide the definition of a two-character set in $\mathsf{PG}(n, q)$, which will play an important role in Chapter 5.

1.3.1 Polarities

> A *collineation* of $\mathsf{PG}(n, q)$, $n \geq 2$, is a permutation of the set of all subspaces of $\mathsf{PG}(n, q)$ that preserves the inclusion. The group of all collineations of $\mathsf{PG}(n, q)$ is denoted by $\mathsf{P\Gamma L}_{n+1}(q)$.

With respect to a given coordinate system in $\mathsf{PG}(n, q)$, such a collineation is determined by a non-singular $(n + 1) \times (n + 1)$ matrix over the finite field $\mathsf{GF}(q)$ and a companion automorphism θ of $\mathsf{GF}(q)$. This is the Fundamental Theorem of Projective Geometry.

> If θ equals the identity, then the corresponding collineation is a so-called *projectivity*. The set of all projectivities also forms a group, denoted by $\mathsf{PGL}_{n+1}(q)$.

A *correlation* α of $\mathsf{PG}(n,q)$, $n \geq 2$, is a permutation of the set of all subspaces of $\mathsf{PG}(n,q)$ that reverses the inclusion between the subspaces of $\mathsf{PG}(n,q)$.

If α, next to being a correlation of $\mathsf{PG}(n,q)$, also satisfies $\alpha^2 = \mathbb{1}$, then α is called a *polarity* of $\mathsf{PG}(n,q)$.

In the same way as applies for a collineation, a correlation α is determined by a non-singular matrix A and an automorphism θ of $\mathsf{GF}(q)$. More explicitly, a point $p(x_0,\ldots,x_n)$ of $\mathsf{PG}(n,q)$ is mapped onto the hyperplane with equation $a_0X_0 + \ldots + a_nX_n = 0$, where

$$\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = A \begin{pmatrix} x_0^\theta \\ \vdots \\ x_n^\theta \end{pmatrix}.$$

Conversely, α maps a hyperplane satisfying the preceding equation onto the point $p(x_0,\ldots,x_n)$ if

$$\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = (A^t)^{-1} \begin{pmatrix} a_0^\theta \\ \vdots \\ a_n^\theta \end{pmatrix}.$$

With this notation, saying that α is a polarity is equivalent to saying that $\theta^2 = \mathbb{1}$ and $(A^t)^\theta = kA$, for some non-zero scalar $k \in \mathsf{GF}(q)$ with $k^{1+\theta} = 1$.

The different types of polarities can be distinguished through the conditions on θ and A given in Table 1.1.

n, q	type	θ	A
$q =$ odd	orthogonal polarity	$\theta = \mathbb{1}$	$A^t = A$
$q =$ even	pseudo polarity	$\theta = \mathbb{1}$	$A^t = A$, $\exists i : a_{ii} \neq 0$
$n =$ odd	symplectic polarity	$\theta = \mathbb{1}$	$A^t = -A$, $\forall i : a_{ii} = 0$
$q =$ square	Hermitian polarity	$\theta : x \to x^{\sqrt{q}}$	$(A^t)^\theta = A$

Table 1.1: Polarities of $\mathsf{PG}(n,q)$

An *absolute point* (respectively *absolute hyperplane*) with respect to a polarity α of $\mathsf{PG}(n,q)$ is a point (respectively hyperplane) which is incident with its image under α. More generally, we call a subspace U of $\mathsf{PG}(n,q)$ *absolute* or *totally isotropic* with respect to α if and only if one of U and U^α is contained in the other.

1.3.2 Quadrics and Hermitian Varieties

A *quadric* of $\mathsf{PG}(n,q)$, $n \geq 1$, is an n-dimensional hypersurface represented by a quadratic equation

$$\sum_{i \leq j=0}^{n} a_{ij} X_i X_j = 0$$

with not all a_{ij} zero. In the same way, the point set of $\mathsf{PG}(n,q^2)$ obtained by considering the form

$$\sum_{i,j=0}^{n} a_{ij} X_i X_j^q = 0 \qquad a_{ij}^q = a_{ji}$$

over $\mathsf{GF}(q^2)$ is called a *Hermitian variety*.

A quadric and a Hermitian variety in a projective plane are often called a *conic* and a *Hermitian curve*, respectively.

Any line of $\mathsf{PG}(n,q)$ intersects a quadric in 0, 1, 2 or $q+1$ points. A line that intersects in 0 or 2 points will be referred to as an *external* or *secant* line respectively, while a line on which one or all points belong to the quadric is a so-called *tangent* line. For a Hermitian variety, however, there are no external lines. Here every line is either a tangent, in which case it contains 1 or q^2+1 points of the variety, or a secant in $q+1$ points.

A quadric or Hermitian variety is said to be *non-singular* if for none of its points all lines through this point are tangent. If such a point does exist, then it is called a *singular point* and the hypersurface it belongs to is said to be *singular*.

The following definitions apply both to a quadric and to a Hermitian variety, hence we consider a non-specified non-singular variety F.

A *generator* of a variety F is a subspace of that variety of maximal dimension. The *projective index* is the dimension of (clearly) all generators.

Let p be a point of F. There is a unique hyperplane, denoted by $T_p(\mathsf{F})$, through p such that the set of all tangent lines on p is exactly the set of lines on p in $T_p(\mathsf{F})$. We call this special hyperplane the *tangent hyperplane* at p. If a point r belongs to the tangent hyperplane at p, then p and r are said to be *conjugate points*. If F is a variety in an n-dimensional space, then the intersection of the tangent hyperplane at p with F is a cone with vertex

p and as base a non-singular variety of the same type as F in a projective $(n-2)$-space.

Concerning the classification of non-singular varieties, we mention the following results.

> In $\mathsf{PG}(2n, q)$, there is, up to isomorphism, a unique non-singular quadric, the *parabolic quadric* $\mathsf{Q}(2n, q)$. In $\mathsf{PG}(2n+1, q)$, there are, up to isomorphism, exactly two non-singular quadrics, the *hyperbolic quadric* $\mathsf{Q}^+(2n+1, q)$ and the *elliptic quadric* $\mathsf{Q}^-(2n+1, q)$.
>
> In $\mathsf{PG}(n, q^2)$, there is, up to isomorphism, a unique non-singular *Hermitian variety* $\mathsf{H}(n, q^2)$.

The number of points and projective index of these different varieties are summarized in Table 1.2.

variety	projective index	number of points
$\mathsf{Q}(2n, q)$	$n-1$	$\frac{q^{2n}-1}{q-1}$
$\mathsf{Q}^+(2n+1, q)$	n	$\frac{(q^n+1)(q^{n+1}-1)}{q-1}$
$\mathsf{Q}^-(2n+1, q)$	$n-1$	$\frac{(q^n-1)(q^{n+1}+1)}{q-1}$
$\mathsf{H}(2n, q^2)$	$n-1$	$\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}$
$\mathsf{H}(2n+1, q^2)$	n	$\frac{(q^{2n+2}-1)(q^{2n+1}+1)}{q^2-1}$

Table 1.2: Non-singular varieties

Note. The hyperbolic quadric $\mathsf{Q}^+(3, q)$ contains $2(q+1)$ lines, which determine 2 *reguli* of $\mathsf{PG}(3, q)$, i.e. a set of $q+1$ skew lines intersecting three mutually skew lines of $\mathsf{PG}(3, q)$. These two reguli of $\mathsf{Q}^+(3, q)$ are said to be *opposite reguli* and one is obtained by taking all *transversals* to the other.

When q is even, every non-singular parabolic quadric $\mathsf{Q}(2n, q)$ has a *nucleus*, i.e. a point contained in every tangent hyperplane. Also, every hyperplane through the nucleus is tangent and every line on this point has exactly one point in common with the quadric.

With a non-singular quadric in $\mathsf{PG}(n, q)$, q odd, one can associate in a natural way an orthogonal polarity β such that it is the set of absolute points with respect to β. Furthermore, a point and its tangent hyperplane are interchanged by β.

In exactly the same way there is a unique Hermitian polarity associated to $\mathsf{H}(n, q^2)$, for all values of n and q. Again, the variety consists of all absolute points of the polarity and a point and its tangent hyperplane are swapped.

To conclude this subsection we display, for further reference, the standard equations of the defined varieties in Table 1.3.

variety	standard equation
$\mathsf{Q}(2n, q)$	$X_0^2 + X_1X_2 + \ldots + X_{2n-1}X_{2n}$
$\mathsf{Q}^+(2n+1, q)$	$X_0X_1 + \ldots + X_{2n}X_{2n+1}$
$\mathsf{Q}^-(2n+1, q)$	$f(X_0, X_1) + X_2X_3 + \ldots + X_{2n}X_{2n+1}$
	f: quadratic polynomial irreducible over $\mathsf{GF}(q)$
$\mathsf{H}(n, q^2)$	$X_0^{q+1} + \ldots + X_n^{q+1}$

Table 1.3: Standard equations of F

1.3.3 m-Systems

Here we introduce *m-systems* on a quadric Q with particular emphasis on those on the quadric $\mathsf{Q}(6, q)$ as this is where, as we will see, the split Cayley hexagon lives. The basic idea behind m-systems is to have a set of m-dimensional spaces that are, in a sense, spread out apart from each other in an optimal way and all over the quadric.

> An m-system $\mathcal{M}$ of Q is a set of mutually disjoint totally singular m-dimensional subspaces of Q of the "maximal theoretical size" with the property that no generator of Q that contains an element of $\mathcal{M}$ intersects another element of $\mathcal{M}$.

For more details on m-systems, see the paper [54].

Note. A 0-system is a set $\mathcal{M}$ of points such that every generator of Q contains exactly one point of $\mathcal{M}$, and an r-system, with r the projective index of Q, is a set of generators that partitions the point set of Q. These particular m-systems, where the value of m equals 0 or r, are better known respectively as *ovoids* and *spreads* of Q. Hence, the concept of m-systems is a generalization of these ovoids and spreads.

> A *derivation* of an m-system $\mathcal{M}$ is obtained by replacing a number of elements of $\mathcal{M}$ by a set of mutually skew subspaces of dimension m, covering the same set of points.

This is inspired on a very similar method known for line spreads of quadrics: if such a spread contains a regulus of lines, then a new spread (isomorphic or not) is obtained by deleting this regulus and replacing it by its opposite regulus (a procedure called *switching* that regulus).

We will be most interested in the quadric $\mathsf{Q}(6,q)$. Besides ovoids and spreads, we also have the definition of a 1-system on this quadric.

> A 1-system of $\mathsf{Q}(6,q)$ is a set of q^3+1 lines such that any plane of the quadric containing one does not intersect any other.

If a 1-system $\mathcal{M}$ of $\mathsf{Q}(6,q)$ contains a regulus $\mathcal{R}$ of lines, then we can *derive* $\mathcal{M}$ at that regulus – namely, we replace all elements of $\mathcal{R}$ by the elements of the opposite regulus – and obtain a 1-system. If we do so at a number of disjoint reguli, then we call the obtained 1-system a *derivation* of the original one.

1.3.4 Unitals, Ovals and Hyperovals

Unitals

> A *unital* $\mathcal{U}$ in $\mathsf{PG}(2,q^2)$ is a set of q^3+1 points such that every line intersects $\mathcal{U}$ in either 1 or $q+1$ points.

The *classical unital* is the set of points of a Hermitian curve.

Ovals and Hyperovals

Let $\mathcal{P}$ be an arbitrary projective plane.

> An *oval* of $\mathcal{P}$ is a non-empty set $\mathcal{O}$ of points no three of which are collinear and such that through any point of $\mathcal{O}$ there is precisely one tangent.

> A hyperoval of $\mathcal{P}$ is a non-empty set $\mathcal{H}$ of points such that any line intersects $\mathcal{H}$ in 0 or exactly 2 points.

The following facts concerning these objects are well known (see e.g. [36]). Say $\mathcal{O}$ is an oval of $\mathsf{PG}(2,q)$. Any point of $\mathcal{O}$ lies on exactly one tangent line. If q is odd, then any point off $\mathcal{O}$ is on either 0 or 2 tangents. If q is even, then all tangents pass through a common point, the *nucleus* of $\mathcal{O}$. In particular, any oval in a projective plane, with q even, can uniquely be extended to a hyperoval (see [48]). Examples of ovals in Desarguesian planes are provided by non-singular quadrics (conics). Conversely, not every oval is a conic. However, from a famous theorem of B. Segre we know that in finite desarguesian planes of odd order there are no other ovals [52].

1.3.5 Grassmann coordinates

Although *Grassmann coordinates* can be introduced for subspaces of arbitrary dimension in $\mathsf{PG}(n,q)$, we opt to give the definition only for lines in $\mathsf{PG}(n,q)$ (in the special case $n=3$: *Plücker coordinates*).

Choose a basis and coordinates and let L be a line of $\mathsf{PG}(n,q)$, $n \geq 2$. Consider two arbitrary points $x=(x_0,\ldots,x_n)$ and $y=(y_0,\ldots,y_n)$. For $0 \leq i < j \leq n$, let p_{ij} denote the element

$$\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} = x_i y_j - x_j y_i$$

of $\mathsf{GF}(q)$. Then one can easily verify that the $\binom{n+1}{2}$-tuple $(p_{ij})_{0\leq i<j\leq n}$ is, up to a non-zero scalar multiple, independent of the points x and y on L. Hence the line L defines a unique point $p_L(p_{ij})_{0\leq i<j\leq n}$ of $\mathsf{PG}(\binom{n+1}{2}-1,q)$ and the coordinates of this point are the so-called Grassmann coordinates of L.

1.3.6 Two-character sets

> A *two-character set* in $\mathsf{PG}(d,q)$ is a set $\mathcal{S}$ of n points together with two constants $w_1>0$ and $w_2>0$ such that every hyperplane meets $\mathcal{S}$ in either $n-w_1$ or $n-w_2$ points.

Embed $\mathsf{PG}(d,q)$ as a hyperplane Π in $\mathsf{PG}(d+1,q)$. The *linear representation graph* $\Gamma_d^*(\mathcal{S})$ is the graph with as vertex set the set of points in $\mathsf{PG}(d+1,q)\setminus\Pi$ and where two vertices are adjacent whenever the line joining them intersects Π in $\mathcal{S}$. Then $\Gamma_d^*(\mathcal{S})$ has $v=q^{d+1}$ vertices and valency $k=(q-1)n$. Delsarte [13] proved that this graph is strongly regular precisely because $\mathcal{S}$ is a two-character set. The other parameters are $\lambda = k-1+(k-qw_1+1)(k-qw_2+1)$

and $\mu = k + (k - qw_1)(k - qw_2)$. Viewing the coordinates of the elements of $\mathcal{S}$ as columns of the generator matrix of a code C of length n and dimension $d+1$, then the property that hyperplanes miss either w_1 or w_2 points of $\mathcal{P}$ translates into the fact that the code C has two weights, namely w_1 and w_2, see [8]. Such a code will be referred to as a *linear projective two-weight code*.

From [8] we know that, if $\mathsf{GF}(q_0)$ is a subfield of $\mathsf{GF}(q)$, with $q_0^r = q$, then the projective two-weight code C (defined over $\mathsf{GF}(q)$) canonically determines a projective two-weight code C' of length n' and dimension $(d+1)r$, with weights w_1' and w_2', where $n' = \frac{(q-1)n}{q_0-1}$, $w_1' = \frac{qw_1}{q_0}$ and $w_2' = \frac{qw_2}{q_0}$.

1.4 Geometries

For the most part the definitions given in this section are ones which the reader might have met before, but we state them so that we can easily use them and refer to them in the later chapters.

In general,

> a *pre-geometry* consists of elements of different types such as points, lines, planes, etc (or vertices, edges, faces, cells or subspaces of dimension i where i is an integer). The *incidence* is a symmetric relation on the set of elements such that no two elements of the same type are incident. The *rank* of a pre-geometry is the number of distinct types of elements.
>
> A pre-geometry of rank n defines an n-partite graph in the obvious way. If every clique of this graph is contained in a clique of size n, then the pre-geometry is called a *geometry*.

In this context, one should abandon the usual physical viewpoint according to which a line is a set of points. The same status will be given to each of the different types of elements, which makes it possible to consider any set of elements as the set of points. In this work, we will be most interested in rank 2 geometries of which we have the following equivalent definition.

> A *point-line incidence structure* or a *geometry (of rank 2)* is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with non-empty *point set* $\mathcal{P}$, non-empty *line set* $\mathcal{L}$, an *incidence relation* $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L} \cup \mathcal{L} \times \mathcal{P}$, and every element is incident with at least one other element.

From here on, the word geometry is used in the sense of a rank 2 geometry. For every other type of geometry we will, in order to avoid any danger of confusion, explicitly mention its rank. We will also assume the geometry to be *connected*, i.e. for every two elements of the geometry there exists a set of consecutively incident elements from the one to the other.

The points of a geometry are often represented by small letters $(x, y, z, p, \ldots)$, while lines are denoted by capital letters $(K, L, M, N, \ldots)$. Both points and lines are called *elements* of the geometry.

A line is often identified with the set of points in $\mathcal{P}$ that are incident with it. In this case incidence becomes symmetrized inclusion.

A point or a line is called *thick* if it is incident with at least 3 elements. We say that Γ itself is thick, if all of its points and lines are. A geometry Γ is called *finite* provided $\mathcal{P}$ and $\mathcal{L}$ are.

A *geometrical hyperplane* of Γ is a set, H, of points such that any line of Γ either has a unique point or all of its points in H.

Two geometries $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ are said to be *isomorphic* if there exists a pair of bijections $\alpha : \mathcal{P} \to \mathcal{P}'$, $\beta : \mathcal{L} \to \mathcal{L}'$ preserving incidence, or non-incidence, i.e. $p \ \mathtt{I} \ L$ if and only if $p^\alpha \ \mathtt{I}' \ L^\beta$ for all $p \in \mathcal{P}, L \in \mathcal{L}$. If Γ and Γ' are isomorphic, then we write $\Gamma \cong \Gamma'$.

If R is a projective or affine space, then Γ is said to be *laxly embedded* in R provided:

1. $\mathcal{P}$ is a subset of the point set of R;
2. $\mathcal{L}$ is a subset of the line set of R;
3. $\forall (p, L) \in \mathcal{P} \times \mathcal{L}$: $p \ \mathtt{I} \ L \Leftrightarrow p \in L$ in R;
4. $\mathcal{P} \not\subseteq \mathcal{H}$, with $\mathcal{H}$ a hyperplane in R.

Moreover, if for every line L of Γ every point of R on L belongs to Γ, then Γ is said to be *fully embedded* in R.

The incidence structure $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathtt{I})$, obtained by interchanging the points and the lines of Γ, is called the *dual* of Γ. When Γ is *self-dual*, i.e. when $\Gamma \cong \Gamma^D$, then the dual of any statement of Γ is another statement that also holds in Γ. This phenomenon is called the *principle of duality*.

A *flag* is a set $\{p, L\}$, with p a point incident with the line L. *Adjacent* flags are distinct flags that have an element in common. The standard notation for the set of flags is $\mathcal{F}$. An *anti-flag* is a set $\{p, L\}$, where the point p and the line L are not incident. A *flag matching* $\mathcal{F}$ of Γ is a set of flags covering

all points and lines of Γ – point or line – is incident with a unique flag of $\mathcal{F}$.

With these definitions we are now ready to define the *Coxeter graph*. Consider $\mathsf{PG}(2,2)$, the projective plane of order 2. The vertices of the Coxeter graph are the anti-flags of $\mathsf{PG}(2,2)$. Two anti-flags $\{p, L\}$, $\{p', L'\}$ form an edge precisely when $p \neq p'$, $L \neq L'$ and the intersection point of the lines L and L' is a point on the line joining p and p'. We have pictured the Coxeter graph and two vertices at maximal distance in Figure 1.1.

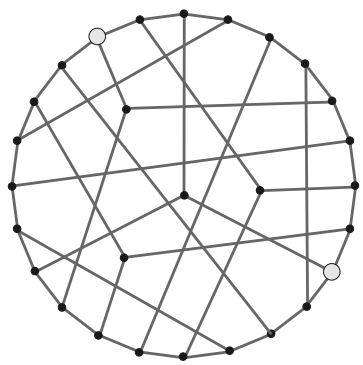

Figure 1.1: Coxeter graph

> The *double* of a geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is the geometry $(\mathcal{P} \cup \mathcal{L}, \mathcal{F}, \in)$ with $\mathcal{F}$ the set of flags and $\in$ the set-theoretic inclusion.

> A *subgeometry* of a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a geometry $(\mathcal{P}', \mathcal{L}', \mathtt{I}')$ where $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathtt{I}' = \mathtt{I} \cap (\mathcal{P}' \times \mathcal{L}' \cup \mathcal{L}' \times \mathcal{P}')$.

For example, a *Baer subplane* is a subgeometry of a projective plane of order q^2, which is itself a plane of order q. If we consider geometries satisfying certain axioms, then we are usually also interested in their subgeometries that satisfy these same axioms.

> The *incidence graph* of a geometry is the graph with vertex set $V = \mathcal{P} \cup \mathcal{L}$ and the flags of the geometry as edges. The adjacency relation $\sim$ of this geometry is, therefore, given by $x \sim y \Leftrightarrow x \,\mathtt{I}\, y$.

For example, the *Heawood graph* is the incidence graph of $\mathsf{PG}(2,2)$. Figure 1.2 pictures the Heawood graph and two vertices at maximal distance.

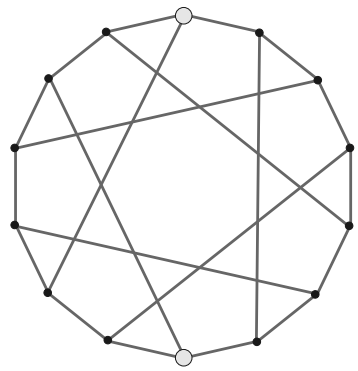

Figure 1.2: Heawood graph

The incidence graph of a geometry is always bipartite and conversely a bipartite graph is always the incidence graph of some geometry.

Note. In this sense, a geometry Γ is said to be connected if its incidence graph is.

Let u, v be two elements (both u and v can be either a point or a line) of the geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$. The *distance* $d(u,v)$ between u and v is measured in the incidence graph, i.e. $d(u,v)$ is the length of the shortest path between u and v (note that $d(u,u) = 0$). In particular, the distance between two elements of the same type (either two points or two lines) will be even.

When two points x, y are at distance 2, we call them *collinear* and write $x \perp y$. If there is only one line containing x and y, then we call $L = xy$ the line *joining* them. Dually, two lines L, M at distance 2 are called *concurrent*; if there is a unique point incident with both, then write $p = L \cap M$ and call p the *intersection point* of L and M. The *perp* of a point x is the set of all elements collinear with x and will be denoted by $x^\perp$. Furthermore, for any set X, the perp of this set is the set of points collinear with every element of X. So in symbols we get $X^\perp = \bigcap_{x \in X} x^\perp$. The set of all elements at distance i (respectively i or j) from an element u is denoted by $\Gamma_i(u)$ (respectively $\Gamma_{i,j}(u)$). For a point p we call $\Gamma_1(p) = \Gamma(p)$ the *line pencil* of p and for a line L the set $\Gamma_1(L) = \Gamma(L)$ will be called the *point row* of this line.

> The *collinearity graph* or *point graph* of a geometry Γ is the graph with vertex set $V = \mathcal{P}$ and in which two distinct vertices are adjacent if they are collinear in Γ.

The *chromatic number* of a graph is the minimum number of colors needed to color the vertices of the graph, with no two adjacent vertices in the same color. When speaking of the chromatic number of a geometry, we in fact implicitly indicate the chromatic number of the point graph of that particular geometry (since the chromatic number of the incidence graph is always equal to 2).

1.5 Designs

We suppose the reader to be familiar with basic design theory. We recall some (to us) useful definitions and notations and introduce the Hölz designs as we will be interested in these in a later chapter.

> A $t-(v,k,\lambda)$ *design* is an incidence structure such that the set of points has cardinality v, every block contains k points and every set of t points is covered by precisely λ blocks.

For example, a *unital* in $\mathsf{PG}(n,q^2)$, as defined in Section 1.3.4, is a $2-(q^3+1,q+1,1)$ design.

> Given a $t-(v,k,\lambda)$ design $(X,\mathcal{B})$, with X the point set and $\mathcal{B}$ the set of all blocks, and a point $x \in X$, we may form its *derived design* (at x) $(X \setminus \{x\}, \{B \setminus \{x\} : x \in B \in \mathcal{B}\})$ which is a $(t-1)-(v-1,k-1,\lambda)$ design. An *extension*, short for a *one-point extension*, of a design D is a design E such that for some point x of E the design D is isomorphic to the derived design of E at x.

The following class of 2-designs, due to G. Hölz, plays an important role in Chapter 3.

Hölz design

In 1981, Hölz [38] constructed a family of $2-(q^3+1,q+1,q+2)$ designs whose point set coincides with the point set of the Hermitian unital over the field $\mathsf{GF}(q^2)$, and with an automorphism group containing $\mathsf{PGU}_3(q)$. In fact,

the blocks of the design are the blocks of the unital and some Baer-conics – those satisfying property (H) (see below) – lying on the unital (viewed as a Hermitian curve in $\mathsf{PG}(2, q^2)$). We will call the blocks corresponding to the conics *Hölz-blocks*.

Let $\mathcal{U}$ be a hermitian curve of $\mathsf{PG}(2, q^2)$.

> A Baer subplane $D \cong \mathsf{PG}(2, q)$ is said to satisfy property (H) if for each point $x \in D \cap \mathcal{U}$ the tangent line L_x to $\mathcal{U}$ at x is a line of D (i.e. $|L_x \cap D| = q + 1$).

If D satisfies property (H), then one can show that if $|D \cap \mathcal{U}| \geq 3$, then $|D \cap \mathcal{U}| = q + 1$. In this case, the points of $D \cap \mathcal{U}$ are collinear if q is even, and the points of $D \cap \mathcal{U}$ are collinear or form an oval in D, if q is odd. If D_1 and D_2 are Baer subplanes satisfying property (H) and if $|D_1 \cap D_2 \cap \mathcal{U}| \geq 3$, then $D_1 \cap \mathcal{U} = D_2 \cap \mathcal{U}$. If moreover $D_i \cap \mathcal{U}$ is an oval of D_i, then $D_1 = D_2$.

Let q be odd. If x and y are distinct points of $\mathcal{U}$, then

(1) there are exactly $q+1$ Baer subplanes D in $\mathsf{PG}(2, q^2)$ that satisfy property (H) and for which $D \cap \mathcal{U} = xy \cap \mathcal{U}$, and

(2) there are exactly $q+1$ Baer subplanes D in $\mathsf{PG}(2, q^2)$ that satisfy property (H) and for which $D \cap \mathcal{U}$ is an oval of D through x and y.

Let B_1 be the set of all intersections $L \cap \mathcal{U}$ with L a non-tangent line of $\mathcal{U}$, and let B' be the set of all intersections $D \cap \mathcal{U}$ with D a Baer subplane of $\mathsf{PG}(2, q^2)$ satisfying property (H) and containing at least three points of $\mathcal{U}$. Finally, let $B^* = B' - B_1$.

Then $S_1 = (\mathcal{U}, B_1, \in)$ is a $2 - (q^3 + 1, q + 1, 1)$ design which we will call the *Hermitian design*; $S' = (\mathcal{U}, B', \in)$ is a $2-(q^3+1, q+1, q+2)$ design, the *Hölz*-design denoted by $\mathsf{D}_{\mathsf{Hölz}}(q)$, and $S^* = (\mathcal{U}, B^*, \in)$ is a $2 - (q^3 + 1, q + 1, q + 1)$ design. Moreover any two distinct blocks of these designs have at most two points in common. We will call the elements of B_1 the *Hermitian blocks*, and the members of B^* the *Hölz-blocks*.

1.6 Generalized Polygons

This Section is for the most part based on *Generalized polygons* by H. Van Maldeghem [67].

An *ordinary n-gon* is a connected geometry with n points and n lines, such that every point is incident with exactly two lines and vice versa. An ordinary 2-gon contains two points incident with each of the two lines.

A *weak generalized n-gon* is a point-line incidence structure such that the incidence graph is a connected bipartite graph with diameter n and girth $2n$, where $n \geq 2$.

If a weak generalized n-gon Γ is thick, then we call it a *generalized n-gon*. As shown in [67] (Lemma 1.3.2), thickness is equivalent with the existence of at least one ordinary $(n+1)$-gon in Γ.

If we do not specify the value of n, then we call this object a *generalized polygon*. If, on the other hand, we specify n, then we are more likely to speak of a *generalized triangle*, *generalized quadrangle*, *generalized hexagon*, and so on, than of a generalized 3-, 4-, 6-gon, respectively.

The dual of a generalized n-gon is also a generalized n-gon. So all definitions and results that hold for points, can be reformulated for lines – and dually.

The definition of Γ implies that any two elements a, b of $\mathcal{P} \cup \mathcal{L}$ are either at distance n from one another, in which case we call them *opposite*, or there exists a unique shortest path from a to b. If, in the latter case, $\gamma = (a, \ldots, b'_a, b)$ denotes this path, then the element b'_a, also denoted by $\mathrm{proj}_b a$, is called the *projection of a onto b*. If two points x and y are at distance 4 and $n > 4$, then the unique point collinear to both is denoted by $x \bowtie y$.

It can be shown, see for instance [67], that in every generalized polygon both the number of points on a line and the number of lines through a point are independent of the choice of the line, respectively the point. For a finite generalized polygon, these numbers are denoted by $s+1$ and $t+1$. With this notation we say that the finite generalized polygon has order (s,t). When dealing with a weak generalized polygon this is not generally true, though some weak polygons do have a constant number of points on a line and a constant number of lines through a point, in which case we still speak of the order of the polygon as defined above.

The integers s and t are also called the *parameters* of the generalized polygon. If s equals t, Γ is also said to have order s. A weak generalized polygon of order (s,t) is called *thin* if one of its parameters equals 1.

A *subpolygon* Γ' of order (s',t') of a generalized n-gon of order (s,t) is a subgeometry of Γ which is itself a generalized n-gon.

If Γ' differs from Γ it is a so-called *proper* subpolygon. A *full* subpolygon is a subpolygon for which $s' = s$, while the term *ideal* subpolygon is reserved

for the situation where $t' = t$. If both parameters s' and t' equal 1, Γ' is an ordinary n-gon and is called an *apartment* of Γ.

We will now briefly explain the geometric structure of some generalized n-gons, that are relevant for this work, short the generalized hexagons. These will be discussed in detail in Chapter 2.

Generalized triangles

In the thick case the generalized triangles are exactly the projective planes.

Generalized quadrangles

One of the most complete references on these objects is *Finite generalized quadrangles* of S.E. Payne and J.A. Thas [45].

For generalized quadrangles we have the following equivalent (more common) definition.

> A finite *generalized quadrangle* Γ *of order* (s,t), with $s,t \geq 1$, is a point-line incidence structure with $s+1$ points on a line and $t+1$ lines through a point, such that for every anti-flag (p,L) there is exactly one flag (M,q) such that $p \mathtt{I} M \mathtt{I} q \mathtt{I} L$.

If $t = 1$ (respectively $s = 1$), then Γ is called a *grid* (respectively *dual grid*). We mention some examples of finite classical generalized quadrangles.

- Let Q be a non-singular quadric in $\mathsf{PG}(d,q)$ of projective index 1. The points and lines of the quadric form a generalized quadrangle that has order $(q,1)$, (q,q) and (q,q^2) if $d = 3,4$ and 5, respectively.
- Let H be a non-singular Hermitian variety in $\mathsf{PG}(d,q^2)$ of projective index 1. The points and lines of the Hermitian variety form a generalized quadrangle that has order (q^2,q) and (q^2,q^3) if $d = 3$ and 4, respectively.
- Let α be a symplectic polarity in $\mathsf{PG}(3,q)$. The points of $\mathsf{PG}(3,q)$ together with the absolute lines of α define a generalized quadrangle $\mathsf{W}(q)$ of order (q,q), called the *symplectic quadrangle*.

Note. $\mathsf{W}(q)$ is a member of a larger class of *symplectic polar spaces* denoted by $\mathsf{W}_n(q)$. In fact, these objects can be described by a generalization of the previous definition of $\mathsf{W}(q)$. Indeed, replace $\mathsf{PG}(3,q)$ by $\mathsf{PG}(n,q)$, with n odd

and $n \geq 3$, consider absolute subspaces of $\mathsf{PG}(n,q)$ of α, with dimension less than or equal to $\frac{n-1}{2}$, instead of absolute lines and obtain a polar space of projective index $\frac{n-1}{2}$.

"The isomorphisms" between the classical generalized quadrangles are the following (see [45]). For all prime powers q, $\mathsf{W}(q) \cong \mathsf{Q}(4,q)^D$ and $\mathsf{Q}(5,q) \cong \mathsf{H}(3,q^2)^D$. If q is even, then we also have that $\mathsf{W}(q) \cong \mathsf{Q}(4,q)$.

In addition to the above mentioned classical generalized quadrangles, we will also be working with the Ahrens-Szekeres generalized quadrangles.
R.W. Ahrens and G. Szekeres [1] constructed, for each prime power q, generalized quadrangles of order $(q-1, q+1)$ as follows. Let $\mathcal{P}$ be the points of the affine 3-space $\mathsf{AG}(3,q)$ over $\mathsf{GF}(q)$. Elements of $\mathcal{L}$ are the following curves of $\mathsf{AG}(3,q)$

(i) $x = \sigma,\ y = a,\ z = b,$

(ii) $x = a,\ y = \sigma,\ z = b,$

(iii) $x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma,$

where the parameter σ ranges over $\mathsf{GF}(q)$ and where a, b, c are arbitrary elements of $\mathsf{GF}(q)$. The incidence $\mathtt{I}$ is the natural one. Then $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized quadrangle of order $(q-1, q+1)$, which will be denoted by $\mathsf{AS}(q)$.

In 1983, J.A. Thas proved that the Hölz designs, as described in Section 1.5, are one-point extensions of these Ahrens-Szekeres generalized quadrangles [62].

Restriction on the parameters

The following theorems collect a number of restrictions on the parameters of finite generalized polygons. We start with a combined result by Feit and Higman [27], Higman [34], Haemers and Roos [32] and Dembowski [14].

Theorem 1.6.1.
Let Γ be a finite generalized n-gon of order (s,t), $s > 1$, $t > 1$, with $n \geq 2$. If Γ is finite, then one of the following holds (with $|\mathcal{P}| = v$ and $|\mathcal{L}| = b$):

- *$n = 2$ with $b = t + 1$, $v = s + 1$;*
- *$n = 3$ and $s = t$ with $v = b = s^2 + s + 1$; Γ is a projective plane;*
- *$n = 4$ and $\frac{st(1+st)}{s+t}$ is an integer; $s \leq t^2$ and $t \leq s^2$;*
- *$n = 6$ and st is a square; $s \leq t^3$ and $t \leq s^3$;*
- *$n = 8$ and $2st$ is a square, in particular $s \neq t$; $s \leq t^2$ and $t \leq s^2$.*

If n is even, then

$$v = (1+s)(s + st + (st)^2 + \ldots + (st)^{\frac{n}{2}-1}),$$

$$b = (1+t)(s + st + (st)^2 + \ldots + (st)^{\frac{n}{2}-1}).$$

Furthermore, from Thas [55], [56], [57], [58] and Van Maldeghem [67] (1.8.8) we know the following.

Theorem 1.6.2.
Let Γ' be a proper ideal sub-n-gon of order (s', t) of a finite thick generalized n-gon Γ of order (s, t). Then one of the following cases occurs.

- $n = 4$ *and* $s \geq s't$ *and* $s \geq t \geq s'$;
- $n = 6$ *and* $s \geq s'^2 t$ *and* $s \geq t \geq s'$;
- $n = 8$ *and* $s' = 1$ *and* $s > t$.

2 Classical generalized hexagons

Having introduced some necessary preliminary concepts, definitions and notations, and having briefly discussed relevant characteristics of generalized polygons in the preceding chapter, we can now turn our attention to generalized hexagons, and, more especially to a particular class of generalized hexagons, the split Cayley hexagons, denoted by $\mathsf{H}(q)$. This class of generalized hexagons can be defined on the parabolic quadric $\mathsf{Q}(6,q)$, a construction due to J. Tits (see [64]), and corresponds to the exceptional groups of Lie type G_2 (Dickson's groups).

The split Cayley hexagon is the standard example of a generalized hexagon. In the finite case there is, up to duality, only one other class of generalized hexagons known, the twisted triality hexagons, introduced in one of the following sections. Both classes of hexagons can be constructed using trialities. One might even remark that it would have been sufficient to give the definition of the triality hexagons and to state $\mathsf{H}(q)$ as a special case of this particular class of hexagons. However, as the split Cayley hexagon and all of its significations will be the main objects of our attention, it would have been brutally unfair not to treat them in a section on its own.

After having sketched the construction of J. Tits, we explain the coordinatization of $\mathsf{H}(q)$ (see [16]). In the rest of this chapter we introduce notions like point and line reguli, tangent, hyperbolic and elliptic hyperplanes, distance-j ovoids and distance-j spreads (with a special interest in Hermitian and Ree-Tits ovoids and spreads), elations, list some relations between objects of $\mathsf{Q}(6,q)$ and objects of $\mathsf{H}(q)$, and investigate some properties of the automorphism groups of generalized hexagons.

A great part of this chapter is based on the monograph *Generalized Polygons* by H. Van Maldeghem [67].

2.1 Introduction

From Section 1.6 we know that

> a *generalized hexagon* (of order (s,t)) is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and every line contains $s+1$ points and every point is incident with $t+1$ lines).

Clearly, the maximal distance between two elements of a generalized hexagon is 6 and such elements are said to be *opposite*.

Up to duality, only two classes of examples of finite generalized hexagons are known: the *split Cayley hexagons* $\mathsf{H}(q)$ of order q and the *twisted triality hexagons* $\mathsf{T}(q^3,q)$ of order (q^3,q). These two are often called *classical* because they naturally live on classical objects (namely quadrics) in projective space. They first appeared in a paper of J. Tits [64]. In the next section we sketch this construction and refer to Section 2.4 of [67] for more details.

2.2 Construction

Let q be a prime power and consider the non-singular hyperbolic quadric $\mathsf{Q}^+(7,q)$ in $\mathsf{PG}(7,q)$. The generators of this particular quadric are three-dimensional subspaces. The set of generators can be divided into two families $\mathcal{P}_1$ and $\mathcal{P}_2$ such that generators belong to the same family if and only if they intersect in a subspace of odd dimension (i.e. in the empty set, in a line or in a three-dimensional subspace (in which case they coincide)).

Let $\mathcal{P}_0$ be the set of points of $\mathsf{Q}^+(7,q)$ and call the elements of $\mathcal{P}_i$ the i-points, $i \in \{0,1,2\}$; let $\mathcal{L}$ denote the set of lines on $\mathsf{Q}^+(7,q)$. A counting argument yields $|\mathcal{P}_0| = |\mathcal{P}_1| = |\mathcal{P}_2|$.

Incidence between a 0-point and an i-point, $i \in \{1,2\}$, or between a line and an i-point, $i \in \{0,1,2\}$, is defined as (reverse) containment, while a 1-point and a 2-point are said to be incident if and only if they intersect in a plane.

A *triality of* $\mathsf{Q}^+(7,q)$ is a bijection θ that maps

$$\mathcal{P}_0 \to \mathcal{P}_1, \qquad \mathcal{P}_1 \to \mathcal{P}_2, \qquad \mathcal{P}_2 \to \mathcal{P}_0, \qquad \mathcal{L} \to \mathcal{L},$$

which is incidence preserving and which satisfies $\theta^3 = \mathbb{1}$.

An *absolute i-point* is an i-point that is incident with its image under the triality θ, $i \in \{0,1,2\}$; an *absolute line* is a line that is fixed by θ.

For certain trialities the incidence structure formed by the absolute i-points, for an $i \in \{0,1,2\}$ and the absolute lines, with given incidence, is a generalized hexagon (see [64]).

Independent of the choice of i the same generalized hexagon (up to isomorphism) is obtained. In [67] Section 2.4, the explicit description of such a triality is given. It involves an automorphism σ of $\mathsf{GF}(q)$ with $\sigma^3 = \mathbb{1}$. The order of the generalized hexagon arising from the triality is (q,q'), where q' is the order of the fixed field $\{x \in \mathsf{GF}(q) \mid x^\sigma = x\}$ of σ. Replacing σ by its inverse yields an isomorphic generalized hexagon.

> By choosing $\sigma = \mathbb{1}$, one finds that the generalized hexagon of order q exists for any prime power q; it is called the *split Cayley hexagon* $\mathsf{H}(q)$.

The reason for that name is that this hexagon can also be constructed using a split Cayley algebra over $\mathsf{GF}(q)$ (see for instance [50] and [51]). Moreover, the associated simple algebraic group is also split.

> Now let σ be the automorphism $x \to x^q$ of $\mathsf{GF}(q^3)$, where q is any prime power. The associated triality of $\mathsf{Q}^+(7,q^3)$ yields a generalized hexagon $\mathsf{T}(q^3,q)$ of order (q^3,q) called the *twisted triality hexagon* and we denote the dual of $\mathsf{T}(q^3,q)$ by $\mathsf{T}(q,q^3)$.

By restricting the coordinates to $\mathsf{GF}(q)$, which is precisely the fixed field of σ, one shows that $\mathsf{H}(q)$ is a subhexagon of $\mathsf{T}(q^3,q)$.

Note. This construction was established for an arbitrary field $\mathbb{K}$. Here and from now on, we restrict ourselves to the Galois field $\mathsf{GF}(q)$.

Tits' description of $\mathsf{H}(q)$

In [64] J. Tits proved that all points of the split Cayley hexagon represented on the quadric $\mathsf{Q}^+(7,q)$ lie in a certain hyperplane of $\mathsf{PG}(7,q)$. This simple

fact allows us to construct $\mathsf{H}(q)$ in a more direct way on a parabolic quadric of $\mathsf{PG}(6,q)$. Choose coordinates in this projective space in such a way that $\mathsf{Q}(6,q)$ has equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$$

and let the points of $\mathsf{H}(q)$ be all points of $\mathsf{Q}(6,q)$. The lines of the hexagon are exactly the lines on $\mathsf{Q}(6,q)$ whose Grassmann coordinates satisfy the following six linear equations

$$p_{12} = p_{34} \qquad p_{54} = p_{32} \qquad p_{20} = p_{35}$$

$$p_{65} = p_{30} \qquad p_{01} = p_{36} \qquad p_{46} = p_{31}$$

This representation of $\mathsf{H}(q)$ is generally known as the *standard embedding* of $\mathsf{H}(q)$.

If q is odd, then the orthogonal polarity associated with this particular $\mathsf{Q}(6,q)$ is determined by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If q is even, the polar space $\mathsf{Q}(6,q)$ is isomorphic to the symplectic polar space $\mathsf{W}_5(q)$ (obtained by projection from the nucleus n of $\mathsf{Q}(6,q)$ onto some hyperplane not containing n). This substantiality results in an embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, where all lines of $\mathsf{H}(q)$ are totally isotropic with respect to a certain symplectic polarity in $\mathsf{PG}(5,q)$ (here $n = (0,0,0,1,0,0,0)$ and choosing the hyperplane with equation $X_3 = 0$, the associated symplectic form is $X_0Y_4 + X_4Y_0 + X_1Y_5 + X_5Y_1 + X_2Y_6 + X_6Y_2$). In this representation $\mathsf{H}(q)$ is sometimes called the *perfect symplectic hexagon*.

The following classification theorem is due to A.M. Cohen and J. Tits.

Theorem 2.2.1 (Cohen, Tits (10)).
A finite generalized hexagon of order (s,t) with $s = 2$ is isomorphic to one of the classical hexagons $\mathsf{H}(2)$, $\mathsf{H}(2)^D$ or $\mathsf{T}(2,8)$.

Meanwhile, when dealing with $\mathsf{H}(q)$ we will see that $\mathsf{H}(3^h)$ plays a special role. The reason for this essentially lies in an unpublished theorem of J. Tits.

Theorem 2.2.2 (Corollary 3.5.7 (67)**).**
The generalized hexagon $\mathsf{H}(q)$ *is self-dual if and only if* $q = 3^h$ *and it is self-polar if and only if* $q = 3^{2e+1}$.

Despite the fact that we do not include the proof of this theorem, the *if*-part will become crystal clear after reading the next section.

2.3 Coordinatization

The coordinatization of generalized $2n$-gons is a generalization of Hall's coordinatization of projective planes (see [33]). The general theory is discussed in [67], and for generalized hexagons in particular we rather refer to a prior paper by V. De Smet and H. Van Maldeghem [16]. Such a coordinatization makes the quite abstract description of Section 2.2 a bit more transparent. However, even though it simplifies some proofs, we do attempt to limit the use of these coordinates as one could easily get lost in tedious calculations and alienate oneself from the geometrical touch of the situation.

In the next few pages we sketch the general rules for coordinatization of a generalized hexagon Γ of order (s, t).

We consider two sets R_1 and R_2 of coordinates, with respective cardinalities s and t, and assume that both of them contain two distinct elements denoted by 0 and 1. Let it be clear that R_1 will be used to label the points on a line except for one, and R_2 to label lines through a point except for one.

Choose an apartment $\mathcal{A}$ and fix a flag in $\mathcal{A}$ labelling it $\{(\infty), [\infty]\}$. The coordinates of the other elements of $\mathcal{A}$ are determined by

$$[\infty] \ \mathtt{I} \ (0) \ \mathtt{I} \ [0,0] \ \mathtt{I} \ (0,0,0) \ \mathtt{I} \ [0,0,0,0] \ \mathtt{I} \ (0,0,0,0,0) \ \mathtt{I}$$

$$[0,0,0,0,0] \ \mathtt{I} \ (0,0,0,0) \ \mathtt{I} \ [0,0,0] \ \mathtt{I} \ (0,0) \ \mathtt{I} \ [0] \ \mathtt{I} \ (\infty)$$

where $\mathtt{I}$ denotes the incidence relation of Γ.

We obtain a labelling of the points and lines of Γ by i-tuples with entries in R_1 and R_2, respectively. An i-tuple associated with a point will be denoted between parentheses, while an i-tuple corresponding to a line will be placed between square brackets. Here the number $i \in \{1, \ldots, 5\}$ is determined by the distance to the special flag $\{(\infty), [\infty]\}$. Namely, if x is an element different from (∞) and $[\infty]$ and

$$i = \min\{d(x, (\infty)), d(x, [\infty])\}$$

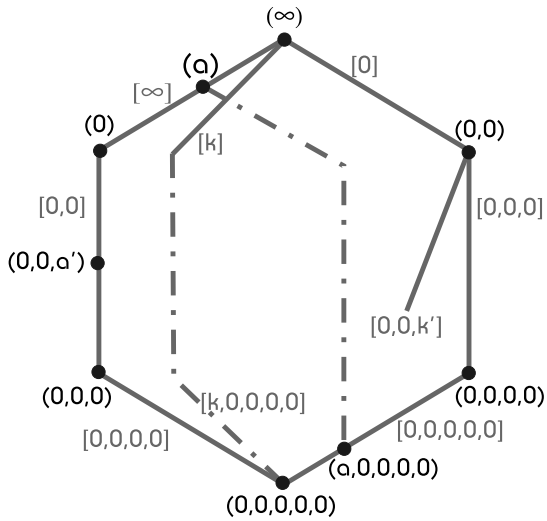

Figure 2.1: Coordinatization of H(q)

then x is represented by an i-tuple.

We now want the incidence to be as clearly visible as possible. This is achieved by the following procedure.

Label the points on $[\infty]$, in a bijective way and consistently with (0), by (a). $a \in R_1$. Dually, the lines through (∞) and distinct of $[\infty]$ are labelled $[k]$, $k \in R_2$. The unique point on $[0,0,0,0,0]$ nearest to (a) we label $(a,0,0,0,0)$ and dually we define $[k,0,0,0,0]$.

From here on, to simplify notation, we will assume a, b, a', b', a'' to be elements of R_1 and k, l, k', l', k'' to be elements of R_2.

Let p_a be the point on $[1,0,0,0,0]$ nearest to (a). The point on $[0]$ at distance 4 from p_a is labelled $(0,a)$, and to the point on $[0,0,0,0]$ nearest to $(0,a)$ we attach the label $(0,0,0,0,a)$. Dually, we get $[0,k]$ through (0) and $[0,0,0,0,k]$ through $(0,0,0,0)$. Next, we bijectively assign to each point on $[0,0]$ apart from (0), a coordinate $(0,0,a')$. Let $(0,0,0,a')$ be the point on $[0,0,0]$ nearest to $(0,0,a')$. Dually, we obtain $[0,0,k']$ and $[0,0,0,k']$.

We have now labelled all elements at distance one of an element of $\mathcal{A}$ (see Figure 2.1 for clarification). With these labels at hand we are now able to complete the coordinatization as follows.

The point on $[k]$ nearest to $(0,0,0,0,b)$ we label (k,b) (dually $[a,l]$ is defined); the point on $[a,l]$ nearest to $(0,0,0,a')$ we label (a,l,a') (dually $[k,b,k']$ is defined); the point on $[k,b,k']$ nearest to $(0,0,b')$ we label (k,b,k',b') (dually $[a,l,a',l']$ is defined); and finally the point on $[a,l,a',l']$ nearest to $(0,a'')$ we label (a,l,a',l',a'') (dually $[k,b,k',b',k'']$ is defined).

If we consider the 1-tuples (∞) and $[\infty]$ formally as (empty) 0-tuples (because they do not contain an element of $R_1 \cup R_2$), then a point, represented by an i-tuple, $0 \leq i \leq 5$, is incident with a line, represented by a j-tuple, $0 \leq j \leq 5$, with $i + j < 10$, if and only if $|i - j| = 1$ and the tuples coincide in the first $\min\{i,j\}$ entries. There is however, sadly enough, no such simple rule to describe the incidence between two 5-tuples of Γ. In that case, we introduce the following algebraic operations

$$\begin{aligned}
S_1(k,a,l,a',l',a'') = b &\Leftrightarrow d((k,b),(a,l,a',l',a'')) = 4,\\
S_2'(k,a,l,a',l',a'') = k' &\Leftrightarrow d([k,S_1(k,a,l,a',l',a''),k'],(a,l,a',l',a'')) = 3,\\
S_2(a,k,b,k',b',k'') = l &\Leftrightarrow d([a,l],[k,b,k',b',k'']) = 4,\\
S_1'(a,k,b,k',b',k'') = a' &\Leftrightarrow d((a,S_2(a,k,b,k',b',k''),a'),[k,b,k',b',k'']) = 3.
\end{aligned}$$

It is now easy to verify that a necessary and sufficient condition for (a,l,a',l',a'') and $[k,b,k',b',k'']$ to be incident is

$$\begin{aligned}
S_1(k,a,l,a',l',a'') &= b\\
S_2'(k,a,l,a',l',a'') &= k'\\
S_2(a,k,b,k',b',k'') &= l\\
S_1'(a,k,b,k',b',k'') &= a'
\end{aligned}$$

for any $a,a',a'',b,b' \in R_1$ and $k,k',k'',l,l' \in R_2$. This final step in the coordinatization completely determines the generalized hexagon.

Split Cayley Hexagon

Although the split Cayley hexagons and twisted triality hexagons could be coordinatized at the same time, we treat only the split Cayley hexagon and omit the coordinatization of the twisted triality hexagon (see Section 3.5.8 in [67] if needed).

Consider $\mathsf{H}(q)$ in its standard embedding. We put $R_1 = R_2 = \mathsf{GF}(q)$ and describe the relation between the coordinates of $\mathsf{H}(q)$ and those of $\mathsf{PG}(6,q)$. Let us start by defining all points of the apartment $\mathcal{A}$,

$$\begin{array}{lcl}
(1,0,0,0,0,0,0) & \to & (\infty), \\
(0,0,0,0,0,1,0) & \to & (0,0), \\
(0,1,0,0,0,0,0) & \to & (0,0,0), \\
(0,0,1,0,0,0,0) & \to & (0,0,0,0), \\
(0,0,0,0,1,0,0) & \to & (0,0,0,0,0),
\end{array}$$

hereby also defining the lines of $\mathcal{A}$ (as in Figure 2.1) and fix the above mentioned bijections as follows

$$\begin{array}{lcl}
(a,0,0,0,0,0,1) & \to & (a), \\
(0,1,0,0,0,0,-a') & \to & (0,0,a'), \\
X_1 = X_2 = X_3 = X_4 = X_6 + kX_5 = 0 & \to & [k], \\
X_0 - k'X_2 = X_1 = X_3 = X_4 = X_6 = 0 & \to & [0,0,k'].
\end{array}$$

Note. When introducing such coordinates for $\mathsf{H}(q)$, there will be no confusion as the standard coordinate tuple of a point in $\mathsf{PG}(6,q)$ has seven entries.

With these coordinates we can rigorously assign labels to every point and line of $\mathsf{H}(q)$. A way to go back and forth between the 6-dimensional space and the hexagon is summarized in Table 2.1.

One can now calculate that a point (a,l,a',l',a'') is incident with a line $[k,b,k',b',k'']$ if and only if

$$\begin{cases}
k'' = a^3k + l - 3a''a^2 + 3aa', \\
b' = a^2k + a' - 2aa'', \\
k' = a^3k^2 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2, \\
b = -ak + a'',
\end{cases}$$

or, equivalently,

$$\begin{cases}
a'' = ak + b, \\
l' = a^3k^2 + k' + kk'' + 3a^2kb + 3bb' + 3ab^2, \\
a' = a^2k + b' + 2ab, \\
l = -a^3k + k'' - 3ba^2 - 3ab'.
\end{cases}$$

As these relations contain many terms with the coefficient 3 they become considerably more simple when the underlying field has characteristic 3. More explicitly, when $q = 3^h$, a point (a,l,a',l',a'') and a line $[k,b,k',b',k'']$ are incident if and only if

$$\begin{cases}
k'' = a^3k + l, \\
b' = a^2k + a' + aa'', \\
k' = a^3k^2 + l' - kl, \\
b = -ak + a'',
\end{cases}$$

POINTS

coordinates in $\mathsf{H}(q)$	coordinates in $\mathsf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,$
	$-a'+aa'',1,l+2aa'-a^2a'',-l'+a'a'')$

LINES

coordinates in $\mathsf{H}(q)$	coordinates in $\mathsf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$
	$(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$
	$(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 2.1: Coordinatization of $\mathsf{H}(q)$

or, equivalently,

$$\begin{cases} a''=ak+b, \\ l'=a^3k^2+k'+kk'', \\ a'=a^2k+b'-ab, \\ l=-a^3k+k''. \end{cases}$$

As noted above, with these formulae, the self-duality of $\mathsf{H}(3^h)$ becomes apparent. Indeed, the mapping

$$\sigma:\begin{cases}(a,l,a',l',a'')\to[a,l,a',l',a''] \\ [k,b,k',b',k'']\to(k^3,b^3,k'^3,b'^3,k''^3)\end{cases}$$

is a duality of order $2h$. Moreover, if h is an odd number, then σ^h is a polarity of $\mathsf{H}(3^h)$.

2.4 Some notations and terminology

The generalized hexagon $\mathsf{H}(q)$ has the following property (see [67], 1.9.17 and 2.4.15). Let x, y be two opposite points and let L, M be two opposite lines at distance 3 from both x, y. All points at distance 3 from both L, M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x, y)$ of $q + 1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L, M)$ of $q + 1$ lines.

> We call $\mathcal{R}(x, y)$ a *point regulus*, and $\mathcal{R}(L, M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus.

Let $\mathcal{H}$ be a hyperplane in $\mathsf{PG}(6, q)$. Then exactly one of the following cases occurs.

(i) The points of $\mathsf{H}(q)$ in $\mathcal{H}$ are the points not opposite a given point x of $\mathsf{H}(q)$; in fact, $\mathcal{H}$ is the *tangent hyperplane* of $\mathsf{Q}(6, q)$ at x.

(ii) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a subhexagon of $\mathsf{H}(q)$ of order $(1, q)$, the points of which evidently are those points of $\mathsf{H}(q)$ that are incident with exactly $q + 1$ lines of $\mathsf{H}(q)$ lying in $\mathcal{H}$. This subhexagon is uniquely determined by any two opposite points x, y it contains and will be denoted by $\Gamma(x, y)$. It contains exactly $2(q^2+q+1)$ points and if collinearity is called adjacency, then it can be viewed as the incidence graph of the Desarguesian projective plane $\mathsf{PG}(2, q)$ of order q. The lines of $\Gamma(x, y)$ can be identified with the incident point-line pairs of that projective plane. We denote $\Gamma(x, y)$ by $2\mathsf{PG}(2, q)$ and call it the *double of* $\mathsf{PG}(2, q)$. The $q^2 + q + 1$ points of $\Gamma(x, y)$ belonging to the same type of elements of $\mathsf{PG}(2, q)$, points or lines, are the points of a projective plane in $\mathsf{PG}(6, q)$. Hence $\mathcal{H} \cap \mathsf{Q}(6, q)$ contains two projective planes Π^+ and Π^-, the points of which are precisely the points of $\Gamma(x, y)$, and which we call the *hexagon twin planes* of $\mathcal{H}$. In this case, we call $\mathcal{H}$ a *hyperbolic hyperplane*. In fact, a hyperbolic hyperplane is a hyperplane that intersects $\mathsf{Q}(6, q)$ in a non-degenerate hyperbolic quadric.

(iii) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a spread, called a *Hermitian* or *classical* spread of $\mathsf{H}(q)$ (which will be defined in one of the next sections). In this case, we call $\mathcal{H}$ an *elliptic hyperplane* (as it intersects $\mathsf{Q}(6, q)$ in an elliptic quadric).

2.5 The underlying quadric $\mathsf{Q}(6, q)$

Throughout we will assume $\mathsf{H}(q)$ to be given by the standard embedding on $\mathsf{Q}(6, q)$. Thus points and lines of $\mathsf{H}(q)$ will be used equally freely as points and lines of $\mathsf{Q}(6, q)$. To avoid confusion, we will make a distinction where necessary by referring to those that are lines of $\mathsf{H}(q)$ as *hexagon lines* and to those that are not as *ideal lines*.

The generators on the quadric $\mathsf{Q}(6, q)$ are planes. Such a plane either contains the $q + 1$ hexagon lines through a point x or contains no hexagon line at all. In the first case we call the plane a *hexagon plane*, and denote it by Π_x. In the second case we call the plane an *ideal plane*.

Note. The same plane Π can never be the hexagon plane of two distinct points x and y as otherwise $\mathsf{H}(q)$ would contain triangles xyz, where z is any point of Π not on the line xy.

For every ideal plane Π, there is a unique ideal plane Π' – called the *hexagon twin* of Π – with the property that $\Pi \cup \Pi'$ is the point set of a subhexagon of order $(1, q)$ of $\mathsf{H}(q)$. Equivalently, every point x of Π is collinear (in $\mathsf{H}(q)$) with exactly $q + 1$ points of Π', and vice versa. These $q + 1$ points form a line L in Π'. The map $x \mapsto L$ from the point set of Π to the line set of Π' defines a unique anti-isomorphism from Π to Π', called the *hexagon twin anti-isomorphism*.

We now list some – easy to prove – relations between objects of $\mathsf{H}(q)$ and objects of $\mathsf{Q}(6, q)$.

> If L is a hexagon line, then the generators of $\mathsf{Q}(6, q)$ on L are precisely the hexagon planes of the points on L.

If L is an ideal line, then there is a unique hexagon plane containing L. The point x associated with that hexagon plane is called the *focus* of L.

> Two points of $\mathsf{H}(q)$ are opposite if and only if they are non-collinear on the quadric.

Two lines of $\mathsf{H}(q)$ are opposite if and only if every generator of $\mathsf{Q}(6, q)$ containing one is disjoint of the other.

> All points of an ideal plane are mutually at distance 4 in the hexagon.

On the quadric $\mathsf{Q}(6,q)$, every line regulus constitutes a hyperbolic quadric isomorphic to $\mathsf{Q}^+(3,q)$. Hence there is a unique *opposite regulus*, which is a set of $q+1$ ideal lines that intersect every line of the given regulus in a unique point.

If q is odd, the quadric $\mathsf{Q}(6,q)$ is associated with a unique non-degenerate polarity ρ of $\mathsf{PG}(6,q)$ and the image under the polarity of the 3-space generated by a line regulus is a plane that meets $\mathsf{Q}(6,q)$ exactly in the complementary point regulus. Point reguli of $\mathsf{H}(q)$ thus are simply (some) conics on $\mathsf{Q}(6,q)$. The plane in which all points of a point regulus $\mathcal{R}$ are contained will be referred to as the *regulus plane* $\alpha_{\mathcal{R}}$.

2.6 Distance-j ovoids and distance-j spreads

2.6.1 General definitions

Let Γ be a generalized hexagon of order (s,t). In general

a *distance-j ovoid*, $1 \leq j \leq 3$, of Γ is a set, $\mathcal{O}_j$, of points such that any two points of $\mathcal{O}_j$ are at distance at least $2j$ and every element of Γ is at distance at most j from at least one element of $\mathcal{O}_j$. The dual objects are called *distance-j spreads*.

For specified values of j there are more common definitions known.

A *distance-2 ovoid* of Γ is a set, $\mathcal{O}_2$, of points such that every line of Γ is incident with exactly one element of $\mathcal{O}_2$. Dually one defines a *distance-2 spread*, which is in fact a partition of the point set of Γ into lines.

It is easy to see that distance-2 ovoids and distance-2 spreads of a generalized hexagon of order (s,t) have s^2t^2+st+1 elements. Despite the fact that this definition is a very natural one, there are no canonical examples of distance-2 ovoids or distance-2 spreads. In fact, for a long time the only known partition into lines of the point set of a finite generalized hexagon occured in the dual of the classical generalized hexagon $\mathsf{H}(2)$. This partition can be constructed as follows. Let Γ' be a full weak subhexagon of order $(2,1)$ of the dual of $\mathsf{H}(2)$. Then one can easily check that the set of lines not belonging to Γ',

but incident with a point of Γ', provides a distance-2 spread of $\mathsf{H}(2)^D$. This example admits a fairly big automorphism group (isomorphic to $\mathsf{Aut}\,\mathsf{PSL}_3(2)$) and has many beautiful geometric properties (see for instance [46]). Moreover it is, up to isomorphism, the unique distance-2 spread of $\mathsf{H}(2)^D$.

Note. A *line spread* of $\mathsf{Q}(6,q)$ is a set of lines that partitions the point set of $\mathsf{Q}(6,q)$. On the other hand, every point of $\mathsf{H}(q)$ is incident with exactly one line of a distance-2 spread. Therefore by construction of $\mathsf{H}(q)$ every distance-2 spread of $\mathsf{H}(q)$ is a line spread of $\mathsf{Q}(6,q)$. The converse is not necessarily true, since not all lines of the quadric belong to the hexagon.

> A so-called *ovoid*, short for distance-3 ovoid, of Γ is a set of $\frac{(1+s)(1+st+s^2t^2)}{1+s+st} = 1 + s^2t$ opposite points (see Proposition 7.2.3 in [67] and [42]). Dually one defines a *spread*.

By A. Offer [42], if Γ admits an ovoid, then Γ has order s and

> an ovoid (respectively spread) of Γ is a set of $s^3 + 1$ opposite points (respectively lines).

Note. All stated specifications for ovoids apply for spreads – by dualization – as well.

From the graph theoretic point of view there exists a one-to-one correspondence between the ovoids of Γ and the perfect codes of the point graph of Γ and this is a direct consequence of the following lemma.

Lemma 2.6.1 (Lemma 7.2.2 of (67)).
Let $\mathcal{O}$ be a set of points of a generalized hexagon Γ. Then $\mathcal{O}$ is an ovoid if and only if every point of Γ lies at distance less than 3 from a unique element of $\mathcal{O}$.

With this lemma, one can easily see that the closed balls of radius 1 centered in the vertices corresponding to the points of $\mathcal{O}$ fill the whole vertex set of the point graph without any overlap.

In $\mathsf{H}(q)$, spreads and ovoids contain exactly $q^3 + 1$ elements, which is also the number of elements in an m-system of the underlying quadric $\mathsf{Q}(6,q)$. This is no coincidence as they are m-systems of $\mathsf{Q}(6,q)$. More precisely, an ovoid of $\mathsf{H}(q)$ is a 0-system (or ovoid) of $\mathsf{Q}(6,q)$ and a spread of $\mathsf{H}(q)$ is a 1-system of the quadric (see [60] and [54], respectively). Furthermore, the hexagon planes of the points of an ovoid $\mathcal{O}$ of $\mathsf{H}(q)$ are two by two disjoint as a point common to two hexagon planes would be at distance 2 from the two corresponding points of $\mathcal{O}$. Since there is the right number of them, these hexagon planes form a 2-system, or a spread, of $\mathsf{Q}(6,q)$. In conclusion,

Lemma 2.6.2.
If $\mathcal{S}$ is a spread of $\mathsf{H}(q)$, then it is also a 1-system of the quadric $\mathsf{Q}(6,q)$. The set $\mathcal{O}$ is an ovoid of $\mathsf{H}(q)$ if and only if it is an ovoid of $\mathsf{Q}(6,q)$. If $\mathcal{O}$ is an ovoid of $\mathsf{H}(q)$, then the hexagon planes of its points form a spread of $\mathsf{Q}(6,q)$.

2.6.2 Hermitian spreads and ovoids

In this subsection we elaborate on the *Hermitian spreads* of $\mathsf{H}(q)$ as constructed by J.A Thas [59]. This type of spread is called *Hermitian* because endowed with the line reguli entirely contained in it, it is isomorphic to a Hermitian unital. Hermitian spreads exist for all values of q, and when q is a power of 3 we obtain (by self-duality of the hexagon) so-called *Hermitian ovoids* of $\mathsf{H}(q)$.

Consider $\mathsf{H}(q)$ in its standard embedding and let $\mathcal{H}$ be a hyperplane of $\mathsf{PG}(6,q)$ meeting $\mathsf{Q}(6,q)$ in an elliptic quadric $\mathsf{Q}^-(5,q)$.

> The hexagon lines of $\mathcal{H}$ form a spread of $\mathsf{H}(q)$, as we will see. Such a spread is called a *Hermitian spread* of $\mathsf{H}(q)$.

First of all, as the projective index of $\mathsf{Q}^-(5,q)$ is 1 it contains no planes. Hence all hexagon lines contained in this elliptic subquadric are opposite lines of $\mathsf{H}(q)$. Take now any line M off $\mathsf{Q}^-(5,q)$ and denote the intersection point of M with $\mathcal{H}$ by x. The hexagon plane Π_x intersects $\mathcal{H}$ in a unique line L. This line is by definition a hexagon line, meaning M is concurrent with a unique hexagon line contained in $\mathsf{Q}^-(5,q)$. Through the dualized version of Lemma 2.6.1 we find that indeed this line set determines a spread of $\mathsf{H}(q)$.

Let L and M be any two lines of a Hermitian spread $\mathcal{S}_\mathsf{H}$ (where H stands for Hermitian). Since both L and M, and hence also the three-space they span, belong to the defining hyperplane of $\mathcal{S}_\mathsf{H}$, all lines of the regulus $\mathcal{R}(L,M)$ are lines of the spread as well. After dualization, this property states that for any two points x and y of a Hermitian ovoid $\mathcal{O}_\mathsf{H}$, all points of the point regulus $\mathcal{R}(x,y)$ belong to $\mathcal{O}_\mathsf{H}$.

With coordinates, a Hermitian spread may be defined by the hyperplane $\Pi : \nu X_1 + X_5 = 0$, which is clearly an elliptic hyperplane of $\mathsf{PG}(6,q)$ if and only if $-\nu$ is a non-square (and q is odd). The lines of the corresponding spread have the following coordinates

$$\mathcal{S}_\mathsf{H} = \{[\infty]\} \cup \{[k,b,k',-\nu k,\nu b] \mid k,b,k' \in \mathsf{GF}(q)\}.$$

As the Hermitian spread is linked to a Hermitian unital in $\mathsf{PG}(2, q^2)$ mapping reguli to blocks (see next paragraph), it follows that the automorphism group of $\mathcal{S}_\mathsf{H}$ is isomorphic to $\mathsf{SU}_3(q) : 2$.

Hermitian spread vs. Hermitian unital

For further reference we will explicitly determine the relationship between the Hermitian spread $\mathcal{S}_\mathsf{H}$ and a Hermitian unital, $\mathcal{U}$, in $\mathsf{PG}(2, q^2)$. Let $\gamma \in \mathsf{GF}(q^2)$ be such that $\gamma^2 = -\nu$. We already established that the lines of $\mathsf{H}(q)$ in the hyperplane Π with equation $X_5 = -\nu X_1$ in $\mathsf{PG}(6, q)$ form the Hermitian spread

$$\mathcal{S}_\mathsf{H} = \{[\infty]\} \cup \{[k, b, k', -\nu k, \nu b] : k, b, k' \in \mathsf{GF}(q)\}$$

in $\mathsf{H}(q)$.

Now we extend $\mathsf{PG}(6, q)$ to $\mathsf{PG}(6, q^2)$, thereby also extending $\mathsf{Q}(6, q)$ to $\mathsf{Q}(6, q^2)$ (having the same equation) and $\mathsf{H}(q)$ to $\mathsf{H}(q^2)$ (the Grassmann coordinates of the lines of $\mathsf{H}(q^2)$ satisfy exactly the same six equations as is the case for $\mathsf{H}(q)$). Let σ be the involution in $\mathsf{H}(q^2)$ defined by applying the map $x \to x^q$ to every coordinate of any element in $\mathsf{H}(q^2)$. It is obvious that σ fixes $\mathsf{H}(q)$ point-wise.

By [68], the hyperplane Π, viewed as a hyperplane of $\mathsf{PG}(6, q^2)$, defines in $\mathsf{H}(q^2)$ the subhexagon $\Gamma(p, p')$ of order $(1, q^2)$ of $\mathsf{H}(q^2)$ (with notation of Section 2.4) and $\Gamma(p, p') \cap \mathsf{H}(q) = \mathcal{S}_\mathsf{H}$, where p is a point of $\mathsf{H}(q^2) \setminus \mathsf{H}(q)$ on $[\infty]$ and p' is the point on $[0, 0, 0, 0, 0]$ at distance 5 from p^σ. We know that $\Gamma(p, p')$ is the double of a Desarguesian projective plane $\Pi_{p,p'}$. Let π^+ (respectively π^-) be the plane of $\mathsf{PG}(6, q^2)$ generated by the points p, p'^σ and $p^\sigma \bowtie p'$ (respectively p^σ, p' and $p'^\sigma \bowtie p$). We know that π^+ and π^- can be thought of as the point set and the line set, respectively, of $\mathsf{PG}(2, q^2)$. And such a point is incident with such a line if they are, as points of $\mathsf{PG}(2, q^2)$, on a line of $\mathcal{S}_\mathsf{H}$.

According to [61], the lines of $\mathcal{S}_\mathsf{H}$ meet the plane π^+ in the points of a Hermitian curve, which we will call $\mathcal{U}$. We now establish an explicit algebraic correspondence. We may choose p to be the point (γ) on the line $[\infty]$. Hence π^+ is generated by the points $p = (\gamma)$, $p'^\sigma = (\gamma, 0, 0, 0, 0)$ and $p' \bowtie p^\sigma = (-\gamma, 0, 0)$ of $\mathsf{H}(q^2)$. Let $\overline{p}_0$ be the fixed coordinate tuple $(\gamma, 0, 0, 0, 0, 0, 1)$ of p. Likewise let $\overline{p}_1 = (0, 0, -\gamma, 0, 1, 0, 0)$, be the fixed representative of p'^σ, and then we have $\overline{p}_2 = (0, 1, 0, \gamma, 0, \gamma^2, 0)$, representing $p' \bowtie p^\sigma$.

We introduce coordinates in π^+ by mapping a point $r_0.\overline{p}_0 + r_1.\overline{p}_1 + r_2.\overline{p}_2$ of $\mathsf{PG}(6, q^2)$ to the point (r_0, r_1, r_2) in $\mathsf{PG}(2, q^2)$.

Lemma 2.6.3.
The equation in π^+ of the Hermitian curve $\mathcal{U}$ corresponding to $\mathcal{S}_{\mathsf{H}}$ is given by

$$\mathcal{U} : -2\gamma X_2 X_2^q = X_0 X_1^q - X_1 X_0^q$$

and the isomorphism $\Phi : \mathcal{S}_{\mathsf{H}} \to \mathcal{U}$ is given by

$$[\infty]^\Phi = (1,0,0), \qquad [k,b,k',-\nu k,\nu b]^\Phi = (\gamma(b^2+\nu k^2)-\nu kb+k',-1,\gamma k+b).$$

Proof. Since the line $[\infty]$ meets π^+ in p it is obvious that we map this line to the point $(1,0,0)$. Consider a general line, $[k,b,k',-\nu k,\nu b]$, of the spread $\mathcal{S}_{\mathsf{H}}$. Using Table 2.1 and the coordinates of points in π^+, a simple calculation yields $\Phi([k,b,k',-\nu k,\nu b]) = (\gamma(b^2+\nu k^2)-\nu kb+k',-1,\gamma k+b)$. The point $(1,0,0)$ clearly satisfies the given equation of $\mathcal{U}$ and therefore it suffices to check whether a general point $(\gamma(b^2+\nu k^2)-\nu kb+k',-1,\gamma k+b)$, with $k,b,k' \in \mathsf{GF}(q)$, is a point on $\mathcal{U}$, and that is an easy calculation.

□

Note. There is a unique polarity ρ of $\mathsf{PG}(2,q^2)$ such that $\mathcal{U}$ is the set of all absolute points of ρ.

2.6.3 Ree-Tits spreads and ovoids

From Theorem 2.2.2 we know that $\mathsf{H}(q)$ is self-dual if and only if the underlying field has characteristic 3 and it admits a polarity δ if q is an odd power of 3.

> If this is the case, then $\mathsf{H}(q)$ contains an ovoid $\mathcal{O}_{\mathsf{RT}}$ (where RT stands for Ree-Tits) arising from that particular polarity, i.e. the set of all absolute points, called a *Ree-Tits ovoid*. Similarly, the set of all absolute lines is known as a *Ree-Tits spread* $\mathcal{S}_{\mathsf{RT}}$.

The automorphism group of $\mathcal{O}_{\mathsf{RT}}$, i.e. the group of permutations of the point set of $\mathsf{H}(q)$ that induce a permutation on the line set of $\mathsf{H}(q)$, and that preserve $\mathcal{O}_{\mathsf{RT}} \cup \mathcal{S}_{\mathsf{RT}}$, is isomorphic to the Ree group $\mathsf{Ree}(q)$.

> If $\mathcal{O}$ is an ovoid of a generalized hexagon, and $\mathcal{S}$ is a spread of that same hexagon, then $(\mathcal{O},\mathcal{S})$ is called an *ovoid-spread pairing* if every element of $\mathcal{O}$ is incident with a (unique) element of $\mathcal{S}$ (see 7.2.6 in [67]).

The class of ovoid-spread pairings arising from polarities in $\mathsf{H}(3^{2e+1})$ is the only infinite class of ovoid-spread pairings known to date. A Ree-Tits spread and its corresponding Ree-Tits ovoid determine a *natural ovoid-spread pairing*. In Chapter 4 of this thesis we are going to construct the first sporadic examples.

Ree unitals

The Ree unitals are defined by J. Tits in [65]. In [17], V. De Smet and H. Van Maldeghem presented a construction of these $2-(q^3+1, q+1, 1)$ designs defined on $\mathsf{H}(q)$.

A block through two points x and y of a Ree-Tits ovoid $\mathcal{O}_{\mathsf{RT}}$ of $\mathsf{H}(q)$ is defined as follows. Let L_x and L_y denote the unique lines through x and y that are at distance 4 from y^δ and x^δ, respectively. Then every line of the regulus $\mathcal{R}(L_x, L_y)$ contains a unique point of $\mathcal{O}_{\mathsf{RT}}$ (see [17]) and this set of $q+1$ points is independent of the choice of any two points in it. This set of $q+1$ points is a so-called *Ree block*. The points of $\mathcal{O}_{\mathsf{RT}}$ and the corresponding Ree blocks forms a design, D_{RT}, isomorphic to the usual Ree unital of [65].

2.6.4 Ovoidal subspaces

Ovoidal subspaces were introduced by Brouns and Van Maldeghem in [4] in order to characterize the finite generalized hexagon $\mathsf{H}(q)$ by means of certain regularity conditions.

> An *ovoidal subspace* in a generalized hexagon is a set of points $\mathcal{O}$ with the property that every point of that hexagon outside $\mathcal{O}$ is collinear with exactly one point of $\mathcal{O}$. Dually, one defines a *dual ovoidal subspace*.

From [29] we know the following theorem.

Theorem 2.6.4.
The intersection of the line sets of two generalized hexagons $\Gamma \cong \mathsf{H}(q)$ and $\Gamma' \cong \mathsf{H}(q)$ on the same quadric $\mathsf{Q}(6,q)$ is a dual ovoidal subspace in both these hexagons.

It follows from [4] that a dual ovoidal subspace, and hence by the previous theorem the line intersection of two hexagons on the same $\mathsf{Q}(6,q)$, is either

the set of lines at distance at most 3 from a given point (type 0), or the set of lines of an ideal non-thick subhexagon (type +), or a distance-3 spread (type −).

Also by [29] we know that there are exactly $q+1$ copies of Γ on $\mathsf{Q}(6,q)$ containing a given subspace of type −, exactly q containing one of type 0 and exactly $q-1$ containing one of type +.

2.7 Groups

Let Γ be a generalized hexagon and γ a fixed path of length 4.

> If a collineation g of Γ fixes all elements incident with at least one element of γ, then we call g a *root elation*, *γ-elation* or briefly an *elation*. We define an *axial elation* (also called an *axial collineation*) g as a collineation fixing all elements at distance at most 3 from a certain line L, which is then called the *axis* of g.

There are two kinds of – potential – elations, namely, a path of length 4 can start and end with a point, or with a line. The first type of elation will be referred to as a *point-elation*, the second as a *line-elation*. In a point-elation we will speak of the *center* of this elation, by which we mean the point in the middle of γ.

From the explicit form of elations of $\mathsf{H}(q)$ given in Section 4.5.6 of [67], we deduce that all *line-elations* of $\mathsf{H}(q)$ are axial collineations (which was already noted by M.A. Ronan in [49]), and all *point-elations* with center p fix all points collinear with p and no other points, and they fix all lines through the points of a unique ideal line contained in the hexagon plane Π_p.

2.7.1 Morphisms of $\mathsf{H}(q)$

Throughout this work we will use group notations as introduced in [11].

To begin with, we quote the following proposition saying that the exceptional group $\mathsf{G}_2(q)$ is basically the automorphism group of $\mathsf{H}(q)$.

Proposition 2.7.1 (Proposition 4.6.7 of (67)**).**
Let $\mathsf{H}(q)$ be embedded naturally in $\mathsf{PG}(6,q)$. Every collineation of $\mathsf{H}(q)$ is induced by a projective semi-linear transformation of $\mathsf{PG}(6,q)$. The full

collineation group of $\mathsf{H}(q)$ *is isomorphic to the semi-direct product* $\mathsf{G}_2(q) : \mathsf{Aut}\,\mathsf{GF}(q)$*, where* $\mathsf{G}_2(q) = \mathsf{Aut}\,\mathsf{H}(q) \cap \mathsf{PGL}_7(q)$ *and*

$$|\mathsf{Aut}\,\mathsf{H}(q)| = q^6(q^6-1)(q^2-1)h,$$

$$|\mathsf{G}_2(q)| = q^6(q^6-1)(q^2-1).$$

Having introduced the coordinatization of $\mathsf{H}(q)$ in Section 2.3, we are now ready to describe some explicit automorphisms of $\mathsf{H}(q)$. Since there is a unique shortest path between any two elements, x and y, at a distance less than 5 from one another, it will be sufficient to know the image of these two elements in order to know the image of every other element on that particular path. Consequently, if an automorphism g of $\mathsf{H}(q)$ fixes the flag $\{(\infty), [\infty]\}$, it suffices to state the action of g on the elements with five coordinates.

For the description of some automorphisms of $\mathsf{H}(q)$, that are relevant to this work, we will restrict ourselves to the case $q = 3^{2h+1}$. To prove that these automorphisms are incidence preserving is a tedious, but elementary, calculation that has been left out.

The automorphism $\theta_{(A,L,A',L',A'')}$ that fixes (∞) and maps the point $(0,0,0,0,0)$ onto the point (A,L,A',L',A'') is determined by the following actions:

$$\begin{aligned}
&(a,l,a',l',a'')\\
&\qquad\qquad \to (a+A, l+L, a'+A'-aA'', l'+L', a''+A'')\\
&[k,b,k',b',k'']\\
&\qquad\qquad \to [k, b-Ak+A'', k'+L'+k^2A^3-kL,\\
&\qquad\qquad\quad b'+A'+A^2k+Ab+AA'', k''+kA^3+L].
\end{aligned}$$

Note. The subgroup of $\mathsf{G}_2(q)$ determined by the set

$$\{\theta_{(A,L,A',L',A'')} \mid A,L,A',L',A'' \in \mathsf{GF}(q)\}$$

has order q^5 and acts regularly on the points of $\mathsf{H}(q)$ that are opposite (∞).

For each element K in $\mathsf{GF}(q)$ there is a collineation of $\mathsf{H}(q)$, denoted by $\theta_{(K)}$, that fixes (∞), $[\infty]$ and $[0,0,0,0,0]$ and whose action is given by

$$\begin{aligned}&(a,l,a',l',a'')\\&\qquad\to(a,l-Ka^3,a'+Ka^2,l'+K^2a^3+Kl,a''+Ka)\\&[k,b,k',b',k'']\\&\qquad\to[k+K,b,k',b',k''].\end{aligned}$$

Note. The set of all such $\theta_{(K)}$, with $K\in\mathsf{GF}(q)$, has order q and acts regularly on the points of $[\infty]$ different from (∞).

The automorphism $\theta'_{(x)}$, $x\neq 0$, fixes the flag $\{(\infty),[\infty]\}$ elementwise and acts on the elements of $\mathsf{H}(q)$ with five coordinates as follows:

$$\begin{aligned}&(a,l,a',l',a'')\to(xa,x^3l,x^2a',x^3l',xa'')\\&[k,b,k',b',k'']\to[k,xb,x^3k',x^2b',x^3k''].\end{aligned}$$

Note. The group element $\theta'_{(x)}$ fixes the apartment $\mathcal{A}$ of Section 2.3 and each line incident with (∞).

The automorphism $\theta''_{(r)}$, $r\in\mathsf{GF}(q)$, that fixes the points (∞) and $(0,0,0,0,0)$ and maps the line $[\infty]$ onto the line $[r]$ is determined by the following actions on elements of five coordinates:

$$\begin{aligned}&(a,l,a',l',a'')\\&\qquad\to(-ar-a'',-a^3r^2+ra''^3-lr(r^2+1)-l'(r^2+1),aa''-a^2r\\&\qquad\quad+a''^2r+a'(r^2+1),-a^3r-r^2a''^3-rl'(r^2+1)+l(r^2+1),\\&\qquad\quad a-a''r)\\&[k,b,k',b',k''],k\neq -r\\&\qquad\to[\frac{kr-1}{k+r},-b\frac{r^2+1}{k+r},-b^3\frac{(r^2+1)^2}{(k+r)^2}-k'\frac{(r^2+1)^2}{k+r},b'(r^2+1)\\&\qquad\quad-b^2\frac{r^2+1}{k+r},b^3\frac{r^2+1}{k+r}-k'(r^2+1)-k''(r^2+1)(k+r)]\end{aligned}$$

$$[-r, b, k', b', k''],$$
$$\rightarrow [-b, -k'(r^2+1) + rb^3, b^2r + b'(r^2+1), k''(r^2+1)^2 - r^2b^3 - rk'(r^2+1)].$$

Note. If we identify $\mathbb{1}$ with $\theta''_{(\infty)}$, then the set $\{\theta''_{(r)} \mid r \in \mathsf{GF}(q) \cup \infty\}$ defines a subgroup of $\mathsf{G}_2(q)$ of order $q+1$, fixing (∞) and $(0,0,0,0,0)$, and acting regularly on the lines through the point (∞).

2.7.2 Maximal subgroups

A *maximal subgroup* H of G is a proper subgroup of G that is not strictly contained in another proper subgroup K of G.

Since this type of subgroups is often related to a geometrically interesting point set, and Chapters 4 and 5 are mainly devoted on substructures of $\mathsf{H}(3)$ and $\mathsf{H}(4)$, we display all maximal subgroups of $\mathsf{G}_2(3)$ and $\mathsf{G}_2(4)$ and their origins in Tables 4.1 and 5.1, respectively (see [11]).

maximal subgroup	order	index	stabilized set
$\mathsf{PSU}_3(3):2$	12096	351	$\mathcal{O}_{\mathsf{H}}$
$\mathsf{PSU}_3(3):2$	12096	351	$\mathcal{S}_{\mathsf{H}}$
$(3_+^{1+2} \times 3^2):2\mathsf{S}_4$	11664	364	p
$(3_+^{1+2} \times 3^2):2\mathsf{S}_4$	11664	364	L
$\mathsf{PSL}_3(3):2$	11232	378	$\Gamma(p,p')$
$\mathsf{PSL}_3(3):2$	11232	378	$\Gamma(L,L')$
$\mathsf{PSL}_2(8):3$	1512	2808	$\mathcal{O}_{\mathsf{RT}}$
$2^3{\cdot}\mathsf{PSL}_3(2)$	1344	3159	
$\mathsf{PSL}_2(13)$	1092	3888	
$2_+^{1+4}:3^2.2$	576	7371	$\mathcal{R}(L,M)$

Table 2.2: Maximal subgroups of $\mathsf{G}_2(3)$

maximal subgroup	order	index	stabilized set
J_2	604800	416	
$2^{2+8}:(3 \times \mathsf{A}_5)$	184320	1365	p
$2^{4+6}:(\mathsf{A}_5 \times 3)$	184320	1365	L
$\mathsf{PSU}_3(4):2$	124800	2016	$\mathcal{S}_{\mathsf{H}}$
$3.\mathsf{PSL}_3(4):2_3$	120960	2080	$\Gamma(p,p')$
$\mathsf{PSU}_3(3):2 \times 2$	12096	20800	$\mathsf{H}(2)$
$\mathsf{A}_5 \times \mathsf{A}_5$	3600	69888	$\mathcal{R}(L,M)$
$\mathsf{PSL}_2(13)$	1092	230400	

Table 2.3: Maximal subgroups of $\mathsf{G}_2(4)$

The main goal of the above mentioned chapters will be to fill in the blanks in the previous two tables.

3 Common point reguli of different H(q)'s on Q(6, q)

In this chapter, we consider any two split Cayley hexagons represented on the parabolic quadric $\mathsf{Q}(6,q)$ and determine their common point reguli. As an application we give an alternative construction of the Hölz design $\mathsf{D}_{\mathsf{Hölz}}(q)$, for $q \not\equiv 2 \bmod 3$. If $q \equiv 2 \bmod 3$, then our construction yields a $2-(q^3+1, q+1, \frac{q+4}{3})$ subdesign of the Hölz-design. With this new construction of $\mathsf{D}_{\mathsf{Hölz}}(3)$, we provide a theoretical proof of the fact that the only unitals contained in the $2-(28,4,5)$ Hölz design are Hermitian and Ree unitals (as was previously proved through a computer search by Tonchev, [66]).

3.1 General Introduction

In Section 2.6.4 we saw, on the basis of papers by E. Govaerts and H. Van Maldeghem [29] and by L. Brouns and H. Van Maldeghem [4], that the line intersection of two hexagons on the same $\mathsf{Q}(6,q)$, is either the set of lines at distance at most 3 from a given point, or the set of lines of an ideal non-thick subhexagon, or a distance-3 spread.

Also, from [29] we know that given a weak subhexagon of order $(1,q)$, a distance-3 spread or a line set at distance at most 3 from a given point, there are respectively $q-1$, $q+1$ and q hexagons on $\mathsf{Q}(6,q)$ that contain this line set as a subset.

In the present chapter, inspired by the results of [29], we study the common point reguli of two generalized hexagons on the same $\mathsf{Q}(6,q)$.

The motivation for this study is two-fold. Firstly, the geometric information can be used in some specific situations to prove other results, for instance on spreads. We demonstrate this in the next chapter by an application to the exceptional spreads of $\mathsf{H}(3)$. We determine by hand all isomorphism classes of 1-systems of $\mathsf{Q}(6,3)$ obtained as a derivation of this exceptional spread.

Secondly, the stabilizer of a dual ovoidal subspace $\mathcal{S}$ of some $\mathsf{H}(q)$ on $\mathsf{Q}(6,q)$ inside the full group of collineations of $\mathsf{Q}(6,q)$ does not always act primitively on the set of generalized hexagons isomorphic to $\mathsf{H}(q)$ naturally embedded on $\mathsf{Q}(6,q)$ and containing $\mathcal{S}$. Indeed, when $\mathcal{S}$ is related to a non-thick ideal subhexagon or a distance-3 spread, this stabilizer is roughly a dihedral group in its natural action, and so the generalized hexagons through $\mathcal{S}$ are paired up in a group-theoretical way. This pairing cannot be explained geometrically by looking at the intersections of line sets, but it can be recovered by considering the common point reguli. Hence, our geometric study provides a finer subdivision that explains the action of certain subgroups.

This result has a nice geometrical application. Namely, if the line intersection of two hexagons on $\mathsf{Q}(6,q)$ is a distance-3 spread, then by looking at their common point reguli we are able to define, for each odd prime power q, a $2-(q^3+1, q+1, 1+\frac{q+1}{(q+1,3)})$ design. We show that these designs are either isomorphic to the Hölz-designs (for $q \not\equiv 2 \bmod 3$) or subdesigns of the Hölz-designs (for $q \equiv 2 \bmod 3$) (for more information on the Hölz design, see Section 1.5). The fact that, for $q \equiv 2 \bmod 3$, the Hölz-design has such large subdesigns is apparently unnoticed in the literature. In fact, these subdesigns emerge as a union of orbits under the subgroup $\mathsf{PSU}_3(q)$, which acts transitively on the Hölz-blocks of the Hölz-design only if $q \not\equiv 2 \bmod 3$. Hence we have an alternative, rather unexpected, construction of the Hölz-designs $\mathsf{D}_{\mathsf{Hölz}}(q)$ for $q \not\equiv 2 \bmod 3$, and we have a geometric explanation of the non-transitivity of the subgroup $\mathsf{PSU}_3(q)$ on the Hölz-blocks if $q \equiv 2 \bmod 3$.

Searching for a use of this alternative construction of the Hölz design, we came across an article of V.D. Tonchev [66]. In this article dated 1991, he shows "by a computer result" that any unital in the $2-(28,4,5)$ design related to the Hermitian unital of order 3, is either a Hermitian or Ree unital. Using the above findings we came up with the following construction of that particular Hölz design. Take Γ the unique generalized quadrangle of order $(2,4)$. We define $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ with point and block set deduced from Γ. Define $\mathcal{P}$ as the point set of Γ to which we add a new point α. The set $\mathcal{B}$ contains two types of blocks: blocks of type (a) contain the point α together with three points of any line of Γ (*Line-block*), and those of type (b) contain the four points of the symmetric difference of two intersecting lines of Γ (*Vee-block*). We define the *vee-point* of a Vee-block V as the intersection point of

the two defining lines, the so-called *legs* of V. It is now routine to check that $\mathcal{D}$ is a $2-(28,4,5)$ design. In the final part of this chapter (Section 3.4) we use this construction to give a computer-free proof of the result in [66].

The results in Section 3.2 and 3.4 are based on [26] and [25], respectively. The results in Section 3.3 are forthcoming in *The Bulletin of the Belgian Mathematical Society – Simon Stevin* [23].

3.1.1 Statement of the Main Results

In this chapter we will prove the following theorems.

Theorem 3.1.1.
Let H_1 and H_2 be two models of $\mathsf{H}(q)$ isomorphic to the standard embedding of $\mathsf{H}(q)$ on $\mathsf{Q}(6,q)$. Denote by S, respectively Ω, the set of common lines, respectively point reguli of H_1 and H_2. If q is even, then H_1 and H_2 share all point reguli. For odd q we have one of the following situations:

(*i*) *S is the set of lines at distance at most 3 from a given point and $|\Omega| = q^3$.*

(*ii*) *S is the set of lines of an ideal subhexagon and $|\Omega| = q^2(q^2+q+1)$ or $|\Omega| = q^3(q^2+q+1)$.*

(*iii*) *S is a distance-3 spread and $|\Omega| = q^2(q^2-q+1)$ or $|\Omega| = q^2(q^2-q+1)(q+2)$.*

Furthermore, in both situations (ii) and (iii) there exists, given H_1 and S, a unique hexagon H_2 such that Ω contains the maximal number of point reguli.

Theorem 3.1.2.
Let H_1 and H_2 be two models of $\mathsf{H}(q)$ isomorphic to the standard embedding of $\mathsf{H}(q)$ on $\mathsf{Q}(6,q)$, with q odd. Define the following incidence structure with point set $\mathcal{P}$ and block set $\mathcal{B}$. The points are the common lines of H_1 and H_2. The blocks are the line reguli entirely contained in $\mathcal{P}$, together with the non-empty sets of elements of $\mathcal{P}$ that are incident with a common point regulus of H_1 and H_2. Then, for each H_1, there exists a suitable choice of H_2 such that, for $q \not\equiv 2 \bmod 3$, this incidence structure is isomorphic to $\mathsf{D}_{\mathsf{Hölz}}(q)$ and, for $q \equiv 2 \bmod 3$, it is a $2-(q^3+1, q+1, 1+\frac{q+1}{3})$ subdesign of $\mathsf{D}_{\mathsf{Hölz}}(q)$, invariant under $\mathsf{PSU}_3(q)$ acting naturally on $\mathsf{D}_{\mathsf{Hölz}}(q)$.

Theorem 3.1.3.
The only unitals contained in the $2-(28,4,5)$ Hölz design are of Hermitian or Ree Type.

In the next section we determine the common point reguli of two distinct split Cayley hexagons embedded into the parabolic quadric $\mathsf{Q}(6,q)$ and prove Theorem 3.1.1. In Section 3.3 we construct a 2-design derived from the split Cayley hexagon and prove Theorem 3.1.2. In Section 3.4 we use previous findings to give an alternative proof of the results in [66].

3.2 Common point reguli

Let us start by proving that for even q any two hexagons embedded on the parabolic quadric $\mathsf{Q}(6,q)$ share all point reguli. This is equivalent to saying that in $\mathsf{W}_5(q)$ all point reguli are the same. In this representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$ a point regulus is the perp of a non-degenerate 3-space determined by the associated line regulus. Hence point reguli are lines dependent only on $\mathsf{W}_5(q)$.

From now on we will be working with odd q.

Consider the standard embedding of $\mathsf{H}(q)$, as described in Section 2.2. We refer to that model as H_1. Say H_2 is another hexagon embedded on $\mathsf{Q}(6,q)$ and denote by S the intersection of the line sets and by Ω the set of all common point reguli of these two hexagons. Let Π_S be the hyperplane generated by S.

Remember. From the introduction we know that S is one of three types, namely 0, $+$ or $-$, corresponding to S being a set of lines at distance at most 3 from a given point, of an ideal non-thick subhexagon or of a distance-3 spread, respectively.

Note. The above symbols 0, $+$ and $-$ are related to the situations where Π_S is a tangent, a hyperbolic or an elliptic hyperplane, hence the choice in notation.

Lemma 3.2.1.
If $\mathcal{R}$ is an element of Ω, then either

- (a) *the complement of $\mathcal{R}$ in H_1 is equal to the one in H_2, and belongs to S or*
- (b) *the complement of $\mathcal{R}$ in H_1 is opposite the one in H_2. Furthermore, every point of $\mathcal{R}$ is incident with a unique line of S.*

Hence Ω is the union of two sets Ω_1 and Ω_2, that correspond to the sets of type (a) and type (b) point reguli, respectively.

Proof. Consider the regulus plane $\alpha_{\mathcal{R}}$ of $\mathcal{R}$. The polar image of $\alpha_{\mathcal{R}}$, say Υ, is a 3-space that has to contain the complementary line regulus of $\mathcal{R}$ both in H_1 and in H_2. Hence these either coincide or are opposite, proving the first part of the lemma. Suppose they are opposite and M_i and L_i, $i \in \{0, \ldots, q\}$, denote the lines of Υ in H_1 and H_2, respectively.

Take a point p of $\mathcal{R}$. Inside H_1, p is collinear with all points of a certain ideal line L_i, $i \in \{0, \ldots, q\}$. In the same way, p is collinear with all points of, say M_j, $j \in \{0, \ldots, q\}$, this time inside H_2. Hence pr_{ij}, with $L_i \mathrel{\mathtt{I}} r_{ij} \mathrel{\mathtt{I}} M_j$, is the unique line on p belonging to S.

□

Note. Lemma 2(b) in fact states that if $\mathcal{R}$ is of type (b), then all of its points belong to Π_S, the hyperplane generated by all lines of S.

From this point on we will denote the complement of a point regulus $\mathcal{R}$ in H_i by $\mathcal{R}_i^c$, for $i = 1, 2$.

Lemma 3.2.2.
If S has type 0, then situation (b) of Lemma 3.2.1 does not occur.

Proof. Let x be the unique point that is at distance at most 3 from every line of S. Suppose by way of contradiction that $\mathcal{R}$ is a point regulus that is completely contained in $T_x\mathsf{Q}(6, q)$. As $\mathcal{R}$ belongs to $T_x\mathsf{Q}(6, q)$, which is the perp of x, we immediately find that x belongs to the 3-space generated by $\mathcal{R}_1^c$. Hence x is incident with one of the lines, say L, of $\mathcal{R}_1^c$. By definition of S we know that L belongs to both hexagons. However, the complement of $\mathcal{R}$ in H_1 should be opposite the one determined by H_2, a contradiction as $L \in \mathcal{R}_1^c$.

□

To prove part (i) of Theorem 3.1.1 it now suffices to show that for given H_1 and S (the lines of H_1 in $T_x\mathsf{Q}(6, q)$ for some point $x \in \mathsf{Q}(6, q)$) the q other hexagons intersecting H_1 in S share q^3 point reguli with H_1. Say $\mathcal{R}$ belongs to Ω_1. Since all lines of $\mathcal{R}_1^c$ belong to S it is easy to see that $\mathcal{R}$ has to have x as one of its points (as $\mathcal{R}_1^c$ does not contain an element of $\Gamma_1(x)$, it belongs to $\Gamma_3(x)$, so $x \in \mathcal{R}$). Conversely, every point regulus on x has a complement which, by definition, consists of $q + 1$ lines at distance 3 from x. In other words, these $q + 1$ lines belong to S and Ω_1 is the set of all point reguli on x. Conclusion: if S is the line set at distance at most 3 from a point, then $|\Omega| = q^3$ (apply a double counting on $(x, \mathcal{R}(x, y))$ to obtain this number).

In order to prove part (ii) of Theorem 3.1.1 we suppose that S has type $+$. Denote by Ω the set of point reguli of H_1 that are also point reguli of

H_2 (another split Cayley hexagon of $\mathsf{Q}(6,q)$ on S). Clearly, Ω contains the subset Ω_1 of all point reguli complementary to the

$$\frac{(q^2+q+1)(q+1)q^3}{(q+1)q} = q^2(q^2+q+1)$$

line reguli in S.

Without loss of generality, we may assume $\Pi_S : X_3 = 0$ to be the hyperplane that determines all lines of S and denote the hexagon twin planes inside Π_S by π^+ and π^-. With this hyperplane we will now determine a unique H_2 through S.

Consider the point $p = \Pi_S^\rho$ with coordinates $(0,0,0,1,0,0,0)$ in $\mathsf{PG}(6,q)$. Every line through p and a point x of $\Pi_S \cap \mathsf{Q}(6,q)$ intersects $\mathsf{Q}(6,q)$ only in x, as x is the radical of that tangent line. Any other line through p and a point y on the quadric intersects $\mathsf{Q}(6,q)$ in a second point y'. The involution g interchanging y and y' and fixing all points of Π_S extends to an involutive collineation of $\mathsf{PG}(6,q)$, which we also denote by g. It is actually easy to see that g does not preserve H_1. Indeed, the set of lines of H_1 through a point x of $(\Pi_S \cap \mathsf{Q}(6,q)) \setminus (\pi^+ \cup \pi^-)$ fill up a plane of $\mathsf{Q}(6,q)$, and this plane is fixed under g only if it contains p or if it is contained in Π_S, which is in contradiction with the fact that, on the one hand, p is a point off $\mathsf{Q}(6,q)$ and, on the other hand, the only hexagon lines of Π_S are the lines incident with some point in $\pi^+ \cup \pi^-$.

Let H_2 be the image of H_1 under g, and note that $\mathsf{H}_1 \neq \mathsf{H}_2$. Henceforth, we will use the convention of writing a point regulus of H_i with a subindex i, $i = 1,2$. So the point regulus determined by two points a, b in H_i is denoted by $\mathcal{R}_i(a,b)$. Since line reguli are determined by $\mathsf{Q}(6,q)$, such a notation for line reguli is superfluous.

We define the following set Ω_3 of point reguli common to H_1 and H_2. Consider two arbitrary but opposite lines L and M of S, let a be any point on $L \setminus (\pi^+ \cup \pi^-)$, and denote by Θ the 3-space generated by L and M. Let b be the point on M collinear with a on $\mathsf{Q}(6,q)$. Then b is at distance 4 from a in the incidence graph of both H_1 and H_2. Put $r = a \bowtie b$ (inside H_1) and denote r^g by r'. Obviously, $r'a$ and $r'b$ are lines of H_2, implying r' belongs to Θ^ρ. Hence both r and r' belong to the point regulus, both in H_1 and H_2, complementary to $\mathcal{R}(L,M)$. Therefore r' is collinear in H_1 with two points a' and b' (obviously distinct from a and b, respectively) on L and M, respectively. Since g is an involution, the lines ra' and rb' of $\mathsf{PG}(6,q)$ are lines of H_2. The point regulus $\mathcal{R}_1(a,b')$ is complementary to $\mathcal{R}(rb, r'a')$ and the point regulus $\mathcal{R}_2(a,b')$ is complementary to $\mathcal{R}(ra', r'b)$. Since rb and $r'a'$

generate the same 3-space Υ in $\mathsf{PG}(6,q)$ as ra' and $r'b$, we conclude that $\mathcal{R}_1(a,b') = \mathcal{R}_2(a,b')$. Moreover, it is clear that Υ is invariant under g, and hence $p \in \Upsilon$. This implies that $\mathcal{R}_1(a,b')$, which belongs to $\Upsilon^\rho \subseteq p^\rho$, is entirely contained in Π_S.

The set Ω_3 consists of all point reguli $\mathcal{R}_1(a,b')$, for all such choices of L, M and a (but a and b' determine L and M uniquely, so there is no need to include L and M in the notation). Remark that one can easily count the number of elements of Ω_3 to be $(q^2+q+1)q^2(q-1)$.

Lemma 3.2.3.
With the above notation, we have $\Omega_3 = \Omega_2$ and $|\Omega| = (q^2+q+1)q^3$.

Proof. From our discussion above we already have $\Omega_3 \subseteq \Omega_2$. Now suppose that $\Omega_3 \neq \Omega_2$ and let $\mathcal{R}$ be a point regulus of $\Omega_2 \setminus \Omega_3$. Let a and b be two points of $\mathcal{R}$. By Lemma 3.2.1 we know that both these points are incident with a unique line of S, say L_a and L_b, respectively. Denote $\mathrm{proj}_{L_a} b$ and $\mathrm{proj}_{L_b} a$ by a' and b', respectively and let x and x' denote $a \bowtie b'$ and $a' \bowtie b$, respectively. We claim that x' is the image of x under g. Indeed, since $\mathcal{R}$ belongs to the set of common point reguli of both hexagons we immediately find that xa' and $x'b'$ are lines of H_2. Suppose x^g is a point x'' distinct of x' then also xa'' (with $a'' = \mathrm{proj}_{L_a} x''$) belongs to the line set of H_2, a contradiction. The claim and the lemma are proved.

□

Lemma 3.2.4.
Every other model of $\mathsf{H}(q)$ (not H_2) that also contains S as a subset of lines, intersects H_1 in a set of $q^2(q^2+q+1)$ point reguli. In other words, Ω equals Ω_1.

Proof. Take $\mathsf{H}_3 \neq \mathsf{H}_2$ a model of $\mathsf{H}(q)$ through S. Obviously, Ω, in the same way as before, contains Ω_1 as a subset. Suppose by way of contradiction that $\mathcal{R}(a,b)$ is a type (b) point regulus of both hexagons. As an immediate consequence of Lemma 3.2.1 we know that $\alpha_{\mathcal{R}}$, the regulus plane of $\mathcal{R}$, is a subspace of Π_S, the hyperplane containing all lines of S. By polarity we thus find $p = \Pi_S^\rho$ to be a point of $\Upsilon = \alpha_{\mathcal{R}}^\rho$. On $\mathsf{Q}(6,q)$ this 3-space constitutes a hyperbolic quadric isomorphic to $\mathsf{Q}^+(3,q)$ which we will denote by Q^+_Υ.

Put L_a, respectively L_b, the unique element of S incident with a, respectively b. Inside H_1 we denote $\mathrm{proj}_{L_b} a$, $\mathrm{proj}_{L_a} b$ by b', a' and $a \bowtie b'$, $a' \bowtie b$ by x, x', respectively.

From our discussion above we know that there exists a unique point regulus $\mathcal{R}'$, on a and a point $b'' \mathrel{\mathtt{I}} L_b$, that belongs to both H_1 and H_2. The point b'' is

the unique point of L_b that is collinear (within H_1) with $x^g = px \cap \mathsf{Q}(6,q) \setminus \{x\}$. However, since $p \in \Upsilon$ and $x \in \Upsilon$ we have that $x^g \in \Upsilon \cap (L_a L_b)^\rho$ and see that x^g should belong to a line of Q^+_Υ and hence equals x'. Furthermore, this implies that H_2 and H_3, apart from having all lines of S in common, share the complementary line regulus of $\mathcal{R}$ (i.e. $\mathcal{R}$ is of type (a) with respect to these two hexagons). In other words, we find that H_3 equals H_2 and we are done.

□

The proof of part (iii) of Theorem 3.1.1 is similar. Here we just point out the main difference.

First of all, the subset Ω_1 contains $q^2(q^2 - q + 1)$ point reguli which are complementary to the line reguli of S. There is, however, a crucial distinction between situations (ii) and (iii) when it comes to defining the set Ω_3. Before, we considered two lines L and M of S and a a point on $L \setminus (\pi^+ \cup \pi^-)$. In the current situation, with S a distance-3 spread, we put no restriction on the choice of a on L and hence end up with

$$\frac{|S|(|S|-1)}{(q+1)q}(q+1)$$

elements of Ω_3 instead of

$$\frac{|S|(|S|-1)}{(q+1)q}(q-1)$$

as we did before.

In order to complete the proof we may carefully copy the above. Indeed, in the same way one can show that Ω_3 consists of all elements of type (b) and hence find $|\Omega| = q^2(q^2-q+1)(q+2)$. Finally by the analogue of Lemma 3.2.4 we may conclude Theorem 3.1.1 to be proved.

Remark. If H_1 and H_2 share point reguli of type (b), meaning Ω_2 is a non-empty set, then (while the set Ω_1 determines all lines of S) the set Ω_2 determines all lines of $\mathsf{H}_1 \setminus S$ (and consequently also all lines of H_2), as we will show.

Denote the set of lines that are in a complementary regulus of some element of Ω_2 by $\mathcal{L}$ and suppose $\mathcal{L}$ has cardinality A. First of all, we determine, given a line L of $\mathcal{L}$ the number n_1 of point reguli $\omega_2 \in \Omega_2$, such that $L \in \omega_2^c$. Since L belongs to $\mathsf{H}_1 \setminus S$, it is concurrent with a unique line L_1 of S (as Π_S is a hyperplane). A point x on L (not on L_1) and a line M through x then completely determine an element $\omega_2 \in \Omega_2$, such that $L \in \omega_2^c$ (this is true

because M is concurrent with a unique line L_2 of S and because x and x^g determine two points a and b of ω_2, on L_2 and L_1, respectively). Hence by an easy counting argument one obtains $n_1 = q$ (consider the triple $(x, M, \mathcal{R})$, with $L_1 \not\mathtt{I}\ x\ \mathtt{I}\ L$, $L \neq M\ \mathtt{I}\ x$, $L \in \mathcal{R}$ and $\mathcal{R}^c \in \Omega_2$).

We are now ready to apply a double counting on the couples (ω_2, L), $\omega_2 \in \Omega_2$ and $L \in {\omega_2}^c$. Here we treat the case where S has type $-$ and omit the counting for S of type $+$, which is similar. Since $|\Omega_2| = (q^3+1)q^2$ we find

$$(q^3+1)q^2(q+1) = Aq$$

and hence

$$A = (q^3+1)q(q+1)$$

This number of lines together with all spread lines add up to

$$A + (q^3+1) = \frac{q^6-1}{q-1}$$

the total amount of lines in H_1.

3.3 2-Designs derived from H(q)

In this section we construct a 2-design derived from the split Cayley hexagon and prove Theorem 3.1.2. To start off the section, we merely recall that a Hölz design is a $2-(q^3+1, q+1, q+2)$ design. For further information on this class of designs we refer to Section 1.5. We remark that throughout this section q is an odd prime power.

First, we construct a 2-design denoted by $\mathsf{D}_{\mathsf{Hex}}(q)$, starting from two hexagons embedded on $\mathsf{Q}(6,q)$ that share a Hermitian spread and intersect in the maximal number of point reguli. We then show that $\mathsf{D}_{\mathsf{Hex}}(q)$ is a subdesign of, or equals $\mathsf{D}_{\mathsf{Hölz}}(q)$ and complete the proof of Theorem 3.1.2.

3.3.1 Construction and first properties

Consider a standard embedding of $\mathsf{H}(q)$ and refer to that model as H_1. Consider the hyperplane Π with equation $\nu X_1 + X_5 = 0$. From the equation of $\mathsf{Q}(6,q)$, it is clear that Π meets $\mathsf{Q}(6,q)$ in an elliptic quadric isomorphic to $\mathsf{Q}^-(5,q)$ if and only if $-\nu$ is a non-square. In this case, by Section 2.6.2, the set $\mathcal{S}_{\mathsf{H}}$ of lines of H_1 in Π is a Hermitian spread

$$\mathcal{S}_{\mathsf{H}} = \{[\infty]\} \cup \{[x, y, z, -\nu x, \nu y] : x, y, z \in \mathsf{GF}(q)\}$$

of H_1. Now consider the point $p = \Pi^\rho$ with coordinates $(0,1,0,0,0,\nu,0)$ in $\mathsf{PG}(6,q)$. In exactly the same way as in the previous section, we can define an involution g that fixes all points of Π and maps a point $y \in \mathsf{Q}(6,q) \setminus \Pi$ to the second point of $\mathsf{Q}(6,q)$ on the line py. We again denote the extension of this involution to an involutive collineation of $\mathsf{PG}(6,q)$ by g.

Let H_2 be the image of H_1 under g and adopt the same convention as used in Section 3.2 as far as the notation of a point regulus is concerned. From the previous section we know that the hexagon H_2 is the unique hexagon on $\mathcal{S}_\mathsf{H}$ that shares $q^2(q^2-q+1)(q+2)$ point reguli with H_1. We recall that this set of common point reguli is the union of two subsets Ω_1 and Ω_2, where Ω_1 consists of all point reguli complementary to the line reguli in $\mathcal{S}_\mathsf{H}$, while Ω_2 contains all point reguli $\mathcal{R}_1(a,b)$, with a and b two points incident with a line, say L_a and L_b respectively, of $\mathcal{S}_\mathsf{H}$, such that $a \bowtie \mathrm{proj}_{L_b} a$ and $b \bowtie \mathrm{proj}_{L_a} b$ are interchanged by g.

Lemma 3.3.1.
The stabilizer G of Π inside the automorphism group of H_1 stabilizes Ω. Also, G acts doubly transitive on $\mathcal{S}_\mathsf{H}$, and the stabilizer of two elements L, M of $\mathcal{S}_\mathsf{H}$ acts transitively on the set of points incident with L.

Proof. In order to prove the first assertion, it suffices to show that any element $h \in G$ stabilizes H_2. Since g is the unique involution stabilizing $\mathsf{Q}(6,q)$ and fixing all points of Π, and since h stabilizes both Π and $\mathsf{Q}(6,q)$, we have $g^h = g$. Hence $h = h^g$ stabilizes $\mathsf{H}_1^g = \mathsf{H}_2$.

Now $\mathsf{SU}_3(q) : 2 \leq G$, with the natural action on $\mathcal{S}_\mathsf{H}$ as a Hermitian unital. Whence the double transitivity. Let $G^* = G \cap \mathsf{SU}_3(q) : 2$. Then, the stabilizer $G^*_{L,M}$ of two lines L, M has order $2(q^2-1)$. Since q is odd and $G^*_{L,M}$ contains a cyclic group of order (q^2-1), the stabilizer in $G^*_{L,M}$ of one point of L has to fix a second point of L, and hence every element σ of that stabilizer is a product of generalized homologies, in the sense of Chapter 4 of [67], see 4.6.6 of [67]. We now consider the explicit form of $\mathcal{S}_\mathsf{H}$ as given in the beginning of the current section. We can take $L = [\infty]$ and $M = [0,0,0,0,0]$. If we assume that σ fixes $[\infty]$ and $[0,0,0,0,0]$ and the point (∞), then σ must be given by the following actions on the coordinates:

$$\begin{array}{rcl} (a,l,a',l',a'') & \mapsto & (\epsilon a, \epsilon K l, K a', \epsilon K^2 l', \epsilon K a''), \\ [k,b,k',b',k''] & \mapsto & [K k, \epsilon K b, \epsilon K^2 k', K b', \epsilon K k''], \end{array}$$

with $K \in \mathsf{GF}(q)$ arbitrary, and $\epsilon \in \{1,-1\}$ (note that (0) is the second point on $[\infty]$ that is stabilized by σ). The group generated by these generalized

homologies has order $2(q-1)$. By the orbit counting formula, the orbit of (∞) under $G^*_{L,M}$ has length $q+1$ and hence constitutes all points of L.

□

Now we define $\mathsf{D}_{\mathsf{Hex}}(q)$ as the incidence geometry with point set the elements of $\mathcal{S}_{\mathsf{H}}$ and block set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, with

$$\mathcal{B}_1 = \{\mathcal{R}(L,M) : L, M \in \mathcal{S}_{\mathsf{H}}, L \neq M\},$$
$$\mathcal{B}_2 = \{\{L \ \mathtt{I} \ p : p \in \omega_2, L \in \mathcal{S}_{\mathsf{H}}\} : w_2 \in \Omega_2\}.$$

Remark. We consider repeated blocks as one block.

We have the following proposition, which is already a substantial part of Theorem 3.1.2.

Proposition 3.3.2.
The incidence geometry $\mathsf{D}_{\mathsf{Hex}}(q)$ *is a* $2-(q^3+1, q+1, q+2)$ *design for* $q \not\equiv 2 \bmod 3$*, and it is a* $2-(q^3+1, q+1, \frac{q+4}{3})$ *design otherwise. In any case, two distinct blocks never meet in more than two points.*

Proof. We determine the number of blocks in $\mathsf{D}_{\mathsf{Hex}}(q)$ through two given points of $\mathsf{D}_{\mathsf{Hex}}(q)$. Note that by the 2-transitivity of G on the point set of $\mathsf{D}_{\mathsf{Hex}}(q)$, this number is already a constant and we may choose these points as the lines $[\infty]$ and $[0,0,0,0,0]$.

First of all there is a unique element of $\mathcal{B}_1$ on any two points of $\mathsf{D}_{\mathsf{Hex}}(q)$. Furthermore, no member of $\mathcal{B}_1$ can meet any member of $\mathcal{B}_2$ in more than two points (by Lemma 3.3.1 we may consider the blocks $\mathcal{R}([\infty],[0,0,0,0,0])$ and B_R defined by $\mathcal{R}((\infty),(0,0,0,0,0))$ and the explicit line set they determine to see this).

So we have to count the number of "repeated blocks" within $\mathcal{B}_2$, i.e. the number of times a block is defined by different point reguli. Again by Lemma 3.3.1, we may take the block B_R defined by the point regulus $R := \mathcal{R}((\infty),(0,0,0,0,0))$ and the block $B_{R'}$ defined by the point regulus $R' := \mathcal{R}((a),(A,0,0,0,0))$, with $(a,0,0)^g = (A,0,0)$ and $a \in \mathsf{GF}(q) \setminus \{0\}$. Suppose some point x of $R \setminus \{(\infty),(0,0,0,0,0)\}$ is on the same line of $\mathcal{S}_{\mathsf{H}}$ as some point $x' \in R'$. Such a point x has coordinates $(0,0,-\nu a',0,0)$, $a' \in \mathsf{GF}(q) \setminus \{0\}$ and is incident with the line $L = [a',0,0,-\nu a',0]$ of $\mathcal{S}_{\mathsf{H}}$.

Now, the point x' has coordinates $(A,0,.,.,.)$ in $\mathsf{H}(q)$, and from the incidence relation described in Section 2.3, we infer that x' is incident with L if and only if $-A^3a' + 3A\nu a' = 0$. This is equivalent to $A^2 = 3\nu$. Hence such a point x' exists if and only if -3 is not a square in $\mathsf{GF}(q)$, i.e. $q \equiv 2 \bmod 3$.

This already shows that, if $q \not\equiv 2 \bmod 3$, then there are no repeated blocks, no two distinct blocks meet in more than two points, and an easy counting argument concludes the proof of the proposition.

Now suppose $q \equiv 2 \bmod 3$. Then the equation $A^2 = 3\nu$ has two solutions in A giving rise to two point x_1', x_2'. This implies that x, x_1' and x_2' are three points of distinct point reguli R, R_1', R_2' (respectively) in Ω_2 that are on the same spread line $L = [a', 0, 0, -\nu a', 0]$. Since the equation $A^2 = 3\nu$ does not depend on x, we see that the point reguli R, R_1', R_2' determine the same element of $\mathcal{B}_2$. By transitivity we thus obtain that any two lines of the spread determine $\frac{q+1}{3}$ elements of $\mathcal{B}_2$. This concludes the proof of the proposition completely.

□

Remark. The previous proof also implies that G acts transitively on the set $\mathcal{B}_2$. If $q \equiv 2 \bmod 3$, then the center of G^*, which has order 3, fixes all points of $\mathsf{D}_{\mathsf{Hex}}(q)$ and permutes the point reguli belonging to Ω_2 in such a way that point reguli that represent same elements of $\mathcal{B}_2$ are permuted. Hence the center acts trivially on $\mathsf{D}_{\mathsf{Hex}}(q)$ and we obtain a faithful action of $\mathsf{PSU}_3(q)$ on $\mathsf{D}_{\mathsf{Hex}}(q)$, transitive on the points and with two orbits on the blocks.

3.3.2 $\mathsf{D}_{\mathsf{Hex}}(q)$ vs. $\mathsf{D}_{\mathsf{Hölz}}(q)$

In this subsection we will show that each block of $\mathsf{D}_{\mathsf{Hex}}(q)$ is a block of $\mathsf{D}_{\mathsf{Hölz}}(q)$. In order to do this, we consider a general block B of $\mathsf{D}_{\mathsf{Hex}}(q)$ and prove that there exists a Baer subplane D that satisfies property (H), i.e. for each point $x \in D \cap \mathcal{U}$ the tangent line to $\mathcal{U}$ at x is a line of D, and such that the points of $B \cap D$ are collinear or form an oval of D. In particular we will prove that a block of $\mathcal{B}_1$ corresponds to a block of S_1, i.e. an intersecting line of $\mathcal{U}$, while a block of $\mathcal{B}_2$ corresponds to a block of S^*, i.e. an oval on $\mathcal{U}$ that determines a Baer subplane of π^+ satisfying property (H). This will complete the proof of Theorem 3.1.2.

Proposition 3.3.3.
$\mathsf{D}_{\mathsf{Hex}}(q)$ *is a subdesign of* $\mathsf{D}_{\mathsf{Hölz}}(q)$. *Moreover, these designs coincide if and only if* $q \not\equiv 2 \bmod 3$.

Proof. It is well known that the line reguli in $\mathcal{S}_{\mathsf{H}}$ correspond to the Hermitian blocks. Hence we only have to prove that any block of type $\mathcal{B}_2$ is a Hölz-block.

By transitivity on the blocks of type $\mathcal{B}_2$ we may only consider the block defined by the point regulus $\mathcal{R}((\infty), (0, 0, 0, 0, 0))$. One easily calculates that

this block equals

$$B = \{[\infty]\} \cup \{[k, 0, 0, -\nu k, 0] : k \in \mathsf{GF}(q)\}.$$

We now need to relate the Hermitian spread $\mathcal{S}_{\mathsf{H}}$ of H_1 to a Hermitian curve, $\mathcal{U}$, in $\mathsf{PG}(2, q^2)$. Let $\gamma \in \mathsf{GF}(q^2)$ be such that $\gamma^2 = -\nu$ (with ν given previously). From Lemma 2.6.3 we know that the Hermitian curve associated with $\mathcal{S}_{\mathsf{H}}$ is given by

$$\mathcal{U} : -2\gamma X_2 X_2^q = X_0 X_1^q - X_1 X_0^q$$

and the isomorphism Φ between the lines of $\mathcal{S}_{\mathsf{H}}$ and the points of $\mathcal{U}$ is given by

$$\begin{aligned}
&[\infty]^\Phi = (1, 0, 0),\\
&[k, b, k', -\nu k, \nu b]^\Phi = (\gamma(b^2 + \nu k^2) - \nu k b + k', -1, \gamma k + b).
\end{aligned}$$

We will now show that B^Φ contains the points of an oval on $\mathcal{U}$, which determine a Baer subplane D satisfying property (H).

By Lemma 2.6.3 the lines of $\mathcal{S}_{\mathsf{H}}$ corresponding to B are mapped onto the points

$$\{(1, 0, 0)\} \cup \{(\gamma\nu k^2, -1, \gamma k) : k \in \mathsf{GF}(q)\},$$

or, since $\gamma^2 = -\nu$,

$$\{(1, 0, 0)\} \cup \{(l^2, \gamma, l) : l \in \mathsf{GF}(q)\}.$$

Now all of these points satisfy the quadratic equation $X_0 X_1 = \gamma {X_2}^2$, which shows that they are contained in a conic of $\mathsf{PG}(2, q^2)$.

We now check Property (H).

Consider the points $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (1, \gamma, 1)$ and $p_4 = (1, \gamma, -1)$ (all of which are points of B^Φ). The Baer subplane through these points contains the following additional points: $p_5 = p_1p_2 \cap p_3p_4 = (1, \gamma, 0)$, $p_6 = p_1p_3 \cap p_2p_4 = (1, -\gamma, -1)$ and $p_7 = p_1p_4 \cap p_5p_6 = (0, -\gamma, 1)$. Now, with these additional points, one can easily see that the Baer subplane through the $q + 1$ points of $\mathcal{U}$ contains $q + 1$ points on $X_1 = 0$, namely

$$\{(1, 0, x) : x \in \mathsf{GF}(q)\} \cup \{(0, 0, 1)\},$$

and its set of q^2 other points is given by:

$$\{(y, \gamma, x) : y, x \in \mathsf{GF}(q)\}.$$

With these explicit coordinates of the points of D we will now check whether this Baer subplane satisfies property (H). Let us first recall that the tangent line at a point (x_0, x_1, x_2) of $\mathcal{U}$ is given by the equation

$$T_p\mathcal{U} : \left(\frac{\partial \mathcal{U}}{\partial X_0}\right)_p X_0 + \left(\frac{\partial \mathcal{U}}{\partial X_1}\right)_p X_1 + \left(\frac{\partial \mathcal{U}}{\partial X_2}\right)_p X_2 = 0$$

with $\left(\frac{\partial \mathcal{U}}{\partial X_i}\right)_p$ the partial derivative with respect to X_i at the point $p = (x_0, x_1, x_2)$. Given the equation of $\mathcal{U}$ we find

$$x_1^q X_0 - x_0^q X_1 + 2\gamma x_2^q X_2 = 0$$

to be the tangent line of $\mathcal{U}$ at the point (x_0, x_1, x_2) over the field $\mathsf{GF}(q^2)$.

To investigate whether D satisfies property (H) we have to consider all points of D on $\mathcal{U}$, determine the tangent line at these points and check that it is a Baer line of D. In particular, as every line of $\mathsf{PG}(2, q^2)$ intersects D in one or in $q + 1$ points, it suffices to find two points of such a tangent line that are in D to conclude that it is a Baer line of the Baer subplane. Now, the points of D on $\mathcal{U}$ are in fact the points of $B^\Phi = \{(1, 0, 0)\} \cup \{(l^2, \gamma, l) : l \in \mathsf{GF}(q)\}$. The tangent line at the first point of this set is the line X_1, which we already know is a Baer line of D. Finally, the tangent line at the point (l^2, γ, l), with $l \in \mathsf{GF}(q)$, is the line

$$\gamma X_0 + l^2 X_1 - 2\gamma l X_2 = 0$$

and this line contains the points (l^2, γ, l) and $(-l^2, \gamma, 0)$ of D.

In conclusion, D satisfies property (H) and consequently $\mathsf{D}_{\mathsf{Hex}}(q)$ is a subdesign of $\mathsf{D}_{\mathsf{Hölz}}(q)$.

□

3.4 Unitals of the Hölz design on 28 points

The goal of this section will be to prove Theorem 3.1.3 using the construction of $\mathcal{D}$ as given in Section 3.1. We start by giving a construction of the generalized quadrangle of order $(2, 4)$ as described in [45]. Next, we provide some elementary properties and lemmas concerning spreads of this particular generalized quadrangle. In Section 3.4.1 we prove that if $\mathcal{U}$ has one derived Hermitian spread, then all other derived spreads have to be Hermitian as well. In Section 3.4.2 we show that $\mathcal{D}$ contains, up to isomorphism, a unique unital that intersects all derived subdesigns in Hermitian spreads.

And, finally, in Section 3.4.3 we complete the proof of Theorem 3.1.3 by showing that $\mathcal{D}$ contains, up to isomorphism, a unique unital that intersects all derived subdesigns in non-Hermitian spreads.

A *duad* is an unordered pair (ij) of distinct integers that belong to $\Omega = \{1, \ldots, 6\}$. A *syntheme* is a triple $(ij)(kl)(mn)$ of duads for which $\{i, j, k, l, m, n\} = \Omega$. The following straight forward construction of the unique generalized quadrangle of order 2, which is denoted by $\mathsf{W}(2)$, is due to J.J. Sylvester: the duads represent the points, the synthemes represent the lines and the incidence is given by simple containment.

Since $\mathsf{W}(2) \cong \mathsf{Q}(4, 2)$ is a subquadrangle of $\mathsf{Q}(5, 2)$, one can extend the above description of $\mathsf{W}(2)$ to obtain the unique generalized quadrangle of order $(2, 4)$, which we denote by Q. The twelve additional points are denoted by $1, \ldots, 6$ and $1', \ldots, 6'$, while the thirty additional lines are denoted by $i(ij)j'$, $1 \leq i, j \leq 6$, $i \neq j$ and the inclusion remains containment. This construction of Q was first discovered by S.E. Payne and J.A. Thas in [45] and it is this construction of Q that we use to construct $\mathcal{D}$. In Figure 3.1 we display – for further reference – two grids and all points of Q.

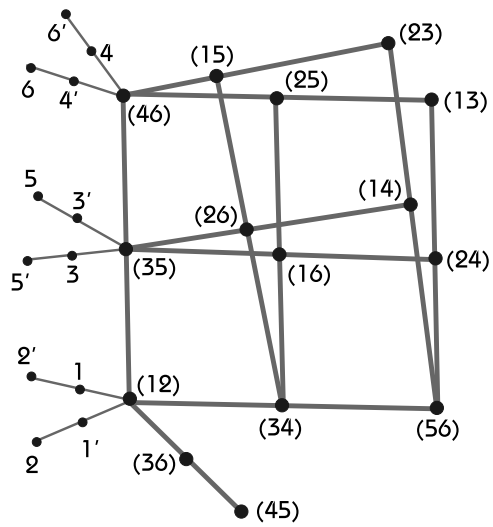

Figure 3.1: Points of Q

A unital $\mathcal{U}$ of $\mathcal{D}$ is by definition a subset of $\mathcal{B}$ such that any two points of $\mathcal{D}$ are contained in a unique block of $\mathcal{U}$ and from now on we let $\mathcal{U}$ be a given

unital. Hence, for every point x in $\mathcal{D}$ such a unital defines a spread, denoted by $\mathcal{S}_x$ and called a *derived spread* of $\mathcal{U}$, in the derived quadrangle $\mathcal{D}_x$ at this particular point. From [5] we know that the generalized quadrangle of order $(2,4)$ has two non-isomorphic spreads: the Hermitian spread (in which any two lines determine a line regulus completely contained in it) and a second spread obtained by switching one of the line reguli of the Hermitian spread. For further reference we will denote this latter spread by non-Hermitian.

Since $\mathsf{Q}(5,q) \cong \mathsf{H}(3,q^2)^D$, a Hermitian spread $\mathcal{S}_{\mathsf{H}}$ of Q determines a so-called Hermitian ovoid $\mathcal{O}_{\mathsf{H}}$ of $\mathsf{H}(3,4)$. The points of $\mathcal{O}_{\mathsf{H}}$ lie in a plane β of $\mathsf{PG}(3,4)$ that intersects the Hermitian variety in the points of a Hermitian curve. If we consider in β any arbitrary point a off $\mathcal{O}_{\mathsf{H}}$, then the points of $\mathcal{O}_{\mathsf{H}}$ belong to the two secant lines on a, say M and N, and to the secant line $a^\perp \cap \beta$, which we denote by L.

Switching a regulus of $\mathcal{S}_{\mathsf{H}}$ to obtain a non-Hermitian spread $\mathcal{S}_{\mathsf{NH}}$ translates dually to replacing one of the secant lines of β by its polar image with respect to the polarity of $\mathsf{H}(3,4)$. Without loss of generality we may consider L as this particular line of β. By polarity we immediately find that $L^\perp$ intersects the plane β in the point a. Denote the intersection point of M, N and $L^\perp$ with $a^\perp$ by m, n and x, respectively. We now claim that these three points are the points of a polar triangle in $a^\perp$. Indeed, since x belongs to $L^\perp$ and the line mn is in fact the line L, we readily see that mn equals $x^\perp \cap a^\perp$ and hence the claim. In other words, a non-Hermitian ovoid is uniquely determined by any point a off $\mathsf{H}(3,4)$ together with any polar triangle in $a^\perp$. Furtheron we will refer to this construction as the *polar triangle construction*. Note that a simple counting argument shows that this construction indeed determines all non-Hermitian ovoids of $\mathsf{H}(3,4)$. We now have the following three lemmas.

Lemma 3.4.1.
A non-Hermitian spread of Q *consists of exactly three line reguli.*

Proof. This lemma readily follows from the above construction of $\mathcal{O}_{\mathsf{NH}}$ (since the three secant lines on a are two by two contained in a plane).

□

Lemma 3.4.2.
Let $\mathcal{S}$ be a non-Hermitian spread of Q*. Suppose $\mathcal{R}$ is one of the three line reguli of $\mathcal{S}$. For any line X of the opposite regulus to $\mathcal{R}$, there are exactly three line reguli on X which contain two lines of $\mathcal{S}$.*

Proof. To prove this lemma we dualize the situation and consider the above construction of $\mathcal{O}_{\mathsf{NH}}$. Without loss of generality we may suppose that the

lines of $\mathcal{R}$ correspond to the points on $L^{\perp}$. Then X corresponds to one of the points, say x, on L. A regulus on X translates into a secant line containing the point x, which obviously intersects $\mathcal{O}_{\mathsf{NH}}$ in no (L) or two points and there are three such secant lines in β.

□

Lemma 3.4.3.
There are exactly two spreads of Q *on a regulus* $\mathcal{R}$ *and a single line off* $\mathcal{R}$*. One of these spreads is Hermitian, the other one is non-Hermitian.*

Proof. To prove this lemma we dualize the given situation and show that an ovoid $\mathcal{O}$ of $\mathsf{H}(3,4)$ is uniquely determined by a secant line and a single point off that line.

Obviously, since a Hermitian ovoid is uniquely determined by a plane, the lemma is trivially met when dealing with the Hermitian case.

If, on the other hand, the ovoid $\mathcal{O}$ is non-Hermitian, then we consider the polar triangle construction to prove the lemma. Let x and L be the given point and secant line. By definition, the line L contains two points off $\mathsf{H}(3,4)$. One of these points is on a tangent line with x, while the other is on a secant line with x. This latter point, that plays the role of a in the polar triangle construction and which we denote by b, together with L and x determines two points of the defining polar triangle in $b^{\perp}$ and hence completely determines $\mathcal{O}$.

□

Let $\mathcal{S}_{\mathsf{H}}$ be the Hermitian spread of Q determined by the line $2(26)6'$ and the regulus $\mathcal{R}$ consisting of the lines

(12) (34) (56)
(35) (16) (24)
(46) (25) (13)

of Q. Then $\mathcal{S}_{\mathsf{H}}$ contains the following set of lines

Blocks of $\mathcal{S}_{\mathsf{H}}$		
(12) (34) (56)	1 (14) 4'	4 (45) 5'
(35) (16) (24)	2 (26) 6'	5 (15) 1'
(46) (25) (13)	3 (23) 2'	6 (36) 3'

Define the non-Hermitian spread $\mathcal{S}_{\mathsf{NH}}$ as the spread that is obtained by switching the regulus (with the obvious notation of rows and columns representing lines)

1'	(15)	5
4	5'	(45)
(14)	1	4'

of $\mathcal{S}_{\mathsf{H}}$. More explicitly,

Blocks of $\mathcal{S}_{\mathsf{NH}}$		
(12) (34) (56)	1 (15) 5'	4 (14) 1'
(35) (16) (24)	2 (26) 6'	5 (45) 4'
(46) (25) (13)	3 (23) 2'	6 (36) 3'

are the nine lines of $\mathcal{S}_{\mathsf{NH}}$. Denote the four grids of Q on the lines (12)(34)(56) and (12)(35)(46) by $\mathcal{G}_i$, $i \in \{a, b, c, d\}$. Each of these grids is determined by a single line on any one of the points of the Vee-block with these two lines as its legs. Consider, for instance, the point (46) and define $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_c$ and $\mathcal{G}_d$ as the grid on (46)(25)(13), (46)(15)(23), $4'(46)6$ and $4(46)6'$, respectively.

Let G_{H} and G_{NH} denote the stabilizer group inside the full automorphism group of Q of $\mathcal{S}_{\mathsf{H}}$ and $\mathcal{S}_{\mathsf{NH}}$, respectively.

We are now ready to state the following two lemmas.

Lemma 3.4.4.
The stabilizer in G_{H} of the point (12)*, the line* (12)(34)(56) *and the grid $\mathcal{G}_a$ acts as S_3 on the set $\{\mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d\}$.*

Proof. We write down the explicit line sets of $\mathcal{G}_i$, with $i \in \{a, b, c, d\}$. Besides the line (12)(34)(56) the grids $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_c$ and $\mathcal{G}_d$ contain the lines

Lines of $\mathcal{G}_a$	Lines of $\mathcal{G}_b$	Lines of $\mathcal{G}_c$	Lines of $\mathcal{G}_d$
(46) (25) (13)	(46) (15) (23)	6 (46) 4'	4 (46) 6'
(35) (16) (24)	(35) (26) (14)	3 (35) 5'	5 (35) 3'

of Q, respectively. It is now easy to verify that the involution

$$g : 1 \leftrightarrow 2, 3 \leftrightarrow 5, 4 \leftrightarrow 6$$

is an element of the group G_{H} that fixes $\mathcal{G}_a$ and $\mathcal{G}_b$, while it interchanges $\mathcal{G}_c$ and $\mathcal{G}_d$. Considering a distinct subquadrangle of order 2 on $\mathcal{G}_a$ and taking the above into account yields the lemma. □

Lemma 3.4.5.
The stabilizer in G_{NH} of the points (12)*,* (35) *and* (46)*, the line* (12)(34)(56) *and the grid $\mathcal{G}_a$ acts transitively on the set $\{\mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d\}$.*

Proof. To begin with, the order of the group G_{NH} is given by

$$\frac{\mathsf{Aut}\ \mathsf{Q}^-(5,2)}{40.4} = 27.12$$

as $|\mathsf{Aut}\ \mathsf{H}(3,q^2)| = q^6(q^4-1)(q^3+1)(q^2-1)2h$ (see Proposition 4.6.3 of [67]) and one can easily count the number of non-Hermitian spreads to be 160 (use the polar triangle construction). Stabilizing the line L and the grid $\mathcal{G}_a$ on L now yields a subgroup K of G_{NH} of order 3.6 (we have 9 lines L in $\mathcal{S}_{\mathsf{NH}}$, a unique regulus $\mathcal{R}$ on L and two remaining lines in $\mathcal{R}$). In other words, the group $K_{(12),(35),(46)}$ contains an order 3 element, say g. But no element of order 3 of S_6 fixes $\mathcal{G}_a$ point-wise. Hence this element g does not fix the subquadrangle $\mathsf{W}(2)$ on $\mathcal{G}_a$ and thus acts transitively on the 3 lines through the point (46) not in $\mathcal{G}_a$, and consequently also on $\mathcal{G}_b$, $\mathcal{G}_c$ and $\mathcal{G}_d$.

□

From now on, we will denote a line or a block by writing down the points it contains. With this notation there will be no possible confusion, as a line is incident with three points, while a block consists of four elements. A Vee-block is denoted in such a way that the first (and consequently also the last) two points belong to one of its legs. We use the convention of denoting the unique line on any two points x and y by xy. Finally, when dealing with a derived quadrangle $\mathcal{D}_x$ or a derived spread $\mathcal{S}_x$ we denote the lines of this substructure simply as the blocks of $\mathcal{D}$ on x, except for x equal to α in which case we use the notations of Q.

Lemma 3.4.6.
Let $\mathcal{R}_x$ be a given regulus of the derived spread $\mathcal{S}_x$ of $\mathcal{U}$, with $x \neq \alpha$. Then $\mathcal{R}_x$ contains at most one Line-block on x. Furthermore, if $\mathcal{R}_x$ contains a Line-block, say αxyz, then the grid $\mathcal{G}$ of Q that is determined by the line xyz and a Vee-block V of $\mathcal{R}_x$ is independent of the choice of V in $\mathcal{R}_x$.

Proof. First of all, since $\mathcal{U}$ contains a unique block on α and x, the regulus $\mathcal{R}_x$ will contain at most one Line-block on x.

To prove the second part of the lemma we define $\mathcal{G}$ as the grid of Q on xyz and a Vee-block V_1 of $\mathcal{R}_x$. Denote the leg of V_1 on x by xst and a point of $\mathcal{G}$ that is incident with i and j by g_{ij}, with $i \in \{s,t\}$ and $j \in \{y,z\}$. Without loss of generality we may choose the vee-point of V_1 as the point t. With these notations we have that the block V_1 is given by $xsg_{ty}g_{tz}$. Obviously, since αxst is a Line-block of $\mathcal{D}_x$, the second block, say V_2, of $\mathcal{R}_x$ has s as its vee-point. A combination of $xyg_{sz}g_{tz}$ being a Vee-block of $\mathcal{D}$ and $xsg_{ty}g_{tz}$

and αxyz being blocks of $\mathcal{R}_x$ forces V_2 to be equal to $xtg_{sz}g_{sy}$, which proves the lemma.

□

3.4.1 One Hermitian spread implies all Hermitian spreads

In this subsection we assume that a given unital $\mathcal{U}$ of $\mathcal{D}$ has at least one derived Hermitian spread and we will show that in this particular case all other derived spreads are Hermitian as well.

Without loss of generality we may assume that we obtain such a Hermitian spread by a one-point derivation in α and that $\mathcal{S}_{\mathsf{H}}$, as defined above, is this derived Hermitian spread.

Let L be any arbitrary line of $\mathcal{S}_{\mathsf{H}}$. We then have the following lemma.

Lemma 3.4.7.
The derived spread $\mathcal{S}_x$, with x a point on L, is completely determined by a line regulus of $\mathcal{S}_x$ on the line L. Furthermore, $\mathcal{S}_x$ is a Hermitian spread.

Proof. Since the automorphism group G_{H} acts transitively on the lines of $\mathcal{S}_{\mathsf{H}}$, we may choose L to be the line (12)(34)(56). Moreover, without loss of generality, we may choose x to be the point (12).

By Lemma 3.4.1 we can consider a regulus $\mathcal{R}_x$ on L regardless of $\mathcal{S}_x$ being Hermitian or not, while by Lemma 3.4.6 we know that this regulus $\mathcal{R}_x$ determines a grid $\mathcal{G}$ of Q on L. As $(G_{\mathsf{H}})_{L,(12)}$ acts transitively on the reguli of $\mathcal{S}_{\mathsf{H}}$ through L, we may suppose that (12)(35)(46) is the second line of $\mathcal{G}$ on (12). We now have $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_c$ and $\mathcal{G}_d$, as defined above, as the four grids on the lines (12)(34)(56) and (12)(35)(46) of Q. Moreover, the grid $\mathcal{G}_a$ contains the lines of $\mathcal{S}_{\mathsf{H}}$. We know that $\mathcal{S}_{\mathsf{H}}$ contains the unique block of $\mathcal{U}$ on (25) and (13), and (by Lemma 3.4.4) that $(G_{\mathsf{H}})_{\mathcal{G}_a}$ acts transitively on the set $\{\mathcal{G}_i \mid i \in \{b, c, d\}\}$. Hence we may choose $\mathcal{G}_b$ as the grid $\mathcal{G}$.

Let $\mathcal{G}'$ be the grid containing the spread line L and the lines $4(45)5'$ and $6(36)3'$ (the two latter lines are the unique lines of $\mathcal{S}_{\mathsf{H}}$ not containing a point of $\mathcal{G}_b$).

Consider the line (12)(45)(36) of $\mathcal{G}'$ on (12). We will now determine the possible spread lines of $\mathcal{S}_x$ on (45) and (36), respectively. To construct a Vee-block V on the points (12) and (45) we have to consider all lines through (36). By definition of a unital the second leg of V cannot be contained in

$\mathcal{G}'$, nor can it intersect $\mathcal{G}_b$ in any one of its points. Consequently, we have a unique choice for V, namely it has to have $3(36)6'$ as its second leg.

Moreover, in the exact same way as for (12) and (45), we find that the Vee-block on (12) and (36) has to have $5(45)4'$ as one of its legs. In other words, both the block on (12) and (45) and the one on (12) and (36) determine the same grid $\mathcal{G}''$ of Q. Hence, since L is now contained in two distinct reguli of $\mathcal{S}_x$, this derived spread is a Hermitian spread. By Lemma 3.4.3 we know that a regulus and a single line determine a Hermitian spread and hence $\mathcal{S}_x$ is completely determined by $\mathcal{R}_x$ and we are done.

□

As L was chosen arbitrary Lemma 3.4.7 holds for every point x in Q.

3.4.2 Hermitian spreads imply uniqueness

In this section we show that

Theorem 3.4.8.
The $2-(28,4,5)$ Hölz design contains, up to isomorphism, a unique unital that intersects all derived subdesigns in Hermitian spreads.

Proof. Without loss of generality we may fix the Hermitian spread $\mathcal{S}_{\mathsf{H}}$ in the derived generalized quadrangle $\mathcal{D}_\alpha$. If the construction of a unital containing this spread is hereby determined, up to isomorphism, then the theorem is proved.

Consider the line (12)(34)(56) of $\mathcal{S}_{\mathsf{H}}$ and define the four grids through this line and the line (12)(35)(46) by $\mathcal{G}_i$, $i \in \{a,b,c,d\}$, in the exact same way as we did before. As any unital on the set of blocks corresponding to the lines of $\mathcal{S}_{\mathsf{H}}$ already contains a block on (25) and (13), a block on (12) and (35) will be determined by a line on (46) off $\mathcal{G}_a$. By Lemma 3.4.4 we may choose $\mathcal{G}_b$ to be the grid containing this particular line. Since, by Lemma 3.4.7, every derived spread has to be Hermitian and, by Lemma 3.4.6, a grid defined by a regulus on a Line-block is independent of the Vee-blocks it contains, we thus find

$$\begin{array}{cccc} (12) & (35) & (15) & (23) \\ (12) & (46) & (26) & (14) \end{array}$$

as blocks of the unital. By Lemma 3.4.7 the derived spread $\mathcal{S}_{(12)}$ is determined.

We now want to determine a block through (34) and (35). As these points are opposite in Q we need to determine a vertex collinear to both, which is then the vee-point of the Vee-block containing these two points. This vertex cannot be the point (12), as otherwise the points (34) and (56) are in two distinct blocks of the unital, nor can (16) (respectively (26)) be that point (two distinct blocks on (35) and (24) (respectively (35) and (15))). Hence this vertex has to be either the point 3 or the point $3'$. Nevertheless, as $\mathcal{S}_{(12)}$ is given by

Blocks of $\mathcal{S}_{(12)}$		
α (12) (34) (56)	(12) (45) 3 6'	(12) 1' (25) 5'
(12) (35) (15) (23)	(12) (36) 4' 5	(12) 2 (16) 6
(12) (46) (26) (14)	(12) 2' (13) 3'	(12) 1 (24) 4

the automorphism

$$g : \begin{cases} \alpha \leftrightarrow (12) & \\ (1a) \leftrightarrow (2a) & \\ (ab) \leftrightarrow (cd) & \{a,b,c,d\} = \{3,4,5,6\} \\ b \leftrightarrow b' & b \notin \{1,2\} \end{cases}$$

swaps the spreads $\mathcal{S}_{\mathsf{H}}$ and $\mathcal{S}_{(12)}$ and interchanges the points 3 and $3'$. Hence, these two situations are equivalent and we may assume

$$(34)4'(35)5'$$

and consequently, by Lemma 3.4.6, also

$$(34)3(46)6$$

to be blocks of the unital. By Lemma 3.4.7 the derived spread $\mathcal{S}_{(34)}$ is determined.

Finally, considering the points (56) and (35) leads to the uniqueness of our unital, as we will see. Indeed, by similar arguments as used above we may exclude the lines of $\mathcal{G}_a$ and $\mathcal{G}_b$ to be legs of the Vee-block on (56) and (35). On the other hand, the lines of $\mathcal{S}_{(34)}$ imply that these legs cannot belong to $\mathcal{G}_c$ either (as otherwise we would have two blocks on the points 6 and 3). Meaning, the choice for a block through (56) and (35), and hence by Lemma 3.4.6 and Lemma 3.4.7, also $\mathcal{S}_{(56)}$ is determined.

To complete the proof of the theorem it now suffices to take a general point p of Q and show that the spread $\mathcal{S}_p$ is determined. Call L_p the line of $\mathcal{S}_{\mathsf{H}}$ on p.

On (12)(34)(56) there is a unique point u collinear to p. As $\mathcal{S}_u$ is determined, we thereby obtain a block on p and u. By Lemma 3.4.6 and Lemma 3.4.7 we obtain that $\mathcal{S}_p$ is determined and we are done.

□

3.4.3 Non-Hermitian spreads imply uniqueness

By a direct consequence of Lemma 3.4.7 we know that if a unital $\mathcal{U}$ of $\mathcal{D}$ has at least one derived non-Hermitian spread, then all other derived spreads have to be non-Hermitian as well.

In this section we start with the fixed non-Hermitian spread $\mathcal{S}_{\mathsf{NH}}$ of the derived generalized quadrangle $\mathcal{D}_\alpha$ and determine, up to isomorphism, all unitals on this spread. We define the grids $\mathcal{G}_i$, $i \in \{a, b, c, d\}$, as in the previous sections and prove the following theorem.

Theorem 3.4.9.
The $2-(28,4,5)$ Hölz design contains, up to isomorphism, a unique unital that intersects all derived subdesigns in non-Hermitian spreads.

Proof. We consider the points (12) and (35) and look at the unital block they determine. In despite of the fact that the group G_{NH} is by far as transitive as G_{H}, Lemma 3.4.5 implies that we are still able to choose the determining leg on (46) in the grid $\mathcal{G}_b$ and hence find the block V_1

$$(12)(35)(15)(23)$$

on these two points.

Taking into account that $\mathcal{S}_{(12)}$ has to be non-Hermitian, the line L can either determine a regulus of $\mathcal{S}_{(12)}$ with V_1 or not. We claim that the former case leads to a contradiction.

Indeed, suppose L is in a regulus with V_1. Then we have

$$\begin{array}{cccc} (12) & (35) & (15) & (23) \\ (12) & (46) & (26) & (14) \end{array}$$

as blocks of the unital.

We now have a unique choice for a block of the unital on the points (12) and (45) as such a block is determined by a line on (36) that cannot intersect $\mathcal{G}_b$,

nor contain the points $3'$ and 6 (these two belong to a common block of $\mathcal{S}_{\mathsf{NH}}$). Hence, by Lemma 3.4.3, we obtain a unique non-Hermitian spread $\mathcal{S}_{(12)}$

Blocks of $\mathcal{S}_{(12)}$		
α (12) (34) (56)	(12) (45) 3 6'	(12) 1 (25) 5
(12) (35) (15) (23)	(12) (36) 4 5'	(12) 2 (16) 6
(12) (46) (26) (14)	(12) 2' (13) 3'	(12) 1' (24) 4'

that is "compatible" with the derived spread $\mathcal{S}_{\mathsf{NH}}$.

We now look at a block on the points (34) and (35) and hence determine a suitable vee-point v. Considering the blocks of $\mathcal{S}_{\mathsf{NH}}$ and the block V_1 immediately yields that v is no point of $\mathcal{G}_a$, nor is it a point of $\mathcal{G}_b$. In other words, v can either be the point 3 or the point $3'$. In the exact same way we obtain two plausible choices for the vee-point of the block on (34) and (46), namely 4 and $4'$.

First suppose that both vee-points belong to $\mathcal{G}_c$. We then have

(34) 4' (35) 5'
(34) 3 (46) 6

as blocks of the derived spread $\mathcal{S}_{(34)}$. Together with $\mathcal{S}_{\mathsf{NH}}$ they immediately force $(34)3'(24)2'$ to be an element of $\mathcal{S}_{(34)}$ as $\{(14), 1'\}, \{(34), 5'\}, \{(34), (46)\}$ are already in blocks of $\mathcal{S}_{\mathsf{NH}}$ and $\mathcal{S}_{(34)}$. However, the pair $\{2', 3'\}$ is already in a block of $\mathcal{S}_{(12)}$, a contradiction.

Now suppose both vee-point lie in $\mathcal{G}_d$, then the following two blocks

(34) 4 (35) 5
(34) 3' (46) 6'

belong to $\mathcal{S}_{(34)}$ and we find a similar contradiction: under this assumption $(34)4'(13)1'$ has to be a block of $\mathcal{S}_{(34)}$, which gives us, next to the element in $\mathcal{S}_{(12)}$, a second block on $\{4', 1'\}$.

If, on the other hand, the blocks

(34) 4 (35) 5
(34) 3 (46) 6

belong to $\mathcal{S}_{(34)}$, then one can readily check that we can find no element of $\mathcal{S}_{(34)}$ on the point 2.

And, finally, if

(34) (35) 4' 5'
(34) (46) 3' 6'

are elements of the unital, then the pair $\{(35), 3'\}$ can never be in an block of $\mathcal{U}$ since $\{(35)(15)\}$, $\{3', 2'\}$, $\{(35), 4'\}$ and $\{(35), 6'\}$ are already in blocks of $\mathcal{S}_{(12)}$, $\mathcal{S}_{(12)}$, $\mathcal{S}_{(34)}$ and $\mathcal{S}_{(34)}$, respectively.

In other words, there exists no $\mathcal{S}_{(34)}$ compatible with $\mathcal{S}_{\mathsf{NH}}$ and the derived spread $\mathcal{S}_{(12)}$ and the claim is proved.

We have shown that (12)(46)(26)(14) cannot be a block of $\mathcal{U}$. Hence, the block on (12) and (46) will therefore be given by

$$(12)(46)3'5$$

or by

$$(12)(46)35'$$

of $\mathcal{D}$. We claim that the latter block cannot occur in $\mathcal{U}$.

An immediate consequence of Lemma 3.4.2 is that every line on (46) (not in $\mathcal{G}_a$) determines a regulus on (12)(35)(46) – and consequently also a grid of Q – containing two lines of $\mathcal{S}_{\mathsf{NH}}$. The line (46)(15)(23) determines such a grid with (12)(35)(46) containing the line $3(35)5'$ as one of its lines and this will be the reason why (12)(35)(15)(23) cannot be in a unital with $(12)(46)35'$ and $\mathcal{S}_{\mathsf{NH}}$.

Suppose, by way of contradiction, that the opposite is true. A block on (12) and (45) is determined by one of the lines

(36) (15) (24) (36) 3 6'
(36) (14) (25) (36) 6 3'

on (36). As, in this particular case, (12) is already in a block of the unital with (15) and with 3, and $(36)3'6$ is an element of $\mathcal{S}_{\mathsf{NH}}$, we conclude that

$$(12)(45)(14)(25)$$

is the only possible block on $\{(12), (45)\}$.

In the same way we find that

$$(12)(36)(26)(13)$$

has to belong to the unital. However the corresponding set of four lines in $\mathcal{D}_{(12)}$ cannot be completed into a spread, as we will show. First of all,

since the blocks on $\{(12),(23)\}$, $\{(12),(25)\}$, $\{(12),(26)\}$ and $\{(12),(13)\}$, $\{(12),(14)\}$, $\{(12),(15)\}$ are already determined in such a spread, we have a unique choice for the blocks on $\{(12),1\}$ and $\{(12),2\}$, namely the blocks (12)1(24)4 and (12)2(16)6. However, this leaves us no further possibilities for a block on $\{(12),1'\}$, nor for a block on $\{(12),2'\}$, a contradiction.

In conclusion, given the fixed non-Hermitian spread $\mathcal{S}_{\mathsf{NH}}$ and the block (12)(35)(15)(23), any unital containing these blocks will also contain the block (12)(46)3′5.

These two blocks in combination with the ones in $\mathcal{S}_{\mathsf{NH}}$ now leave us two possibilities both for the block on $\{(12),2\}$ and for the block on $\{(12),1'\}$. Two out of four combinations, however, lead to a contradiction and the remaining two combinations will be shown to be isomorphic. The block on (12) and 2 can either be determined by 1′(13)3 or by 1′(16)6, whereas the one on (12) and 1′ is determined by 2(25)5′ or by 2(24)4′. In chronological order these situations will be denoted by increasing numbers 1 to 4.

A combination of the first and third situation leads to a contradiction as there remains no acceptable block on $\{(12),(36)\}$. In the same way the second and fourth situation allows no block on $\{(12),2'\}$.

The first and fourth situation and the second and third situation, on the other hand, lead to unique non-Hermitian spreads $\mathcal{S}_{(12)}$ and $\mathcal{S}'_{(12)}$, respectively. Indeed, by Lemma 3.4.3, the line (12)(46)3′5 and the regulus

	(12)	(12)	(12)
(12)	α	(34)	(56)
(12)	1'	(24)	3
(12)	2	(13)	4'

completely fix all lines of the spread $\mathcal{S}_{(12)}$, while in $\mathcal{S}'_{(12)}$

	(12)	(12)	(12)
(12)	α	(34)	(56)
(12)	1'	5'	(25)
(12)	2	6	(16)

and (12)(46)3′5 yield uniqueness of the spread. The following tables list the blocks of these two derived spreads.

Blocks of $\mathcal{S}_{(12)}$			
(12)	α	(34)	(56)
(12)	(35)	(15)	(23)
(12)	(46)	3'	5
(12)	2'	(14)	4'
(12)	1	(24)	4
(12)	2	(16)	6
(12)	1'	(25)	5'
(12)	(45)	6'	3
(12)	(36)	(26)	(13)

Blocks of $\mathcal{S}'_{(12)}$			
(12)	α	(34)	(56)
(12)	(35)	(15)	(23)
(12)	(46)	3'	5
(12)	2'	(16)	6'
(12)	1	(26)	6
(12)	2	(13)	3
(12)	1'	(24)	4'
(12)	(45)	(14)	(25)
(12)	(36)	4	5'

Nevertheless, it is routine to check that

$$g : \begin{cases} 1 \leftrightarrow 2, 3 \leftrightarrow 5, 4 \leftrightarrow 6 & \\ i \leftrightarrow i' & \forall i \in \Omega \end{cases}$$

is an involution of G_{NH} that interchanges $\mathcal{S}_{(12)}$ to $\mathcal{S}'_{(12)}$. Hence it suffices to proceed using $\mathcal{S}_{(12)}$ as the non-Hermitian spread of $\mathcal{D}_{(12)}$ in the unital.

We will now determine the blocks of $\mathcal{S}_{(34)}$ and $\mathcal{S}_{(56)}$. First of all, wanting compatibility with the blocks of $\mathcal{S}_{\mathsf{NH}}$ and $\mathcal{S}_{(12)}$, the block on $\{(34),(35)\}$ and also the one on $\{(56),(35)\}$ (respectively on $\{(34),(46)\}$ and $\{(56),(46)\}$) has to have its vee-point in $\mathcal{G}_c$ or in $\mathcal{G}_d$ (respectively $\mathcal{G}_b$ or $\mathcal{G}_c$).

Suppose the vee-point of the block on $\{(34),(35)\}$ belongs to $\mathcal{G}_c$, while the one corresponding to $\{(34),(46)\}$ belongs to $\mathcal{G}_b$. More explicitly, suppose

(34) 4' (35) 5'
(34) (26) (46) (23)

are blocks of the unital. This choice of blocks immediately forces

(34) (25) 1 6'
(34) 3' (24) 4'

and consequently also

(34) (16) (14) (36)
(34) (15) (13) (45)

to be elements of the unital. However, this leaves us no choice for a block on $\{(34),4\}$, a contradiction.

In the same way the combination

(34) 4 (35) 5
(34) 3 (46) 6

of vee-points in $\mathcal{G}_d$ and $\mathcal{G}_c$ yields a situation where there is no acceptable block on $\{(34), 2\}$.

If both the block on $\{(34), (35)\}$ and the one on $\{(34), (46)\}$ are determined by vee-points in $\mathcal{G}_c$, then we are able to complete this set of blocks on (34) into a spread $\mathcal{S}_{(34)}$. Nevertheless, these two blocks, i.e.

(34) 4' (35) 5'
(34) 3 (46) 6

force us to take

(56) (35) 6' 3
(56) (46) (15) (14)

as blocks on (56) and we claim that this combination of blocks cannot be in $\mathcal{U}$. Indeed, a block on (35) and (26) is determined by one of the non-spread lines on the point (14), namely (14)(25)(36), (14)(23)(56) or $1(14)4'$, and each of these lines gives a contradiction with the known blocks of $\mathcal{S}_{(12)}$, $\mathcal{S}_{(56)}$ and $\mathcal{S}_{(34)}$, respectively. The claim is proved.

Hence the blocks on $\{(34), (35)\}$ and $\{(34), (46)\}$ are uniquely determined as the blocks

(34) 4 (35) 5
(34) (26) (46) (23)

with vee-points in $\mathcal{G}_d$ and $\mathcal{G}_b$, respectively. Since now $\{(34), (23)\}$ and $\{(34), (35)\}$ are in a block of $\mathcal{S}_{(34)}$, and $\{(25), 1'\}$ is in a block of $\mathcal{S}_{(12)}$ we find

$$(34)(25)16'$$

as the only possible block of the unital on $\{(34), (25)\}$. Given the set of blocks of $\mathcal{S}_{\mathsf{NH}}$ and $\mathcal{S}_{(12)}$ one can now carefully determine the compatible derived spread $\mathcal{S}_{(34)}$, which has the following block set

Blocks of $\mathcal{S}_{(34)}$		
(34) α (12) (56)	(34) (25) 1 6'	(34) 3' (45) 5'
(34) 4 (35) 5	(34) 4' (13) 1'	(34) (16) (36) (14)
(34) (26) (46) (23)	(34) 3 (24) 2	(34) (15) 2' 6

of $\mathcal{D}$. Note that the previous set of blocks are denoted in the order that they are forced to belong to $\mathcal{S}_{(34)}$.

We now have a unique choice for a unital block on $\{(56),(13)\}$. The Vee-blocks of $\mathcal{D}$ on these two points are determined by the lines on (24) and since $\{(13),4'\}$, $\{(13),(36)\}$ and $\{(16),(35)\}$ are in blocks of $\mathcal{S}_{(34)}$, $\mathcal{S}_{(12)}$ and $\mathcal{S}_{\mathsf{NH}}$, respectively, we can only consider $4(24)2'$ to be this line. In the mean time, carefully considering all blocks of $\mathcal{D}$ on $\{(56),1'\}$ results in another unique choice, namely the block $(56)6'(15)1'$. This forces us to take $(56)6(35)3$ and not $(56)6'(35)3'$ as the element on $\{(56),(35)\}$. One can readily check that these blocks determine a unique compatible $\mathcal{S}_{(56)}$.

Blocks of $\mathcal{S}_{(56)}$		
(56) α (12) (34)	(56) 6 (35) 3	(56) (14) 2 3'
(56) (13) 4 2'	(56) 5 (16) 1	(56) (24) (45) (26)
(56) 6' (15) 1'	(56) 5' (46) 4'	(56) (23) (36) (25)

To end the proof of the theorem it suffices to take a general point p of Q and show that the spread $\mathcal{S}_p$ is fixed. We first claim that $\mathcal{S}_p$ is fixed for all $p \in (12)^{\perp}$. Take p equal to (45). From the previously obtained derived spreads, we already know four out of the nine spread lines of $\mathcal{S}_{(45)}$, say $L_1,\ldots,L_4$. Showing that these four lines are as such that both L_1 and L_2 are not in a regulus contained in $\mathcal{S}_{(45)}$ with L_3 and L_4; nor is L_3 with L_4 implies the uniqueness of $\mathcal{S}_{(45)}$. Indeed, if this is true, then L_1 and L_2 necessarily determine a regulus of the spread and hence, by Lemma 3.4.3, $\mathcal{S}_{(45)}$ is fixed. Consider

$$\begin{array}{rcccccc} L_1 & = & (34) & 3' & (45) & 5' \\ L_2 & = & (56) & (24) & (45) & (26) \\ L_3 & = & \alpha & 5 & (45) & 4' \\ L_4 & = & (12) & (45) & 5 & 4' \end{array}$$

as the four known lines. After some calculations we find the following lines M_{ij}

$$\begin{array}{rcccccc} M_{13} & = & (35) & 3 & (45) & 4 \\ M_{14} & = & (45) & 5 & (46) & 6 \\ M_{23} & = & (45) & (13) & 6' & 2 \\ M_{24} & = & (15) & 1' & (45) & 4' \\ M_{34} & = & (45) & (36) & (34) & (56) \\ M_{12} & = & (45) & (36) & 2' & 1 \end{array}$$

as third lines in $\mathcal{R}(L_i, L_j)$. Since $\{(35), 3\}$, $\{(46), 5\}$, $\{(45), 6'\}$, $\{(15), 1'\}$ and finally $\{(34), (56)\}$ are already in blocks of $\mathcal{S}_{(56)}$, $\mathcal{S}_{(12)}$, $\mathcal{S}_{(12)}$, $\mathcal{S}_{(56)}$ and $\mathcal{S}_{\mathsf{NH}}$, respectively, we find on the one hand that $(45)(36)2'1$ is a block of the unital and on the other hand that $\mathcal{S}_{(45)}$ is fixed.

For p equal to (36) we immediately find, in addition to the line $(34)(36)(16)(14)$, a regulus of $\mathcal{S}_{(36)}$, namely

$$\begin{array}{cccc} & (36) & (36) & (36) \\ (36) & \alpha & 3' & 6 \\ (36) & (12) & (26) & (13) \\ (36) & (45) & 2' & 1 \end{array}$$

and hence, by Lemma 3.4.3, also $\mathcal{S}_{(36)}$ is fixed.

For $p \in \{(35), (46), 2, 1'\}$ we know that $(35)(46)21'$ determines a first line of the spread $\mathcal{S}_p$. Apart from this line we have six other, two by two distinct, lines (corresponding to $\mathcal{S}_{\mathsf{NH}}$, $\mathcal{S}_{(12)}$, $\mathcal{S}_{(34)}$, $\mathcal{S}_{(56)}$, $\mathcal{S}_{(45)}$ and $\mathcal{S}_{(36)}$) and obviously seven out of nine lines of the spread completely determine the spread.

If p equals $2'$ or 1, then we obtain at least seven distinct lines of $\mathcal{S}_p$ when considering all previous constructed spreads. Hence the derived spread $\mathcal{S}_p$ is fixed.

Finally, consider any point p of Q that is non-collinear to (12). Then $\mathcal{S}_p$ is determined by the unique elements of $\mathcal{S}_{\mathsf{NH}}$, $\mathcal{S}_{(12)}$ and $\mathcal{S}_{M_i^p}$ (with M_i, $i \in \{1, \ldots, 5\}$, a line on (12) and M_i^p the projection of p onto M_i) it belongs to. One can easily see that we thus establish a line set which uniquely determines all lines of $\mathcal{S}_p$ and we are done.

□

3.5 Conclusion

To conclude this chapter we note that the common point reguli in two generalized hexagons provides a sharper subdivision of the mutual position of two hexagons on the same $\mathsf{Q}(6, q)$ than the common lines do. Indeed, if two couples of generalized hexagons have the same point reguli intersection, they will also have the same line intersection, while the converse is not necessarily true. In fact, this is the sharpest subdivision possible considering two generalized hexagons on the same $\mathsf{Q}(6, q)$. For an improvement of this result one has to investigate the mutual position of three or more hexagons.

4 Distance-j ovoids and distance-j spreads of $\mathsf{H}(3)$

In this chapter, we construct and classify all distance-j ovoids and distance-j spreads of the known generalized hexagon of order 3 (the split Cayley hexagon $\mathsf{H}(3)$).

4.1 General Introduction

This chapter will be divided into two main parts.

In the first part, we construct a new type of ovoids, called the exceptional ovoids, and hereby classify all ovoids and spreads of $\mathsf{H}(3)$. We exhibit some unexpected and geometric properties of this new type of ovoids. As an application, we provide an elementary and geometric construction of a $\mathsf{G}_2(3)$-GAB of type $\widetilde{\mathsf{G}}_2$, exhibit new ovoid-spread pairings and investigate which 1-systems of $\mathsf{Q}(6,3)$ that are a derivation of the exceptional spread of $\mathsf{H}(3)$ are a spread of some hexagon on this quadric. This part of the chapter is based on two papers [19] and [26] that are joined works with J. Huizinga and H. Van Maldeghem, and H. Van Maldeghem, respectively.

In the second part, we construct a distance-2 spread of $\mathsf{H}(3)$. Furthermore we prove the uniqueness of this distance-2 spread in $\mathsf{H}(3)$ and show that its automorphism group is the linear group $\mathsf{PSL}_2(13)$. We observe that a distance-2 spread in any split Cayley hexagon $\mathsf{H}(q)$ is a line spread of the underlying polar space $\mathsf{Q}(6,q)$ and we construct a line spread of $\mathsf{Q}(6,2)$ that is not a distance-2 spread in any $\mathsf{H}(2)$ defined on $\mathsf{Q}(6,2)$. To conclude this part of the chapter we provide some applications concerning the chromatic

number of the point graph of $\mathsf{H}(3)$, the existence of distance-2 spreads in $\mathsf{T}(27,3)$, and embeddings of $\mathsf{H}(3)$. This result is joined work with H. Van Maldeghem and has been published in *Annals of Combinatorics* [20].

4.2 Part I: Ovoids and Spreads of H(3)

4.2.1 Introduction

It is well known that the parabolic quadric $\mathsf{Q}(6,3)$ admits, up to projectivity, a unique ovoid, the automorphism group of which is isomorphic to the symplectic group $\mathsf{S}_6(2) \cong \mathsf{O}_7(2)$, acting 2-transitively on the 28 points of the ovoid (see [43, 53]). Since every ovoid of $\mathsf{Q}(6,3)$ is an ovoid of every split Cayley hexagon $\mathsf{H}(3)$ naturally embedded in $\mathsf{Q}(6,3)$, and conversely (see [59]), we obtain all ovoids of $\mathsf{H}(3)$ by fixing one particular $\mathsf{H}(3)$ on $\mathsf{Q}(6,3)$ and considering the images of that ovoid under the full group of $\mathsf{Q}(6,3)$. This exercise can be done with a computer, but we carry it out by hand and hence establish geometric relationships between the ovoids and spreads. Note that $\mathsf{H}(3)$ is self-dual, hence by classifying the ovoids, we also classify the spreads. It turns out that there are exactly 3 non-isomorphic ovoids (and 3 non-isomorphic spreads): the classical Hermitian one, the Ree-Tits one, and a new one which has particularly nice properties. The first motivation for this research is exactly the discovery of the geometric properties of this new spread, which brings up the question whether generalization is possible. The second motivation is to provide a reference for the classification of all ovoids and spreads of $\mathsf{H}(3)$, since people in finite geometry sometimes want to use this, but cannot refer to the existing literature. Finally, a third motivation is that we provide an additional geometric interpretation of a maximal subgroup of the group $\mathsf{G}_2(3)$, and usually such interpretations can be used to construct other geometries. We illustrate this with a purely geometric construction of a Geometry that is Almost a Building (GAB) related to $\mathsf{G}_2(3)$, see Kantor [41].

As a second application we show that with this new type of spreads arises a new type of ovoid-spread pairing in $\mathsf{H}(3)$. As a consequence it will follow that the generalized hexagon $\mathsf{H}(3)$ does not admit the projective plane $\mathsf{PG}(2,3)$ as a locally isomorphic epimorphic image (this question remains open in general, i.e. for those $\mathsf{H}(q)$ that are known to admit both ovoids and spreads – hence $q = 3^n$, with n a positive integer – but is now answered for the smallest open case).

Finally we mention that the spreads of $\mathsf{H}(3)$ define 1-systems of $\mathsf{Q}(6,3)$, and the ones related to the exceptional spread of $\mathsf{H}(3)$ are new. They can be derived in 2^7-1 ways and we classify them as a final application of Part I.

We now state the main result of Part I of this chapter.

Theorem 4.2.1.
The hexagon $\mathsf{H}(3)$ contains, up to automorphisms, exactly three different spreads. One of them is a Hermitian spread, the second one is a Ree-Tits spread. The third one, which we will denote by $\mathcal{S}_{\mathsf{E}}$, consists of 7 disjoint line reguli the complementary reguli of which form the dual of the same spread. Its automorphism group G acts transitively on its elements, and 2-transitively on the seven line reguli, equivalent as permutation group to the action of $\mathsf{PSL}_3(2)$ on the seven points of the projective plane $\mathsf{PG}(2,2)$. Also, $G \equiv 2^3{\cdot}\mathsf{PSL}_3(2)$ is a maximal subgroup of the automorphism group $\mathsf{G}_2(3)$ of $\mathsf{H}(3)$.

We will call $\mathcal{S}_{\mathsf{E}}$ the *exceptional spread*, because it does not seem to belong to a family of spreads.

In the next section we will construct $\mathcal{S}_{\mathsf{E}}$. In Section 4.2.3, we carry out the classification of all spreads of $\mathsf{H}(3)$ and prove a large part of Theorem 4.2.1. In Section 4.2.4, we prove some beautiful geometric properties of $\mathcal{S}_{\mathsf{E}}$ (completing the proof of Theorem 4.2.1), and finally, in Section 4.2.5 we give some applications.

4.2.2 Construction of the spread $\mathcal{S}_{\mathsf{E}}$

Since -1 is a non-square over $\mathsf{GF}(3)$ we obtain, by Section 2.6.2,

$$\mathcal{S}_{\mathsf{H}} = \{[\infty]\} \cup \{[k,b,k',-k,b] \mid k,b,k' \in \mathsf{GF}(3)\}$$

as a Hermitian spread of $\mathsf{H}(3)$, which is determined by the hyperplane $\Pi : X_1 + X_5 = 0$. We immediately apply Lemma 2.6.3 onto the current situation, namely $q = 3$, and find that the equation of the Hermitian curve $\mathcal{U}$ corresponding to $\mathcal{S}_{\mathsf{H}}$ is given by

$$\mathcal{U} : \gamma X_2 X_2^3 = X_0 X_1^3 - X_1 X_0^3$$

with γ an element of $\mathsf{GF}(9)$ such that $\gamma^2 = -1$. The isomorphism $\Phi : \mathcal{S}_{\mathsf{H}} \to \mathcal{U}$ is given by

$$\begin{aligned}
&[\infty]^{\Phi} = (1,0,0),\\
&[k,b,k',-k,b]^{\Phi} = (\gamma b^2 + \gamma k^2 - kb + k', -1, \gamma k + b).
\end{aligned}$$

Any element x of $\mathsf{GF}(9)$ can be written as $x = I(x)\gamma + R(x)$, with $I(x), R(x) \in \mathsf{GF}(3)$. We call $I(x)$ the *imaginary* part of x and $R(x)$ the *real* part. With this notation, a point

$$r = (x_0, -1, x_2)$$

on $\mathcal{U}$ corresponds to the line

$$[I(x_2), R(x_2), R(x_0) + I(x_2)R(x_2), -I(x_2), R(x_2)]$$

of $\mathcal{S}_{\mathsf{H}}$. This is a direct consequence of $x_2 = \gamma k + b$ and $x_0 = \gamma k^2 - kb + k'$, and it provides a rather explicit form of the inverse Φ^{-1}.

Let ρ be the unique polarity of $\mathsf{PG}(2, 9)$ such that $\mathcal{U}$ is the set of absolute points of ρ.

Definition of the exceptional spread $\mathcal{S}_{\mathsf{E}}$

Let $\mathcal{R}_0$ be the regulus determined by the lines $[\infty]$ and $[0, 0, 0, 0, 0]$. The image of this regulus under Φ is the intersection of $\mathcal{U}$ with the line L having equation $X_2 = 0$.

Roughly, the construction of $\mathcal{S}_{\mathsf{E}}$ goes as follows. There are three Hermitian spreads containing $\mathcal{R}_0$. In each of these, we choose appropriately two additional reguli in such a way that, together with $\mathcal{R}_0$, these three reguli form, viewed as blocks of $\mathcal{U}$, a *polar triangle*, i.e. three blocks the corresponding lines in $\mathsf{PG}(2, 9)$ of which are pairwise conjugate under the polarity ρ (which means that the intersection of any two of these lines is the image of the third line under ρ).

First, we describe all Hermitian spreads containing $\mathcal{R}_0$.

From [17], we infer that an arbitrary Hermitian spread of $\mathsf{H}(q)$ through $[\infty]$ can be written in a unique way as $\mathcal{S}_{y,K,L,L'}$ and contains the lines

$$\{[\infty]\} \cup \{[k, yb^3 - yk^3K + L, yk' + yk^2K, -y^2k^3 + L' + y^2k^3K^2 + y^2b^3K, \\ yb + ykK] \mid k, b, k' \in \mathsf{GF}(q)\}$$

with $K, L, L' \in \mathsf{GF}(q)$ and y a non-zero square in the field $\mathsf{GF}(q)$. Applying this in the case $q = 3$ (hence $y = 1$), and noting that $\mathcal{R}_0$ belongs to the spread if and only if $L = L' = 0$ we find the three spreads

$$\mathcal{S}_{\mathsf{H}}^{(K)} = \{[\infty]\} \cup \{[k, b - kK, k' + k^2K, -k + kK^2 + bK, b + kK] \mid \\ k, b, k' \in \mathsf{GF}(q)\}$$

with $K \in \mathsf{GF}(3)$, through $\mathcal{R}_0$, with $\mathcal{S}_{\mathsf{H}}^{(0)} = \mathcal{S}_{\mathsf{H}}$. In fact, $\mathcal{S}_{\mathsf{H}}^{(K)}$ is the image of $\mathcal{S}_{\mathsf{H}}$ under the automorphism $\theta_{(K)}$ of $\mathsf{H}(3)$ determined by its image on the lines with 5 coordinates as

$$[k, b, k', b', k"]^{\theta_{(K)}} = [k, b - kK, k' + k^2K, b' + kK^2 + bK, k" + kK].$$

Note. This is in fact the dualized version of $\theta_{(K)}$ as defined in Section 2.7.1.

Now, every point $p_x = (1, x, 0)$, $x \in \mathsf{GF}(9) \setminus \mathsf{GF}(3)$, on L in $\mathsf{PG}(2, 9) \setminus \mathsf{PG}(2, 3)$ determines a unique polar triangle $\{L, p_xL^\rho, p_x^\rho\}$. Every line p_xL^ρ in $\mathsf{PG}(2, 9)$ defines a unique regulus $\mathcal{R}_{x,K}$ in $\mathcal{S}_{\mathsf{H}}^{(K)}$ with the property that $\mathcal{R}_{x,K}^{\theta_{(-K)}\Phi}$ is the block of $\mathcal{U}$ determined by intersecting $\mathcal{U}$ with p_xL^ρ.

We can now define

$$\mathcal{S}_{\mathsf{E}} = \mathcal{R}_0 \cup \mathcal{R}_{\gamma,0} \cup \mathcal{R}_{-\gamma,0} \cup \mathcal{R}_{\gamma+1,-1} \cup \mathcal{R}_{-\gamma+1,-1} \cup \mathcal{R}_{\gamma-1,1} \cup \mathcal{R}_{-\gamma-1,1}.$$

$\mathcal{S}_{\mathsf{E}}$ is a spread

In order to simplify the proof of Theorem 4.2.1 we will determine an explicit description of the lines of $\mathsf{H}(3)$ that are contained in $\mathcal{R}_{\epsilon,K}$, $\epsilon \in \mathsf{GF}(9) \setminus \mathsf{GF}(3)$, $K \in \mathsf{GF}(3)$. To do this, one must first determine the intersection in $\mathsf{PG}(2, 9)$ of $\mathcal{U}$ with the line $p_\epsilon L^\rho$, then apply Φ^{-1} and finally apply $\theta_{(K)}$. Every step has explicit formulae mentioned above, except for the first one. We indicate how to determine the intersection in $\mathsf{PG}(2, 9)$ of $\mathcal{U}$ with $p_\epsilon L^\rho$.

Note first that L^ρ has coordinates $(0, 0, 1)$. Hence $p_\epsilon L^\rho$ has equation $\epsilon X_0 = X_1$, and substituting this in the equation of $\mathcal{U}$, we find

$$\gamma = I(\epsilon)\gamma X_0^4.$$

As $I(\epsilon)$ equals either 1 or -1, this defines the four points

$$q_i = (x_i, \epsilon x_i, 1), \quad x_i^4 = I(\epsilon), \; i = 1, \ldots, 4$$

of $\mathcal{U} \cap p_\epsilon L^\rho$.

We want to write these points as $(x_0, -1, x_2)$ so that we can apply Φ^{-1}. To achieve this particular form we will consider two distinct cases.

- $R(\epsilon)$ equals zero.
  As ϵ is an element of $\mathsf{GF}(9) \setminus \mathsf{GF}(3)$, the imaginary part, $I(\epsilon)$, is non-zero and consequently $I(\epsilon)^2$ equals 1. Multiplying q_i with x_i^3 yields $q_i = (I(\epsilon), \gamma, x_i^3)$. Therefore, since $\epsilon = I(\epsilon)\gamma$ and $\gamma^4 = 1$, we find

  $$q_i = (\epsilon, -1, x_i'), x_i'^4 = I(\epsilon), i = 1, \ldots, 4$$

  as intersection points of $p_\epsilon L^\rho$ with $\mathcal{U}$.

- $R(\epsilon)$ is non-zero.
  A multiplication of $q_i = (I(\epsilon), \gamma + I(\epsilon)R(\epsilon), x_i^3)$ with the conjugate of the x_1-coordinate results in

  $$q_i = (\epsilon^3, -1, x_i'), x_i'^4 = -I(\epsilon), i = 1, \ldots, 4$$

  as intersection points.

Now we are able to map these points to their corresponding lines of $\mathsf{H}(3)$. In the first case

$$[I(x_i), R(x_i), R(\epsilon) + I(x_i)R(x_i), -I(x_i), R(x_i)], i = 1, \ldots, 4$$

is the corresponding regulus $\mathcal{R}_{\epsilon,0}$, where $x_i^4 = I(\epsilon)$.
In the second case we get the same equation, but now $x_i^4 = -I(\epsilon)$.

In conclusion, $\theta_{(K)}$ maps these lines onto the lines

$$\begin{aligned}&[I(x_i), R(x_i) - KI(x_i), R(\epsilon) + I(x_i)R(x_i) + KI(x_i)^2,\\ &- I(x_i) + K^2 I(x_i) + KR(x_i), R(x_i) + KI(x_i)]\end{aligned}$$

of $\mathcal{R}_{\epsilon,K}$, with $x_i^4 = I(\epsilon)(1 + R(\epsilon)^2)$, $i = 1, \ldots, 4$.

Lemma 4.2.2.
The set $\mathcal{S}_{\mathsf{E}}$ is a spread of $\mathsf{H}(3)$ and contains exactly seven line reguli, which are moreover pairwise disjoint.

Proof. The automorphism ψ ($= \theta_{(0,0,0,L,0)}$ as defined in Section 2.7.1) of $\mathsf{H}(3)$ determined by

$$\begin{aligned}(a, l, a', l', a'')^\psi &= (a, l, a', l' + L, a''),\\ [k, b, k', b', k'']^\psi &= [k, b, k' + L, b', k''],\end{aligned}$$

with $L \in \mathsf{GF}(3)$ preserves the regulus $\mathcal{R}_0$ and is compatible with the collineation

$$\psi' : \mathsf{PG}(2, 9) \to \mathsf{PG}(2, 9) : (x, y, z) \mapsto (x - Ly, y, z)$$

(meaning that for any element a of $\mathcal{S}_{\mathsf{H}}$ $a^{\psi\Phi} = a^{\Phi\psi'}$, which is easy to see). Putting $L = 1$ we obtain $(1, \gamma, 0)^{\psi'} = (1, -\gamma + 1, 0)$, and we will see that $\mathcal{R}_{\gamma,0}^{\psi\theta_{(-1)}} = \mathcal{R}_{-\gamma+1,-1}$. Indeed, this is true if $\mathcal{R}_{\gamma,0}^{\psi\theta_{(-1)}\theta_{(1)}\Phi}$ is the block of $\mathcal{U}$ determined by the intersection of $\mathcal{U}$ with the line $p_{-\gamma+1}L^\rho$. Obviously, since $\theta_{(-1)}\theta_{(1)} = \mathbb{1}$, this is equivalent with saying that $\mathcal{R}_{\gamma,0}^{\psi\Phi}$ has to be the block

corresponding to the line $p_{-\gamma+1}L^\rho$. As ψ is compatible with ψ' we first obtain $\mathcal{R}_{\gamma,0}^{\psi\Phi} = \mathcal{R}_{\gamma,0}^{\Phi\psi'}$ and next find that

$$\mathcal{R}_{\gamma,0}^{\psi\theta_{(-1)}\theta_{(1)}\Phi} = (p_\gamma L^\rho)^{\psi'} = p_{-\gamma+1}L^\rho,$$

which is exactly what we had to prove. Similarly, we deduce that $\psi\theta_{(-1)}$ maps $\mathcal{R}_{-\gamma,0}$ to $\mathcal{R}_{\gamma+1,-1}$.

In order to determine the respective images of $\mathcal{R}_{\gamma+1,-1}$ and $\mathcal{R}_{-\gamma+1,-1}$ under $\psi\theta_{(-1)}$ we note that $\theta_{(K)}^2 = \theta_{(-K)}$ and that ψ and $\theta_{(K)}$ commute in their action on elements with 5 coordinates. Now, since

$$\begin{aligned}(1,\gamma+1,0)^{\psi'} =&(-\gamma,\gamma+1,0)\\ =&(1,\gamma^2+\gamma,0)\\ =&(1,\gamma-1,0)\end{aligned}$$

we find that $\mathcal{R}_{\gamma+1,-1}^{\psi\theta_{(-1)}} = \mathcal{R}_{\gamma-1,1}$ if and only if

$$\mathcal{R}_{\gamma+1,-1}^{\psi\theta_{(-1)}\theta_{(-1)}\Phi} = \mathcal{R}_{\gamma+1,0}^{\Phi\psi'},$$

which is readily checked by taking the above remarks into account. Similarly, since

$$p_{-\gamma+1}^{\psi'} = p_{-\gamma-1},$$

we see that $\mathcal{R}_{-\gamma+1,-1}^{\psi\theta_{(-1)}} = \mathcal{R}_{-\gamma-1,1}$.

In the exact same way, $p_{\pm\gamma-1}^{\psi'} = p_{\mp\gamma}$ yields $\mathcal{R}_{\pm\gamma-1,1}^{\psi\theta_{(-1)}} = \mathcal{R}_{\mp\gamma,0}$.

In conclusion, $\psi\theta_{(-1)}$ is an automorphism of $\mathsf{H}(3)$ that fixes $\mathcal{R}_0$, preserves $\mathcal{S}_\mathsf{E}$ and acts as an element of order 3 on the remaining 6 line reguli of $\mathcal{S}_\mathsf{E}$. More precisely,

$$\begin{array}{lllll}\psi\theta_{(-1)}: & \mathcal{R}_{\pm\gamma,0} & \to & \mathcal{R}_{\mp\gamma+1,-1},\\ & \mathcal{R}_{\mp\gamma+1,-1} & \to & \mathcal{R}_{\mp\gamma-1,1},\\ & \mathcal{R}_{\mp\gamma-1,1} & \to & \mathcal{R}_{\pm\gamma,0}.\end{array}$$

Similarly for $\psi^2\theta_{(1)}$ we find

$$\begin{array}{lllll}\psi^2\theta_{(1)}: & \mathcal{R}_{\pm\gamma,0} & \to & \mathcal{R}_{\mp\gamma-1,1},\\ & \mathcal{R}_{\mp\gamma-1,1} & \to & \mathcal{R}_{\mp\gamma+1,-1},\\ & \mathcal{R}_{\mp\gamma+1,-1} & \to & \mathcal{R}_{\pm\gamma,0}.\end{array}$$

Moreover, the mapping $\varphi : \mathsf{H}(3) \to \mathsf{H}(3)$ that fixes the lines $[\infty]$ and $[0,0,0,0,0]$, that maps the point (∞) onto the point (0) and for which

$$\begin{aligned}
&[k,b,k',b',k'']^\varphi = [-k'',-b',kk''+k',b,k]\\
&(a,l,a',l',a'')^\varphi = (-\frac{1}{a},-\frac{b}{a},-\frac{b}{a^2}-\frac{a'}{a},l'-\frac{l^2}{a},\frac{l}{a}-a'-a''a), a\neq 0\\
&(0,l,a',l',a'')^\varphi = (-l,-a',l',a'')
\end{aligned}$$

is an automorphism of $\mathsf{H}(3)$ (see Section 2.7.1, put $r=0$ in $\theta''_{(r)}$ and use the fact that $\mathsf{H}(3)$ is self-dual). Furthermore, we claim that φ fixes $\mathcal{R}_0$, $\mathcal{R}_{\gamma,0}$ and $\mathcal{R}_{-\gamma,0}$ and preserves all three the Hermitian spreads on $\mathcal{R}_0$. Clearly φ fixes $\mathcal{R}_0$. Now let

$$\{[I(x_i),R(x_i),I(x_i)R(x_i),-I(x_i),R(x_i)] \mid x_i^4=\epsilon\}$$

be the set of lines in $\mathcal{R}_{\epsilon\gamma,0}$, with $\epsilon^2=1$. Since either $x_i\in\{\gamma+1,\gamma-1,-\gamma+1,-\gamma-1\}$ or $x_i\in\{1,-1,\gamma,-\gamma\}$, replacing $I(x_i)$ and $R(x_i)$ by $-R(x_i)$ and $I(x_i)$, respectively, does not alter the resulting set of lines. Hence $\mathcal{R}_{\gamma,0}$ and $\mathcal{R}_{-\gamma,0}$ and consequently also $\mathcal{S}_\mathsf{H}$ are preserved by φ.

Let $L=[k,b,k',b+k,b-k]$ be any line of $\mathcal{S}_\mathsf{H}^{(1)}$. The image of this line under φ is given by $[k-b,-b-k,kb-k^2+k',b,k]$. A simple renaming of the entries of the latter shows us that L^φ belongs to $\mathcal{S}_\mathsf{H}^{(1)}$ as well. In the same way, one readily shows that also $\mathcal{S}_\mathsf{H}^{(-1)}$ is fixed under φ and the claim is proved.

Hence φ preserves $\mathcal{S}_\mathsf{E}$. Moreover, it is an easy calculation to see that $\mathcal{R}_{\gamma-1,1}$ and $\mathcal{R}_{-\gamma-1,1}$ (and also $\mathcal{R}_{\gamma+1,-1}$ and $\mathcal{R}_{-\gamma+1,-1}$) are interchanged by φ. We conclude that the automorphism group of $\mathcal{S}_\mathsf{E}$ acts transitively on the six line reguli $\mathcal{R}_{\pm\gamma+\epsilon,-\epsilon}$, with $\epsilon\in\mathsf{GF}(3)$.

So it suffices to show that the regulus $\mathcal{R}_{-\gamma,0}$ is opposite every other regulus in the definition of $\mathcal{S}_\mathsf{E}$. For this, we claim that it suffices to show that the subspace of $\mathsf{PG}(6,3)$ spanned by two lines of $\mathcal{R}_{-\gamma,0}$ and two lines of any of the line reguli $\mathcal{R}_{\gamma+1,-1}$, $\mathcal{R}_{-\gamma+1,-1}$, $\mathcal{R}_{\gamma-1,1}$, $\mathcal{R}_{-\gamma-1,1}$, is an elliptic hyperplane. Indeed, in that case, the regulus $\mathcal{R}_{-\gamma,0}$ and each one of the reguli $\mathcal{R}_{\gamma+1,-1}$, $\mathcal{R}_{-\gamma+1,-1}$, $\mathcal{R}_{\gamma-1,1}$, $\mathcal{R}_{-\gamma-1,1}$ are contained in a Hermitian spread, and consequently must be opposite, hence the claim.

To calculate the coordinates of two points on the lines of a regulus $\mathcal{R}_{\epsilon,K}$, we use the last line of Table 2.1.

One can now check with an elementary calculation that the line reguli $\mathcal{R}_{-\gamma,0}$, $\mathcal{R}_{\gamma+1,-1}$ and $\mathcal{R}_{\gamma-1,1}$ are contained in the hyperplane with equation $X_2+X_6=0$, which is clearly an elliptic hyperplane. By way of example, we

show that $\mathcal{R}_{\gamma+1,-1}$ is contained in this particular hyperplane. First of all, by the preliminary calculations above, we know that a line of $\mathcal{R}_{\gamma+1,-1}$ has coordinates

$$[I(x_i), R(x_i) + I(x_i), 1 + I(x_i)R(x_i) - I(x_i)^2, -R(x_i), R(x_i) - I(x_i)],$$

with $x_i^4 = -1$, hence $x_i = \pm\gamma \pm 1$. In other words, $R(x_i)^2 = I(x_i)^2 = 1$ for all $i \in \{1, 2, 3, 4\}$. Using the last line of Table 2.1 we find that a first point of such a line has $x_2 = 1$ and $x_6 = b^2 - b'k = I(x_i)^2 + R(x_i)^2 = -1$, while a second point has $x_2 = 0$ and $x_6 = -kk'' - k' + bb' = -1 - I(x_i)^2 - R(x_i)^2 = 0$. Either way, both these points belong to $X_2 + X_6 = 0$.

Likewise, the reguli $\mathcal{R}_{-\gamma,0}$, $\mathcal{R}_{-\gamma+1,-1}$ and $\mathcal{R}_{-\gamma-1,1}$ are contained in the – elliptic – hyperplane with equation $X_0 + X_4 = 0$.

This proves the first part of the lemma.

Now suppose that $\mathcal{S}_{\mathsf{E}}$ contains a line regulus $\mathcal{R}$ not contained in one of the $\mathcal{S}_{\mathsf{H}}^{(K)}$, $K \in \mathsf{GF}(3)$. Since $\mathcal{R}$ contains 4 lines, two of them must be contained in the same $\mathcal{S}_{\mathsf{H}}^{(K)}$, for some $K \in \mathsf{GF}(3)$. But then $\mathcal{R}$ is entirely contained in $\mathcal{S}_{\mathsf{H}}^{(K)}$, a contradiction. The second part of the lemma now follows easily.

The proof of the lemma is complete.

□

4.2.3 Classification of all ovoids and spreads of H(3)

We will now show our main result. We classify all ovoids of $\mathsf{H}(3)$. Dualizing, we obtain a classification for spreads.

First we note that all ovoids of $\mathsf{H}(3)$ are isomorphic, on $\mathsf{Q}(6,3)$, to each other, and they have automorphism group $\mathsf{S}_6(2) = \mathsf{O}_7(2)$. Hence $\mathsf{H}(3)$ contains exactly

$$\frac{|\mathsf{O}_7(3)|}{|\mathsf{O}_7(2)|} = \frac{2^{10}.3^9.5.7.13}{2^9.3^4.5.7} = 2.3^5.13$$

ovoids. Every Hermitian ovoid has automorphism group isomorphic to $\mathsf{G}_2(2)$. Hence the number of Hermitian ovoids equals

$$\frac{|\mathsf{G}_2(3)|}{|\mathsf{G}_2(2)|} = \frac{2^6.3^6.7.13}{2^6.3^3.7} = 3^3.13.$$

Next, the number of Ree-Tits ovoids is equal to

$$\frac{|\mathsf{G}_2(3)|}{|\mathsf{Ree}(3)|} = \frac{2^6.3^6.7.13}{2^3.3^3.7} = 2^3.3^3.13.$$

Hence there are exactly $3^5.13$ ovoids of $\mathsf{H}(3)$ not isomorphic to a Hermitian or a Ree-Tits one. Let $\mathcal{O}$ be any such ovoid. Its automorphism group has size at least $|\mathsf{G}_2(3)|/3^5.13 = 2^6.3.7$. Now let $\mathcal{O}_\mathsf{E}$ be the dual of $\mathcal{S}_\mathsf{E}$.

We remark that the stabilizer in $\mathsf{G}_2(3)$ of $\mathcal{S}_\mathsf{H}$, the line $[\infty]$, the line reguli $\mathcal{R}_0$, $\mathcal{R}_{\gamma,0}$ and $\mathcal{R}_{-\gamma,0}$ and an arbitrary line of $\mathcal{R}_{\gamma,0}$ has order at most 2. Indeed, this follows from translating the situation to the plane $\mathsf{PG}(2,9)$, where one fixes a quadrangle, and hence only a Baer involution can be involved. Moreover, any isomorphism of $\mathsf{H}(3)$ fixing all lines of $\mathcal{S}_\mathsf{H}$ is trivial.

Also, if we fix the line $[\infty]$ and preserve $\mathcal{S}_\mathsf{E}$, then we must preserve the line regulus $\mathcal{R}_0$. Hence, from all this follows immediately that the automorphism group of $\mathcal{S}_\mathsf{E}$, and hence of $\mathcal{O}_\mathsf{E}$, has at most $28.24.2 = 2^6.3.7$ elements. Hence it has exactly $2^6.3.7$ elements and there are exactly $3^5.13$ ovoids isomorphic to $\mathcal{O}_\mathsf{E}$.

It now follows from the above that the automorphism group G of $\mathcal{S}_\mathsf{E}$ acts transitively on its elements, and doubly transitive on its line reguli. From the list of maximal subgroups of $\mathsf{G}_2(3)$, see [11], we easily deduce from the order of G that G must be isomorphic to the non-split extension $2^3{\cdot}\mathsf{PSL}_3(2)$, which is itself a maximal subgroup of $\mathsf{G}_2(3)$.

We will prove that the action of G on the reguli of $\mathcal{S}_\mathsf{E}$ is as described in Theorem 4.2.1 in the next section in a geometric way.

4.2.4 Some geometric properties of $\mathcal{S}_\mathsf{E}$

Recall from the proof of Lemma 4.2.2 that the line reguli $\mathcal{R}_{-\gamma,0}$, $\mathcal{R}_{\gamma+1,-1}$ and $\mathcal{R}_{\gamma-1,1}$ are contained in an elliptic hyperplane, and so are the line reguli $\mathcal{R}_{-\gamma,0}$, $\mathcal{R}_{-\gamma+1,-1}$ and $\mathcal{R}_{-\gamma-1,1}$. Using the transitivity of G, we now conclude that the geometry with point set the line reguli of $\mathcal{S}_\mathsf{E}$, and line set the sets of three reguli $\mathcal{S}_\mathsf{E}$ contained in a common elliptic hyperplane, is a projective plane. This provides a geometric interpretation of the subgroup $\mathsf{PSL}_3(2)$ of $\mathsf{G}_2(3)$.

We will derive a rather strange and unusual geometric property of $\mathcal{S}_\mathsf{E}$. Namely, we construct a dual of $\mathcal{S}_\mathsf{E}$ by taking the complementary point reguli of the line reguli of $\mathcal{S}_\mathsf{E}$. This is the content of the next proposition.

Proposition 4.2.3.
The set of points $\mathcal{O}$ obtained by taking all complementary point reguli of all line reguli contained in $\mathcal{S}_\mathsf{E}$ is an ovoid isomorphic to $\mathcal{O}_\mathsf{E}$.

Proof. Consider the reguli $\mathcal{R}_0$ and $\mathcal{R}_{-\gamma,0}$. The former one is contained in the 3-space with equations $X_1 = X_3 = X_5 = 0$; the latter is contained in the 3-space with equations $X_0 + X_4 = X_1 + X_5 = X_2 + X_6 = 0$. The complementary reguli are thus contained in the planes $\pi_1 = \langle e_1, e_3, e_5 \rangle$ and $\pi_2 = \langle e_0 + e_4, e_1 + e_5, e_2 + e_6 \rangle$, respectively, where e_i denotes the element of $\mathsf{PG}(6,2)$ with 1 in its ith entry and 0 in all other entries (counting positions from 0 to 6).

The points of $\mathsf{Q}(6,q)$ in the plane π_1 have coordinates $(0, u^2, 0, uv, 0, v^2, 0)$, with $u, v \in \mathsf{GF}(3)$. Those in π_2 have coordinates $(a, b, c, 0, a, b, c)$, with $-b^2 = a^2 + c^2$, $a, b, c \in \mathsf{GF}(3)$. Note that automatically $b \neq 0$, otherwise $a = b = c = 0$, a contradiction. Now, to check whether two points of $\mathsf{H}(3)$ are opposite is equivalent to checking whether they are not conjugate with respect to $\mathsf{Q}(6,3)$. But the point $(0, u^2, 0, uv, 0, v^2, 0)$ is conjugate to the point $(a, b, c, 0, a, b, c)$ if and only if $b(u^2 + v^2) = 0$, or, equivalently, $u^2 + v^2 = 0$. This can only happen when $u = v = 0$, a contradiction.

By the double transitivity of G on the reguli of $\mathcal{S}_{\mathsf{E}}$, we have shown that $\mathcal{O}$ is an ovoid. Since clearly, $\mathcal{O}$ and $\mathcal{S}_{\mathsf{E}}$ have the same automorphism group, we conclude that $\mathcal{O}$ is isomorphic in $\mathsf{H}(3)$ to $\mathcal{O}_{\mathsf{E}}$.

□

4.2.5 Some applications

We first provide a geometric construction of a GAB of type $\widetilde{\mathsf{G}}_2$ related to $\mathsf{G}_2(3)$. Further on, we classify all types of ovoid-spread pairings in $\mathsf{H}(3)$ and investigate all possible derivations of the 1-system associated to $\mathcal{S}_{\mathsf{E}}$.

A GAB of type $\widetilde{\mathsf{G}}_2$

Before coming to the actual proposition, we need some preliminary definitions.

> A GAB – which stands for *a* G*eometry that is* A*lmost a* B*uilding* – of type $\widetilde{\mathsf{G}}_2$ is a 4-tuple $\Delta = (\mathcal{H}, \mathcal{L}, \mathcal{P}, *)$, where $*$ is a symmetric incidence relation in $\mathcal{H} \cup \mathcal{L} \cup \mathcal{P}$, satisfying (G1) and (G2):
>
> (G1) If $x * y$ and $x \in \mathcal{H}$ (respectively $\mathcal{L}$, $\mathcal{P}$), then $y \notin \mathcal{H}$ (respectively $\mathcal{L}$, $\mathcal{P}$).

For (G2) we need a definition. Let $x \in \mathcal{H} \cup \mathcal{L} \cup \mathcal{P}$ be arbitrary. Then $\mathrm{Res}_\Delta(x)$ is the geometry consisting of all elements of Δ incident with x and with induced incidence relation.

(G2) For all $h \in \mathcal{H} : \mathrm{Res}_\Delta(h)$ is a generalized hexagon.
For all $l \in \mathcal{L} : \mathrm{Res}_\Delta(h)$ is a complete bipartite graph.
For all $p \in \mathcal{P} : \mathrm{Res}_\Delta(p)$ is a projective plane.

In [12], Cooperstein constructs a GAB of type $\widetilde{\mathsf{G}}_2$, with automorphism group $\mathsf{G}_2(3)$, using the split octaves over the field with three elements. We will not repeat this construction here, since it would consume too much space. We will not explicitly prove that our construction yields the same GAB as in Cooperstein [12], since we did not explain Cooperstein's construction. The equivalence is, however, apparent from the two constructions. In Δ, the residues that are generalized hexagons are – up to duality – isomorphic to $\mathsf{H}(2)$ and the projective planes are $\mathsf{PG}(2,2)$. The following proposition provides an alternative construction of Δ.

Proposition 4.2.4.
Define the following rank 3 *geometry* Δ*. The elements of type* 1 *of* Δ *are the Hermitian ovoids of* $\mathsf{H}(3)$*, the elements of type* 2 *are the point reguli of* $\mathsf{H}(3)$*, and the elements of type* 3 *of* Δ *are the exceptional ovoids of* $\mathsf{H}(3)$*. Incidence is defined as follows. Between elements of type* 2 *and* 1*, respectively* 3*, incidence is symmetrized containment; a Hermitian ovoid is incident with an exceptional one if they share exactly three point reguli. Then* Δ *is a geometry that is a* GAB *of type* $\widetilde{\mathsf{G}}_2$ *for the group* $\mathsf{G}_2(3)$*.*

Proof. This follows from the following observations. Counting the number of Hermitian ovoids and of exceptional ovoids containing a fixed point regulus, one obtains three in both cases. Each of the three Hermitian ovoids meets each of the exceptional ovoids in three reguli.

Now fix a Hermitian ovoid $\mathcal{O}_\mathsf{H}$. There is a bijection between the set of exceptional ovoids meeting $\mathcal{O}_\mathsf{H}$ in three point reguli and the set of polar triangles of the corresponding Hermitian curve in $\mathsf{PG}(2,9)$. Moreover, the generalized hexagon $\mathsf{H}(2)$ can be constructed as follows: the lines are the reguli of $\mathcal{O}_\mathsf{H}$ and the points are the polar triangles (this follows from applying ρ to the construction of Tits in [64]). The proposition is now clear.

□

In Cooperstein's construction, the exceptional spreads were not recognized or identified. In fact, no object was interpreted inside the generalized hexagon $\mathsf{H}(3)$. In this respect, our construction is complementary to the one of Cooperstein.

Ovoid-spreads pairings

Ovoid-spread pairings were introduced in [67] and used in [30, 31] to characterize local isomorphisms of generalized hexagons to projective planes (a *local isomorphism* is an epimorphism that maps each point row and each line pencil isomorphically onto its image). Every such local isomorphism, however, gives rise to a partition of the point set into ovoids, and a partition of the line set into spreads such that these ovoids and spreads pair off into ovoid-spread pairings.

Now it is easy to find an ovoid-spread pairing in $\mathsf{H}(3)$ using a Ree-Tits ovoid $\mathcal{O}_{\mathsf{RT}}$ and an exceptional spread $\mathcal{S}_{\mathsf{E}}$. Indeed, explicitly, let ρ be the polarity of $\mathsf{H}(3)$ induced by interchanging the parentheses with the square brackets and let $\mathcal{O}_{\mathsf{RT}}$ be the corresponding Ree-Tits ovoid. In coordinates we find

$$\begin{aligned}\mathcal{O}_{\mathsf{RT}} = \{&(\infty), && (0,0,0,0,0), && (0,0,1,1,0),\\ &(0,0,-1,-1,0), && (0,1,0,0,1), && (0,1,1,1,1),\\ &(0,1,-1,-1,1), && (0,-1,0,0,-1), && (0,-1,1,1,-1),\\ &(0,-1,-1,-1,-1), && (1,0,0,-1,1), && (1,0,1,0,1),\\ &(1,0,-1,1,1), && (1,1,0,0,-1), && (1,1,1,1,-1),\\ &(1,1,-1,-1,-1), && (1,-1,0,1,0), && (1,-1,1,-1,0),\\ &(1,-1,-1,0,0), && (-1,0,0,1,1), && (-1,0,1,-1,1),\\ &(-1,0,-1,0,1), && (-1,1,0,0,-1), && (-1,1,1,1,-1),\\ &(-1,1,-1,-1,-1), && (-1,-1,0,-1,0), && (-1,-1,1,0,0),\\ &(-1,-1,-1,1,0)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_{\mathsf{E}} = \{&\mathcal{R}([0],[0,0,0,0]), && \mathcal{R}([1,0,0,-1],[-1,0,0,1]),\\ &\mathcal{R}([1,1,-1,-1],[-1,1,1,1]), && \mathcal{R}([1,-1,1,-1],[-1,-1,-1,1]),\\ &\mathcal{R}([1,0,1,-1,0],[1,-1,0,-1,-1]), && \mathcal{R}([1,0,-1,-1,0],[1,1,0,-1,1]),\\ &\mathcal{R}([1,1,1,-1,1],[1,-1,-1,-1,-1])\}.\end{aligned}$$

In fact, from this we can derive an alternative description of the exceptional spreads. Indeed, one can see that the four points of $\mathcal{O}_{\mathsf{RT}}$ incident with the respective lines of a line regulus contained in $\mathcal{S}_{\mathsf{E}}$ form a block of the corresponding Ree-Tits unital (this follows immediately from the geometric construction of the blocks of this unital from a Ree-Tits ovoid as explained in Section 2.6.3). Conversely, every block of the unital defines a unique line

regulus (again by Section 2.6.3). Now the above ovoid-spread pairing defines seven pair-wise disjoint blocks of the Ree-Tits unital on 28 points. We call this a *block spread* of the unital. An example of a block spread is the smallest orbit on the block set of the Ree-Tits unital of the maximal subgroup of $\mathsf{Ree}(3) \cong \mathsf{P\Gamma L}_2(8)$ corresponding to the one point stabilizer of the projective line $\mathsf{PG}(1,8)$. Let us call such a block spread *exceptional*. Then the block spread defined by the pairing above is exceptional. Since its stabilizer has index 9 in $\mathsf{Ree}(3)$, there arise nine ovoid-spread pairings using the fixed ovoid $\mathcal{O}_{\mathsf{RT}}$. By the two-transitive action of $\mathsf{P\Gamma L}_2(8)$ on $\mathsf{PG}(1,8)$, every two such spreads have at least one line in common (as otherwise there arise 9 lines through every point of the ovoid!).

Remark. Without proof we mention that there are no other ovoid-spread pairings at all (one can check this with the aid of a computer, or even by hand). It now easily follows that no locally isomorphic epimorphic image of $\mathsf{H}(3)$ on $\mathsf{PG}(2,3)$ exists.

The construction of the blocks of the Ree-Tits unital now imply the following elegant description of the exceptional spreads. Let $\mathcal{O}$ be any Ree-Tits ovoid in $\mathsf{H}(3)$, and let $\mathcal{S}$ be the spread naturally associated with $\mathcal{O}$ (i.e. there is a polarity ρ of $\mathsf{H}(3)$ with $\mathcal{O}$ as set of absolute points and $\mathcal{S}$ as set of absolute lines). Choose an exceptional block spread $\mathcal{B}$ in $\mathcal{O}$. For each block B in $\mathcal{B}$, we consider the set of lines obtained by projecting any line $L \in B^\rho$ onto any point $x \in B \setminus \{L^\rho\}$. According to [17] this gives us a line regulus $\mathcal{L}_B$. The seven line reguli thus obtained form an exceptional spread.

Alternatively, for each block B as above, one can consider the set of points obtained by projecting any point $x \in B$ onto any line y^ρ, $y \in B \setminus \{x\}$. Again by [17] this gives us a point regulus $\mathcal{P}_B$. The seven point reguli thus obtained form an exceptional ovoid.

Note. As the point regulus $\mathcal{P}_B$ and the line regulus $\mathcal{L}_B$ are clearly complementary reguli, these respective descriptions of the exceptional spread and ovoid provide a geometrical clarification of Proposition 4.2.3.

It is rather remarkable and funny to notice that to define the GAB described above, we can start with a Ree-Tits ovoid, construct an exceptional spread from it (using a maximal subgroup of the corresponding Ree group), and then use all exceptional spreads and Hermitian spreads to construct the GAB. This way, all classes of spreads of $\mathsf{H}(3)$ are involved in this GAB.

We observe that the exceptional block spreads of the Ree-Tits unital on 28 points are not the only block spreads. As we do not need this, we do not prove this either.

A transitive 1-system of $\mathsf{Q}(6,3)$

If π^+ and π^- are the two hexagon twin planes of Π (as defined in Section 4.2.2) over $\mathsf{GF}(9)$, then according to Section 2.6.2 the lines of $\mathcal{S}_\mathsf{H}$ meet both planes in the points of a Hermitian curve, which we will call $\mathcal{U}^\pm$, with $\mathcal{U}^+ = \mathcal{U}$. Put $\Phi^\pm$ the corresponding isomorphism between the lines of $\mathcal{S}_\mathsf{H}$ and the points of $\mathcal{U}^\pm$. The image of $\mathcal{R}_0$ under $\Phi^\pm$ is then the intersection of $\mathcal{U}^\pm$ with a line $L_0^\pm$.

In this subsection we will make use of the observation that when considering the set of all line reguli of $\mathcal{S}_\mathsf{E}$ as a point set and the sets of three reguli contained in a common elliptic hyperplane as line set one obtains a geometry isomorphic to the projective plane of order 2.

Let Γ be this geometry, isomorphic to $\mathsf{PG}(2,2)$, with as point set the reguli of $\mathcal{S}_\mathsf{E}$ and where three points are on a line if they – as line reguli of $\mathcal{S}_\mathsf{E}$ – belong to the same Hermitian spread. From now on we will denote the line reguli of $\mathcal{S}_\mathsf{E}$ by $\mathcal{R}_0, \ldots, \mathcal{R}_6$ and use the standard description of Γ as a difference set. Namely, for each $i \in \mathbb{Z}$ mod 7, there is a line $\{\mathcal{R}_i, \mathcal{R}_{i+1}, \mathcal{R}_{i+3}\}$.

Note. It now makes sense to talk about switching points of Γ when meaning to switch the corresponding line reguli.

By [18] we know that switching any number of line reguli of $\mathcal{S}_\mathsf{E}$ yields a 1-system of $\mathsf{Q}(6,3)$. It is now straightforward to see that the 1-system obtained by a "full"derivation of $\mathcal{S}_\mathsf{E}$ (i.e. switch all seven reguli) admits $2^3 \cdot \mathsf{PSL}_3(2)$ acting transitively.

We start by determining when a derivation belongs to some $\mathsf{H}_2 \neq \mathsf{H}_1$.

Take H_1 and H_2, two models of $\mathsf{H}(3)$ on $\mathsf{Q}(6,3)$, and let $\mathcal{S}_\mathsf{E}$ be an exceptional spread of the hexagon H_1. Denote a derivation of $\mathcal{S}_\mathsf{E}$ by $\mathcal{S}'_\mathsf{E}$, the common line set of those two hexagons by S and the set of switched line reguli by χ. Since $\mathcal{S}'_\mathsf{E}$ is a proper derivation of $\mathcal{S}_\mathsf{E}$, the set Ω_2 – as defined in Section 3.2 – will be non-empty. By Lemma 3.2.4, of the previous chapter, we may thus conclude that $\mathsf{H}_2 = \mathsf{H}_1^g$, where g is the involutive collineation linked to Π_S, the hyperplane generated by S. Furthermore we know that p, the polar point of Π_S, belongs to every 3-space generated by any one of the switched reguli. In other words, if the lines of $\mathcal{R}_i$ determine the 3-space Υ_i then p belongs to Υ_i when $\mathcal{R}_i$ is one of the line reguli that we switch.

Lemma 4.2.5.
As soon as χ contains the points of a line of Γ as a subset, the thus obtained derivation of $\mathcal{S}_\mathsf{E}$ can never be a spread of any other hexagon H_2.

Proof. Suppose $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_3$ are three such line reguli. These three determine, by definition of the exceptional spread, a polar triangle Δ^+ in π^+. In the exact same way they also determine a polar triangle Δ^- in π^-. Denote a point of these respective polar triangles by $r_{ij}^{\pm}$ if it is the intersection point of $L_i^{\pm}$ and $L_j^{\pm}$, where $L_i^{\pm}$ is the line containing all points of $\mathcal{R}_i^{\Phi^{\pm}} \cap \mathcal{U}^{\pm}$.

With this notation at hand it is easy to see that $\Upsilon_0 \cap \Upsilon_1 = r_{01}^{+} r_{01}^{-}$. However, these two points (r_{01}^{+} and r_{01}^{-}) belong to Υ_3^{ρ} which is a plane disjoint of Υ_3. Hence Υ_0, Υ_1 and Υ_3 have an empty intersection, which is contradictory to the fact that p belongs to this intersection.

□

Lemma 4.2.6.
If there is a line of Γ on which exactly one point is switched, then $\mathcal{S}'_{\mathsf{E}}$ can never be a spread of any other hexagon.

Proof. Suppose by way of contradiction that there exists a line L on which we switch a single point t and suppose that the hexagon H_2 contains $\mathcal{S}'_{\mathsf{E}}$. The two remaining points on L correspond to two reguli of $\mathcal{S}_{\mathsf{E}}$ that are not switched and are hence entirely contained in Π_S, as defined above. Since these two reguli generate Π_S, the lines of t belong to S and the set of lines opposite the lines of t belong to H_2, a contradiction.

□

As an immediate consequence of Lemma 4.2.5 and 4.2.6 we see that the only possibility for a derivation of $\mathcal{S}_{\mathsf{E}}$ to be a spread of H_2 is obtained by not switching all points on a line of Γ and switching the rest.

Proposition 4.2.7.
The 1-system obtained by not switching all points on a line of Γ and switching the rest is a spread of H_2.

Proof. To prove this theorem, a crucial property will be that the regulus $\mathcal{R}_0$ is contained in three distinct Hermitian spreads. This phenomenon can be explained by looking at the extension of $\mathsf{PG}(6,3)$ to $\mathsf{PG}(6,9)$. Indeed, take any two lines M and N of $\mathcal{R}_0$ and consider these lines in $\mathsf{H}(9)$. On those lines we have three pairs of conjugate points and each of these pairs determines a hyperbolic hyperplane of $\mathsf{Q}(6,9)$ (the focus of each transversal to M and N in these two points, together with $\mathcal{R}_0$ completely determines this hyperplane), which is an elliptic hyperplane of $\mathsf{Q}(6,3)$.

For instance, if $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_3$ are the three reguli contained in $\mathcal{S}_{\mathsf{H}}$, the Hermitian spread we started from, then the intersection of $L_0^{\pm}$, where $L_0^{\pm} =$

$\mathcal{R}_0^{\Phi^\pm} \cap \mathcal{U}^\pm$, with M and N gives us the corresponding conjugate pair of points. Furthermore r_{13}^-, respectively r_{13}^+, is the point of $\mathsf{H}(9)$ collinear to both intersection points of L_0^+, respectively L_0^-, with M and N.

Let α be the polar image of Υ_0 and put $\overline{\alpha}$ the extension of α over $\mathsf{GF}(9)$. The intersection of α with $\mathsf{Q}(6,3)$ is a conic, say C. The extension of C over $\mathsf{GF}(9)$, denoted by $\overline{C}$, contains six additional points of which r_{13}^+ and r_{13}^- are a conjugate pair. As this first pair corresponds to $\mathcal{S}_\mathsf{H} = \mathcal{S}_\mathsf{H}^0$ we will, from now on, denote them by r_0^+ and r_0^-. In general we put (r_i^+, r_i^-) as the conjugate pair of points corresponding to the Hermitian spread $\mathcal{S}_\mathsf{H}^i$, with $i = 0, 1, -1$.

We are now ready to start the actual proof of the theorem.

Suppose $\overline{\chi} = \{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_3\}$ and switch all other points of Γ. Without loss of generality we may suppose $\mathcal{R}_2$, $\mathcal{R}_6$ and $\mathcal{R}_4$, $\mathcal{R}_5$ together with $\mathcal{R}_0$ to be reguli of the Hermitian spreads $\mathcal{S}_\mathsf{H}^1$ and $\mathcal{S}_\mathsf{H}^{-1}$, respectively. If $p = \Pi_S^\rho$, with $\Pi_S = \langle \Upsilon_0, \Upsilon_1, \Upsilon_3 \rangle$, is a point of every one of the 3-spaces $\Upsilon_2, \Upsilon_4, \Upsilon_5, \Upsilon_6$, then the obtained 1-system is a spread of $\mathsf{H}_2 (= \mathsf{H}_1^g)$.

First of all, one can easily see that $\Upsilon_2 \cap \Upsilon_6 = r_1^+ r_1^-$ while $\Upsilon_4 \cap \Upsilon_5 = r_{-1}^+ r_{-1}^-$. As both these lines contain two conjugate points they are lines that belong to α and hence intersect in a point, say $t \in \alpha$. It now suffices to show that t equals p to complete the proof.

As we already know two lines of α on t and as every line on t and a point of C is either a tangent or an intersection line, the two remaining lines of α through t are intersection lines of C.

Since $r_0^+ r_0^-$ is a line of α, it does not pass through t. Hence tr_0^+ and tr_0^- are tangent lines of $\overline{C}$, meaning $r_0^+ r_0^-$ belongs to t^ρ. On the other hand, a simple application of the polarity gives us, as t belongs to Υ_0^ρ $(= \alpha)$, that also Υ_0 is a subspace of t^ρ. Hence the span of this 3-space and the line $r_0^+ r_0^-$ belongs to the polar hyperplane of t. As both Υ_0 and $r_0^+ r_0^-$ are disjoint subspace of Π_S, they in fact span this particular hyperplane. In other words, we have that $t^\rho = p^\rho$ or that t equals p and we are done.

□

A nice consequence of this theorem is that there are only 4 equivalence classes of derived 1-systems of $\mathcal{S}_\mathsf{E}$ on $\mathsf{Q}(6,3)$, as we will show. We can also determine the automorphism groups in each case. Suppose $\mathcal{S}_\mathsf{E}$ is an exceptional spread of H_1.

Corollary 4.2.8.
If $\mathcal{S}_\mathsf{E}'$ is a derivation of $\mathcal{S}_\mathsf{E}$, then, up to isomorphism, either

(*i*) *the set χ is empty;*

(*ii*) *the set* χ *contains a unique point;*

(*iii*) *the set* χ *contains two points;*

(*iv*) *the set* χ *contains three points on a line of* Γ.

In cases (*i*) *and* (*iv*)*, the automorphism group of* $\mathcal{S}'_{\mathsf{E}}$ *has the structure* $2^3 \cdot \mathsf{PSL}_3(2)$*; in the other two cases we have* $2^3.\mathsf{S}_4$*, where* S_4 *is a maximal subgroup of* $\mathsf{PSL}_3(2)$ *corresponding to a point stabilizer in* $\mathsf{PG}(2,2)$.

Proof. The first three equivalence classes are obtained trivially (the automorphism group acts 2-transitively on the set of line reguli).

Now suppose χ contains three points that are not on a line. Without loss of generality we may assume $\mathcal{R}_0, \mathcal{R}_3, \mathcal{R}_6$ to be these three points (with the labelling of Γ as described above). By additionally switching $\mathcal{R}_5$, we obtain (as a result of Theorem 4.2.7) a derivation of $\mathcal{S}_{\mathsf{E}}$ that is contained in another hexagon H_2. Mapping H_1 to H_2 and switching only $\mathcal{R}_5$ shows the equivalence between this situation and situation (*ii*).

When χ contains three points on a line a similar technique always results in the switching of three points on a line in the new hexagon. Hence this is an isomorphism class which is non-equivalent with the classes (*i*) up to (*iii*).

For the remaining situations, the cardinality of χ will be greater than or equal to 4 and it will be advisable to look at the complement of this set of points, denoted by $\overline{\chi}$.

Suppose $\overline{\chi}$ contains three points in a triangle. Again without loss of generality we may assume this triangle to contain the points $\mathcal{R}_0, \mathcal{R}_3, \mathcal{R}_6$. This situation is equivalent to situation (*iii*), as switching $\mathcal{R}_2$ and $\mathcal{R}_3$ gives us a derivation contained in some H_2.

Not switching the points on a line and switching the rest is just how we obtain a derivation that yields a spread of H_2, hence this is equivalent with situation (*i*).

Switching the unique third point on the line containing both non-switched points of $\overline{\chi}$ shows the equivalence between $|\chi| = 5$ and situation (*ii*).

While switching two points on any line through the unique element of $\overline{\chi}$ demonstrates the equivalence between $|\chi| = 6$ and situation (*iii*).

Finally, suppose we have switched all points of Γ. Switching back three points on a line results in the spread of Theorem 4.2.7. Hence switching all points of Γ can be reduced to situation (*iv*) and we are done.

Regarding the automorphism groups, the one of case (*i*) follows from Theorem 4.2.1. It is clear that, if all reguli are switched, then we obtain again the

same automorphism group; whence case (iv). In case (ii), the regulus that can be switched in order to obtain a spread in a hexagon is unique; clearly the automorphisms of the bottom group 2^3 also act on every derivation. This proves the case (ii). For case (iii), it is similarly enough to remark that the third point on the line joining the two points that are switched is unique with respect to the following geometric property: if it is switched, then we obtain a spread of type (iv).

This completes the proof of the corollary.

□

The following theorem states a general result concerning similar questions starting from a Hermitian spread $\mathcal{S}_{\mathsf{H}}$ of H_1, a model of $\mathsf{H}(q)$.

Proposition 4.2.9.
If we switch disjoint blocks of the Hermitian spread $\mathcal{S}_{\mathsf{H}}$ of H_1, then the following statements are equivalent.

- (i) *The obtained set of lines is a spread of some hexagon $\mathsf{H}_2 \neq \mathsf{H}_1$.*
- (ii) *The obtained set of lines is isomorphic to $\mathsf{H}(2, q^2)$ on $\mathsf{Q}^-(5, q)$.*
- (iii) *All blocks conjugate to some given block B are switched.*

Proof. We start by proving the equivalence between (i) and (ii). Suppose $\mathcal{S}'_{\mathsf{H}}$, the derived set of lines, is a spread of H_2. This fact, together with the knowledge that switching lines does not alter the space they are in, easily implies that $\mathcal{S}'_{\mathsf{H}}$ is a Hermitian spread. Thus situation (i) implies situation (ii).

If $\mathcal{S}'_{\mathsf{H}}$ is isomorphic to $\mathsf{H}(2, q^2)$ on $\mathsf{Q}^-(5, q)$, then there are $q + 1$ hexagons – hence at least one – containing this set of lines as a subset. Hence the equivalence between the first two cases is shown.

In order to prove the equivalence of (ii) and (iii), we dualize the situation and consider ovoids of the Hermitian generalized quadrangle $\mathsf{H}(3, q^2)$. Note that an ovoid of $\mathsf{H}(3, q^2)$ is Hermitian if and only if all points of it are contained in a plane of the ambient projective space $\mathsf{PG}(3, q^2)$. Also, a regulus is here a set of points on a secant line, and "switching a regulus" corresponds to "substituting a secant line all of whose points are contained in the ovoid by the conjugate line with respect to the (unitary) polarity of $\mathsf{PG}(3, q^2)$ defined by $\mathsf{H}(3, q^2)$"; we will briefly say that we "replace a block (by its conjugate)". It is now clear that, in a Hermitian ovoid $\mathcal{O}$ generating the plane π, replacing one block by its conjugate does not produce a Hermitian ovoid because the unchanged points still span π. Also, if we replace two blocks B_1 and B_2

by their conjugates B_1' and B_2', respectively, then clearly all conjugates of lines in the plane π' spanned by B_1' and B_2' are incident with the intersection point z of the lines defined by the blocks B_1 and B_2. Hence, in order to obtain an ovoid contained in π', we have to get rid of every point x not in the intersection of π and π' by replacing the unique block on the line xz. The block B defined by $\pi \cap \pi'$ must remain unchanged, and that is exactly the block all of whose points are conjugate to z with respect to the polarity in π corresponding to the Hermitian curve $\mathcal{O}$. The equivalence between (ii) and (iii) now follows.

□

4.3 Part II: Distance-2 ovoids and distance-2 spreads of H(3)

4.3.1 Introduction

The only known partition into lines of the point set of a finite generalized hexagon happens for the dual of the classical generalized hexagon $\mathsf{H}(2)$ (see Chapter 2, Section 2.6). This distance-2 spread is unique, up to isomorphism, in $\mathsf{H}(2)^D$.

Since no other construction of distance-2 spreads is known, this example seems to be a coincidence due to the small parameters of $\mathsf{H}(2)$. In the present chapter, we construct another example, namely in $\mathsf{H}(3)$. It will turn out that also this example has a fairly big (transitive) automorphism group, and many nice geometric properties, as we will see. In this chapter, we will prove:

Theorem 4.3.1.
The hexagon $\mathsf{H}(3)$ *contains, up to automorphism, a unique distance-*2 *spread (and by the self-duality of* $\mathsf{H}(3)$*, a unique distance-*2 *ovoid). Its automorphism group* G *is isomorphic to the projective special linear group* $\mathsf{PSL}_2(13)$*, which is a maximal subgroup of the automorphism group* $\mathsf{G}_2(3)$ *of* $\mathsf{H}(3)$*.*

As stated in Section 2.5 of Chapter 2 every distance-2 spread of $\mathsf{H}(q)$ is a line spread of $\mathsf{Q}(6,q)$. The converse is not necessarily true, since not all lines of the quadric belong to the hexagon. We will give an example of this phenomenon below. In particular, we will prove:

Theorem 4.3.2.
The polar space $\mathsf{Q}(6,2)$ *contains a line spread, while the generalized hexagon* $\mathsf{H}(2)$ *does not admit any distance-*2 *spread.*

In the next subsection we will give a geometrical construction of a distance-2 spread and a distance-2 ovoid. In Section 4.3.3 we determine all flag matchings of the projective plane $\mathsf{PG}(2,3)$. In Section 4.3.4 we use this classification in order to show the uniqueness of the distance-2 spread constructed in Section 4.3.2. In Section 4.3.5 we will give an example of a line spread in $\mathsf{Q}(6,2)$ that cannot be induced by a distance-2 spread of $\mathsf{H}(2)$. Finally, in Section 4.3.6, we present some applications. Namely, we prove the non-existence of a $q+1$-coloring in the point graph of $\mathsf{H}(q)$ for $q=2,3$, state some results on the distance-2 spreads of small twisted triality hexagons and show that a distance-2 ovoid is not a good starting point to lift the standard embedding of $\mathsf{H}(3)$ to an embedding in $\mathsf{PG}(7,3)$.

4.3.2 Construction of a distance-2 spread and distance-2 ovoid

Consider the set Ω consisting of the following 14 points in $\mathsf{PG}(6,3)$, not belonging to $\mathsf{H}(3)$:

$$\begin{aligned}
\overline{\infty} &= (0,0,0,1,0,0,0), & \overline{0} &= (1,0,0,-1,0,0,1),\\
\overline{1} &= (0,0,1,1,-1,0,0), & \overline{2} &= (1,-1,0,1,-1,-1,1),\\
\overline{3} &= (0,1,-1,-1,1,-1,-1), & \overline{4} &= (1,1,-1,1,1,0,1),\\
\overline{5} &= (1,1,0,-1,1,-1,1), & \overline{6} &= (0,1,0,-1,1,0,-1),\\
\overline{7} &= (0,1,1,-1,0,1,-1), & \overline{8} &= (1,0,1,-1,0,1,0),\\
\overline{9} &= (1,-1,-1,-1,1,1,0), & \overline{10} &= (1,0,-1,1,1,1,1),\\
\overline{11} &= (1,-1,1,-1,0,1,1), & \overline{12} &= (1,1,1,-1,-1,1,0).
\end{aligned}$$

This set of points is stabilized by the group elements φ_∞ and φ_0 of $\mathsf{G}_2(3)$ with respective matrices

$$A_\infty = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & -1 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & -1 & 1\\
0 & 0 & 0 & 0 & -1 & -1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

and consequently also by the group G generated by these elements.

Note. The image of a point $\overline{x}$ is obtained by right multiplication, i.e. $\overline{x}^{\varphi_\infty} = \overline{x}A_\infty$ and $\overline{x}^{\varphi_0} = \overline{x}A_0$.

We now consider the elements $\overline{x}$, $x \in \{0, 1, 2, \ldots, 12\}$, as elements of $\mathsf{GF}(13)$ in the obvious way. We then note that φ_∞ fixes $\overline{\infty}$ and maps $\overline{x}$ to $\overline{x+1}$, while φ_0 fixes $\overline{0}$ and maps $\overline{x}$ to $\overline{x/(x+1)}$, with usual multiplication and addition laws if ∞ is involved. This easily implies that G is isomorphic to $\mathsf{PSL}_2(13)$, and that we may identify Ω with the points of the projective line $\mathsf{PG}(1, 13)$ (in the natural way), at least concerning the action of G.

Now consider the line through the points $\overline{\infty}$ and $\overline{0}$. The two other points on this line have coordinates $(1, 0, 0, 1, 0, 0, 1)$ and $(1, 0, 0, 0, 0, 0, 1)$. Meaning, this line intersects $\mathsf{H}(3)$ in a unique point, namely the point with coordinates $(1, 0, 0, 0, 0, 0, 1)$ in $\mathsf{PG}(6, 3)$. As G acts 2-transitively on Ω, we conclude that every two points in Ω determine a unique point of $\mathsf{H}(3)$, which we call the *peak* of the corresponding pair of points. The set of peaks will be denoted by $\mathcal{O}_\Omega$. We will prove below:

Proposition 4.3.3.
The set $\mathcal{O}_\Omega$ contains exactly 91 *points and constitutes a distance-*2 *ovoid of* $\mathsf{H}(3)$.

Now we consider the set Ω^* of polar hyperplanes of the points of Ω. We denote the polar hyperplane of the point $\overline{x}$ by $\overline{x}^*$, $x \in \mathsf{GF}(13) \cup \{\infty\}$. In

coordinates, we have

$$\begin{aligned}
\overline{\infty}^* &= [0,0,0,-1,0,0,0], & \overline{0}^* &= [0,0,-1,1,-1,0,0],\\
\overline{1}^* &= [1,0,0,-1,0,0,-1], & \overline{2}^* &= [1,1,-1,-1,-1,1,0],\\
\overline{3}^* &= [-1,1,1,1,0,-1,1], & \overline{4}^* &= [-1,0,-1,-1,-1,-1,1],\\
\overline{5}^* &= [-1,1,-1,1,-1,-1,0], & \overline{6}^* &= [-1,0,1,1,0,-1,0],\\
\overline{7}^* &= [0,-1,1,1,0,-1,-1], & \overline{8}^* &= [0,-1,0,1,-1,0,-1],\\
\overline{9}^* &= [-1,-1,0,1,-1,1,1], & \overline{10}^* &= [-1,-1,-1,-1,-1,0,1],\\
\overline{11}^* &= [0,-1,-1,1,-1,1,-1], & \overline{12}^* &= [1,-1,0,1,-1,-1,-1].
\end{aligned}$$

where $[x_0,\ldots,x_6]$ represents the hyperplane $x_0X_0+\ldots+x_6X_6=0$.

It is clear that all these hyperplanes are hyperbolic (as $\overline{\infty}^*$ is, and as G acts transitively on Ω, and hence also on Ω^*).

This time we choose two elements of Ω^* and look at their intersection with the line set of $\mathsf{H}(3)$. By the double transitivity it suffices to consider the two hyperplanes $\overline{\infty}^*$ and $\overline{0}^*$. Their intersection contains only the lines

$$[\infty],[0],[1],[-1],[0,0],[0,1],[0,-1]$$

(in coordinates of $\mathsf{H}(3)$) and some more points of $\mathsf{H}(3)$ (as is easily checked). The line $[\infty]$ is the unique line in this intersection that is concurrent with all lines of the intersection. We call it the *ridge* of the intersection, and we consider the set $\mathcal{S}_{\Omega^*}$ of ridges of the intersection of all pairs of elements of Ω^*. We will prove below:

Proposition 4.3.4.
The set $\mathcal{S}_{\Omega^*}$ contains exactly 91 *lines and constitutes a distance-*2 *spread of* $\mathsf{H}(3)$.

Since the peak of the pair $\{\overline{\infty},\overline{0}\}$ is the point (1), since the ridge of the pair $\{\overline{\infty}^*,\overline{0}^*\}$ is the line $[\infty]$ and since G acts 2-transitively on both Ω and Ω^*, we have that the peak of a pair of points is incident with the ridge of the corresponding pair of hyperplanes, but it is not incident with any other line of $\mathsf{H}(3)$ in the intersection of those hyperplanes. Moreover, one can readily check that (-1) does not belong to $\mathcal{O}_\Omega$. Hence, the ridge of any two hyperplanes contains a unique peak, namely the one corresponding to the associated points of Ω.

Now we consider the action of φ_∞ on the space $\overline{\infty}^*$. Since φ_∞ stabilizes $\mathsf{H}(3)$ and has odd order, it also stabilizes the two hexagon twin planes of $\overline{\infty}^*$. Since

these planes have exactly 13 points, since the order of φ_∞ is equal to 13, and since φ_∞ does not fix all points of the hexagon twin planes (the latter would imply that φ_∞ fixes all point of $\overline{\infty}^*$, a contradiction; this is also clear from the matrix of φ_∞ which does not act trivially on the planes with equations $X_0 = X_1 = X_2 = 0$ and $X_4 = X_5 = X_6 = 0$), the collineation φ_∞ induces a Singer cycle in both hexagon twin planes. Hence the ridges L_x of the pairs $\{\overline{\infty}^*, \overline{x}^*\}$, with $x \in \mathsf{GF}(13)$, form an orbit under this Singer cycle and they cover all points of the subhexagon Γ' defined by the hyperplane $\overline{\infty}^*$.

By the primitive action of $\mathsf{PSL}_2(13)$ on the pairs of points of $\mathsf{PG}(1,13)$ it already follows that $|\mathcal{O}_\Omega| = |\mathcal{S}_{\Omega^*}| = 91$, since the previous paragraph implies that they contain at least two elements.

Up to this point, we have that none of the lines of $\mathcal{S}_{\Omega^*}$ in $\overline{\infty}^*$ are concurrent. In other words, the ridge of the pair $\{\overline{\infty}^*, \overline{x}^*\}$ never intersects the ridge of the pair $\{\overline{\infty}^*, \overline{y}^*\}$, with $x, y \in \mathsf{GF}(13)$. Therefore, by 2-transitivity of G, we immediately have that the ridge of the pair $\{\overline{a}^*, \overline{b}^*\}$ is non-concurrent with the ridge of the pair $\{\overline{a}^*, \overline{c}^*\}$, nor is it concurrent with the ridge of the pair $\{\overline{b}^*, \overline{c}^*\}$, and this for all choices $a, b, c \in \mathsf{GF}(13) \cup \{\infty\}$.

An easy counting argument shows that there are six ridges L_x, $x \neq 0$, not opposite L_0, and six ridges L_y opposite L_0. Considering the group action of G, it is clear that these sets correspond with the squares and non-squares in $\mathsf{GF}(13)$. Let L_x be a ridge not opposite L_0. Let p_i be the peak of the pair $\{\overline{\infty}, \overline{i}\}$, $i \in \mathsf{GF}(13)$. Then obviously p_0 and p_x are opposite points in $\mathsf{H}(3)$. Now consider the points p_0, p_1 and the peak $p_{0,1}$ of the pair $\{\overline{0}, \overline{1}\}$. Notice that $p_1 = p_0^{\varphi_\infty}$ and $p_{0,1} = p_0^{\varphi_0}$. In coordinates we have $p_0 = (1,0,0,0,0,0,1)$, $p_1 = (0,0,1,0,-1,0,0)$ and $p_{0,1} = (1,0,1,0,-1,0,1)$. Clearly these three points are collinear, hence p_0 is not opposite p_1. Consequently L_0 is opposite L_1, and hence also opposite L_y, for y a non-zero square in $\mathsf{GF}(13)$. It follows that L_0 is not opposite L_z, for z a non-square in $\mathsf{GF}(13)$.

Now suppose, by way of contradiction, that the ridge $L_{a,b}$ of some pair $\{\overline{a}, \overline{b}\}$ meets L_0 (by the foregoing we know $a, b \neq 0$). There are two possibilities.

First, L_a and L_b are not opposite. Then there is a unique line M of Γ' meeting both L_a and L_b. If M were the ridge of $\{\overline{a}^*, \overline{b}^*\}$, then L_a would be concurrent with $L_{a,b}$, a contradiction. Since $L_a \in \overline{a}^* \cap \overline{\infty}^*$ and $L_b \in \overline{b}^* \cap \overline{\infty}^*$ and given the explicit form of these intersections, we easily see that M belongs to $\overline{a}^* \cap \overline{b}^*$. Hence $L_{a,b}$ meets $\overline{\infty}^*$ in a unique point on M off Γ' and so $L_{a,b}$ is not concurrent with L_0.

Secondly, L_a and L_b are opposite. By a foregoing argument, the peak $p_{a,b}$ of $\{\overline{a}, \overline{b}\}$ is contained in $\overline{\infty}^*$, and so clearly $p_{a,b}$ is the intersection of L_0 and $L_{a,b}$

(indeed, $L_{a,b}$ meets $\overline{\infty}^*$ in a unique point, which must necessarily be $p_{a,b}$). This contradicts the fact that there is a unique peak on every ridge.

Since L_0 is essentially arbitrary, the proof of Proposition 4.3.4 is complete.

We now prove Proposition 4.3.3. In principle, this could be done by examining the peaks of some representative in each orbital of the action of G on the pairs of Ω. We here present a much nicer argument.

The group G acts transitively on the 91 elements of $\mathcal{S}_{\Omega^*}$. Furthermore, the stabilizer $G_{\overline{\infty}}$ in G of $\overline{\infty}$ acts transitively on the lines of Γ' not contained in $\mathcal{S}_{\Omega^*}$ (this follows immediately from the fact that the stabilizer in $G_{\overline{\infty}}$ of the line L_0 acts transitively on the six ridges in Γ' not at maximal distance of L_0). Let Θ be the set of lines of $\mathsf{H}(3)$ contained in some $\overline{x}^*$, $x \in \mathsf{GF}(13) \cup \{\infty\}$, but not contained in $\mathcal{S}_{\Omega^*}$. We claim that each of these lines is contained in exactly three such hyperplanes since in every such hyperplane, it is concurrent with exactly two ridges. Indeed, let L be such a line in a fixed hyperplane $\overline{a}^*$ and denote the two concurrent ridges by R_1 and R_2. Suppose $\overline{b}^*$ is another such a hyperplane containing L, then either $L_{a,b}$ equals R_1 or R_2, or $L_{a,b}$ intersects one of these two ridges, which is in contradiction with Theorem 4.3.4. Therefore L is contained in exactly three such hyperplanes and hence the claim. Counting the pairs (L, x), with $L \in \Theta$, $x \in \mathsf{GF}(13) \cup \{\infty\}$ and L in $\overline{x}^*$, we obtain $|\Theta| = 182$. But now the remaining 91 lines (not belonging to either Θ or $\mathcal{S}_{\Omega^*}$) must also form an orbit, as G does not fix any line (its order does not divide the order of the stabilizer of a line in the full automorphism group of $\mathsf{H}(3)$, see Section 2.7.2, Table 4.1), and as the only primitive actions of $\mathsf{PSL}_2(13)$ on sets with less than 91 elements happen on sets with 14 and 78 elements.

From a foregoing argument follows that the peaks of the pairs $\{\overline{a}, \overline{b}\}$, with $a, b \in \mathsf{GF}(13)$ and $a - b$ a square of $\mathsf{GF}(13)$ are contained in Γ', as well as the peaks of the pairs $\{\overline{\infty}, \overline{a}\}$, $a \in \mathsf{GF}(13)$ of course. The remaining peaks are not contained in Γ' as their corresponding ridges meet $\overline{\infty}^*$ in a point not contained in Γ' (see above). Hence $\overline{\infty}^*$ contains exactly 52 peaks, which means that every line of Γ' is incident with exactly one peak (as this is obviously true for the orbit $\mathcal{S}_{\Omega^*}$, and hence also for the lines of Γ' not contained in $\mathcal{S}_{\Omega^*}$). Hence every element of $\mathcal{S}_{\Omega^*} \cup \Theta$ is incident with a unique element of $\mathcal{O}_\Omega$. If every remaining line is incident with n points of $\mathcal{O}_\Omega$ (n is constant since the remaining lines form an orbit under G), then a simple counting argument shows that $n = 1$ and we are done.

Remarks.

(1) As we will prove in the next sections there is a unique distance-2 spread, up to isomorphism, of $\mathsf{H}(3)$. Since $\mathsf{H}(3)$ is self-dual, the previous theorems

imply two different constructions of this distance-2 spread.

(2) The sets Ω and Ω^* have the following maximality property. It is clear that no element of Ω is incident with an element of Ω^*. But it can also be proved that Ω is precisely the set of points of $\mathsf{PG}(6,3)$ not contained in any member of Ω^*.

4.3.3 Flag matchings of $\mathsf{PG}(2,3)$

Since every distance-2 spread of $\mathsf{H}(q)$ intersects a hyperbolic hyperplane in a distance-2 spread of the weak subhexagon it contains, we must classify the distance-2 spreads of the generalized hexagon arising as the double of the projective plane $\mathsf{PG}(2,3)$. Such a distance-2 spread is nothing else than a matching of the incidence graph of $\mathsf{PG}(2,3)$, and hence our proof implies a classification of all matchings of that graph!

In this subsection we show the following proposition.

Proposition 4.3.5.
There are, up to isomorphism, exactly five flag matchings in $\mathsf{PG}(2,3)$.

In fact, this proposition can easily be proved with a computer, but we prefer to give an explicit theoretic proof. The advantage is that such a proof potentially can be generalized to larger classes of projective planes, possibly with some additional assumptions on the flag matchings (such as hypothesizing a big automorphism group). Another motivation is that it gives more insight in the structure of flag matchings. For instance, Lemma 4.3.6 below shows a structural property that, by computer, only comes out after the classification, and as such does not tell why the property is true.

In the sequel we will denote the point set of $\mathsf{PG}(2,3)$ by $\mathbb{Z}$ mod 13 and the lines are the subsets $\{i, i+1, i+4, i+6\}$, $i \in \mathbb{Z}$ mod 13. We briefly denote a flag by its line, where we underline the point. For instance, the flag $\{1, \{1,2,5,7\}\}$ is denoted by $\{\underline{1}, 2, 5, 7\}$.

To improve the readability of the proofs of the upcoming lemmas we include the following picture of $\mathsf{PG}(2,3)$.

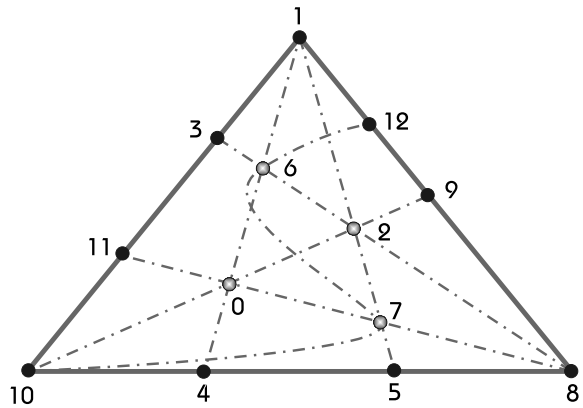

Figure 4.1: Coordinatization of $\mathsf{PG}(2,3)$

Note. There are only four lines of $\mathsf{PG}(2,3)$ missing in Figure 4.1, namely

12 **0** 3 5,	11 12 **2** 4,
3 4 **7** 9,	5 **6** 9 11.

Each of these four lines is incident with a **unique point** that is not on a line of the triangle 1 10 8. Such a point and line will be called an *interior point* and a *secant line* of that particular triangle. The dotted lines in Figure 4.1 are so-called *interior lines* and contain two interior points.

We now start the proof. A *triangle* in a flag matching $\mathcal{F}$ of $\mathsf{PG}(2,3)$ is a subset of three flags of $\mathcal{F}$ whose union constitutes an ordinary triangle in $\mathsf{PG}(2,3)$. We have a first lemma.

Lemma 4.3.6.
Every flag matching of $\mathsf{PG}(2,3)$ *contains a triangle.*

Proof. Suppose by way of contradiction that $\mathcal{F}$ is a flag matching without any triangle. Without loss of generality, we may assume that $\mathcal{F}$ contains $\{\underline{1},3,10,11\}$ and $\{4,5,8,\underline{10}\}$. Since $\mathcal{F}$ does not contain triangles, the three flags with points $4,5,8$ have lines not incident with 1, hence at least two of these lines meet in a point on $\{1,3,10,11\}$. But they cannot all meet in the same point for otherwise there is no line through that point available to form a flag of $\mathcal{F}$. So, without loss of generality, we may assume that $\mathcal{F}$

contains the flags $\{2,3,6,\underline{8}\}$, $\{3,\underline{4},7,9\}$ and $\{\underline{5},6,9,11\}$. If some flag of $\mathcal{F}$ containing a line through 10 contains one of the points $2,6,7$ or 9, then this flag, together with $\{4,5,8,\underline{10}\}$ and one of the three foregoing flags form a triangle. It follows that $\mathcal{F}$ contains the flags $\{\underline{0},2,9,10\}$ and $\{6,7,10,\underline{12}\}$. Now all lines through the point 9 except for one are already contained in one of the seven flags of $\mathcal{F}$ we mentioned yet. Hence the flag $\{1,8,\underline{9},12\}$ belongs to $\mathcal{F}$. In the same way the flags $\{0,1,4,\underline{6}\}$ and $\{0,\underline{3},5,12\}$ belong to $\mathcal{F}$. The three points $2,7,11$ and the three lines $\{1,2,5,7\}$, $\{0,7,8,11\}$, $\{2,4,11,12\}$ that do not belong to the ten flags of $\mathcal{F}$ we mentioned yet, form a triangle, a contradiction.

The lemma is proved.

□

Let $\mathcal{F}$ be a flag matching of $\mathsf{PG}(2,3)$, and let T be an ordinary triangle of $\mathsf{PG}(2,3)$ that is the union of three elements of $\mathcal{F}$. For each line L of T we count the number of flags of $\mathcal{F}$ with a point not belonging to T but on L, and with a line through the point of T opposite L. We obtain three numbers between 0 and 2. Written in increasing order we call this sequence the *type* of T. Since there are only four interior points of T, we have $i+j+k \geq 2$, for (ijk) the type of T. Since the unique secant line through an interior point x of T does not contain any of the points $ax \cap A$, with a a vertex of T and A the line of T opposite a, we easily see that the type of T cannot be (122). Hence only the types (002), (011), (012), (111), (022), (112) and (222) possibly occur. Also, we call (ijk) a *type* of $\mathcal{F}$ if (ijk) is the type of some triangle of $\mathcal{F}$ with $i+j+k$ maximal. Of the seven potential types of flag matchings of $\mathsf{PG}(2,3)$ there are two that can never occur, as the following lemma states.

Lemma 4.3.7.
No flag matching of $\mathsf{PG}(2,3)$ *has type* (002) *or* (012).

Proof. Suppose first that the flag matching $\mathcal{F}$ contains a triangle T of type (002). Without loss of generality $\mathcal{F}$ contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{0,2,\underline{9},10\}$ and $\{6,7,10,\underline{12}\}$. Let L be any line through the point 8, but not contained in T. Let x be such that $\{x,L\}$ belongs to $\mathcal{F}$. We may assume that x belongs to $\{0,2,6,7\}$, in which case the triangle T' with vertices $8,10,12$ has type $(i,j,2)$, with $j>0$. Hence (002) is not the type of $\mathcal{F}$.

Suppose now that the flag matching $\mathcal{F}$ contains a triangle T of type (012). First we assume that $\mathcal{F}$ contains the flags $\{1,3,\underline{10},11\}$, $\{4,5,\underline{8},10\}$, $\{\underline{1},8,9,12\}$ (these three flags form the triangle T), $\{0,2,\underline{9},10\}$, $\{6,7,10,\underline{12}\}$ and $\{1,2,\underline{5},7\}$.

Since the flag of $\mathcal{F}$ containing the line $\{0,1,4,6\}$ does not contain the point 4, we may assume without loss of generality that $\{\underline{0},1,4,6\}$ belongs to $\mathcal{F}$. If at least one of the flags of $\mathcal{F}$ through the points 3 and 11 has a line incident with 9, then the triangle with vertices $1,9,10$ has type (112) (remember that (122) is impossible). Otherwise, the triangle with vertices $1,10,12$ has type (022).

Hence we may assume that $\mathcal{F}$ contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{0,2,\underline{9},10\}$, $\{6,7,10,\underline{12}\}$ and $\{1,2,\underline{5},7\}$. As above, we may also assume without loss of generality that the flag $\{0,1,4,\underline{6}\}$ belongs to $\mathcal{F}$. It follows that the flag of $\mathcal{F}$ that contains the line $\{2,3,6,8\}$ is $\{\underline{2},3,6,8\}$. If the triangle T' with vertices $8,9,10$ has type (012), then $\mathcal{F}$ contains the flags $\{2,\underline{4},11,12\}$ and $\{\underline{0},3,5,12\}$. If now the triangle T'' with vertices $8,10,12$ has type (012), then the flag $\{3,4,\underline{7},9\}$ belongs to $\mathcal{F}$, and hence there can be no flag in $\mathcal{F}$ with the point 3, a contradiction. Hence one of the flags T' or T'' has type $(i,j,2)$ with $i+j>1$. Consequently $\mathcal{F}$ does not have type (012).

The lemma is proved.

□

We now deal with the remaining types, and prove that there is, up to isomorphism, a unique flag matching in each case.

Lemma 4.3.8.
There is, up to isomorphism, exactly one flag matching of type (011).

Proof. Let $\mathcal{F}$ be a flag matching of type (011). Let T be a triangle of $\mathcal{F}$ of type (011) (actually, every triangle of $\mathcal{F}$ must have type (011)). Without loss of generality we may assume that $\mathcal{F}$ contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{0,1,\underline{4},6\}$ and $\{2,\underline{3},6,8\}$. Since there remain exactly four interior lines that have to be in a flag with an interior point, we see that no internal point forms a flag of $\mathcal{F}$ with a secant line. It follows that the flag of $\mathcal{F}$ containing $\{3,4,7,9\}$ is $\{3,4,7,\underline{9}\}$ and the flag containing $\{6,7,10,12\}$ is $\{\underline{6},7,10,12\}$. Considering the triangle with vertices $1,3,8$, which must have type (011), we see that $\{1,2,5,\underline{7}\}$, $\{2,4,11,\underline{12}\}$ and therefore also $\{0,\underline{2},9,10\}$ belong to $\mathcal{F}$. Since the flag of $\mathcal{F}$ through 11 does not contain the line through 8, the flag $\{5,6,9,\underline{11}\}$ belongs to $\mathcal{F}$. Now by a similar argument the flags $\{\underline{0},7,8,11\}$ and $\{0,3,\underline{5},12\}$ complete $\mathcal{F}$. One can now check that $\mathcal{F}$ indeed has type (011).

The lemma is proved.

□

Lemma 4.3.9.
There is, up to isomorphism, exactly one flag matching of type (111).

Proof. Up to isomorphism there are two possibilities for triangles of type (111). Either (i) the lines of the flags that have a point on the sides of the triangle (but do not belong to the triangle) are concurrent, or (ii) not. First suppose that $\mathcal{F}$ (assumed to be of type (111)) contains a triangle T such that these lines are not concurrent (Case (ii)). Then without loss of generality the following flags belong to $\mathcal{F}$: $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{0,1,\underline{4},6\}$, $\{2,\underline{3},6,8\}$ and $\{0,2,\underline{9},10\}$. It follows easily that the flag of $\mathcal{F}$ containing the line $\{3,4,7,9\}$ is $\{3,4,\underline{7},9\}$. But now since the flag $\{1,2,\underline{5},7\}$ does not belong to $\mathcal{F}$, the flag of $\mathcal{F}$ containing the line $\{1,2,5,7\}$ is $\{1,\underline{2},5,7\}$. Likewise the flags $\{\underline{6},7,10,12\}$ and $\{\underline{0},7,8,11\}$ belong to $\mathcal{F}$. If the flag $\{0,3,\underline{5},12\}$ belonged to $\mathcal{F}$, then the triangle with vertices $2,3,7$ would have type (112), a contradiction. Consequently $\mathcal{F}$ contains the flag $\{\underline{5},6,9,11\}$. Similarly, the flags $\{0,3,5,\underline{12}\}$ and $\{2,4,\underline{11},12\}$ belong to $\mathcal{F}$. Hence $\mathcal{F}$ is uniquely determined. Conversely, one readily checks that $\mathcal{F}$ actually is of type (111).

Suppose now that T is a triangle of $\mathcal{F}$ for which (i) above holds. Then without loss of generality the following flags belong to $\mathcal{F}$: $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{0,1,\underline{4},6\}$, $\{0,2,\underline{9},10\}$ and $\{0,7,8,\underline{11}\}$. Clearly the point 0 gives rise to $\{\underline{0},3,5,12\} \in \mathcal{F}$. The lines of the flags of $\mathcal{F}$ containing the points $3,5,12$ being secants, forces the flags $\{2,4,11,\underline{12}\}$, $\{\underline{3},4,7,9\}$ and $\{6,\underline{5},9,11\}$ to belong to $\mathcal{F}$. But now the triangle with vertices $2,6,7$ is of type (111) and satisfies (ii) above. The first part of the proof completes the proof of the lemma.

□

Lemma 4.3.10.
There is, up to isomorphism, exactly one flag matching of type (022)*, and it does not have type* (112).

Proof. Let $\mathcal{F}$ be a flag matching of type (022) and let T be a triangle of type (022). Without loss of generality, we may assume that $\mathcal{F}$ contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{6,7,10,\underline{12}\}$, $\{0,2,\underline{9},10\}$, $\{0,7,8,\underline{11}\}$ and $\{2,\underline{3},6,8\}$. Since $\mathsf{PG}(2,3)$ admits an involutory automorphism fixing the points $1,4,5,8,10$, and interchanging the points 12 and 9, we may assume that $\mathcal{F}$ contains the flag $\{0,3,\underline{5},12\}$. This easily implies that $\{\underline{0},1,4,6\} \in \mathcal{F}$, and also $\{5,\underline{6},9,11\} \in \mathcal{F}$. There are now two possibilities. Either $\{2,\underline{4},11,12\} \in \mathcal{F}$ or $\{3,\underline{4},7,9\} \in \mathcal{F}$. In the first

case, one checks that $\{1, \underline{2}, 5, 7\} \in \mathcal{F}$ and $\{3, 4, \underline{7}, 9\} \in \mathcal{F}$. In the second case, we similarly have $\{1, 2, 5, \underline{7}\} \in \mathcal{F}$ and $\{\underline{2}, 4, 11, 12\} \in \mathcal{F}$. But the permutation $0 \mapsto 12 \mapsto 1 \mapsto 7 \mapsto 11 \mapsto 4 \mapsto 6 \mapsto 10 \mapsto 3 \mapsto 9 \mapsto 5 \mapsto 8 \mapsto 2 \mapsto 0$ preserves the lines and maps the second possibility onto the first. Hence the two flag matchings obtained are projectively equivalent. One can check that the type of $\mathcal{F}$ is really (022), and not (112).

□

Lemma 4.3.11.
There is, up to isomorphism, exactly one flag matching of type (112)*, and it does not have type* (022)*.*

Proof. As before, we may assume that our flag matching of type (112) containing the triangle T of type (112) contains the flags $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{6, 7, 10, \underline{12}\}$, $\{0, 1, \underline{4}, 6\}$, $\{1, 2, \underline{5}, 7\}$ and $\{2, \underline{3}, 6, 8\}$. As before this implies that $\{\underline{0}, 3, 5, 12\} \in \mathcal{F}$ and $\{5, \underline{6}, 9, 11\} \in \mathcal{F}$. But now the flags through 9 and 11 are determined (because the lines of these flags are secants): $\{3, 4, 7, \underline{9}\} \in \mathcal{F}$ and $\{2, 4, \underline{11}, 12\} \in \mathcal{F}$. Consequently $\{0, \underline{2}, 9, 10\}$ and $\{0, \underline{7}, 8, 11\}$ belong to $\mathcal{F}$ and $\mathcal{F}$ is uniquely determined. One can now check that $\mathcal{F}$ is really of type (112), and not of type (022).

□

Lemma 4.3.12.
There is, up to isomorphism, exactly one flag matching of type (222)*.*

Proof. This is obvious as every internal point of a triangle of type (222) is incident with only one secant line.

□

The assemblage of the previous lemmas results in the proof of Proposition 4.3.5.

4.3.4 Uniqueness of the distance-2 spread of H(3)

In this subsection, we prove the uniqueness of the distance-2 spread in $\mathsf{H}(3)$.

We know that every hyperbolic hyperplane, $\mathcal{H}$, of $\mathsf{PG}(6, 3)$ determines a unique weak subhexagon Γ in $\mathsf{H}(3)$. This weak subhexagon, in its turn, can be represented as the double of a projective plane $\mathsf{PG}(2, 3)$, see Section 2.4. The points and lines of the latter are the points of the two disjoint hexagon

twin planes in $\mathcal{H}$, denoted by π^+ and π^-. We can choose for $\mathcal{H}$ the hyperplane with equation $X_3 = 0$. The two planes within this hyperplane, which contain the points of Γ, are the planes with equations

$$\pi^+ \leftrightarrow X_0 = X_1 = X_2 = X_3 = 0,$$

$$\pi^- \leftrightarrow X_3 = X_4 = X_5 = X_6 = 0,$$

where the points of π^+ (respectively π^-) represent the points (respectively lines) of the projective plane $\mathsf{PG}(2,3)$.

A natural and legible choice for an isomorphism between the points (respectively lines) of $\mathsf{PG}(2,3)$ and those of π^+ (respectively π^-) is given by

$$\begin{aligned}
&\phi^+ : (x,y,z) \to (0,0,0,0,x,y,z),\\
&\phi^- : [x,y,z] \to (x,y,z,0,0,0,0),\\
&\Phi : \{(x,y,z),[x',y',z']\} \to \{(x,y,z)^{\phi^+},[x',y',z']^{\phi^-}\}.
\end{aligned}$$

To simplify and shorten the notation we denote all points and all lines of $\mathsf{H}(3)$ with an index from 0 to 363. This is done in the following way. We coordinatize $\mathsf{H}(3)$ as explained in Section 2.3 of Chapter 2, and write for each point and line the coordinate tuple in such a way that every entry is a member of $\{0,1,2\}$. We conceive the entries as natural numbers and then, if the coordinate tuple (e.g. of a point) is equal to $(a_0,\ldots,a_k)$, we label the point with the index

$$I(a_0,\ldots,a_k) = \frac{3^{k+1}-1}{2} + a_k + 3a_{k-1} + \ldots + 3^k a_0.$$

Similarly for lines. The elements $[\infty]$ and (∞) have labels 0.

Obviously a line obtained by interchanging the parentheses of a point, with index i, with square brackets, will have the same index i as that corresponding point.

Note. This is actually a lexicographic ordering on the points and lines.

A point with index i will be denoted p_i and a line with index j will be denoted L_j.

Concerning incidence we note that, equivalently to the situation with coordinates of $\mathsf{H}(3)$, the elements incident with an element of index i, where $i < 121$, is very easy. Indeed, the point p_i, $i = 3 \cdot i' + r$, with $r \in \{1,2,3\}$, is incident with $L_{i'}$ and with $L_{3\cdot i+r'}$, $r' = 1,2,3$. Similarly for the points incident with a given line with index < 121.

When $i \geq 121$, one has to translate back and forth to coordinates and make explicit calculations.

In this notation we obtain

$$\{p_1, p_4, p_{10}, p_7, p_{121}, p_{127}, p_{124}, p_{148}, p_{154}, p_{151}, p_{175}, p_{181}, p_{178}\},$$
$$\{p_{40}, p_{13}, p_{67}, p_{94}, p_0, p_{43}, p_{46}, p_{19}, p_{70}, p_{100}, p_{16}, p_{97}, p_{73}\},$$

as the points of π^+ and π^-. One can now check that the following sets $\mathcal{M}_{(ijk)}$ of pairs of points are the image of a flag matching of $\mathsf{PG}(2,3)$ of type (ijk), $i, j, k \in \{0, 1, 2\}$.

$$\begin{aligned}
\mathcal{M}_{(222)} = \{&\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{127}, p_{46}\}, \{p_{124}, p_{13}\}, \{p_{10}, p_{94}\},\\
&\{p_7, p_{67}\}, \{p_{148}, p_{16}\}, \{p_{154}, p_{70}\}, \{p_{151}, p_{100}\}, \{p_{175}, p_{19}\},\\
&\{p_{181}, p_{97}\}, \{p_{178}, p_{73}\}\};\\
\mathcal{M}_{(112)} = \{&\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{127}, p_{46}\}, \{p_{124}, p_{13}\}, \{p_{10}, p_{94}\},\\
&\{p_7, p_{67}\}, \{p_{148}, p_{97}\}, \{p_{154}, p_{70}\}, \{p_{151}, p_{16}\}, \{p_{175}, p_{100}\},\\
&\{p_{181}, p_{19}\}, \{p_{178}, p_{73}\}\};\\
\mathcal{M}_{(022)} = \{&\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{127}, p_{46}\}, \{p_{124}, p_{13}\}, \{p_{10}, p_{94}\},\\
&\{p_7, p_{67}\}, \{p_{148}, p_{97}\}, \{p_{154}, p_{16}\}, \{p_{151}, p_{100}\}, \{p_{175}, p_{70}\},\\
&\{p_{181}, p_{19}\}, \{p_{178}, p_{73}\}\};\\
\mathcal{M}_{(011)} = \{&\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{154}, p_{46}\}, \{p_{10}, p_{94}\}, \{p_7, p_{67}\},\\
&\{p_{127}, p_{13}\}, \{p_{124}, p_{70}\}, \{p_{148}, p_{97}\}, \{p_{151}, p_{16}\}, \{p_{175}, p_{100}\},\\
&\{p_{181}, p_{19}\}, \{p_{178}, p_{73}\}\};\\
\mathcal{M}_{(111)} = \{&\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{154}, p_{46}\}, \{p_{10}, p_{97}\}, \{p_7, p_{67}\},\\
&\{p_{127}, p_{100}\}, \{p_{124}, p_{13}\}, \{p_{148}, p_{73}\}, \{p_{151}, p_{16}\}, \{p_{175}, p_{70}\},\\
&\{p_{181}, p_{19}\}, \{p_{178}, p_{94}\}\}
\end{aligned}$$

and therefore, to prove Theorem 4.3.1, it suffices to prove that the line sets

$$\begin{aligned}
\mathcal{L}_{(222)} &= \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{49}, L_{212}, L_{302}, L_{58}, L_{294}, L_{222}\};\\
\mathcal{L}_{(112)} &= \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{293}, L_{212}, L_{50}, L_{303}, L_{60}, L_{222}\};\\
\mathcal{L}_{(022)} &= \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{293}, L_{51}, L_{302}, L_{213}, L_{60}, L_{222}\};\\
\mathcal{L}_{(011)} &= \{L_{121}, L_0, L_{14}, L_{140}, L_{31}, L_{22}, L_{42}, L_{211}, L_{293}, L_{50}, L_{303}, L_{60}, L_{222}\};\\
\mathcal{L}_{(111)} &= \{L_{121}, L_0, L_{14}, L_{140}, L_{32}, L_{22}, L_{301}, L_{41}, L_{221}, L_{50}, L_{213}, L_{60}, L_{285}\}
\end{aligned}$$

are, up to isomorphism, contained in at most one distance-2 spread of $\mathsf{H}(3)$.

We will do this as follows. For each set $\mathcal{L}_{(ijk)}$ as above, we denote by $V_{(ijk)}$ the set of points covered by the lines of $\mathcal{L}_{(ijk)}$. We consider points p of $\mathsf{H}(3)$

outside $V_{(ijk)}$, that are at distance 3 from *exactly three* lines in $\mathcal{L}_{(ijk)}$. For such a point p, the unique line L incident with it and not meeting any line of $\mathcal{L}_{(ijk)}$ must be contained in any distance-2 spread containing $\mathcal{L}_{(ijk)}$. Hence we may add L to $\mathcal{L}_{(ijk)}$ and start this procedure again. For further reference, we will call this procedure the *point-at-distance*-3-*procedure*.

This procedure runs very smoothly for the sets $\mathcal{L}_{(112)}$, $\mathcal{L}_{(022)}$ and $\mathcal{L}_{(011)}$ in that for the first two sets, we find a unique distance-2 spread, and for the third set, we run into a contradiction (finding a point outside the set that is at distance 3 from *four* lines of the set). We will show how this happens below. We will also give some more information on the cases $\mathcal{L}_{(222)}$ and $\mathcal{L}_{(111)}$. The two smooth cases will be left to the reader.

We have made the computations by computer, but they can just as well be checked by hand. The upshot of this method is that we detect crucial computer errors, but the ones that we would not detect (for instance because the computer "forgets" to mention a point that has distance 3 to exactly three lines of our line set) do not harm the proof.

- Let us start by considering the 13 lines of $\mathcal{L}_{(222)}$.

  The first point (with minimal index) not in $V_{(222)}$ and at distance 3 from exactly three lines of $\mathcal{L}_{(222)}$ is the point p_{22}. This point is incident with the three lines L_7, L_{67} and L_{69} which intersect $V_{(222)}$ in points on respective lines L_0, L_{121} and L_{139}. Hence the fourth line though p_{22}, namely L_{68}, must be added to $\mathcal{L}_{(222)}$ (and we accordingly add the four points incident with L_{68} to $V_{(222)}$).

  By a similar argument the points $p_{31}, p_{244}, p_{280}, p_{331}, p_{349}$ (not in $V_{(222)}$) force $L_{95}, L_{143}, L_{147}, L_{247}, L_{355}$ to be distance-2 spread lines, and we add them to $\mathcal{L}_{(222)}$.

  But now the point-at-distance-3-procedure runs into trouble, as there are no suitable points available anymore. We therefore consider the point p_8, which is at distance 3 from exactly two lines of $\mathcal{L}_{(222)}$ (namely, L_{247} and L_0), and distinguish between the two cases where either L_{25} or L_{26} is added to $\mathcal{L}_{(222)}$. These two cases separately give rise to unique distance-2 spreads by the point-at-distance-3-procedure. Here are the details.

  First suppose we added L_{25}. Then the points $p_{79}, p_{81}, p_{161}, p_{200}, p_{210}, p_{219}$ give us $L_{239}, L_{246}, L_{53}, L_{174}, L_{199}, L_{72}$, respectively, as lines to be added.

Furthermore, we get the lines

$$L_{65}, L_{98}, L_{118}, L_{168}, L_{304}, L_{360}, L_{363}, L_{251}, L_{159}, L_{63},$$
$$L_{281}, L_{82}, L_{313}, L_{87}, L_{276}, L_{296}, L_{127}, L_{333}, L_{310},$$

as the only possible lines through the points

$$p_{21}, p_{32}, p_{39}, p_{55}, p_{101}, p_{119}, p_{120}, p_{158}, p_{179}, p_{190},$$
$$p_{214}, p_{248}, p_{250}, p_{264}, p_{284}, p_{290}, p_{301}, p_{327}, p_{345},$$

respectively, followed by another 41 lines

$$L_{16}, L_{35}, L_{44}, L_{47}, L_{75}, L_{79}, L_{91}, L_{109}, L_{112}, L_{115},$$
$$L_{162}, L_{164}, L_{19}, L_{216}, L_{225}, L_{226}, L_{29}, L_{308}, L_{329}, L_{334},$$
$$L_{37}, L_{124}, L_{259}, L_{169}, L_{316}, L_{262}, L_{194}, L_{350}, L_{299}, L_{219},$$
$$L_{354}, L_{254}, L_{78}, L_{185}, L_{197}, L_{88}, L_{256}, L_{189}, L_{103}, L_{106}, L_{242},$$

determined by the respective corresponding points

$$p_5, p_{11}, p_{14}, p_{15}, p_{24}, p_{26}, p_{30}, p_{36}, p_{37}, p_{38},$$
$$p_{53}, p_{54}, p_{59}, p_{71}, p_{74}, p_{75}, p_{88}, p_{102}, p_{109}, p_{111},$$
$$p_{114}, p_{130}, p_{132}, p_{137}, p_{140}, p_{141}, p_{156}, p_{159}, p_{166}, p_{193},$$
$$p_{221}, p_{225}, p_{237}, p_{243}, p_{255}, p_{267}, p_{277}, p_{279}, p_{312}, p_{319}, p_{340}.$$

Finally, $L_{57}, L_{101}, L_{191}, L_{277}, L_{348}$ are the only possible lines to complete the set. By construction, we obtain a line set consisting of 91 lines, such that every point of $\mathsf{H}(3)$ is incident with exactly one of these lines, i.e. a distance-2 spread

$$\begin{aligned}\mathcal{S}_2 = \{&L_0, L_{14}, L_{16}, L_{19}, L_{22}, L_{25}, L_{29}, L_{31}, L_{35}, L_{37}, L_{41}, L_{44}, L_{47},\\ &L_{49}, L_{53}, L_{57}, L_{58}, L_{63}, L_{65}, L_{68}, L_{72}, L_{75}, L_{78}, L_{79}, L_{82}, L_{87},\\ &L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{103}, L_{106}, L_{109}, L_{112}, L_{115}, L_{118}, L_{121},\\ &L_{124}, L_{127}, L_{139}, L_{143}, L_{147}, L_{159}, L_{162}, L_{164}, L_{168}, L_{169}, L_{174}, L_{185},\\ &L_{189}, L_{191}, L_{194}, L_{197}, L_{199}, L_{212}, L_{216}, L_{219}, L_{222}, L_{225}, L_{226}, L_{239},\\ &L_{242}, L_{246}, L_{247}, L_{251}, L_{254}, L_{256}, L_{259}, L_{262}, L_{276}, L_{277}, L_{281}, L_{294},\\ &L_{296}, L_{299}, L_{302}, L_{304}, L_{308}, L_{310}, L_{313}, L_{316}, L_{329}, L_{333}, L_{334}, L_{348},\\ &L_{350}, L_{354}, L_{355}, L_{360}, L_{363}\}.\end{aligned}$$

Taking L_{26} as the spread line incident with p_8 and consecutively repeating the same procedure – we mention the points and corresponding lines of each step –

Step 1.

$$p_{155}, p_{204}, p_{213}, p_{298}$$
$$\longrightarrow$$
$$L_{167}, L_{297}, L_{175}, L_{99}$$

Step 2.

$$p_{60}, p_{99}, p_{132}, p_{187}, p_{192}, p_{210}, p_{242}, p_{277}, p_{281}, p_{290},$$
$$p_{297}, p_{317}, p_{353}$$
$$\longrightarrow$$
$$L_{182}, L_{298}, L_{43}, L_{62}, L_{261}, L_{230}, L_{80}, L_{92}, L_{214}, L_{172},$$
$$L_{314}, L_{105}, L_{117}$$

Step 3.

$$p_{15}, p_{23}, p_{35}, p_{36}, p_{39}, p_{52}, p_{77}, p_{84}, p_{85}, p_{93},$$
$$p_{101}, p_{103}, p_{109}, p_{120}, p_{122}, p_{134}, p_{135}, p_{136}, p_{143}, p_{144},$$
$$p_{165}, p_{166}, p_{182}, p_{184}, p_{195}, p_{200}, p_{225}, p_{240}, p_{250}, p_{260},$$
$$p_{276}, p_{285}, p_{310}, p_{311}, p_{344}, p_{362}$$
$$\longrightarrow$$
$$L_{46}, L_{71}, L_{107}, L_{111}, L_{119}, L_{17}, L_{233}, L_{255}, L_{257}, L_{30},$$
$$L_{305}, L_{312}, L_{36}, L_{361}, L_{148}, L_{322}, L_{268}, L_{223}, L_{325}, L_{271},$$
$$L_{341}, L_{128}, L_{321}, L_{126}, L_{64}, L_{237}, L_{74}, L_{307}, L_{83}, L_{86},$$
$$L_{180}, L_{228}, L_{345}, L_{155}, L_{114}, L_{171}$$

Step 4.

$$p_{12}, p_{17}, p_{25}, p_{29}, p_{33}, p_{50}, p_{61}, p_{72}, p_{83}, p_{87},$$
$$p_{88}, p_{105}, p_{112}, p_{119}, p_{138}, p_{169}, p_{183}, p_{237}$$
$$\longrightarrow$$
$$L_{38}, L_{52}, L_{77}, L_{89}, L_{102}, L_{153}, L_{20}, L_{218}, L_{250}, L_{264},$$
$$L_{266}, L_{317}, L_{339}, L_{359}, L_{196}, L_{56}, L_{195}, L_{200}$$

also leads to a distance-2 spread

$$\begin{aligned}\mathcal{S}_2{}' = \{&L_0, L_{14}, L_{17}, L_{20}, L_{22}, L_{26}, L_{30}, L_{31}, L_{36}, L_{38}, L_{41}, L_{43}, L_{46},\\ &L_{49}, L_{52}, L_{56}, L_{58}, L_{62}, L_{64}, L_{68}, L_{71}, L_{74}, L_{77}, L_{80}, L_{83}, L_{86},\\ &L_{89}, L_{92}, L_{95}, L_{99}, L_{102}, L_{105}, L_{107}, L_{111}, L_{114}, L_{117}, L_{119}, L_{121},\\ &L_{126}, L_{128}, L_{139}, L_{143}, L_{147}, L_{148}, L_{153}, L_{155}, L_{167}, L_{171}, L_{172}, L_{175},\\ &L_{180}, L_{182}, L_{195}, L_{196}, L_{200}, L_{212}, L_{214}, L_{218}, L_{222}, L_{223}, L_{228}, L_{230},\\ &L_{233}, L_{237}, L_{247}, L_{250}, L_{255}, L_{257}, L_{261}, L_{264}, L_{266}, L_{268}, L_{271}, L_{294},\\ &L_{297}, L_{298}, L_{302}, L_{305}, L_{307}, L_{312}, L_{314}, L_{317}, L_{321}, L_{322}, L_{325}, L_{339},\\ &L_{341}, L_{345}, L_{355}, L_{359}, L_{361}\}.\end{aligned}$$

These two distance-2 spreads, however, are isomorphic and the element

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

of the automorphism group $G_2(3)$ of $\mathsf{H}(3)$, maps $\mathcal{S}_2$ to $\mathcal{S}_2{}'$.

Note. This element can be identified with the dual version of $\theta''_{(r)}$, where $r = 0$ (defined in Section 2.7.1), as it fixes the lines $[\infty]$ and $[0, 0, 0, 0, 0]$, and maps the point (∞) onto the point (0). In other words, the action of this matrix on the lines of $\mathsf{H}(3)$ is completely determined by

$$[k, b, k', b', k''] \to [-k'', -b', kk'' + k', b, k]$$

or, when using the indices of the lines of $\mathsf{H}(3)$, by

$$L_{\frac{3^5-1}{2}+k''+3b'+3^2k'+3^3b+3^4k} \to L_{\frac{3^5-1}{2}+k+3b+3^2(k'+kk'')+3^3(-b')+3^4(-k'')}.$$

For example, consider L_{126} of $\mathcal{S}_2$. Since

$$126 = 121 + 2 + 3.1 + 3^2.0 + 3^3.0 + 3^4.0$$

we find that L_{126} is mapped onto $L_{121+3^3.2+3^4.1} = L_{256}$, which indeed belongs to $\mathcal{S}_2'$.

- The point-at-distance-3-procedure repeatedly applied to the 13 lines of $\mathcal{L}_{(112)}$ leads directly to the distance-2 spread

$$\begin{aligned}\mathcal{S}_2{}'' = \{&L_0, L_{14}, L_{18}, L_{20}, L_{22}, L_{27}, L_{30}, L_{31}, L_{34}, L_{38}, L_{41}, L_{45}, L_{46},\\ &L_{50}, L_{52}, L_{57}, L_{60}, L_{63}, L_{66}, L_{68}, L_{71}, L_{73}, L_{77}, L_{80}, L_{82}, L_{85},\\ &L_{88}, L_{93}, L_{95}, L_{98}, L_{101}, L_{104}, L_{107}, L_{110}, L_{114}, L_{117}, L_{120}, L_{121},\\ &L_{124}, L_{129}, L_{139}, L_{143}, L_{145}, L_{150}, L_{151}, L_{155}, L_{159}, L_{162}, L_{165}, L_{176},\\ &L_{180}, L_{182}, L_{194}, L_{197}, L_{199}, L_{212}, L_{216}, L_{217}, L_{222}, L_{225}, L_{228}, L_{229},\\ &L_{233}, L_{236}, L_{239}, L_{242}, L_{246}, L_{256}, L_{259}, L_{263}, L_{267}, L_{268}, L_{271}, L_{293},\\ &L_{295}, L_{299}, L_{303}, L_{305}, L_{308}, L_{320}, L_{322}, L_{325}, L_{328}, L_{333}, L_{334}, L_{339},\\ &L_{342}, L_{345}, L_{355}, L_{359}, L_{363}\}.\end{aligned}$$

This distance-2 spread is just like $\mathcal{S}_2'$ isomorphic to $\mathcal{S}_2$ and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & -1\\ -1 & 1 & 1 & 0 & 0 & 1 & -1\\ 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & 1 & -1 & -1\\ 0 & 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

maps all lines of $\mathcal{S}_2$ to those of $\mathcal{S}_2''$.

Note. One can easily see that this element of $\mathsf{G}_2(3)$ fixes (∞) and maps the point $(0,0,0,0,0)$ onto the point $(0,-1,-1,1,0)$. In other words, it is a morphism of type $\theta_{(0,-1,-1,1,0)}$ (see Section 2.7.1).

- The lines of $\mathcal{L}_{(022)}$, determine a unique distance-2 spread

$$\begin{aligned}\mathcal{S}_2{}''' = \{&L_0, L_{14}, L_{16}, L_{19}, L_{22}, L_{27}, L_{30}, L_{31}, L_{36}, L_{39}, L_{41}, L_{45}, L_{48},\\ &L_{51}, L_{52}, L_{55}, L_{60}, L_{61}, L_{64}, L_{68}, L_{71}, L_{74}, L_{76}, L_{79}, L_{82}, L_{85},\\ &L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{103}, L_{107}, L_{111}, L_{112}, L_{117}, L_{119}, L_{121},\\ &L_{124}, L_{127}, L_{139}, L_{143}, L_{147}, L_{159}, L_{161}, L_{165}, L_{166}, L_{171}, L_{174}, L_{185},\\ &L_{188}, L_{192}, L_{193}, L_{197}, L_{200}, L_{213}, L_{214}, L_{218}, L_{222}, L_{225}, L_{228}, L_{230},\\ &L_{232}, L_{236}, L_{240}, L_{242}, L_{245}, L_{256}, L_{259}, L_{263}, L_{267}, L_{268}, L_{271}, L_{293},\\ &L_{297}, L_{298}, L_{302}, L_{305}, L_{308}, L_{310}, L_{315}, L_{316}, L_{320}, L_{322}, L_{325}, L_{339},\\ &L_{342}, L_{343}, L_{347}, L_{351}, L_{354}\}\end{aligned}$$

and the isomorphism with $\mathcal{S}_2$ is shown by the following element of

$\mathsf{G}_2(3)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note. This matrix element corresponds to the element $\theta_{(-1,0,-1,0,1)}$ as defined in Section 2.7.1.

- Opposed to the previous cases the lines of $\mathcal{L}_{(011)}$ will not be extendable to a distance-2 spread.

  As there are no points outside $V_{(011)}$ at distance 3 from three lines of $\mathcal{L}_{(001)}$, we are forced to look for a point at distance 3 from two lines of this set.

  The point p_{22} is incident with the lines L_{67} and L_7 intersecting $V_{(011)}$ in points on the lines L_{121} and L_0, respectively. Therefore we have a choice between the lines L_{68} and L_{69} as the line to add. We will consider the case where we add L_{68}; the other choice is completely similar and also leads to a contradiction.

  Adding the line L_{68} gives us 6 points

  $$p_{118}, p_{209}, p_{210}, p_{317}, p_{336}, p_{353}$$

  outside the (modified) point set $V_{(011)}$, that force the lines

  $$L_{39}, L_{339}, L_{199}, L_{256}, L_{194}, L_{117}$$

  to be added. By repeatedly applying the point-at-distance-3-procedure we need to add the lines L_{98}, L_{271}, L_{245} incident with p_{32}, p_{271}, p_{349}, respectively, and the lines

  $$L_{30}, L_{46}, L_{75}, L_{110}, L_{34}, L_{217}, L_{163}, L_{344}, L_{66}, L_{224}, L_{76}, L_{228}$$

  incident with the respective points

  $$p_9, p_{15}, p_{24}, p_{36}, p_{104}, p_{142}, p_{143}, p_{174}, p_{200}, p_{224}, p_{230}, p_{285}.$$

  These 32 lines, however, can never be in a distance-2 spread. Consider the points p_{241} and p_{238} outside the point set $V_{(011)}$ (which now

contains 128 points). The point p_{241} (respectively p_{238}) is incident with the lines $L_{134}, L_{273}, L_{310}$ (respectively $L_{328}, L_{125}, L_{264}$) intersecting $V_{(011)}$ in the points p_{44}, p_{308}, p_{103} (respectively $p_{128}, p_{319}, p_{201}$) on the lines L_{14}, L_{293}, L_{34} (respectively L_{42}, L_{339}, L_{66}) of our yet to be completed set. Therefore the concurrent lines L_{80}, as the line through p_{241}, and L_{79}, as the line through p_{238}, should be in the set, and this is in contradiction with the definition of a distance-2 spread.

- Starting from the lines of the set $\mathcal{L}_{(111)}$ we have to consider several intermediate points or, in other words, several possible ways to complete the set. Nevertheless only one of these distinct cases will lead to a distance-2 spread, given by

$$\begin{aligned}\mathcal{S}_2{}^{(iv)} = \{&L_0, L_{14}, L_{17}, L_{19}, L_{22}, L_{25}, L_{30}, L_{32}, L_{35}, L_{37}, L_{41}, L_{45}, L_{47},\\ &L_{50}, L_{54}, L_{56}, L_{60}, L_{61}, L_{66}, L_{69}, L_{70}, L_{74}, L_{76}, L_{81}, L_{83}, L_{85},\\ &L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{104}, L_{107}, L_{110}, L_{112}, L_{115}, L_{118}, L_{121},\\ &L_{125}, L_{128}, L_{140}, L_{144}, L_{146}, L_{148}, L_{151}, L_{155}, L_{166}, L_{171}, L_{172}, L_{186},\\ &L_{189}, L_{192}, L_{193}, L_{198}, L_{200}, L_{213}, L_{214}, L_{219}, L_{221}, L_{225}, L_{228}, L_{239},\\ &L_{241}, L_{245}, L_{248}, L_{251}, L_{255}, L_{256}, L_{259}, L_{262}, L_{266}, L_{268}, L_{273}, L_{285},\\ &L_{286}, L_{289}, L_{301}, L_{305}, L_{307}, L_{312}, L_{315}, L_{316}, L_{330}, L_{332}, L_{336}, L_{347},\\ &L_{350}, L_{353}, L_{357}, L_{359}, L_{361}\},\end{aligned}$$

and every other possibility yields a contradiction. Again $\mathcal{S}_2^{(iv)}$ is isomorphic to $\mathcal{S}_2$ as the group element

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

($= \theta_{(1,-1,1,1,1)}$, Section 2.7.1) shows.

This completes the proof of the uniqueness part of Theorem 4.3.1.

4.3.5 A line spread in Q(6, 2)

The fact that H(2) does not admit a distance-2 spread can easily be deduced from the classification of geometric hyperplanes of the dual of H(2), as done

in [28] (partly using the computer). We present a short and independent proof.

The generalized hexagon $\mathsf{H}(2)$ has an easy construction using the projective plane $\mathsf{PG}(2,2)$, see [69]. However, here we will rephrase this construction thereby considerably simplifying the construction of the ambient quadric $\mathsf{Q}(6,2)$. In fact, we make the construction of [69] more algebraic.

The point set $\mathcal{P}$ of the projective plane $\mathsf{PG}(2,2)$ can be viewed as the set of non-zero vectors of a 3-dimensional vector space $V(3,2)$ over the field with two elements. Likewise, the line set $\mathcal{L}$ of $\mathsf{PG}(2,2)$ can be viewed as the non-zero vectors of the dual space $V(3,2)^*$ of $V(3,2)$, and a point v (viewed as non-zero vector of $V(3,2)$) is incident with a line ℓ (a non-zero vector of $V(3,2)^*$) precisely if $\ell(v)=0$. Now consider the set $\Pi = V(3,2)\times V(3,2)^* \setminus \{\vec{o},\vec{o^*}\}$ (where $\vec{o^*}$ denotes the zero vector of $V(3,2)^*$). If $(v,\ell)\in\Pi$, then we define the *type* of (v,ℓ) as P (from Point); L (from Line); F (from Flag); A (from Antiflag), according to $\ell=\vec{o^*}$; $v=\vec{o}$; $\ell(v)=0$, $v\neq\vec{o}$ and $\ell\neq\vec{o^*}$; $\ell(v)=1$, respectively. If the elements of a 3-subset of Π have type X, Y and Z, respectively, then we say that the subset has type XYZ.

Let Λ_1 be the set of 3-subsets $\{(v_1,\ell_1),(v_2,\ell_2),(v_3,\ell_3)\}$ of Π of type PLF, PPP, LLL, PAA, LAA, PFF, LFF, FFF or FAA and such that $v_1+v_2+v_3=\vec{o}$ and $\ell_1+\ell_2+\ell_3=\vec{o^*}$. Then it follows from [69] that $(\Pi,\Lambda_1,\in)$ is an incidence structure isomorphic to $\mathsf{Q}(6,2)$ (or, equivalently, to $\mathsf{W}_5(2)$).

Let Λ_2 be the set of 3-subsets $\{(v_1,\ell_1),(v_2,\ell_2),(v_3,\ell_3)\}$ of Π of type PLF such that $v_1+v_2+v_3=\vec{o}$ and $\ell_1+\ell_2+\ell_3=\vec{o^*}$, and the ones of FAA with the additional condition that $\ell_i(v_1)=\ell_1(v_i)=0$, for all $i\in\{1,2,3\}$. Then, again by [69], $(\Pi,\Lambda_2,\in)$ is an incidence structure isomorphic to $\mathsf{H}(2)$.

We can now label the non-zero vectors of $V(3,2)$ with $\mathbb{Z}$ mod 7 such that the lines of $\mathsf{PG}(2,2)$ correspond to the 3-subsets $\{i,i+1,i+3\}$, with i varying over $\mathbb{Z}$ mod 7 (we will write the zero element of $\mathbb{Z}$ mod 7 as 7 to avoid confusion with the zero vector $\vec{o}$). Consequently, we will denote a point of type P as $(i,\vec{o^*})$ (with $i\in\mathbb{Z}$ mod 7), one of type L as $(\vec{o},i/j/k)$ (with $i=j-1=k-3\in\mathbb{Z}$ mod 7), one of type F or A as $(n,i/j/k)$ (with $n\in\mathbb{Z}$ mod 7 and $i=j-1=k-3\in\mathbb{Z}$ mod 7).

The points of type P and L form the point set of a subhexagon of order $(1,2)$, which is in a natural way isomorphic to the double of $\mathsf{PG}(2,2)$ (that we started from above). Hence a hypothetical distance-2 spread $\mathcal{S}$ of $\mathsf{H}(2)$ induces a flag matching in $\mathsf{PG}(2,2)$. It is now very easy to prove that, up to isomorphism, there is a unique flag matching of $\mathsf{PG}(2,2)$. Hence we may assume that $\mathcal{S}$ contains, for each $i\in\mathbb{Z}$ mod 7, the lines of type PLF containing the points $(i,\vec{o^*})$, $(\vec{o},i/i+1/i+3)$ and $(i,i/i+1/i+3)$ (because these lines clearly

partition the point set of the subhexagon of order $(1,2)$). We apply the same method as in the previous section. The point $(i+4, i/i+1/i+3)$ is at distance 3 from two lines of $\mathcal{S}$ of type PLF, namely those containing the points $(i+1, i+1/i+2/i+4)$ and $(i+3, i+3/i+4/i+6)$. This implies that the line L_i of $\mathsf{H}(2)$ through $(i+4, i/i+1/i+3)$ not meeting the two above mentioned lines, belongs to $\mathcal{S}$. One easily computes that L_i contains the points $(i+4, i/i+1/i+3)$, $(i+5, i-1/i/i+2)$ and $(i, i+4/i+5/i)$. Rewrite both anti-flags of L_i as $(j, j+3/j+4/j+6)$ and $(k, k+1/k+2/k+4)$, with $j = i+4$ and $k = i+5$, and consider the point $p = (j, j+2/j+3/j+5)$, which is a point of type A that is not yet incident with a line of $\mathcal{S}$. It is now easy to see that each of the lines of $\mathsf{H}(2)$ on p, i.e.

$$\begin{array}{lcl} (j+2, j+6/j/j+2) & p & (j+6, j+1/j+2/j+4), \\ (j+3, j/j+1/j+3) & p & (j+1, j+3/j+4/j+6), \\ (j+5, j+4/j+5/j) & p & (j+4, j+5/j+6/j+1), \end{array}$$

meets a line of the 14 above mentioned lines of $\mathcal{S}$, a contradiction.

Now we construct a partition $\mathcal{S}$ of $\mathsf{Q}(6,2)$ by lines. We define

$$\begin{aligned} \mathcal{S}_1 &= \{\{(i, i/i+1/i+3), (i, \vec{o^*}), (\vec{o}, i/i+1/i+3)\} \mid i \in \mathbb{Z} \bmod 7\}, \\ \mathcal{S}_2 &= \{\{(i, i-3/i-2/i), (i+1, i+2/i+3/i-2), (i+3, i-2/i-1/i+1)\} \mid \\ &\qquad i \in \mathbb{Z} \bmod 7\}, \\ \mathcal{S}_3 &= \{\{(i, i-2/i-1/i+1), (i+1, i+4/i-2/i), (i+3, i+2/i+3/i-2)\} \mid \\ &\qquad i \in \mathbb{Z} \bmod 7\}. \end{aligned}$$

Then clearly, all elements of $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are lines of $\mathsf{Q}(6,2)$, and every point of $\mathsf{Q}(6,2)$ is on one of the elements of $\mathcal{S}$. Hence $\mathcal{S}$ is the desired partition.

Note. The existence of a line spread of $\mathsf{Q}(6,2)$ was questioned by E.E. Shult (private communication to J.A. Thas) many years ago. In fact, J.A. Thas constructed in some unpublished notes a set of 20 disjoint lines of $\mathsf{Q}(6,2)$. By a theorem of R.C. Bose and R.C. Burton [3], one can always complete such a set to a line spread. Our result above gives a simple and direct construction of a line spread of $\mathsf{Q}(6,2)$ and answers the question of E.E. Shult affirmatively.

4.3.6 Some applications

In this subsection we present some geometric and graph-theoretical consequences. First of all, our results imply that the collinearity graph of $\mathsf{H}(3)$ is

not 4-colorable. It is still an open question whether, in general, the collinearity graph of $\mathsf{H}(q)$ is $(1+q)$-colorable. In this subsection we show that the answer is negative for $q \in \{2,3\}$. Next, we prove that the twisted triality hexagon $\mathsf{T}(27,3)$ has no distance-2 spread. Previously, this was known only for the hexagon $\mathsf{T}(8,2)$. It may indicate that no triality hexagon $\mathsf{T}(q^3,q)$ has a distance-2 spread. And finally, we explain why one cannot lift the standard embedding of $\mathsf{H}(3)$ into $\mathsf{PG}(7,3)$ by using as geometrical hyperplane this distance-2 ovoid to look for new embeddings.

Coloring the point graphs of small hexagons

For the generalized hexagon $\mathsf{H}(q)$, the point graph has maximal cliques of size $1+q$, hence its chromatic number is at least $1+q$. The problem is now: is this bound sharp? It is very easy to see that a coloring of the point graph of $\mathsf{H}(q)$, or of its dual, using only $1+q$ colors amounts to a partition of the point set into $1+q$ distance-2 ovoids. For the dual of $\mathsf{H}(2)$ there does not even exist a single distance-2 ovoid, hence the chromatic number cannot be equal to 3. Consider now $\mathsf{H}(2)$. If we remove a distance-2 ovoid from $\mathsf{H}(2)$, then we are left with a geometry with two points per line, hence a graph. This graph has two connected components, one of which is the Coxeter graph, the other being the Heawood graph; see [46]. If the chromatic number of the point graph of $\mathsf{H}(2)$ were equal to 3, then we could divide the vertices of the Coxeter graph into two sets of 14 such that no two vertices of the same set belong to the same edge. Clearly, this is equivalent to saying that the Coxeter graph is bipartite, which is however not the case. Hence also in this case, the bound $1+q$ is certainly not sharp.

For $q=3$, we now know that there is a unique distance-2 ovoid, and so if the chromatic number of the point graph of $\mathsf{H}(3)$ were equal to 4, then there would exist 4 mutually disjoint distance-2 ovoids, all isomorphic to one another. To prove the non-existence of four disjoint distance-2 ovoids, we will start by determining the possible intersection numbers of such a distance-2 ovoid with a general hyperbolic hyperplane.

As the fixed distance-2 ovoid we will use the set $\mathcal{O}_\Omega$ as obtained in Section 4.3.2 (with $\Omega = \{\overline{\infty}, \overline{1}, \ldots, \overline{12}\}$ and where $\mathcal{O}_\Omega$ is the set of all peaks determined by any two distinct elements of Ω). Keep in mind that we may identify Ω with the points of the projective line $\mathsf{PG}(1,13)$ (in the natural way) concerning the action of G (the automorphism group of $\mathcal{O}_\Omega$).

First consider the hyperbolic hyperplanes of the set Ω^* (of polar hyperplanes of the points of Ω). As G acts transitively on these hyperplanes, each of them

will have the same intersection number with $\mathcal{O}_\Omega$. By construction of $\mathcal{O}_\Omega$, it is easy to see that this number equals 52.

For the hyperplane $\mathcal{H}_2$ with equation $X_3 + X_4 = 0$ we can, given the explicit description of $\mathcal{O}_\Omega$, simply count the number of points in the intersection. There are 6 elements of Ω, namely the points $\overline{1}, \overline{2}, \overline{3}, \overline{5}, \overline{6}, \overline{9}$, contained in this hyperplane, which determine the first 15 points of $\mathcal{O}_\Omega$ in $\mathcal{H}_2$. Furthermore the following pairs of Ω

$$\begin{array}{ccccc} \{\overline{\infty}, \overline{0}\}; & \{\overline{\infty}, \overline{7}\}; & \{\overline{\infty}, \overline{8}\}; & \{\overline{\infty}, \overline{11}\}; & \{\overline{7}, \overline{8}\}; \\ \{\overline{7}, \overline{11}\}; & \{\overline{8}, \overline{11}\}; & \{\overline{4}, \overline{11}\}; & \{\overline{7}, \overline{12}\}; & \{\overline{8}, \overline{10}\}; \end{array}$$

determine peaks that are contained in $\mathcal{H}_2$. This gives us a total of 25 intersection points.

The hyperplane $\mathcal{H}_2$ is generated by the six elements of Ω it contains, as is easily verified. Hence its stabilizer inside $\mathsf{PSL}_2(13)$ coincides with the set-wise stabilizer in $\mathsf{PSL}_2(13)$ of those six points. The following permutations

$$s = (\overline{\infty})(\overline{0})(\overline{1}, \overline{3}, \overline{9})(\overline{2}, \overline{6}, \overline{5})(\overline{4}, \overline{12}, \overline{10})(\overline{7}, \overline{8}, \overline{11})$$

and

$$t = (\overline{\infty}, \overline{8})(\overline{0}, \overline{10})(\overline{1})(\overline{2})(\overline{3}, \overline{6})(\overline{4}, \overline{12})(\overline{5}, \overline{9})(\overline{7}, \overline{11})$$

of G stabilize the six points. Consequently the group generated by these two elements fixes $\mathcal{H}_2$. Since $\mathsf{A}_4 = \langle s, t \,|\, s^3 = 1, t^2 = 1, (st)^3 = 1 \rangle$ is a maximal subgroup of $\mathsf{PSL}_2(13)$ and st is in this case equal to

$$(\overline{\infty}, \overline{8}, \overline{7})(\overline{0}, \overline{10}, \overline{12})(\overline{1}, \overline{6}, \overline{9})(\overline{2}, \overline{3}, \overline{5})(\overline{4})(\overline{11})$$

we see that the stabilizer group of $\mathcal{H}_2$ in G is isomorphic to the alternating group A_4. As a result we have 91 hyperbolic hyperplanes, in the orbit of $\mathcal{H}_2$ under $G_{\mathcal{H}_2}$, that intersect $\mathcal{O}_\Omega$ in 25 points.

In a third hyperplane $\mathcal{H}_3$, given by equation $X_2 + X_3 + X_4 = 0$, one can count, similarly to the previous case, 34 points of the distance-2 ovoid. Also this hyperplane contains six elements of Ω, namely the points $\overline{2}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{11}$, and is generated by them. These six elements are fixed by the group elements

$$u = (\overline{\infty}, \overline{0})(\overline{1}, \overline{10})(\overline{2}, \overline{5})(\overline{3}, \overline{12})(\overline{4}, \overline{9})(\overline{8}, \overline{11})(\overline{7})(\overline{6})$$

and

$$v = (\overline{\infty}, \overline{0})(\overline{1})(\overline{2}, \overline{7})(\overline{3}, \overline{9})(\overline{4}, \overline{10})(\overline{5}, \overline{8})(\overline{6}, \overline{11})(\overline{12})$$

and thus also by the group generated by u and v. We know that $\mathsf{D}_{12} = \langle s, t \,|\, s^2 = 1, t^2 = 1, (st)^6 = 1 \rangle$ is a maximal subgroup of $\mathsf{PSL}_2(13)$ and

therefore (since $uv = (\overline{\infty})(\overline{0})(\overline{1},\overline{4},\overline{3},\overline{12},\overline{9},\overline{10})(\overline{2},\overline{8},\overline{6},\overline{11},\overline{5},\overline{7})$) the group $G_{\mathcal{H}_3}$ is isomorphic to D_{12}. Hence we obtain 91 hyperbolic hyperplanes, now as the orbit of $\mathcal{H}_3$ under $G_{\mathcal{H}_3}$, with 34 as intersection number.

Finally consider the hyperplane $\mathcal{H}_4$ with equation $X_3 - X_4 = 0$. This hyperplane, just as $\mathcal{H}_3$, contains 34 points of the distance-2 ovoid. Now one checks that $\mathcal{H}_4$ contains 3 points of Ω. If $\mathcal{H}_4$ were stabilized by a subgroup of $\mathsf{PSL}_2(13)$ of order strictly greater than 6, then a non-trivial element of $\mathsf{PSL}_2(13)$ would fix these 3 points, a contradiction. Thus the orbit of $\mathcal{H}_4$ under G contains at least $|G|/6 = 182$ elements. Since there are only this many hyperbolic hyperplanes remaining, we conclude that $G_{\mathcal{H}_4}$ is isomorphic to the symmetric group S_3 and determines a set of 182 hyperbolic hyperplanes intersecting $\mathcal{O}_\Omega$ in 34 points.

These four orbits thus contain all 378 hyperbolic hyperplanes of $\mathsf{PG}(6,3)$. This results into three possible intersection numbers, namely 25, 34 or 52, of a hyperbolic hyperplane with a distance-2 ovoid.

Suppose now, by way of contradiction, that there exists a 4-coloring on the point graph of $\mathsf{H}(3)$. Then these four distance-2 ovoids will partition the point set of any hyperbolic subquadrangle $\mathsf{Q}^+(5,3)$ of $\mathsf{Q}(6,3)$ into four sets, where each of these sets contains either 25, 34 or 52 points. We thereby get two equations in x, y and z, which are the respective number of distance-2 ovoids intersecting $\mathsf{Q}^+(5,3)$ in 25, 34 or 52 points, namely

$$\begin{aligned} 130 &= 25x + 34y + 52z; \\ 4 &= x + y + z. \end{aligned}$$

One can easily check that this system of equations has no solution for x, y and z positive integers between 0 and 4. Hence the chromatic number of the point graph of $\mathsf{H}(3)$ is bigger than 4.

Remark. In fact, one can show by computer that any two distance-2 ovoids of $\mathsf{H}(3)$ meet non-trivially. Since we were not able to show this theoretically, we do not elaborate on it.

Distance-2 spreads of small twisted triality hexagons

From Section 2.2 we know that the twisted triality hexagon contains $\mathsf{H}(q)$ as a subhexagon. Hence, every distance-2 spread of $\mathsf{T}(q^3, q)$ induces a distance-2 spread in $\mathsf{H}(q)$. Since for $q = 2$, there are no distance-2 spreads of $\mathsf{H}(2)$, it immediately follows that $\mathsf{T}(8,2)$ has no distance-2 spreads. If $q = 3$, then we may assume that the distance-2 spread induced in a subhexagon $\mathsf{H}(3)$ of

$\mathsf{T}(27,3)$ by a hypothetical distance-2 spread of $\mathsf{T}(27,3)$, is the spread $\mathcal{S}_{\Omega^*}$ as constructed in Section 4.2.2. But then we can apply the point-at-distance-3-procedure. This procedure runs into a contradiction very soon. We omit the details. Hence we can state the following proposition.

Proposition 4.3.13.
The twisted triality hexagons $\mathsf{T}(8,2)$ *and* $\mathsf{T}(27,3)$ *do not contain any distance-*2 *spread.*

It is now tempting to conjecture that $\mathsf{T}(q^3,q)$ does not admit distance-2 spreads, for arbitrary q. But we do not have a clue for a general proof. The present subsection, however, settles the two smallest cases.

Lifting of H(3)

A distance-2 ovoid of $\mathsf{H}(3)$ is nothing else than a geometric hyperplane not containing lines. But in all known embeddings of $\mathsf{H}(3)$ this geometric hyperplane spans the whole space, and hence is not induced by any ordinary hyperplane. This might indicate that there could be more embeddings of $\mathsf{H}(3)$, but we show that it is impossible to lift the standard embedding of $\mathsf{H}(3)$ into a higher dimension starting from a distance-2 ovoid.

The general idea behind the attempt to lift the standard embedding of $\mathsf{H}(3)$ into $\mathsf{PG}(7,3)$ using a distance-2 ovoid is the following: given a geometrical hyperplane that spans the whole six dimensional space $\mathsf{PG}(6,3)$, we can try and identify these points with the points of a hyperplane of $\mathsf{PG}(7,3)$, say $X_0=0$, and lift a point $p=(x_0,\ldots,x_6)$ outside this geometrical hyperplane to the point $(1,x_0,\ldots,x_6)$. As every line of $\mathsf{H}(3)$ intersects the geometrical hyperplane in a unique point, the coordinates of a point collinear to p will be a linear combination of the coordinates of p and the coordinates of a point in $X_0=0$. In this way we can try to coordinatize all points of $\mathsf{H}(3)$ in $\mathsf{PG}(7,3)$.

Consider the points of the distance-2 ovoid $\mathcal{O}_\Omega$ as constructed in Section 4.3.2. In coordinates of $\mathsf{H}(3)$ the points of $\mathcal{O}_\Omega$ are the points of the set

$$\{p_2,p_6,p_9,p_{12},p_{14},p_{17},p_{20},p_{33},p_{35},p_{37},p_{41},p_{44},p_{48},p_{50},p_{53},p_{55},$$
$$p_{68},p_{72},p_{74},p_{77},p_{79},p_{83},p_{95},p_{98},p_{102},p_{104},p_{107},p_{109},p_{122},p_{126},p_{129},$$
$$p_{139},p_{144},p_{147},p_{149},p_{153},p_{156},p_{166},p_{171},p_{174},p_{177},p_{179},p_{183},p_{195},p_{196},p_{201},$$
$$p_{202},p_{205},p_{209},p_{213},p_{215},p_{218},p_{222},p_{223},p_{228},p_{231},p_{232},p_{235},p_{239},p_{242},p_{244},$$
$$p_{249},p_{251},p_{255},p_{256},p_{259},p_{262},p_{266},p_{269},p_{272},p_{276},p_{279},p_{282},p_{284},p_{288},p_{289},$$
$$p_{294},p_{296},p_{298},p_{310},p_{315},p_{317},p_{330},p_{331},p_{335},p_{347},p_{351},p_{352},p_{355},p_{360},p_{362}\}$$

and these will, as we stated before, be mapped onto points of the hyperplane $\mathcal{H}$, with equation $X_0 = 0$, by the isomorphism $\Phi : \mathsf{PG}(6,3) \to \mathsf{PG}(7,3)$ given by the map

$$x = (x_0, \ldots, x_6) \to x^\Phi = (0, x_0, \ldots, x_6).$$

As the point (∞) does not belong to the distance-2 ovoid, we will choose this as our special point p that will be lifted to the point $(1,1,0,0,0,0,0,0)$, which we also denote by p.

The coordinates of a general point (a, l, a', l', a'') opposite (∞) can now be determined as follows (see Figure 4.2 for clarification):

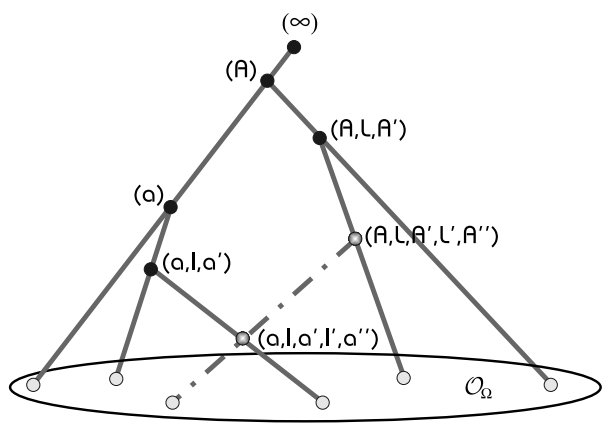

Figure 4.2: Lifting of H(3)

First, the intersection of the line $[\infty]$ with the set of points of the distance-2 ovoid gives us the unique point p_r. Therefore any other point on this line will have coordinates $x_i = (1,1,0,0,0,0,0) + \epsilon^i p_r^\Phi$, with $\epsilon = -1$ and $i \in \{0,1\}$. These coordinates are precisely the only possible coordinates of the point (a) in the lifted embedding of $\mathsf{H}(3)$ in $\mathsf{PG}(7,3)$.

Then, the point (a, l, a') is a point on the line $[a, l]$ through (a) and $[a, l] \cap \mathcal{O}_\Omega$. The former point has coordinates $x_i, i = 0, 1$, the latter point is p_m^Φ with $m = 3.I([a, l]) + r$, $r = 1, 2$ or 3, depending on which value of m gives a point in $\mathcal{O}_\Omega$. In other words, a point on the line $[a, l]$ of type (a, l, a') will have coordinates $y_{ij} = x_i + \epsilon^j p_m^\Phi$, $i, j \in \{0, 1\}$.

Finally, by a similar argument, we find that the point (a, l, a', l', a'') has coordinates $z_{ijk} = y_{ij} + \epsilon^k p_n^\Phi$, $i, j, k \in \{0, 1\}$, and $n = 3.I([a, l, a, l']) + r$ such that $r = 1, 2$ or 3 and $p_n \in \mathcal{O}_\Omega$.

We will now consider two points (a, l, a', l', a'') and (A, L, A', L', A'') that are collinear in the standard embedding of $\mathsf{H}(3)$ and check whether they are collinear in coordinates of $\mathsf{PG}(7, 3)$ as well.

Consider $p_{121} = (0, 0, 0, 0, 0)$ and $p_{283} = (-1, 0, 0, 0, 0)$, two collinear points off $\mathcal{O}_\Omega$ that are on the line $L_{121} = [0, 0, 0, 0, 0]$. For each of these two points we will obtain 8 possible coordinates in $\mathsf{PG}(7, 3)$ (since none of the points (0), (-1), $(0, 0, 0)$ and (-1,0,0) belongs to $\mathcal{O}_\Omega$). As we will see, none of the 64 possible lines in $\mathsf{PG}(7, 3)$, that ought to represent the line L_{121}, will contain the point p_{202}^Φ, a contradiction.

Let us start with the point $p_{121} = (0, 0, 0, 0, 0)$. The coordinates of the point (0) in $\mathsf{PG}(7, 3)$ are $x_i = (1, 1, 0, 0, 0, 0, 0, 0) + \epsilon^i p_2^\Phi$, with $p_2 = (1)$ and hence $p_2^\Phi = (0, 1, 0, 0, 0, 0, 0, 1)$:

$$(1, -1, 0, 0, 0, 0, 0, 1), \qquad (1, 0, 0, 0, 0, 0, 0, -1).$$

Next we have 4 possibilities for $(0, 0, 0)$, $y_{ij} = x_i + \epsilon^j p_{14}^\Phi$, with $p_{14} = (0, 0, 1)$ and $p_{14}^\Phi = (0, 0, 1, 0, 0, 0, 0, -1)$:

$$\begin{array}{ll} (1, -1, 1, 0, 0, 0, 0, 0), & (1, -1, -1, 0, 0, 0, 0, -1), \\ (1, 0, 1, 0, 0, 0, 0, 1), & (1, 0, -1, 0, 0, 0, 0, 0). \end{array}$$

And finally we obtain the following 8 coordinates for $(0, 0, 0, 0, 0)$, $z_{ijk} = y_{ij} + \epsilon^k p_{122}^\Phi$, with $p_{122} = (0, 0, 0, 0, 1)$ and thus $p_{122}^\Phi = (0, 0, 1, 0, 0, -1, 0, 0)$:

$$\begin{array}{ll} (1, -1, -1, 0, 0, -1, 0, 0), & (1, -1, 0, 0, 0, 1, 0, 0), \\ (1, -1, 0, 0, 0, -1, 0, -1), & (1, -1, 1, 0, 0, 1, 0, -1), \\ (1, 0, -1, 0, 0, -1, 0, 1), & (1, 0, 0, 0, 0, 1, 0, 1), \\ (1, 0, 0, 0, 0, -1, 0, 0), & (1, 0, 1, 0, 0, 1, 0, 0). \end{array}$$

Now consider the point $p_{283} = (-1, 0, 0, 0, 0)$. The two possible coordinates for the point (-1) in this hypothetical embedding of $\mathsf{H}(3)$ in $\mathsf{PG}(7, 3)$ are the same as the above coordinates for the point (0). Hence, we immediately turn to the coordinates of the point $(-1, 0, 0)$, which are given by $x_i + \epsilon^j p_{33}^\Phi$, with $p_{33} = (-1, 0, -1)$ and hence $p_{33}^\Phi = (0, 1, -1, 0, -1, 0, -1, -1)$:

$$\begin{array}{ll} (1, 0, -1, 0, -1, 0, -1, 0), & (1, 1, 1, 0, 1, 0, 1, -1), \\ (1, 1, -1, 0, -1, 0, -1, 1), & (1, -1, 1, 0, 1, 0, 1, 0). \end{array}$$

Finally, we have the following 8 coordinates for $(-1,0,0,0,0)$, $z_{ijk} = y_{ij} + \epsilon^k p^{\Phi}_{284}$, with $p_{284} = (-1,0,0,0,1)$ and $p^{\Phi}_{284} = (0,0,1,-1,1,-1,1,0)$:

$$\begin{array}{ll}
(1,0,0,-1,0,-1,0,0), & (1,0,1,1,1,1,1,0), \\
(1,1,-1,-1,-1,-1,-1,-1), & (1,1,0,1,0,1,0,-1), \\
(1,1,0,-1,0,-1,0,1), & (1,1,1,1,1,1,1,1), \\
(1,-1,-1,-1,-1,-1,-1,0), & (1,-1,0,1,0,1,0,0).
\end{array}$$

As p^{Φ}_{202} has coordinates $(0,0,0,1,0,-1,0,0)$ it is routine to check that indeed none of these possible line contains this point. Consequently there is no embedding in $\mathsf{PG}(7,3)$ of $\mathsf{H}(3)$, obtained by using $\mathcal{O}_\Omega$ as a geometrical hyperplane to lift all points of $\mathsf{H}(3)$.

4.4 Conclusion

To conclude this chapter we remark that all maximal subgroups of $\mathsf{G}_2(3)$ now have an easy geometric interpretation inside the generalized hexagon $\mathsf{H}(3)$. Indeed, they are either the stabilizer of a point, a line, a Hermitian ovoid, a Hermitian spread, a Ree-Tits ovoid, an exceptional ovoid, a subhexagon of order $(1,3)$, a subhexagon of order $(3,1)$, a line regulus or a distance-2 ovoid. Notice that the stabilizer of a Ree-Tits ovoid automatically stabilizes a Ree-Tits spread; similarly for an exceptional ovoid (take the complementary one), for a line regulus (taking the complementary point regulus) and for a distance-2 spread (take the unique 14 points of $\mathsf{PG}(6,3)$ that are in none of the 14 defining hyperplanes and determine the corresponding distance-2 ovoid). This explains geometrically why these maximal subgroups have no "dual".

maximal subgroup	order	index	stabilized set
$\mathsf{PSU}_3(3):2$	12096	351	$\mathcal{O}_{\mathsf{H}}$
$\mathsf{PSU}_3(3):2$	12096	351	$\mathcal{S}_{\mathsf{H}}$
$(3_+^{1+2}\times 3^2):2\mathsf{S}_4$	11664	364	p
$(3_+^{1+2}\times 3^2):2\mathsf{S}_4$	11664	364	L
$\mathsf{PSL}_3(3):2$	11232	378	$\Gamma(p,p')$
$\mathsf{PSL}_3(3):2$	11232	378	$\Gamma(L,L')$
$\mathsf{PSL}_2(8):3$	1512	2808	$\mathcal{O}_{\mathsf{RT}}$
$2^3\cdot\mathsf{PSL}_3(2)$	1344	3159	$\mathcal{O}_{\mathsf{E}}$
$\mathsf{PSL}_2(13)$	1092	3888	$\mathcal{O}_{\Omega}$
$2_+^{1+4}:3^2.2$	576	7371	$\mathcal{R}(L,M)$

Table 4.1: Maximal subgroups of $\mathsf{G}_2(3)$

5 Distance-2 ovoids of $\mathsf{H}(4)$

In this chapter, we construct two non-isomorphic distance-2 ovoids of the known generalized hexagon of order 4 (the split Cayley hexagon $\mathsf{H}(4)$).

5.1 General Introduction

This chapter, just as the previous one, will be divided into several main parts.

In the first part, we construct a distance-2 ovoid of $\mathsf{H}(4)$ and prove that the automorphism group is isomorphic to the linear group $\mathsf{PSL}_2(13)$. With this distance-2 ovoid we construct a new type of two-weight codes. Also the corresponding strongly regular graph is new.

In the second part, we construct a new infinite class of two-character sets in $\mathsf{PG}(5, q^2)$ and determine their automorphism groups. From this construction arise new infinite classes of two-weight codes and strongly regular graphs, and a new distance-2 ovoid of the split Cayley hexagon of order 4.

In the third and final part, we provide a joint application for the first two parts. Namely, we show that none of the two distance-2 ovoids that were constructed in the previous sections can be used to lift the known embeddings of $\mathsf{H}(4)$ to higher dimensional embeddings.

The contents of the first two parts of this chapter await publication in *Designs, Codes and Cryptography* [21] and in *Discrete Mathematics* [22].

5.2 Part I: A distance-2 ovoid of H(4)

5.2.1 Introduction

It is well known that distance-3 ovoids in generalized hexagons define perfect codes in the collinearity graph. In the present part of this chapter, we show that distance-2 ovoids in the classical hexagons $\mathsf{H}(2^e)$ define two-weight codes.

Theorem 5.2.1.
Every distance-2 ovoid $\mathcal{O}$ of the generalized hexagon $\mathsf{H}(q)$, with q even, defines a two-character set of $\mathsf{PG}(5,q)$ of size q^4+q^2+1 and constants q^4-q^3 and $q^4-q^3+q^2$.

Consequently, each such distance-2 ovoid defines a linear projective q-ary two-weight code of length q^4+q^2+1 and dimension 6 with weights q^4-q^3 and $q^4-q^3+q^2$. Also, the associated strongly regular graph has parameters

$$\begin{aligned} v &= q^6, \\ k &= (q-1)(q^4+q^2+1), \\ \lambda &= q^4-q^3+q-2, \\ \mu &= q(q-1)(q^2-q+1). \end{aligned}$$

In this section we construct a distance-2 ovoid in $\mathsf{H}(4)$, and we show that the corresponding codes and strongly regular graph are new.

The reason why the projective two-character set arising from the new distance-2 ovoid was not discovered before, is probably because it does not admit a transitive group. In fact, one has to take two orbits of a subgroup of $\mathsf{G}_2(4) \leq \mathsf{PSp}_5(4) \leq \mathsf{PGL}_5(4)$ isomorphic to $\mathsf{PSL}_2(13)$. It is very likely that there are more distance-2 ovoids to be found like this, but apart from the one that will be described in the next section we were unable to continue this positive trend of finding new distance-2 ovoids of $\mathsf{H}(q)$.

We now state the main result of this section.

Theorem 5.2.2.
The generalized hexagon $\mathsf{H}(4)$ contains a distance-2 ovoid $\mathcal{O}_\Omega$ on which the group $\mathsf{PSL}_2(13)$ acts in two orbits as its full automorphism group inside $\mathsf{H}(4)$. One orbit contains 91 points and the action of $\mathsf{PSL}_2(13)$ is equivalent to the primitive action of $\mathsf{PSL}_2(13)$ on the polar triangles of $\mathsf{PG}(2,13)$ when $\mathsf{PSL}_2(13)$ is considered as stabilizer of a conic in $\mathsf{PG}(2,13)$. The other orbit contains 182 points and the action of $\mathsf{PSL}_2(13)$ is imprimitive with 91 classes of length 2 on which $\mathsf{PSL}_2(13)$ acts as on the pairs of points of $\mathsf{PG}(1,13)$ (in the natural action of $\mathsf{PSL}_2(13)$ on the projective line $\mathsf{PG}(1,13)$).

As a consequence we have

Corollary 5.2.3.
The distance-2 ovoid of $\mathsf{H}(4)$ *of* Theorem 5.2.2 *defines a new strongly regular graph with parameters* $(v, k, \lambda, \mu) = (4096, 819, 194, 156)$ *and two new linear projective two-weight codes, one* 4*-ary code of length* 273*, dimension* 6 *and weights* $208, 192$*, and one binary code of length* 819*, dimension* 12 *and weights* $384, 416$.

Remarks. (1) The codes of the previous corollary are most likely new because of their parameters, see [8].

(2) The unique distance-2 ovoid of the generalized hexagon $\mathsf{H}(2)$ defines a two-character set $\mathcal{S}$ of $\mathsf{PG}(5, 2)$ with 21 points and constants 8 and 12. The linear representation graph $\Gamma^*(\mathcal{S})$ is a well known strongly regular (rank 3) graph with parameters $(v, k, \lambda, \mu) = (64, 21, 8, 6)$. The corresponding binary two-weight code has length 21, dimension 6 and weights 8 and 12. This two-weight code is described in [8] as a two-weight code of type $SU2$, where $q = 2$, $l = 3$ and $i = 3$.

In the next section we prove Theorem 5.2.1. In Section 5.2.3, we present a geometrical construction of a distance-2 ovoid of $\mathsf{H}(4)$ and prove Theorem 5.2.2.

5.2.2 Two-weight codes from distance-2 ovoids

In this section we prove Theorem 5.2.1. Let $\mathcal{O}$ be a distance-2 ovoid in $\mathsf{H}(q)$, with q even. From Section 2.2 we know that in this case the quadric $\mathsf{Q}(6, q)$ has a nucleus and that we may project the points and lines of $\mathsf{H}(q)$ from that nucleus to obtain a representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5, q)$. The distance-2 ovoid $\mathcal{O}$ can thus be viewed as a set of $q^4 + q^2 + 1$ points in $\mathsf{PG}(5, q)$. Now every hyperplane H of $\mathsf{PG}(5, q)$ is the projection of a tangent hyperplane to $\mathsf{Q}(6, q)$, hence the points in H are precisely the points not opposite a certain point x of $\mathsf{H}(q)$. We now have to count the number of points in $H \cap \mathcal{O}$.

There are two cases to consider. Either x is contained in $\mathcal{O}$, or x is not contained in $\mathcal{O}$. Note that H is the union of all lines of $\mathsf{H}(q)$ at distance at most 3 from x.

In the first case each line at distance 3 from x contains a unique point of $\mathcal{O}$ and no point of $\mathcal{O}$ is contained in two distinct such lines. Since there are $q^2(q + 1)$ such lines, H contains precisely $q^3 + q^2 + 1$ elements of $\mathcal{O}$.

In the second case, every line incident with x contains a point of $\mathcal{O}$ – and there are $q+1$ such lines – and every line at distance 3 not incident with a point of $\mathcal{O}$ that is collinear with x contains a unique point of $\mathcal{O}$ at distance 4 from x. It follows that H contains

$$(q+1)+(q-1)q(q+1)=q^3+1$$

elements of $\mathcal{O}$.

Hence $\mathcal{O}$ is a two-character set of size q^4+q^2+1 with constants

$$(q^4+q^2+1)-(q^3+q^2+1)=q^4-q^3$$

and

$$(q^4+q^2+1)-(q^3+1)=q^4-q^3+q^2.$$

Theorem 5.2.1 is now clear.

5.2.3 Construction of a distance-2 ovoid in H(4)

In this section, we prove Theorem 5.2.2.

Consider the following set, Ω, of 14 hyperplanes of $\mathsf{PG}(6,4)$

$$\begin{aligned}
\overline{\infty} &= [0,1,0,\sigma,0,1,0], & \overline{0} &= [0,1,0,1,0,\sigma^2,0],\\
\overline{1} &= [1,1,0,1,\sigma,1,0], & \overline{2} &= [1,\sigma^2,\sigma^2,1,\sigma^2,\sigma,0],\\
\overline{3} &= [1,\sigma,\sigma^2,\sigma,1,\sigma,0], & \overline{4} &= [1,0,1,\sigma^2,1,\sigma^2,\sigma],\\
\overline{5} &= [1,\sigma,0,\sigma^2,1,1,0], & \overline{6} &= [1,1,1,\sigma,\sigma,\sigma^2,0],\\
\overline{7} &= [1,0,1,\sigma,0,\sigma^2,\sigma], & \overline{8} &= [1,0,0,\sigma^2,1,1,\sigma],\\
\overline{9} &= [1,\sigma^2,1,\sigma^2,\sigma^2,\sigma^2,0], & \overline{10} &= [1,0,\sigma^2,\sigma,0,\sigma,\sigma],\\
\overline{11} &= [1,0,\sigma^2,1,\sigma,\sigma,\sigma], & \overline{12} &= [1,0,0,1,\sigma,1,\sigma],
\end{aligned}$$

with $\sigma \in \mathsf{GF}(4) \setminus \mathsf{GF}(2)$.

This set of hyperplanes is stabilized by the group elements φ_∞ and φ_0 of $\mathsf{PGL}_7(4)$ with respective matrices

$$A_\infty = \begin{bmatrix}
0 & \sigma & 1 & \sigma^2 & 1 & \sigma^2 & \sigma^2\\
1 & 1 & \sigma^2 & \sigma^2 & \sigma & \sigma^2 & 1\\
\sigma^2 & 0 & 0 & \sigma^2 & \sigma^2 & 1 & 1\\
0 & 0 & 0 & \sigma^2 & 0 & 0 & 0\\
1 & \sigma^2 & \sigma^2 & 1 & 1 & 1 & 1\\
\sigma^2 & \sigma^2 & 0 & \sigma & \sigma & \sigma^2 & 0\\
\sigma^2 & \sigma^2 & 1 & \sigma^2 & \sigma^2 & \sigma & 1
\end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} \sigma^2 & \sigma^2 & 1 & 1 & 1 & \sigma^2 & 0 \\ 1 & \sigma^2 & \sigma & 0 & 1 & \sigma^2 & \sigma \\ \sigma^2 & 0 & \sigma^2 & 1 & 0 & \sigma & \sigma \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sigma^2 & 1 & 0 & \sigma & \sigma^2 & 1 & 0 \\ 1 & 1 & 0 & \sigma & \sigma^2 & 0 & 0 \\ 0 & 1 & \sigma & \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \end{bmatrix}$$

and consequently also by the group G generated by these elements.

Note. Both φ_∞ and φ_0 stabilize the split Cayley hexagon $\mathsf{H}(4)$ as can be checked with an elementary but tedious computation.

By definition of a hyperplane, we easily find that the image of a hyperplane H of Ω is obtained by right multiplication with the matrices

$$A_\infty^* = (A_\infty^T)^{-1} = \begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & 0 & \sigma^2 & \sigma & \sigma \\ 1 & \sigma & 0 & 0 & \sigma & \sigma & 0 \\ \sigma & 1 & \sigma^2 & 0 & \sigma & \sigma & \sigma^2 \\ 0 & 1 & \sigma & \sigma & \sigma^2 & \sigma & 1 \\ \sigma^2 & \sigma & \sigma & 0 & 0 & 1 & \sigma^2 \\ 1 & \sigma & \sigma^2 & 0 & \sigma^2 & \sigma^2 & \sigma \\ \sigma & \sigma^2 & \sigma^2 & 0 & \sigma & 0 & 0 \end{bmatrix}$$

and

$$A_0^* = (A_0^T)^{-1} = \begin{bmatrix} \sigma^2 & 1 & 0 & 0 & \sigma^2 & 1 & 0 \\ \sigma^2 & 0 & 0 & 0 & 1 & 1 & 0 \\ \sigma^2 & \sigma^2 & \sigma^2 & 0 & 0 & 1 & \sigma \\ 0 & \sigma^2 & 1 & 1 & \sigma & 0 & 1 \\ 1 & \sigma^2 & 0 & 0 & \sigma^2 & \sigma^2 & 1 \\ 1 & \sigma^2 & \sigma & 0 & 1 & \sigma^2 & \sigma \\ 0 & \sigma & \sigma & 0 & \sigma^2 & 0 & \sigma^2 \end{bmatrix},$$

i.e. $H^{\varphi_\infty} = HA_\infty^*$ and $H^{\varphi_0} = HA_0^*$.

We now consider the elements $\overline{x}$, $x \in \{0, 1, 2, \ldots, 12\}$, as elements of $\mathsf{GF}(13)$ in the obvious way. We then note that φ_∞ fixes $\overline{\infty}$ and maps $\overline{x}$ to $\overline{x+1}$, while φ_0 fixes $\overline{0}$ and maps $\overline{x}$ to $\overline{x/x+1}$, with usual multiplication and addition laws if ∞ is involved. This easily implies that G is isomorphic to $\mathsf{PSL}_2(13)$, and that we may identify Ω with the points of the projective line $\mathsf{PG}(1, 13)$ (in the natural way), at least concerning the action of G. The fact that G is

not isomorphic to a non-trivial central extension of $\mathsf{PSL}_2(13)$ can easily be derived from the list of maximal subgroups of $\mathsf{G}_2(4)$ (see [11]). Alternatively, the point-wise stabilizer of Ω in $\mathsf{PGL}_7(4)$ is the identity.

As the coordinates of points of $\overline{\infty} \cap \mathsf{Q}(6,4)$ satisfy the equation

$$X_0X_4 + X_2X_6 = X_3^2 + \sigma X_3X_5 + X_5^2,$$

and as $X_3^2+\sigma X_3X_5+X_5^2$ is irreducible over $\mathsf{GF}(4)$, it is clear that all elements of Ω are elliptic (as $\overline{\infty}$ is, and as G acts transitively on Ω).

We will now choose two elements of Ω and look at their intersection. By the double transitivity it suffices to consider the two hyperplanes $\overline{\infty}$ and $\overline{0}$. Their intersection is the 4-dimensional subspace S of $\mathsf{PG}(6,4)$ determined by the equations $X_1 = 0$ and $\sigma X_3 + X_5 = 0$. Intersecting S with $\mathsf{Q}(6,4)$ gives us the parabolic quadric $\mathsf{Q}(4,4)$ the points of which satisfy the equation $X_0X_4 + X_2X_6 = X_3^2$ (besides $X_1 = 0$ and $\sigma X_3 + X_5 = 0$). There is now a unique hyperplane H tangent to $\mathsf{Q}(6,4)$ and containing $\mathsf{Q}(4,4)$, namely the one with equation $X_1 = 0$. This hyperplane contains a unique corresponding tangent point p, which has coordinates $(0,0)$. The hexagon plane Π_p intersects $\mathsf{Q}(4,4)$ in an ideal line, which we denote by $L(\overline{\infty},\overline{0})$, and that contains the points (∞) and $(0,0,k',0)$, for all $k' \in \mathsf{GF}(4)$. For each point x of $L(\overline{\infty},\overline{0})$, the hyperplane $\overline{\infty}$ meets the hexagon plane Π_x in a line through x distinct from px (since $p \notin \overline{\infty}$). Since $\overline{\infty} \cap H = \overline{\infty} \cap \overline{0}$, we see that the lines of $\mathsf{H}(4)$ in $\mathsf{Q}(4,4)$ form a regulus $\mathcal{R}_{\overline{\infty},\overline{0}}$ with transversal $L(\overline{\infty},\overline{0})$. It is easy to see that the elements of $\mathcal{R}_{\overline{\infty},\overline{0}}$ are the lines $[\infty]$ and $[0,0,k',0,0]$.

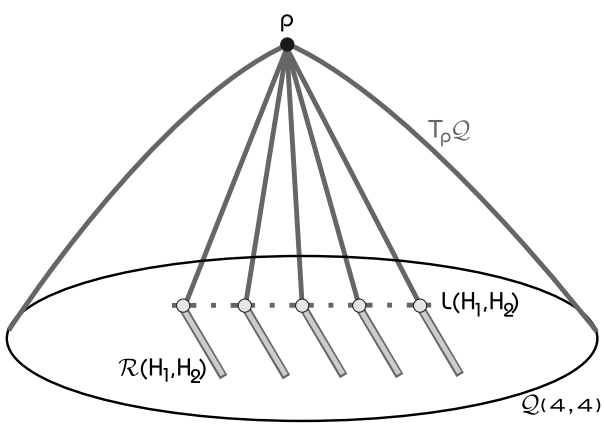

Figure 5.1: Intersection of H_1 and H_2

As G acts 2-transitively on Ω, we conclude that every two hyperplanes $H_1, H_2 \in \Omega$ determine a unique line $L(H_1, H_2)$ on $\mathsf{Q}(6,4)$, not contained in $\mathsf{H}(4)$. We call it the *tangent line* of the corresponding pair of hyperplanes. The union of the points on all tangent lines will be denoted by $\mathcal{O}_\Omega$. We will now prove the following proposition.

Proposition 5.2.4.
The set $\mathcal{O}_\Omega$ contains exactly 273 *points and constitutes a distance-*2 *ovoid of* $\mathsf{H}(4)$.

Looking at the points of the tangent line, $L(\overline{\infty}, \overline{0})$, it is easy to check that (∞) and $(0,0,\sigma,0)$ are in no other hyperplane of Ω, while the remaining 3 points of $L(\overline{\infty}, \overline{0})$ are, each of them, in 4 other hyperplanes, which gives us the remaining 12 hyperplanes of Ω. By double transitivity of G, the orbit $\mathcal{O}_1 := \{(\infty), (0,0,\sigma,0)\}^G \subseteq \mathcal{O}_\Omega$ contains 182 elements.

We now claim that any point of $L(\overline{\infty}, \overline{0}) \setminus \mathcal{O}_1$ is determined by 3 pairs of Ω. This will lead to a second set $\mathcal{O}_2$ of 91 distinct points, that is disjoint from $\mathcal{O}_1$ (because each element of $\mathcal{O}_2$ is contained in 6 hyperplanes of Ω, while the points of $\mathcal{O}_1$ are in exactly two such hyperplanes).

In order to prove the above claim, we now write down the respective hyperplanes containing the points $(0,0,k',0)$, $k' \in \{0,1,\sigma^2\}$. For $k' = 0,1,\sigma^2$, these points respectively belong to the hyperplanes $\{\overline{1}, \overline{5}, \overline{8}, \overline{12}\}$, $\{\overline{4}, \overline{6}, \overline{7}, \overline{9}\}$ and $\{\overline{2}, \overline{3}, \overline{10}, \overline{11}\}$.

Now consider the element g of $\mathsf{PSL}_2(13)$ mapping $\overline{x}$ to $\overline{-x}$. Note that g fixes $\overline{\infty}$ and $\overline{0}$ and hence also stabilizes $L(\overline{\infty}, \overline{0})$, the point $(0,0)$ and the regulus $\mathcal{R}_{\overline{\infty},\overline{0}}$. As the hyperplane $\overline{1}$, respectively $\overline{6}$, is mapped onto the hyperplane $\overline{12}$, respectively $\overline{7}$, we have that both points $(0,0,0,0)$ and $(0,0,1,0)$ of $L(\overline{\infty}, \overline{0})$ are fixed. Thus, since $g \in \mathsf{PGL}_7(4)$ and since g is an involution, the latter line is fixed point-wise, and hence the regulus $\mathcal{R}_{\overline{\infty},\overline{0}}$ is fixed line-wise. Similarly all points collinear with the point $(0,0)$ and all (hexagon) lines incident with the points of $L(\overline{\infty}, \overline{0})$ are fixed by g.

Hence g is a point-elation with center $p = (0,0)$, the tangent point of the fixed pair of hyperplanes.

The involution g is uniquely determined by the pair $\{\overline{\infty}, \overline{0}\}$. In general, there is such an involution corresponding to every pair $\{H_1, H_2\}$ of Ω, and we denote it by $\sigma[H_1, H_2]$.

Now consider $g' = \sigma[\overline{1}, \overline{12}]$. Its action on Ω is given by $g' : \overline{x} \mapsto \overline{1/x}$. It is also a point-elation, having as center the tangent point q corresponding to the pair $(\overline{1}, \overline{12})$. But g' stabilizes the set $\{\overline{\infty}, \overline{0}\}$ and consequently fixes $L(\overline{\infty}, \overline{0})$,

p and $\mathcal{R}_{\overline{\infty},\overline{0}}$. However, this time, the line $L(\overline{\infty},\overline{0})$ is not fixed point-wise. In fact, only the point $(0,0,0,0)$ is fixed. This implies that all points on the line L through p and $(0,0,0,0)$ are fixed points. Since the center of a point-elation obviously is incident with every point-wise fixed line, the point q is a point on L different from p and $(0,0,0,0)$ (as all tangent points are distinct and do not belong to the pair of hyperplanes they correspond to).

Since g' fixes all lines through the point $(0,0,0,0)$, the latter belongs to the tangent line corresponding to the pair $(\overline{1},\overline{12})$. Similar arguments show that it also belongs to the tangent line corresponding to the pair $(\overline{5},\overline{8})$.

The same thing can be shown for the points $(0,0,1,0)$ and $(0,0,\sigma^2,0)$. Hence, in order to prove the claim, it suffices to show that for no other pair of hyperplanes H_1 and H_2 the line $L(H_1,H_2)$ contains a point of $L(\overline{\infty},\overline{0})$. Suppose by way of contradiction that $L(H_1,H_2)$ contains a point of $L(\overline{\infty},\overline{0})$. By the 2-transitivity, we may assume that $H_1=\overline{\infty}$.

First of all, before coming to the actual contradiction of this statement, we claim that the intersection of any three hyperplanes that are contained in Ω share no hexagon lines. Indeed, by 2-transitivity we may assume $\overline{\infty}$ and $\overline{0}$ to be two of these three hyperplanes. The stabilizer of the set $\{\overline{\infty},\overline{0}\}$ in G acts transitively on two sets of 6 hyperplanes. Considering the group action of G, it is clear these sets correspond with the squares and non-squares in $\mathsf{GF}(13)$. Let the hyperplane $\overline{1}$, respectively $\overline{5}$, be an element related to the squares, respectively non-squares, of $\mathsf{GF}(13)$. Both these hyperplanes intersect the line $L(\overline{\infty},\overline{0})$ in the point $(0,0,0,0)$. The only possible line in the intersection of $\overline{\infty}$, $\overline{0}$ and $\overline{1}$, respectively $\overline{5}$, is therefore the line $[0,0,0,0,0]=\langle(0,0,0,0),(0,0,0,0,0)\rangle$. As the point $(0,0,0,0,0)$ does not belong to neither $\overline{1}$ nor to $\overline{5}$, the intersection of any three hyperplanes contains no hexagon lines and hence the claim.

Now, if $L(\overline{\infty},\overline{0})$ were to intersect the line $L(\overline{\infty},H_2)$ in the point x, then we would have $M_x\in\mathcal{R}_{\overline{\infty},\overline{0}}$ and $N_x\in\mathcal{R}_{\overline{\infty},H_2}$ as regulus lines on x. Since $\overline{\infty}$ is an elliptic hyperplane and every such a hyperplane determines a spread of $\mathsf{H}(q)$, we immediately find that $M_x=N_x$. But then the line M_x would be a line of the intersection of these three hyperplanes, a contradiction. The claim is proved.

As mentioned above, this implies that $|\mathcal{O}_2|=91$ and $|\mathcal{O}_\Omega|=273$.

To prove that the set $\mathcal{O}_\Omega$ constitutes a distance-2 ovoid it now suffices to show that first, the point (∞) (as a 2-hyperplane point) and secondly, the point $(0,0,0,0)$ (as a 6-hyperplane point) are not collinear to other points of O_Ω.

Lemma 5.2.5.
A regulus line, say $R \in \mathcal{R}_{H_1,H_2}$, never contains O_Ω-points other than its point, say x, on $L(H_1, H_2)$.

Proof. Suppose by way of contradiction that R contains a second point y of O_Ω, say on $L(H_3, H_4)$. First of all, it is impossible that R belongs to $\mathcal{R}_{H_3,H_4}$, as no three hyperplanes share a line. Since y belongs to $H_1 \cap H_2$ as well as to $H_3 \cap H_4$, it must be a 6-hyperplane point. In other words, there are two more regulus lines through y, namely a line in $\mathcal{R}_{H_1,H_5}$ and one in $\mathcal{R}_{H_2,H_6}$. This means, since no three hyperplanes share a line, that y is on two distinct lines of $H_1 \cap \mathsf{Q}(6,4)$, a contradiction (as these lines are the lines of a spread).

□

Consider the point (∞). By Lemma 5.2.5, the line $[\infty]$ contains no other O_Ω-points. As $\sigma[\overline{\infty}, \overline{0}]$ fixes the lines incident with (∞), we have to check whether $[a]$, with $a \in \mathsf{GF}(4)$, contains points of our set. We will start by looking at the points on the line $[0]$. This line is incident with the points $(0, b)$, with $b \in \mathsf{GF}(4)$ and (∞). First, the point p belongs to none of the 14 hyperplanes and can therefore never belong to O_Ω. Secondly, with the coordinatization given in Table 2.1, it is easy to see that each point $(0, b)$, $b \neq 0$, belongs to exactly 4 hyperplanes ($(0,1) \in \overline{1}, \overline{5}, \overline{8}, \overline{12}$; $(0,\sigma) \in \overline{2}, \overline{3}, \overline{10}, \overline{11}$; $(0,\sigma^2) \in \overline{4}, \overline{6}, \overline{7}, \overline{9}$). Thus, the line $[0]$ contains no points of O_Ω other than (∞).

The points $(1, b)$ of $[1]$, with $b \in \mathsf{GF}(4)$, belong to (in order of increasing b) the hyperplanes $\overline{10}, \overline{11}$; $\overline{4}, \overline{5}, \overline{7}, \overline{1}$; $\overline{2}, \overline{3}$ and $\overline{6}, \overline{8}, \overline{9}, \overline{12}$, respectively. This immediately tells us that neither $(1,1)$ nor $(1,\sigma^2)$ belongs to O_Ω. For $(1,0)$ (respectively $(1,\sigma)$) not to be in O_Ω, it must not be incident with the line $L(\overline{10}, \overline{11})$ (respectively $L(\overline{2}, \overline{3})$). As the tangent point determined by these two hyperplanes equals $(\sigma, 1, \sigma^2, \sigma, \sigma)$ (respectively $(0, 1, \sigma^2, \sigma, \sigma)$) and this point is at distance 4 from $(1,0)$ (respectively $(1,\sigma)$), this is indeed not the case.

The same arguments can be used to show that $[\sigma]$ nor $[\sigma^2]$ contain points of $\mathcal{O}_\Omega$ other than (∞). Consequently a 2-hyperplane point is not collinear to other points of the set.

Finally, we will show that $(0,0,0,0)$ is not collinear to any point of O_Ω and thus prove that O_Ω constitutes a distance-2 ovoid.

The line $[0,0,0]$ contains, besides p, the points $(0,0,0,b')$. We know that $(0,0,0,0)$ belongs to 6 hyperplanes of Ω. The 8 remaining hyperplanes of Ω intersect the line $[0,0,0]$ in the point $(0,0,0,\sigma)$, which implies that none of these points belongs to O_Ω.

As we have shown above, the point $(0,0,0,0)$ is determined by 3 pairs of hyperplanes, each of them determines a distinct regulus line through $(0,0,0,0)$, namely $[0,0,0,0,0] \in \mathcal{R}(\overline{\infty},\overline{0})$, $[0,0,0,0,1] \in \mathcal{R}(\overline{5},\overline{8})$, $[0,0,0,0,\sigma] \in \mathcal{R}(\overline{1},\overline{12})$. This leaves us, by Lemma 5.2.5, with $[0,0,0,0,\sigma^2]$ as the only possible line to contain O_Ω-points. This line is incident with the points $(a,\sigma^2,0,0,0)$, $a \in \mathsf{GF}(4)$, and $(0,0,0,0)$. For every $a \in \mathsf{GF}(4)$ these points belong to two hyperplanes of Ω, namely for $a=0$ to $\overline{3},\overline{6}$; for $a=1$ to $\overline{9},\overline{11}$; for $a=\sigma$ to $\overline{7},\overline{10}$ and for $a=\sigma^2$ to $\overline{2},\overline{4}$. Since we already know that 2-hyperplane points are not collinear to any points of O_Ω, none of these points belongs to our set and we are done.

□

Also Theorem 5.2.2 follows from the above proof.

Hence $\mathcal{O}_\Omega$ can be viewed as a two-character set of $\mathsf{PG}(5,4)$. The corresponding linear representation graph $\Gamma_4^*(\mathcal{O}_\Omega)$ is an $srg(4096,819,194,156)$. This strongly regular graph, however, does not contain maximal cliques of size 64 (which we checked by computer). Therefore this graph is not isomorphic to the known strongly regular graphs with the above parameters, which all arise from the point graph of nets of order 64 and degree 13.

5.3 Part II: Two-character sets in $\mathsf{PG}(5,q^2)$

5.3.1 Introduction

In 1986 R. Calderbank and W.M. Kantor examined all two-character sets that were known at that time, see [8]. Some new examples of two-character sets arose since then, mainly from constructions of interesting geometric objects such as pseudo-ovoids (or eggs) and m-systems. In the present section, we will define a new two-character set in $\mathsf{PG}(5,q^2)$, for every prime power q. Our construction uses an unexpected idea, namely, the idea of an anti-isomorphism between two skew planes of $\mathsf{PG}(5,q^2)$, combined with the consideration of a Baer subplane.

As an application, we will embed the two-character set thus obtained in $\mathsf{PG}(5,4)$ into the split Cayley hexagon $\mathsf{H}(4)$ in such a way that it becomes a distance-2 ovoid of $\mathsf{H}(4)$. This is, next to the known example in $\mathsf{H}(2)$, the one in $\mathsf{H}(3)$ as described in Chapter 4 and the one defined in the previous section, the fourth example of a distance-2 ovoid. But our attempts to define an infinite family this way were not successful; in fact we proved that the –

otherwise natural – construction of this distance-2 ovoid does not generalize to other values of q distinct from 4.

We will also prove that every member of the new family of two-character sets has a fairly big (though intransitive) automorphism group, and some nice geometric properties. The intransitivity of the automorphism group is probably the major reason why this class has not been noticed before.

5.3.2 Statement of the main results

The new two-character sets

Let q be a prime power and let $\mathsf{PG}(5, q^2)$ denote the 5-dimensional projective space over the Galois field $\mathsf{GF}(q^2)$ of order q^2. Let Π and Π' be two skew planes in $\mathsf{PG}(5, q^2)$, and choose any anti-isomorphism θ from Π onto Π' (so θ maps the point set of Π onto the line set of Π' and the line set of Π onto the point set of Π', thereby preserving the incidence relation). Let B be a Baer subplane of Π and denote its image under θ by B'. We will call the points and lines of both B and B' *Baer points* and *Baer lines*, respectively. The other points and lines of Π and Π' will be referred to as *non-Baer points* and *non-Baer lines*.

We now construct the set $S(\Pi, \Pi', B, \theta)$ that will turn out to be a two-character set. The points of $S(\Pi, \Pi', B, \theta)$ are of three types. Notice first that every point x of $\mathsf{PG}(5, q^2)$, not in the union of Π and Π', lies on a unique line $L(x)$ that meets both Π and Π' in points x_Π and $x_{\Pi'}$, respectively. Namely, $x_\Pi = \langle \Pi', x\rangle \cap \Pi$ and $x'_\Pi = \langle \Pi, x\rangle \cap \Pi'$ and hence the line $L(x)$ is unique as the intersection of $\langle \Pi, x\rangle$ and $\langle \Pi', x\rangle$. Moreover, for three collinear points x, y, z in $\mathsf{PG}(5, q^2) \setminus (\Pi \cup \Pi')$ the three points x_Π, y_Π, z_Π of Π (respectively $x_{\Pi'}$, $y_{\Pi'}$, $z_{\Pi'}$ of Π') are collinear (or coincide). We will use this notation below.

(PI) The PI points are the points of Π.

(BB) The BB points are the points x of $\mathsf{PG}(5, q^2)$ not in $\Pi \cup \Pi'$ such that both x_Π and $x_{\Pi'}$ are Baer points with $x_{\Pi'}$ not incident with x_Π^θ.

(NB) The NB points are the points x of $\mathsf{PG}(5, q^2)$ not in $\Pi \cup \Pi'$ such that both x_Π and $x_{\Pi'}$ are non-Baer points with $x_{\Pi'}$ incident with x_Π^θ.

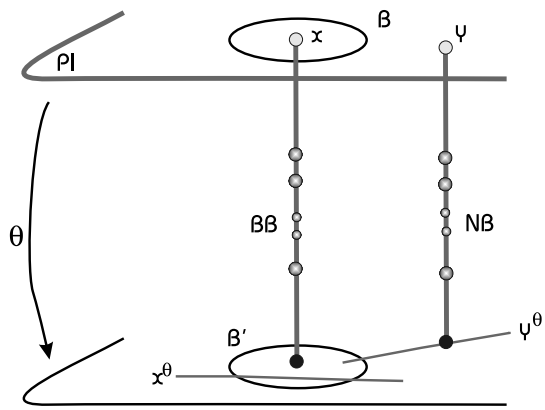

Figure 5.2: Two-character set in $\mathsf{PG}(5, q^2)$

Call a line uu', with $u \in \Pi$ and $u' \in \Pi'$, an S-line if either both u and u' are Baer points and u^θ is not incident with u', or both u and u' are non-Baer points and u^θ is incident with u'. Note that all q^2 points of an S-line distinct from its intersection with Π' belong to S.

Remark. For any point x of $\mathsf{PG}(5, q^2) \setminus \Pi \cup \Pi'$ the line $L(x)$ is an S-line if and only if $x \in S$.

The first main result of this section reads:

Proposition 5.3.1.
The set $S(\Pi, \Pi', B, \theta)$ contains exactly $q^8 + q^4 + 1$ points of $\mathsf{PG}(5, q^2)$ and constitutes a two-character set with constants $q^8 - q^6$ and $q^8 - q^6 + q^4$. The automorphism group of $S(\Pi, \Pi', B, \theta)$ in $\mathsf{PG}(5, q^2)$ is a non-split extension $(q+1) \cdot (\Gamma\mathsf{L}_3(q) \times 2)$. The planes Π and Π' have the following characterizing properties with respect to $S(\Pi, \Pi', B, \theta)$ (and hence are fixed under the automorphism group of $S(\Pi, \Pi', B, \theta)$):

- (i) *the plane Π is the unique plane of $\mathsf{PG}(5, q^2)$ entirely contained in $S(\Pi, \Pi', B, \theta)$;*
- (ii) *the plane Π' is the only plane of $\mathsf{PG}(5, q^2)$ all of whose points x have the following property: x is not contained in $S(\Pi, \Pi', B, \theta)$, but x is incident with a line L all other points of which are contained in $S(\Pi, \Pi', B, \theta)$, and such that L meets the unique plane Π entirely contained in $S(\Pi, \Pi', B, \theta)$.*

Let $S =: S(\Pi, \Pi', B, \theta)$ be the two-character set as described in the previous proposition. Now embed $\mathsf{PG}(5, q^2)$ as a hyperplane $\mathcal{H}$ in $\mathsf{PG}(6, q^2)$. Let $\Gamma_5^*(S)$ be the linear representation graph associated with S as described in Section 1.3.6. Then, again by Section 1.3.6, we immediately have the following corollary to our proposition above:

Corollary 5.3.2.
The two-character set S in $\mathsf{PG}(5, q^2)$ described above defines a new strongly regular graph with parameters

$$\begin{aligned} v &= q^{12}, \\ k &= (q^2-1)(q^8+q^4+1), \\ \lambda &= q^8 - q^6 + q^2 - 2, \\ \mu &= q^2(q^2-1)(q^4-q^2+1) \end{aligned}$$

and new linear projective two-weight codes: one q^2-ary code of length q^8+q^4+1, dimension 6 and weights q^8-q^6, $q^8-q^6+q^4$, and, for each prime power q_0 such that $q_0^r = q$, one q_0-ary code of length $(q^8+q^4+1)\frac{q^2-1}{q_0-1}$, dimension $6 \times 2r$ and weights ${q_0}^{2r-1}(q^8-q^6)$ and ${q_0}^{2r-1}(q^8-q^6+q^4)$.

Generalized hexagons and distance-2 ovoids

It is shown in Section 5.2.2 that every distance-2 ovoid of $\mathsf{H}(q)$, represented in $\mathsf{PG}(5, q)$, is a two-character set in $\mathsf{PG}(5, q)$. We now consider the question whether the two-character set of Proposition 5.3.1 could arise in this way from $\mathsf{H}(q^2)$. Of course, to answer this, one has to investigate all possible situations for the two planes Π and Π', the anti-isomorphism θ, and the Baer subplane B. We will only consider the most natural situation (and we conjecture that the other situations never give rise to a distance-2 ovoid of $\mathsf{H}(q^2)$).

So consider $\mathsf{H}(q^2)$ represented in $\mathsf{PG}(5, q^2)$ as explained in Chapter 2, Section 2.2. Choose an arbitrary ideal plane Π and let the ideal plane Π' be its hexagon twin. Let τ be the associated hexagon twin anti-isomorphism. Choose a Baer subplane B in Π and let β denote the associated semi-linear involution in Π whose fixed point set is exactly the set of points of B. Put $\theta = \beta\tau$. Then we will prove below:

Proposition 5.3.3.
The two-character set $S(\Pi, \Pi', B, \theta)$, with Π, Π', B and θ as just described, is a distance-2 ovoid of $\mathsf{H}(q^2)$, q even, if and only if $q = 2$ and B is not contained in a subhexagon of order $(2, 2)$. In that case, it is a new distance-2 ovoid.

Hence we now have four distance-2 ovoids: one in each of $\mathsf{H}(2)$ and $\mathsf{H}(3)$ (and these are unique, see Chapter 4), and two in $\mathsf{H}(4)$.

5.3.3 Two-character sets in $\mathsf{PG}(5, q^2)$

In this section, we prove Proposition 5.3.1. We set $S := S(\Pi, \Pi', B, \theta)$. The proof of this proposition comes in two parts. First (and this is the easy part) we will show that the size of S is in fact q^8+q^4+1 and secondly we will show that every hyperplane intersects in one of two intersection numbers, being q^6+1 and q^6+q^4+1.

To start with, S contains q^4+q^2+1 points of Π. Each one of these points, say for instance a point x, now determines $q^2(q^2-1)$ BB or NB (depending on x being a Baer point or not) points of S. Furthermore, any one of those BB or NB points is collinear to a unique point of Π. Hence none of the points thus obtained are counted double and a simple calculation

$$|S| = (q^4+q^2+1)(1+q^2(q^2-1))$$

yields the above stated.

Before coming to the actual proof of the second part, we note a certain symmetry in the construction of S. Indeed, one easily checks that the set $S(\Pi', \Pi, B^\theta, \theta^{-1})$ is equal to $(S \cup \Pi') \setminus \Pi$. We will refer to this as the *Symmetry Property.*

Now let H be any hyperplane of $\mathsf{PG}(5, q^2)$. We must show that H intersects S in either q^6+1 or q^6+q^4+1 points.

There are three distinct situations to consider.

H contains Π′

In this case we show that $|H \cap S| = q^6+1$.

The intersection of H with Π is here some line L. Note that for every point p of Π, there are exactly $q^2(q^2-1)$ points x of $\mathsf{PG}(5, q^2) \setminus (\Pi \cup \Pi')$ with $x_\Pi = p$, which are partitioned into q^2 lines through p. Hence it is clear that, if a point $x \notin \Pi$ belongs to $H \cap S$, then the point x_Π is incident with L. There are q^2+1 possibilities for this. The construction of S implies that there are now exactly q^2 possibilities for $x_{\Pi'}$. In conclusion, $H \cap S$ contains $(q^2+1)+(q^2+1)q^2(q^2-1) = q^6+1$ points.

H contains Π

In this case we show that $|H \cap S| = q^6 + q^4 + 1$.

Indeed, by the Symmetry Property, $H \cap S$ contains $(q^6+1)-(q^2+1)+(q^4+q^2+1)$ points.

The general situation: H meets Π in a line L and Π′ in a line L′

Denote by ℓ the number of S-lines intersecting both L and L' non-trivially. Since there are exactly $(q^2+1)q^2$ S-lines intersecting L, respectively L', non-trivially, and since every S-line meeting neither L nor L' intersects H in a unique point of S, we obtain

$$\begin{aligned} |H \cap S| &= (q^2-1)\ell + (q^2+1) + ((q^6+q^4+q^2) - 2(q^4+q^2-\ell) - \ell) \\ &= q^6 - q^4 + \ell q^2 + 1 \end{aligned}$$

Hence it suffices to determine ℓ. For this, we need to distinguish the cases where L is a Baer line or not, L' is a Baer line or not, and L^θ is incident with L' or not. Using the Symmetry Property, we thus obtain six different cases. For each case one performs an elementary counting.

L and L′ are non-Baer lines and L^θ belongs to L′.

Note that L^θ is a non-Baer point.

Let x be the unique Baer point on L. Then x^θ is incident with L^θ and different from L'. Hence it is not incident with the unique Baer point x' on L'. So the line xx' is an S-line.

Since L^θ is on L', there exists a unique non-Baer point y on L with $y^\theta = L'$. Every line yy', with y' any non-Baer point of L', is an S-line.

Symmetrically, putting $z' = L^\theta = z^\theta \cap L'$, for every other point z on L, the line zz' is an S-line.

Hence we count $1 + q^2 + (q^2 - 1) = 2q^2$ S-lines meeting both L and L' non-trivially.

L and L′ are non-Baer lines and L^θ is a point off L′.

In this particular situation we have to distinguish two subsituations, namely the Baer line through L^θ can intersect L' in its Baer point or not. However, in both cases we will end up with l equal to q^2, as we will show.

Suppose x', the Baer point on L', is incident with x^θ, where x is the Baer point on L. Then yy', with $y' = y^\theta \cap L'$ and y ranging over the q^2 non-Baer points of L, exhaust all S-lines.

If x^θ does not go through x', then the S-lines in question are xx' and yy', with $y' = y^\theta \cap L'$ and y ranging over the $q^2 - 1$ non-Baer points on L for which y^θ does not contain x'.

Thus in either case, we have $\ell = q^2$.

L and L′ are Baer lines and L^θ belongs to L′.

Let x be the point on L that is mapped onto L' by θ. Obviously, by definition of the S-lines, the point x will not contribute to the value of ℓ.

If y is one of the q remaining Baer points of L, then yy', with y' a Baer point on L' different from the point L^θ, determine all S-lines of H.

For z a non-Baer point on L, the line z^θ contains the point L^θ as unique point of L' and hence none of the S-lines on z belong to the hyperplane.

In other words, we obtain $\ell = q^2$.

L and L′ are Baer lines and L^θ is a point off L′.

Since for every point p on L, the line p^θ is incident with the Baer point L^θ, since a non-Baer line contains a unique Baer point and since both L and L' contain the exact same number of non-Baer points, the image of a non-Baer point of L will intersect L' in a non-Baer point.

For every such a non-Baer point y on L we hence have a unique S-line that is contained in H.

Consequently x^θ, with x a Baer point, intersects L' in a Baer point, which leaves us with q other Baer points x' on L' that are on a S-line xx' with x.

These two numbers add up to $\ell = q^2 - q + (q+1)q = 2q^2$.

L is a Baer line, while L′ is a non-Baer line and L^θ belongs to L′.

If x is a Baer point on L, then x^θ is a Baer line through L^θ, which is in fact the unique Baer point on L'. Hence none of the S-lines through x belong to H.

Since L^θ belongs to L', there exists a unique non-Baer point y on L with $y^\theta = L'$. Every line yy', with y' any non-Baer point of L', is an S-line.

If z is a non-Baer point that is distinct of y, then z^θ intersects L' in L^θ and consequently no S-line on z is completely contained in H.

All together, this particular situation leads to $\ell = q^2$.

L is a Baer line, while L′ is a non-Baer line and L^θ is a point off L′.
First of all, we know that there is a unique Baer point b incident with L'. As also L^θ is Baer, the point x for which x^θ intersects L' in this point b, is a Baer point. Hence x does not contribute to the set of S-lines that intersect both L and L' non-trivially.

For y any other Baer point on L, the S-line yb belongs to H.

If, on the other hand, z is a non-Baer point on L, then zz', with $z' = z^\theta \cap L'$, is a S-line of H.

Since there are q and $q^2 - q$ choices for y and z, respectively, we obtain $\ell = q^2$.

We summarize the counting results for all the six different cases in the following table.

L	$\in B$	$\in B$	$\in B$	$\in B$	$\notin B$	$\notin B$
L'	$\notin B$	$\in B$	$\notin B$	$\in B$	$\notin B$	$\notin B$
L^θ	$\in L'$	$\in L'$	$\notin L'$	$\notin L'$	$\in L'$	$\notin L'$
ℓ	q^2	q^2	q^2	$2q^2$	$2q^2$	q^2

Hence S is a two-character set with weights $q^8 - q^6 + q^4$ and $q^8 - q^6$.

We now show that Π is the only plane of $\mathsf{PG}(5, q^2)$ all of whose points belong to S.

Let, by way of contradiction, Π^* be another plane all of whose points are contained in S. Note that $\Pi' \cap \Pi^* = \emptyset$.

If $\Pi \cap \Pi^*$ is a line, then let the point x' be the intersection of Π' with the space generated by Π and Π^*. Since for every point x of $\Pi^* \setminus \Pi$, the S-line containing x also contains x', there are q^4 S-lines through x', a contradiction (every point of Π' is on exactly q^2 S-lines by construction).

If $\Pi \cap \Pi^*$ is a point p, then let the line M' be the intersection of Π' with the space generated by Π and Π^*. Let x' be a Baer point on M'. The 3-space generated by x' and Π^* meets Π in a line M and clearly all lines xx', with $x \in M \setminus \{p\}$, are S-lines. This contradicts the construction of S and the fact that at least one such point x is non-Baer.

If $\Pi \cap \Pi^*$ is empty, then the correspondence $\xi : x \mapsto x'$, with $x \in \Pi$, $x' \in \Pi'$ and $|xx' \cap \Pi^*| = 1$, is an isomorphism from Π to Π'. Note that every line xx^ξ is an S-line. Let L be any non-Baer line of Π. If $L^\theta \in L^\xi$, then all

S-lines containing non-Baer points of both L and L^ξ pass through either L^θ or $(L^\xi)^{\theta^{-1}}$, which is in contradiction with the definition of ξ. Hence $L^\theta \notin L^\xi$ and the correspondence $\zeta : L \to L^\xi : x \mapsto L^\xi \cap x^\theta$ is an isomorphism of projective lines. By the construction of S, this isomorphism now coincides with the restriction of ξ to L on all non-Baer points of L. Hence it must also coincide on the unique Baer point z of L, and we obtain $z^\xi = z^\zeta \in z^\theta$, contradicting the fact that zz^ξ is an S-line.

This completes the proof of the fact that Π is the unique plane entirely contained in S.

We call a point x of $\mathsf{PG}(5, q^2)$ an *anti-point* (with respect to S) if it does not belong to S, but if it is contained in a line L of $\mathsf{PG}(5, q^2)$ meeting Π in some point and such that all points of L except for x are contained in S.

We now show that Π' is the unique plane of $\mathsf{PG}(5, q^2)$ all of whose points are anti-points. Note first that, if x is an anti-point not contained in Π', then the point x_Π is a Baer point. Indeed, let L be a line incident with x and meeting Π in some point y such that all points of L except for x are points of S. If z' is the intersection of Π' with the space spanned by Π and x, then for every point $u \in L \setminus \{x, y\}$, the line $z'u$ contains points of S, and hence z' is a non-Baer point (note that z' is in fact the point $x_{\Pi'}$). Since $z'x$ does not contain points of S, the point $x_\Pi = z'x \cap \Pi$ is a Baer point (by construction of S). Moreover, as the argument shows, $x_{\Pi'}$ is non-Baer for any anti-point x not in Π'.

Now let Π^* be a plane all of whose points are anti-points, and assume $\Pi^* \neq \Pi'$. Then Π^* cannot be disjoint from Π' since the previous paragraph would imply that every point of Π is a Baer point. Also, $\Pi^* \cap \Pi'$ cannot be a point since this would imply that all points of the line obtained by intersecting Π with the space spanned by Π^* and Π' are Baer points. Hence $\Pi^* \cap \Pi'$ is a line L'. The intersection point, say x, of Π with the space spanned by $\Pi^* \cup \Pi'$ is a Baer point lying on the inverse image of θ of at least q^4 points of Π', clearly a contradiction!

So we have shown that Π and Π' are unique in S with respect to a geometric property. Consequently the automorphism group G of S fixes both planes Π and Π'. Since the image under θ of any point of Π is geometrically determined by S, we deduce that the restriction of every element of G to $\Pi \cup \Pi'$ commutes with θ. It follows that the automorphism group of S consists of those elements of $\mathsf{P\Gamma L}_6(q^2)$ having any companion field automorphism of $\mathsf{GF}(q^2)$ and a block matrix

$$\begin{pmatrix} M & 0 \\ 0 & kM^{-t} \end{pmatrix},$$

where $k \in \mathsf{GF}(q^2)$ and M is an arbitrary non-singular 3×3 matrix over $\mathsf{GF}(q)$. Including those k that belong to $\mathsf{GF}(q)$ in $\Gamma\mathsf{L}_3(q)$ and noting that the Baer involution (identity matrix and involutive field automorphism) commutes with $\Gamma\mathsf{L}_3(q)$, we obtain the structure of G as stated in the proposition.

Proposition 5.3.1 is completely proved.

□

Remarks. (1) The two-character set $S = S(\Pi, \Pi', B, \theta)$ in $\mathsf{PG}(5, 4)$ is distinct from the one described in the previous section as the full automorphism group in $\mathsf{PGL}_6(4)$ of the latter contains $\mathsf{PSL}_2(13)$ and as 13 does not divide the order of the full automorphism group of S in $\mathsf{PG}(5, 4)$.

(2) The codes of Corollary 5.3.2 are most likely new because of their parameters, see [8]. Also, the smallest graphs of that corollary are new in view of Brouwer's list on the internet. Although it is probably hard to check with a certainty of 100 percent, it is quite likely that all the graphs are new.

5.3.4 A distance-2 ovoid in $\mathsf{H}(4)$

In this section we prove Proposition 5.3.3.

The proof is completely algebraic and consists of writing down explicit coordinates for the points of the two-character set.

So consider $\mathsf{H}(q^2)$, q even, represented in $\mathsf{PG}(5, q^2)$ as explained at the end of Section 2.2. Let Π be an arbitrary ideal plane and let Π' be its hexagon twin. Let τ be the associated hexagon twin anti-isomorphism. Let B be a Baer subplane in Π and let β denote the associated semi-linear involution in Π whose fixed point set is exactly the set of points of B. Put $\theta = \beta\tau$ and set $S = S(\Pi, \Pi', B, \theta)$, with notations as in the previous section; when referring to BB or NB points.

Note. Considering the standard embedding of $\mathsf{H}(q^2)$, we define Π, Π', B, B' and θ in the exact same way as above, while a BB or NB point is defined to belong to $\mathcal{R}(p, q) \setminus \{p, q\}$ instead of to the line on these two Baer or respectively non-Baer points. We refer to this representation of S in $\mathsf{PG}(6, q^2)$ as the standard representation of S. Let $\mathcal{H}$ be the hyperbolic hyperplane of $\mathsf{PG}(6, q^2)$ containing both Π and Π' and denote by Γ the weak subhexagon determined by $\mathcal{H}$. One can easily see that a projection of this standard representation of S onto the hyperplane $\mathcal{H}$ yields the two-character set of the previous section. Moreover, if p is a BB or NB point, then it belongs to $\mathcal{R}(p_\Pi, p_{\Pi'})$, with p_Π and $p_{\Pi'}$ the two points of Γ collinear to p after projection.

First we note that for $q \equiv 1 \bmod 3$, the points of B are necessarily contained in a subhexagon of order (q,q) (and isomorphic to $\mathsf{H}(q)$), while for $q \equiv 2 \bmod 3$, there are two orbits of such B in the full automorphism group of $\mathsf{H}(q^2)$. This follows from the orbit counting theorem, noting that the stabilizer within $\mathsf{G}_2(s)$ of an ideal plane P in $\mathsf{H}(s)$ is isomorphic to $\mathsf{SL}_3(s)$, and hence the point-wise stabilizer of P has order 3 for $s \equiv 1 \bmod 3$ and is trivial for $s \equiv 2 \bmod 3$. Using this observation for $s = q$ and $s = q^2$ yields that the number of Baer subplanes of some ideal plane that are contained in some subhexagon of order (q,q) is one third of the total number of Baer subplanes of some ideal plane, in the case $q \equiv 2 \bmod 3$ (and then the fact that the other two-thirds are all equivalent follows from the fact that the Frobenius map in $\mathsf{GF}(q^2)$ interchanges the two cosets of the subgroup of third powers of the multiplicative group of $\mathsf{GF}(q^2)$). First suppose that B is not contained in a subhexagon of order (q,q).

To check whether or not the points of S constitute a distance-2 ovoid, it now suffices to examine possible collinearity within this set.

Obviously a point p of Π is never collinear to a BB or NB point q, as (in the standard representation of S) q does not belong to $\mathcal{H}$, while any point collinear to p does.

Suppose p and q are two collinear points of type BB or NB and suppose they belong to $\mathcal{R}(p_\Pi, p_{\Pi'}) = \mathcal{R}$ and $\mathcal{R}(q_\Pi, q_{\Pi'}) = \mathcal{R}'$, respectively. By definition of a point regulus, the line pq has to be at distance 2 from a unique line, say L, of $\mathcal{R}^c$. In the same way, the line pq has to be at distance 2 from a line of $\mathcal{R}'^c$. Furthermore, since pq does not belong to Γ, it will intersect a unique line of this weak subhexagon and this in a point off Γ. In other words, the line L has to be an element of both $\mathcal{R}^c$ and $\mathcal{R}'^c$. In conclusion, if p and q are collinear, then there exist a point t in Π and a point t' in Π' that is collinear to $p_{\Pi'}$ and $q_{\Pi'}$; and p_Π and q_Π, respectively.

We can now choose coordinates in $\mathsf{H}(q^2)$ in such a way that t (respectively t') is equal to the point (∞) (respectively (0)). With these special coordinates, the line containing those two regulus points is the line $[a,l]$. Moreover, the point (a,l,x) should belong to the regulus $\mathcal{R}((0,y,0),(z,0))$. Using the coordinatization of $\mathsf{H}(q^2)$ in $\mathsf{PG}(6,q^2)$, as described in Table 2.1, we find

$$\begin{aligned}(a,l,x) &= (l+ax,1,0,a,0,a^2,x)\\ (0,y,0) &= (y,1,0,0,0,0,0,0)\\ (z,0) &= (0,0,0,0,0,1,z)\end{aligned}$$

or after projection from $n = (0,0,0,1,0,0,0)$ onto the hyperbolic hyperplane

$X_3 = 0$ we obtain

$$\begin{aligned}&(l + ax, 1, 0, 0, a^2, x)\\&(y, 1, 0, 0, 0, 0)\\&(0, 0, 0, 0, 1, z)\end{aligned}$$

as respective points in $\mathsf{PG}(5, q^2)$. Wanting (a, l, x) to be a point of $\mathcal{R}((0, y, 0), (z, 0))$ comes down to saying that $(l + ax, 1, 0, 0, a^2, x)$ is a point on the line

$$\langle (y, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, z)\rangle.$$

We thus obtain the following equation

$$y = l + a^3 z$$

expressing (a, l, x) to be in $\mathcal{R}((0, y, 0), (z, 0))$. In other words, if we want two such points on that line $[a, l]$ the quotient $\frac{y+y'}{z+z'}$ has to be a third power in $\mathsf{GF}(q^2)$.

We will now choose our Baer subplane B in such a way that there are no collinear points within the set of BB points.

Consider the standard embedding of $\mathsf{H}(q^2)$ in $\mathsf{PG}(6, q^2)$. For Π we can take the plane $X_3 = X_4 = X_5 = X_6 = 0$, and then the points of Π' have coordinates $(0, 0, 0, 0, x_4, x_5, x_6)$, with x_4, x_5, x_6 in $\mathsf{GF}(q^2)$. We choose $\sigma \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ such that σ is not a third power in $\mathsf{GF}(q^2)$. Then we may choose the points of B as the points with coordinates

$$(\infty), \quad (0, \sigma r, 0), \quad (r, 0, \sigma r', 0),$$

with $r, r' \in \mathsf{GF}(q)$. In $\mathsf{PG}(6, q^2)$ these points have coordinates

$$(\sigma r_0, r_1, r_2, 0, 0, 0, 0),$$

with $(r_0, r_1, r_2) \neq (0, 0, 0)$ and $r_i \in \mathsf{GF}(q)$, $i = 0, 1, 2$. The Baer involution fixing all points of B is then given by

$$(x_0, x_1, x_2, 0, 0, 0, 0) \mapsto (\sigma^{1-q} x_0^q, x_1^q, x_2^q, 0, 0, 0, 0).$$

From now on, we denote x^q by $\overline{x}$. One easily checks that the points of B^θ are the points

$$(0), \quad (r, 0), \quad (0, \sigma r, 0, \sigma r', 0),$$

and have coordinates

$$(0, 0, 0, 0, r_4, \sigma r_5, \sigma r_6),$$

with $(r_4, r_5, r_6) \neq (0,0,0)$ and $r, r', r_4, r_5, r_6 \in \mathsf{GF}(q)$.

It is now a straightforward exercise to check that for any even q, two BB points of S are never collinear in $\mathsf{H}(q^2)$, as we will see. Indeed, for this particular choice of B the quotient $\frac{y+y'}{z+z'}$ equals $\sigma\frac{r_1+r'_1}{r_2+r'_2}$, which is a third power in $\mathsf{GF}(q^2)$ if and only if σ is, and this is by definition not true.

Within the plane Π we now have the following situation: for any two Baer points p_Π and q_Π and for any two Baer lines L and M that intersect in a point on $p_\Pi q_\Pi$, no point of $\mathcal{R}(p_\Pi, L^\theta)$ is collinear to a point of $\mathcal{R}(q_\Pi, M^\theta)$.

We will now investigate whether or not a BB point p can be collinear to a NB point q. In every one of the following situations we suppose that p and q belong to $\mathcal{R} = \mathcal{R}(p_\Pi, p_{\Pi'})$ and $\mathcal{R}' = \mathcal{R}(q_\Pi, q_{\Pi'})$, respectively and when collinear they uniquely determine the line $L \in \mathcal{R}^c \cap \mathcal{R}'^c$ of Γ.

Let p_Π be a Baer point and q_Π be a non-Baer point of Π. Then $p_\Pi q_\Pi$ can either be a Baer line or not. Let us start with the former case and denote by t and t' the points of Γ incident with L. We already know that p_Π and q_Π are collinear to t', as $p_{\Pi'}$ and $q_{\Pi'}$ should be to t. Now, since $p_\Pi q_\Pi$ is a Baer line, the conjugate q_Π^β of q_Π also belongs to this line and by definition of the NB points, $q_{\Pi'}$ is collinear to q_Π^β. Therefore, as $\mathsf{H}(q^2)$ contains no ordinary quadrangles, we may conclude t to be equal to q_Π^β. Without loss of generality we may choose the point p_Π as the point (∞) and q_Π as the point $(0,k,0)$, with $k \neq \sigma r$ for all $r \in \mathsf{GF}(q)$. Since $t = q_\Pi^\beta = (0, \sigma^{1-q}\overline{k}, 0)$ and $p \in \mathcal{R}((\infty), p_{\Pi'})$, $p_{\Pi'}$ should be a Baer point that is at the same time opposite to (∞) and collinear to $(0, \sigma^{1-q}\overline{k}, 0)$. This is impossible, as all Baer points of Π' that are opposite to (∞), have coordinates $(0, \sigma r, 0, \sigma r', 0)$ and $\sigma^{1-q}\overline{k} = \sigma r$ would imply that q_Π^β is a Baer point.

Suppose, on the other hand, that $p_\Pi q_\Pi$ is a non-Baer line. Since p_Π is the unique Baer point on this line, we immediately find that t is a non-Baer point and consequently t^τ is a non-Baer line. In other words, $p_{\Pi'}$ is the unique Baer point on t^τ. To prove this situation to be contradictory we consider the standard Baer subplane of Π (namely the one with as points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$) and, for the sake of simplicity, we denote the associated involution and anti-isomorphism also by β and θ. Using coordinates in $\mathsf{PG}(2,q^2)$ we may choose p_Π and t to be the points $(1,0,0)$ and $(0,y,1)$, with $y \in \mathsf{GF}(q^2)\backslash\mathsf{GF}(q)$, respectively. The point $p_{\Pi'}$ then corresponds to the line $\langle(0,y,1),(0,\overline{y},1)\rangle$ through t in Π. The point $q_{\Pi'}$ is by construction the intersection point of t^τ with q_Π^θ and if we choose q_Π to be the point $(z,y,1)$ we thus obtain $q_{\Pi'}^{\tau^{-1}} = \langle t, (\overline{z}, \overline{y}, 1)\rangle$. We will now try to map the standard Baer subplane onto the Baer subplane determined by the points t, p_Π, q_Π and the lines $p_{\Pi'}^{\tau^{-1}}$ and $q_{\Pi'}^{\tau^{-1}}$. If that particular element is an element of $\mathsf{PSL}_3(q^2)$

there will be no collinearity in this case. We therefore determine the matrix M mapping the standard Baer subplane of Π onto the one with as triangle points the points $(1,0,0)$, $(0,y,1)$ and $(0,\overline{y},1)$ and as lines $\langle(0,y,1),(1,0,0)\rangle$, $\langle(0,y,1),(0,\overline{y},1)\rangle$ and $\langle(0,y,1),(\overline{z},\overline{y},1)\rangle$. To completely fix our image Baer subplane and consequently uniquely determine our matrix M, we need a unit point u. Let u be the intersection point of the lines $\langle(0,y,1),(\overline{z},\overline{y},1)\rangle$ and $\langle(0,\overline{y},1),(z,y,1)\rangle$. We now determine

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ 0 & \overline{y} & 1 \\ z & y & 1 \end{vmatrix} = 0 \qquad \cap \qquad \begin{vmatrix} x_0 & x_1 & x_2 \\ 0 & y & 1 \\ \overline{z} & \overline{y} & 1 \end{vmatrix} = 0$$

and (note we are working in even characteristic) find

$$u = (z\overline{z}(\overline{y}+y), (\overline{y}+y)(z\overline{y}+y\overline{z}), (\overline{y}+y)(z+\overline{z}))$$

or (as we are working with projective points)

$$u = (z\overline{z}, z\overline{y}+y\overline{z}, z+\overline{z})$$

as the unit point of our new Baer subplane, say B_2. This unit point will now uniquely determine the constants K_1, K_2, K_3 such that

$$M = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 y & K_2 \\ 0 & K_3\overline{y} & K_3 \end{pmatrix}$$

maps the standard Baer subplane to B_2. We now immediately obtain $K_1 = z\overline{z}$, $K_2 = \overline{z}$ and $K_3 = z$ and consequently find the element $z^2\overline{z}^2(y+\overline{y})$, which is obviously an element of $\mathsf{GF}(q)$, as the determinant of M. In other words, M belongs to $\mathsf{PSL}_3(q^2)$ and there is no collinearity between BB and NB points.

Hence we consider two generic NB points p and q and deduce a necessary and sufficient condition for p and q to be collinear in $\mathsf{H}(q^2)$.

Just as in the previous situations, we have to consider two possibilities. Either $p_\Pi q_\Pi$ is a Baer line or not.

First suppose it is and use the same definitions of t and t' as are given above. As always the point t is collinear to both $p_{\Pi'}$ and $q_{\Pi'}$, two points that, on their turn, are collinear to p_Π^β and q_Π^β, respectively. In other words, since p_Π^β belongs to $p_\Pi q_\Pi$ and $\mathsf{H}(q^2)$ contains no ordinary quadrangles, t has to be equal to p_Π^β. However, in the exact same way we obtain that t equals q_Π^β, a contradiction as $p_\Pi \neq q_\Pi$.

Now suppose that $p_\Pi q_\Pi$ is a non-Baer line and that t is the unique Baer point on this line. The point $p_{\Pi'}$ is a point that is both collinear to t and to p_Π^β, while p_Π^β is a point on an ideal line through t containing also the point q_Π^β. This implies that $p_{\Pi'} = t \bowtie p_\Pi^\beta = t \bowtie q_\Pi^\beta = q_{\Pi'}$. We will now show that, under this assumption, it is impossible for p to be collinear to q. We choose coordinates in $\mathsf{PG}(6, q^2)$ in such a way that L equals the line $[\infty]$, meaning $t = (\infty)$ and $t' = (0)$. As we have shown above, for the point (a, l, x) to be a point of $\mathcal{R}((0, y, 0), (z, 0))$ the constant y should be equal to $l + a^3 z$. If we then want a second point (a, l, x') on $\mathcal{R}((0, y', 0), (z, 0))$, we find $y' = l + a^3 z$, or in other words $y = y'$ and hence $p_\Pi = q_\Pi$, a contradiction.

Finally say $p_\Pi q_\Pi$ is a non-Baer line and L intersects Π in a non-Baer point t, which has coordinates (a, b, c) in $\mathsf{PG}(2, q^2)$. This implies that t differs from the unique Baer point on $p_\Pi q_\Pi$, which we may choose to be the point $(1, 0, 0)$. The points $p_{\Pi'}$ and $q_{\Pi'}$ are mapped by the inverse of τ onto two lines in Π through the point t. We will now examine when these two points, which we choose to have respective coordinates $(a + x, b, c)$ and $(a + y, b, c)$, and lines determine collinear NB points. We will treat this particular situation by using the same general idea as in the case where $p_\Pi q_\Pi$ was a non-Baer line and p_Π was the unique Baer point on this line. If we are able to map the standard Baer subplane to the Baer subplane determined by these points and lines, and this by using an element of $\mathsf{PSL}_3(q^2)$, there will be no such collinear points. We therefore determine the matrix M mapping the standard Baer subplane of Π to the one with as triangle points the points (a, b, c), $(a + x, b, c)$ and $(\overline{a+x}, \overline{b}, \overline{c})$ and as lines $\langle (a, b, c), (1, 0, 0)\rangle$, $\langle (a, b, c), (\overline{a+x}, \overline{b}, \overline{c})\rangle$ and $\langle (a, b, c), (\overline{a+y}, \overline{b}, \overline{c})\rangle$. To completely fix our image Baer subplane and consequently uniquely determine our matrix M, we need a unit point u. Let u be the intersection point of the lines $\langle (a, b, c), (\overline{a+y}, \overline{b}, \overline{c})\rangle$ and $\langle (a + y, b, c), (\overline{a+x}, \overline{b}, \overline{c})\rangle$. We now determine

$$\left[\begin{vmatrix} \overline{b} & \overline{c} \\ b & c \end{vmatrix}, \begin{vmatrix} \overline{c} & \overline{a}+\overline{x} \\ c & a+y \end{vmatrix}, \begin{vmatrix} \overline{a}+\overline{x} & \overline{b} \\ a+y & b \end{vmatrix}\right] \cap \left[\begin{vmatrix} \overline{b} & \overline{c} \\ b & c \end{vmatrix}, \begin{vmatrix} \overline{c} & \overline{a}+\overline{y} \\ c & a \end{vmatrix}, \begin{vmatrix} \overline{b} & \overline{a}+\overline{y} \\ b & a \end{vmatrix}\right] = (x, y, z)$$

and find

$$\begin{aligned} x =& (\overline{c}a + \overline{c}y + c\overline{a} + c\overline{x})(\overline{b}a + b\overline{a} + b\overline{y}) + (\overline{a}b + b\overline{x} + \overline{b}a + \overline{b}y)(\overline{c}a + c\overline{a} + c\overline{y}) \\ =& (b\overline{c} + \overline{b}c)(a\overline{x} + \overline{y}a + \overline{a}y + y\overline{y}) \\ y =& (b\overline{c} + \overline{b}c)(\overline{x}b + y\overline{b} + \overline{y}b) \\ z =& (b\overline{c} + \overline{b}c)(y\overline{c} + \overline{x}c + c\overline{y}) \end{aligned}$$

or

$$u = (a\overline{x} + \overline{y}a + \overline{a}y + y\overline{y}, \overline{x}b + y\overline{b} + \overline{y}b, y\overline{c} + \overline{x}c + c\overline{y})$$

as the unit point of our new Baer subplane, say B_2. This unit point will now uniquely determine the constants K_1, K_2, K_3 such that

$$M = \begin{pmatrix} K_1 a & K_1 b & K_1 c \\ K_2(a+x) & K_2 b & K_2 c \\ K_3(\overline{a+x}) & K_3 \overline{b} & K_3 \overline{c} \end{pmatrix}$$

maps the standard Baer subplane to B_2. Looking at the second and third entry of u, suggests to take $K_1 = \overline{x}$, $K_2 = \overline{y}$ and $K_3 = y$. Nevertheless, with this choice of constants the first entry in u differs from the one in $(1,1,1)M$. However, altering K_1 and K_2 by adding a constant k does not change a thing in the last two entries of the image of $(1,1,1)$. We can now choose a constant k in such a way that also the first entry in $(1,1,1)M$ is correct. Namely, if

$$(\overline{x}+k)a + (\overline{y}+k)(a+x) + y(\overline{a}+\overline{x})$$

is equal to

$$a\overline{x} + \overline{y}a + \overline{a}y + y\overline{y}$$

then we should have

$$xk = \overline{y}x + y\overline{y} + y\overline{x}$$

or in other words (as x is a non-zero element of $\mathsf{GF}(q^2)$) we find

$$k = x^{-1}(\overline{y}x + y\overline{y} + y\overline{x})$$

To simplify further calculations we multiply our constants by x and obtain $K_1 = x\overline{x} + \overline{y}x + y\overline{y} + y\overline{x}$, $K_2 = \overline{y}y + y\overline{x}$ and $K_3 = yx$. For M to be an element of $\mathsf{PSL}_3(q^2)$ its determinant has to be a third power over $\mathsf{GF}(q^2)$. Now, this determinant equals

$$y^2x^2(x\overline{x} + \overline{y}x + y\overline{y} + y\overline{x})(\overline{y}+\overline{x})(b\overline{c} + \overline{b}c)$$

which is a third power if and only if

$$y^2x^2(\overline{y}+\overline{x}) \qquad (*)$$

is. As this should be true for all x, y in $\mathsf{GF}(q^2)$, we claim that q has to equal 2. Indeed, a substitution of $y = \frac{1}{x}$ in equation $(*)$ yields $\frac{1}{x} + x$ is a third power and this for all $x \in \mathsf{GF}(q^2) \setminus \{0\}$. As we have at most two solutions in x for every a in $a = \frac{1+x^2}{x}$, we find at least $\frac{q^2-2}{2}$ values for a if we let x run through $\mathsf{GF}(q^2) \setminus \{0,1\}$. Hence, since there are exactly $\frac{q^2-1}{3}$ third powers in $\mathsf{GF}(q^2)$, we obtain

$$\frac{q^2-2}{2} \leq \frac{q^2-1}{3}$$

or $q^2 \leq 4$ and the claim is proved. Hence by property $(*)$ the set S constitutes a distance-2 ovoid only for q equal to 2.

The case where B is contained in a subhexagon of order (q, q) never leads to a distance-2 ovoid and this, opposed to the previous situation, can be shown by a geometrical argument. First of all, note that $B \cup B'$ is the point set of a weak subhexagon Γ' of order $(1, q)$. If now B and consequently also this weak subhexagon were to be in a subhexagon Γ of order (q, q), then all BB points of S belong to Γ. However, this would imply that the alleged distance-2 ovoid contains

$$(1 + q + q^2)[q^2(q - 1) + 1]$$

points of Γ, which is far to big a number for a set of non-collinear points inside a generalized hexagon of order (q, q).

The proposition is proved. □

5.4 Part III: Lifting of H(4)

5.4.1 Introduction

In this section we show that nor the distance-2 ovoid, say $\mathcal{O}_\Omega$, as constructed in Section 5.2.3, nor the distance-2 ovoid, say $\mathcal{O}_2$, as constructed in Section 5.3.4, is a suitable choice for a geometrical hyperplane to lift the standard embedding of $\mathsf{H}(4)$ into $\mathsf{PG}(7, 4)$.

Furthermore, since q is even, we can use one or the other to try and lift the representation of $\mathsf{H}(4)$ in $\mathsf{PG}(5, 4)$ to a new embedding of $\mathsf{H}(4)$ in $\mathsf{PG}(6, 4)$.

The general idea behind the concept of lifting has already been explained in Section 4.3.6 of Chapter 4.

In the exact same way as we indexed the elements of $\mathsf{H}(3)$, we will now denote all points and all lines of $\mathsf{H}(4)$ with an index from 0 to 1364. For each point and line the coordinate tuple is written in such a way that every entry is a member of $\{0, 1, 2, 3\}$. We conceive the entries as natural numbers and then, if the coordinate tuple (e.g. of a point) is equal to $(a_0, \ldots, a_k)$, we label the point with the index

$$I(a_0, \ldots, a_k) = \frac{4^{k+1} - 1}{3} + a_k + 4a_{k-1} + \ldots + 4^k a_0.$$

Similarly for lines. The elements $[\infty]$ and (∞) have labels 0.

A point with index i will be denoted p_i and a line with index j will be denoted L_j.

Concerning incidence a point p_i of index less than or equal to 341 and $i = 4{\cdot}i'{+}r$, with $r \in \{1,2,3,4\}$, is incident with $L_{i'}$ and with $L_{4\cdot i+r'}$, $r' = 1,2,3,4$. Similarly for the points incident with a given line with index < 341.

When $i \geq 341$, one has to translate back and forth to coordinates and make explicit calculations.

In the next section we start with $\mathcal{O}_\Omega$ as a geometrical hyperplane, while in Section 5.4.3 we use $\mathcal{O}_2$ as a starting point.

5.4.2 Using $\mathcal{O}_\Omega$

Consider the points of the distance-2 ovoid $\mathcal{O}_\Omega$ as constructed in Section 5.2.3. In coordinates of H(4) the points of $\mathcal{O}_\Omega$ are the points of the set

$$\begin{aligned}
\{&p_0, p_{21}, p_{25}, p_{29}, p_{33}, p_{37}, p_{44}, p_{46}, p_{51}, p_{53}, p_{57}, p_{61}, p_{65}, p_{69}, p_{76},\\
&p_{78}, p_{83}, p_{85}, p_{89}, p_{93}, p_{97}, p_{101}, p_{107}, p_{109}, p_{115}, p_{120}, p_{124}, p_{125}, p_{129}, p_{134},\\
&p_{137}, p_{141}, p_{146}, p_{152}, p_{156}, p_{160}, p_{161}, p_{168}, p_{169}, p_{174}, p_{178}, p_{181}, p_{185}, p_{192}, p_{193},\\
&p_{197}, p_{204}, p_{207}, p_{211}, p_{214}, p_{217}, p_{222}, p_{226}, p_{230}, p_{236}, p_{240}, p_{241}, p_{245}, p_{251}, p_{255},\\
&p_{258}, p_{261}, p_{265}, p_{270}, p_{273}, p_{277}, p_{283}, p_{287}, p_{291}, p_{293}, p_{297}, p_{303}, p_{305}, p_{310}, p_{313},\\
&p_{318}, p_{323}, p_{328}, p_{331}, p_{336}, p_{337}, p_{358}, p_{361}, p_{368}, p_{371}, p_{373}, p_{380}, p_{382}, p_{387}, p_{390},\\
&p_{396}, p_{399}, p_{401}, p_{422}, p_{428}, p_{431}, p_{433}, p_{438}, p_{444}, p_{447}, p_{449}, p_{456}, p_{458}, p_{461}, p_{467},\\
&p_{487}, p_{492}, p_{493}, p_{498}, p_{502}, p_{505}, p_{512}, p_{515}, p_{518}, p_{521}, p_{528}, p_{531}, p_{549}, p_{556}, p_{558},\\
&p_{563}, p_{566}, p_{571}, p_{573}, p_{580}, p_{581}, p_{588}, p_{590}, p_{595}, p_{614}, p_{617}, p_{621}, p_{627}, p_{629}, p_{636},\\
&p_{637}, p_{643}, p_{646}, p_{652}, p_{653}, p_{657}, p_{662}, p_{668}, p_{671}, p_{675}, p_{678}, p_{683}, p_{687}, p_{689}, p_{695},\\
&p_{700}, p_{703}, p_{705}, p_{726}, p_{732}, p_{736}, p_{739}, p_{760}, p_{761}, p_{768}, p_{771}, p_{774}, p_{777}, p_{784}, p_{788},\\
&p_{790}, p_{796}, p_{798}, p_{803}, p_{805}, p_{810}, p_{814}, p_{819}, p_{837}, p_{844}, p_{846}, p_{850}, p_{870}, p_{873}, p_{880},\\
&p_{883}, p_{885}, p_{892}, p_{894}, p_{899}, p_{902}, p_{908}, p_{911}, p_{913}, p_{934}, p_{940}, p_{943}, p_{945}, p_{950}, p_{956},\\
&p_{959}, p_{961}, p_{968}, p_{970}, p_{973}, p_{979}, p_{999}, p_{1004}, p_{1005}, p_{1010}, p_{1014}, p_{1017}, p_{1024}, p_{1027},\\
&p_{1030}, p_{1033}, p_{1040}, p_{1043}, p_{1061}, p_{1068}, p_{1070}, p_{1075}, p_{1078}, p_{1083}, p_{1085}, p_{1092}, p_{1093},\\
&p_{1100}, p_{1102}, p_{1107}, p_{1126}, p_{1129}, p_{1133}, p_{1139}, p_{1141}, p_{1148}, p_{1149}, p_{1155}, p_{1158}, p_{1164},\\
&p_{1165}, p_{1169}, p_{1174}, p_{1180}, p_{1183}, p_{1187}, p_{1190}, p_{1195}, p_{1199}, p_{1201}, p_{1207}, p_{1212}, p_{1215},\\
&p_{1217}, p_{1238}, p_{1244}, p_{1248}, p_{1251}, p_{1272}, p_{1273}, p_{1280}, p_{1283}, p_{1286}, p_{1289}, p_{1296}, p_{1300},\\
&p_{1302}, p_{1308}, p_{1310}, p_{1315}, p_{1317}, p_{1322}, p_{1326}, p_{1331}, p_{1349}, p_{1356}, p_{1358}, p_{1362}\}.
\end{aligned}$$

Since the point (∞) belongs to $\mathcal{O}_\Omega$ and the lifting procedure becomes noticeably easier when we consider (∞) as the special point p, we will map $\mathcal{O}_\Omega$ onto an isomorphic distance-2 ovoid denoted by $\mathcal{O}'_\Omega$ and consisting of the

following set of points

$$\{p_1, p_7, p_{11}, p_{15}, p_{19}, p_{39}, p_{41}, p_{46}, p_{52}, p_{54}, p_{60}, p_{63}, p_{65}, p_{70}, p_{76},$$
$$p_{79}, p_{81}, p_{87}, p_{90}, p_{96}, p_{97}, p_{101}, p_{107}, p_{112}, p_{114}, p_{135}, p_{140}, p_{141}, p_{146}, p_{152},$$
$$p_{154}, p_{157}, p_{163}, p_{167}, p_{169}, p_{174}, p_{180}, p_{198}, p_{204}, p_{207}, p_{209}, p_{216}, p_{219}, p_{222}, p_{225},$$
$$p_{230}, p_{233}, p_{240}, p_{243}, p_{262}, p_{265}, p_{272}, p_{275}, p_{280}, p_{281}, p_{287}, p_{290}, p_{296}, p_{297}, p_{303},$$
$$p_{306}, p_{326}, p_{331}, p_{333}, p_{340}, p_{343}, p_{347}, p_{350}, p_{354}, p_{359}, p_{364}, p_{368}, p_{371}, p_{375}, p_{379},$$
$$p_{383}, p_{387}, p_{391}, p_{393}, p_{399}, p_{401}, p_{407}, p_{410}, p_{415}, p_{419}, p_{422}, p_{427}, p_{429}, p_{433}, p_{438},$$
$$p_{442}, p_{446}, p_{451}, p_{456}, p_{460}, p_{462}, p_{467}, p_{472}, p_{473}, p_{477}, p_{483}, p_{487}, p_{491}, p_{495}, p_{500},$$
$$p_{504}, p_{507}, p_{512}, p_{516}, p_{518}, p_{523}, p_{528}, p_{530}, p_{533}, p_{540}, p_{543}, p_{548}, p_{549}, p_{554}, p_{559},$$
$$p_{562}, p_{567}, p_{569}, p_{573}, p_{577}, p_{581}, p_{587}, p_{591}, p_{595}, p_{597}, p_{603}, p_{606}, p_{611}, p_{616}, p_{618},$$
$$p_{623}, p_{627}, p_{648}, p_{651}, p_{655}, p_{657}, p_{678}, p_{684}, p_{686}, p_{689}, p_{694}, p_{700}, p_{703}, p_{706}, p_{711},$$
$$p_{713}, p_{718}, p_{722}, p_{727}, p_{732}, p_{736}, p_{738}, p_{760}, p_{761}, p_{768}, p_{771}, p_{773}, p_{778}, p_{784}, p_{788},$$
$$p_{790}, p_{793}, p_{797}, p_{804}, p_{805}, p_{811}, p_{813}, p_{820}, p_{823}, p_{826}, p_{829}, p_{833}, p_{856}, p_{857}, p_{864},$$
$$p_{866}, p_{888}, p_{892}, p_{895}, p_{898}, p_{903}, p_{908}, p_{912}, p_{913}, p_{917}, p_{922}, p_{925}, p_{931}, p_{936}, p_{938},$$
$$p_{941}, p_{945}, p_{951}, p_{953}, p_{957}, p_{964}, p_{982}, p_{987}, p_{991}, p_{996}, p_{997}, p_{1003}, p_{1008}, p_{1011}, p_{1030},$$
$$p_{1033}, p_{1039}, p_{1043}, p_{1063}, p_{1068}, p_{1070}, p_{1074}, p_{1078}, p_{1083}, p_{1086}, p_{1089}, p_{1093}, p_{1098},$$
$$p_{1102}, p_{1108}, p_{1112}, p_{1114}, p_{1117}, p_{1123}, p_{1143}, p_{1145}, p_{1150}, p_{1156}, p_{1158}, p_{1164}, p_{1167},$$
$$p_{1169}, p_{1174}, p_{1177}, p_{1184}, p_{1187}, p_{1192}, p_{1195}, p_{1198}, p_{1201}, p_{1207}, p_{1212}, p_{1213}, p_{1218},$$
$$p_{1237}, p_{1244}, p_{1246}, p_{1251}, p_{1253}, p_{1259}, p_{1264}, p_{1266}, p_{1285}, p_{1290}, p_{1295}, p_{1300}, p_{1318},$$
$$p_{1323}, p_{1325}, p_{1332}, p_{1335}, p_{1338}, p_{1344}, p_{1345}, p_{1352}, p_{1353}, p_{1359}, p_{1362}\}.$$

Standard embedding

In the standard embedding of $\mathsf{H}(4)$ the points of $\mathcal{O}'_\Omega$ will be mapped to points of the hyperplane $\mathcal{H} : X_0 = 0$ by the isomorphism $\Phi : \mathsf{PG}(6,4) \to \mathsf{PG}(7,4)$ given by the map

$$x = (x_0, \ldots, x_6) \to x^\Phi = (0, x_0, \ldots, x_6).$$

We now choose the point (∞) as our special point p, and lift it to the point $(1,1,0,0,0,0,0,0)$, which we also denote by p.

The coordinates of a general point (a, l, a', l', a'') opposite (∞) can now be determined as follows:

> First, the intersection of the line $[\infty]$ with the set of points of the distance-2 ovoid gives us the unique point p_r. Therefore any other point on this line will have coordinates $x_i = (1,1,0,0,0,0,0) +$

$\sigma^i p_r^\Phi$, with $\sigma^2 + \sigma = 1$ and $i \in \{0, 1, 2\}$. These coordinates are precisely the only possible coordinates of the point (a) in the lifted embedding of $\mathsf{H}(4)$ in $\mathsf{PG}(7, 4)$.

Then, the point (a, l, a') is a point on the line $[a, l]$ through (a) and $[a, l] \cap \mathcal{O}'_\Omega$. The former point has coordinates $x_i, i \in \{0, 1, 2\}$, the latter point is p_m^Φ with $m = 4.I([a, l]) + r$, $r = 1, 2, 3$ or 4, depending on which value of m gives a point in $\mathcal{O}'_\Omega$. In other words, a point on the line $[a, l]$ of type (a, l, a') will have coordinates $y_{ij} = x_i + \sigma^j p_m^\Phi$, $i, j \in \{0, 1, 2\}$.

Finally, by a similar argument, we find that the point (a, l, a', l', a'') has coordinates $z_{ijk} = y_{ij} + \sigma^k p_n^\Phi$, $i, j, k \in \{0, 1, 2\}$, and $n = 4.I([a, l, a, l']) + r$ such that $r = 1, 2, 3$ or 4 and $p_n \in \mathcal{O}'_\Omega$.

The determination of these 27 points is completely algebraic and consists of writing down explicit coordinates for the points (a), (a, l, a') and (a, l, a', l', a''). We are not going to perform all calculations in detail here, but give a single example.

We will now consider two points (a, l, a', l', a'') and (A, L, A', L', A'') that are collinear in the standard embedding of $\mathsf{H}(4)$ and check whether they are collinear in coordinates of $\mathsf{PG}(7, 4)$ as well.

Consider p_{598} and p_{994} two collinear points off $\mathcal{O}'_\Omega$ that are on the line L_{422}. The points incident with L_{422} are the points p_{105}, p_{410}, p_{598}, p_{994} and p_{1310} and thus L_{422} intersects $\mathcal{O}_2$ in p_{410}. For each of these two points, p_{598} and p_{994}, we obtain 27 possible coordinates. As we will see, none of the 729 possible lines in $\mathsf{PG}(7, 3)$, that ought to represent the line L_{422}, will contain the point p_{410}^Φ, a contradiction.

By way of example we determine, step by step, the 27 hypothetical coordinates of $p_{598} = (1, 0, 0, 0, 1)$. The coordinates of the point (1) in $\mathsf{PG}(7, 4)$ are $x_i = (1, 1, 0, 0, 0, 0, 0) + \sigma^i p_1^\Phi$, with $p_1 = (0, 0, 0, 0, 0, 0, 1)$ and hence $p_1^\Phi = (0, 0, 0, 0, 0, 0, 0, 1)$:

$$(1, 1, 0, 0, 0, 0, 1), \qquad (1, 1, 0, 0, 0, 0, \sigma), \qquad (1, 1, 0, 0, 0, 0, \sigma^2).$$

Next we have 9 possibilities for $(1, 0, 0)$, $y_{ij} = x_i + \sigma^j p_{39}^\Phi$, with $p_{39} = (\sigma, 1, 0, 1, 0, 1, \sigma)$ and $p_{39}^\Phi = (0, \sigma, 1, 0, 1, 0, 1, \sigma)$:

$$\begin{array}{lll}
(1, \sigma^2, 1, 0, 1, 0, 1, \sigma^2), & (1, \sigma, \sigma, 0, \sigma, 0, \sigma, \sigma), & (1, 0, \sigma^2, 0, \sigma^2, 0, \sigma^2, 0), \\
(1, \sigma^2, 1, 0, 1, 0, 1, 0), & (1, \sigma, \sigma, 0, \sigma, 0, \sigma, 1), & (1, 0, \sigma^2, 0, \sigma^2, 0, \sigma^2, \sigma^2), \\
(1, \sigma^2, 1, 0, 1, 0, 1, 1), & (1, \sigma, \sigma, 0, \sigma, 0, \sigma, 0), & (1, 0, \sigma^2, 0, \sigma^2, 0, \sigma^2, \sigma).
\end{array}$$

And finally we obtain the following 27 coordinates for $(1,0,0,0,1)$, $z_{ijk} = y_{ij} + \sigma^k p^{\Phi}_{597}$, with $p_{597} = (0,0,1,0,1,0,0)$ and thus $p^{\Phi}_{597} = (0,0,0,1,0,1,0,0)$:

$$
\begin{array}{lll}
(1,\sigma^2,1,1,1,1,1,\sigma^2), & (1,\sigma^2,1,\sigma,1,\sigma,1,\sigma^2), & (1,\sigma^2,1,\sigma^2,1,\sigma^2,1,\sigma^2), \\
(1,\sigma,\sigma,1,\sigma,1,\sigma,\sigma), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma), & (1,\sigma,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,\sigma), \\
(1,0,\sigma^2,1,\sigma^2,1,\sigma^2,0), & (1,0,\sigma^2,\sigma,\sigma^2,\sigma,\sigma^2,0), & (1,0,\sigma^2,\sigma^2,\sigma^2,\sigma^2,\sigma^2,0), \\
(1,\sigma^2,1,1,1,1,1,0), & (1,\sigma^2,1,\sigma,1,\sigma,1,0), & (1,\sigma^2,1,\sigma^2,1,\sigma^2,1,0), \\
(1,\sigma,\sigma,1,\sigma,1,\sigma,1), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma,1), & (1,\sigma,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,1), \\
(1,0,\sigma^2,1,\sigma^2,1,\sigma^2,\sigma^2), & (1,0,\sigma^2,\sigma,\sigma^2,\sigma,\sigma^2,\sigma^2), & (1,0,\sigma^2,\sigma^2,\sigma^2,\sigma^2,\sigma^2,\sigma^2), \\
(1,\sigma^2,1,1,1,1,1,1), & (1,\sigma^2,1,\sigma,1,\sigma,1,1), & (1,\sigma^2,1,\sigma^2,1,\sigma^2,1,1), \\
(1,\sigma,\sigma,1,\sigma,1,\sigma,0), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma,0), & (1,\sigma,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,0), \\
(1,0,\sigma^2,1,\sigma^2,1,\sigma^2,\sigma), & (1,0,\sigma^2,\sigma,\sigma^2,\sigma,\sigma^2,\sigma), & (1,0,\sigma^2,\sigma^2,\sigma^2,\sigma^2,\sigma^2,\sigma).
\end{array}
$$

Completely analogously to the latter case, we obtain

$$
\begin{array}{lll}
(1,0,\sigma,\sigma,\sigma^2,1,\sigma^2,0), & (1,0,0,\sigma^2,0,\sigma,\sigma^2,\sigma), & (1,0,\sigma^2,1,1,\sigma^2,\sigma^2,1), \\
(1,\sigma^2,1,\sigma,\sigma,1,1,1), & (1,\sigma^2,\sigma^2,\sigma^2,1,\sigma,1,\sigma^2), & (1,\sigma^2,0,1,0,\sigma^2,1,0), \\
(1,\sigma,0,\sigma,0,1,\sigma,\sigma^2), & (1,\sigma,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,1), & (1,\sigma,1,1,\sigma,\sigma^2,\sigma,\sigma), \\
(1,0,\sigma,\sigma,\sigma^2,1,\sigma^2,\sigma^2), & (1,0,0,\sigma^2,0,\sigma,\sigma^2,1), & (1,0,\sigma^2,1,1,\sigma^2,\sigma^2,\sigma), \\
(1,\sigma^2,1,\sigma,\sigma,1,1,\sigma), & (1,\sigma^2,\sigma^2,\sigma^2,1,\sigma,1,0), & (1,\sigma^2,0,1,0,\sigma^2,1,\sigma^2), \\
(1,\sigma,0,\sigma,0,1,\sigma,0), & (1,\sigma,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,\sigma), & (1,\sigma,1,1,\sigma,\sigma^2,\sigma,1), \\
(1,0,\sigma,\sigma,\sigma^2,1,\sigma^2,\sigma), & (1,0,0,\sigma^2,0,\sigma,\sigma^2,0), & (1,0,\sigma^2,1,1,\sigma^2,\sigma^2,\sigma^2), \\
(1,\sigma^2,1,\sigma,\sigma,1,1,\sigma^2), & (1,\sigma^2,\sigma^2,\sigma^2,1,\sigma,1,1), & (1,\sigma^2,0,1,0,\sigma^2,1,\sigma), \\
(1,\sigma,0,\sigma,0,1,\sigma,1), & (1,\sigma,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,\sigma^2), & (1,\sigma,1,1,\sigma,\sigma^2,\sigma,0)
\end{array}
$$

as the 27 possible coordinates for $p_{994} = (\sigma,\sigma,0,\sigma^2,1)$. One can now show that, since p^{Φ}_{410} has coordinates $(0,1,1,0,0,1,1,1)$, indeed none of these possible lines contains this point. Consequently there is no embedding in $\mathsf{PG}(7,4)$ of $\mathsf{H}(4)$, obtained by using $\mathcal{O}'_{\Omega}$ as a geometrical hyperplane to lift all points of $\mathsf{H}(4)$.

Perfect symplectic representation

When considering the representation of $\mathsf{H}(4)$ in $\mathsf{PG}(5,4)$, all points of $\mathsf{H}(4)$ in fact belong to the hyperplane $X_3 = 0$ in $\mathsf{PG}(6,4)$ and these are the coordinates we will be working with. We lift (∞) out of this hyperplane to the point $(1,0,0,1,0,0,0)$ of $\mathsf{PG}(6,4)$, hereby also lifting all other points not belonging to the considered distance-2 ovoid. The main idea in the lifting procedure

remains the same as in the previous paragraph. The only difference is that we are now working with coordinates obtained after projection from the nucleus onto the hyperplane $X_3 = 0$ and that we consider this hyperplane of $\mathsf{PG}(6,4)$ to be the hyperplane the distance-2 ovoid belongs to.

However, under these presumptions, the two collinear points p_{598} and p_{994} again lead to a contradiction. Indeed, for p_{598} and p_{994} we obtain the following 27 possible coordinates

$$\begin{array}{lll}
(0,1,1,\sigma,1,1,0), & (0,1,1,\sigma,1,1,1), & (0,1,1,\sigma,1,1,\sigma^2),\\
(0,1,\sigma,\sigma,\sigma,1,0), & (0,1,\sigma,\sigma,\sigma,1,1), & (0,1,\sigma,\sigma,\sigma,1,\sigma^2),\\
(0,1,\sigma^2,\sigma,\sigma^2,1,0), & (0,1,\sigma^2,\sigma,\sigma^2,1,1), & (0,1,\sigma^2,\sigma,\sigma^2,1,\sigma^2),\\
(1,1,1,\sigma^2,1,1,0), & (1,1,1,\sigma^2,1,1,1), & (1,1,1,\sigma^2,1,1,\sigma^2),\\
(1,1,\sigma,\sigma^2,\sigma,1,0), & (1,1,\sigma,\sigma^2,\sigma,1,1), & (1,1,\sigma,\sigma^2,\sigma,1,\sigma^2),\\
(1,1,\sigma^2,\sigma^2,\sigma^2,1,0), & (1,1,\sigma^2,\sigma^2,\sigma^2,1,1), & (1,1,\sigma^2,\sigma^2,\sigma^2,1,\sigma^2),\\
(1,\sigma,1,\sigma,1,\sigma,0), & (1,\sigma,1,\sigma,1,\sigma,1), & (1,\sigma,1,\sigma,1,\sigma,\sigma),\\
(1,\sigma,\sigma,\sigma,\sigma,\sigma,0), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,1), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma),\\
(1,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,0), & (1,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,1), & (1,\sigma,\sigma^2,\sigma,\sigma^2,\sigma,\sigma)
\end{array}$$

and

$$\begin{array}{lll}
(0,0,1,\sigma,\sigma^2,1,0), & (0,0,1,\sigma,\sigma^2,1,\sigma), & (0,0,1,\sigma,\sigma^2,1,\sigma^2),\\
(0,1,1,\sigma^2,\sigma^2,\sigma,0), & (0,1,1,\sigma^2,\sigma^2,\sigma,1), & (0,1,1,\sigma^2,\sigma^2,\sigma,\sigma),\\
(0,1,\sigma,\sigma,1,1,1), & (0,1,\sigma,\sigma,1,1,\sigma), & (0,1,\sigma,\sigma,1,1,\sigma^2),\\
(1,0,1,\sigma^2,\sigma^2,1,0), & (1,0,1,\sigma^2,\sigma^2,1,\sigma), & (1,0,1,\sigma^2,\sigma^2,1,\sigma^2),\\
(1,0,\sigma,\sigma,1,\sigma,0), & (1,0,\sigma,\sigma,1,\sigma,1), & (1,0,\sigma,\sigma,1,\sigma,\sigma^2),\\
(1,1,1,\sigma,\sigma^2,\sigma,0), & (1,1,1,\sigma,\sigma^2,\sigma,1), & (1,1,1,\sigma,\sigma^2,\sigma,\sigma),\\
(1,1,\sigma,\sigma^2,1,1,1), & (1,1,\sigma,\sigma^2,1,1,\sigma), & (1,1,\sigma,\sigma^2,1,1,\sigma^2),\\
(1,\sigma,\sigma^2,\sigma,\sigma,\sigma,1), & (1,\sigma,\sigma^2,\sigma,\sigma,\sigma,\sigma), & (1,\sigma,\sigma^2,\sigma,\sigma,\sigma,\sigma^2),\\
(1,\sigma^2,\sigma^2,\sigma^2,\sigma,1,0), & (1,\sigma^2,\sigma^2,\sigma^2,\sigma,1,1), & (1,\sigma^2,\sigma^2,\sigma^2,\sigma,1,\sigma^2)
\end{array}$$

respectively. Knowing that p_{410} belongs to $X_3 = 0$ in the first place, one readily checks that this point, with explicit coordinates $(1,1,0,0,1,1,1)$, does not belong to any one of the 729 combinations of which at least one should represent the line L_{422}. Hence there exists no such representation of $\mathsf{H}(4)$ inside $\mathsf{PG}(6,4)$ starting from a distance-2 ovoid isomorphic to $\mathcal{O}_\Omega$.

5.4.3 Using $\mathcal{O}_2$

Just as we did not want (∞) to belong to our first geometrical hyperplane, we again opt to choose a distance-2 ovoid isomorphic to $\mathcal{O}_2$ that does not contain (∞), rather than using $\mathcal{O}_2$ itself.

However, in Section 5.3.3 we noticed that $(S \cup \Pi' \setminus \Pi)$ is a two-character set of $\mathsf{PG}(5, q^2)$ isomorphic to S and hence we may replace all points of Π with those of Π' to obtain a distance-2 ovoid isomorphic to $\mathcal{O}_2$ not containing (∞). Choosing the plane Π and the Baer subplane B as in Section 5.3.4 we find, in coordinates of $\mathsf{H}(4)$,

$$\begin{aligned}\mathcal{O}_2' = \{&p_1, p_5, p_9, p_{13}, p_{17}, p_{37}, p_{42}, p_{45}, p_{50}, p_{53}, p_{60}, p_{61}, p_{68}, p_{69}, p_{75},\\
&p_{77}, p_{83}, p_{101}, p_{106}, p_{109}, p_{114}, p_{117}, p_{124}, p_{125}, p_{132}, p_{133}, p_{139}, p_{141}, p_{147}, p_{165},\\
&p_{170}, p_{173}, p_{178}, p_{181}, p_{188}, p_{189}, p_{196}, p_{197}, p_{203}, p_{205}, p_{211}, p_{230}, p_{234}, p_{238}, p_{242},\\
&p_{248}, p_{252}, p_{256}, p_{260}, p_{263}, p_{267}, p_{271}, p_{275}, p_{294}, p_{298}, p_{302}, p_{306}, p_{312}, p_{316}, p_{320},\\
&p_{324}, p_{327}, p_{331}, p_{335}, p_{339}, p_{341}, p_{345}, p_{349}, p_{353}, p_{357}, p_{364}, p_{365}, p_{372}, p_{373}, p_{379},\\
&p_{381}, p_{387}, p_{389}, p_{394}, p_{397}, p_{402}, p_{405}, p_{409}, p_{413}, p_{417}, p_{424}, p_{428}, p_{432}, p_{436}, p_{439},\\
&p_{443}, p_{447}, p_{451}, p_{454}, p_{458}, p_{462}, p_{466}, p_{469}, p_{473}, p_{477}, p_{481}, p_{485}, p_{492}, p_{493}, p_{500},\\
&p_{501}, p_{507}, p_{509}, p_{515}, p_{517}, p_{522}, p_{525}, p_{530}, p_{533}, p_{537}, p_{541}, p_{545}, p_{552}, p_{556}, p_{560},\\
&p_{564}, p_{567}, p_{571}, p_{575}, p_{579}, p_{582}, p_{586}, p_{590}, p_{594}, p_{616}, p_{619}, p_{624}, p_{627}, p_{630}, p_{636},\\
&p_{638}, p_{644}, p_{645}, p_{649}, p_{653}, p_{657}, p_{663}, p_{668}, p_{671}, p_{676}, p_{694}, p_{698}, p_{702}, p_{706}, p_{711},\\
&p_{713}, p_{719}, p_{721}, p_{744}, p_{747}, p_{752}, p_{755}, p_{758}, p_{764}, p_{766}, p_{772}, p_{773}, p_{777}, p_{781}, p_{785},\\
&p_{791}, p_{796}, p_{799}, p_{804}, p_{822}, p_{826}, p_{830}, p_{834}, p_{839}, p_{841}, p_{847}, p_{849}, p_{871}, p_{874}, p_{879},\\
&p_{882}, p_{885}, p_{889}, p_{893}, p_{897}, p_{902}, p_{908}, p_{910}, p_{916}, p_{920}, p_{922}, p_{928}, p_{930}, p_{935}, p_{939},\\
&p_{943}, p_{947}, p_{952}, p_{953}, p_{960}, p_{961}, p_{999}, p_{1002}, p_{1007}, p_{1010}, p_{1013}, p_{1017}, p_{1021}, p_{1025},\\
&p_{1030}, p_{1036}, p_{1038}, p_{1044}, p_{1048}, p_{1050}, p_{1056}, p_{1058}, p_{1063}, p_{1067}, p_{1071}, p_{1075}, p_{1080},\\
&p_{1081}, p_{1088}, p_{1089}, p_{1125}, p_{1129}, p_{1133}, p_{1137}, p_{1143}, p_{1146}, p_{1151}, p_{1154}, p_{1160}, p_{1163},\\
&p_{1168}, p_{1171}, p_{1174}, p_{1179}, p_{1182}, p_{1187}, p_{1190}, p_{1193}, p_{1198}, p_{1201}, p_{1224}, p_{1228}, p_{1232},\\
&p_{1236}, p_{1253}, p_{1257}, p_{1261}, p_{1265}, p_{1271}, p_{1274}, p_{1279}, p_{1282}, p_{1288}, p_{1291}, p_{1296}, p_{1299},\\
&p_{1302}, p_{1307}, p_{1310}, p_{1315}, p_{1318}, p_{1321}, p_{1326}, p_{1329}, p_{1352}, p_{1356}, p_{1360}, p_{1364}\}.\end{aligned}$$

as this particular distance-2 ovoid.

In the exact same way as in the previous section, the points p_{613} and p_{962}, which are on the line L_{1044} that intersects $\mathcal{O}_2'$ in the point p_{260}, lead to a contradiction both in the standard and in the perfect symplectic representation of $\mathsf{H}(4)$.

Standard embedding

When starting with the standard embedding of $\mathsf{H}(4)$, the coordinates of p_{613} in $\mathsf{PG}(7,4)$ would be

$$
\begin{array}{lll}
(1,0,0,\sigma^2,\sigma^2,\sigma^2,0,0), & (1,0,0,\sigma^2,\sigma^2,\sigma^2,0,1), & (1,0,0,\sigma^2,\sigma^2,\sigma^2,0,\sigma^2),\\
(1,0,1,\sigma^2,\sigma,\sigma^2,1,0), & (1,0,1,\sigma^2,\sigma,\sigma^2,1,1), & (1,0,1,\sigma^2,\sigma,\sigma^2,1,\sigma^2),\\
(1,0,\sigma^2,\sigma^2,0,\sigma^2,\sigma^2,0), & (1,0,\sigma^2,\sigma^2,0,\sigma^2,\sigma^2,1), & (1,0,\sigma^2,\sigma^2,0,\sigma^2,\sigma^2,\sigma^2),\\
(1,\sigma,0,\sigma,\sigma,\sigma,0,0), & (1,\sigma,0,\sigma,\sigma,\sigma,0,\sigma), & (1,\sigma,0,\sigma,\sigma,\sigma,0,\sigma^2),\\
(1,\sigma,\sigma,\sigma,0,\sigma,\sigma,0), & (1,\sigma,\sigma,\sigma,0,\sigma,\sigma,\sigma), & (1,\sigma,\sigma,\sigma,0,\sigma,\sigma,\sigma^2),\\
(1,\sigma,\sigma^2,\sigma,1,\sigma,\sigma^2,0), & (1,\sigma,\sigma^2,\sigma,1,\sigma,\sigma^2,\sigma), & (1,\sigma,\sigma^2,\sigma,1,\sigma,\sigma^2,\sigma^2),\\
(1,\sigma^2,0,1,1,1,0,0), & (1,\sigma^2,0,1,1,1,0,1), & (1,\sigma^2,0,1,1,1,0,\sigma),\\
(1,\sigma^2,1,1,0,1,1,0), & (1,\sigma^2,1,1,0,1,1,1), & (1,\sigma^2,1,1,0,1,1,\sigma),\\
(1,\sigma^2,\sigma,1,\sigma^2,1,\sigma,0), & (1,\sigma^2,\sigma,1,\sigma^2,1,\sigma,1), & (1,\sigma^2,\sigma,1,\sigma^2,1,\sigma,\sigma)
\end{array}
$$

while those of p_{962} would be

$$
\begin{array}{lll}
(1,0,1,1,\sigma^2,\sigma^2,0,0), & (1,0,1,1,\sigma^2,\sigma^2,0,\sigma), & (1,0,1,1,\sigma^2,\sigma^2,0,\sigma^2),\\
(1,0,\sigma,1,\sigma,\sigma^2,\sigma,0), & (1,0,\sigma,1,\sigma,\sigma^2,\sigma,1), & (1,0,\sigma,1,\sigma,\sigma^2,\sigma,\sigma),\\
(1,0,\sigma^2,1,0,\sigma^2,1,1), & (1,0,\sigma^2,1,0,\sigma^2,1,\sigma), & (1,0,\sigma^2,1,0,\sigma^2,1,\sigma^2),\\
(1,\sigma,1,\sigma^2,1,\sigma,1,0), & (1,\sigma,1,\sigma^2,1,\sigma,1,1), & (1,\sigma,1,\sigma^2,1,\sigma,1,\sigma^2),\\
(1,\sigma,\sigma,\sigma^2,0,\sigma,\sigma^2,1), & (1,\sigma,\sigma,\sigma^2,0,\sigma,\sigma^2,\sigma), & (1,\sigma,\sigma,\sigma^2,0,\sigma,\sigma^2,\sigma^2),\\
(1,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,0,0), & (1,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,0,1), & (1,\sigma,\sigma^2,\sigma^2,\sigma,\sigma,0,\sigma),\\
(1,\sigma^2,1,\sigma,0,1,\sigma,1), & (1,\sigma^2,1,\sigma,0,1,\sigma,\sigma), & (1,\sigma^2,1,\sigma,0,1,\sigma,\sigma^2),\\
(1,\sigma^2,\sigma,\sigma,1,1,0,0), & (1,\sigma^2,\sigma,\sigma,1,1,0,1), & (1,\sigma^2,\sigma,\sigma,1,1,0,\sigma^2),\\
(1,\sigma^2,\sigma^2,\sigma,\sigma^2,1,\sigma^2,0), & (1,\sigma^2,\sigma^2,\sigma,\sigma^2,1,\sigma^2,\sigma), & (1,\sigma^2,\sigma^2,\sigma,\sigma^2,1,\sigma^2,\sigma^2).
\end{array}
$$

After some tedious calculations one can conclude that, since

$$p_{260}^{\Phi} = (0,1,1,\sigma^2,1,0,\sigma,1),$$

none of the possible lines intersects $\mathcal{O}_2'^{\Phi}$ in the point p_{260}^{Φ}. In other words, the distance-2 ovoid $\mathcal{O}_2'$ is not a suitable choice for a geometrical hyperplane to lift the embedding of $\mathsf{H}(4)$ into a higher dimensional embedding.

Perfect symplectic representation

In the lifted embedding of $\mathsf{H}(4)$ in $\mathsf{PG}(6,4)$ we obtain that p_{613} should have one of the following 27 coordinates

$$\begin{array}{lll}
(0,0,1,\sigma,1,0,0), & (0,0,1,\sigma,1,0,1), & (0,0,1,\sigma,1,0,\sigma),\\
(0,1,1,\sigma,1,1,0), & (0,1,1,\sigma,1,1,1), & (0,1,1,\sigma,1,1,\sigma),\\
(0,1,\sigma^2,1,\sigma^2,1,0), & (0,1,\sigma^2,1,\sigma^2,1,1), & (0,1,\sigma^2,1,\sigma^2,1,\sigma^2),\\
(1,0,1,\sigma^2,1,0,0), & (1,0,1,\sigma^2,1,0,1), & (1,0,1,\sigma^2,1,0,\sigma),\\
(1,0,\sigma,\sigma,\sigma,0,0), & (1,0,\sigma,\sigma,\sigma,0,\sigma), & (1,0,\sigma,\sigma,\sigma,0,\sigma^2),\\
(1,1,1,\sigma^2,1,1,0), & (1,1,1,\sigma^2,1,1,1), & (1,1,1,\sigma^2,1,1,\sigma),\\
(1,\sigma,1,\sigma^2,1,\sigma,0), & (1,\sigma,1,\sigma^2,1,\sigma,1), & (1,\sigma,1,\sigma^2,1,\sigma,\sigma),\\
(1,\sigma,\sigma,\sigma,\sigma,\sigma,0), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma), & (1,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma^2),\\
(1,\sigma^2,\sigma,\sigma,\sigma,\sigma^2,0), & (1,\sigma^2,\sigma,\sigma,\sigma,\sigma^2,\sigma), & (1,\sigma^2,\sigma,\sigma,\sigma,\sigma^2,\sigma^2),
\end{array}$$

that p_{962} should belong to

$$\begin{array}{lll}
(0,1,1,1,\sigma^2,0,0), & (0,1,1,1,\sigma^2,0,\sigma), & (0,1,1,1,\sigma^2,0,\sigma^2),\\
(0,1,\sigma,\sigma,1,\sigma,1), & (0,1,\sigma,\sigma,1,\sigma,\sigma), & (0,1,\sigma,\sigma,1,\sigma,\sigma^2),\\
(0,1,\sigma^2,\sigma^2,\sigma,1,0), & (0,1,\sigma^2,\sigma^2,\sigma,1,1), & (0,1,\sigma^2,\sigma^2,\sigma,1,\sigma^2),\\
(1,1,\sigma,\sigma^2,1,\sigma,1), & (1,1,\sigma,\sigma^2,1,\sigma,\sigma), & (1,1,\sigma,\sigma^2,1,\sigma,\sigma^2),\\
(1,1,\sigma^2,\sigma,\sigma,1,0), & (1,1,\sigma^2,\sigma,\sigma,1,1), & (1,1,\sigma^2,\sigma,\sigma,1,\sigma^2),\\
(1,\sigma,\sigma,\sigma^2,1,0,0), & (1,\sigma,\sigma,\sigma^2,1,0,1), & (1,\sigma,\sigma,\sigma^2,1,0,\sigma^2),\\
(1,\sigma,\sigma^2,\sigma,\sigma,\sigma^2,1), & (1,\sigma,\sigma^2,\sigma,\sigma,\sigma^2,\sigma), & (1,\sigma,\sigma^2,\sigma,\sigma,\sigma^2,\sigma^2),\\
(1,\sigma^2,\sigma,\sigma^2,1,\sigma^2,0), & (1,\sigma^2,\sigma,\sigma^2,1,\sigma^2,\sigma), & (1,\sigma^2,\sigma,\sigma^2,1,\sigma^2,\sigma^2),\\
(1,\sigma^2,\sigma^2,\sigma,\sigma,0,0), & (1,\sigma^2,\sigma^2,\sigma,\sigma,0,1), & (1,\sigma^2,\sigma^2,\sigma,\sigma,0,\sigma)
\end{array}$$

and that one of the 729 lines of $\mathsf{PG}(6,4)$ containing a point of each of these two sets has to contain the point $(1,1,\sigma^2,0,0,\sigma,1)$, which represents p_{260} in the perfect symplectic representation of $\mathsf{H}(4)$. As this final condition is not fulfilled, there exists no such embedding of $\mathsf{H}(4)$.

5.5 Conclusion

We would like to conclude this chapter with some final remarks.

First of all, we remark that, apart from J_2, all maximal subgroups of $\mathsf{G}_2(4)$ now have an easy geometric interpretation inside the

generalized hexagon $\mathsf{H}(4)$. They are either the stabilizer of a point, a line, a Hermitian spread, a subhexagon of order $(1,4)$, a subhexagon of order 2, a line regulus or a distance-2 ovoid.

maximal subgroup	order	index	stabilized set
J_2	604800	416	
$2^{2+8}:(3\times\mathsf{A}_5)$	184320	1365	p
$2^{4+6}:(\mathsf{A}_5\times 3)$	184320	1365	L
$\mathsf{PSU}_3(4):2$	124800	2016	$\mathcal{S}_{\mathsf{H}}$
$3.\mathsf{PSL}_3(4):2_3$	120960	2080	$\Gamma(p,p')$
$\mathsf{PSU}_3(3):2\times 2$	12096	20800	$\mathsf{H}(2)$
$\mathsf{A}_5\times\mathsf{A}_5$	3600	69888	$\mathcal{R}(L,M)$
$\mathsf{PSL}_2(13)$	1092	230400	$\mathcal{O}_\Omega$

Table 5.1: Maximal subgroups of $\mathsf{G}_2(4)$

Desperately wanting to complete Table 5.1 we looked at the orbits of J_2 within $\mathsf{H}(4)$. However, the sizes of both the orbits on the points and those on the lines do not seem interesting. More explicitly, J_2 acts transitively on two sets of points of $\mathsf{H}(4)$, one of which has size 525, the other having size 840. Considering the line set yields an orbit of size 315 and one of size 1050.

Secondly, even though we are not yet able to prove this theoretically, we would like to conjecture (based on an extensive computer search) that the two distance-2 ovoids constructed in this chapter, are, up to isomorphism, the only types of distance-2 ovoids living inside the split Cayley hexagon of order 4. A similar search in the dual of $\mathsf{H}(4)$ lets believe that there exists no distance-2 spread in $\mathsf{H}(4)$, but as we were so far unable to obtain a theoretical explanation, we do not elaborate on this matter.

Also, under the assumption that the above conjecture is in fact true, we have shown (again by aid of the computer) that there exists no 5-coloring of the point graph of $\mathsf{H}(4)$. And this fact in addition to the results of the previous chapter, makes it now very plausible to conjecture that the chromatic number of the point graph of $\mathsf{H}(q)$ has to be strictly greater than $q+1$.

As a final conclusion to Section 4.3.6 and Section 5.4 we would like to say that, concerning the subject of lifting the known embeddings of $\mathsf{H}(q)$, it seems to be that the promising distance-2 ovoids

are in fact of no use. If $q = 3^h$, then by a result of J.A. Thas and H. Van Maldeghem [63] there exists a unique embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(13, q)$ and hence we have lots of geometrical hyperplanes of $\mathsf{H}(q)$ coming from this 13-dimensional embedding that do not arise from hyperplanes of $\mathsf{PG}(6, q)$ (namely all projections of ordinary hyperplanes in $\mathsf{PG}(13, q)$ that do not contain the 6-dimensional *nucleus* as a subspace; the latter is the 6-dimensional space that yields the standard embedding by projection from it). Call these geometrical hyperplanes, together with the ordinary hyperplanes in the standard embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(6, q)$, classical. Since the distance-2 ovoids are in fact the only non-classical geometrical hyperplanes known to date, it seems like the embeddings of $\mathsf{H}(q)$ are not liftable. However, we are far from ready to conjecture this hunch!

6 Eigenvalues and codes

In this chapter we provide a geometrical interpretation of the eigenvalues of the point graph of a generalized hexagon of order q and construct some codes that arise from generalized hexagons with small parameters.

6.1 General Introduction

This chapter comes in two parts, each divided into a number of short sections.

The first part of this chapter is concerned with the determination of the eigenvalues and their respective eigenvectors and multiplicities of the point graph of a generalized hexagon of order q. First we use some basic algebraic graph theory to come to these values. Once we have done this, we look for a geometrical explanation of these results. As an application we use these eigenvectors to determine the cardinality of some intersections of the corresponding substructures.

In the second part we consider the generalized hexagons $\mathsf{H}(2)$ and its dual $\mathsf{H}(2)^D$ and construct many very symmetric binary codes from the characteristic vectors of certain natural geometric configurations of points of these hexagons. We list the results in some tables in Section 6.3. These results were generated by computer. For the moment it seems pointless to produce long and tiresome theoretic proofs for these tables.

6.2 Part I: Eigenvalues

6.2.1 Introduction

Let Δ be a generalized hexagon of order q. In this section we list all eigenvalues and their eigenvectors and multiplicities of the point graph of Δ. The first part deals with the algebraic approach, while the second part deals with the geometrical point of view.

We start by explaining some generalities.

Let G be a distance regular graph with diameter d. Next to the constants b_j and c_j, as defined in Section 1.1 of Chapter 1, we define for any two vertices x and y, with $d(x,y) = j$, the constants

$$a_j = |\Gamma_j(y) \cap \Gamma_1(x)|$$

for $0 \leq j \leq d$.

The *intersection matrix* of G is then given by the following matrix

$$B = \begin{pmatrix} 0 & 1 & & & & \\ k & a_1 & c_2 & & & \\ & b_1 & a_2 & . & & \\ & & b_2 & . & . & \\ & & & . & . & . \\ & & & & . & . & c_d \\ & & & & & . & a_d \end{pmatrix}$$

and from [2] we have

Theorem 6.2.1.
Let G be a distance regular graph with valency k and diameter d. Then G has $d+1$ eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$, which are the eigenvalues of the intersection matrix B.

We now introduce *left* and *right eigenvectors* of B corresponding to the eigenvalue λ_i as the solution of the systems $\mathbf{u}B = \lambda_i \mathbf{u}$ and $B\mathbf{v} = \lambda_i \mathbf{v}$, respectively. Again from [2] we know that

Theorem 6.2.2.
The multiplicity of the eigenvalues λ_i of a distance regular graph with n vertices is

$$m(\lambda_i) = \frac{n}{(\mathbf{u_i}, \mathbf{v_i})}$$

for $0 \leq i \leq d$.

6.2.2 Algebraic approach

We are now ready to list all eigenvalues plus related eigenvectors and multiplicities of the point graph G of Δ. One readily verifies that for this graph G the intersection matrix B is given by the following matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q(q+1) & q-1 & 1 & 0 \\ 0 & q^2 & q-1 & q+1 \\ 0 & 0 & q^2 & q^2-1 \end{pmatrix}.$$

After some tedious calculations we find

$$q(q+1) \qquad 2q-1 \qquad -(q+1) \qquad -1$$

as the eigenvalues of B. With these respective eigenvalues correspond

$$\begin{pmatrix} 1 \\ q(q+1) \\ q^3+q^4 \\ q^5 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2q-1 \\ q^2-2q \\ \frac{q^3-q^2}{q+1} \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -(q+1) \\ q(q+1) \\ -q^2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ -q^2 \\ q^2 \end{pmatrix}$$

and

$$\begin{array}{ll} \begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix} & \begin{pmatrix} 1, & \frac{2q-1}{q(q+1)}, & \frac{q-2}{q^2(q+1)}, & -\frac{1}{q^3} \end{pmatrix} \\ \begin{pmatrix} 1, & -\frac{1}{q}, & \frac{1}{q^2}, & -\frac{1}{q^3} \end{pmatrix} & \begin{pmatrix} 1, & -\frac{1}{q(q+1)}, & -\frac{q}{q^2(q+1)}, & \frac{1}{q^3} \end{pmatrix} \end{array}$$

as the respective right and left eigenvectors, while a simple substitution of these values into the multiplicity-formula of Theorem 6.2.2 yields

$$\begin{aligned} m(q(q+1)) &= 1 \\ m(2q-1) &= \tfrac{q(q+1)^2(q^2+q+1)}{6} \\ m(-(q+1)) &= \tfrac{q(q^2+q+1)(q^2-q+1)}{3} \\ m(-1) &= \tfrac{q(q+1)^2(q^2-q+1)}{2} \end{aligned}$$

as the respective multiplicities.

6.2.3 Geometric approach

We will now provide a geometrical clarification of these eigenvalues by considering some substructures of Δ. We will study several theoretical substructures, not knowing whether Δ actually contains such substructures or not.

Ovoid

One of these uncertainties is an ovoid of Δ. Say Δ does contain an ovoid $\mathcal{O}$. Define a vector $\mathbf{v}(a,b)$ of length $n = \frac{q^6-1}{q-1}$ and attach an index $1 \leq i \leq n$ to every point of Δ. Put an element a in all entries of $\mathbf{v}(a,b)$ corresponding to the points of the ovoid, an element b in all other entries. Since no ovoid contains collinear points and every point off $\mathcal{O}$ is collinear to a unique element of $\mathcal{O}$ we obtain the following system of equations to determine the eigenvalues corresponding to the eigenvector $\mathbf{v}(a,b)$

$$\begin{cases} ra &= 0 &+ & q(q+1)b \\ rb &= a &+ & ((q+1)q-1)b \end{cases}$$

In other words, the eigenvalues of $\mathbf{v}(a,b)$ are the solutions of

$$r^2 - (q^2+q-1)r - q(q+1) = 0$$

and thus -1 and $q(q+1)$ are eigenvalues that are determined by any ovoid of Δ. Using the above system of equations we can also determine the eigenvectors corresponding to these respective eigenvalues. First, we see that the eigenvector corresponding to $q(q+1)$ is the vector $\mathbf{v}(1,1)$ and secondly, we find $\mathbf{v}(q(q+1),-1)$ as the eigenvector of -1.

Spread

Consider $\mathcal{S}$ a spread of Δ (in $\mathsf{H}(q)$ one can always consider a Hermitian spread). We define $\mathbf{v}(a,b)$ in a way similar to the previous situation, we put a in the position of a point on a spread line, b otherwise. If x is a point on a spread line, then only the q other points on this particular line are both collinear to x and belong to V, the set of points incident with a line of $\mathcal{S}$. If x, on the other hand, is a point outside V, then every line through x intersects the set V in a unique point. In conclusion, we obtain

$$\begin{cases} ra &= qa &+ & q^2b \\ rb &= (q+1)a &+ & (q^2-1)b \end{cases}$$

as the corresponding system of equations. This system yields the exact same quadratic equation in r as obtained when dealing with the points of an ovoid. Hence, in $\mathsf{H}(q)$ the eigenvalues -1 and $q(q+1)$ can always be seen as the eigenvalues linked to a Hermitian spread. While the eigenvector corresponding to $q(q+1)$ remains the same, the one associated with -1, namely $\mathbf{v}(q^2,-(q+1))$, differs from the one obtained above.

Weak subhexagon

Besides the existence of a Hermitian spread in every $\mathsf{H}(q)$, we also know that every $\mathsf{H}(q)$ contains a weak subhexagon of order $(1, q)$ as a substructure. Suppose Γ is a weak subhexagon of order $(1, q)$ of Δ. In such a weak subhexagon we can either consider the lines or the points.

First, assume Δ is self-dual and consider the set of points of Γ^D in the same way as we did for an ovoid of Δ. In order to determine the corresponding eigenvalues we can just as well consider the lines of Γ and work with concurrency instead of working with collinearity. Meaning, we put an element a in every entry of $\mathbf{v}(a, b)$ corresponding to the lines of Γ – which we labeled $1 \leq i \leq n$ – and put an element b otherwise. Since every line of Γ is concurrent with $2q$ other weak subhexagon lines and a line off Γ is concurrent with a unique such a line, we immediately find

$$\left\{ \begin{array}{rcccc} ra & = & 2qa & + & q(q-1)b \\ rb & = & a & + & (q(q+1)-1)b \end{array} \right.$$

and consequently see that the eigenvalues associated with a weak subhexagon of order $(q, 1)$, with $q = 3^h$, satisfy the following equation in r

$$r^2 - (q^2 + 3q - 1)r + q(q+1)(2q-1) = 0.$$

In other words, if Δ is self-dual, then Γ^D geometrically determines the eigenvalues $q(q+1)$ and $2q-1$. A substitution of $r = 2q-1$ in one of these equations, yields the eigenvector $\mathbf{v}(q(q-1), -1)$ to this specific eigenvalue.

On the other hand, if we consider the points of this substructure, then we have three possibilities for any arbitrary point. Either it is a point of Γ (attach a); or it is a point off Γ that is incident with a line of Γ (attach b) or it is none of the above (attach c). By arguments used in the previous situation we immediately find that the thus obtained vector $\mathbf{v}(a, b, c)$ determines the following system of equations

$$\left\{ \begin{array}{rcccccc} ra & = & (q+1)a & + & (q+1)(q-1)b & + & 0 \\ rb & = & 2a & + & (q-2)b & + & q^2c \\ rc & = & 0 & + & (q+1)b & + & (q^2-1)c \end{array} \right.$$

which, in turn, gives us a cubic equation in r with as solutions $q(q+1)$, $-(q+1)$ and $2q-1$. For the second and third value of r, we obtain $\mathbf{v}(q(q-1), -2q, 2)$ and $\mathbf{v}(-q(q^2-1), (2-q)q, q+1)$ as respective corresponding eigenvectors.

Distance-2 ovoid

Finally, suppose Δ contains a distance-2 ovoid and define $\mathbf{v}(a,b)$ as above (see ovoid). For this particular set of points we know that there is no collinearity within the set and since every line intersects in a unique point we obtain the following system of equations

$$\begin{cases} ra &= 0 &+ q(q+1)b \\ rb &= (q+1)a &+ (q-1)(q+1)b \end{cases}$$

in r and find

$$r^2 - (q^2-1)r - (q+1)^2 q = 0$$

or $q(q+1)$ and $-(q+1)$ as the associated eigenvalues. The respective eigenvectors to these eigenvalues are given by $\mathbf{v}(1,1)$ and $\mathbf{v}(q,-1)$.

6.2.4 Applications

With these geometric eigenvectors at hand we now have a powerful tool to determine the intersection number of the corresponding subsets of points. Indeed, as the eigenvectors of distinct eigenvalues have to be orthogonal vectors we can determine the number of points in the intersections

$\mathcal{O} \cap \Gamma^D$	$\mathcal{S} \cap \Gamma^D$	$\Gamma^D \cap \mathcal{O}_2$
$\mathcal{O} \cap \Gamma$	$\mathcal{S} \cap \Gamma$	$\Gamma \cap \mathcal{O}_2$
$\mathcal{O} \cap \mathcal{O}_2$	$\mathcal{S} \cap \mathcal{O}_2$	

where $\mathcal{O}$, $\mathcal{S}$, Γ, Γ^D and $\mathcal{O}_2$ represent an ovoid, a spread, a weak subhexagon of order $(1,q)$, a weak subhexagon of order $(q,1)$ and a distance-2 ovoid, respectively. Some of these intersection numbers have a very easy geometrical explanation, for instance a spread will intersect a distance-2 ovoid in q^3+1 points simply by definition of a distance-2 ovoid, but others are new and have not been determined in a geometrical manner so far.

$\mathcal{O} \cap \Gamma^D$

We consider $\mathbf{v}(q(q+1),-1)$ and $\mathbf{v}(q(q-1),-1)$ as the respective eigenvectors belonging to the eigenvalues -1 and $2q-1$ of an ovoid and a dual weak subhexagon, respectively. The former eigenvector has $q(q+1)$ in q^3+1 entries and -1 in $q^5+q^4+q^2+q$ entries, while the latter one has $q(q-1)$ in $(q^2+q+1)(q+1)$ entries and -1 in $q^5+q^4-q^2-q$ entries. Suppose k is

the number of entries in which the symbols of these vectors are $q(q+1)$ and $q(q-1)$, respectively. In other words, suppose the dual weak subhexagon contains k points of the ovoid. Then we have the following equation

$$kq^2(q+1)(q-1) - (q^3+1-k)q(q+1) - (2q^2+2q+q^3+1-k)q(q-1) + q^5+q^4-q^2-q-q^3-1+k = 0$$

to determine the value of the constant k, which yields

$$k = q+1.$$

Therefore any arbitrary ovoid intersects any arbitrary weak subhexagon of order $(q,1)$ in $q+1$ of its points.

$\mathcal{O} \cap \Gamma$

Consider the eigenvector $\mathbf{v}(q(q+1),-1)$ of $\mathcal{O}$ and the eigenvector $\mathbf{v}(-q(q^2-1),(2-q)q,q+1)$ of Γ. Since there are three types of points in the description of the weak subhexagon Γ, we have to be a bit more careful considering the inner product with a eigenvector corresponding to this subset of points. With the definition of type a, b and c points as given above, we first of all have $A = 2(q^2+q+1)$ type a points, $B = (q^2+q+1)(q^2-1)$ type b points and $C = q^5-q^2$ type c points. We will now determine the cardinality of the intersection of each of these subsets of points with any ovoid $\mathcal{O}$. Suppose there are $\chi_a = k$ type a points that belong to $\mathcal{O}$. Then by definition of an ovoid, i.e. every point off $\mathcal{O}$ is collinear to a unique point in $\mathcal{O}$, one can easily see that there will be $\chi_b = \frac{2(q^2+q+1)-k(q+2)}{2}$ type b points in $\mathcal{O}$. Finally, the number of type c ovoid points, say χ_c, is now simply given by the cardinality of the remaining set of points in $\mathcal{O}$. A substitution of these explicit values into the following equation

$$\begin{aligned}&\chi_a(-q(q^2-1))q(q+1) + (A-\chi_a)(-q(q^2-1))(-1) + \\ &\chi_b(2-q)q^2(q+1) + (B-\chi_b)(2-q)q(-1) + \\ &\chi_c(q+1)q(q+1) + (C-\chi_c)(q+1)(-1) = 0\end{aligned}$$

yields $k=2$. Hence, if a generalized hexagon of order q contains an ovoid and a weak subhexagon of order $(1,q)$, then these two substructures share 2 points. This result is an improvement to a previous result obtained by V. De Smet and H. Van Maldeghem, ([15] Lemma 2.4).

$\mathcal{O} \cap \mathcal{O}_2$

As both eigenvectors $\mathbf{v}(q(q+1), -1)$ and $\mathbf{v}(q, -1)$ of the eigenvalues -1 and $-(q+1)$ of $\mathcal{O}$ and $\mathcal{O}_2$, respectively, just have two possible values in all of their entries, we immediately find

$$kq^2(q+1) - (q^3+1-k)q(q+1) - (q^4+q^2+1-k)q + q^5 + q - 1 + k = 0$$

where k again stands for the number of points in the intersection. We thus obtain

$$k = q^2 - q + 1$$

as the number of points that any ovoid and any distance-2 ovoid of a generalized hexagon of order q have in common.

$\mathcal{S} \cap \Gamma^{\mathbf{D}}$

Considering the eigenvectors $\mathbf{v}(q^2, -(q+1))$ and $\mathbf{v}(q(q-1), -1)$ that correspond to the eigenvalues -1 and $2q-1$ of $\mathcal{S}$ and Γ^D, respectively, we now find – in the exact same way as before –

$$k = (q+1)^2$$

points that are both on a spread line of $\mathcal{S}$ and in Γ^D.

Note. This number of points geometrically follows from the fact that an ovoid and a weak subhexagon of order $(1, q)$ share 2 points.

$\mathcal{S} \cap \Gamma$

Similar to the situation in "ovoid $\cap$ weak subhexagon of order $(1, q)$", we determine the number of type a, type b and type c points that are on a spread line. Let us first note that any line of $\mathcal{S}$ either belongs to Γ or intersects in a unique point of type b and conversely, every line of Γ either belongs to $\mathcal{S}$ or is concurrent with a unique element of $\mathcal{S}$. Suppose there are k lines of $\mathcal{S}$ that belong to Γ, then we immediately see that $\mathcal{S}$ contains

$$(q^2+q+1)(q+1) - 2kq - k \qquad (*)$$

lines that intersect Γ in a type b point. Hence, in total we find $\chi_a = 2k$ type a points, $\chi_b = (q^2+q+1)(q+1) - 2kq - k + k(q-1)$ type b points and $\chi_c = (q^3+1)(q+1) - \chi_a - \chi_b$ type c points on a spread line. Given the

eigenvectors $\mathbf{v}(q^2, -(q+1))$ and $\mathbf{v}(-q(q^2-1), (2-q)q, q+1)$ of $\mathcal{S}$ and Γ, respectively, we now substitute these numbers into the equation

$$\begin{aligned}&\chi_a(-q(q^2-1))q^2 + (A-\chi_a)(-q(q^2-1))(-(q+1)) + \\ &\chi_b(2-q)q^3 + (B-\chi_b)(2-q)q(-(q+1)) + \\ &\chi_c(q+1)q^2 + (C-\chi_c)(q+1)(-(q+1)) = 0\end{aligned}$$

and find $k = q+1$.

Note. We can immediately deduce k from equation $(*)$ as this number of lines equals $q^3 + 1 - k$.

$\mathcal{S} \cap \mathcal{O}_2$

As we already mentioned above this intersection number is by geometric arguments trivially equal to q^3+1, which is indeed the value of k that we obtain if we algebraically determine the inner product of $\mathbf{v}(q^2, -(q+1))$ and $\mathbf{v}(q, -1)$.

$\Gamma^D \cap \mathcal{O}_2$

The inner product of $\mathbf{v}(q(q-1), -1)$ of Γ^D with $\mathbf{v}(q, -1)$ of $\mathcal{O}_2$ yields the equation

$$\begin{aligned}&kq^2(q-1) - (q^4+q^2+1-k)q - (2q^2+2q+q^3+1-k)q(q-1) + \\ &q^5 - q - 2q^2 - 1 + k = 0\end{aligned}$$

and hence we obtain q^2+q+1 intersection points.

Note. This number can also be found through a double counting of the pairs (L, p), with $L \in \Gamma^D$, $p \in \mathcal{O}_2 \cap \Gamma^D$ and $p \mathrel{\mathtt{I}} L$.

$\Gamma \cap \mathcal{O}_2$

We use the same notations as introduced above. If $\chi_a = k$, then one readily checks that $\chi_b = (q^2+q+1-k)(q+1)$ and hence $\chi_c = q^4+q^2+1-\chi_a-\chi_b$. A substitution of χ_a, χ_b and χ_c together with the respective values of the entries in $\mathbf{v}(-q(q^2-1), (2-q)q, q+1)$ and $\mathbf{v}(q, -1)$, into an equation similar to the ones before yields $0 = 0$. Meaning, we can draw no conclusion on the intersection number of a distance-2 ovoid with a weak subhexagon of order $(1, q)$.

Note. The fact that this particular intersection number is not always a constant can easily be seen in the construction of the known distance-2 ovoid in $\mathsf{H}(2)$, which has a zero-intersection number.

6.3 Part II: Binary codes

6.3.1 Introduction

In this section we present the parameters of some binary codes arising from various substructures of $\mathsf{H}(q)$, mainly with $q = 2$. We have calculated the invariants with the use of a straightforward computer program. The general idea is as follows: we consider the $\mathsf{GF}(2)$-vector space of characteristic functions of all subsets of the point set of $\mathsf{H}(q)$. Then we fix a subset S of points (arising from a certain geometric substructure) and define $\mathcal{C}(S)$ as the linear code generated by all elements of the orbit of S under $\mathsf{Aut}\,\mathsf{H}(q)$. Such a code has length $1+q+q^2+q^3+q^4+q^5$ and its automorphism group contains the group $\mathsf{G}_2(q)$.

We start with a code that we consider for all q.

6.3.2 Codes from subhexagons

Let $\mathsf{H}(1,q)$ be a subhexagon of order $(1,q)$ of $\mathsf{H}(q)$ and let S be the point set of this subhexagon. The code $\mathcal{C}(S)$ has minimum distance d less than or equal to $2(1+q+q^2)$, the number of points of $\mathsf{H}(1,q)$. We now show that d equals $2(1+q+q^2)$ and that every codeword with minimal weight is the characteristic function of the point set of a subhexagon of order $(1,q)$. In the sequel, we will identify the characteristic function of some subset with that subset. This way, codewords are just subsets of points of $\mathsf{H}(q)$ (like S, for instance).

The codeword S meets every line of $\mathsf{H}(q)$ in an even number of points. Hence every codeword of $\mathcal{C}(S)$ meets every line in an even number of points. Now let S' be a codeword with minimal weight, and let p be a point of S'. Since every line has an even number of points of S', there are at least $1+q$ points of S' collinear with p. Similarly, there are at least $q(1+q)$ points of S' at distance 4 from p in $\mathsf{H}(q)$. Consider any arbitrary point r of S' that is collinear to p. There are now, besides the elements collinear to p, at least q^2 points of S' at distance 4 from r and none of these points has been counted before. This way, we already have at least $1+(1+q)+q(1+q)+q^2 = 2(1+q+q^2)$ points.

This implies that $d = 2(1+q+q^2)$. Now suppose that S' has weight d. Then "at least" in the previous argument becomes "exactly". If A is the set of points at distance 4 from p, then there are exactly q^2 points at distance 4 from a fixed point $x \in A$. Moreover, the set of points of S' at distance 4 from some point y of A is independent of $y \in A$. Also, this holds for arbitrary $p \in S'$. This now implies that for any two points of S' at distance 4 from each other, the unique point collinear with both points also belongs to S'. So S' is the point set of a subhexagon of order $(1, q)$.

In the case where q equals 2 (respectively 3) we obtain a binary code of length 63 (respectively 364), dimension 14 (respectively 91) and minimal distance 14 (respectively 26). We do not know the dimension for general q.

6.3.3 Codes of length 63 from $\mathsf{H}(2)$ and $\mathsf{H}(2)^D$

In Tables 6.1 and 6.2, we present a list of codes arising from configurations in $\mathsf{H}(2)$ and $\mathsf{H}(2)^D$. In the first column we assign a reference number to the different codes . In the second column, each line corresponds to a different set of generators. The next two columns contain the dimension and the minimal weight. In the last column we mention the minimal weight codewords (or rather the configuration that is responsible for it).

We now explain the different configurations. We mention the cardinality (or, equivalently, the weight as codewords).

Note that for each configuration, we can also consider the complementary set of points. The weight of a complementary set is of course 63 minus the weight of the set. We will not mention this explicitly in the following list.

Distance-2 ovoid in $\mathsf{H}(2)$. There are exactly 36 of these. They are all isomorphic. Their weight is 21.

Distance-3 spread in $\mathsf{H}(2)$. There are exactly 28 of these. As corresponding point set (or codeword) we consider the union of these lines, as set of points incident with them. Their weight is 27.

Distance-3 ovoid in $\mathsf{H}(2)^D$. This is dual to the previous case. Here, the weight is 9.

Coxeter graph in $\mathsf{H}(2)$. This is the set of points not incident with any line of a given subhexagon of order $(1, 2)$. This point set, endowed with the lines of $\mathsf{H}(2)$ meeting this set non-trivially, is isomorphic with the Coxeter graph, see Polster [47]. The weight is 28. There are 36 such substructures in $\mathsf{H}(2)$.

Subhexagon. Here we consider the point set of a subhexagon of order $(1,2)$ of $\mathsf{H}(2)$, or dually, of order $(2,1)$ of $\mathsf{H}(2)^D$. There are 36 such structures, all isomorphic. The weights are 14 and 21, respectively.

Ideal set. This is defined by an ideal line $\{x,y,z\}$. The points of the ideal set are the points collinear to one of x,y,z that are not collinear to $x \bowtie y$, together with all points that are at distance 4 from all of x,y,z.

Spheres. Here, there are several possibilities. In general, we look at the set of points whose distance to a certain object (point, line, flag) is in a certain set J. We denote this by $\mathcal{P}_J$(object). For instance, $\mathcal{P}_4$(flag) is the set of points at distance 4 from the point of a fixed flag and at the same time at distance 5 from the line of the flag.

Opposite connected component. In $\mathsf{H}(2)^D$, the graph with point set $\mathcal{P}_6$(point) and adjacency induced by collinearity, has two connected components. We here consider one of these components. The weight is 16.

	generated by	dim	min wght	min wght codewords
(C1)	Ideal set Compl of subhexagon D-2 ovoid	14	20	Ideal set
(C2)	Subhexagon Compl of D-2 ovoid	14	14	Subhexagon
(C3)	Coxeter graph Compl of D-3 spread	7	28	Coxeter graph
(C4)	D-3 spread Compl of Coxeter graph	7	27	D-3 spread
(C5)	$\mathcal{P}_{\{0,3\}}$(line)	21	15	$\mathcal{P}_{\{0,3\}}$(line)
(C6)	$\mathcal{P}_4$(flag) Compl of $\mathcal{P}_{\{0,3\}}$(line)	20	16	$\mathcal{P}_4$(flag)
(C7)	$\mathcal{P}_6$(point)	6	32	$\mathcal{P}_6$(point)
(C8)	Compl of $\mathcal{P}_6$(point)	7	31	Compl of $\mathcal{P}_6$(point)

Table 6.1: Binary codes arising from substructures in $\mathsf{H}(2)$

We remark that only the codes (C5), (C8), (C12) and (C14) contain the all-one-vector $(1,1,1,\ldots,1)$.

The three codes (C3), (C4) and (C8) lie in a common 8-dimensional code (C15) with minimal weight 27 (distance-3 spreads), and is generated by every pair of these three codes. It does slightly better dimension-wise than (C4).

Likewise, the three codes (C1), (C2) and (C4) lie in a common 15-dimensional code (C16) with minimal weight 14 (subhexagons), and is generated by every pair of these three codes. It is also generated by (C1) together with (C8), and by (C2) together with (C8). This code does slightly better dimension-wise than (C2).

	generated by	dim	min wght	min wght codewords
(C9)	D-3 ovoid Subhexagon	21	9	D-3 ovoid
(C10)	$\mathcal{P}_4$(flag) Compl of $\mathcal{P}_{\{0,3\}}$(line)	20	16	$\mathcal{P}_4$(flag)
(C11)	Compl of subhexagon Compl of D-3 ovoid	21	16	$\mathcal{P}_4$(flag)
(C12)	$\mathcal{P}_{\{0,3\}}$(line)	21	15	$\mathcal{P}_{\{0,3\}}$(line)
(C13)	Opp conn component $\mathcal{P}_6$(point)	14	16	Opp conn component
(C14)	Compl of $\mathcal{P}_6$(point)	15	16	Opp conn component

Table 6.2: Binary codes arising from substructures in $\mathsf{H}(2)^{\mathsf{D}}$

Finally, the three codes (C9), (C11) and (C12) lie in a common 22-dimensional code (C17) with minimal weight 9 (distance-3 ovoids), and is generated by every pair of these three codes. This code does slightly better dimension-wise than (C9).

6.3.4 Codes of length 28 and 36 from $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathsf{D}}$

We get codes of smaller length than 63 by considering some codes dual to the previous ones. This goes as follows. Some substructures appear only 28 or 36 times in $\mathsf{H}(2)$ or in $\mathsf{H}(2)^D$. The rows of the matrix whose rows are indexed by the points x of the appropriate hexagon, and whose columns are indexed by the appropriate substructures S, and whose (x, S)-entry is equal to 1 if $x \in S$, and 0 otherwise, generate over $\mathsf{GF}(2)$ a binary code of length the number of substructures S.

The codes we obtain are listed in Table 6.3 below. Every code is also obtained by considering the corresponding complementary substructure (which we do not mention in the table).

	hexagon	substructure	length	dim	min wght
(C18)	$\mathsf{H}(2)$	D-2 ovoid Subhexagon	36	14	8
(C19)	$\mathsf{H}(2)$	Coxeter graph	36	7	16
(C20)	$\mathsf{H}(2)$	D-3 spread	28	7	12
(C21)	$\mathsf{H}(2)^D$	Subhexagon	36	21	6
(C22)	$\mathsf{H}(2)^D$	D-3 Ovoid	28	21	4

Table 6.3: Dual codes related to the $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathsf{D}}$

The codes (C18) and (C21) lie in a common 28-dimensional code (C23) with minimal weight 4.

Of course, the code (C7) is well-known and arises from $\mathsf{PG}(5,2)$ by taking the complements of the hyperplanes as codewords. Likewise, the code (C8) arises from $\mathsf{PG}(5,2)$ by taking as codewords the hyperplanes (and their complements).

In the tables of Brouwer and Verhoeff

see `http://www.win.tue.nl/~aeb/voorlincod.html`

we find lower and upper bounds for the minimal distance of an optimal linear code with word length n, dimension k over a field $\mathsf{GF}(q)$. These bounds are reached by the codes (C7), (C8), (C19), (C20), (C22) and (C23).

6.4 Conclusion

Looking at the tables above, one might try to generalize most of these results to arbitrary $\mathsf{H}(q)$ and/or its dual. A first step could be to consider higher values of q. An ideal scenario would then be that these results trigger an interaction/collaboration between coding theory and the study of generalized hexagons in the same way as this is currently going on between coding theory and Galois geometry.

7 One-point extensions of generalized hexagons

In this chapter, we prove the uniqueness of the one-point extension $\mathcal{S}$ of a generalized hexagon of order 2 provided the following property holds: for any three points x, y and z of $\mathcal{S}$, the graph theoretic distance from y to z in the derived generalized hexagon $\mathcal{S}_x$ is the same as the distance from x to z in $\mathcal{S}_y$.

7.1 General Introduction

As we know from Section 1.6 the Hölz designs are one-point extensions of the Ahrens-Szekeres generalized quadrangles $\mathsf{AS}(q)$ of order $(q-1, q+1)$ (see [1]).

Besides the well known situation in generalized triangles (giving rise to Matthieu groups) and the above infinite family of one-point extensions of generalized quadrangles, there are only two sporadic examples of one-point extensions of finite generalized polygons known. The first one is the unique one-point extension of the unique generalized quadrangle of order 2, and the second one is a one-point extension of the split Cayley hexagon of order 2. The existence of these sporadic examples is due to the fact that the point sets of the corresponding polygons can be identified with the non-zero vectors of some vector space over $\mathsf{GF}(2)$, while the lines can be identified with some *special* 2-spaces. To obtain a one-point extension, one adds the zero vector and all translates of the special 2-spaces.

The one-point extension of the split Cayley hexagon $\mathsf{H}(2)$ of order 2 has been characterized by Cuypers using a combinatorial property. In the present

chapter we prove a geometric characterization and show that Cuypers' result is a consequence of it. Other consequences will also be proved. For instance, we will show that this one-point extension is the unique flag-transitive one of any hexagon of order 2.

More exactly, we will show that $\mathsf{H}(2)$ (respectively $\mathsf{H}(2)^D$) has a unique (respectively, no) one-point extension provided this extension, $\mathcal{S}$, satisfies the following *distance property*: for any three points x, y and z, the graph theoretic distance from y to z in the derived generalized hexagon $\mathcal{S}_x$ is the same as from x to z in $\mathcal{S}_y$. From this point on we will denote the distance in the derived geometry at x by d_x.

Furthermore we prove that $\mathsf{H}(2)$ is in fact the only finite generalized hexagon of classical order that is extendable under this assumption. In other words, we give a characterization of the one-point extension of $\mathsf{H}(2)$ and prove the following theorem.

Theorem 7.1.1.
There exists a unique one-point extension of a generalized hexagon of classical order (s,t) under the assumption that this extension satisfies the property

$$\forall x, y, z \ : \ d_x(y,z) = d_y(x,z).$$

The content of this chapter is work in progress jointly with H. Cuypers and H. Van Maldeghem.

7.2 Proof of the main result

The main goal of this section is to prove Theorem 7.1.1. We start with a, what will turn out to be very useful, lemma on the parameters of a generalized hexagon having the distance property. Next, we show that if there exists an extension of a generalized hexagon of classical order (s,t) that satisfies the distance property, then s has to be equal to 2. In Section 7.2.2 we prove Theorem 7.1.1. And finally, in Section 7.2.3 we prove that Cuypers' result is a consequence of Theorem 7.1.1 and show that the obtained extension of $\mathsf{H}(2)$ is the unique flag-transitive one of any hexagon of order 2.

7.2.1 Preliminary results

Lemma 7.2.1.
Suppose $\mathcal{S}$ is a one-point extension of a generalized hexagon of order (s,t) having the distance property. Then

(*a*) $s+2 \mid 2t(t+1)$.

(*b*) $t \geq s/2$.

(*c*) *s is even.*

Proof. Let us first remark that this lemma is trivially satisfied for s equal to 2. Thus, for the rest of the proof we may assume s to be greater than 2. Fix the points x, y and suppose $B_1, \dots B_{t+1}$ are the $t+1$ blocks on x, y. Let

$$X = X_{x,y} := \bigcup_{i=1}^{t+1} B_i.$$

Set $B_1 = \{x, y, b_0, b_1, \dots, b_{s-1}\}$ and $B_2 = \{x, y, c_0, c_1, \dots, c_{s-1}\}$. Inside $\mathcal{S}_{b_0}$ we see that the point c_i is at distance 4 from the points x and y and therefore collinear to some point of B_1 different from x, y, b_0. Assume that inside $\mathcal{S}_{b_0}$ the point c_1 is collinear to b_1. Then inside $\mathcal{S}_{c_1}$ the points b_0 and b_1 are collinear. Moreover, they are both at distance 3 from the line $B_2 \setminus \{c_1\}$. Hence, inside $\mathcal{S}_{c_1}$ the line through b_0 and b_1 meets $B_2 \setminus \{c_1\}$ in a point which we may assume to be c_0. So c_0, c_1, b_0, b_1 are contained in some block B of $\mathcal{S}$.

Fix a point $b \in B$ different from c_0, c_1, b_0, b_1. Inside $\mathcal{S}_z$, with $z = c_0, c_1, b_1$, we see that x is at distance 4 from b. So, in $\mathcal{S}_x$ we see that b is at distance 4 from c_0, c_1 and b_1. This implies that b is collinear to y inside $\mathcal{S}_x$. In particular, $b \in X$. As the point b was arbitrarily chosen in B, the block B is contained in X.

The above implies that b_0 and each point $a \in X \setminus B_1$ are in a block B, moreover, such block meets B_1 in two points. Inside $\mathcal{S}_{b_0}$ we see that there is a unique block on b_0 and a. But this implies that b_0 is in

$$1 + st/s = 1 + t$$

blocks contained in X. Counting incident point block pairs in X we obtain $(2 + s(t+1))(t+1) = \beta(2+s)$, where β is the number of blocks in X. This implies that $s+2 \mid (2+st+s)(t+1)$ from which we deduce that

$$s+2 \mid 2t(t+1).$$

This proves (*a*).

Now consider the point c_j, $j \geq 2$. In $\mathcal{S}_{c_0}$ we find the point c_j to be at distance 2 from c_1 and hence at distance 4 from b_0 and b_1. Thus inside $\mathcal{S}_{b_0}$ the point c_j is at distance 4 from c_0. On the other hand, it is collinear with some point

on the line through x and y. This implies that c_j is also collinear to b_1. In particular, all points on B_2 are collinear with b_1.

As above we find that each block on b_0 and b_1 meeting B_2 in a point meets it in two points. So, there are $1 + s/2$ such blocks proving (b) and (c).

□

As a direct consequence of this lemma we find that

Corollary 7.2.2.
If there exists a one-point extension of a generalized hexagon of classical order (s, t), with $s \geq 2$, that satisfies the distance property, then $s = 2$.

Proof. By (b) of the above lemma $t \neq \sqrt[3]{s}$. If $t = s$, then by (a) we have $s + 2 \mid 4$ and $s = 2$. If $t = s^3$, then (a) implies $s + 2 \mid 2^4 \cdot 7$. Using that s is a power of 2 yields the corollary.

□

In other words, by Theorem 2.2.1 there are only three finite classical generalized hexagons which hypothetically can have a one-point extension satisfying the distance property, namely the split Cayley hexagon $\mathsf{H}(2)$, its dual and the dual of twisted triality hexagon $\mathsf{T}(8, 2)$.

7.2.2 Proof

Say Γ is a generalized hexagon of order $(2, t)$. We will now construct a unique extension, $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$, of Γ only using the distance property. First of all, we define $\mathcal{P}$ as the point set of Γ union a new point α. Wanting $\mathcal{S}_\alpha$ to be isomorphic to Γ yields that $\{\alpha, x, y, z\}$ is a block of $\mathcal{S}$ for all sets $\{x, y, z\}$ that are the point sets of a line of Γ.

Now suppose x, y and u are three points of a block, with $d_\alpha(x, y) = 2$ and u a point off the line xy. Inside $\mathcal{S}_x$ we immediately obtain $d_x(\alpha, u) = 4$ and therefore $d(u, x) = 4$ (here and from now on the distance function in $\mathcal{S}_\alpha$ will be denoted by d). Similarly $d(u, y) = 4$ and hence u is collinear in Γ to the unique third point z on the line xy. The fourth point v of the block containing x, y and u can now either be the third point on the line uz of Γ or be incident with the third line in Γ through z. In the latter case, looking in $\mathcal{S}_u$ leads to

$$d_u(v, \alpha) = d(u, v) = 4,$$

$$d_u(x, \alpha) = d(u, x) = 4,$$

$$d_u(y, \alpha) = d(u, y) = 4.$$

This, together with the fact that x, y and v are collinear points in $\mathcal{S}_u$, yields a contradiction.

> In other words, the four points of the symmetric difference of any two intersecting lines of Γ are the points of a block in $\mathcal{S}$. Such a block will be referred to as a *Vee-block*.

An easy counting now shows that this type of blocks together with the first type covers all pairs of collinear points in $\mathcal{S}_\alpha$.

We now prove the following lemma.

Lemma 7.2.3.
If $B = \{a, b, c, d\}$ is a block of $\mathcal{S}$ and $d(a, b) = 6$, then one of the two points c or d is at distance 4 from a and opposite b, while the other is opposite a and at distance 4 from b (all distances in $\mathcal{S}_\alpha$).

Proof. From $d(a, b) = 6$ we know that $d_a(\alpha, b) = 6$ and thus find in $\mathcal{S}_a$ that either c or d is the closest point to α on the line bcd. Without loss of generality we may assume that $d_a(\alpha, c) = 4$ and consequently also that $d_a(\alpha, d) = 6$. An easy application of the distance property immediately leads to the first part of the lemma. In a similar way $d(a, b) = 6$ tells us that $d(b, r) = 4$ with $r \in \{c, d\}$. Suppose r is equal to the point c, which is then at distance 4 from both a and b. We have one of two situations: first, c can be on the line L in the path $(a, M, a_1, L, a_2, N, b)$ connecting a to b; secondly, c can be the third point of the point regulus through a_2 and a_5, with $a = a_0, a_1, a_2, a_3 = b, a_4, a_5, a_6 = a$ the points of an ordinary hexagon through a and b (see Figure 7.1 for clarification).

However, both of these cases lead to a contradiction in $\mathcal{S}_c$ as we will show.

Suppose c is the point on L distinct from a_1 and a_2 and denote the third point on M (respectively N) by e (respectively f). We successively find adb, bfa_1, $a_1\alpha a_2$, a_2ea as lines in $\mathcal{S}_c$ and consequently obtain a quadrangle in a generalized hexagon, a contradiction.

Suppose c is the third point in $\mathcal{R}(a_2, a_5)$. Call P and Q the lines of Γ incident with c concurrent with aa_1 and ba_4, respectively. Denote by p (respectively q) the point on P (respectively Q) distinct from c and not on aa_1 (respectively ba_4). Similar to the previous situation we now obtain a pentagon in $\mathcal{S}_c$ (ab, ba_4q, $q\alpha$, αp, pa_1a), which again is in contradiction with the assumption that $\mathcal{S}_c$ is a generalized hexagon.

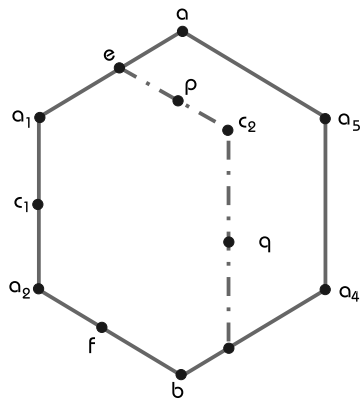

Figure 7.1

Therefore r is equal to d and we are done.

□

Note that it is now easy to show that c and d from the previous lemma are opposite points in Γ (look for instance in $\mathcal{S}_c$ and apply the distance property).

Suppose that $B = \{a, b, c, d\}$ is a block of $\mathcal{S}$, with a and b opposite points in Γ. We will now determine the relative position of c (and d) to $a = a_0$ and $b = a_3$ under the assumption that c (respectively d) is at distance 4 from a (respectively b).

Denote the projection of b onto the line az (with $z = a \bowtie c$) by a_1 and $b \bowtie a_1$ by a_2. Finally, denote the third point on any line $a_i a_j$ by a_{ij}, for i and j elements of $\{0, 1, 2, 3\}$.

With this notation, let us first assume that z differs from a_1 (as in Figure 7.2) and call p_{cz} the third point on the line cz (from now on we will always use this convention).

Looking in $\mathcal{S}_b$ we will show that this situation can never occur. First of all, we find $a_1 a_{12} a_{23}$ as a line in $\mathcal{S}_b$ corresponding to a Vee-block through b. On this line the point a_{23} is collinear to α, as it is collinear to b in Γ. On the other hand, we defined B as a block of $\mathcal{S}$ and thus find a, c and d also to be the points of a line in $\mathcal{S}_b$. We now claim that the point d is collinear to a_1 in $\mathcal{S}_b$. Indeed, in $\mathcal{S}_a$ the point c is both collinear to b and d (by definition of B) and to a_1 and p_{cz} (Vee-block). Therefore we find $d_a(b, a_1) = 4$ and thus $d_b(a_1, a) = 4$. In a similar way (by using the distance property in $\mathcal{S}_c$) one can find that $d_b(a_1, c) = 4$, which leads to the claim.

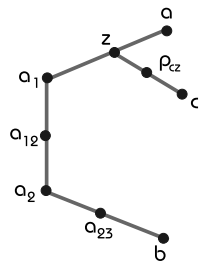

Figure 7.2

Now, as $d(b,d) = 4$, we obtain $d_b(d,\alpha) = 4$. This together with the fact that d is collinear to a_1 in $\mathcal{S}_b$, implies that d has to be the point a_{12}, contradicting the fact that d is opposite a in Γ.

In other words, c has to be collinear to the projection from b onto a certain line through a.

Before starting to determine the actual structure of such a block we first show that, next to B being a block, a point p collinear to a_1 distinct of c can never be in a block with a and b.

Indeed, if not, then inside $\mathcal{S}_a$ we obtain a triangle pbc when p is on the line a_1c or a quadrangle $bca_{01}p$ if p is on another line (distinct of aa_1, a_1c and a_1a_2) through a_1 (note that this situation only occurs when $t > 2$). This simple result implies that for every one of the paths from a to b there exists a block containing these two points and a point with the same relative position to a and b as c. In other words, for every path

$$a \ \mathtt{I} \ L \ \mathtt{I} \ a_1 \ \mathtt{I} \ M \ \mathtt{I} \ a_2 \ \mathtt{I} \ N \ \mathtt{I} \ b$$

one of the points collinear to a_1, at distance 4 from a and opposite b, is in a block with a and b.

We now put this path $(a_0, \ldots, a_3)$ into an ordinary hexagon $(a_0, \ldots, a_6 = a_0)$ and suppose that d is, in the same way as c is to a_1, collinear to a_4. As we mentioned above, c and d are opposite points. We now have one of three situations (exhibited as (c_i, d_i), $i = 1, 2, 3$, in Figure 7.3) each of which will be shown to be contradictory: first of all, p_{ca_1} can be collinear to p_{da_4} (this is the only possibility for $t = 2$); secondly, p_{ca_1} can be at distance 4 from a_4 as is a_1 to d; and finally, a_1 and a_4 are at distance 4 from d and c, respectively.

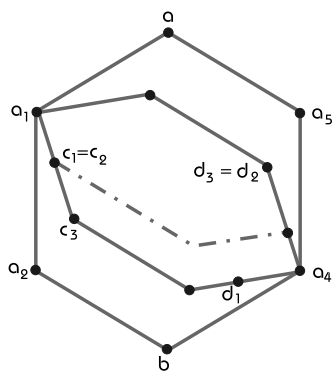

Figure 7.3

In the first case, we look inside $\mathcal{S}_a$ and obtain either a triangle, a quadrangle or a pentagon in this derived hexagon, as we will show. First of all bcd is a line of $\mathcal{S}_a$. Furthermore we have that a, d and a_5' (a point collinear to a_5) belong to a block of this last type. As do a, b and a_5'' (collinear to a_5). Now, if a_5' equals a_5'' we get a triangle; if they are collinear we obtain a quadrangle and otherwise we obtain a pentagon where in both latter cases we use the fact that every point that is collinear to a_5 is in a Vee block with a and a_{05}.

Both the second and third situation do not occur when $t = 2$. Nevertheless, when $t > 2$ these situations as well as the previous one lead to the following contradictions.

If p_{ca_1} projects onto a_4 as does a_1 onto d, we obtain a pentagon inside $\mathcal{S}_d$. First of all, the point b is in a Vee-block with d and p_{da_4}, hence inside $\mathcal{S}_d$ these two points, b and p_{da_4}, are collinear. As p_{ca_1} projects onto a_4 and a_1 projects onto d, the point c has to project onto the point p_{da_4}. Therefore c is in a block with d and a point x collinear to p_{da_4}. In other words, c is collinear to x in $\mathcal{S}_d$ and as x is also in a Vee-block with a_4 we obtain a pentagon $(bp_{da_4}a_4xc)$ inside what ought to be a generalized hexagon, a contradiction.

Finally, when a_1 and a_4 project onto d and c, respectively, there exists a quadrangle inside $\mathcal{S}_d$, a contradiction as we will show. Just as in the latter case b is collinear to p_{da_4} in this particular derived hexagon. However, since there exists a path from c to d passing through the point a_4, the point c is in a block with d and a point collinear to a_4. This last point is also in a Vee-block with p_{da_4}, hence obtaining a quadrangle.

> Conclusion: if B is a block of $\mathcal{S}$, then c has to be collinear to a_1 and d has to be collinear to a_2 and this for some path $a \perp a_1 \perp a_2 \perp b$ from a to b.

For Γ isomorphic to $\mathsf{H}(2)$ we will show that c can not be on the ideal line through a and a_2, with notations as described above. On the other hand, for Γ isomorphic to the dual of $\mathsf{H}(2)$, or later on also for Γ isomorphic to $\mathsf{T}(2,8)$, the following will lead to a contradiction and we will be able to conclude that these geometries are non-extendable under the given assumption.

Let us first assume t to be equal to 2 and suppose, by way of contradiction, that the point c does belong to the ideal line aa_2 and denote $\{a, b, c, d\}$ by B. Let $a_1 \perp a_2 \perp d \perp x \perp y \perp a \perp a_1$ be the points of an ordinary hexagon (where $\perp$ denotes collinearity). The points x and y depend on the choice of line through a. As c is on an ideal line with a and a_2, every such a line L through a determines a line regulus with a_2d with as third line, N, a line incident with c. Call z the point on N that is collinear to x (as in Figure 7.4).

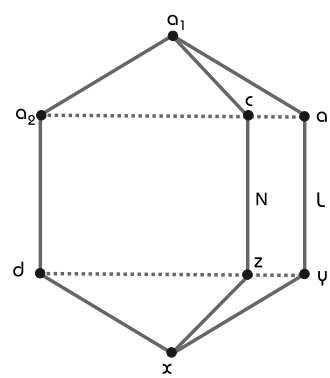

Figure 7.4

Within $\mathcal{S}_d$ we obtain the following path

$$abc \ \mathtt{I} \ a \ \mathtt{I} \ ay'(z/p_{xz}) \ \mathtt{I} \ z/p_{xz} \ \mathtt{I} \ (z/p_{xz})(p_{xz}/z)p_{dx} \ \mathtt{I} \ p_{dx} \ \mathtt{I} \ p_{dx}yp_{xy}$$

where y' is a point collinear to y and the first and second line of this path are of the-to-be-described type, while the third and fourth correspond to Vee-blocks. Now, as c and d also define a block with either y or p_{xy} we obtain a pentagon in $\mathcal{S}_d$, a contradiction and we are done.

To complete the construction of the extension of Γ we used the regulus $\mathcal{R}(L, a_2d)$ to exclude the point on N collinear to a_1 from this type of blocks. However, in $\mathsf{H}(2)^D$, opposed to the situation in $\mathsf{H}(2)$, that point depends on the choice of L and therefore none of the points collinear to a_1 can be in a block with a and b. In other words there does not exist a one-point extension of $\mathsf{H}(2)^D$ under the assumption of the distance property, while in $\mathsf{H}(2)$ the extension is unique.

When dealing with $\mathsf{T}(2,8)$ we have to be a bit more careful. Let $\{a, d, x', y'\}$ be a block of the extension, where x' and y' are points collinear to x and y, respectively. Applying the exact same technique as above we rule out the point m_0 collinear to a_1 that is collinear to x' (if $d(a_1, x') = 4$) or is at distance 4 from x' (if $d(a_1, x') = 6$). Now, consider m_i a point collinear to a_1 and incident with one of the regulus lines (of $\mathcal{R}(a_2d, ay)$) distinct of the one through m_0 (see Figure 7.5) and suppose this point is in a block with a, b and d.

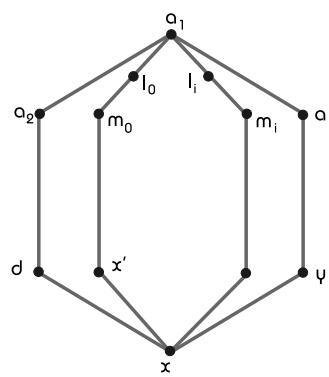

Figure 7.5

Inside $\mathcal{S}_d$ we then obtain an n-gon, with $n \leq 5$

$$abm_i \ \mathtt{I} \ a \ \mathtt{I} \ ax' \ \mathtt{I} \ x' \ \mathtt{I} \ p_{dx}x' \ \mathtt{I} \ p_{dx} \ \mathtt{I} \ p_{dx}x''$$

where x'' is in a block of the yet to be described type with d and m_i. Hence none of the points m_i, $i = 0, \ldots, 6$, collinear to a_1 and on a line of $\mathcal{R}(a_2d, ay) \setminus \{a_2d, ay\}$, can be in such a block. However, for every point $l_i \neq a_1, m_i$, $i = 0, \ldots, 6$, on a_1m_i there exists a unique subhexagon of order 2 in $\mathsf{T}(2,8)$ (which consequently is isomorphic to $\mathsf{H}(2)^D$) containing l_i in addition to that

fixed ordinary hexagon. Meaning, there is a unique third line M through a (namely within that subhexagon) that determines a line regulus with a_2d having a regulus line incident with l_i.

Summarizing all of this, we can exclude all points collinear to a_1 from a block through a and b, which is in contradiction with previous findings and we are done.

7.2.3 Some consequences

Lemma 7.2.4.
If a one-point extension $\mathcal{S}$ of a generalized hexagon Γ of order $(2,t)$ contains all Vee-blocks in every one of its derivations, then it satisfies the distance property.

Proof. To prove the lemma we have to show that for all x, y, z in $\mathcal{S}$

$$d_x(y,z) = d_y(x,z).$$

Without loss of generality, we may assume x to be the point α of $\mathcal{S}$. It now suffices to prove that if y and z have distance $d \in \{2,4\}$ in Γ, then α and z have distance d in $\mathcal{S}_y$ (and consequently $d = 6$ follows). First, suppose y and z are collinear points of Γ. Then α, y and z are in a line block of $\mathcal{S}$ and hence α and z are collinear in $\mathcal{S}_y$. If, on the other hand, y and z are at distance 4 from one another, then they belong to a unique Vee-block $yuzv$ of $\mathcal{S}$. Within the derived generalized hexagon $\mathcal{S}_y$ we now have a path

$$z \ \mathtt{I} \ zuv \ \mathtt{I} \ u \ \mathtt{I} \ u\alpha \ \mathtt{I} \ \alpha$$

of length 4 from z to α and we are done.

□

The following theorem, which is due to H. Cuypers, is an immediate consequence of Theorem 7.1.1.

Theorem 7.2.5.
There exists a unique one-point extension of the split Cayley hexagon of order 2 under the assumption that this extension satisfies the following block property: for every two blocks B and B' with $|B \cap B'| = 2$ the set $(B \cup B') \setminus (B \cap B')$ is the point set of a block of the extension.

Proof. Indeed, starting with this block property we can deduce that, considering two intersecting lines in the hexagon, such an extension has to contain all Vee-blocks, as defined above. However, by Lemma 7.2.4 such a one-point extension consequently satisfies the distance property and hence Theorem 7.2.5 is a direct result from Theorem 7.1.1.

□

Another consequence is the following result.

Theorem 7.2.6.
There exists a unique flag-transitive one-point extension of a hexagon of order 2*, which is also the unique doubly transitive one-point extension of such a hexagon.*

Proof. A flag is here an incident point-block pair. To prove this theorem one shows that every derived hexagon Γ contains all Vee-blocks. This goes as follows. First we remark that every point- or line-transitive group of $\mathsf{H}(2)$ or its dual contains the simple group $\mathsf{G}_2'(2)$. Then it is a tedious exercise to show that there is only one orbit of size ≤ 2 of the stabilizer of a pair of collinear points x, y in Γ on the pairs of remaining points not collinear with x nor y. By way of example we treat some of the to be considered situations and note that the remaining cases can be done in a similar way. Let x, y, u, v be the points of a block B of $\mathcal{S}$. First of all, if both u and v are at distance 4 from both x and y, but are non-collinear, then an axial elation g with as axis any line on u, that is not incident with $x \bowtie u$, maps B onto a distinct block B' that shares three points – x or y, u and v – with B, a contradiction. In the same way, when u is at distance 4 from both x and y and v is opposite x or y, we consider an axial elation with axis xy. This elation fixes the points x, y and u and does not fix the point v. Hence we again obtain a block that intersects B in three of its points. Now suppose u and v are collinear points that are opposite x, y and denote by p_{ab} the unique third point on the line ab, for collinear points a, b. With this notation we have that p_{xy} is collinear to p_{uv}. Let g and g' be axial elations with axes xy and L, with $xy \neq L \mathrel{\mathtt{I}} x$, respectively. Then both B^g and $B^{g'}$ differ from B, are blocks of $\mathcal{S}$ and have x, y as two of their points. Hence we obtain, besides the line block on x, y, at least three blocks on x, y, a contradiction. Next to these three particular cases there are a lot of other situations to consider, but every one of these, besides the one corresponding to the Vee-blocks on x, y, can be excluded using methods similar to the previous ones. Since there are at most three blocks through x and y, one of which determined by the line xy of Γ, these orbits give rise to two Vee-blocks, and we are done by Lemma 7.2.4.

□

Known construction of the one-point extension of $\mathsf{H}(2)$

In this subsection we link the know construction of the extension of $\mathsf{H}(2)$ to the one described in previous section.

When projecting from the nucleus n of $\mathsf{Q}(6,2)$ onto a hyperplane Π, not containing this point, one obtains the perfect symplectic representation Γ_Π of $\mathsf{H}(2)$ in $\mathsf{PG}(5,2)$. From this representation we can go to a representation in the affine space $\mathsf{AG}(6,2)$ by fixing a point o and mapping each point X of Π to the unique affine point, say x, on the line Xo distinct of o. The points of a line of Γ_Π will as such be mapped onto an affine plane through o. The generalized hexagon $\mathsf{H}(2)$ can thus be seen as the incidence geometry, Γ, with as points all points of $\mathsf{AG}(6,2) \setminus \{o\}$ and as lines the planes through o intersecting Π in a line of Γ_Π.

The one-point extension of Γ contains all points of $\mathsf{AG}(6,2)$, we thus add o to the point set of Γ, and all translations in $\mathsf{AG}(6,2)$ of the lines of Γ (i.e. planes of $\mathsf{AG}(6,2)$).

The point o corresponds to our point α and the first type of blocks are the planes of Γ to which one adds o or in other words the lines of $\mathsf{H}(2)$ to which one adds α. The second and third type of our construction correspond to different types of translations of a plane in $\mathsf{AG}(6,2)$ containing o. Indeed, suppose the plane $\{o,a,b,c\}$ is mapped onto the plane $\{o',a',b',c'\}$ (in the obvious way). These two planes span a 3-space Υ, that intersects Π in a plane β. The hyperplane Π contains three types of planes: first of all, those that contain no lines of Γ_Π; secondly, those that contain three lines through a point of Γ_Π and finally, those that contain a unique line of Γ_Π. As we started from a line $\{o,a,b,c\}$ in Γ the first type of plane can not occur as the projection of that 3-space. The second type of planes translates into the Vee-blocks, while the last type corresponds to the final type of blocks, as we will show. Let us start with the situation where β contains 3 lines through a point. Without loss of generality we may assume this special point to be the point A, which is on the line ABC of Γ_Π. Looking at the planes through o and a we find $AA'O'$ and $AB'C'$ as the other two lines through A. In other words, the points of $\{o',a',b',c'\}$ correspond to four points of $\mathsf{H}(2)$ which are the points of the disjoint union of two concurrent lines.

Suppose β contains a unique line of Γ_Π. Take any point X off ABC in β. This point is by definition of a generalized hexagon closest to a unique point of that line ABC. Again, without loss of generality we may assume this point to be the point A. Take $Y \in \{B,C\}$, then apart from X and Y also the third point, say Z, of $\mathcal{R}(X,Y)$ is a point of β (as the points of a regulus

of $\mathsf{H}(2)$ are projected onto a line of Π). As a result, both AX and AZ are ideal lines of Γ_Π and thus O', A', B', C' have the same relative position to one another as the points of a blocks of $\mathcal{S}$ of the final type.

7.3 Conclusion

It seems like one-point extensions of generalized polygons are rare objects. Within the known generalized quadrangles these objects have been classified to be the Hölz designs and a unique one-point extension of the unique generalized quadrangle of order 2 (see [6]). In the current chapter we showed that – under the distance property assumption – there is also a unique one-point extension of $\mathsf{H}(2)$ and that this is moreover the only known finite generalized hexagon to have such a one-point extension. For the moment, we are investigating possible one-point extensions of the known finite generalized octagons that satisfy this distance property and our first findings tend to a negative answer to the existence question of these objects.

A Nederlandstalige samenvatting

> Het doel van deze appendix is om een overzicht te geven van de behaalde resultaten uit deze thesis. Daar waar we in de eerste twee hoofdstukken van de Engelstalige versie uitvoerig terugkomen op een reeks gekende definities en notaties, zullen we hier slechts een opsomming geven van de voor ons belangrijkste begrippen. Verder doorlopen we de hoofdstukken in chronologische volgorde en geven we een voorstelling van de belangrijkste constructies, resultaten en stellingen die – afhankelijk van hun complexiteit – al dan niet gevolgd worden door een summiere schets van een bewijs.

Tenslotte merken we op dat we, om ludieke benamingen te vermijden, sommige wiskundige termen uit het Engels overnemen.

A.1 Inleiding

Voor meer informatie over *grafen*, *codes*, *designs* en *projectieve ruimten over eindige velden* (met onder meer definities van *polariteiten*, *kwadrieken en Hermitische variëteiten*, *unitalen, ovalen en hyperovalen*, *Grassmanncoördinaten* en *twee-karakterverzamelingen*) verwijzen we naar het eerste hoofdstuk van deze thesis.

A.1.1 Meetkundes

In dit werk zullen we vooral geïnteresseerd zijn in rang 2 meetkundes die als volgt gedefinieerd kunnen worden.

> Een *punt-rechte incidentiemeetkunde* is een drietal $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, bestaande uit een niet-ledige puntenverzameling $\mathcal{P}$, een niet-ledige rechtenverzameling $\mathcal{L}$ en een incidentierelatie $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L}$, waarbij elk element incident is met tenminste één ander element.

Verder veronderstellen we tevens dat de meetkunde *samenhangend* is, i.e. voor elke twee elementen bestaat er een ketting van opeenvolgende incidente elementen die het eerste met het tweede verbindt.

> Het *incidentiegraaf* van een meetkunde Γ is het graaf met als toppenverzameling $V = \mathcal{P} \cup \mathcal{L}$ en waarbij de vlaggen van de meetkunde de bogen definiëren.

De *afstand* tussen twee elementen van Γ wordt gemeten in het corresponderend incidentiegraaf.

A.1.2 Veralgemeende veelhoeken

Zoals de naam reeds doet vermoeden is het begrip *veralgemeende veelhoek* afkomstig van de gewone veelhoeken.

Een *gewone n-hoek* is hierbij een samenhangende meetkunde bestaande uit n punten en n rechten, zodanig dat elk punt incident is met juist twee rechten en vice versa.

> Een *veralgemeende n-hoek*, $n \geq 2$, is een meetkunde Γ bestaande uit een puntenverzameling $\mathcal{P}$, een rechtenverzameling $\mathcal{L}$ en een incidentierelatie $\mathtt{I}$, waarbij aan de volgende axioma's voldaan is:
>
> (*i*) in de meetkunde Γ zijn geen gewone k-hoeken, met $k < n$, te vinden,
> (*ii*) door elke twee elementen is steeds een gewone n-hoek te vinden,
> (*iii*) elke rechte en elk punt is incident met ten minste 3 elementen.
>
> Axioma (*iii*) kan men ook vervangen door het volgende axioma:
>
> (*iii*)' er is ergens in de meetkunde een gewone $(n+1)$-hoek te vinden.

Aan de hand van deze axioma's kan men duidelijk zien dat elke twee elementen van Γ hoogstens op afstand n van elkaar verwijderd zijn. In dat geval zeggen we dat dit *diametraal tegenovergestelde elementen* zijn.

Verder kan men aantonen dat elke rechte een constant aantal punten bevat, stel $s+1$, en dat er door elk punt een constant aantal rechten gaat, stel $t+1$. We zeggen dan dat Γ *orde* (s,t) heeft.

In het verdere verloop van deze thesis komen vooral de veralgemeende vierhoeken en zeshoeken aan bod. Op deze laatste komen we in de volgende sectie uitvoerig terug. Voor veralgemeende vierhoeken hebben we de volgende equivalente definitie.

> Een *orde* (s,t) *veralgemeende vierhoek* Γ, met $s,t \geq 1$, is een punt-rechte incidentiestructuur met $s+1$ punten op een rechte en $t+1$ rechten door een punt, zodanig dat voor elke anti-vlag (p,L) een unieke vlag (q,M) bestaat waarvoor $p \mathtt{\ I\ } M \mathtt{\ I\ } q \mathtt{\ I\ } L$.

A.2 Klassieke veralgemeende zeshoeken

Op dualiteiten na zijn er slechts twee eindige veralgemeende zeshoeken gekend: de *split Cayley veralgemeende zeshoek* $\mathsf{H}(q)$ van orde q en de *twisted triality veralgemeende zeshoek* $\mathsf{T}(q^3,q)$ van orde (q^3,q). Deze twee klassieke veralgemeende zeshoeken werden het eerst vernoemd in een artikel van J. Tits [64]. We geven hier enkel Tits' constructie van $\mathsf{H}(q)$, voor deze van $\mathsf{T}(q^3,q)$ verwijzen we naar Hoofdstuk 2.

Tits' constructie van $\mathsf{H}(q)$

Coördinatizeer $\mathsf{PG}(6,q)$ op een zodanige manier dat de parabolische kwadriek $\mathsf{Q}(6,q)$ de volgende standaardvergelijking heeft

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$$

en definieer de punten van $\mathsf{H}(q)$ als deze van $\mathsf{Q}(6,q)$. De rechten van de zeshoek zijn precies die rechten van $\mathsf{Q}(6,q)$ wiens Grassmanncoördinaten voldoen aan

$$p_{12} = p_{34} \qquad p_{54} = p_{32} \qquad p_{20} = p_{35}$$

$$p_{65} = p_{30} \qquad p_{01} = p_{36} \qquad p_{46} = p_{31}$$

en omgekeerd, elke rechte van $\mathsf{Q}(6,q)$ die hieraan voldoet is een rechte van $\mathsf{H}(q)$. Deze constructie is gekend als de *standaard inbedding* van $\mathsf{H}(q)$.

Voor even q bestaat er een representatie van $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, de zogenaamde *perfecte symplectische inbedding*. Deze inbedding bekomt men na projectie vanuit de kern van $\mathsf{Q}(6,q)$ op het hypervlak $X_3 = 0$.

In 1985 bewezen A.M. Cohen en J. Tits de volgende klassificatiestelling.

Stelling A.2.1 (Cohen, Tits (10)**).**
Een eindige veralgemeende zeshoek van orde (s,t) met $s = 2$ is noodzakelijk isomorf met één van de klassieke zeshoeken $\mathsf{H}(2)$, $\mathsf{H}(2)^D$ of $\mathsf{T}(2,8)$.

Verder zullen we bij het beschouwen van $\mathsf{H}(q)$ zien dat het geval $q = 3^h$ een speciale rol speelt. De verklaring hiervoor ligt in de volgende ongepubliceerde stelling van J. Tits.

Stelling A.2.2 (Corollary 3.5.7 (67)**).**
De veralgemeende zeshoek $\mathsf{H}(q)$ is zelf-duaal als en slechts als $q = 3^h$ en is zelf-polair als en slechts als $q = 3^{2e+1}$.

A.2.1 Coördinatizering

In [67] beschrijft H. Van Maldeghem een algemene methode om veralgemeende veelhoeken te coördinatizeren. De coördinatizering van veralgemeende zeshoeken, zoals beschreven in [16], wordt uitvoerig behandeld in Sectie 2.3 van Hoofdstuk 2.

A.2.2 Notaties en terminologie

> Een puntenregulus van $\mathsf{H}(q)$ is een verzameling van $q+1$ punten die allemaal op afstand 3 van twee gegeven diametraal tegenovergestelde rechten gelegen zijn. Duaal definiëren we een rechtenregulus van $\mathsf{H}(q)$.

Merk op. Zowel een punten-, als een rechtenregulus is bepaald door twee van diens elementen. In dit opzicht noteren we $\mathcal{R}(x,y)$ en $\mathcal{R}(L,M)$ als de respectieve punten- en rechtenregulus door x,y en L,M.

Als $\mathcal{H}$ een hypervlak is van $\mathsf{PG}(6,q)$ dan hebben we één van de volgende situaties.

(*i*) $\mathcal{H}$ is een *raakhypervlak* en bevat alle punten van $\mathsf{H}(q)$ die niet diametraal tegenovergesteld zijn aan een zeker punt x.

(*ii*) $\mathcal{H}$ is een *hyperbolisch hypervlak* en bevat de rechten van een dunne deelzeshoek van orde $(1, q)$ van $\mathsf{H}(q)$.

(*iii*) $\mathcal{H}$ is een *elliptisch hypervlak* en bevat de rechten van een Hermitische spread van $\mathsf{H}(q)$.

A.2.3 Afstands-j ovoïdes en afstands-j spreads

Een *afstands*-2 *ovoïde* van $\mathsf{H}(q)$ is een puntenverzameling $\mathcal{O}_2$ met de eigenschap dat elke rechte van $\mathsf{H}(q)$ een uniek punt van $\mathcal{O}_2$ bevat. Duaal definieert men een *afstands*-2 *spread* van $\mathsf{H}(q)$.

Een *ovoïde*, kort voor *afstands*-3 *ovoïde*, van $\mathsf{H}(q)$ is een verzameling van $q^3 + 1$ diametraal tegenovergestelde punten. Duaal definieert men een *spread* of *afstands*-3 *spread* van $\mathsf{H}(q)$.

In Sectie 2.6 van Hoofdstuk 2 wordt verdere informatie gegeven over de Hermitische spread en ovoïde, over het verband tussen deze spread en de Hermitische unitaal in $\mathsf{PG}(2, q^2)$, over de Ree-Tits spread en ovoïde en definiëren we de ovoïdale deelruimten zoals ze door L. Brouns en H. Van Maldeghem in [4] werden ingevoerd.

A.2.4 De onderliggende kwadriek $\mathsf{Q}(6, q)$

Algemeen zullen we werken met de standaard inbedding van $\mathsf{H}(q)$. Om verwarring te vermijden tussen de rechten van $\mathsf{H}(q)$ en die van $\mathsf{Q}(6, q)$ spreken we, daar waar nodig, over *zeshoeksrechten* voor rechten van $\mathsf{H}(q)$ en over *ideale rechten* voor rechten van de kwadriek die niet tot de rechtenverzameling van $\mathsf{H}(q)$ behoren.

Op dezelfde manier noemen we een vlak van $\mathsf{Q}(6, q)$ dat $q+1$ zeshoeksrechten door een punt x bevat een *zeshoeksvlak*, dat we noteren als Π_x, en een vlak zonder zeshoeksrechten een *ideaal vlak*.

Voor enkele eenvoudige relaties tussen objecten van $\mathsf{H}(q)$ en objecten van $\mathsf{Q}(6, q)$ verwijzen we opnieuw naar Hoofdstuk 2.

A.2.5 Groepen

Neem γ een pad van lengte 4 in Γ, een veralgemeende zeshoek.

> Als een collineatie g van Γ alle elementen incident met tenminste één element van γ fixeert, dan noemen we g een *wortelelatie*, γ-*elatie* of kort *elatie* van Γ.

Aangezien zo een pad ofwel met een punt, ofwel met een rechte kan beginnen en eindigen, maken we een onderscheid door de eerste soort *punt-elatie* en de tweede soort *rechte-elatie* te noemen. Het *centrum* van een punt-elatie is dan het unieke punt in het midden van γ.

Voor enkele expliciete beschrijvingen van automorfismen van $\mathsf{H}(q)$ verwijzen we naar Hoofdstuk 2.

Tenslotte besluiten we deze sectie met een verwijzing naar Tabel 4.1 en Tabel 5.1 waarin alle maximale deelgroepen van $\mathsf{G}_2(3)$ en $\mathsf{G}_2(4)$, de respectieve automorfismegroepen van $\mathsf{H}(3)$ en $\mathsf{H}(4)$, terug te vinden zijn.

A.3 Gemeenschappelijke puntenreguli in twee $\mathsf{H}(q)$'s op $\mathsf{Q}(6,q)$

In Hoofdstuk 3 beschouwen we twee split Cayley veralgemeende zeshoeken, beide gerepresenteerd op dezelfde parabolische kwadriek $\mathsf{Q}(6,q)$ en bepalen we hun gemeenschappelijke puntenreguli. Als toepassing geven we een alternatieve constructie van de Hölz designs $\mathsf{D}_{\mathsf{Hölz}}(q)$ voor $q \not\equiv 2 \bmod 3$. Als $q \equiv 2 \bmod 3$, dan levert onze constructie een $2-(q^3+1, q+1, \frac{q+4}{3})$ deeldesign van de Hölz design op. Verder bewijzen we in dit hoofdstuk theoretisch wat V. Tonchev aan de hand van de computer aantoonde, namelijk dat de Hölz design op 28 punten enkel de Hermitische en de Ree unitaal als unitalen bevat.

A.3.1 Inleiding

In 2002 bewezen E. Govaert en H. Van Maldeghem, steunend op een resultaat uit [4], dat de gemeenschappelijke rechtenverzameling van twee zeshoeken op dezelfde $\mathsf{Q}(6,q)$ gegeven wordt door een verzameling rechten op afstand 3 van een vast punt (type 0), door de rechtenverzameling van een dunne deelzeshoek

van orde $(1,q)$ (type $+$) of door de rechtenverzameling van een Hermitische spread (type $-$).
Geïnspireerd door dit resultaat bekijken we de gemeenschappelijke puntenreguli van twee zulke zeshoeken en bewijzen we – gegeven, H_1 en H_2, twee modellen van $\mathsf{H}(q)$ isomorf met de standaard inbedding van $\mathsf{H}(q)$ op $\mathsf{Q}(6,q)$ en met S en Ω de respectieve gemeenschappelijke rechten- en puntenreguliverzameling van H_1 en H_2 – de volgende resultaten.

Stelling A.3.1.
Als q even is, dan hebben H_1 en H_2 alle puntenreguli gemeen. Voor oneven q hebben we één van de volgende situaties:

(i) S heeft type 0 en $|\Omega| = q^3$,
(ii) S heeft type $+$ en $|\Omega| = q^2(q^2+q+1)$ of $|\Omega| = q^3(q^2+q+1)$ of
(iii) S heeft type $-$ en $|\Omega| = q^2(q^2-q+1)$ of $|\Omega| = q^2(q^2-q+1)(q+2)$.

Bovendien bestaat er, gegeven een vaste H_1 en S, zowel in situatie (ii) als in situatie (iii), een unieke H_2 waarvoor de kardinaliteit van Ω maximaal is.

Stelling A.3.2.
Definieer $\Gamma = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ als volgt: $\mathcal{P}$ bestaat uit alle rechten in S; $\mathcal{B}$ bestaat enerzijds uit alle rechtenreguli die volledig in S bevat zijn en anderzijds uit de niet-ledige verzamelingen van rechten van S die incident zijn met een element van Ω. Voor elke H_1 bestaat er een geschikte H_2 zodanig dat voor $q \not\equiv 2 \bmod 3$, $\Gamma \cong \mathsf{D}_{\mathsf{Hölz}}(q)$ en voor $q \equiv 2 \bmod 3$, Γ een $2-(q^3+1, q+1, \frac{q+4}{3})$ deeldesign van $\mathsf{D}_{\mathsf{Hölz}}(q)$ is, met Γ invariant onder $\mathsf{PSU}_3(q)$ met een natuurlijke werking op $\mathsf{D}_{\mathsf{Hölz}}(q)$.

En tenslotte hebben we het volgende computerresultaat van V.D. Tonchev met de hand bewezen.

Stelling A.3.3.
De enige unitalen van de $2-(28,4,5)$ Hölz design zijn de Hermitische en de Ree unitaal.

A.3.2 Gemeenschappelijke puntenreguli

Eerst en vooral is het geval q even van Stelling A.3.1 eenvoudig af te leiden uit de perfecte symplectische representatie van $\mathsf{H}(q)$.
Voor het oneven geval, daarentegen, tonen we eerst volgend lemma aan.

Lemma A.3.4.
Als $\mathcal{R}$ een element van Ω is, dan

(*a*) *behoort het complement van $\mathcal{R}$ tot S of*

(*b*) *is het complement van $\mathcal{R}$ in H_1 diametraal tegenovergesteld aan het complement van $\mathcal{R}$ in H_2. Bovendien is elk punt van $\mathcal{R}$ incident met een unieke rechte van $\mathcal{S}$.*

Bijgevolg is Ω de unie van twee verzamelingen Ω_1 en Ω_2 die corresponderen met de verzamelingen van type (a) en type (b) puntenreguli.

Vervolgens kan men bewijzen dat als S type 0 heeft ten opzichte van een punt x er geen type (b) elementen in Ω bevat zijn en dat Ω_1 hier bestaat uit alle puntenreguli door het punt x.
We construeren nu, zowel voor een S van type $+$, als voor een S van type $-$, een unieke H_2 als volgt. Noem Π het hypervlak opgespannen door de rechtenverzameling S en stel p het poolpunt van Π. Een rechte door p en een punt $x \in \Pi \cap \mathsf{Q}(6,q)$ snijdt de kwadriek per definitie enkel in x. Elke andere rechte door p en een punt $y \in \mathsf{Q}(6,q) \setminus \Pi$ snijdt $\mathsf{Q}(6,q)$ in een uniek tweede punt $y' \neq y$. De involutie g die alle punten van Π fixeert en zo een punt $y \in \mathsf{Q}(6,q) \setminus \Pi$ op het punt y' afbeeldt, breidt uit tot een involutieve collineatie van $\mathsf{PG}(6,q)$, die we opnieuw als g benoemen. Men kan nu aantonen dat $\mathsf{H}_2 = \mathsf{H}_1^g$ de unieke zeshoek is op $\mathsf{Q}(6,q)$ die H_1 snijdt in S en waarvoor $|\Omega|$ maximaal is. Bovendien is dit de enige zeshoek door S die type (b) puntenreguli met H_1 gemeen heeft.

A.3.3 2-designs uit H(q)

Noem H_1 een model van de standaard inbedding van $\mathsf{H}(q)$ op $\mathsf{Q}(6,q)$ en beschouw een Hermitische spread $\mathcal{S}_\mathsf{H}$ in H_1. Stel $\mathsf{H}_2 = \mathsf{H}_1^g$, waarbij g de hierboven gedefinieerde involutie corresponderend met $\mathcal{S}_\mathsf{H}$ is. We definiëren nu de incidentiemeetkunde $\mathsf{D}_{\mathsf{Hex}}(q)$ met als puntenverzameling de elementen van $\mathcal{S}_\mathsf{H}$ en als blokkenverzameling $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, waarbij

$$\mathcal{B}_1 = \{\mathcal{R}(L,M) : L, M \in \mathcal{S}_\mathsf{H}, L \neq M\},$$
$$\mathcal{B}_2 = \{\{L \ \mathtt{I} \ p : p \in \omega_2, L \in \mathcal{S}_\mathsf{H}\} : w_2 \in \Omega_2\}.$$

Een eerste grote stap in het bewijs van Stelling A.3.2 is het volgende lemma.

Lemma A.3.5.
De incidentiemeetkunde $\mathsf{D}_{\mathsf{Hex}}(q)$ is een $2-(q^3+1, q+1, 1+\frac{q+1}{(q+1,3)})$ design. Twee verschillende blokken snijden nooit in meer dan twee punten.

Om tenslotte via volgend lemma het bewijs van Stelling A.3.2 te vervolledigen.

Lemma A.3.6.
$\mathsf{D}_{\mathsf{Hex}}(q)$ *is een deeldesign van* $\mathsf{D}_{\mathsf{Hölz}}(q)$. *In het bijzonder vallen deze twee designs samen als en slechts als* $q \not\equiv 2 \mod 3$.

A.3.4 Unitalen van $\mathsf{D}_{\mathsf{Hölz}}(3)$

We kunnen de Hölz design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ op 28 punten aan de hand van de veralgemeende vierhoek Γ van orde $(2,4)$ als volgt beschrijven: definieer $\mathcal{P}$ als de puntenverzameling van Γ waaraan we een punt α toevoegen; $\mathcal{B}$ bevat enerzijds *rechtenblokken*, die α en 3 punten op een rechte van Γ bevatten, en anderzijds *Vee-blokken*, die de 4 punten in het symmetrisch verschil van 2 snijdende rechten bevatten; $\mathtt{I}$ is de natuurlijke incidentie.

Een unitaal van een design is per definitie een deelverzameling $\mathcal{U}$ van $\mathcal{B}$ met de eigenschap dat elke 2 punten van $\mathcal{D}$ bevat zijn in een uniek blok van $\mathcal{U}$. Bijgevolg bepaalt zo een unitaal in elke afgeleide deeldesign $\mathcal{D}_x$ van $\mathcal{D}$ ($\mathcal{D}_x$ is een veralgemeende vierhoek van orde $(2,4)$) een spread, die we een afgeleide spread van $\mathcal{U}$ noemen. Via [5] weten we dat de veralgemeende vierhoek van orde $(2,4)$ slechts 2 soorten spreads bevat, namelijk de *Hermitische spread*, waarin elke 2 rechten een regulus bepalen die volledig in de spread bevat is, en een *niet-Hermitische spread*, die men bekomt na switching van een regulus uit de Hermitische spread. We starten Sectie 3.4 dan ook met een handvol hulpstellingen betreffende deze 2 soorten spreads.

Het bewijs van Stelling A.3.3 verloopt in enkele opeenvolgende stappen. Vooreerst bewijzen we dat als een unitaal van $\mathcal{D}$ tenminste één Hermitische spread bepaalt, alle afgeleide spreads Hermitisch zijn. Met andere woorden, ofwel zijn alle afgeleide spreads van een unitaal van $\mathcal{D}$ Hermitisch ofwel zijn ze alle niet-Hermitisch. Vervolgens tonen we aan dat $\mathcal{D}$, op een isomorfisme na, een unieke unitaal bevat die alle afgeleide deeldesigns in Hermitische spreads snijdt. En tenslotte vervolledigen we het bewijs door hetzelfde aan te tonen voor een unitaal die alle afgeleide deeldesigns snijdt in niet-Hermitische spreads. Voor de expliciete uitwerking van elk van deze stappen verwijzen we naar Hoofdstuk 3 van de thesis.

A.4 Afstands-j ovoïdes en afstands-j spreads in $\mathsf{H}(3)$

In Hoofdstuk 4 construeren en classificeren we alle afstands-j ovoïdes en afstands-j spreads van $\mathsf{H}(3)$.

A.4.1 Inleiding

In het eerste deel van dit hoofdstuk construeren we een nieuwe soort ovoïdes, de *exceptionele ovoïdes* genoemd, en classificeren we alle ovoïdes en spreads van $\mathsf{H}(3)$. Als toepassing leveren we een elementaire meetkundige constructie van de $\mathsf{G}_2(3)$-GAB van type $\widetilde{\mathsf{G}}_2$, geven we nieuwe voorbeelden van ovoïde-spread paringen, tonen we aan dat $\mathsf{PG}(2,3)$ niet het lokaal isomorfe epimorfe beeld is van $\mathsf{H}(3)$ en onderzoeken we welke derivaties van de exceptionele spread niet enkel een 1-systeem van $\mathsf{Q}(6,3)$, maar tevens een spread van een zeshoek op $\mathsf{Q}(6,3)$ zijn.
In het tweede deel construeren we de unieke afstands-2 ovoïde en afstands-2 spread van $\mathsf{H}(3)$ en bewijzen we dat $\mathsf{PSL}_2(13)$ diens automorfismegroep is. Verder construeren we – wetende dat elke afstands-2 spread van $\mathsf{H}(q)$ een rechtenspread is van $\mathsf{Q}(6,q)$ – een rechtenspread van $\mathsf{Q}(6,2)$ die geen afstands-2 spread is van $\mathsf{H}(2)$. Als toepassing tonen we aan dat de puntgraaf van $\mathsf{H}(3)$ niet 4-kleurbaar is; dat $\mathsf{T}(8,2)$ en $\mathsf{T}(27,3)$ geen afstands-2 spread bevatten en dat de standaardinbedding van $\mathsf{H}(3)$ niet via de unieke afstands-2 ovoïde kan gelift worden naar een hoger dimensionele inbedding.

A.4.2 Deel I: Ovoïdes en spreads van $\mathsf{H}(3)$

De hoofdstelling van dit eerste deel stelt het volgende.

Stelling A.4.1.
De zeshoek $\mathsf{H}(3)$ bevat, op een isomorfisme na, juist 3 verschillende spreads. De eerste is de Hermitische spread, de tweede is de Ree-Tits spread en de derde, die we noteren als $\mathcal{S}_{\mathsf{E}}$, bevat 7 disjuncte rechtenreguli waarvan de complementaire puntenreguli het duale vormen van dezelfde spread. De automorfismegroep G van $\mathcal{S}_{\mathsf{E}}$ werkt transitief op diens rechten en 2-voudig transitief op de 7 rechtenreguli, en is equivalent als permutatiegroep met de werking van $\mathsf{PSL}_3(2)$ op de 7 punten van het projectieve vlak $\mathsf{PG}(2,2)$. Bovendien is $G \equiv 2^3 \cdot \mathsf{PSL}_3(2)$ een maximale deelgroep van de automorfismegroep $\mathsf{G}_2(3)$ van $\mathsf{H}(3)$.

Constructie van de spread $\mathcal{S}_{\mathsf{E}}$

We zullen hier slechts een schets van de constructie geven. Voor de eigenlijke uitwerking verwijzen we naar Hoofdstuk 4.
Neem een vaste Hermitische spread $\mathcal{S}_{\mathsf{H}}$ van $\mathsf{H}(3)$ en stel $\mathcal{U}$ de corresponderende Hermitische kromme in $\mathsf{PG}(2,9)$. Noem Φ het isomorfisme tussen de rechten van $\mathcal{S}_{\mathsf{H}}$ en de punten van $\mathcal{U}$. Beschouw een vaste rechtenregulus $\mathcal{R}_0$ in $\mathcal{S}_{\mathsf{H}}$. Het beeld van $\mathcal{R}_0$ onder Φ wordt gegeven door de doorsnede van de rechte L met $\mathcal{U}$. De constructie van de spread $\mathcal{S}_{\mathsf{E}}$ gaat als volgt: er zijn 3 Hermitische spreads van $\mathsf{H}(3)$ die de rechtenregulus $\mathcal{R}_0$ bevatten; in elk van deze 3 kiezen we twee geschikte rechtenreguli zodanig dat deze twee samen met $\mathcal{R}_0$ als blokken van $\mathcal{U}$ de rechten van een pooldriehoek vormen.

Classificatie van alle ovoïdes en spreads van $\mathsf{H}(3)$

Aangezien alle ovoïdes van $\mathsf{H}(3)$ isomorf zijn op $\mathsf{Q}(6,3)$ en de automorfismegroep hiervan gekend is, kennen we het exacte aantal ovoïdes op $\mathsf{H}(3)$. Dit in combinatie met de gekende automorfismegroepen van de Hermitische en Ree-Tits ovoïde leidt tot het vastleggen van de automorfismegroep van $\mathcal{S}_{\mathsf{E}}$ en de volledige classificatie van de ovoïdes en spreads van $\mathsf{H}(3)$.

Meetkundige eigenschappen en Toepassingen

Met de exceptionele spread $\mathcal{S}_{\mathsf{E}}$ als startobject kunnen we de volgende meetkundige structuren construeren:

het projectieve vlak van orde 2;
een ovoïde isomorf met $\mathcal{O}_{\mathsf{E}}$;
een GAB van type $\widetilde{\mathsf{G}_2}$;
een ovoïde-spread paring en
een transitief 1-systeem van $\mathsf{Q}(6,3)$.

Voor meer informatie over deze structuren en over hun constructiewijze refereren we naar de Engelstalige versie van de thesis. We vermelden hier enkel kort dat we een projectieve vlak $\Gamma \cong \mathsf{PG}(2,2)$ bekomen door als puntenverzameling de rechtenreguli te nemen en elke 3 reguli in dezelfde Hermitische spread als rechte te definiëren.
We kunnen nu alle mogelijke derivaties van $\mathcal{S}_{\mathsf{E}}$ beschouwen en ons afvragen wanneer het aldus bekomen 1-systeem van $\mathsf{Q}(6,3)$ een spread zal zijn van één of andere $\mathsf{H}(3)$ op $\mathsf{Q}(6,3)$. Hieromtrent hebben we het volgende resultaat.

Stelling A.4.2.
Het 1-systeem dat we bekomen door alle punten op een rechte van Γ te fixeren en de rest te switchen is een spread in een standaard inbedding van $\mathsf{H}(3)$.

Als gevolg hiervan (zie Gevolg 4.2.8) kan men aantonen dat er slechts 4 equivalentieklassen zijn van afgeleide 1-systemen van $\mathcal{S}_{\mathsf{E}}$ op $\mathsf{Q}(6,3)$.
Een soortgelijk, meer algemeen, resultaat wordt gegeven in volgende stelling.

Stelling A.4.3.
Als we twee disjuncte blokken van de Hermitische spread $\mathcal{S}_{\mathsf{H}}$ in H_1 switchen, dan zijn volgende uitspraken equivalent

(*i*) *de bekomen verzameling rechten is een spread in een zeshoek $\mathsf{H}_2 \neq \mathsf{H}_1$,*
(*ii*) *de bekomen verzameling rechten is isomorf met $\mathsf{H}(2,q^2)$ op $\mathsf{Q}^-(5,q)$,*
(*iii*) *we hebben alle blokken toegevoegd aan een gegeven blok B geswitched.*

A.4.3 Deel II: Afstands-2 ovoïdes en afstands-2 spreads van H(3)

Beschouw $\mathsf{H}(3)$ in diens standaard inbedding. In $\mathsf{PG}(6,3) \setminus \mathsf{H}(3)$ bestaat er een welbepaalde verzameling van 14 punten (zie Sectie 4.3.2 voor de expliciete coördinaten) die gestabilizeerd worden door een groep $G \cong \mathsf{PSL}_2(13)$. Wetende dat $\mathsf{PSL}_2(13)$ 2-transitief werkt en gegeven de expliciete coördinaten van de punten van Ω, kan men besluiten dat elke 2 punten van Ω een rechte bepalen die $\mathsf{H}(3)$ in een uniek punt, de *piek* van dit koppel, snijdt. De verzameling van alle pieken noemen we $\mathcal{O}_\Omega$. Men kan nu aantonen dat

Stelling A.4.4.
De verzameling $\mathcal{O}_\Omega$ bevat juist 91 punten en bepaalt een afstands-2 ovoïde van $\mathsf{H}(3)$.

Beschouw vervolgens de verzameling Ω^* bestaande uit de 14 poolhypervlakken van de punten uit Ω. Elke twee hypervlakken uit Ω^* snijden elkaar in 7 rechten van $\mathsf{H}(3)$, waarvan er juist één rechte concurrent is met alle andere rechten uit de doorsnede. Deze rechten noemen we de *rand* van de doorsnede, en we definiëren $\mathcal{S}_{\Omega^*}$ als de verzameling van alle randen van alle doorsnedes van de paren hypervlakken uit Ω^*. Met deze definitie kan men bewijzen dat

Stelling A.4.5.
De verzameling $\mathcal{S}_{\Omega^*}$ bevat juist 91 rechten en bepaalt een afstands-2 spread van $\mathsf{H}(3)$.

Om te bewijzen dat deze afstands-2 spread uniek is in $\mathsf{H}(3)$, maken we gebruik van het feit dat elke afstands-2 spread een dunne deelzeshoek snijdt in 13 rechten, die op hun beurt een *vlagkoppeling* bepalen in $\mathsf{PG}(2,3)$. In Sectie 4.3.3 van de thesis tonen we aan dat

Lemma A.4.6.
Er bestaan, op isomorfismes na, juist 5 vlagkoppelingen in $\mathsf{PG}(2,3)$.

Corresponderend met deze 5 vlagkoppelingen bepalen we 5 verzamelingen van 13 rechten uit $\mathsf{H}(3)$ die we al dan niet tot een afstands-2 spread kunnen vervolledigen. Om dit te doen, maken we gebruik van een eenvoudige procedure, die we de *punt-op-afstand-3-procedure* noemen: stel $\mathcal{L}$ een rechtenverzameling die tot een afstands-2 spread $\mathcal{S}_2$ moet behoren en noem V de verzameling van punten bedekt door de rechten van $\mathcal{L}$; beschouw nu de punten van $\mathsf{H}(3) \setminus V$ die op afstand 3 van juist 3 rechten van $\mathcal{L}$ gelegen zijn; voor elk zo een punt ligt de afstands-2 spreadrechte erdoor vast; we mogen bijgevolg deze rechte toevoegen aan de verzameling $\mathcal{L}$ en kunnen de procedure herbeginnen. Op deze manier bekomen we in elk van de 5 desbetreffende startsituaties ofwel een tegenstrijdigheid ofwel een afstands-2 spread isomorf aan $\mathcal{S}_{\Omega^*}$.

Een rechtenspread in $\mathsf{Q}(6,2)$

Op een analoge manier kunnen we, vertrekkende van de unieke vlagkoppeling in $\mathsf{PG}(2,2)$ en via de punt-op-afstand-3-procedure, aantonen dat $\mathsf{H}(2)$ geen afstands-2 spreads bevat. Aangezien elke afstands-2 spread van $\mathsf{H}(q)$ een rechtenspread is van de onderliggende kwadriek $\mathsf{Q}(6,q)$, kan men zich ook afvragen of $\mathsf{Q}(6,2)$ wel degelijk rechtenspreads bevat. Als positief antwoord op deze vraag geven we in Sectie 4.3.5 een zeer eenvoudige constructie van een rechtenspread van $\mathsf{Q}(6,2)$.

Toepassingen

Hier vermelden we enkel dat we aan de hand van de uniciteit van de afstands-2 spread en afstands-2 ovoïde kunnen aantonen dat

- de puntgraaf van $\mathsf{H}(3)$ niet 4-kleurbaar is,
- de twisted triality zeshoek $\mathsf{T}(27,3)$ geen afstands-2 spread bevat en
- de standaard inbedding van $\mathsf{H}(3)$ niet via een afstands-2 ovoïde kan gelift worden.

De achterliggende redeneringen hiervoor zijn in het kort:

> opdat er een 4-kleuring van de puntgraaf zou bestaan, moeten er 4 disjuncte afstands-2 ovoïdes bestaan,
>
> elke afstands-2 spread van $\mathsf{T}(q^3, q)$ snijdt $\mathsf{H}(q)$ in een afstands-2 spread van deze zeshoek en
>
> een afstands-2 ovoïde $\mathcal{O}_2$ van $\mathsf{H}(q)$ is een meetkundig hypervlak dat niet in een hypervlak van $\mathsf{PG}(6, q)$ gelegen is; we kunnen bijgevolg een punt $(x_0, \ldots, x_6)$ van $\mathsf{H}(q) \setminus \mathcal{O}_2$ liften naar het punt $(1, x_0, \ldots, x_6)$, terwijl we de punten van $\mathcal{O}_2$ in het hypervlak $X_0 = 0$ van $\mathsf{PG}(7, q)$ laten liggen; op die manier pogen we alle punten van $\mathsf{H}(q)$ te coördinatizeren in $\mathsf{PG}(7, q)$.

A.5 Afstands-2 ovoïdes in $\mathsf{H}(4)$

In Hoofdstuk 5 construeren we 2 niet-isomorfe afstands-2 ovoïdes van $\mathsf{H}(4)$.

A.5.1 Inleiding

Dit hoofdstuk wordt, net zoals het vorige, opgedeeld in een aantal onderverdelingen.
In het eerste deel construeren we een afstands-2 ovoïde van $\mathsf{H}(4)$ die $\mathsf{PSL}_2(13)$ als automorfismegroep heeft. Zowel de hieraan gelinkte twee-gewichtscode als de sterk reguliere graaf zijn nieuw.
In het tweede deel construeren we in $\mathsf{PG}(5, q^2)$ een oneindige klasse twee-karakterverzamelingen en bepalen we diens automorfismegroep. Uit deze constructie ontstaat bovendien een oneindige klasse twee-gewichtscodes en sterk reguliere grafen, en een nieuwe afstands-2 ovoïde van $\mathsf{H}(4)$.
In het derde en laatste deel tonen we aan dat geen van beide afstands-2 ovoïdes een geschikt meetkundig hypervlak is om tot een nieuwe inbedding van $\mathsf{H}(4)$ te komen.

A.5.2 Deel I: Een afstands-2 ovoïde van $\mathsf{H}(4)$

Als eerste algemeen resultaat in dit deel van de thesis hebben we de volgende stelling.

Stelling A.5.1.
Elke afstands-2 ovoïde $\mathcal{O}$ van $\mathsf{H}(q)$, met q even, definieert een twee-karakter-verzameling in $\mathsf{PG}(5,q)$ bestaande uit q^4+q^2+1 punten en met constanten q^4-q^3 en $q^4-q^3+q^2$.
Bijgevolg definieert $\mathcal{O}$ een projectieve lineaire q-aire twee-gewichtscode van lengte q^4+q^2+1, dimensie 6 en met gewichten q^4-q^3 en $q^4-q^3+q^2$. De geassocieerde sterk reguliere graaf heeft parameters

$$(v,k,\lambda,\mu)=(q^6,(q-1)(q^4+q^2+1),q^4-q^3+q-2,q(q-1)(q^2-q+1)).$$

Als hoofdresultaat van dit eerste deel bewijzen we hetvolgende.

Stelling A.5.2.
$\mathsf{H}(4)$ *bevat een afstands-2 ovoïde $\mathcal{O}_\Omega$ waarop de groep $\mathsf{PSL}_2(13)$ als automorfismegroep in 2 banen werkt. De ene baan bevat 91 punten en de actie van $\mathsf{PSL}_2(13)$ is equivalent met de primitieve actie hiervan op de pooldriehoeken van $\mathsf{PG}(2,13)$ als $\mathsf{PSL}_2(13)$ beschouwd wordt als de stabilizator van een kegelsnede in $\mathsf{PG}(2,13)$. De andere baan bevat 182 punten en de actie van $\mathsf{PSL}_2(13)$ is imprimitief met 91 klassen van lengte 2 waarop $\mathsf{PSL}_2(13)$ werkt als op de paren punten van $\mathsf{PG}(1,13)$.*

Als direct gevolg van deze stelling en Stelling A.5.1 bekomen we een sterk reguliere graaf en een quaternaire en een binaire twee-gewichtscode waarvan de parameters gegeven worden in Gevolg 5.2.3.

Constructie van $\mathcal{O}_\Omega$

In $\mathsf{PG}(6,4)$ bestaat er een verzameling Ω van 14 hypervlakken (zie Sectie 5.2.3 voor de expliciete vergelijkingen van de hypervlakken) die gestabilizeerd worden door een groep $G\cong\mathsf{PSL}_2(13)$. Net als in het vorige hoofdstuk kunnen we wegens de 2-transitiviteit van $\mathsf{PSL}_2(13)$ en de expliciete vergelijkingen van de hypervlakken besluiten dat elke 2 hypervlakken in Ω een unieke ideale rechte van $\mathsf{Q}(6,4)$ bepalen, en dit op de volgende manier: twee zulke hypervlakken H_1 en H_2 snijden elkaar in een $\mathsf{Q}(4,4)\subseteq\mathsf{Q}(6,4)$; er bestaat nu een uniek raakhypervlak H aan $\mathsf{Q}(6,4)$ dat deze $\mathsf{Q}(4,4)$ als deelruimte bevat; met dit hypervlak correspondeert een uniek punt p zodanig dat $H=T_p\mathsf{Q}(6,4)$; het hexagonvlak Π_p snijdt $\mathsf{Q}(4,4)$ in een ideale rechte, $L(H_1,H_2)$, de zogenaamde raakrechte van H_1 en H_2. De verzameling van punten die bedekt worden door alle raakrechten van paren hypervlakken noemen we $\mathcal{O}_\Omega$. Men kan nu aantonen dat

Stelling A.5.3.
De verzameling $\mathcal{O}_\Omega$ bevat juist 273 punten en bepaalt een afstands-2 ovoïde van $\mathsf{H}(4)$.

Voor het eigenlijke bewijs van deze stelling verwijzen we naar Hoofdstuk 5.

A.5.3 Deel II: Twee-karakterverzamelingen in $\mathsf{PG}(5, q^2)$

In $\mathsf{PG}(5, q^2)$ beschouwen we twee kruisende vlakken Π en Π', een anti-isomorfisme θ tussen deze vlakken en een Baer deelvlak B in Π.

Constructie van de twee-karakterverzameling

De punten en rechten van B en B^θ noemen we Baer punten en rechten. De overige punten en rechten uit $\Pi \cup \Pi'$ zijn de zogenaamde niet-Baer punten en rechten. Merk vooreerst op dat een gegeven punt x van $\mathsf{PG}(5, q^2) \setminus (\Pi \cup \Pi')$ op een unieke rechte $L(x)$, die zowel Π als Π' snijdt, gelegen is. De respectieve punten van Π en Π' op $L(x)$ noteren we als x_Π en $x_{\Pi'}$.
We definiëren de drie soorten punten van de verzameling $S(\Pi, \Pi', B, \theta)$ als volgt:

(PI) De PI punten zijn alle punten in het vlak Π.

(BB) De BB punten zijn de punten x van $\mathsf{PG}(5, q^2)$, niet in $\Pi \cup \Pi'$ gelegen, waarvoor zowel x_Π als $x_{\Pi'}$ Baer punten zijn en $x_{\Pi'}$ niet incident is met x_Π^θ.

(NB) De NB punten zijn de punten x van $\mathsf{PG}(5, q^2)$, niet in $\Pi \cup \Pi'$ gelegen, waarvoor zowel x_Π als $x_{\Pi'}$ niet-Baer punten zijn en $x_{\Pi'}$ incident is met x_Π^θ.

Aan de hand van deze definities kunnen we de volgende stelling formuleren.

Stelling A.5.4.
De verzameling $S(\Pi, \Pi', B, \theta)$ bevat juist $q^8 + q^4 + 1$ punten van $\mathsf{PG}(5, q^2)$ en bepaalt een twee-karakterverzameling met gewichten $q^8 - q^6$ en $q^8 - q^6 + q^4$. De automorfismegroep van $S(\Pi, \Pi', B, \theta)$ in $\mathsf{PG}(5, q^2)$ is een non-split extensie $(q+1) \cdot (\Gamma\mathsf{L}(3, q) \times 2)$. De vlakken Π en Π' hebben volgende karakteriserende eigenschap ten opzichte van $S(\Pi, \Pi', B, \theta)$ (en worden bijgevolg gefixeerd door de automorfismegroep van $S(\Pi, \Pi', B, \theta)$):

(*i*) Π *is het unieke vlak van* $\mathsf{PG}(5, q^2)$ *dat volledig bevat is in* $S(\Pi, \Pi', B, \theta)$*;*

(*ii*) Π' *is het enige vlak van* $\mathsf{PG}(5, q^2)$ *waarvan alle punten* x *voldoen aan de volgende eigenschap:* x *is niet bevat in* $S(\Pi, \Pi', B, \theta)$*, maar* x *is incident met een rechte* L *waarvan alle andere punten tot* $S(\Pi, \Pi', B, \theta)$ *behoren, en zodanig dat* L *het vlak* Π *snijdt.*

Deze twee-karakterverzamelingen definiëren opnieuw sterk reguliere grafen en projectieve twee-gewichtscodes. De parameters hiervan kan men terugvinden in Gevolg 5.3.2.

Veralgemeende zeshoeken en afstands-2 ovoïdes

Uit Stelling A.5.1 weten we dat elke afstands-2 ovoïde van $\mathsf{H}(q)$ in diens perfecte symplectische representatie een twee-karakterverzameling in $\mathsf{PG}(5, q)$ is. We vragen ons nu af of de twee-karakterverzameling uit Stelling A.5.4 op deze manier van een afstands-2 ovoïde van $\mathsf{H}(q^2)$ afkomstig kan zijn. Om deze vraag te kunnen beantwoorden moeten we in principe alle mogelijkheden voor Π en Π', θ en B overlopen. In Hoofdstuk 5 van de thesis beschouwen we echter enkel de meest natuurlijke keuze: we beschouwen $\mathsf{H}(q^2)$ in de perfecte symplectische representatie, nemen Π een willekeurig ideaal vlak en definiëren Π' als diens hexagon tweeling; we noemen τ het geassocieerde hexagon tweeling anti-isomorfisme; we kiezen B een Baer deelvlak in Π en noemen β de geassocieerde semi-lineaire involutie die B als fixpuntverzameling heeft; en tenslotte stellen we $\theta = \beta\tau$.

Stelling A.5.5.
De twee-karakterverzameling $S(\Pi, \Pi', B, \theta)$*, met* Π*,* Π'*,* B *en* θ *zoals net beschreven, is een afstands-2 ovoïde van* $\mathsf{H}(q^2)$*,* q *even, als en slechts als* $q = 2$ *en* B *niet bevat is in een deelzeshoek van orde 2. In dat geval, is het een nieuwe afstands-2 ovoïde.*

We hebben dus in het totaal 4 afstands-2 ovoïdes: één in zowel $\mathsf{H}(2)$ als in $\mathsf{H}(3)$ en twee in $\mathsf{H}(4)$.

A.5.4 Deel III: Lifting van H(4)

In Sectie 5.4 van de thesis tonen we aan dat noch de afstands-2 ovoïde $\mathcal{O}_\Omega$, zoals geconstrueerd in Sectie 5.2.3, noch de afstands-2 ovoïde $\mathcal{O}_2$, zoals geconstrueerd in Sectie 5.3.4, een geschikt meetkundige hypervlak is om de standaard inbedding van $\mathsf{H}(4)$ te liften naar $\mathsf{PG}(7, 4)$. Bovendien kunnen we, aangezien q hier even is, deze twee afstands-2 ovoïdes pogen te gebruiken

om, vertrekkende van de perfecte symplectische representatie van $\mathsf{H}(4)$, een nieuwe inbedding in $\mathsf{PG}(6,4)$ te construeren. Maar ook deze pogingen blijken vruchteloos.

A.6 Eigenwaarden en codes

In Hoofdstuk 6 geven we een meetkundige interpretatie van de eigenwaarden van een orde q veralgemeende zeshoek en construeren we enkele codes uit kleine veralgemeende zeshoeken.

A.6.1 Inleiding

In het eerste deel van dit hoofdstuk bepalen we de eigenwaarden en hun respectieve eigenvectoren en multipliciteiten van de puntgraaf van Δ, een orde q veralgemeende zeshoek. We beginnen met de algebraïsche aanpak, om vervolgens een meetkundige verklaring te geven. Als toepassing bepalen we, aan de hand van de bekomen eigenvectoren, de kardinaliteit van de doorsnedes van welbepaalde puntenverzamelingen.
In het tweede deel beschouwen we $\mathsf{H}(2)$ en $\mathsf{H}(2)^D$ en construeren we een aantal binaire codes uit de karakteristieke vectoren corresponderend met een deelverzameling punten van Δ.

A.6.2 Deel I: Eigenwaarden

Algebraïsche aanpak

Na enkele secure berekeningen vinden we

$$q(q+1) \qquad 2q-1 \qquad -(q+1) \qquad -1$$

als eigenwaarden voor de puntgraaf van Δ.

Meetkundige aanpak

We beschouwen enkele deelstructuren van Δ, niet wetende of deze deelstructuren ook effectief in Δ bestaan. De algemene methode om, gegeven een puntenverzameling V, tot de corresponderende eigenwaarden te komen is de volgende: we definiëren een vector $\mathbf{v}(a,b)$, die lengte $n = \frac{q^6-1}{q-1}$ heeft, en

nummeren de punten van Δ van 1 tot en met n; de vector $\mathbf{v}(a,b)$ krijgt in de posities van de punten van V de waarde a en in alle overige posities de waarde b toegekend; via de meetkundige karakterisering van de punten van V kunnen we dan een stelsel vergelijkingen opstellen waaruit we a en b en de geassocieerde eigenvectoren berekenen.

Merk op. Bij een rechtenverzameling beschouwen de bedekte puntenverzameling om de vector $\mathbf{v}(a,b)$ te definiëren.

Elk van de beschouwde puntenverzamelingen heeft $q(q+1)$ als 1 van diens eigenwaarden en de 1-vector als corresponderende eigenvector.

Let op. Als we een dunne deelzeshoek Γ van orde $(1,q)$ beschouwen, krijgen we 3 soorten punten: de punten van Γ, de punten niet in Γ maar op een rechte van Γ en de punten niet in Γ, noch op een rechte van Γ.

De niet-triviale eigenwaarden en -vectoren worden samengevat in onderstaande tabel

punten	eigenwaarde	eigenvector
$\mathcal{O}$	-1	$\mathbf{v}(q(q+1),-1)$
$\mathcal{S}$	-1	$\mathbf{v}(q^2,-(q+1))$
Γ	$-(q+1)$	$\mathbf{v}(q(q-1),-2q,2)$
	$2q-1$	$\mathbf{v}(-q(q^2-1),(2-q)q,q+1)$
Γ^D	$2q-1$	$\mathbf{v}(q(q-1),-1)$
$\mathcal{O}_2$	$-(q+1)$	$\mathbf{v}(q,-1)$

Table A.1: Eigenwaarden en -vectoren van Δ

waarbij $\mathcal{O}$, $\mathcal{S}$, Γ, Γ^D en $\mathcal{O}_2$ respectievelijk een ovoïde, een spread, een dunne deelzeshoek van orde $(1,q)$, een dunne deelzeshoek van orde $(q,1)$ en een afstands-2 ovoïde voorstellen.

Toepassing

Aangezien twee eigenvectoren van verschillende eigenwaarden altijd orthogonaal staan op elkaar, kunnen we de kardinaliteit k bepalen van de doorsnede van twee zulke puntenverzamelingen. Wanneer we de doorsnede met Γ beschouwen, moeten we enkele meetkundige eigenschappen in acht nemen. Voor de overige doorsnedes levert simpelweg het inproduct van de twee eigenvectoren de waarde van k op. Samengevat vinden we

	$\mathcal{O}$	$\mathcal{S}$	Γ	Γ^D	$\mathcal{O}_2$
$\mathcal{O}$	$*$	?	2	$q+1$	q^2-q+1
$\mathcal{S}$		$*$	$q+1$	$(q+1)^2$	q^3+1
Γ			$*$	?	$k \neq$ constant
Γ^D				$*$	q^2+q+1
$\mathcal{O}_2$					$*$

Table A.2: Doorsnedes

A.6.3 Deel II: Binaire codes

In Sectie 6.3 van de thesis construeren we enkele binaire codes uit $\mathsf{H}(q)$, met vooral $q = 2$. We beschouwen de $\mathsf{GF}(2)$-vectorruimte van karakteristieke functies van allerhande deelverzamelingen van de puntenverzameling van $\mathsf{H}(q)$. Gegeven zo een deelverzameling S, definiëren we $\mathcal{C}(S)$ als de lineaire code, voortgebracht door alle elementen in de baan van S onder $\mathsf{Aut}\,\mathsf{H}(q)$. Elke van deze codes heeft lengte $\frac{q^6-1}{q-1}$. We beginnen met een code die we beschouwen voor algemene q.

Codes uit deelzeshoeken

Veronderstel dat Γ een dunne deelzeshoek van orde $(1, q)$ is van $\mathsf{H}(q)$. De code $\mathcal{C}(S)$ heeft dan een minimumgewicht d kleiner dan of gelijk aan $2(1+q+q^2)$, het aantal punten van S. In Sectie 6.3.2 tonen we aan dat d effectief gelijk is aan $2(1+q+q^2)$ en dat elk codewoord met minimaal gewicht de karakteristieke functie is van de puntenverzameling van een dunne deelzeshoek van orde $(1, q)$.

Codes van lengte 63 van $\mathsf{H}(2)$ en $\mathsf{H}(2)^D$

In Sectie 6.3.3 beschouwen we een hele reeks deelstructuren van $\mathsf{H}(2)$ en $\mathsf{H}(2)^D$. Voor elk van deze deelstructuren kan men eveneens de complementaire puntenverzameling in $\mathsf{H}(2)$ of $\mathsf{H}(2)^D$ beschouwen. Alle geassocieerde codes plus hun dimensie, minimum gewicht en voortbrengende puntenverzameling kan men terugvinden in Tabel 6.1 en Tabel 6.2.

Codes van lengte 28 en 36 van $\mathsf{H}(2)$ en $\mathsf{H}(2)^{\mathsf{D}}$

Sommige deelstructuren van $\mathsf{H}(2)$ en $\mathsf{H}(2)^D$ komen slechts $A = 28$ of 36 keer voor. De matrix met als rijen de geïndexeerde punten x van de desbetreffende zeshoek, met als kolommen de deelstructuur S en waarin de (x, S)-positie 1 is als $x \in S$ en 0 is anders, brengt over $\mathsf{GF}(2)$ een binaire code van lengte A voort. De codes die we aldus bekomen vindt men terug in Tabel 6.3.

A.7 Eén-puntsextensies van veralgemeende zeshoeken

In het laatste hoofdstuk van de thesis bewijzen we de uniciteit van de 1-puntsextensie $\mathcal{S}$ van $\mathsf{H}(2)$ onder de voorwaarde dat: voor elke 3 punten x, y en z van $\mathcal{S}$, de afstand tussen y en z in de afgeleide zeshoek $\mathcal{S}_x$ gelijk is aan die tussen x en z in $\mathcal{S}_y$.

A.7.1 Inleiding

Naast de 1-puntsextensies van $\mathsf{PG}(2, 2)$ en $\mathsf{PG}(2, 4)$ en de Hölz designs, die 1-puntsextensies zijn van de Ahrens-Szekeres veralgemeende vierhoeken van orde $(q - 1, q + 1)$, zijn er slechts 2 voorbeelden van 1-puntsextensies van eindige veralgemeende n-hoeken, $n > 3$, gekend: een unieke in $\mathsf{W}(2)$ en één in $\mathsf{H}(2)$. Deze laatste werd reeds gekarakteriseerd door een combinatorische eigenschap van H. Cuypers. In Hoofdstuk 7 geven we een meetkundige karakterisering en tonen we aan dat het resultaat van H. Cuypers hiervan een rechtstreeks gevolg is. Bovendien tonen we aan dat $\mathsf{H}(2)$ de enige eindig veralgemeende zeshoek met klassieke orde is die een 1-puntsextensie bevat onder deze afstands-voorwaarde. In het bijzonder bewijzen we dat

Stelling A.7.1.
Er bestaat een unieke 1-puntsextensie van een veralgemeende zeshoek met klassieke orde (s, t) onder de voorwaarde dat deze extensie voldoet aan:

$$\forall x, y, z \ : \ d_x(y, z) = d_y(x, z).$$

A.7.2 Bewijs

In Sectie 7.2 bewijzen we Stelling A.7.1. Hiervoor maken we gebruik van volgend lemma.

Lemma A.7.2.
Veronderstel dat $\mathcal{S}$ een 1-puntsextensie is van een veralgemeende zeshoek van orde (s,t) die voldoet aan de afstands-voorwaarde. Dan hebben we dat

(a) $s+2 \mid 2t(t+1)$.
(b) $t \geq s/2$.
(c) *s is even.*

Als direct gevolg van bovenstaand lemma vinden we dat

Gevolg A.7.3.
Als er een 1-puntsextensie, die voldoet aan de afstands-voorwaarde, van een veralgemeende zeshoek met klassieke orde (s,t) bestaat, met $s \geq 2$, dan is $s = 2$.

Met andere woorden, er zijn slechts 3 eindig klassieke veralgemeende zeshoeken die in aanmerking komen om zo een 1-puntsextensie te hebben: $\mathsf{H}(2)$, $\mathsf{H}(2)^D$ en $\mathsf{T}(2,8)$.

Gegeven Γ, een veralgemeende zeshoek van orde $(2,t)$, construeren we een 1-puntsextensie $\mathcal{S}$ van Γ door enkel gebruik te maken van de afstands-voorwaarde. Op die manier vinden we dat $\mathcal{S}$, naast het extra punt α, de volgende 3 soorten blokken moet bevatten:

rechtenblokken, i.e. $\{x,y,z,\alpha\}$ met xyz een rechte van Γ;

Vee-blokken, i.e. $\{x,y,u,v\}$ met xyz en uvz rechten van Γ;

Wee-blokken, i.e. $\{x,y,u,v\}$ met $x \bowtie y = a$, $u \bowtie v = b$, abc een rechte van Γ, en xyc en uvc twee ideale rechten van Γ.

Via de constructie van de Wee-blokken kunnen we de gevallen $\Gamma \cong \mathsf{H}(2)^D$ en $\Gamma \cong \mathsf{T}(2,8)$ uitsluiten.

A.7.3 Enkele gevolgen

Lemma A.7.4.
Als een 1-puntsextensie $\mathcal{S}$ van een veralgemeende zeshoek Γ van orde $(2,t)$ in elke afleiding alle Vee-blokken bevat, dan voldoet $\mathcal{S}$ aan de afstands-voorwaarde.

Met behulp van Lemma A.7.4 kunnen we nu enerzijds aantonen dat volgende stelling van H. Cuypers een direct gevolg is van Stelling A.7.1.

Stelling A.7.5.
Er bestaat een unieke 1-puntsextensie $\mathcal{S}$ *van* $\mathsf{H}(2)$ *op voorwaarde dat* $\mathcal{S}$ *voldoet aan de volgende eigenschap: voor elke 2 blokken* B *en* B' *met* $|B \cap B'| = 2$ *behoort de verzameling* $(B \cup B') \setminus (B \cap B')$ *tot de blokkenverzameling van* $\mathcal{S}$.

en anderzijds dat

Stelling A.7.6.
Er bestaat een unieke vlag-transitieve 1-puntsextensie van een veralgemeende zeshoek van orde 2*, die eveneens de unieke 2-voudig transitieve 1-puntsextensie van deze zeshoek is.*

Tenslotte eindigen we Hoofdstuk 7 met een link tussen de blokken van de gekende constructie van de 1-puntsextensie van $\mathsf{H}(2)$ en de rechten-, Vee- en Wee-blokken. Hiervoor verwijzen we naar de Engelstalige tekst.

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