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Rotation-invariant t-norms

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Dankwoord

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Koen C. Maes

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Preface

In many fields of science, monotone functions are used to aggregate multiple numerical inputs into a single numerical output. These numerical inputs can represent physical observations, biological criteria, preferences, statistical data, economical and/or financial data, probabilities, etc. The output enables us to explain and predict physical, biological and economical phenomena, to classify objects and species or to make well-founded decisions. The aggregation process often requires that the input values as well as the output value must belong to a same numerical interval. Due to the monotonicity of the aggregation functions involved, it is often possible to rescale the input values as well as the output value into the unit interval.

In this work we mainly focus on the description of monotone $[0,1]^2 \to [0,1]$ functions F in terms of monotone $[0,1] \to \mathbb{R}$ functions. In the literature there largely exist two possible approaches. Firstly, F is sometimes expressed by means of a monotone $[0,1] \to \mathbb{R}$ function (the generator) and its (pseudo-)inverse. A predetermined external $[0,1]^2 \to [0,1]$ function combines the generator and its inverse. Additive and multiplicative generators of t-norms are by far the best known examples of this approach (see e.g. [51]). Unfortunately, the method is not applicable to all monotone functions F. Additional conditions such as associativity are often difficult to grasp. In this respect, when dealing with t-norms, additive and multiplicative generators produce only Archimedean t-norms. The second approach consists in fixing F on a subset $\{(x, f(x)) \mid x \in [0,1]\}$ of $[0,1]^2$ that is determined by a given monotone $[0,1] \to [0,1]$ function f. Invoking some additional (required) properties on F, $F|_{\{(x,f(x)) \mid x\in[0,1]\}}$ can then be used to define F on the whole unit square. In this respect, continuous t-norms and copulas have been constructed with fixed diagonal sections ($f = \mathbf{id}$) (see e.g. [20] and [51]). Note that often multiple functions F can coincide on $\{(x, f(x)) \mid x \in [0,1]\}$.

We contribute to a less known third approach in which F is described by means of its contour lines. Contour lines are decreasing $[0,1] \to [0,1]$ functions determining the limits of the horizontal cuts of F. The dissertation is organized as follows.

1. In Part I we focus on the symmetry aspects of monotone $[0,1] \to [0,1]$ functions. In Chap-

ter 1 we generalize the classical inverse of a monotone function. We use this generalization to describe the symmetry of monotone $[0,1] \to [0,1]$ functions in Chapter 2. Finally, we invoke the new insights to investigate the invariance (Chapter 3) and the orthosymmetry aspects (Chapter 4) of more general monotone $[0,1]^n \to [0,1]$ functions.

2. The results from Part I lay the foundation for a more profound study of rotation-invariant t-norms in terms of contour lines (Part II). We express first the characteristic properties of uninorms in terms of contour lines (Chapter 5). Special attention goes to the existence of a continuous contour line. In particular, we try to understand how the continuity of a contour line influences the structure of a uninorm. In Chapter 6 we focus on left-continuous t-norms. We introduce the companion and zooms as additional tools to lay bare the geometrical structure of a left-continuous t-norm T. Finally, we introduce brand new and natural methods for decomposing (Chapter 7) and constructing (Chapter 8) rotation-invariant t-norms.

3. Finally, we invoke our knowledge on the structure of rotation-invariant t-norms to perform a comparative study between the disjunctive and conjunctive fuzzified normal forms (Part III). These fuzzified normal forms are rooted in a straightforward adjustment of the disjunctive and conjunctive Boolean normal forms. In Chapter 9 we set out the framework in which the fuzzified normal forms occur. We explore for which continuous De Morgan triplets the disjunctive fuzzified normal form is smaller than or equal to the conjunctive fuzzified normal form. A system of functional equations turns up if some functional independence of the difference between both fuzzified normal forms is demanded. In Chapter 10 we inquire which De Morgan triplets, based on a left-continuous (rotation-invariant) t-norm T, solve this system.

Most of the work presented in this dissertation has already been published in peer reviewed international journals. Chapter 1 has been described in [63]. The results from Chapter 2 can be found in [62]. The work presented in Chapter 3 has been published in [61]. Chapter 4 contains a lot of new, yet unpublished work. However, the results from Section 4.2 and Chapter 5 have been described in [60]. Most of our work stated in Chapters 6–8 can be found in [64], [65] and [66]. Part III of the dissertation contains the oldest material. Chapter 9 has been described in [57]. Finally, our results from [58] and [59] have been given a face-lift and have been summarized in Chapter 10.

Part I

Monotone functions

CHAPTER 1

Inverses of monotone functions

1.1 Introduction

In the unit square $[0,1]^2$, the inverse A^{-1} of a set $A \subseteq [0,1]^2$ is defined as $A^{-1} := \{(x,y) \in [0,1]^2 \mid (y,x) \in A\}$. Geometrically, we obtain A^{-1} by reflecting A about the graph of the *identity function* $\mathbf{id} : [0,1] \to [0,1] : x \mapsto x$. For a function f (*i.e.* every element x in the domain of f is mapped to a unique image $f(x)$), its inverse $f^{-1} = \{(x,y) \in [0,1]^2 \mid x = f(y)\}$ is again a function if and only if f is injective. A set A is symmetrical w.r.t. the identity function if $(x,y) \in A$ whenever $(y,x) \in A$, meaning that the set and its inverse coincide. Analogously, A is symmetrical w.r.t. the *standard negator* $\mathcal{N} : [0,1] \to [0,1] : x \mapsto 1-x$ if it holds that $(x,y) \in A$ whenever $(1-y, 1-x) \in A$. Hence, $A^{\mathcal{N}} := \{(x,y) \in [0,1]^2 \mid (1-y,1-x) \in A\}$ can be understood as the inverse of A w.r.t. the standard negator. In particular, $A^{\mathcal{N}}$ is the reflection of A about the graph of $\mathcal{N}$. However, reflections are not always apt to define the inverse of a set w.r.t. a given monotone $[0,1] \to [0,1]$ bijection Φ. For instance, suppose that Φ contains part of a circle with center (x,y) belonging to A (see Fig. 1.1(a)). There does not exist a unique straight line perpendicular to Φ that contains (x,y). This observation forces us to approach the inverse of A in a different way.

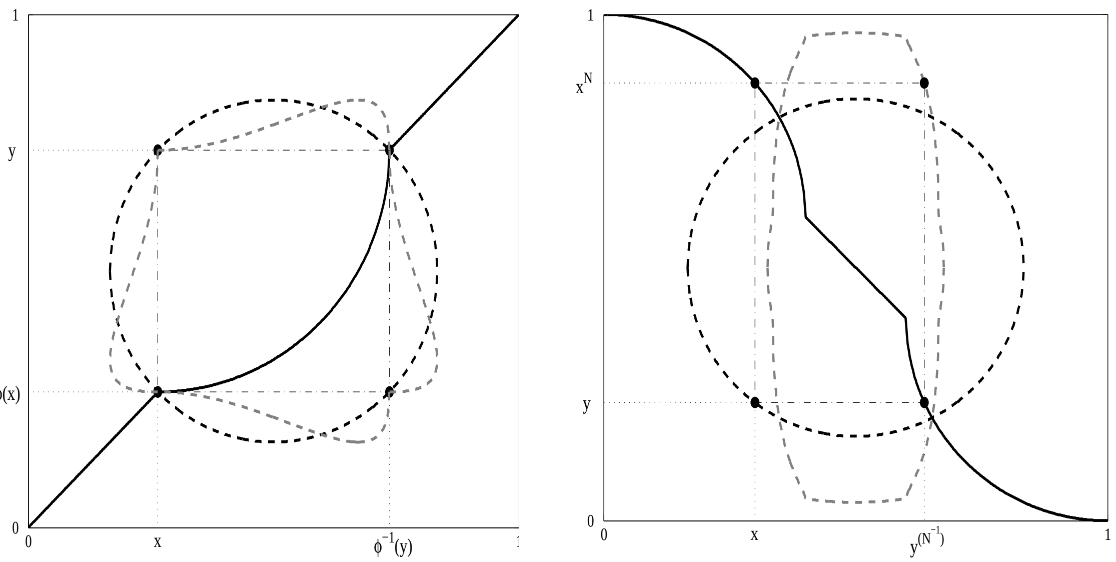
We introduce a new type of inverse w.r.t. monotone bijections Φ. Inverting a monotone function in the unit square, however, does not necessarily result in a function. Extending the approach of Schweizer and Sklar [85] we associate to each monotone function f a set $Q(f,\Phi)$ containing the 'inverse' functions of f w.r.t. a given monotone $[0,1] \to [0,1]$ bijection Φ. By far the most attention goes to exposing the geometrical and algebraical properties of $Q(f,\Phi)$. The study of the set $Q(f,\mathbf{id})$ is crucial as each set $Q(f,\Phi)$ is either isomorphic or antimorphic with $Q(f,\mathbf{id})$.

1.2 Inverse functions

Monotone $[0,1] \to [0,1]$ bijections can be either increasing or decreasing. We will use the following terminology to indicate the type of monotonicity:

Definition 1.1 An increasing $[0,1] \to [0,1]$ bijection ϕ is called an *automorphism*; a decreasing $[0,1] \to [0,1]$ bijection N is called a *strict negator*. The image of x under ϕ is denoted as $\phi(x)$. For a strict negator we (usually) use the exponential notation x^N.

The identity function **id** is a prototypical automorphism while the standard negator $\mathcal{N}$ is the prototype of a strict negator. Given a monotone bijection Φ, we introduce now an alternative way to invert a set $A \subseteq [0,1]^2$ w.r.t. Φ. Through every point $(x,y) \in A$ we draw a line parallel to the X-axis and a line parallel to the Y-axis. These lines intersect the graph of Φ in the points $(\Phi^{-1}(y), y)$ and $(x, \Phi(x))$, respectively. $(\Phi^{-1}(y), \Phi(x))$ is the fourth point of the rectangle defined by (x,y), $(\Phi^{-1}(y), y)$ and $(x, \Phi(x))$. The set of all these points $(\Phi^{-1}(y), \Phi(x))$, with $(x,y) \in A$, can be understood as the inverse of A w.r.t. the bijection Φ. Figure 1.1 illustrates this procedure.



(a) ϕ-inverse of a circle (b) N-inverse of a circle

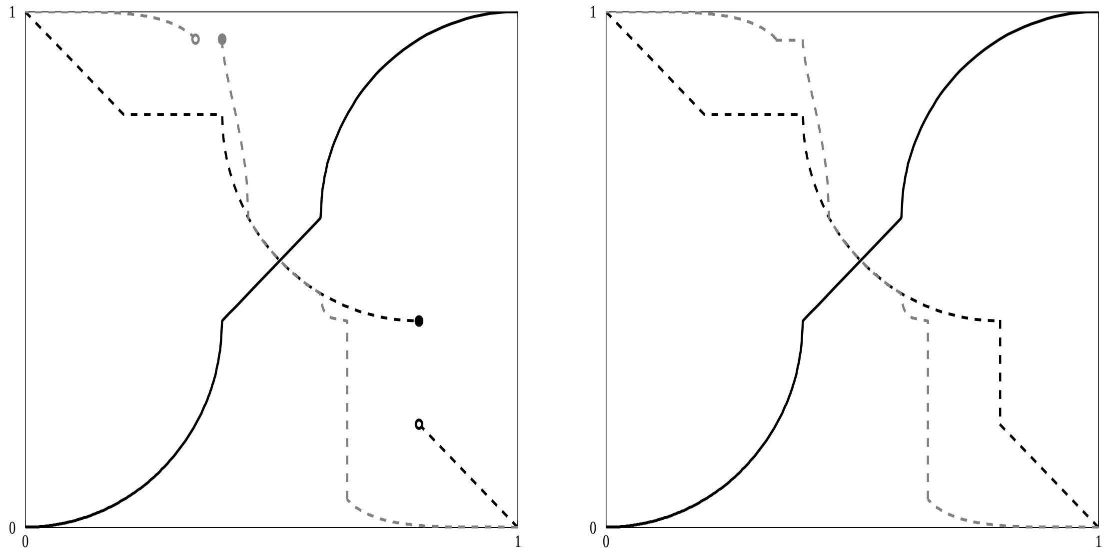
Figure 1.1: The ϕ-inverse and N-inverse (dashed gray lines) of a circle (dashed black line), with ϕ the automorphism (solid line) depicted in Fig. 1.1(a) and N the strict negator (solid line) depicted in Fig. 1.1(b).

Definition 1.2 Let Φ be a monotone $[0,1] \to [0,1]$ bijection. The Φ-*inverse* of a set A$\subseteq [0,1]^2$ is given by $A^\Phi := \{(x,y) \in [0,1]^2 \mid (\Phi^{-1}(y), \Phi(x)) \in A\}$.

It holds that $(x,y) \in A^\Phi$ if and only if $(\Phi(x), \Phi^{-1}(y)) \in A^{-1}$. In case Φ is the identity function **id**, $A^{\mathbf{id}}$ equals A^{-1} and will still be referred to as the inverse of A. The Φ-inverse of a function f is again a function if and only if f is injective. Moreover, in this case $f^\Phi = \Phi \circ f^{-1} \circ \Phi$. Note also that $(A^\Phi)^\Phi = A$.

From now on, let f be a monotone $[0,1] \to [0,1]$ function and Φ be a monotone $[0,1] \to [0,1]$ bijection. If f is not bijective, its Φ-inverse f^Φ cannot be seen as a $[0,1] \to [0,1]$ function (see e.g. Fig. 1.2(a)). There are various ways to adjust this Φ-inverse, ensuring that it becomes a $[0,1] \to [0,1]$ function. Given an increasing $[0,1] \to [0,1]$ function f, Schweizer and Sklar geometrically construct a set of '**id**-inverse' functions [85]. Some additional results for monotone functions are due to Klement *et al.* [50, 51]. We will largely extend these results and associate to each monotone function f a set of $[0,1] \to [0,1]$ functions: the 'Φ-inverse' functions of f.

Definition 1.3 A *completion* f^* of a monotone $[0,1] \to [0,1]$ function f is a continuous line from the point $(0,0)$ to the point $(1,1)$, whenever f is increasing, and/or from the point $(0,1)$ to the point $(1,0)$, whenever f is decreasing. f^* is obtained by adding vertical segments to the graph of f.



(a) ϕ-inverse f^ϕ of f (b) ϕ-inverse $(f^*)^\phi$ of f^*

Figure 1.2: The ϕ-inverse (dashed gray lines) of a decreasing function f and of its completion f^* (dashed black lines), with ϕ the automorphism depicted by the solid line.

For example, Fig. 1.2(b) depicts the completion f^* (dashed black line) of the decreasing function f (dashed black line) from Fig. 1.2(a). Clearly, every non-constant monotone function f has a unique completion. As a constant $[0,1] \to [0,1]$ function $\boldsymbol{\alpha}$ ($\boldsymbol{\alpha}(x) = \alpha$), with $\alpha \in [0,1]$, is both increasing and decreasing, it has an increasing completion as well as a decreasing completion.

Definition 1.4 Let Φ be a monotone $[0,1] \to [0,1]$ bijection and f be a monotone $[0,1] \to [0,1]$ function. $Q(f, \Phi)$ is the set of all $[0,1] \to [0,1]$ functions that have a completion coinciding with the Φ-inverse $(f^*)^\Phi$ of a completion f^* of f.

All decreasing $[0,1] \to [0,1]$ functions that are covered by the dashed gray line in Fig. 1.2(b) constitute the set $Q(f,\phi)$, with f the decreasing function from Fig. 1.2(a) (dashed black line) and ϕ the automorphism from Figs. 1.2(a) and 1.2(b) (solid black line). For a non-constant function f, the members of $Q(f,\Phi)$ are constructed from $(f^*)^\Phi$ by deleting from any vertical segment all but one point. For a constant $[0,1] \to [0,1]$ function $\boldsymbol{\alpha}$, the set $Q(\boldsymbol{\alpha},\Phi)$ contains functions constructed from the increasing completion of $\boldsymbol{\alpha}$ as well as functions constructed from the decreasing completion of $\boldsymbol{\alpha}$. The following theorem shows that the injectivity and/or surjectivity of f is reflected in the set $Q(f,\Phi)$.

Theorem 1.5 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *f is injective if and only if $|Q(f,\Phi)| = 1$.*
2. *f is surjective if and only if $Q(f,\Phi)$ contains injective functions only.*
3. *f is bijective if and only if $f^\Phi \in Q(f,\Phi)$.*

For a bijective function f it clearly holds that $Q(f,\Phi) = \{f^\Phi\}$.

Proof Follows immediately from the definition of $Q(f,\Phi)$. $\square$

We can introduce an equivalence relation on the class of monotone $[0,1] \to [0,1]$ functions by calling two functions f and h equivalent if their completed curves coincide, or equivalently, if the sets $Q(f,\Phi)$ and $Q(h,\Phi)$ coincide. The monotone bijection Φ can be chosen arbitrarily. The equivalence class containing a function f is then given by $Q(g,\Phi)$, with $g \in Q(f,\Phi)$.

Theorem 1.6 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *For every $g \in Q(f,\Phi)$ it holds that $f \in Q(g,\Phi)$.*
2. *For every $g_1, g_2 \in Q(f,\Phi)$ it holds that $Q(g_1,\Phi) = Q(g_2,\Phi)$.*
3. *For every $g \in Q(f,\Phi)$ it holds that $h \in Q(g,\Phi) \;\Leftrightarrow\; Q(h,\Phi) = Q(f,\Phi)$.*

Proof Follows immediately from the definition of $Q(f,\Phi)$. $\square$

In order to describe the members of $Q(f,\Phi)$ mathematically, we first have to introduce four $[0,1] \to [0,1]$ functions $\overline{f}^\Phi$, $\overline{f}_\Phi$, $\underline{f}^\Phi$ and $\underline{f}_\Phi$:

$$\begin{aligned}
\overline{f}^\Phi(x) &= \sup\{t \in [0,1] \mid f(\Phi^{-1}(t)) < \Phi(x)\} \\
\overline{f}_\Phi(x) &= \inf\{t \in [0,1] \mid f(\Phi^{-1}(t)) > \Phi(x)\} \\
\underline{f}^\Phi(x) &= \sup\{t \in [0,1] \mid f(\Phi^{-1}(t)) > \Phi(x)\} \\
\underline{f}_\Phi(x) &= \inf\{t \in [0,1] \mid f(\Phi^{-1}(t)) < \Phi(x)\}
\end{aligned}$$

($\sup\emptyset = 0$ and $\inf\emptyset = 1$). In the following theorem we lay bare the tight connection between the above functions constructed from a monotone bijection Φ and those constructed from the identity function $\mathbf{id}$.

Theorem 1.7 *Consider a monotone $[0,1] \to [0,1]$ function f.*

1. *For an automorphism ϕ the following identities hold:*

$$\begin{aligned}
\overline{f}^{\phi} &= \overline{f\circ\phi^{-1}}^{\mathbf{id}}\circ\phi = \phi\circ\overline{\phi^{-1}\circ f}^{\mathbf{id}} = \overline{\phi^{-1}\circ f\circ\phi^{-1}}^{\mathbf{id}} = \phi\circ\overline{f}^{\mathbf{id}}\circ\phi\,;\\
\overline{f}_{\phi} &= \overline{f\circ\phi^{-1}}_{\mathbf{id}}\circ\phi = \phi\circ\overline{\phi^{-1}\circ f}_{\mathbf{id}} = \overline{\phi^{-1}\circ f\circ\phi^{-1}}_{\mathbf{id}} = \phi\circ\overline{f}_{\mathbf{id}}\circ\phi\,;\\
\underline{f}^{\phi} &= \underline{f\circ\phi^{-1}}^{\mathbf{id}}\circ\phi = \phi\circ\underline{\phi^{-1}\circ f}^{\mathbf{id}} = \underline{\phi^{-1}\circ f\circ\phi^{-1}}^{\mathbf{id}} = \phi\circ\underline{f}^{\mathbf{id}}\circ\phi\,;\\
\underline{f}_{\phi} &= \underline{f\circ\phi^{-1}}_{\mathbf{id}}\circ\phi = \phi\circ\underline{\phi^{-1}\circ f}_{\mathbf{id}} = \underline{\phi^{-1}\circ f\circ\phi^{-1}}_{\mathbf{id}} = \phi\circ\underline{f}_{\mathbf{id}}\circ\phi\,.
\end{aligned}$$

2. *For a strict negator N the following identities hold:*

$$\begin{aligned}
\overline{f}^{N} &= \overline{f\circ N^{-1}}^{\mathbf{id}}\circ N = N\circ\overline{N^{-1}\circ f}_{\mathbf{id}} = \underline{N^{-1}\circ f\circ N^{-1}}^{\mathbf{id}} = N\circ\underline{f}_{\mathbf{id}}\circ N\,;\\
\overline{f}_{N} &= \overline{f\circ N^{-1}}_{\mathbf{id}}\circ N = N\circ\overline{N^{-1}\circ f}^{\mathbf{id}} = \underline{N^{-1}\circ f\circ N^{-1}}_{\mathbf{id}} = N\circ\underline{f}^{\mathbf{id}}\circ N\,;\\
\underline{f}^{N} &= \underline{f\circ N^{-1}}^{\mathbf{id}}\circ N = N\circ\underline{N^{-1}\circ f}_{\mathbf{id}} = \overline{N^{-1}\circ f\circ N^{-1}}^{\mathbf{id}} = N\circ\overline{f}_{\mathbf{id}}\circ N\,;\\
\underline{f}_{N} &= \underline{f\circ N^{-1}}_{\mathbf{id}}\circ N = N\circ\underline{N^{-1}\circ f}^{\mathbf{id}} = \overline{N^{-1}\circ f\circ N^{-1}}_{\mathbf{id}} = N\circ\overline{f}^{\mathbf{id}}\circ N\,.
\end{aligned}$$

Proof We will prove the theorem for $\underline{f}_{\phi}$ and $\underline{f}_{N}$ the other cases being similar. On the one hand, for an automorphism ϕ, we obtain that

$$\begin{aligned}
\underline{f}_{\phi}(x) &= \inf\{t\in[0,1] \mid f(\phi^{-1}(t)) < \phi(x)\} = \underline{f\circ\phi^{-1}}_{\mathbf{id}}(\phi(x))\,;\\
&= \phi(\inf\{s\in[0,1] \mid f(s) < \phi(x)\}) = \phi(\underline{f}_{\mathbf{id}}(\phi(x)))\,;\\
&= \inf\{t\in[0,1] \mid \phi^{-1}(f(\phi^{-1}(t))) < x\} = \underline{\phi^{-1}\circ f\circ\phi^{-1}}_{\mathbf{id}}(x)\,;\\
&= \phi(\inf\{s\in[0,1] \mid \phi^{-1}(f(s)) < x\}) = \phi(\underline{\phi^{-1}\circ f}_{\mathbf{id}}(x))\,,
\end{aligned}$$

for every $x\in[0,1]$. On the other hand, for a strict negator N, we obtain that

$$\begin{aligned}
\underline{f}_{N}(x) &= \inf\bigl\{t\in[0,1] \mid f(t^{(N^{-1})}) < x^{N}\bigr\} = \underline{f\circ N^{-1}}_{\mathbf{id}}(x^{N})\,;\\
&= \bigl(\sup\{s\in[0,1] \mid f(s) < x^{N}\}\bigr)^{N} = (\overline{f}^{\mathbf{id}}(x^{N}))^{N}\,;\\
&= \inf\bigl\{t\in[0,1] \mid (f(t^{(N^{-1})}))^{(N^{-1})} > x\bigr\} = \overline{N^{-1}\circ f\circ N^{-1}}_{\mathbf{id}}(x)\,;\\
&= \bigl(\sup\bigl\{s\in[0,1] \mid (f(s))^{(N^{-1})} > x\bigr\}\bigr)^{N} = (\underline{N^{-1}\circ f}^{\mathbf{id}}(x))^{N}\,,
\end{aligned}$$

for every $x\in[0,1]$. $\square$

Thanks to this theorem, properties of $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ are easily translated to properties of $\overline{f}^{\Phi}$, $\overline{f}_{\Phi}$, $\underline{f}^{\Phi}$ and $\underline{f}_{\Phi}$.

Corollary 1.8 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ and a monotone $[0,1] \to [0,1]$ function f. Both functions $\overline{f}^{\Phi}$ and $\overline{f}_{\Phi}$ have the same type of monotonicity as Φ. The monotonicity of the functions $\underline{f}^{\Phi}$ and $\underline{f}_{\Phi}$ is opposite to the monotonicity of Φ.*

Proof It is easily verified that $\overline{f}^{\mathbf{id}}$ and $\overline{f}_{\mathbf{id}}$ are always increasing and that $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ are always decreasing. Taking into account Theorem 1.7 yields the postulate. $\square$

We will now show that both sets $Q(f, \Phi)$ and $Q(f, \mathbf{id})$ are either *isomorphic* or *antimorphic*, (*i.e.* there exists an isomorphism or antimorphism from $Q(f, \Phi)$ to $Q(f, \mathbf{id})$). To this end we first recall some well-known definitions.

Definition 1.9 A *partially ordered set* or *poset* $(P, \leqslant)$ consists of a non-empty set P and a binary relation $\leqslant$ on P that satisfies the following properties:

(PS1) *Reflexivity*: $x \leqslant x$, for every $x \in P$.
(PS2) *Antisymmetry*: $x \leqslant y \ \wedge\ y \leqslant x \ \Rightarrow\ x = y$, for every $(x, y) \in P^2$.
(PS3) *Transitivity*: $x \leqslant y \ \wedge\ y \leqslant z \ \Rightarrow\ x \leqslant z$, for every $(x, y, z) \in P^3$.

A binary relation on P satisfying the above properties is called a *partial order*. If the partial order is clear from the context, we briefly use P to denote the poset $(P, \leqslant)$.

Definition 1.10 Let $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ be two posets. An *isomorphism* is an order-preserving $P_1 \to P_2$ bijection. An order-reversing $P_1 \to P_2$ bijection is called an *antimorphism*.

Automorphisms are those isomorphisms that map the unit interval $[0,1]$ to itself. Strict negators are in fact $[0,1] \to [0,1]$ antimorphisms. Note that we equip the set of all $[0,1] \to [0,1]$ functions with an elementwise partial ordering. Explicitly, for two $[0,1] \to [0,1]$ functions g_1 and g_2, $g_1 \leqslant g_2$ holds if $g_1(x) \leqslant g_2(x)$ is satisfied for every $x \in [0,1]$.

Theorem 1.11 *Consider two monotone $[0,1] \to [0,1]$ bijections Φ and Ψ. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold*

1. *$Q(f, \Phi)$ and $Q(f, \Psi)$ are isomorphic in case Φ and Ψ have the same type of monotonicity.*
2. *$Q(f, \Phi)$ and $Q(f, \Psi)$ are antimorphic in case Φ and Ψ have opposite types of monotonicity.*

In particular, for every $g \in Q(f, \Phi)$ there exists a unique function $h \in Q(f, \Psi)$ such that $\Phi^{-1} \circ g \circ \Phi^{-1} = \Psi^{-1} \circ h \circ \Psi^{-1}$.

Proof It suffices to prove that

$$\mathcal{I}_\Phi : Q(f, \Phi) \to Q(f, \mathbf{id}) : g \mapsto \mathcal{I}_\Phi(g) := \Phi^{-1} \circ g \circ \Phi^{-1}$$

is an isomorphism for every automorphism Φ and an antimorphism whenever Φ is a strict negator. Recall the geometrical construction of $Q(f, \Phi)$ and $Q(f, \mathbf{id})$. For every $g \in Q(f, \Phi)$ we know that $(\Phi^{-1}(g(x)), \Phi(x))$, with $x \in [0,1]$, is covered by a completion f^* of f. Hence, the set

$$\{(\Phi(x), \Phi^{-1}(g(x))) \mid x \in [0,1]\} = \{(u, \mathcal{I}_\Phi(g)(u)) \mid u \in [0,1]\}$$

indeed defines a $[0,1] \to [0,1]$ function belonging to $Q(f, \mathbf{id})$. Conversely, consider a function $k \in Q(f, \mathbf{id})$, then $\{(k(x), x) \mid x \in [0,1]\}$ is a subset of a completion f^*. It is clear that the set

$$\{(\Phi^{-1}(x), \Phi(k(x))) \mid x \in [0,1]\} = \{(u, \Phi(k(\Phi(u)))) \mid u \in [0,1]\}$$

defines a function belonging to $Q(f, \Phi)$ and thus $k = \mathcal{I}_\Phi(\Phi \circ k \circ \Phi)$. We conclude that $\mathcal{I}_\Phi$ is a surjection. The bijectivity of Φ ensures that $\mathcal{I}_\Phi$ is also injective. It clearly holds that $\mathcal{I}_\Phi$ is increasing whenever Φ is increasing and that $\mathcal{I}_\Phi$ is decreasing whenever Φ is decreasing. Therefore, depending on the monotonicity of Φ, $\mathcal{I}_\Phi$ is indeed an order-preserving or order-reversing bijection. $\square$

Corollary 1.12 *Consider two monotone $[0,1] \to [0,1]$ bijections Φ and Ψ, and a monotone $[0,1] \to [0,1]$ function f. Then for every $g \in Q(f, \Phi)$ there exists a unique function $h \in Q(f, \Psi)$ such that*

$$\overline{g}^\Phi = \overline{h}^\Psi,\ \overline{g}_\Phi = \overline{h}_\Psi,\ \underline{g}^\Phi = \underline{h}^\Psi \text{ and } \underline{g}_\Phi = \underline{h}_\Psi\,,$$

whenever Φ and Ψ have the same monotonicity and

$$\overline{g}^\Phi = \underline{h}^\Psi,\ \overline{g}_\Phi = \underline{h}_\Psi,\ \underline{g}^\Phi = \overline{h}^\Psi \text{ and } \underline{g}_\Phi = \overline{h}_\Psi\,,$$

whenever Φ and Ψ have opposite types of monotonicity.

Proof From Theorem 1.11 we know that, given a function $g \in Q(f, \Phi)$, there exists a unique function $h \in Q(f, \Psi)$ such that $\Phi^{-1} \circ g \circ \Phi^{-1} = \Psi^{-1} \circ h \circ \Psi^{-1}$. The statements then follow immediately from Theorem 1.7. $\square$

Taking a closer look at functions of the form $\Phi^{-1} \circ f \circ \Phi^{-1}$ we obtain the following result:

Theorem 1.13 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. For every monotone $[0,1] \to [0,1]$ function f it holds that $Q(f, \Phi) = Q(\Phi^{-1} \circ f \circ \Phi^{-1}, \mathbf{id})$.*

Proof Because $f(\Phi^{-1}(y)) = \Phi(x) \;\Leftrightarrow\; \Phi^{-1} \circ f \circ \Phi^{-1}(y) = x$ it holds that $(\Phi^{-1}(y), \Phi(x)) \in f^*$ if and only if $(y, x) \in (\Phi^{-1} \circ f \circ \Phi)^*$, with f^* a completion of f and $(\Phi^{-1} \circ f \circ \Phi)^*$ a completion of $\Phi^{-1} \circ f \circ \Phi$. Hence, $(x, y) \in (f^*)^\Phi \;\Leftrightarrow\; (x, y) \in ((\Phi^{-1} \circ f \circ \Phi)^*)^{\mathbf{id}}$. In view of the geometrical construction of the sets $Q(f, \Phi)$ and $Q(\Phi^{-1} \circ f \circ \Phi^{-1}, \mathbf{id})$, their equality follows. $\square$

1.3 The set $Q(f, \mathbf{id})$

The mathematical description of the set $Q(f, \mathbf{id})$ originates from the following observations dealing with monotone $[0,1] \to [0,1]$ functions f:

(Q1) If $x \in f([0,1])$, then $f^{-1}(x) := \{y \in [0,1] \mid f(y) = x\}$ is an interval.

(Q2a) If f is increasing and $x \in [0,1] \setminus f([0,1])$, then $\overline{f}^{\mathbf{id}}(x) = \overline{f}_{\mathbf{id}}(x)$.

(Q2b) If f is decreasing and $x \in [0,1] \setminus f([0,1])$, then $\underline{f}^{\mathbf{id}}(x) = \underline{f}_{\mathbf{id}}(x)$.

As shown by Schweizer and Sklar [85], the set $Q(f, \mathbf{id})$ can be described as the set of $[0,1] \to [0,1]$ functions g fulfilling the following conditions:

$\mathbf{(Q1)_{id}}$ $\big(\forall x \in f([0,1])\big)\big(g(x) \in \big[\inf(f^{-1}(x)), \sup(f^{-1}(x))\big]\big)$.

$\mathbf{(Q2a)_{id}}$ If f is increasing: $\big(\forall x \in [0,1] \setminus f([0,1])\big)\big(g(x) = \overline{f}^{\mathbf{id}}(x) = \overline{f}_{\mathbf{id}}(x)\big)$.

$\mathbf{(Q2b)_{id}}$ If f is decreasing: $\big(\forall x \in [0,1] \setminus f([0,1])\big)\big(g(x) = \underline{f}^{\mathbf{id}}(x) = \underline{f}_{\mathbf{id}}(x)\big)$.

Special attention is drawn to the constant functions $\boldsymbol{\alpha}$. $Q(\boldsymbol{\alpha}, \mathbf{id})$ contains functions fulfilling condition $\mathbf{(Q2a)_{id}}$ as well as functions fulfilling condition $\mathbf{(Q2b)_{id}}$. Whenever $f(0) \neq f(1)$, all elements of $Q(f, \mathbf{id})$ fulfill the same condition: either $\mathbf{(Q2a)_{id}}$ or $\mathbf{(Q2b)_{id}}$. According to Klement *et al.* [51], in this case we can merge conditions $\mathbf{(Q2a)_{id}}$ and $\mathbf{(Q2b)_{id}}$ as follows:

$\mathbf{(Q2)_{id}}$
$$\big(\forall\, x \in [0,1] \setminus f([0,1])\big)\big(g(x) = \sup\{t \in [0,1] \mid (f(t) - x) \cdot (f(1) - f(0)) < 0\}$$
$$= \inf\{t \in [0,1] \mid (f(t) - x) \cdot (f(1) - f(0)) > 0\}\big)\,.$$

In case $f(0) < f(1)$, resp. $f(1) < f(0)$, the function $\overline{f}^{\mathbf{id}}$, resp. $\underline{f}^{\mathbf{id}}$, is known as the *pseudo-inverse* $f^{(-1)}$ of f [51]. For a constant $[0,1] \to [0,1]$ function $\boldsymbol{\alpha}$, Klement *et al.* [51] define the pseudo-inverse as $\boldsymbol{\alpha}^{(-1)} := \mathbf{0}$. This pseudo-inverse does not necessarily coincide with $\overline{\boldsymbol{\alpha}}^{\mathbf{id}}$ or $\underline{\boldsymbol{\alpha}}^{\mathbf{id}}$, which can easily be verified by considering the $[0,1] \to [0,1]$ function $\mathbf{\frac{1}{2}}$. The authors were clearly inspired by the 'supremum expression' in condition $\mathbf{(Q2)_{id}}$. However, when dealing with constant functions, condition $\mathbf{(Q2)_{id}}$ can never hold as $\sup\emptyset = 0 < 1 = \inf\emptyset$. The 'supremum expression' in condition $\mathbf{(Q2)_{id}}$ is then neither related to condition $\mathbf{(Q2a)_{id}}$ nor to condition $\mathbf{(Q2b)_{id}}$. Pseudo-inverses are often used in the construction of triangular norms and conorms (see [50, 51, 90, 91]). They have been studied extensively in that context. Some of our results concerning the pseudo-inverse of non-constant monotone functions can be (partially) found in [50, 51, 91]. Our goal was not only to extend the existing knowledge, but also to purify the theorems from superfluous conditions and to rearrange the results in a more insightful way. We also clarified the inversion of constant functions.

We now try to figure out the significance of the four functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$. In the following theorem we investigate which of these functions belongs to $Q(f, \mathbf{id})$ and can therefore be understood as some kind of inverse of f.

Theorem 1.14 *For a monotone* $[0,1] \to [0,1]$ *function* f *the following assertions hold:*

1. *If* $f(0) < f(1)$*, then a* $[0,1] \to [0,1]$ *function* g *belongs to* $Q(f, \mathbf{id})$ *if and only if* $\overline{f}^{\mathbf{id}} \leqslant g \leqslant \overline{f}_{\mathbf{id}}$.
2. *If* $f(1) < f(0)$*, then a* $[0,1] \to [0,1]$ *function* g *belongs to* $Q(f, \mathbf{id})$ *if and only if* $\underline{f}^{\mathbf{id}} \leqslant g \leqslant \underline{f}_{\mathbf{id}}$.
3. *If* $f(0) = f(1)$*, then a* $[0,1] \to [0,1]$ *function* g *belongs to* $Q(f, \mathbf{id})$ *if and only if* $\overline{f}^{\mathbf{id}} \leqslant g \leqslant \overline{f}_{\mathbf{id}}$ *or* $\underline{f}^{\mathbf{id}} \leqslant g \leqslant \underline{f}_{\mathbf{id}}$.

Proof Can be shown easily by considering conditions **(Q1)$_{\mathbf{id}}$**, **(Q2a)$_{\mathbf{id}}$** and **(Q2b)$_{\mathbf{id}}$** and by recalling the definitions of the functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$. $\square$

The structural difference between $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ and $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$ implies the following corollary:

Corollary 1.15 *For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *If $f(0) < f(1)$, then $Q(f, \mathbf{id})$ contains increasing functions only and $\{\underline{f}^{\mathbf{id}}, \underline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$.*
2. *If $f(1) < f(0)$, then $Q(f, \mathbf{id})$ contains decreasing functions only and $\{\overline{f}^{\mathbf{id}}, \overline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$.*
3. *If $f(0) = f(1)$, then $Q(f, \mathbf{id})$ contains increasing and decreasing functions.*

Proof Consider arbitrary $(x, y) \in [0,1]^2$ such that $x < y$. Because $\{t \in [0,1] \mid f(t) \leqslant x\} \subseteq \{t \in [0,1] \mid f(t) < y\}$, it holds that

$$\begin{aligned} \overline{f}_{\mathbf{id}}(x) &= \sup\{t \in [0,1] \mid f(t) \leqslant x\} \leqslant \sup\{t \in [0,1] \mid f(t) < y\} = \overline{f}^{\mathbf{id}}(y) \\ \underline{f}_{\mathbf{id}}(y) &= \inf\{t \in [0,1] \mid f(t) < y\} \leqslant \inf\{t \in [0,1] \mid f(t) \leqslant x\} = \underline{f}^{\mathbf{id}}(x) \,. \end{aligned}$$

It is now easily verified that every function located between $\overline{f}^{\mathbf{id}}$ and $\overline{f}_{\mathbf{id}}$ is increasing and that every function located between $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ is decreasing. By definition it holds that $\overline{f}^{\mathbf{id}}(0) = \underline{f}^{\mathbf{id}}(1) = 0$ and $\overline{f}_{\mathbf{id}}(1) = \underline{f}_{\mathbf{id}}(0) = 1$. Furthermore, $\underline{f}^{\mathbf{id}}(0) = 1$ and $\underline{f}_{\mathbf{id}}(1) = 0$, resp. $\overline{f}^{\mathbf{id}}(1) = 1$ and $\overline{f}_{\mathbf{id}}(0) = 0$, if $f(0) < f(1)$, resp. $f(1) < f(0)$. Taking into account the monotonicity of the members of $Q(f, \mathbf{id})$ yields that $\{\underline{f}^{\mathbf{id}}, \underline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$ whenever $f(0) < f(1)$ and $\{\overline{f}^{\mathbf{id}}, \overline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$ whenever $f(1) < f(0)$. $\square$

Depending on the monotonicity of f, the functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ or $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$ do not only constitute the boundaries of $Q(f, \mathbf{id})$, they can also be sifted out of $Q(f, \mathbf{id})$ on the basis of their continuity.

Theorem 1.16 *Consider a monotone $[0,1] \to [0,1]$ function f satisfying $f \notin \{\mathbf{0}, \mathbf{1}\}$.*

1. *If f is increasing, then*
   a) *$\overline{f}^{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is left continuous and maps 0 to 0.*
   b) *$\overline{f}_{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is right continuous and maps 1 to 1.*
2. *If f is decreasing, then*
   a) *$\underline{f}^{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is right continuous and maps 1 to 0.*
   b) *$\underline{f}_{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is left continuous and maps 0 to 1.*

Proof Follows immediately from Theorem 1.14 and the fact that $\overline{f}^{\mathbf{id}}(0) = 0$, $\overline{f}_{\mathbf{id}}(1) = 1$, $\underline{f}^{\mathbf{id}}(1) = 0$, $\underline{f}_{\mathbf{id}}(0) = 1$. $\square$

The set $Q(\mathbf{0}, \mathbf{id})$, resp. $Q(\mathbf{1}, \mathbf{id})$, contains exactly two continuous functions: $\underline{\mathbf{0}}^{\mathbf{id}} = \mathbf{0}$ and $\overline{\mathbf{0}}_{\mathbf{id}} = \mathbf{1}$, resp. $\overline{\mathbf{1}}^{\mathbf{id}} = \mathbf{0}$ and $\underline{\mathbf{1}}_{\mathbf{id}} = \mathbf{1}$. The above theorem has to be adjusted as follows.

Theorem 1.17 *The following assertions hold:*

1. a) $\overline{\mathbf{0}}^{\mathbf{id}}$ *and* $\underline{\mathbf{0}}^{\mathbf{id}}$ *are the only members of* $Q(\mathbf{0}, \mathbf{id})$ *that are left continuous and map* 0 *to* 0.
   b) $\overline{\mathbf{0}}_{\mathbf{id}}$ *is the only member of* $Q(\mathbf{0}, \mathbf{id})$ *that is right continuous and maps* 1 *to* 1.
   c) $\underline{\mathbf{0}}^{\mathbf{id}}$ *is the only member of* $Q(\mathbf{0}, \mathbf{id})$ *that is right continuous and maps* 1 *to* 0.
   d) $\overline{\mathbf{0}}_{\mathbf{id}}$ *and* $\underline{\mathbf{0}}_{\mathbf{id}}$ *are the only members of* $Q(\mathbf{0}, \mathbf{id})$ *that are left continuous and map* 0 *to* 1.

2. a) $\overline{\mathbf{1}}^{\mathbf{id}}$ *is the only member of* $Q(\mathbf{1}, \mathbf{id})$ *that is left continuous and maps* 0 *to* 0.
   b) $\overline{\mathbf{1}}_{\mathbf{id}}$ *and* $\underline{\mathbf{1}}_{\mathbf{id}}$ *are the only members of* $Q(\mathbf{1}, \mathbf{id})$ *that are right continuous and map* 1 *to* 1.
   c) $\overline{\mathbf{1}}^{\mathbf{id}}$ *and* $\underline{\mathbf{1}}^{\mathbf{id}}$ *are the only members of* $Q(\mathbf{1}, \mathbf{id})$ *that are right continuous and map* 1 *to* 0.
   d) $\underline{\mathbf{1}}_{\mathbf{id}}$ *is the only member of* $Q(\mathbf{1}, \mathbf{id})$ *that is left continuous and maps* 0 *to* 1.

Note that Theorem 1.16 remains applicable to the other constant functions $\boldsymbol{\alpha}$, with $\alpha \in \,]0, 1[$. The boundary conditions ensure the unicity.

We now focus on the characteristic properties of the classical inverse and figure out under which conditions these properties are preserved in the new framework. Firstly, we deal with the *involutivity* of the 'inverse', *i.e.* $(f^{-1})^{-1} = f$. From Theorem 1.6 we know that $f \in Q(g, \mathbf{id})$, for every $g \in Q(f, \mathbf{id})$. Therefore, interpreting g as an inverse of f and f as an inverse of g, we obtain that in some sense inverting some inverse yields the original function. For monotone bijections f this reasoning is sound as $Q(f, \mathbf{id}) = \{f^{\mathbf{id}}\} = \{f^{-1}\}$ (Theorem 1.5). Otherwise, whenever f is not bijective, we know that $|Q(f, \mathbf{id})| > 1$ and/or $|Q(g, \mathbf{id})| > 1$, for some $g \in Q(f, \mathbf{id})$ (Theorem 1.5). We need to find out how the inverse g of f, resp. the inverse of g, should be selected from the set $Q(f, \mathbf{id})$, resp. $Q(g, \mathbf{id})$. Special attention is drawn here to the boundary functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ and $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$.

Theorem 1.18 *For a monotone* $[0, 1] \to [0, 1]$ *function* f *the following assertions hold:*

1. *If there exists a function* $g \in Q(f, \mathbf{id})$ *such that* $\overline{g}^{\mathbf{id}} = f$, *then* f *must be increasing, left continuous and* $f(0) = 0$.
2. *If there exists a function* $g \in Q(f, \mathbf{id})$ *such that* $\overline{g}_{\mathbf{id}} = f$, *then* f *must be increasing, right continuous and* $f(1) = 1$.
3. *If there exists a function* $g \in Q(f, \mathbf{id})$ *such that* $\underline{g}^{\mathbf{id}} = f$, *then* f *must be decreasing, right continuous and* $f(1) = 0$.
4. *If there exists a function* $g \in Q(f, \mathbf{id})$ *such that* $\underline{g}_{\mathbf{id}} = f$, *then* f *must be decreasing, left continuous and* $f(0) = 1$.

Proof Consider a monotone function f and suppose that there exists a function $g \in Q(f, \mathbf{id})$ such that $\overline{g}^{\mathbf{id}} = f$. The increasingness of f is an immediate consequence of Corollary 1.8. From Corollary 1.15 we know that g must be increasing whenever $f(0) < f(1)$. In case $f(0) = f(1)$, it follows from $f(0) = \overline{g}^{\mathbf{id}}(0) = \sup\{t \in [0,1] \mid g(t) < 0\} = 0$ that $f = \mathbf{0}$. Therefore, $\overline{g}^{\mathbf{id}}(x) = \sup\{t \in [0,1] \mid g(t) < x\} = 0$, for every $x \in [0,1]$. The latter can only hold if $g(t) = 1$, for every $t \in]0,1]$. We conclude that g must be increasing. Theorems 1.16 and 1.17 then ensure the left continuity of $f = \overline{g}^{\mathbf{id}}$. The other cases are proven in the same way. $\square$

Note that neither $\underline{g}^{\mathbf{id}} = f$ nor $\underline{g}_{\mathbf{id}} = f$ can hold if $f(0) < f(1)$ and $g \in Q(f, \mathbf{id})$. Indeed, in contrast to f, both functions $\underline{g}^{\mathbf{id}}$ and $\underline{g}_{\mathbf{id}}$ are decreasing (Corollary 1.8). Similarly, if $f(1) < f(0)$, there does not exist a function $g \in Q(f, \mathbf{id})$ such that $\overline{g}^{\mathbf{id}} = f$ or $\overline{g}_{\mathbf{id}} = f$. Also the converse of the previous theorem holds.

Theorem 1.19 *Consider a non-constant monotone $[0,1] \to [0,1]$ function f.*

1. *For an increasing function f it holds that:*

   a) *If f is left continuous and $f(0) = 0$, then $\overline{g}^{\mathbf{id}} = f$, for every $g \in Q(f, \mathbf{id})$.*
   b) *If f is right continuous and $f(1) = 1$, then $\overline{g}_{\mathbf{id}} = f$, for every $g \in Q(f, \mathbf{id})$.*

2. *For a decreasing function f it holds that:*

   a) *If f is right continuous and $f(1) = 0$, then $\underline{g}^{\mathbf{id}} = f$, for every $g \in Q(f, \mathbf{id})$.*
   b) *If f is left continuous and $f(0) = 1$, then $\underline{g}_{\mathbf{id}} = f$, for every $g \in Q(f, \mathbf{id})$.*

Proof Consider a left-continuous, increasing function f for which $f(0) = 0$ and take $g \in Q(f, \mathbf{id})$. Theorem 1.14 and the left continuity of f ensure that

$$g(f(x) - \varepsilon) \leqslant \overline{f}_{\mathbf{id}}(f(x) - \varepsilon) = \inf\{t \in [0,1] \mid f(t) > f(x) - \varepsilon\} < x\,, \tag{1.1}$$

for every $x \in [0,1]$ such that $0 < f(x)$ and with $\varepsilon \in]0, f(x)]$. Moreover, it holds that

$$g(f(x) + \varepsilon) \geqslant \overline{f}^{\mathbf{id}}(f(x) + \varepsilon) = \sup\{t \in [0,1] \mid f(t) < f(x) + \varepsilon\} \geqslant x\,, \tag{1.2}$$

for every $x \in [0,1]$ such that $f(x) < 1$ and with $\varepsilon \in]0, 1 - f(x)]$. Consider arbitrary $x \in [0,1]$ such that $f(x) \in]0,1[$ and let $\varepsilon \in]0, \min(f(x), 1 - f(x))]$. As g is increasing (Corollary 1.15), combining Eqs. (1.1) and (1.2) leads to

$$\overline{g}^{\mathbf{id}}(x) = \sup\{t \in [0,1] \mid g(t) < x\} = f(x)\,. \tag{1.3}$$

In case $f(x) = 0$, then Eq. (1.2), with arbitrary $\varepsilon \in]0,1]$, also implies Eq. (1.3). In a similar way, Eq. (1.1) implies Eq. (1.3) whenever $f(x) = 1$. We conclude that $\overline{g}^{\mathbf{id}} = f$. The other cases are proven in a similar way. $\square$

For the constant functions $\mathbf{0}$ and $\mathbf{1}$ we obtain the following result.

Theorem 1.20 *For a $[0,1] \to [0,1]$ function g the following assertions hold:*

1. $\overline{g}^{\mathbf{id}} = \mathbf{0}$ *if and only if* $\overline{\mathbf{0}}^{\mathbf{id}} \leqslant g \leqslant \overline{\mathbf{0}}_{\mathbf{id}}$.

2. $\overline{g}_{\mathbf{id}} = \mathbf{1}$ *if and only if* $\overline{\mathbf{1}}^{\mathbf{id}} \leqslant g \leqslant \overline{\mathbf{1}}_{\mathbf{id}}$.
3. $\underline{g}^{\mathbf{id}} = \mathbf{0}$ *if and only if* $\underline{\mathbf{0}}^{\mathbf{id}} \leqslant g \leqslant \underline{\mathbf{0}}_{\mathbf{id}}$.
4. $\underline{g}_{\mathbf{id}} = \mathbf{1}$ *if and only if* $\underline{\mathbf{1}}^{\mathbf{id}} \leqslant g \leqslant \underline{\mathbf{1}}_{\mathbf{id}}$.

Proof As $\overline{\mathbf{0}}^{\mathbf{id}}(x) = \overline{\mathbf{0}}_{\mathbf{id}}(x) = 1$ whenever $x \in]0,1]$ and $\overline{\mathbf{0}}^{\mathbf{id}}(0) = 0 < 1 = \overline{\mathbf{0}}_{\mathbf{id}}(0)$, it holds that $\overline{\mathbf{0}}^{\mathbf{id}} \leqslant g \leqslant \overline{\mathbf{0}}_{\mathbf{id}}$ is equivalent with $g(x) = 1$, for every $x \in]0,1]$. The latter is also equivalent with $\overline{g}^{\mathbf{id}}(x) = \sup\{t \in [0,1] \mid g(t) < x\} = 0$, for every $x \in [0,1]$. This proves the first assertion. The other assertions are proven in the same way. $\square$

Note that if, for example, $\overline{g}^{\mathbf{id}}$ equals a non-constant left continuous function f fulfilling $f(0) = 0$, then it does not necessarily hold that $g \in Q(f, \mathbf{id})$ (e.g. $g = \overline{\mathbf{0}}^{\mathbf{id}}$). This prevents us from further generalizing Theorem 1.19.

In classical analysis it holds that $f^{-1} \circ f = \mathbf{id}$ if and only if f is injective. It is easily verified that $\overline{f}^{\mathbf{id}} \circ f \leqslant \mathbf{id}_{[0,1]} \leqslant \overline{f}_{\mathbf{id}} \circ f$ whenever f is increasing and $\underline{f}^{\mathbf{id}} \circ f \leqslant \mathbf{id} \leqslant \underline{f}_{\mathbf{id}} \circ f$ whenever f is decreasing.

Theorem 1.21 *A monotone $[0,1] \to [0,1]$ function f is injective if and only if there exists a function $g \in Q(f, \mathbf{id})$ such that $g \circ f = \mathbf{id}$.*

Proof We present the proof for an increasing function f. If $g \circ f = \mathbf{id}$ holds for some $g \in Q(f, \mathbf{id})$, then g must be surjective. From Theorem 1.5 it then follows that $Q(g, \mathbf{id})$ contains only injective functions. Since $f \in Q(g, \mathbf{id})$ (Theorem 1.6), this means that f must be injective. Conversely, assume that f is an injective increasing $[0,1] \to [0,1]$ function. Expressing the injectivity of f

$$(\forall x \in]0,1])(\forall \varepsilon \in]0,x])(f(x-\varepsilon) < f(x))$$

is equivalent with

$$\overline{f}^{\mathbf{id}}(f(x)) = \sup\{t \in [0,1] \mid f(t) < f(x)\} = x\,,$$

for every $x \in [0,1]$. Recall from Theorems 1.5 and 1.14 that $Q(f, \mathbf{id}) = \{\overline{f}^{\mathbf{id}}\}$. Hence, $g \circ f = \mathbf{id}$, if $g \in Q(f, \mathbf{id})$. $\square$

For monotone $[0,1] \to [0,1]$ functions f, it holds that $f \circ f^{-1} = \mathbf{id}$ if and only if f is bijective. The injectivity of f ensures that f^{-1} is a function. Since $Q(f, \mathbf{id})$ only contains functions, the injectivity of f will become superfluous when replacing f^{-1} by some $g \in Q(f, \mathbf{id})$.

Theorem 1.22 *A monotone $[0,1] \to [0,1]$ function f is surjective if and only if there exists a function $g \in Q(f, \mathbf{id})$ such that $f \circ g = \mathbf{id}$. In particular, $f \circ g = \mathbf{id}$, for every $g \in Q(f, \mathbf{id})$.*

Proof We present the proof for an increasing function f. Clearly $f \circ g = \mathbf{id}$, for some $g \in Q(f, \mathbf{id})$, requires the surjectivity of f. Conversely, suppose that f is surjective, then f is continuous, $f(0) = 0$ and $f(1) = 1$. By definition it then holds that

$$f(\overline{f}^{\mathbf{id}}(x)) = f(\sup\{t \in [0,1] \mid f(t) < x\}) = x = f(\inf\{t \in [0,1] \mid f(t) > x\}) = f(\overline{f}_{\mathbf{id}}(x))\,,$$

for every $x \in [0,1]$. Taking into account that $\overline{f}^{\mathbf{id}} \leqslant g \leqslant \overline{f}_{\mathbf{id}}$, for every $g \in Q(f,\mathbf{id})$ (Theorem 1.14), this leads to $f \circ \underline{f}^{\mathbf{id}} = f \circ g = f \circ \underline{f}_{\mathbf{id}} = \mathbf{id}$. $\square$

Combining Theorems 1.21 and 1.22, we obtain the following corollary.

Corollary 1.23 *A monotone $[0,1] \to [0,1]$ function f is bijective if and only if there exists a function $g \in Q(f,\mathbf{id})$ such that $g \circ f = \mathbf{id}$ and $f \circ g = \mathbf{id}$.*

Recall that in this case necessarily $g = f^{-1}$ (Theorem 1.5).

1.4 The set $Q(f,\Phi)$

In this section we generalize our previous results concerning the set $Q(f,\mathbf{id})$, to properties of the set $Q(f,\Phi)$ where f is a monotone $[0,1] \to [0,1]$ function and Φ is a monotone $[0,1] \to [0,1]$ bijection. The correlation between $Q(f,\mathbf{id})$ and $Q(f,\Phi)$ (see Theorem 1.11) allows a straightforward conversion of the properties of $Q(f,\mathbf{id})$ to those of $Q(f,\Phi)$: for every $g \in Q(f,\Phi)$, we know that $\Phi^{-1} \circ g \circ \Phi^{-1}$ belongs to $Q(f,\mathbf{id})$. Throughout this translation process we make extensively use of Theorem 1.7 and Corollary 1.12, where $\Psi = \mathbf{id}$. The proofs are elementary and therefore left out.

A. Φ is an automorphism ϕ

The set $Q(f,\phi)$ can be described as the set of all $[0,1] \to [0,1]$ functions g satisfying the following conditions:

(Q1)$_\phi$ $\big(\forall\, x \in \phi^{-1}(f([0,1]))\big)\Big(g(x) \in \phi\big(\big[\inf(f^{-1}(\phi(x))), \sup(f^{-1}(\phi(x)))\big]\big)\Big)$.

(Q2a)$_\phi$ If f is increasing: $\big(\forall\, x \in \phi^{-1}\left([0,1] \setminus f([0,1])\right)\big)\big(g(x) = \overline{f}^{\phi}(x) = \overline{f}_{\phi}(x)\big)$.

(Q2b)$_\phi$ If f is decreasing: $\big(\forall\, x \in \phi^{-1}\left([0,1] \setminus f([0,1])\right)\big)\big(g(x) = \underline{f}^{\phi}(x) = \underline{f}_{\phi}(x)\big)$.

For a constant function $\boldsymbol{\alpha}$, with $\alpha \in [0,1]$, the set $Q(\boldsymbol{\alpha},\phi)$ contains functions satisfying **(Q2a)$_\phi$** as well as functions satisfying **(Q2b)$_\phi$**. The following theorems point out the significance and importance of the functions $\overline{f}^{\phi}, \overline{f}_{\phi}, \underline{f}^{\phi}$ and $\underline{f}_{\phi}$.

Theorem 1.24 *Consider an automorphism ϕ. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *If $f(0) < f(1)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,\phi)$ if and only if $\overline{f}^{\phi} \leqslant g \leqslant \overline{f}_{\phi}$.*
2. *If $f(1) < f(0)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,\phi)$ if and only if $\underline{f}^{\phi} \leqslant g \leqslant \underline{f}_{\phi}$.*
3. *If $f(0) = f(1)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,\phi)$ if and only if $\overline{f}^{\phi} \leqslant g \leqslant \overline{f}_{\phi}$ or $\underline{f}^{\phi} \leqslant g \leqslant \underline{f}_{\phi}$.*

It is clear that $Q(f, \phi)$ only contains increasing, resp. decreasing, functions provided that $f(0) < f(1)$, resp. $f(1) < f(0)$. Depending on the monotonicity of f, the functions $\overline{f}^{\phi}$, $\overline{f}_{\phi}$ and $\underline{f}^{\phi}$, $\underline{f}_{\phi}$ can also be characterized by means of some continuity conditions.

Theorem 1.25 *Consider an automorphism ϕ and a monotone $[0,1] \to [0,1]$ function f satisfying $f \notin \{\mathbf{0}, \mathbf{1}\}$.*

1. *If f is increasing, then*

   a) *$\overline{f}^{\phi}$ is the only member of $Q(f, \phi)$ that is left continuous and maps 0 to 0.*
   b) *$\overline{f}_{\phi}$ is the only member of $Q(f, \phi)$ that is right continuous and maps 1 to 1.*

2. *If f is decreasing, then*

   a) *$\underline{f}^{\phi}$ is the only member of $Q(f, \phi)$ that is right continuous and maps 1 to 0.*
   b) *$\underline{f}_{\phi}$ is the only member of $Q(f, \phi)$ that is left continuous and maps 0 to 1.*

Dealing with the constant functions $\mathbf{0}$ and $\mathbf{1}$, we have to reformulate Theorem 1.17 in a similar way. This adjustment has been omitted since it is straightforward yet lengthy. Next, we show under which conditions 'inverting' some inverse of f yields the original function.

Theorem 1.26 *Consider an automorphism ϕ. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *If there exists a function $g \in Q(f, \phi)$ such that $\overline{g}^{\phi} = f$, then f must be increasing, left continuous and $f(0) = 0$.*
2. *If there exists a function $g \in Q(f, \phi)$ such that $\overline{g}_{\phi} = f$, then f must be increasing, right continuous and $f(1) = 1$.*
3. *If there exists a function $g \in Q(f, \phi)$ such that $\underline{g}^{\phi} = f$, then f must be decreasing, right continuous and $f(1) = 0$.*
4. *If there exists a function $g \in Q(f, \phi)$ such that $\underline{g}_{\phi} = f$, then f must be decreasing, left continuous and $f(0) = 1$.*

Also the converse property holds.

Theorem 1.27 *Consider an automorphism ϕ and a non-constant monotone $[0,1] \to [0,1]$ function f.*

1. *For an increasing function f it holds that:*

   a) *If f is left continuous and $f(0) = 0$, then $\overline{g}^{\phi} = f$, for every $g \in Q(f, \phi)$.*
   b) *If f is right continuous and $f(1) = 1$, then $\overline{g}_{\phi} = f$, for every $g \in Q(f, \phi)$.*

2. *For a decreasing function f it holds that:*

   a) *If f is right continuous and $f(1) = 0$, then $\underline{g}^{\phi} = f$, for every $g \in Q(f, \phi)$.*

b) If f is left continuous and $f(0)=1$, then $\underline{g}_\phi = f$, for every $g \in Q(f,\phi)$.

The results for the constant functions $\mathbf{0}$ and $\mathbf{1}$ are easily obtained from Theorem 1.20. Although Theorems 1.21, 1.22 and Corollary 1.23 can also be easily transformed to properties on the set $Q(f,\phi)$, it still remains unclear what the meaning is of $g \circ f$ and $f \circ g$ with $g \in Q(f,\phi)$. Also, $f^\phi \circ f$ and $f \circ f^\phi$ have no straightforward interpretation.

B. Φ is a strict negator N

The set $Q(f,N)$ can be described as the set of all $[0,1] \to [0,1]$ functions g satisfying the following conditions:

(Q1)$_N$ $\big(\forall\, x \in (f([0,1]))^{(N^{-1})}\big)\Big(g(x) \in \big([\inf(f^{-1}(x^N)), \sup(f^{-1}(x^N))]\big)^N\Big)$.

(Q2a)$_N$ if f is increasing: $\big(\forall\, x \in ([0,1] \setminus f([0,1]))^{(N^{-1})}\big)\big(g(x) = \underline{f}^N(x) = \underline{f}_N(x)\big)$.

(Q2b)$_N$ if f is decreasing: $\big(\forall\, x \in ([0,1] \setminus f([0,1]))^{(N^{-1})}\big)\big(g(x) = \overline{f}^N(x) = \overline{f}_N(x)\big)$.

Working with decreasing bijections instead of increasing bijections interchanges the role of the functions $\overline{f}^N$ and $\underline{f}^N$ and of the functions $\overline{f}_N$ and $\underline{f}_N$.

Theorem 1.28 *Consider a strict negator N. For a monotone $[0,1] \to [0,1]$ function f the following assertions hold:*

1. *If $f(0) < f(1)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,N)$ if and only if $\underline{f}^N \leqslant g \leqslant \underline{f}_N$.*
2. *If $f(1) < f(0)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,N)$ if and only if $\overline{f}^N \leqslant g \leqslant \overline{f}_N$.*
3. *If $f(0) = f(1)$, then a $[0,1] \to [0,1]$ function g belongs to $Q(f,N)$ if and only if $\underline{f}^N \leqslant g \leqslant \underline{f}_N$ or $\overline{f}^N \leqslant g \leqslant \overline{f}_N$.*

Every function located between $\underline{f}^N$ and $\underline{f}_N$ is increasing and every function located between $\overline{f}^N$ and $\overline{f}_N$ is decreasing. In the following theorem we try to pinpoint the functions $\overline{f}^N$, $\overline{f}_N$ and $\underline{f}^N$, $\underline{f}_N$ by means of their continuity.

Theorem 1.29 *Consider a strict negator N and a monotone $[0,1] \to [0,1]$ function f satisfying $f \notin \{\mathbf{0},\mathbf{1}\}$.*

1. *If f is increasing, then*

   a) $\underline{f}^N$ is the only member of $Q(f,N)$ that is left continuous and maps 0 to 0.
   b) $\underline{f}_N$ is the only member of $Q(f,N)$ that is right continuous and maps 1 to 1.

2. *If f is decreasing, then*

a) $\overline{f}^N$ *is the only member of* $Q(f,N)$ *that is right continuous and maps* 1 *to* 0.
b) $\overline{f}_N$ *is the only member of* $Q(f,N)$ *that is left continuous and maps* 0 *to* 1.

The next two theorems we can be used to more profoundly study the existence of the identities

$$\overline{\overline{f}^N}^N = f, \ \overline{\overline{f}_N}_N = f, \ \underline{\underline{f}^N}^N = f \text{ and } \underline{\underline{f}_N}_N = f\,.$$

Theorem 1.30 *Consider a strict negator* N. *For a monotone* $[0,1] \to [0,1]$ *function* f *the following assertions hold:*

1. *If there exists a function* $g \in Q(f,N)$ *such that* $\underline{g}^N = f$, *then* f *must be increasing, left continuous and* $f(0) = 0$.
2. *If there exists a function* $g \in Q(f,N)$ *such that* $\underline{g}_N = f$, *then* f *must be increasing, right continuous and* $f(1) = 1$.
3. *If there exists a function* $g \in Q(f,N)$ *such that* $\overline{g}^N = f$, *then* f *must be decreasing, right continuous and* $f(1) = 0$.
4. *If there exists a function* $g \in Q(f,N)$ *such that* $\overline{g}_N = f$, *then* f *must be decreasing, left continuous and* $f(0) = 1$.

Theorem 1.31 *Consider a strict negator* N *and a non-constant monotone* $[0,1] \to [0,1]$ *function* f.

1. *For an increasing function* f *it holds that:*

   a) *If* f *is left continuous and* $f(0) = 0$, *then* $\underline{g}^N = f$, *for every* $g \in Q(f,N)$.
   b) *If* f *is right continuous and* $f(1) = 1$, *then* $\underline{g}_N = f$, *for every* $g \in Q(f,N)$.

2. *For a decreasing function* f *it holds that:*

   a) *If* f *is right continuous and* $f(1) = 0$, *then* $\overline{g}^N = f$, *for every* $g \in Q(f,N)$.
   b) *If* f *is left continuous and* $f(0) = 1$, *then* $\overline{g}_N = f$, *for every* $g \in Q(f,N)$.

CHAPTER 2

Orthosymmetry of monotone functions

2.1 Introduction

The identity function **id** and symmetrical strict negators are the only monotone $[0,1] \to [0,1]$ functions f that coincide with their inverse. However, a monotone function f can have some symmetrical behaviour w.r.t. a monotone $[0,1] \to [0,1]$ bijection Φ different from **id**. Introducing two new kinds of symmetry, one based on the Φ-inverse of f and one based on its associated set $Q(f,\Phi)$ of Φ-inverse functions, we reveal some unknown symmetry aspects of monotone functions. Special attention goes to symmetrical pairs. The study of these pairs, consisting of two monotone $[0,1] \to [0,1]$ bijections that are symmetrical w.r.t. each other, will provide new insights into the class of monotone $[0,1] \to [0,1]$ bijections. Composing two strict negators yields an automorphism and no composition of two automorphisms results in a strict negator. This observation indicates that the class of all monotone $[0,1] \to [0,1]$ bijections can at best be described in terms of strict negators. Note that the latter contrasts the approach of Trillas [87] and Fodor [25]. Their characterization of involutive, resp. strict, negators is based on automorphisms. More profoundly studying symmetrical pairs sheds a new light on some mathematical folklore: every monotone $[0,1] \to [0,1]$ bijection can be built from at most four involutive negators [24, 38, 73]. This allows us to partition the class of monotone bijections into four subclasses. We present a geometrical way of determining the subclass in which a monotone bijection Φ can be classified. Although our method is very similar to the approaches of Young [99], Jarczyk [39] and O'Farrell [73], we largely focus on the geometrical and symmetry aspects of the construction instead of recalling its topological background.

Besides involutive negators several other types of *negators* (*i.e.* decreasing $[0,1] \to [0,1]$ functions N satisfying $0^N = 1$ and $1^N = 0$) have been studied in the literature: the intuitionistic negation [97] and its dual [75], fractal negations [68], Sugeno negations [67], negations generated by compensations [93], contracting and expanding negations [4, 2], sub-involutive and super-involutive negators [12, 21, 96], etc. Fixed point properties of negators have been investigated in [37] and [92]. Batyrshin and Wagenknecht [3] laid bare the overall structure of a non-involutive strict negator N.

2.2 Orthosymmetry

Generalizing the classical notion of symmetry w.r.t. the identity function **id**, we now study sets that are symmetrical w.r.t. a given monotone $[0,1] \to [0,1]$ bijection Φ.

Definition 2.1 Let Φ be a monotone $[0,1] \to [0,1]$ bijection. A set $A \subseteq [0,1]^2$ is Φ-*symmetrical* if it satisfies $A = A^\Phi$.

Explicitly, A is Φ-symmetrical if it holds that $(\Phi^{-1}(y), \Phi(x)) \in A \;\Leftrightarrow\; (x,y) \in A$. Unfortunately, when dealing with a monotone $[0,1] \to [0,1]$ function f only bijections can coincide with their Φ-inverse. Indeed, if f has discontinuity points, its Φ-inverse f^Φ will not be defined on the entire unit interval $[0,1]$. Otherwise, if f is not injective, its Φ-inverse will not be a function. To overcome these problems we will further generalize the concept of symmetry in terms of the set $Q(f,\Phi)$ which contains the Φ-inverse functions associated with f.

Definition 2.2 Let Φ be a monotone $[0,1] \to [0,1]$ bijection. A monotone $[0,1] \to [0,1]$ function f is Φ-*orthosymmetrical* if $f \in Q(f,\Phi)$.

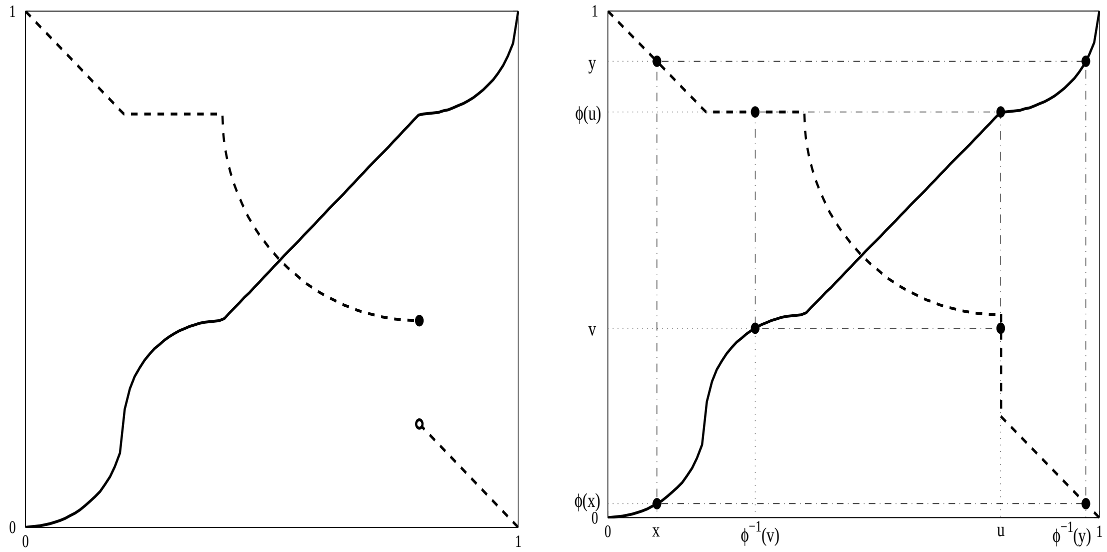
The prefix 'ortho' refers to the rectangle-based construction of $Q(f,\Phi)$ (see Section 1.2). By definition of the set $Q(f,\Phi)$ it holds that a monotone function f is Φ-orthosymmetrical if it has a completion f^* such that its Φ-inverse $(f^*)^\Phi$ is again a completion of f. Due to the uniqueness of its completion f^*, a non-constant monotone function f is Φ-orthosymmetrical if it has a Φ-symmetrical completion (*i.e.* $(f^*)^\Phi = f^*$). Figure 2.1 depicts an automorphism ϕ (solid line) for which the decreasing function f (dashed black line) from Fig. 1.2 is ϕ-orthosymmetrical.

Theorem 2.3 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. If a monotone $[0,1] \to [0,1]$ function f is Φ-orthosymmetrical, then every member of $Q(f,\Phi)$ is Φ-orthosymmetrical.*

Proof If f is Φ-orthosymmetrical, then there exists a function $g \in Q(f,\Phi)$ such that $f = g$. Consider then $h \in Q(g,\Phi)$. Based on Theorem 1.6, we know that $Q(h,\Phi) = Q(f,\Phi) = Q(g,\Phi)$. Therefore $h \in Q(h,\Phi)$, for every $h \in Q(f,\Phi)$. $\square$

From Theorem 1.5 we know that $Q(f,\Phi) = \{f^\Phi\}$ holds whenever f is bijective. This observation straightforwardly allows us to link Φ-symmetry to Φ-orthosymmetry.

Theorem 2.4 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. A monotone $[0,1] \to [0,1]$ bijection Ψ is Φ-symmetrical if and only if it is Φ-orthosymmetrical. Monotone $[0,1] \to [0,1]$ bijections are the only monotone $[0,1] \to [0,1]$ functions that can be both Φ-symmetrical and Φ-orthosymmetrical.*



(a) ϕ-orthosymmetrical function f (b) $(f^*)^{\phi} = f^*$

Figure 2.1: A ϕ-orthosymmetrical decreasing function f (dashed line), with ϕ the automorphism depicted by the solid line.

Proof The first statement follows immediately from $Q(\Psi, \Phi) = \{\Psi^{\Phi}\}$. Furthermore, if a monotone $[0,1] \to [0,1]$ function f is Φ-symmetrical and Φ-orthosymmetrical, then necessarily $f = f^{\Phi} \in Q(f, \Phi)$. Theorem 1.5 states that in this case f must be bijective. $\square$

As Φ itself is clearly Φ-symmetrical this leads to the following corollary.

Corollary 2.5 *Every monotone $[0,1] \to [0,1]$ bijection Φ is Φ-orthosymmetrical.*

The following theorem yields necessary and sufficient conditions for Φ-orthosymmetry in terms of the boundary functions $\overline{f}^{\Phi}$, $\overline{f}_{\Phi}$, $\underline{f}^{\Phi}$ and $\underline{f}_{\Phi}$.

Theorem 2.6 *Consider a monotone $[0,1] \to [0,1]$ bijection Φ. Then a non-constant monotone $[0,1] \to [0,1]$ function f is Φ-orthosymmetrical if and only if*

1. *$\overline{f}^{\Phi} \leqslant f \leqslant \overline{f}_{\Phi}$ in case f and Φ have the same type of monotonicity.*
2. *$\underline{f}^{\Phi} \leqslant f \leqslant \underline{f}_{\Phi}$ in case f and Φ have opposite types of monotonicity.*

The only Φ-orthosymmetrical constant functions are $\mathbf{0}$ *and* $\mathbf{1}$.

Proof The first part is an immediate consequence of Theorems 1.24 and 1.28. Consider now a Φ-orthosymmetrical, constant function $\boldsymbol{\alpha}$ and suppose that $\alpha \in \,]0,1[$. By definition, it holds that $\overline{\boldsymbol{\alpha}}^{\Phi}(0) = \overline{\boldsymbol{\alpha}}_{\Phi}(0) \in \{0,1\}$ and $\underline{\boldsymbol{\alpha}}^{\Phi}(0) = \underline{\boldsymbol{\alpha}}_{\Phi}(0) \in \{0,1\}$. However, we know from Theorems 1.24 and 1.28 that $\overline{\boldsymbol{\alpha}}^{\Phi}(0) \leqslant \boldsymbol{\alpha}(0) \leqslant \overline{\boldsymbol{\alpha}}_{\Phi}(0)$ or $\underline{\boldsymbol{\alpha}}^{\Phi}(0) \leqslant \boldsymbol{\alpha}(0) \leqslant \underline{\boldsymbol{\alpha}}_{\Phi}(0)$, which leads to the contradiction $\alpha = \boldsymbol{\alpha}(0) \in \{0,1\}$. Our supposition $\alpha \in \,]0,1[$ is false and hence, $\alpha \in \{0,1\}$.

Because $\mathbf{0} = \underline{\mathbf{0}}^\Phi \in Q(\mathbf{0}, \Phi)$ and $\mathbf{1} = \underline{\mathbf{1}}_\Phi \in Q(\mathbf{1}, \Phi)$, the constant functions $\mathbf{0}$ and $\mathbf{1}$ are indeed Φ-orthosymmetrical. $\square$

The geometrical construction of a Φ-inverse (Section 1.2) implies that a monotone bijection Ψ can only be Φ-symmetrical if it coincides with the bijection Φ itself or if Ψ and Φ have opposite types of monotonicity. Similar results hold when considering Φ-orthosymmetry.

Theorem 2.7 *Consider a monotone* $[0,1] \to [0,1]$ *bijection* Φ *and a non-constant monotone* $[0,1] \to [0,1]$ *function* f*. If* f *is* Φ*-orthosymmetrical, then one of the following assertions holds:*

1. $f = \Phi$*.*
2. f *and* Φ *have opposite types of monotonicity.*

Proof It suffices to prove that $f = \Phi$ whenever f and Φ have the same type of monotonicity. From Theorem 2.6 we know that

$$\sup\{t \in [0,1] \mid f(\Phi^{-1}(t)) < \Phi(x)\} \leqslant f(x) \leqslant \inf\{t \in [0,1] \mid f(\Phi^{-1}(t)) > \Phi(x)\}\,,$$

for every $x \in [0,1]$. In particular this means that $\Phi(x) \leqslant f(\Phi^{-1}(t))$ whenever $t \in \,]f(x), 1]$ and that $f(\Phi^{-1}(t)) \leqslant \Phi(x)$ whenever $t \in [0, f(x)[$. Suppose that there exists a number $x \in [0,1]$ such that $f(x) < \Phi(x)$. If we choose arbitrary $t \in \,]f(x), \Phi(x)[$, then the increasingness of $f \circ \Phi^{-1}$ implies that $f(\Phi^{-1}(t)) \leqslant f(x) < \Phi(x)$, which contradicts $\Phi(x) \leqslant f(\Phi^{-1}(t))$. Similarly, suppose that there exists a number $x \in [0,1]$ such that $\Phi(x) < f(x)$, then for every $t \in \,]\Phi(x), f(x)[$, we obtain the contradiction $\Phi(x) < f(x) \leqslant f(\Phi^{-1}(t))$. We conclude that $f = \Phi$. $\square$

For a non-constant function f it is imposible that $\underline{f}^\Phi \leqslant f \leqslant \underline{f}_\Phi$ if f and Φ have the same type of monotonicity. This is easily illustrated by evaluating the functions in $x = 0$. It enables us to simplify the previous theorem. Combining Theorems 2.6 and 2.7 leads to the following result

Corollary 2.8 *Consider a monotone* $[0,1] \to [0,1]$ *bijection* Φ*. A monotone* $[0,1] \to [0,1]$ *function* f *is* Φ*-orthosymmetrical if and only if either* $f \in \{\mathbf{0}, \Phi, \mathbf{1}\}$ *or* $\underline{f}^\Phi \leqslant f \leqslant \underline{f}_\Phi$*.*

Based on Theorems 1.25 and 1.29 we can provide simple methods to verify whether a non-constant, left- or right-continuous, monotone $[0,1] \to [0,1]$ function $f \neq \Phi$ is Φ-orthosymmetrical or not. Depending on the monotonicity of f and Φ, the continuity of f, and given some additional boundary conditions, we have to verify whether $f = \underline{f}^\Phi$ or $f = \underline{f}_\Phi$ holds. Moreover, given the bijection Φ, these equalities fix the monotonicity and continuity of f, and imply its Φ-orthosymmetry. The explicit formulation of these results has been omitted as they are straightforwardly obtained by combining Corollary 1.8, Theorems 1.24 and 1.26, and Corollary 2.8 and by combining Corollary 1.8, Theorems 1.28 and 1.30, and Corollary 2.8. Note that every $f \in \{\mathbf{0}, \Phi, \mathbf{1}\}$ besides being Φ-orthosymmetrical is trivially continuous. In the following theorem we look for all other continuous, Φ-orthosymmetrical, monotone functions. For uninorms, the result will facilitate the description of continuous contour lines (Chapter 5).

Theorem 2.9 *Consider a monotone bijection Φ and a non-constant, monotone $[0,1] \to [0,1]$ function f that has the opposite type of monotonicity as Φ. Then f is Φ-orthosymmetrical and continuous if and only if $f(\Phi^{-1}(f(x))) = \Phi(x)$ holds for every $x \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$ with either $f(0) \in \{0,1\}$ or $f(1) \in \{0,1\}$.*

Proof $\Rightarrow$ Suppose that f is Φ-orthosymmetrical and continuous. Then necessarily $\underline{f}^{\Phi} \leqslant f \leqslant \underline{f}_{\Phi}$ (Corollary 2.8). Take $x \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$ and denote $l := \min\{t \in [0,1] \mid f(\Phi^{-1}(t)) = \Phi(x)\}$ and $u := \max\{t \in [0,1] \mid f(\Phi^{-1}(t)) = \Phi(x)\}$. The decreasingness of $f \circ \Phi^{-1}$ ensures that $l = \underline{f}^{\Phi}(x) \leqslant f(x) \leqslant \underline{f}_{\Phi}(x) = u$. Hence, $f(\Phi^{-1}(f(x))) = \Phi(x)$, for every $x \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$. Suppose now that $f(0) \in\,]0,1[$, then, taking into account that $f \circ \Phi^{-1}$ is decreasing and f is non-constant, it is easily verified that $\underline{f}^{\Phi}(1) = \underline{f}_{\Phi}(1) = 0$ whenever Φ is increasing and that $\underline{f}^{\Phi}(1) = \underline{f}_{\Phi}(1) = 1$ whenever Φ is decreasing. Invoking Corollary 2.8 this leads to $f(1) \in \{0,1\}$. In a similar way $f(1) \in\,]0,1[$ implies that $f(0) \in \{0,1\}$.

$\Leftarrow$ Assume that f satisfies $f(\Phi^{-1}(f(x))) = \Phi(x)$, for every $x \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$, and that either $f(0) \in \{0,1\}$ or $f(1) \in \{0,1\}$. Then, $f(x) = f(y)$, with $(x,y) \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]^2$, can only occur if $x = y$ and f must reach every number in $[\min(f(0), f(1)), \max(f(0), f(1))]$. As f and Φ have opposite types of monotonicity, we also know that either $\Phi^{-1}(f(0)) = 1$ or $\Phi^{-1}(f(1)) = 0$. Hence, $f(1) = f(\Phi^{-1}(f(0)))$ or $f(0) = f(\Phi^{-1}(f(1)))$ which leads to resp. $f(\Phi^{-1}(f(1))) = f(\Phi^{-1}(f(\Phi^{-1}(f(0))))) = f(0)$ and $f(\Phi^{-1}(f(0))) = f(\Phi^{-1}(f(\Phi^{-1}(f(1))))) = f(1)$. We conclude that $f(\Phi^{-1}(f(0))) = f(1)$ and $f(\Phi^{-1}(f(1))) = f(0)$. The restriction of f to $[\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$ is a $[\Phi^{-1}(f(1)), \Phi^{-1}(f(0))] \to [\min(f(0), f(1)), \max(f(0), f(1))]$ bijection. Taking into account that $f([0, \Phi^{-1}(f(1))]) = \{f(0)\}$ and $f([\Phi^{-1}(f(0)), 1]) = \{f(1)\}$, it follows that f must be continuous on $[0,1]$.

To illustrate the Φ-orthosymmetry of f it suffices to show that $\underline{f}^{\Phi} \leqslant f \leqslant \underline{f}_{\Phi}$ (Corollary 2.8). For every $x \in [\Phi^{-1}(f(1)), \Phi^{-1}(f(0))]$ the decreasingness of $f \circ \Phi^{-1}$ ensures that

$$
\begin{aligned}
\underline{f}^{\Phi}(x) &= \sup\{t \in [0,1] \mid f(\Phi^{-1}(t)) > f(\Phi^{-1}(f(x)))\} \leqslant f(x) \\
&\leqslant \inf\{t \in [0,1] \mid f(\Phi^{-1}(t)) < f(\Phi^{-1}(f(x)))\} = \underline{f}_{\Phi}(x)\,.
\end{aligned}
$$

We now have to figure out what happens if $x \in [0, \Phi^{-1}(f(1))[\, \cup\,]\Phi^{-1}(f(0)), 1]$. Recall that either $\Phi^{-1}(f(0)) = 1$ or $\Phi^{-1}(f(1)) = 0$. In case $0 < \Phi^{-1}(f(1))$, then $\Phi^{-1}(f(0)) = 1$. Take arbitrary $x \in [0, \Phi^{-1}(f(1))[$. If Φ is increasing then $f(0) = 1$ and $\Phi(x) < f(1) = f(\Phi^{-1}(f(0))) = f(\Phi^{-1}(1))$. For a decreasing bijection Φ we obtain that $f(0) = 0$ and $f(\Phi^{-1}(0)) = f(\Phi^{-1}(f(0))) = f(1) < \Phi(x)$. Recall from the discussion above that $f(x) = f(0)$. By definition it then holds that $f(x) = 1 = \underline{f}^{\Phi}(x) = \underline{f}_{\Phi}(x)$ if Φ is increasing and $f(x) = 0 = \underline{f}^{\Phi}(x) = \underline{f}_{\Phi}(x)$ if Φ is decreasing. A similar reasoning applies to $\Phi^{-1}(f(0)) < 1$. This finishes the proof. $\square$

To conclude this section we investigate the convergence of a sequence of Φ-orthosymmetrical, monotone functions. Consider the family $(\phi_n)_{n \in \mathbb{N}_0}$ of automorphisms defined by $\phi_n(x) = \sqrt[n]{1 - (1-x)^n}$. It is easily verified that all these automorphisms are $\mathcal{N}$-symmetrical, with $\mathcal{N}$

the standard negator. Unfortunately, the limit function $\phi_\infty = \lim_{n\to\infty} \phi_n$, given by

$$\phi_\infty(x) = \begin{cases} 0, & \text{if } x = 0\,, \\ 1, & \text{if } x \in \,]0,1]\,, \end{cases}$$

is not $\mathcal{N}$-symmetrical. Nevertheless, as $(\phi^*_\infty)^{\mathcal{N}} = \phi^*_\infty$, the $\mathcal{N}$-orthosymmetry of the automorphisms ϕ_n is passed on to ϕ_∞.

Theorem 2.10 *Consider a monotone* $[0,1] \to [0,1]$ *bijection* Φ*. The limit of a pointwisely converging sequence of* Φ*-orthosymmetrical, monotone* $[0,1] \to [0,1]$ *functions* $(f_n)_{n\in\mathbb{N}}$ *is always a* Φ*-orthosymmetrical, monotone* $[0,1] \to [0,1]$ *function.*

Proof Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Φ-orthosymmetrical, monotone $[0,1] \to [0,1]$ functions pointwisely converging to a function f. Clearly, f is a monotone $[0,1] \to [0,1]$ function. If $f \in \{\mathbf{0}, \mathbf{1}\}$ then it follows from Theorem 2.6 that f is Φ-orthosymmetrical. Furthermore, $f = \Phi$ trivially ensures the Φ-orthosymmetry of f. Suppose now that $f \notin \{\mathbf{0}, \Phi, \mathbf{1}\}$, then there exits a number $n_0 \in \mathbb{N}$ such that all functions f_n, with $n \geqslant n_0$, differ from $\mathbf{0}$, Φ and $\mathbf{1}$. From Corollary 2.8 we then know that

$$\sup\{t \in [0,1] \mid f_n(\Phi^{-1}(t)) > \Phi(x)\} \leqslant f_n(x) \leqslant \inf\{t \in [0,1] \mid f_n(\Phi^{-1}(t)) < \Phi(x)\}\,,$$

for every $x \in [0,1]$ and every $n \geqslant n_0$. The latter implies that for $n \geqslant n_0$ it holds that $f_n(\Phi^{-1}(t)) \leqslant \Phi(x)$ whenever $t \in \,]f_n(x), 1]$ and that $\Phi(x) \leqslant f_n(\Phi^{-1}(t))$ whenever $t \in [0, f_n(x)[$. Suppose now that there exists a number $t \in \,]f(x), 1]$ such that $f(\Phi^{-1}(t)) > \Phi(x)$. Because $\lim_{n\to\infty} f_n = f$, there exists a natural number $n_1 > n_0$ such that for every $n \geqslant n_1$ it holds that $t \in \,]f_n(x), 1]$. Furthermore, there exists a second natural number $n_2 > n_1$ such that $f_n(\Phi^{-1}(t)) > \Phi(x)$, for every $n \geqslant n_2$. Combining both results we obtain the contradiction that there exists for every $n \geqslant n_2$ a number $t \in \,]f_n(x), 1]$ such that $f_n(\Phi^{-1}(t)) > \Phi(x)$. Consequently, it necessarily holds that $f(\Phi^{-1}(t)) \leqslant \Phi(x)$ whenever $t \in \,]f(x), 1]$. In a similar way, it is shown that $\Phi(x) \leqslant f(\Phi^{-1}(t))$ whenever $t \in [0, f(x)[$. Hence,

$$\sup\{t \in [0,1] \mid f(\Phi^{-1}(t)) > \Phi(x)\} \leqslant f(x) \leqslant \inf\{t \in [0,1] \mid f(\Phi^{-1}(t)) < \Phi(x)\}\,,$$

for every $x \in [0,1]$, or, in other words $\underline{f}^{\Phi} \leqslant f \leqslant \underline{f}_{\Phi}$. Applying Corollary 2.8 finishes the proof. $\square$

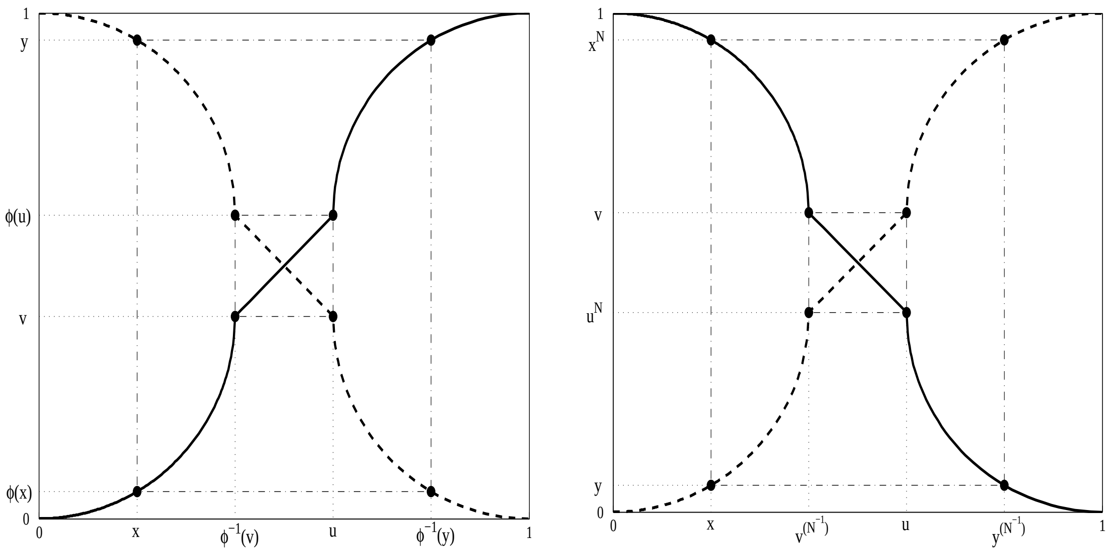
From Theorem 2.6, it then follows that a sequence of Φ-orthosymmetrical, monotone $[0,1] \to [0,1]$ functions $(f_n)_{n\in\mathbb{N}}$ can never converge to $\boldsymbol{\alpha}$ if $\alpha \in \,]0,1[$.

2.3 Symmetrical pairs

Dealing with the Φ-orthosymmetry of a monotone $[0,1] \to [0,1]$ bijection Ψ we know that it suffices to investigate its Φ-symmetry only (Theorem 2.4). Explicitly, Ψ is Φ-symmetrical if and only if $\Psi = \Psi^{\Phi} = \Phi \circ \Psi^{-1} \circ \Phi$ or equivalently $\Phi = \Psi \circ \Phi^{-1} \circ \Psi$, which expresses the Ψ-symmetry of Φ. This interchangeability between Φ and Ψ supports the following definition.

Definition 2.11 Two monotone $[0,1] \to [0,1]$ bijections Φ and Ψ form a *symmetrical pair* $\{\Phi, \Psi\}$ if Ψ is Φ-symmetrical.

In particular, $\{\Phi, \Phi\}$ is a (trivial) symmetrical pair for every monotone $[0,1] \to [0,1]$ bijection Φ (Corollary 2.5). Figure 2.2 illustrates that the automorphism ϕ (solid line) displayed in Fig. 1.2(a) and strict negator N (solid line) displayed in Fig. 1.1(b) form a symmetrical pair.



(a) N is ϕ-symmetrical (b) ϕ is N-symmetrical

Figure 2.2: A symmetrical pair $\{\phi, N\}$, with ϕ the automorphism depicted by the solid in Fig. 2.2(a) and N the strict negator depicted by the solid line in Fig. 2.2(b).

In the following theorem we present a method for constructing a symmetrical pair $\{\Phi, \Psi\}$, given one of its components.

Theorem 2.12 *Two monotone $[0,1] \to [0,1]$ bijections Φ and Ψ form a symmetrical pair if and only if $\Psi = \Phi$ or there exists a number $\beta \in \,]0,1[$ and a monotone $[0,\beta] \to \Phi([\beta,1])$ bijection Γ with the opposite type of monotonicity as Φ such that*

$$\Psi(x) = \begin{cases} \Gamma(x), & \textit{if } x \in [0,\beta]\,, \\ \Phi(\Gamma^{-1}(\Phi(x))), & \textit{if } x \in [\beta,1]\,. \end{cases} \tag{2.1}$$

Proof In case $\Psi = \Phi$ or Eq. (2.1) holds, we immediately obtain that $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$. The latter expresses the Φ-symmetry of Ψ and, hence, $\{\Phi, \Psi\}$ is a symmetrical pair. Conversely, if $\Psi \neq \Phi$ and $\{\Phi, \Psi\}$ is a symmetrical pair, then $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$. Since in this case Ψ and Φ must have opposite types of monotonicity (Theorem 2.7), it holds that $\Psi(0) = \Phi(1)$ and $\Psi(1) = \Phi(0)$. Furthermore, there exists a unique $\beta \in \,]0,1[$ such that $\Psi(\beta) = \Phi(\beta)$. Hence, $\Psi([0,\beta]) = \Phi([\beta,1])$ and $\Psi([\beta,1]) = \Phi([0,\beta])$. It is then clear that $\Gamma := \Psi|_{[0,\beta]}$ is a $[0,\beta] \to \Phi([\beta,1])$ bijection. Note that Γ has the same type of monotonicity as Ψ and that $\Gamma^{-1} = \Psi^{-1}|_{\Phi([\beta,1])}$. Taking into account that $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$, Eq. (2.1) is easily verified. $\square$

Studying symmetrical pairs, involutive bijections and in particular involutive negators will play a profound role.

Definition 2.13 Let $A \subseteq [0,1]$. A monotone $A \to A$ function f is *involutive* if $f \circ f = \mathbf{id}|_A$.

Obviously, every involutive monotone function f must be bijective. Its surjectivity is straightforward and its injectivity is required as $f(x) = f(y)$, for some $(x,y) \in A^2$ implies that $x = f(f(x)) = f(f(y)) = y$. Geometrically, involutive monotone functions are exactly those monotone $A \to A$ bijections that are $\mathbf{id}$-symmetrical (see e.g. also [2]). Hence, the identity function $\mathbf{id}$ is the only involutive automorphism. All other involutive monotone $[0,1] \to [0,1]$ functions are involutive strict negators. They will be briefly referred to as *involutive negators*. The standard negator $\mathcal{N}$ is the prototype of such an involutive negator. The importance of involutive negators already shows from the observation that they link the components of symmetrical pairs.

Theorem 2.14 *Two monotone* $[0,1] \to [0,1]$ *bijections* Φ *and* Ψ *form a symmetrical pair if and only if* $\Psi = \Phi$ *or there exists an involutive negator* N *such that* $\Psi = \Phi \circ N$.

Proof For a symmetrical pair $\{\Phi, \Psi\}$ it holds by definition that $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$. Rewriting this equality as $(\Phi^{-1} \circ \Psi) \circ (\Phi^{-1} \circ \Psi) = \mathbf{id}$, it follows that $\Phi^{-1} \circ \Psi$ must be involutive. Hence, $\Phi^{-1} \circ \Psi = \mathbf{id}$ or $\Phi^{-1} \circ \Psi$ defines an involutive negator N. Conversely, as $\{\Phi, \Phi\}$ is a trivial symmetrical pair, we only have to consider $\Psi = \Phi \circ N$, for some involutive negator N. Expressing the involutivity of N leads to $N = \Phi^{-1} \circ \Psi = \Psi^{-1} \circ \Phi$. The latter implies that $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$. We conclude that Ψ is indeed Φ-symmetrical and, thus, forms a symmetrical pair with Φ. $\square$

Definition 2.15 Let Φ and Ψ be two monotone $[0,1] \to [0,1]$ bijections and consider $A \subseteq [0,1]$. If $\Phi(x) < \Psi(x)$ whenever $x \in A$, or $\Phi(x) = \Psi(x)$ whenever $x \in A$, or $\Psi(x) < \Phi(x)$ whenever $x \in A$, we say that the *mutual position* of Φ and Ψ is *fixed on* A. Otherwise, we say that the *mutual position* of Φ and Ψ *on* A is *undetermined*.

For instance, the mutual position of an automorphism ϕ and a strict negator N is fixed on the sets $[0,\beta[$, $\{\beta\}$ and $]\beta,1]$, with β the unique point satisfying $\phi(\beta) = \beta^N$. In the following theorem we present a sufficient condition such that an automorphism ϕ is N-symmetrical, with N some involutive negator. For every monotone $[0,1] \to [0,1]$ bijection Φ we denote $\Phi \circ \ldots \circ \Phi$ (j times), resp. $\Phi^{-1} \circ \ldots \circ \Phi^{-1}$ (j times), as Φ^j, resp. Φ^{-j}. By convention, $\Phi^0 = \mathbf{id}$.

Theorem 2.16 *Consider an automorphism* ϕ*. If the mutual position of* ϕ *and* $\mathbf{id}$ *is fixed on* $]0,1[$*, then* ϕ *is symmetrical w.r.t. an involutive negator.*

Proof As stated before, every involutive negator forms a symmetrical pair with $\phi = \mathbf{id}$. We present the proof for an automorphism ϕ satisfying $\mathbf{id}|_{]0,1[} < \phi|_{]0,1[}$, the case $\phi|_{]0,1[} < \mathbf{id}|_{]0,1[}$ being similar. Choose $a \in]0,1[$ and let N_1 be an arbitrary decreasing $[a,\phi(a)] \to [a,\phi(a)]$ bijection fulfilling $N_1 \circ N_1 = \mathbf{id}|_{[a,\phi(a)]}$ (a rescaled involutive negator will do). Figure 2.3 illustrates how we can build, starting from N_1, an involutive negator that is ϕ-symmetrical. Part I of N is the **id**-symmetrical bijection N_1. Drawing the ϕ-inverse of part I, we obtain part II. Reflecting part II

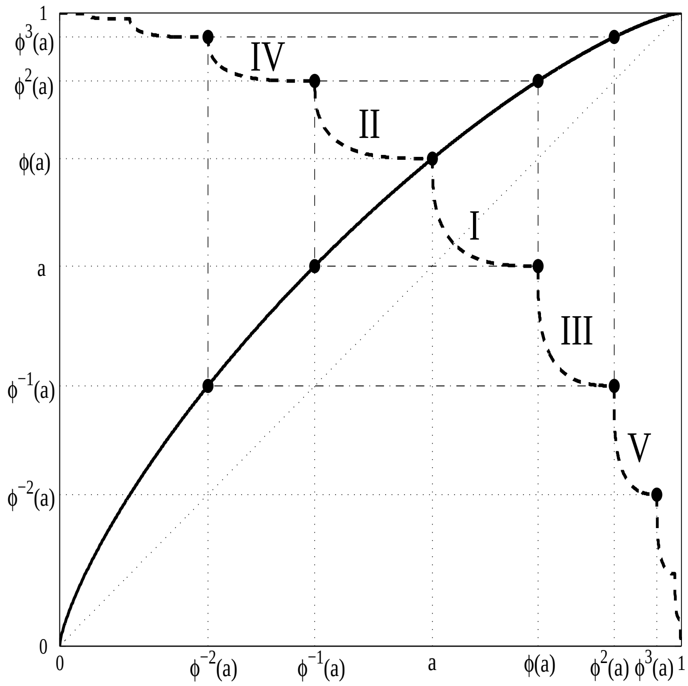


Figure 2.3: Construction of an involutive negator N (dashed line) that forms a symmetrical pair with a given automorphism ϕ (solid line) that satisfies $x < \phi(x)$, for every $x \in]0,1[$.

about the identy function yields part III. Part IV is established by expressing that it must be the ϕ-inverse of part III. Part V is the reflection (about **id**) of part IV. Pursuing this procedure of alternately implementing ϕ-symmetry and **id**-symmetry, yields an appropriate negator N. Mathematically, we can describe N as follows:

$$x^N = \begin{cases} 1, & \text{if } x = 0\,, \\ \phi^{-i} \circ N_1 \circ \phi^{-i}(x), & \text{if } x \in [\phi^i(a), \phi^{i+1}(a)], \text{ with } i \in \mathbb{Z}\,, \\ 0, & \text{if } x = 1\,. \end{cases}$$

Because $a < \phi(a)$, we know that $\phi^i(a) < \phi^{i+1}(a)$, for every $i \in \mathbb{Z}$. Note also that $(\phi^i(a))^N = \phi^{-(i-1)}(a)$. The function N is clearly continuous and strictly decreasing on $]\phi^{-\infty}(a), \phi^{\infty}(a)[$. Consider the equality $\phi^{i+1}(a) = \phi(\phi^i(a))$. Taking the limits $i \to -\infty$ and $i \to \infty$ it follows from the continuity of ϕ that $\phi^{-\infty}(a) = \phi(\phi^{-\infty}(a))$ and $\phi^{\infty}(a) = \phi(\phi^{\infty}(a))$. The latter is only possible if $\phi^{-\infty}(a) = 0$ and $\phi^{\infty}(a) = 1$. Therefore, N is indeed a strict negator. Due to the observation that

$$x^N \in \begin{cases} [1,1], & \text{if } x = 0\,, \\ [\phi^{-i}(a), \phi^{-i+1}(a)], & \text{if } x \in [\phi^i(a), \phi^{i+1}(a)], \text{ with } i \in \mathbb{Z}\,, \\ [0,0], & \text{if } x = 1\,. \end{cases}$$

it is now easily verified that $N \circ N = \mathbf{id}$ and $\phi = N \circ \phi^{-1} \circ N$. We conclude that N is indeed an involutive negator and that ϕ is N-symmetrical (*i.e.* $\{\phi, N\}$ is a symmetrical pair). □

The proof of Theorem 2.16 provides a method for constructing an involutive negator that is ϕ-symmetrical, with ϕ a given automorphism whose position on the interval $]0,1[$ is fixed w.r.t. the identity function **id**. As can be seen from the construction method, there exist infinitely many appropriate involutive negators. Combining Theorems 2.14 and 2.16, we obtain the following result.

Corollary 2.17 *Consider an automorphism ϕ. If the mutual position of ϕ and* **id** *is fixed on $]0,1[$, then there exist two involutive negators N_1 and N_2 such that $\phi = N_1 \circ N_2$.*

Unfortunately, we cannot extend this corollary to all automorphisms ϕ. For example, let ϕ be an automorphism fulfilling $\phi(a) = a$, for some $a \in]0,1[$, $\phi(x) < x$, whenever $x < a$, and $x < \phi(x)$, whenever $a < x$. Suppose that $\phi = N_1 \circ N_2$, where N_1 and N_2 are involutive negators. Then $a^{N_1} = a^{N_2}$, $x^{N_1} < x^{N_2}$, whenever $x < a$ and $x^{N_2} < x^{N_1}$, whenever $a < x$. For arbitrary $y < \min(a, a^{N_1})$ it holds that $\max(a, a^{N_1}) < y^{N_1}$. Hence, $y^{N_2} < y^{N_1}$ and $(y^{N_1})^{N_1} < (y^{N_1})^{N_2}$. The second inequality is equivalent with $y^{N_1} < y^{N_2}$, a contradiction. The automorphism ϕ can never be written as a composition of two involutive negators.

Theorem 2.18
1. *For every strict negator N there exist three involutive negators N_1, N_2 and N_3 such that $N = N_1 \circ N_2 \circ N_3$.*
2. *For every automorphism ϕ there exist four involutive negators N_1, N_2, N_3 and N_4 such that $\phi = N_1 \circ N_2 \circ N_3 \circ N_4$.*

Proof Consider an arbitrary strict negator N and choose an involutive negator N_1 such that $x^N < x^{N_1}$ holds for every $x \in]0,1[$. If $x = a_N$ is the unique number in $[0,1]$ satisfying $x^N = x$, it suffices to define N_1 as follows (see Theorem 2.12 with $\Phi = \mathbf{id}$ and $\Psi = N_1$):

$$x^{N_1} = \begin{cases} x^M, & \text{if } x \in [0,\beta]\,, \\ x^{(M^{-1})}, & \text{if } x \in [\beta, 1]\,, \end{cases}$$

where $\beta \in]a_N, 1[$ and M is a decreasing $[0,\beta] \to [\beta, 1]$ bijection satisfying $x^M > \max(x^N, x^{(N^{-1})})$, for every $x \in]0,\beta]$. As $N < N_1$ on $]0,1[$, we also know that $x < (x^N)^{N_1}$, for every $x \in]0,1[$. The automorphism $N_1 \circ N$ fulfills the conditions of Theorem 2.16: $\mathbf{id}|_{]0,1[} < N_1 \circ N|_{]0,1[}$. There exist now two involutive negators N_2 and N_3 such that $N_1 \circ N = N_2 \circ N_3$, or equivalently $N = N_1 \circ N_2 \circ N_3$. Note that every automorphism ϕ can be written as $\phi = N \circ N_4$, with N_4 an arbitrary involutive negator and $N := \phi \circ N_4$ a strict negator. This completes the proof. $\square$

Theorem 2.18 enables us to partition the set of monotone $[0,1] \to [0,1]$ bijections. There exist two types of strict negators: involutive ones and non-involutive ones. Every non-involutive strict negator is a composition of three involutive negators and can never be represented as a single involutive negator. The set of automorphisms can also be divided into two parts. On the one hand, we distinguish automorphisms that are symmetrical w.r.t. an involutive negator N. These automorphisms can be expressed as a composition of two involutive negators. On the other hand, we group those automorphisms that are not symmetrical w.r.t. any involutive negator. They can never be composed out of two involutive negators and are always written as the composition of four involutive negators. As can be seen from the proofs of Theorems 2.16 and 2.18, the set of involutive negators generating a given monotone bijection Φ is not unique.

Remarks 2.19

1. Interpreting $[0,1]$ as a topological space, with the open subintervals of $[0,1]$ as its open subsets, every monotone $[0,1] \to [0,1]$ bijection Φ that forms a symmetrical pair with an involutive negator N is said to be *conjugate* to Φ^{-1}. Monotone $[0,1] \to [0,1]$ bijections Φ constitute the set of *homeomorphisms* (*i.e.* continuous bijections between two topological spaces that have a continuous inverse) on the unit interval $[0,1]$. In this context, Theorem 2.16, Corollary 2.17 and Theorem 2.18 reproduce some less-known historical mathematical results (see e.g. [24, 38, 73]). Unaware of their existence we rediscovered these results by studying symmetrical pairs. Several months after the acceptance of our work for publication [62], E. Walker brought the matter to our attention. Nevertheless, we opted to explicitly present here the proofs of the results as our approach additionally provides a simple method for constructing an appropriate set of involutive negators generating a given monotone $[0,1] \to [0,1]$ bijection Φ and displays very clearly the symmetry aspects of this building process.
2. From a group-theoretical point of view the set $\mathcal{G}$, of all monotone $[0,1] \to [0,1]$ bijections, equipped with the composition $\circ$ forms a group $\mathbb{G} := (\mathcal{G}, \circ)$ that has neutral element $\mathbf{id}$ [71]. The set of all automorphisms forms a non-trivial subgroup of $\mathbb{G}$ [71]. Therefore, no set of automorphisms can generate $\mathbb{G}$. Moreover, $\mathbb{G}$ is an example of a *Coxeter group* as it is generated by its involutive elements (Theorem 2.18).

2.4 Automorphisms that have an alternating behaviour

Given an automorphism ϕ it is yet unclear how to determine whether it can be written as a composition of two involutive negators or not. Young [99] and O'Farrell [73] give a characterization of such automorphisms by using an appropriate signature concept. Jarczyk [39] already recognizes some symmetrical behaviour in ϕ. As argued in Remark 2.19, we also independently described automorphisms that are the composition of two involutive negators. Our approach is slightly more elaborated as it focuses more profoundly on the geometrical aspects of such an automorphism ϕ and provides an easy method for constructing two involutive negators generating ϕ.

Definition 2.20 Let $A \subset [0,1]$. A number $x \in A$ is a *fixpoint* of an $A \to [0,1]$ function f if $f(x) = x$.

It is obvious that every strict negator N has a unique fixpoint. Every automorphism ϕ has at least two fixpoints: 0 and 1. Denote the set of all fixpoints of an automorphism ϕ by $\mathcal{F}_\Phi$. The continuity of an automorphism ϕ ensures that $\mathcal{F}_\phi$ is the union of closed disjoint subintervals of $[0,1]$. As the total number of these intervals can never exceed the cardinality of $\mathbb{Q}$, we know that their number is countable. Note that intervals containing only a single point are also possible. Let B_ϕ be the set containing all the endpoints of the intervals constituting $\mathcal{F}_\phi$:

$$B_\phi := \left\{ x \in \mathcal{F}_\phi \mid (\forall \varepsilon \in]0, \min(x, 1-x)])(\exists\, y \in [x-\varepsilon, x+\varepsilon])(y \notin \mathcal{F}_\phi) \right\}.$$

Note that $\{0,1\} \subseteq B_\phi$. The following properties will be crucial for the overall structure of automorphisms that are the composition of two involutive negators.

Property 2.21 *For an automorphism ϕ the following properties hold:*

(A1) $|B_\phi| \leqslant \aleph_0 (= |\mathbb{N}|)$.
(A2) $\phi|_{B_\phi} = \mathbf{id}|_{B_\phi}$.
(A3) *The mutual position of ϕ and* **id** *is fixed on $]x, y[$, for every pair of consecutive elements $(x, y) \in B_\phi^2$.*
(A4) $\inf(A) \in B_\phi$ *and* $\sup(A) \in B_\phi$*, for every set $A \subseteq B_\phi$.*
(A5) $[0,1]$ *is the union of B_ϕ and all open intervals $]x, y[$, with (x, y) a pair of consecutive elements of B_ϕ.*

Proof **(A1)&(A2):** As $B_\phi \subseteq \mathcal{F}_\phi$, the cardinality of B_ϕ must be countable and $\phi|_{B_\phi} = \mathbf{id}|_{B_\phi}$.

(A3): Let x and y be two consecutive elements in B_ϕ $(x < y)$ and suppose that the mutual position of ϕ and **id** is not fixed on $]x, y[$. Then, due to the continuity of ϕ there exists a subinterval $]u, v[\subset]x, y[$ such that $]u, v[\cap \mathcal{F}_\phi = \emptyset$ and $(u, v) \in \mathcal{F}_\phi^2$. The latter is only possible if either $u \in]x, y[\cap B_\phi$ or $v \in]x, y[\cap B_\phi$, a contradiction.

(A4): Consider a set $A \subseteq B_\phi$ and suppose that $x := \inf(A) \notin B_\phi$. Then, by definition, $x \notin \{0,1\}$ and there must exist a number $\varepsilon \in]0, \min(x, 1-x)]$ such that $[x-\varepsilon, x+\varepsilon] \subseteq \mathcal{F}_\phi$. Hence, $]x-\varepsilon, x+\varepsilon[\cap B_\phi = \emptyset$ which contradicts $x = \inf(A)$. In a similar way it is shown that $\sup(A) \in B_\phi$.

(A5): Take arbitrary $z \in [0,1] \setminus B_\phi \subseteq]0,1[$. We now need to prove that there exists a couple of consecutive elements $(x, y) \in B_\phi$ such that $z \in]x, y[$. Suppose that the latter does not hold, then there must exist an increasing or decreasing sequence $(a_i)_{i \in \mathbb{N}}$ in B_ϕ such that $\lim_{i \to \infty} a_i = z$. From property **(A4)** we obtain the contradiction $z \in B_\phi$. $\square$

Note that property **(A1)** also states that B_ϕ can never contain an interval.

Definition 2.22 An automorphism ϕ has an *alternating behaviour* if there exists an involutive $B_\phi \to B_\phi$ antimorphism N such that, for any pair of consecutive elements $(x, y) \in B_\phi^2$, it holds that the mutual position of ϕ and **id** is fixed on $]x, y[\cup]y^N, x^N[$.

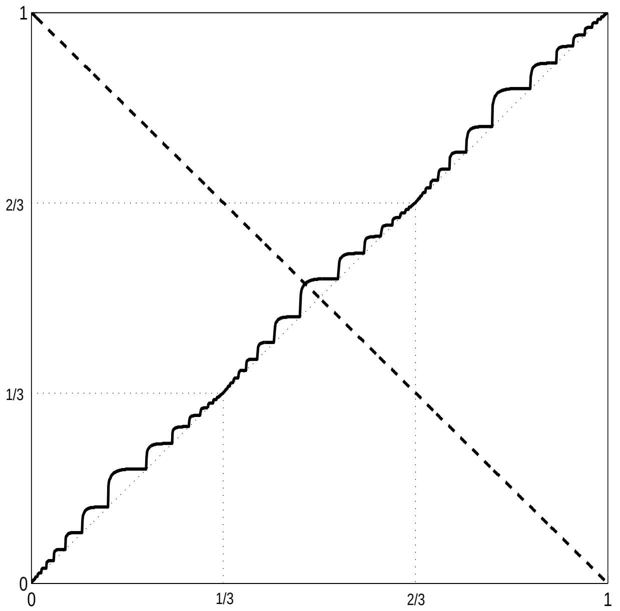


Figure 2.4: An $\mathcal{N}$-symmetrical automorphism ϕ (solid line).

In Fig. 2.4 we give an example of such an automorphism ϕ that has an alternating behaviour. Its set B_ϕ contains four accumulation points: 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1. In general, the alternating behaviour of an automorphism partitions B_ϕ in two sets.

Theorem 2.23 *An automorphism ϕ has an alternating behaviour if and only if we can select from B_ϕ two sequences $(\alpha_i)_{i\in I_\phi}$ and $(\beta_i)_{i\in I_\phi}$ that fulfill the following conditions:*

(B1) *All elements of $(\alpha_i)_{i\in I_\phi}$, resp. $(\beta_i)_{i\in I_\phi}$, are different.*
(B2) $B_\phi = \{\alpha_i \mid i \in I_\phi\} \cup \{\beta_i \mid i \in I_\phi\}$.
(B3) $\sup\{\alpha_i \mid i \in I_\phi\} \leqslant \inf\{\beta_i \mid i \in I_\phi\}$.
(B4) $\alpha_i < \alpha_j \quad \Leftrightarrow \quad \beta_j < \beta_i$*, for every* $i, j \in I_\phi$.
(B5) *The mutual position of ϕ and* **id** *is fixed on $]\alpha_i, \alpha_j[\cup]\beta_j, \beta_i[$ whenever $]\alpha_i, \alpha_j[\cap B_\phi = \emptyset$.*

Proof $\Rightarrow$ Let N be an involutive $B_\phi \to B_\phi$ antimorphism as in Definition 2.22. Denote $L := \{x \in B_\phi \mid x \leqslant x^N\}$ and $U := \{x \in B_\phi \mid x^N \leqslant x\}$. The involutivity of N ensures that $x \in L$ if and only if $x^N \in U$. Consider now an arbitrary index set I_ϕ such that $|I_\phi| = |L| = |U| = \lceil |B_\phi|/2 \rceil$ (*i.e.* $|I_\phi|$ must be the smallest integer that is larger or equal than $|B_\phi|/2$). By means of this index set we form with all elements of L a sequence $(\alpha_i)_{i\in I_\phi}$. Defining $\beta_i = \alpha_i^N$, for every $i \in I_\phi$, we obtain a second sequence $(\beta_i)_{i\in I_\phi}$ containing all elements of U. As N is involutive and satisfies the conditions stated in Definition 2.22, both sequences $(\alpha_i)_{i\in I_\phi}$ and $(\beta_i)_{i\in I_\phi}$ must fulfill conditions **(B1)**, **(B2)**, **(B4)** and **(B5)**. Condition **(B3)** follows from the observation that $x \leqslant y$, for every $(x, y) \in L \times U$. Indeed, $y < x$ would yield the contradiction $y^N \leqslant y < x \leqslant x^N$.

$\Leftarrow$ Suppose that we can select from B_ϕ two sequences $(\alpha_i)_{i\in I_\phi}$ and $(\beta_i)_{i\in I_\phi}$ satisfying conditions **(B1)**–**(B5)**. From property **(A4)** and condition **(B2)** we know that $\sup\{\alpha_i \mid i \in I_\phi\}$ equals either α_j or β_j, for some $j \in I_\phi$. In case $\sup\{\alpha_i \mid i \in I_\phi\} = \beta_j$, it must hold that $\beta_j \leqslant \beta_i$, for every $i \in I_\phi$ (condition **(B3)**). Condition **(B4)** then implies that $\alpha_i \leqslant \alpha_j$, for every $i \in I_\phi$, and thus, $\sup\{\alpha_i \mid i \in I_\phi\} = \alpha_j$. Invoking condition **(B4)** it also holds that $\inf\{\beta_i \mid i \in I_\phi\} = \beta_j$. Therefore, without loss of generality, we may assume that $0 \in I_\phi$, $\alpha_0 = \sup\{\alpha_i \mid i \in I_\phi\}$ and $\beta_0 = \inf\{\beta_i \mid i \in I_\phi\}$. It suffices now to define N as follows: $\alpha_i^N = \beta_i$ and $\beta_i^N = \alpha_i$, for every $i \in I_\phi$. Note that $\beta_i^N = \alpha_i < \alpha_0 = \beta_0^N \leqslant \alpha_0^N = \beta_0 < \beta_i = \alpha_i^N$, for every $i \in I_\phi \setminus \{0\}$. N is by definition involutive and, hence, bijective. Its decreasingness is implied by conditions **(B3)** and **(B4)**. For two consecutive fixpoints $(x, y) \in B_\phi^2$ $(x < y)$ it necessarily holds that either $(x, y) = (\alpha_0, \beta_0)$, $(x, y) = (\alpha_i, \alpha_j)$ or $(x, y) = (\beta_j, \beta_i)$, for some $(i, j) \in I_\phi^2$ such that $]\alpha_i, \alpha_j[\cap B_\phi = \emptyset$. Note that, due to condition **(B4)**, the latter ensures that also $]\beta_i, \beta_j[\cap B_\phi = \emptyset$. Property **(A3)** and condition **(B5)** yield that the mutual position of ϕ and **id** is then fixed on $]x, y[\cup]x^N, y^N[$. $\square$

Using both sequences $(\alpha_i)_{i\in I_\phi}$ and $(\beta_i)_{i\in I_\phi}$ we will show in the proof of the following theorem how to construct an involutive negator N that forms a symmetrical pair with a given automorphism ϕ that has an alternating behaviour.

Theorem 2.24 *An automorphism ϕ is symmetrical w.r.t. an involutive negator if and only if it has an alternating behaviour.*

Proof $\Rightarrow$ Consider an automorphism ϕ that is symmetrical w.r.t. an involutive negator N (*i.e.* $\phi = N \circ \phi^{-1} \circ N$). It is easily verified that the N-symmetry of ϕ implies that

$$\phi(x) < x \quad \Leftrightarrow \quad \phi(x^N) < x^N\,, \tag{2.2}$$

$$\phi(x) = x \quad \Leftrightarrow \quad \phi(x^N) = x^N\,, \tag{2.3}$$

$$\phi(x) > x \quad \Leftrightarrow \quad \phi(x^N) > x^N\,, \tag{2.4}$$

for every $x \in [0, 1]$. Combining property **(A3)** with Eqs. (2.2)–(2.4) we immediately obtain that $x \in B_\phi \;\Leftrightarrow\; x^N \in B_\phi$ and that the mutual position of ϕ and **id** is fixed on $]x, y[\cup]y^N, x^N[$, for any pair of consecutive fixpoints $(x, y) \in B_\phi^2$. Therefore, $N|_{B_\phi}$ is an involutive $B_\phi \to B_\phi$ antimorphism and Definition 2.22 states that ϕ has indeed an alternating behaviour.

$\Leftarrow$ Suppose that ϕ has an alternating behaviour and let N be the involutive $B_\phi \to B_\phi$ antimorphism from Definition 2.22. We will now extend N to an involutive negator that forms a symmetrical pair with ϕ. Use N to select from B_ϕ two sequences $(\alpha_i)_{i\in I_\phi}$ and $(\beta_i)_{i\in I_\phi}$ as described in the proof of Theorem 2.23. Recall that $\alpha_i^N = \beta_i$ and $\beta_i^N = \alpha_i$, for every $i \in I_\phi$. Furthermore, we may assume that $0 \in I_\phi$, $\alpha_0 = \sup\{\alpha_i \mid i \in I_\phi\}$ and $\beta_0 = \inf\{\beta_i \mid i \in I_\phi\}$. The definition of both sequences ensures that, for two consecutive fixpoints $(x, y) \in B_\phi^2$ $(x < y)$ one can always find $(i, j) \in I_\phi^2$ such that $]\alpha_i, \alpha_j[\cap B_\phi = \emptyset$ and $(x, y) \in \{(\alpha_i, \alpha_j), (\alpha_0, \beta_0), (\beta_j, \beta_i)\}$. Due to property **(A5)** it then suffices to define N on $]\alpha_0, \beta_0[$ and on all open sets $]\alpha_i, \alpha_j[\cup]\beta_j, \beta_i[$, where $]\alpha_i, \alpha_j[\cap B_\phi = \emptyset$. If $\alpha_0 < \beta_0$, we first deal with the interval $]\alpha_0, \beta_0[$. Rescale $\phi|_{[\alpha_0,\beta_0]}$ to the unit interval by means of a $[\alpha_0, \beta_0] \to [0,1]$ isomorphism σ. As the mutual position of ϕ and $\mathbf{id}$ is fixed on $]\alpha_0, \beta_0[$ (property **(A3)**), the mutual position of the rescaled automorphism $\sigma \circ \phi|_{[\alpha_0,\beta_0]} \circ \sigma^{-1}$ and $\mathbf{id}$ is fixed on $]0,1[$. Applying Theorem 2.16, there exists an involutive negator N_1 such that $\sigma \circ \phi|_{[\alpha_0,\beta_0]} \circ \sigma^{-1} = N_1 \circ \sigma \circ \phi^{-1}|_{[\alpha_0,\beta_0]} \circ \sigma^{-1} \circ N_1$. If we define $N|_{[\alpha_0,\beta_0]} := \sigma^{-1} \circ N_1 \circ \sigma$, the latter implies that $\phi(x) = (\phi^{-1}(x^N))^N$, for every $x \in [\alpha_0, \beta_0]$. The involutivity of N_1 ensures that $(x^N)^N = \sigma^{-1}[((\sigma[x])^{N_1})^{N_1}] = x$, for every $x \in [\alpha_0, \beta_0]$. Note that, as required, $\alpha_0^N = \beta_0$ and that $N|_{[\alpha_0,\beta_0]}$ is a $[\alpha_0, \beta_0] \to [\alpha_0, \beta_0]$ antimorphism.

Consider two arbitrary indices $(i,j) \in I_\phi^2$ such that $]\alpha_i, \alpha_j[\cap B_\phi = \emptyset$. Recall from condition **(B5)** that in this case the mutual position of ϕ and $\mathbf{id}$ is fixed on $]\alpha_i, \alpha_j[\cup]\beta_j, \beta_i[$. If $\phi(x) = x$, for every $x \in [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$, then it suffices to take for $N|_{[\alpha_i,\alpha_j]}$ an arbitrary $[\alpha_i, \alpha_j] \to [\beta_j, \beta_i]$ antimorphism N_1. If we put $N|_{[\beta_j,\beta_i]} = N_1^{-1}$, then $(x^N)^N = x$ and $\phi(x) = x = (\phi^{-1}(x^N))^N$ is trivially fulfilled for every $x \in [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$. Suppose now that $x < \phi(x)$, for every $x \in]\alpha_i, \alpha_j[\cup]\beta_j, \beta_i[$. Take arbitrary $(a,b) \in]\alpha_i, \alpha_j[\times]\beta_j, \beta_i[$ and let N_1 be a $[a, \phi(a)] \to [\phi^{-1}(b), b]$ antimorphism. Recall that $\alpha_j \leqslant \beta_j$ (condition **(B3)**). In Figure 2.5 we illustrate how to build $N|_{[\alpha_i,\alpha_j]\cup[\beta_j,\beta_i]}$ from N_1. Part I depicts N_1. Two operations are possible: reflecting part I about the first bisector yields part II and drawing the ϕ-inverse of part I results in part III. On parts II and III we can apply once again both operations. However, in each case one action

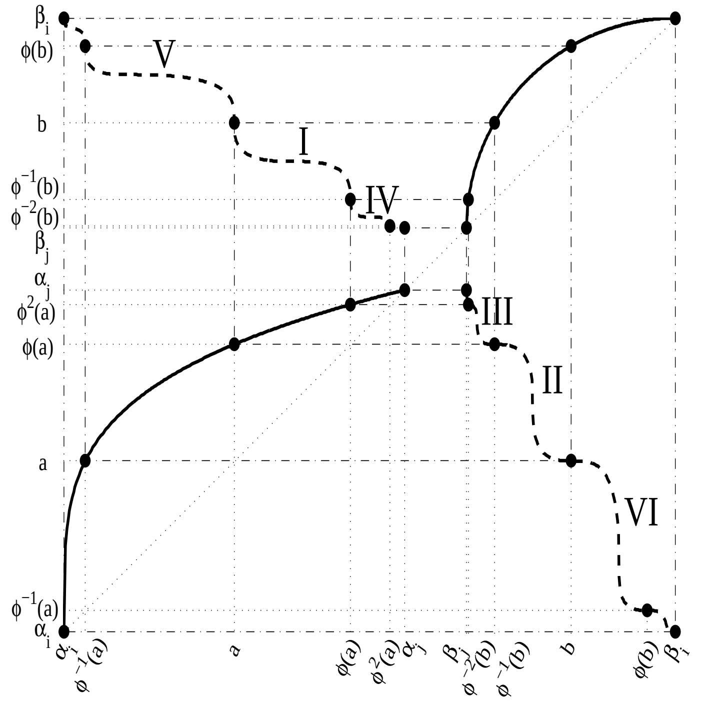


Figure 2.5: Construction of a $[\alpha_i, \alpha_j] \cup [\beta_j, \beta_i] \to [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$ antimorphism N (dashed line) that satisfies $N \circ N = \mathbf{id}$ and $\phi = N \circ \phi^{-1} \circ N$, where ϕ (solid line) is an $[\alpha_i, \alpha_j] \cup [\beta_j, \beta_i] \to [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$ isomorphism that satisfies $x < \phi(x)$, for every $x \in]\alpha_i, \alpha_j[\cup]\beta_j, \beta_i[$.

only will provide new information. Part IV is the reflection of part III and part V is obtained by expressing that it must be the ϕ-inverse of part II. Repeating this procedure (part VI is the reflection of part V, etc.), we construct $N|_{[\alpha_i,\alpha_j]\cup[\beta_j,\beta_i]}$. Mathematically, $N|_{[\alpha_i,\alpha_j]\cup[\beta_j,\beta_i]}$ can be expressed as follows:

$$x^N = \begin{cases} \beta_i, & \text{if } x = \alpha_i\,, \\ \phi^{-k} \circ N_1 \circ \phi^{-k}(x), & \text{if } x \in [\phi^k(a), \phi^{k+1}(a)], \text{ with } k \in \mathbb{Z}\,, \\ \beta_j, & \text{if } x = \alpha_j\,, \\ \alpha_j, & \text{if } x = \beta_j\,, \\ \phi^k \circ N_1^{-1} \circ \phi^k(x), & \text{if } x \in [\phi^{-(k+1)}(b), \phi^{-k}(b)], \text{ with } k \in \mathbb{Z}\,, \\ \alpha_i, & \text{if } x = \beta_i\,. \end{cases}$$

As $a < \phi(a)$ and $\phi^{-1}(b) < b$, the strict increasingness of ϕ and ϕ^{-1} imply that $\phi^k(a) < \phi^{k+1}(a)$ and $\phi^{-(k+1)}(b) < \phi^{-k}(b)$, for every $k \in \mathbb{Z}$. Denote $L = \lim_{k\to-\infty} \phi^k(a)$. Due to the continuity of ϕ, we obtain from $\phi(\phi^k(a)) = \phi^{k+1}(a)$ that $\phi(L) = L$. Taking into account $\alpha_i \leqslant \phi^k(a) < a$, for every $k \in \mathbb{Z}_0^-$ it holds that $\alpha_i \leqslant L \leqslant a < \alpha_j$. As α_i and α_j are the only fixpoints of $\phi|_{[\alpha_i,\alpha_j]}$ we conclude that $L = \alpha_i$. In a similar way it can be shown that $\lim_{k\to\infty} \phi^k(a) = \alpha_j$, $\lim_{k\to-\infty} \phi^k(b) = \beta_j$ and $\lim_{k\to\infty} \phi^k(b) = \beta_i$. Hence, N is indeed defined for every $x \in [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$. Furthermore, $N|_{[\alpha_i,\alpha_j]\cup[\beta_j,\beta_i]}$ is a decreasing bijection. Taking into account that

$$x^N \in \begin{cases} [\beta_i, \beta_i], & \text{if } x = \alpha_i\,, \\ [\phi^{-(k+1)}(b), \phi^{-k}(b)], & \text{if } x \in [\phi^k(a), \phi^{k+1}(a)], \text{ with } k \in \mathbb{Z}\,, \\ [\beta_j, \beta_j], & \text{if } x = \alpha_j\,, \\ [\alpha_j, \alpha_j], & \text{if } x = \beta_j\,, \\ [\phi^k(a), \phi^{k+1}(a)], & \text{if } x \in [\phi^{-(k+1)}(b), \phi^{-k}(b)], \text{ with } k \in \mathbb{Z}\,, \\ [\alpha_i, \alpha_i], & \text{if } x = \beta_i\,, \end{cases}$$

it follows that $(x^N)^N = x$ and $\phi(x) = (\phi^{-1}(x^N))^N$, for every $x \in [\alpha_i, \alpha_j] \cup [\beta_j, \beta_i]$. Repeating this construction for every pair of indices $(i, j) \in I_\phi^2$ such that $]\alpha_i, \alpha_j[\, \cap B_\phi = \emptyset$, we obtain a strict negator N that satisfies $N \circ N = \mathbf{id}$ and $\phi = N \circ \phi^{-1} \circ N$. $\square$

Combining Theorems 2.14 and 2.24 leads to the following result.

Corollary 2.25 *An automorphism ϕ is composed of two involutive negators if and only if it has an alternating behaviour.*

CHAPTER 3

Invariance of monotone functions

3.1 Introduction

Monotone $[0,1] \to [0,1]$ bijections can be used to transform a monotone $[0,1]^n \to [0,1]$ function F into a new monotone function. Although there are several ways to perform this transformation, properties such as monotonicity, commutativity, assocociativity, etc., of the original function are preserved if we first apply a monotone $[0,1] \to [0,1]$ bijection Φ to the arguments of F and then use Φ^{-1} to adjust the image. For a fixed bijection Φ there always exists a monotone function F that remains invariant under this transformation. In this chapter we mainly focus on those monotone functions that are invariant under a given involutive negator N. These functions ensure that complementary inputs result in a complementary output and are therefore extremely suited to be used in real life applications. In preference modeling for example, $[0,1]$-valued binary relations R can be used to render the individual intensity of preference. Consider a finite set of alternatives $A = \{a_1, \ldots, a_m\}$ and n experts. The opinion of expert k is represented by a relation $R_k : A^2 \to [0,1]$, such that $R_k(a_i, a_j)$ expresses the degree to which expert k prefers alternative a_i to alternative a_j (see e.g. [8, 31, 32]). In order to rule out incomparability, it is often required that the degree to which a_i is preferred to a_j is in some sense complementary to the degree to which a_j is preferred to a_i. This naturally leads to the use of *reciprocal preference relations* R_k, *i.e.* $R_k(a_i, a_j) + R_k(a_j, a_i) = 1$. In this setting, two alternatives a_i and a_j are indifferent if $R_k(a_i, a_j) = R_k(a_j, a_i) = \frac{1}{2}$. These individual preferences can be merged by means of an increasing $[0,1]^n \to [0,1]$ function F. The relation R is defined by $R(a_i, a_j) = F(R_1(a_i, a_j), \ldots, R_n(a_i, a_j))$ and represents the collective preference. It was soon noticed that R is reciprocal provided F fulfills $1 - F(x_1, \ldots, x_n) = F(1 - x_1, \ldots, 1 - x_n)$ for every $(x_1, \ldots, x_n) \in [0,1]^n$ [31, 33]. The latter expresses that F must be invariant under the standard negator $\mathcal{N}$.

Unless stated differently, we work in this section with some fixed dimension $n \in \mathbb{N}_0$.

3.2 Invariant monotone functions

By means of a monotone $[0,1] \to [0,1]$ bijection Φ we reshape a monotone $[0,1]^n \to [0,1]$ function F in the following way.

Definition 3.1 Let Φ be a monotone $[0,1] \to [0,1]$ bijection and consider a monotone $[0,1]^n \to [0,1]$ function F. The Φ-*transform* of F is the monotone $[0,1]^n \to [0,1]$ function F_Φ defined by

$$F_\Phi(x_1, \ldots, x_n) = \Phi^{-1}(F(\Phi(x_1), \ldots, \Phi(x_n)))\,.$$

Obviously, F_Φ must have the same type of monotonicity as the original function F. The **id**-transform of a monotone $[0,1]^n \to [0,1]$ function is trivially the function itself. Trillas [87] has proven that every involutive negator is a transformed standard negator.

Theorem 3.2 [87] *A strict negator N is involutive if and only if there exits an automorphism ϕ such that $N = \mathcal{N}_\phi$.*

The following theorem states that transforming a monotone function F by means of a bijection that is composed out of two bijections entails two consecutive tranformations.

Theorem 3.3 [7] *Consider two monotone $[0,1] \to [0,1]$ bijections Φ and Ψ. For every monotone $[0,1]^n \to [0,1]$ function F it holds that $F_{\Phi\circ\Psi} = (F_\Phi)_\Psi$.*

Definition 3.4 Let Φ be a monotone $[0,1] \to [0,1]$ bijection. A monotone $[0,1]^n \to [0,1]$ function F is Φ-*invariant* if $F_\Phi = F$. In case $F_\Phi = F$ holds for every monotone bijection Φ we call F *invariant*.

In measurement theory a Φ-invariant function F is also called *stable* for the monotone bijection Φ [27, 82]. A monotone $[0,1]^n \to [0,1]$ function F is then called *ordinally stable* if it is stable for all automorphisms (*i.e.* if it is invariant under all automorphisms). Due to the generating character of involutive negators (Theorem 2.18), we are able to reduce the conditions for calling a monotone $[0,1]^n \to [0,1]$ function invariant.

Theorem 3.5 *Consider a monotone $[0,1]^n \to [0,1]$ function F. There exists a monotone $[0,1]^n \to [0,1]$ function G such that $F_N = G$ holds for every involutive negator N if and only if F is invariant under all automorphisms. In this case it also holds that $F_N = G$, for every strict negator N.*

Proof Suppose that $F_N = G$ holds for every involutive negator N. In particular, we obtain that $F_{M_1} = F_{M_2}$, for every pair of involutive negators (M_1, M_2). Due to Theorem 3.3 and the involutivity of M_2 this implies that $F_{M_1\circ M_2} = (F_{M_1})_{M_2} = (F_{M_2})_{M_2} = F_{M_2\circ M_2} = F$, for every pair of involutive negators (M_1, M_2). Now consider an arbitrary automorphism ϕ. From Theorem 2.18 we know that there exist four involutive negators N_1, N_2, N_3 and N_4 such that $\phi = N_1 \circ N_2 \circ N_3 \circ N_4$. Invoking Theorem 3.3 once again leads to $F_\phi = F_{N_1\circ N_2\circ N_3\circ N_4} = (F_{N_1\circ N_2})_{N_3\circ N_4} = F_{N_3\circ N_4} = F$.

Conversely, assuming that F is invariant under all automorphisms, we know that $F_{M_1 \circ \mathcal{N}} = F$ holds for every strict negator M_1. Based on Theorem 3.3, the latter implies $F_{M_1} = F_{M_1 \circ \mathcal{N} \circ \mathcal{N}} = (F_{M_1 \circ \mathcal{N}})_{\mathcal{N}} = F_{\mathcal{N}}$. It now suffices to denote $F_{\mathcal{N}}$ as G. $\square$

Hence, a monotone function F is invariant if and only if it is invariant under all involutive negators. Studying N-transforms, with N an involutive negator, it suffices to consider increasing $[0,1]^n \to [0,1]$ functions only. The N-transform G_N of a decreasing $[0,1]^n \to [0,1]$ function G can be understood as the negation $N \circ F$ of the increasing function F, defined by $F(x_1, \ldots, x_n) = G(x_1^N, \ldots, x_n^N)$. If there exists a function H such that $G_N = H$ holds for every involutive negator N then $F_N = K$ holds for every involutive negator N, with $K(x_1, \ldots, x_n) = H(x_1^N, \ldots, x_n^N)$. Also the converse is true and, hence, G is invariant under all automorphisms if and only if F is invariant under all automorphisms (Theorem 3.5). Furthermore, G is N-invariant if and only if F is N-invariant.

In various fields such as fuzzy logic, fuzzy set-theory, decision making and preference modeling a special type of increasing $[0,1]^n \to [0,1]$ functions is used to combine different input values into a single output value.

Definition 3.6 [7] An *n-ary aggregation operator* F is an increasing $[0,1]^n \to [0,1]$ function that satisfies the following boundary conditions:

(AO1) $F(0, \ldots, 0) = 0$ and $F(1, \ldots, 1) = 1$.
(AO2) $F = \mathbf{id}$ if $n = 1$.

It is evident that the Φ-transform of an n-ary aggregation operator is again an n-ary aggregation operator [7]. In the literature, the $\mathcal{N}$-transform $F_{\mathcal{N}}$ of F is known as the *dual* of F (see e.g. [7]). An aggregation operator F is called *self-dual* if it is $\mathcal{N}$-invariant. Several other terms are used for expressing self-duality: *neutrality* [32], *reciprocity* [31, 33], etc. Examples of self-dual aggregation operators are [7]:

- The *arithmetic mean* $\mathbf{M}(x_1, \ldots, x_n) = \sum_{i=1}^n x_i/n$;
- *Quasi-arithmetic means* $\mathbf{M}_f(x_1, \ldots, x_n) = f^{-1}(\sum_{i=1}^n f(x_i)/n)$ for which the strictly monotone continuous function $f : [0,1] \to [-\infty, \infty]$ is reciprocal (*i.e.* $f(1-x) = 1 - f(x)$);
- *Weighted means* $\mathbf{W}(x_1, \ldots, x_n) = \sum_{i=1}^n w_i \cdot x_i$, where $\sum_{i=1}^n w_i = 1$ and $w_i \geqslant 0$;
- *OWA operators* $\mathbf{W}'(x_1, \ldots, x_n) = \sum_{i=1}^n w_i \cdot x_i'$, with $(x_1', \ldots, x_n')$ an increasing permutation of $(x_1, \ldots, x_n)$, $\sum_{i=1}^n w_i = 1$, $w_i \geqslant 0$ and $(w_1, \ldots, w_n) = (w_n, \ldots, w_1)$.

Mesiar and Rückschlossová [69] showed that invariant aggregation operators are exactly those self-dual aggregation operators that are invariant under all automorphisms. These aggregation operators can be described by means of the *Choquet integral* [9, 69, 76] and are tedious patchworks of the constant functions $\mathbf{0}$ and $\mathbf{1}$ and of the projections $P_i : [0,1]^n \to [0,1] : (x_1, \ldots, x_n) \to x_i$, $i \in \{1, \ldots, n\}$. Theorem 3.5 puts these results in a more general framework.

Corollary 3.7 *An n-ary aggregation operator is invariant if and only if it is invariant under all involutive negators.*

3.3 N-Invariant increasing functions

Given a monotone $[0,1] \to [0,1]$ bijection Φ, it remains an intriguing problem how to characterize all Φ-invariant monotone $[0,1]^n \to [0,1]$ functions F. Clearly, a first subset of solutions consists of all invariant functions. As indicated in the previous section, it suffices to study increasing functions F only. In view of Theorems 2.18 and 3.5 we will focus here on the characterization of all N-invariant increasing $[0,1]^n \to [0,1]$ functions, where N is a given involutive negator. Explicitly, the N-invariance of an aggregation operator F means that

$$F(x_1, \ldots, x_n) = F(x_1^N, \ldots, x_n^N)^N \,, \tag{3.1}$$

for every $(x_1, \ldots, x_n) \in [0,1]^n$. Let β be the unique fixpoint of N. From a geometrical point of view, Eq. (3.1) enforces some kind of point symmetry w.r.t. $(\beta, \ldots, \beta)$ upon the aggregation operator F. For the point of symmetry $(\beta, \ldots, \beta)$ it holds that $F(\beta, \ldots, \beta) = \beta$. Once $F(x_1, \ldots, x_n)$ is known, Eq. (3.1) fixes $F(x_1^N, \ldots, x_n^N)$.

Two alternative characterizations for self-dual aggregation operators are available in the literature. The *symmetric sums* of Sivert [86] have been the source of inspiration for Calvo *et al.* [7]. In general, symmetric sums are continuous, commutative, binary, self-dual aggregation operators [18, 27, 86].

Proposition 3.8 [7] *An n-ary aggregation operator F is self-dual if and only if there exists an n-ary aggregation operator G such that*

$$F(x_1, \ldots, x_n) = \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_n) + G(1 - x_1, \ldots, 1 - x_n)} \,, \tag{3.2}$$

with $\frac{0}{0+0} := \frac{1}{2}$.

Whenever F is self-dual it is enough to choose $G = F$ to obtain Eq. (3.2). Besides the approach of Calvo *et al.*, García–Lapresta and Marques Pereira provided a different characterization based on the arithmetic mean.

Proposition 3.9 [33] *An n-ary aggregation operator F is self-dual if and only if there exists an n-ary aggregation operator G such that*

$$F(x_1, \ldots, x_n) = \frac{G(x_1, \ldots, x_n) + G_{\mathcal{N}}(x_1, \ldots, x_n)}{2} \,. \tag{3.3}$$

For each self-dual F we can again choose $G = F$. Rewriting Eq. (3.2) as

$$F(x_1, \ldots, x_n) = \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_n) + 1 - G_{\mathcal{N}}(x_1, \ldots, x_n)} \,, \tag{3.4}$$

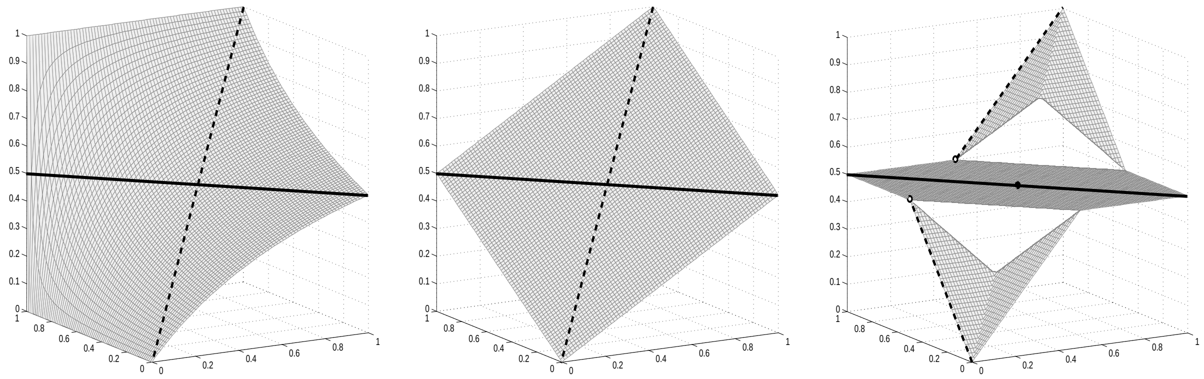
it strikes that both expressions Eqs. (3.3) and (3.4) are of the form

$$F(x_1, \ldots, x_n) = \boldsymbol{\mathcal{C}}\big(G(x_1, \ldots, x_n), G_N(x_1, \ldots, x_n)\big) \,, \tag{3.5}$$

for some $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ and a given involutive negator N. The first two plots of Fig. 3.1 illustrate $\boldsymbol{\mathcal{C}}$ for Eqs. (3.3) and (3.4). The third plot in the figure visualizes the $[0,1]^2 \to [0,1]$ function $\widehat{\boldsymbol{\mathcal{C}}}$, defined by

$$\widehat{\boldsymbol{\mathcal{C}}}(x,y) = \begin{cases} \max(x,y), & \text{if } x+y < \frac{1}{2}\,, \\ \min(x,y), & \text{if } \frac{3}{2} < x+y\,, \\ \frac{1}{2}, & \text{elsewhere}\,. \end{cases}$$

As will be shown further, also $\widehat{\boldsymbol{\mathcal{C}}}$ is a valid choice for $\boldsymbol{\mathcal{C}}$. Eq. (3.5) can be used to embed Propositions 3.8 and 3.9 into a much more general framework. In particular, we intend to sift out those functions $\boldsymbol{\mathcal{C}}$ that allow to characterize the class of N-invariant increasing $[0,1]^n \to [0,1]$ functions.



(a) $\boldsymbol{\mathcal{C}}(x,y) = \dfrac{x}{x+1-y}$ (b) $\boldsymbol{\mathcal{C}}(x,y) = \dfrac{x+y}{2}$ (c) $\boldsymbol{\mathcal{C}}(x,y) = \widehat{\boldsymbol{\mathcal{C}}}(x,y)$

Figure 3.1: Possible choices for $\boldsymbol{\mathcal{C}}$ if $N = \mathcal{N}$. The black solid lines reflect that $\boldsymbol{\mathcal{C}}(x, x^{\mathcal{N}}) = \beta$. The dashed black lines visualize the curve $\boldsymbol{\mathcal{C}}(f(x), f(x^{\mathcal{N}})^{\mathcal{N}})$.

Definition 3.10 Let N be an involutive negator. We say that a $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ enables a *full characterization* of all N-invariant, increasing $[0,1]^n \to [0,1]$ functions if the following equivalence holds:

> *A $[0,1]^n \to [0,1]$ function F is increasing and N-invariant if and only if there exists an increasing $[0,1]^n \to [0,1]$ function G such that Eq. (3.5) holds for every $(x_1,\ldots,x_n) \in [0,1]^n$.*

Before continuing the search for suitable $\boldsymbol{\mathcal{C}}$'s we would like to remark that our starting point slightly differs from Propositions 3.8 and 3.9 as we do not assume F to be increasing from the beginning. Let $\boldsymbol{\mathcal{C}}_G$ be the $[0,1]^n \to [0,1]$ function determined by the right-hand side of Eq. (3.5):

$$\boldsymbol{\mathcal{C}}_G(x_1,\ldots,x_n) := \boldsymbol{\mathcal{C}}\big(G(x_1,\ldots,x_n), G_N(x_1,\ldots,x_n)\big)\,.$$

Then $\boldsymbol{\mathcal{C}}$ enables a full characterization of all N-invariant increasing $[0,1]^n \to [0,1]$ functions if and only if the following properties hold:

(C1) $\boldsymbol{\mathcal{C}}_G$ is increasing for every increasing $[0,1]^n \to [0,1]$ function G.
(C2) $\boldsymbol{\mathcal{C}}_G$ is N-invariant for every increasing $[0,1]^n \to [0,1]$ function G.
(C3) For every N-invariant, increasing $[0,1]^n \to [0,1]$ function F there exists an increasing $[0,1]^n \to [0,1]$ function G such that $F = \boldsymbol{\mathcal{C}}_G$.

The following three lemmata tackle these conditions.

Lemma 3.11 *Consider an involutive negator N and a $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$. $\boldsymbol{\mathcal{C}}_G$ is increasing for every increasing $[0,1]^n \to [0,1]$ function G if and only if $\boldsymbol{\mathcal{C}}$ is increasing.*

Proof Suppose that $\boldsymbol{\mathcal{C}}_G$ is increasing for every increasing function G. It is well know that the increasingness of $\boldsymbol{\mathcal{C}}$ is equivalent with the increasingness of all its *partial functions* $\boldsymbol{\mathcal{C}}(x,\bullet)$ and $\boldsymbol{\mathcal{C}}(\bullet,x)$ (*i.e.* the functions obtained by fixing the first, resp. the second argument of $\boldsymbol{\mathcal{C}}$). We will prove that the partial functions $\boldsymbol{\mathcal{C}}(x,\bullet)$ are indeed increasing. A similar reasoning applies to the partial functions $\boldsymbol{\mathcal{C}}(\bullet,x)$. Consider arbitrary $(x,y,z) \in [0,1]^3$ such that $y \leqslant z$. Let β be the fixpoint of N and choose $(u,v) \in]0,\beta[^2$ such that $u < v$. We distinguish three cases:

1. If $x \leqslant z^N \leqslant y^N$, then take an arbitrary increasing $[0,1]^n \to [0,1]$ function G satisfying $G(u,\beta,\ldots,\beta) = x$, $G(v,\beta,\ldots,\beta) = x$, $G(v^N,\beta,\ldots,\beta) = z^N$ and $G(u^N,\beta,\ldots,\beta) = y^N$. We obtain the following chain of inequalities:
$$\begin{aligned}\boldsymbol{\mathcal{C}}(x,y) &= \boldsymbol{\mathcal{C}}\big(G(u,\beta,\ldots,\beta), G_N(u,\beta,\ldots,\beta)\big) = \boldsymbol{\mathcal{C}}_G(u,\beta,\ldots,\beta)\\ &\leqslant \boldsymbol{\mathcal{C}}_G(v,\beta,\ldots,\beta) = \boldsymbol{\mathcal{C}}\big(G(v,\beta,\ldots,\beta), G_N(v,\beta,\ldots,\beta)\big) = \boldsymbol{\mathcal{C}}(x,z)\,.\end{aligned}$$
2. If $z^N \leqslant y^N \leqslant x$, then take an arbitrary increasing $[0,1]^n \to [0,1]$ function G satisfying $G(u,\beta,\ldots,\beta) = z^N$, $G(v,\beta,\ldots,\beta) = y^N$, $G(v^N,\beta,\ldots,\beta) = x$ and $G(u^N,\beta,\ldots,\beta) = x$. We obtain the following chain of inequalities:
$$\begin{aligned}\boldsymbol{\mathcal{C}}(x,y) &= \boldsymbol{\mathcal{C}}\big(G(v^N,\beta,\ldots,\beta), G_N(v^N,\beta,\ldots,\beta)\big) = \boldsymbol{\mathcal{C}}_G(v^N,\beta,\ldots,\beta)\\ &\leqslant \boldsymbol{\mathcal{C}}_G(u^N,\beta,\ldots,\beta) = \boldsymbol{\mathcal{C}}\big(G(u^N,\beta,\ldots,\beta), G_N(u^N,\beta,\ldots,\beta)\big) = \boldsymbol{\mathcal{C}}(x,z)\,.\end{aligned}$$
3. If $z^N \leqslant x \leqslant y^N$, then in particular $z^N \leqslant (x^N)^N = x$ and $x = (x^N)^N \leqslant y^N$. We know from the first two cases that $\boldsymbol{\mathcal{C}}(x,x^N) \leqslant \boldsymbol{\mathcal{C}}(x,z)$ and $\boldsymbol{\mathcal{C}}(x,y) \leqslant \boldsymbol{\mathcal{C}}(x,x^N)$. Therefore, $\boldsymbol{\mathcal{C}}(x,y) \leqslant \boldsymbol{\mathcal{C}}(x,z)$.

Conversely, as G_N and G must have the same type of monotonicity it is clear that the increasingness of $\boldsymbol{\mathcal{C}}$ is passed on to $\boldsymbol{\mathcal{C}}_G$. $\square$

Lemma 3.12 *Consider an involutive negator N and a $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$. $\boldsymbol{\mathcal{C}}_G$ is N-invariant for every increasing $[0,1]^n \to [0,1]$ function G if and only if*
$$\boldsymbol{\mathcal{C}}(x,y) = \boldsymbol{\mathcal{C}}(y^N,x^N)^N \tag{3.6}$$
holds for every $(x,y) \in [0,1]^2$.

Proof Suppose that $\mathcal{C}_G$ is N-invariant for every increasing $[0,1]^n \to [0,1]$ function G. Let β be the unique fixpoint of N. For each couple $(x,y) \in [0,1]^2$, there exists an increasing function G such that $x = G(u,\beta,\ldots,\beta)$ and $y^N = G(u^N,\beta,\ldots,\beta)$, with $u \in \,]0,\beta[$ whenever $x \leqslant y^N$ and $u \in \,]\beta,1[$ whenever $y^N < x$. Expressing the N-invariance of $\mathcal{C}_G$ then leads to

$$\begin{aligned}
\mathcal{C}(y^N,x^N) &= \mathcal{C}\big(G(u^N,\beta,\ldots,\beta),G(u,\beta,\ldots,\beta)^N\big)\\
&= \mathcal{C}_G(u^N,\beta,\ldots,\beta) = \mathcal{C}_G(u,\beta,\ldots,\beta)^N\\
&= \mathcal{C}\big(G(u,\beta,\ldots,\beta),G_N(u,\beta,\ldots,\beta)\big)^N = \mathcal{C}(x,y)^N\,.
\end{aligned}$$

Given Eq. (3.6), the N-invariance of $\mathcal{C}_G$ is trivially obtained by expressing $\mathcal{C}_G(x_1^N,\ldots,x_n^N)$ and $\mathcal{C}_G(x_1,\ldots,x_n)^N$ in terms of $\mathcal{C}$ $\square$

Putting $y = x^N$ in Eq. (3.6), we see that $\mathcal{C}(x,x^N) = \beta$. The black solid lines in Figure 3.1 reflect this property. Geometrically, Eq. (3.6) expresses a kind of symmetry of $\mathcal{C}$ w.r.t. the involutive negator N. Once $\mathcal{C}(x,y)$ is known, Eq. (3.6) fixes the value of $\mathcal{C}$ in (y^N,x^N), the N-inverse of the point (x,y) (see also Section 4.3). If $\mathcal{C}$ is *commutative* (*i.e.* $\mathcal{C}(x,y) = \mathcal{C}(y,x)$, for every $(x,y) \in [0,1]^2$), Eqs. (3.1) ($n=2$) and (3.6) are identical and hence Eq. (3.6) will be trivially fulfilled when considering a commutative, N-invariant $[0,1]^2 \to [0,1]$ function $\mathcal{C}$. If $\mathcal{C}$ is not commutative, Eq. (3.6) substantially differs from Eq. (3.1) ($n=2$).

Definition 3.13 Let F be a monotone $[0,1]^n \to [0,1]$ function. A number $x \in [0,1]$ is called an *idempotent element* of F if $F(x,\ldots,x) = x$ holds. F is *idempotent* if all numbers $x \in [0,1]$ are idempotent elements of F.

Idempotent N-invariant functions will allow us to reformulate property **(C3)**.

Lemma 3.14 *Consider an involutive negator N and a $[0,1]^2 \to [0,1]$ function $\mathcal{C}$. For every N-invariant increasing $[0,1]^n \to [0,1]$ function F there exists an increasing $[0,1]^n \to [0,1]$ function G such that $F = \mathcal{C}_G$ if and only if there exists an increasing $[0,1] \to [0,1]$ function f satisfying*

$$\mathcal{C}(f(x),f_N(x)) = x\,, \tag{3.7}$$

for every $x \in [0,1]$.

Proof Suppose that for every N-invariant, increasing $[0,1]^n \to [0,1]$ function F it is possible to find an increasing $[0,1]^n \to [0,1]$ function G such that $F = \mathcal{C}_G$. As N is an involutive negator, we know from Theorem 3.2 that there exists an automorphism ϕ such that $N = \mathcal{N}_\phi$. If we use ϕ to transform the arithmetic mean $\mathbf{M}$ into $\mathbf{M}_\phi$ then it follows from Theorem 3.3 that $(\mathbf{M}_\phi)_N = (\mathbf{M}_\phi)_{\phi^{-1}\circ\mathcal{N}\circ\phi} = \mathbf{M}_{\mathcal{N}\circ\phi} = (\mathbf{M}_\mathcal{N})_\phi$. Recall that $\mathbf{M}$ is self-dual. Hence, $\mathbf{M}_\mathcal{N} = \mathbf{M}$ and $(\mathbf{M}_\phi)_N = \mathbf{M}_\phi$ which expresses the N-invariance of $\mathbf{M}_\phi$. Further, consider an arbitrary increasing $[0,1]^n \to [0,1]$ function G such that $\mathbf{M}_\phi = \mathcal{C}_G$. Since for every $x \in [0,1]$ it holds that

$$x = \mathbf{M}_\phi(x,\ldots,x) = \mathcal{C}\big(G(x,\ldots,x),G_N(x,\ldots,x)\big)\,,$$

it suffices to define $f(x) := G(x, \ldots, x)$, for every $x \in [0,1]$. Clearly, f is increasing and fulfills Eq. (3.7).

Conversely, suppose that there exists an increasing function f, fulfilling the conditions of this lemma. For each F it is then sufficient to define G as follows

$$G(x_1, \ldots, x_n) = f(F(x_1, \ldots, x_n)) .$$

The increasingness of both f and F ensure that G is an increasing $[0,1]^n \to [0,1]$ function. Replacing x by $F(x_1, \ldots, x_n)$ in Eq. (3.7) and taking into account that F is N-invariant, immediately leads to $F = \boldsymbol{\mathcal{C}}_G$. $\square$

The dashed black lines in Figure 3.1 visualize $\boldsymbol{\mathcal{C}}(f(x), f(x^N)^N) = x$ for some suitable increasing function f. For Figs. 3.1(a) and 3.1(b) we used $f = \mathbf{id}$. The function f used in the Fig. 3.1(c) is given by $f(x) = x$ whenever $x \in [\frac{1}{2}, 1]$ and $f(x) = 0$ elsewhere. The proof of Lemma 3.14 also ensures that, for every suitable f and every N-invariant increasing function F, $f(F(x_1, \ldots, x_n))$ defines an increasing function G that generates F. The three increasing functions G_1, G_2 and G_3 depicted in Figs. 3.2(a)–3.2(c) were created as such and generate the arithmetic mean ($n = 2$). They correspond to the different settings in Fig. 3.1 (e.g. $\widehat{\boldsymbol{\mathcal{C}}}_{G_3} = \mathbf{M}$ for $N = \mathcal{N}$ and $n = 2$). Note that $G_1 = G_2 = \mathbf{M}$ ($n = 2$). For aesthetic reasons we have always rotated the unit cube in Fig. 3.2 60 degrees to the right in comparison with the plots in Fig. 3.1. Joining the previous lemmata finally leads to the following theorem.

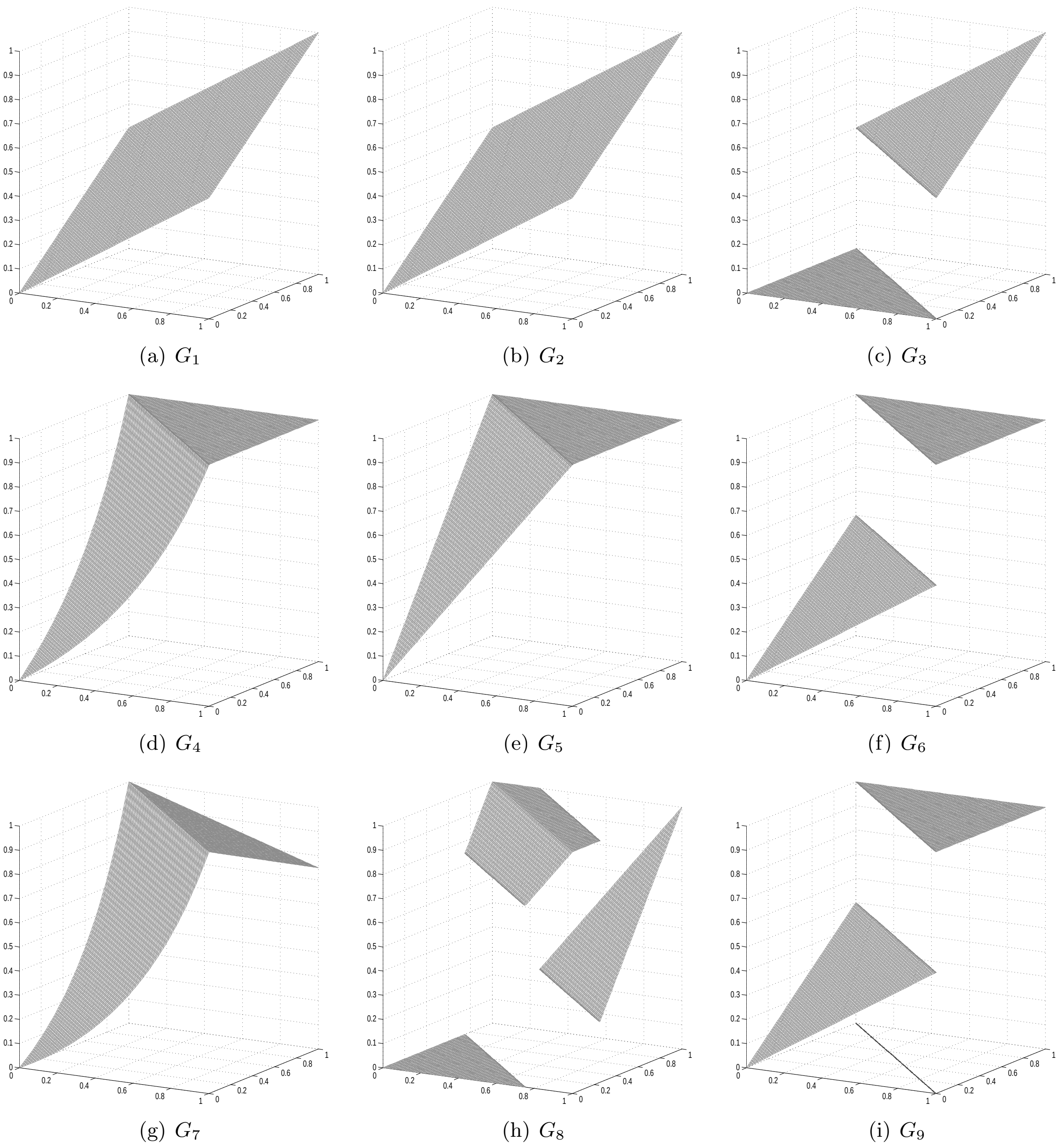
Theorem 3.15 *Consider an involutive negator N. A $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ enables a full characterization of all N-invariant, increasing $[0,1]^n \to [0,1]$ functions if and only if the following assertions hold*

1. *$\boldsymbol{\mathcal{C}}$ is an aggregation operator.*
2. *$\boldsymbol{\mathcal{C}}(x, y) = \boldsymbol{\mathcal{C}}(y^N, x^N)^N$ holds for every $(x, y) \in [0,1]^2$.*
3. *The graph of $\boldsymbol{\mathcal{C}}$ contains an increasing (w.r.t. the three space coordinates) curve whose Z-coordinate reaches every number of $[0,1]$.*

Proof From Lemmata 3.11–3.14 and properties **(C1)**–**(C3)** we know that a $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ enables a full characterization of all N-invariant increasing $[0,1]^n \to [0,1]$ functions if and only if

1. $\boldsymbol{\mathcal{C}}$ is increasing.
2. $\boldsymbol{\mathcal{C}}(x, y) = \boldsymbol{\mathcal{C}}(y^N, x^N)^N$ holds for every $(x, y) \in [0,1]^2$.
3. $\boldsymbol{\mathcal{C}}(f(x), f_N(x)) = x$ holds for some increasing $[0,1] \to [0,1]$ function f and for every $x \in [0,1]$.

The third property requires that $\boldsymbol{\mathcal{C}}$ reaches every number of $[0,1]$. In combination with the first property this means that $\boldsymbol{\mathcal{C}}(0,0) = 0$ and $\boldsymbol{\mathcal{C}}(1,1) = 1$. Hence, $\boldsymbol{\mathcal{C}}$ must be a $[0,1]^2 \to [0,1]$ aggregation operator. As f, f_N and C are increasing, we can extend $\{(f(x), f_N(x), \boldsymbol{\mathcal{C}}(f(x), f_N(x))) \mid$



(a) G_1 (b) G_2 (c) G_3

(d) G_4 (e) G_5 (f) G_6

(g) G_7 (h) G_8 (i) G_9

Figure 3.2: Increasing $[0,1]^2 \to [0,1]$ functions generating the arithmetic mean $\mathbf{M}$ ($n = 2$) by means of the resp. functions $\boldsymbol{\mathcal{C}}$ from Fig. 3.1 ($N = \mathcal{N}$). In particular, the left subfigures correspond to Fig. 3.1(a), the middle subfigures correspond to Fig. 3.1(b) and the right subfigures correspond to Fig. 3.1(c).

$x \in [0,1]\}$ to an increasing (w.r.t. the three space coordinates) curve on the graph of $\boldsymbol{\mathcal{C}}$. Invoking that $\boldsymbol{\mathcal{C}}(f(x), f_N(x)) = x$, for every $x \in [0,1]$, the Z-coordinate of this curve reaches every number of $[0,1]$.

Conversely, suppose that the three assertions of the theorem hold, then $\boldsymbol{\mathcal{C}}$ is clearly increasing. It remains to prove that there exists an increasing function f such that $\boldsymbol{\mathcal{C}}(f(x), f_N(x)) = x$, for every $x \in [0,1]$. Consider an increasing (w.r.t. the three space coordinates) curve whose Z-coordinate reaches every number of $[0,1]$. Mathematically, the graph of this curve contains a set of points $\{\big(g(x), h(x), x\big) \mid x \in [0,1]\}$, with g and h two increasing $[0,1] \to [0,1]$ functions and $\boldsymbol{\mathcal{C}}(g(x), h(x)) = x$, for every $x \in [0,1]$. The second assertion in the theorem ensures that also $\boldsymbol{\mathcal{C}}(h_N(x), g_N(x)) = x$, for every $x \in [0,1]$. Let β be the unique fixpoint of N. If we define $f(x) := g(x)$, for every $x \in [0,\beta[$, and $f(x) := h_N(x)$, for every $x \in]\beta,1]$, then $\boldsymbol{\mathcal{C}}(f(x), f_N(x)) = x$ holds for every $x \in [0,1] \setminus \{\beta\}$. Since $\boldsymbol{\mathcal{C}}(x, x^N) = \beta$ whenever $x \in [0,1]$, we know that $g(x) < h(x)^N$, for every $x \in [0,\beta[$. Indeed, $h(x)^N \leqslant g(x)$ would imply the contradiction $\beta = \boldsymbol{\mathcal{C}}(h(x)^N, h(x)) \leqslant \boldsymbol{\mathcal{C}}(g(x), h(x)) = x$. Choose arbitrarily

$$f(\beta) \in \left[\lim_{x\nearrow\beta} g(x), \lim_{x\nearrow\beta} h(x)^N\right] = \left[\lim_{x\nearrow\beta} g(x), \lim_{x\searrow\beta} h_N(x)\right] = \left[\lim_{x\nearrow\beta} f(x), \lim_{x\searrow\beta} f(x)\right].$$

We obtain that $x = \boldsymbol{\mathcal{C}}(f(x), f(x^N)^N) \leqslant \boldsymbol{\mathcal{C}}(f(\beta), f(\beta)^N) \leqslant \boldsymbol{\mathcal{C}}(f(x^N), f(x)^N) = x^N$, for every $x \in [0,\beta[$. Hence, $\boldsymbol{\mathcal{C}}(f(\beta), f_N(\beta)) = \beta$. $\square$

It is now easily checked that the third plot in Figure 3.1 indeed enables a full characterization of all N-invariant aggregation operators. Unfortunately, no binary aggregation operator $\boldsymbol{\mathcal{C}}$ enables for every involutive negator N a full characterization of all N-invariant increasing $[0,1]^n \to [0,1]$ functions. For example, consider the two involutive negators N_1 and N_2 defined by

$$x^{N_1} = \sqrt{1-x^2} \quad \text{and} \quad x^{N_2} = \begin{cases} -\dfrac{x}{3} + 1, & x \in [0, \frac{3}{4}]\,, \\ -3\,x + 3, & x \in [\frac{3}{4}, 1]\,, \end{cases}$$

and with fixpoints $\beta_1 = \sqrt{\frac{1}{2}}$ and $\beta_2 = \frac{3}{4}$. Obviously, $(\frac{3}{5})^{N_1} = (\frac{3}{5})^{N_2} = \frac{4}{5}$ and therefore $\boldsymbol{\mathcal{C}}\big(\frac{3}{5}, (\frac{3}{5})^{N_1}\big) = \boldsymbol{\mathcal{C}}\big(\frac{3}{5}, (\frac{3}{5})^{N_2}\big)$. The second assertion of Theorem 3.15, however, implies that

$$\boldsymbol{\mathcal{C}}\big(\tfrac{3}{5}, (\tfrac{3}{5})^{N_1}\big) = \beta_1 = \sqrt{\tfrac{1}{2}} < \tfrac{3}{4} = \beta_2 = \boldsymbol{\mathcal{C}}\big(\tfrac{3}{5}, (\tfrac{3}{5})^{N_2}\big)\,,$$

a contradiction.

Once $\boldsymbol{\mathcal{C}}$ is fixed in accordance with Theorem 3.15, every increasing $[0,1]^n \to [0,1]$ function G will provide an N-invariant increasing $[0,1]^n \to [0,1]$ function F and, conversely, with every N-invariant increasing function F there corresponds at least one increasing function G such that $F = \boldsymbol{\mathcal{C}}_G$. Usually, multiple suchlike functions G generate the same F. The set of all increasing $[0,1]^n \to [0,1]$ functions G can be partitioned into equivalence classes, each containing those functions determining a given $[0,1]^n \to [0,1]$ function F.

Example 3.16 The increasing $[0,1]^2 \to [0,1]$ functions depicted in Figs. 3.2(a)–3.2(f) generate the arithmetic mean $\mathbf{M}$ ($n=2$) by means of the resp. aggregation operators $\boldsymbol{\mathcal{C}}$ from Fig. 3.1 and with $N = \mathcal{N}$. The functions G_4, G_5 and G_6 depicted in Figs. 3.2(d)–3.2(f) have been obtained by fixing $G_4(x,y) = G_5(x,y) = G_6(x,y) = 1$, for every $\{(x,y) \in [0,1]^2 \mid 1 \leqslant x+y\}$. The equalities $\boldsymbol{\mathcal{C}}_{G_4} = \mathbf{M}$, $\boldsymbol{\mathcal{C}}_{G_5} = \mathbf{M}$ and $\boldsymbol{\mathcal{C}}_{G_6} = \mathbf{M}$, with $\boldsymbol{\mathcal{C}}$ the resp. functions from Fig. 3.1 and $N = \mathcal{N}$, have been used to compute the values of G_4, G_5 and G_6 on $\{(x,y) \in [0,1]^2 \mid x+y < 1\}$. Due to their maximality w.r.t. the set $\{(x,y) \in [0,1]^2 \mid 1 \leqslant x+y\}$, these three functions can be used to represent the equivalence class they belong to.

Figs. 3.2(g)–3.2(i) depict three non-monotone $[0,1]^2 \to [0,1]$ functions that generate the arithmetic mean $\mathbf{M}$ ($n=2$). Also here we use the resp. aggregation operators $\boldsymbol{\mathcal{C}}$ from Fig. 3.1 and take $N = \mathcal{N}$ to compute $\boldsymbol{\mathcal{C}}_{G_7}$, $\boldsymbol{\mathcal{C}}_{G_8}$ and $\boldsymbol{\mathcal{C}}_{G_9}$. G_9 has been obtained from $G_3 = G_6$ by lowering its values to zero on the diagonal $\{(x,y) \in [0,1]^2 \mid x+y = 1\}$. The functions G_7 and G_8 are defined as follows:

$$G_7(x,y) = \begin{cases} \dfrac{(x+y)(3+x+y)}{8-4\,(x+y)}, & \text{if } x+y \leqslant 1\,, \\ \dfrac{5-(x+y)}{4}, & \text{if } 1 < x+y\,, \end{cases} \qquad G_8(x,y) = \begin{cases} 0, & \text{if } x+y < \frac{3}{4}\,, \\ x+y-1, & \text{if } \frac{5}{4} < x+y\,, \\ \min(x+y,1), & \text{elsewhere}\,. \end{cases} \quad \triangle$$

As illustrated in the example, G itself does not need to be increasing to generate an N-invariant, increasing $[0,1]^n \to [0,1]$ function F. The minimal conditions on a $[0,1]^n \to [0,1]$ function G such that $\boldsymbol{\mathcal{C}}_G$ yields an N-invariant increasing function are inextricably bound up with the choice of $\boldsymbol{\mathcal{C}}$ and N. Therefore, general results are not to be expected.

It is worthwhile noting that, for every self-dual n-ary aggregation operator F, $G = F$ fulfills Eqs. (3.2) and (3.3).

Theorem 3.17 *Consider an involutive negator N. Then $F = \boldsymbol{\mathcal{C}}_F$ holds for every N-invariant, increasing $[0,1]^n \to [0,1]$ function F if and only if $\boldsymbol{\mathcal{C}}$ is idempotent.*

Proof If $\boldsymbol{\mathcal{C}}$ is idempotent, it is trivially verified that $F = \boldsymbol{\mathcal{C}}_F$, for every N-invariant function F. To obtain the converse we consider $F = \mathbf{M}$. Then

$$x = \mathbf{M}(x,\ldots,x) = \boldsymbol{\mathcal{C}}\big(\mathbf{M}(x,\ldots,x), \mathbf{M}(x^N,\ldots,x^N)^N\big) = \boldsymbol{\mathcal{C}}(x,x)\,,$$

for every $x \in [0,1]$. $\square$

Note that the third assertion in Theorem 3.15 is trivially fulfilled whenever $\boldsymbol{\mathcal{C}}$ is idempotent. To conclude this section, we give some general comments on the presented techniques and results.

Remarks 3.18 1. A similar approach (as Theorem 3.15) for describing all Φ-invariant aggregation operators, with Φ a non-involutive monotone $[0,1] \to [0,1]$ bijection, cannot be expected. Without the involutivity property, no combination of B, B_Φ, $B_{\Phi^{-1}}$, $B_{\Phi\circ\Phi}$, etc., will yield an expression similar to Eq. (3.6) that ensures Φ-invariance.

2. Theorem 3.15 remains valid if we consider only n-ary aggregation operators instead of increasing $[0,1]^n \to [0,1]$ functions. The proofs of Lemmata 3.11 and 3.12 need no adjustments when dealing with n-ary aggregation operators G. To ensure the boundary conditions $F(0,\ldots,0) = G(0,\ldots,0) = 0$ and $F(1,\ldots,1) = G(1,\ldots,1) = 1$ in the proof of Lemma 3.14 we need to require that $f(0) = 0$ and $f(1) = 1$. These additional conditions do not affect the (re)formulation of Theorem 3.15 for n-ary aggregation operators.

3. An increasing $[0,1]^n \to [0,1]$ function F can also be N-invariant on

$$A = [0,1]^n \setminus \{(x_1,\ldots,x_n) \mid \min(x_1,\ldots,x_n) = 0 \ \wedge \ \max(x_1,\ldots,x_n) = 1\}\,.$$

For example, the *conjunctive* $3\,\Pi$*–operator* E [12, 17, 28, 49], defined by

$$E(x_1,\ldots,x_n) = \begin{cases} \dfrac{x_1\cdot\ldots\cdot x_n}{x_1\cdot\ldots\cdot x_n + (1-x_1)\cdot\ldots\cdot(1-x_n)}, & \text{if } (x_1,\ldots,x_n)\in A\,,\\ 0, & \text{elsewhere}\,, \end{cases}$$

is an n-ary aggregation operator that is $\mathcal{N}$-invariant on A. The convention $E(x_1,\ldots,x_n) = \frac{0}{0+0} := 0$, whenever $(x_1,\ldots,x_n) \in [0,1]^n \setminus A$ prevents E from being self-dual. As indicated in [7], under the alternative convention $\frac{0}{0+0} := \frac{1}{2}$ and with $G(x_1,\ldots,x_n) := x_1\cdot\ldots\cdot x_n$, the $3\,\Pi$–operator can be constructed by means of Eq. (3.2).

3.4 Shift invariance

Comparing Eq. (3.2) with Eq. (3.3), García–Lapresta and Marques Pereira [33] argue that their approach (Eq. (3.3)), in contrast to Eq. (3.2), preserves shift invariance.

Definition 3.19 [55] An increasing $[0,1]^n \to [0,1]$ function F is *shift invariant* if it holds that

$$F(x_1+t,\ldots,x_n+t) = F(x_1,\ldots,x_n) + t\,, \tag{3.8}$$

for every $t \in [0,1]$ and all $(x_1,\ldots,x_n) \in [0,1-t]^n$.

Interpreting the translations in question as $[0,1-t] \to [t,1]$ isomorphisms Φ_t (*i.e.* $\Phi_t(x) = x+t$), with $t \in [0,1]$, Eq. (3.8) expresses some kind of 'Φ_t-invariance' of F. In measurement-theoretic frameworks a shift-invariant function F is called *stable for any admissible translation* [27, 70]. The arithmetic mean $\mathbf{M}$, the *minimum operator* $T_{\mathbf{M}}$ ($T_{\mathbf{M}}(x_1,\ldots,x_n) = \min(x_1,\ldots,x_n)$) and the *maximum operator* $S_{\mathbf{M}}$ ($S_{\mathbf{M}}(x_1,\ldots,x_n) = \max(x_1,\ldots,x_n)$) are all examples of shift-invariant, increasing $[0,1]^n \to [0,1]$ functions. A full characterization of shift-invariant, binary aggregation operators can be found in [55]. It is clear that the identity function **id** is the only shift-invariant, increasing $[0,1] \to [0,1]$ function. Hence, every shift-invariant, increasing $[0,1]^n \to [0,1]$ function F must be idempotent (take $x_1 = \ldots = x_n$ in Eq. (3.8)) [55]. Denoting $y_i = x_i + t$ in Eq. (3.8) it also follows that every shift-invariant function F must be invariant under 'negative translations' (*i.e.* $t \in [-1,0]$) [33].

We will contribute to the existing knowledge by further exploring the argument of García–Lapresta and Marques Pereira [33]. In particular, we look for those increasing $[0,1]^2 \to [0,1]$ functions $\boldsymbol{\mathcal{C}}$ that enable a full characterization of all $\mathcal{N}$-invariant, increasing functions F and that, in combination with the standard negator $\mathcal{N}$, preserve shift invariance.

Definition 3.20 Let N be an involutive negator and consider an increasing $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$. If $\boldsymbol{\mathcal{C}}_G$ is shift invariant for every shift-invariant, increasing $[0,1]^n \to [0,1]$ function G, we say that, the couple $(\boldsymbol{\mathcal{C}}, N)$ *preserves shift invariance*.

Although it is not explicitly visible, the involutive negator N in this definition is required to formulate the functions $\boldsymbol{\mathcal{C}}_G$.

Theorem 3.21 *Consider an involutive negator N and an increasing $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$. If $n > 1$ and $(\boldsymbol{\mathcal{C}}, N)$ preserves shift invariance, then $\boldsymbol{\mathcal{C}}$ is shift invariant.*

Proof Let N and $\boldsymbol{\mathcal{C}}$ be as described in the statement. Recall that $T_{\mathbf{M}}$ is a shift-invariant, increasing $[0,1]^2 \to [0,1]$ function. Take arbitrary $(x,y,t) \in [0,1]^3$ such that $(x+t, y+t) \in [0,1]^2$. If $x \leqslant y$ then

$$\begin{aligned}\boldsymbol{\mathcal{C}}(x+t,y+t) &= \boldsymbol{\mathcal{C}}\left(\min(x+t,\ldots,x+t,y+t), \min\left((x+t)^N,\ldots,(x+t)^N,(y+t)^N\right)^N\right)\\ &= \boldsymbol{\mathcal{C}}_{T_{\mathbf{M}}}(x+t,\ldots,x+t,y+t) = \boldsymbol{\mathcal{C}}_{T_{\mathbf{M}}}(x,\ldots,x,y)+t\\ &= \boldsymbol{\mathcal{C}}\left(\min(x,\ldots,x,y), \min\left(x^N,\ldots,x^N,y^N\right)^N\right)+t = \boldsymbol{\mathcal{C}}(x,y)+t\,.\end{aligned}$$

By replacing in the above chain of equalities min by max and $T_{\mathbf{M}}$ by $S_{\mathbf{M}}$ it is shown that $\boldsymbol{\mathcal{C}}(x+t,y+t) = \boldsymbol{\mathcal{C}}(x,y)+t$ whenever $y < x$. We conclude that $\boldsymbol{\mathcal{C}}$ itself is shift invariant. □

The above theorem does not hold for $n = 1$. As **id** is the only shift-invariant, increasing $[0,1] \to [0,1]$ function, a couple $(\boldsymbol{\mathcal{C}}, N)$ preserves shift invariance if and only if $\boldsymbol{\mathcal{C}}(x+t,x+t) = \boldsymbol{\mathcal{C}}(x,x)+t$, for every $t \in [0,1]$ and $x \in [0,1-t]$. The latter requires that $\boldsymbol{\mathcal{C}}$ itself must be idempotent. Therefore, for $n = 1$, preserving shift invariance is equivalent with idempotency. Furthermore, also the converse of the theorem is not always true. Consider, for example, the involutive negator N defined by $x^N = \sqrt{1-x^2}$ and take $\boldsymbol{\mathcal{C}} = G = \mathbf{M}$. Clearly, $\boldsymbol{\mathcal{C}}$ is shift invariant but $\boldsymbol{\mathcal{C}}_G = \mathbf{M}_{\mathbf{M}}$ is not:

$$\mathbf{M}_{\mathbf{M}}\left(0,\frac{1}{2}\right)+\frac{1}{2} \approx 0.8049 < 0.8257 \approx \mathbf{M}_{\mathbf{M}}\left(0+\frac{1}{2},\frac{1}{2}+\frac{1}{2}\right)\,.$$

Nevertheless, if $N = \mathcal{N}$, then the $\mathcal{N}$-transform $G_{\mathcal{N}}$ of a shift-invariant, increasing $[0,1]^n \to [0,1]$ function G is also shift invariant:

$$\begin{aligned}G_{\mathcal{N}}(x_1+t,\ldots,x_n+t) &= G(1-x_1-t,\ldots,1-x_n-t)^{\mathcal{N}} = (G(1-x_1,\ldots,1-x_n)-t)^{\mathcal{N}}\\ &= G(1-x_1,\ldots,1-x_n)^{\mathcal{N}}+t = G_{\mathcal{N}}(x_1,\ldots,x_n)+t\end{aligned}$$

whenever $(x_1+t,\ldots,x_n+t) \in [0,1]^n$. Therefore, for every shift-invariant, increasing $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$, the couple $(\boldsymbol{\mathcal{C}}, \mathcal{N})$ preserves shift invariance. If we additionally want that $\boldsymbol{\mathcal{C}}$

enables a full characterization of all $\mathcal{N}$-invariant, increasing $[0,1]^n \to [0,1]$ functions, the arithmetic mean is the only good choice for $n > 1$.

Theorem 3.22 *If $n > 1$, then the arithmetic mean $\mathbf{M}$ is the only increasing $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ that enables a full characterization of all $\mathcal{N}$-invariant, increasing $[0,1]^n \to [0,1]$ functions and for which $(\boldsymbol{\mathcal{C}}, \mathcal{N})$ preserves shift invariance.*

Proof From Theorem 3.21 it follows that $\boldsymbol{\mathcal{C}}$ must be shift invariant in order to preserve shift invariance. Aczél [1] showed that the general solution of Eq. (3.8) ($n = 2$) is given by $\boldsymbol{\mathcal{C}}(x, y) = x + f(y - x)$, for some function $f : [-1, 1] \to [0, 1]$ such that $x + f(y - x) \in [0, 1]$. Expressing that Eq. (3.6) must hold for $N = \mathcal{N}$ leads to $f(y-x) = (y-x)/2$. Consequently, $\boldsymbol{\mathcal{C}}$ must be the arithmetic mean. From Theorem 3.15 and from the discussion preceding this theorem it follows that the arithmetic mean $\mathbf{M}$ indeed enables a full characterization of all $\mathcal{N}$-invariant increasing $[0,1]^n \to [0,1]$ functions and that $(\mathbf{M}, \mathcal{N})$ preserves shift invariance. $\square$

In case $n = 1$ it follows from the discussion above that every idempotent $\boldsymbol{\mathcal{C}}$ satisfying the assertions of Theorem 3.15 will do.

CHAPTER 4

Traces of orthosymmetry

4.1 Introduction

Functions that fuse multiple input values into a single output value are indispensable tools for various sciences such as pure and applied mathematics, computer science, economics and psychology. It is often the case that all inputs as well as the output belong to the same domain. Usually, also some monotonic behaviour is required. Studying the properties of an increasing $[0,1]^n \to [0,1]$ function F, with $n \geqslant 2$, requires some basic geometrical insight into the structure of its partial functions, obtained by fixing $n-2$ input values. As shown in the previous chapter, increasing $[0,1]^2 \to [0,1]$ functions are also indispensable tools for describing the set of N-invariant monotone $[0,1]^n \to [0,1]$ functions, with N a fixed involutive negator. For these reasons we now direct our attention to the study of increasing $[0,1]^2 \to [0,1]$ functions satisfying one or more properties. It is often worthwhile to observe these functions from a different point of view. Describing an increasing $[0,1]^2 \to [0,1]$ function F in terms of contour lines yields several new insights into its geometrical structure. Throughout Chapters 7 and 8 contour lines will prove to be indispensable for the decomposition and construction of rotation-invariant t-norms. In this chapter, however, we describe some orthosymmetrical aspects of contour lines.

It should be noted that also decreasing $[0,1]^2 \to [0,1]$ functions can be described in terms of their contour lines. Clearly, for every decreasing $[0,1]^2 \to [0,1]$ function G and every strict negator N, $N \circ G$ is an increasing $[0,1]^2 \to [0,1]$ function. Therefore, results concerning increasing functions can easily be translated into results for decreasing functions. Since our goal is to better understand rotation-invariant t-norms (which are increasing), we focus here on increasing $[0,1]^2 \to [0,1]$ functions.

4.2 Countour lines

Each increasing $[0,1]^2 \to [0,1]$ function F is totally determined by its *horizontal cuts* (*i.e.* the intersections of its graph by planes parallel to the domain $[0,1]^2$). The contour lines of F are those $[0,1] \to [0,1]$ functions determining the upper, lower, right or left limits of its horizontal cuts.

Definition 4.1 We associate with an increasing $[0,1]^2 \to [0,1]$ function F four types of *contour lines* ($a \in [0,1]$):

$$\begin{aligned}
C_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(x,t) \leqslant a\} \\
D_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(x,t) \geqslant a\} \\
\widetilde{C}_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(t,x) \leqslant a\} \\
\widetilde{D}_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(t,x) \geqslant a\}
\end{aligned}$$

(with $\sup \emptyset = 0$ and $\inf \emptyset = 1$). It will be clear from the context which function F we are considering. Considering the ensemble of contour lines, we can associate an additional function to each type of contour line. For example, the contour lines of the type C_a are totally determined by the $[0,1]^2 \to [0,1]$ function C that maps a couple (x,a) to $C_a(x)$. Hence, contour lines of the type C_a are partial functions of C, obtained by fixing its second argument. The partial functions obtained by fixing the first argument of C will be denoted $C_\bullet(x)$, with $x \in [0,1]$. A similar argument applies to the other types of contour lines.

Property 4.2 *The contour lines of an increasing $[0,1]^2 \to [0,1]$ function F satisfy the following properties:*

(D1) *C_a, D_a, $\widetilde{C}_a$ and $\widetilde{D}_a$ are decreasing, for every $a \in [0,1]$.*
(D2) *$D_a \leqslant C_a$ and $\widetilde{D}_a \leqslant \widetilde{C}_a$, for every $a \in [0,1]$.*
(D3) *$C_{a_1} \leqslant C_{a_2}$, $D_{a_1} \leqslant D_{a_2}$, $\widetilde{C}_{a_1} \leqslant \widetilde{C}_{a_2}$ and $\widetilde{D}_{a_1} \leqslant \widetilde{D}_{a_2}$, for every $(a_1,a_2) \in [0,1]^2$ such that $a_1 \leqslant a_2$.*

Proof Properties **(D1)** and **(D3)** follow immediately from the definition of the four different types of contour lines. Thanks to the increasingness of F we know that also

$$\begin{aligned}
C_a(x) &= \inf\{t \in [0,1] \mid F(x,t) > a\}\,, \\
D_a(x) &= \sup\{t \in [0,1] \mid F(x,t) < a\}\,, \\
\widetilde{C}_a(x) &= \inf\{t \in [0,1] \mid F(t,x) > a\}\,, \\
\widetilde{D}_a(x) &= \sup\{t \in [0,1] \mid F(t,x) < a\}\,,
\end{aligned}$$

for every $(x,a) \in [0,1]^2$. In combination with the definition of contour lines, this yields property **(D2)**. $\square$

Before studying the symmetrical aspects of contour lines, we first discuss some continuity conditions that are crucial for our further results. Note that F will be called *left continuous*, resp.

right continuous, if all of its partial functions $F(x, \bullet)$ and $F(\bullet, x)$ are left continuous, resp. right continuous (see e.g. [51]).

Definition 4.3 [6] Two monotone $[0,1] \to [0,1]$ functions f and g form a *Galois connection* (f, g) if $f(x) \leqslant y \;\Leftrightarrow\; x \leqslant g(y)$ holds for every $(x, y) \in [0,1]^2$.

Dealing with an arbitrary increasing $[0,1]^2 \to [0,1]$ function F, we obtain the following characterization.

Theorem 4.4 *Consider an increasing $[0,1] \to [0,1]$ function F. For every $x \in [0,1]$ the following assertions hold:*

1. *$F(x, \bullet)$ is left continuous if and only if*
$$F(x, y) \leqslant a \quad \Leftrightarrow \quad y \leqslant C_a(x) \tag{4.1}$$
holds for every $(y, a) \in [0,1]^2$, with $0 < y$.
2. *$F(x, \bullet)$ is right continuous if and only if*
$$D_a(x) \leqslant y \quad \Leftrightarrow \quad a \leqslant F(x, y) \tag{4.2}$$
holds for every $(y, a) \in [0,1]^2$, with $y < 1$.
3. *$F(\bullet, x)$ is left continuous if and only if*
$$F(y, x) \leqslant a \quad \Leftrightarrow \quad y \leqslant \widetilde{C}_a(x) \tag{4.3}$$
holds for every $(y, a) \in [0,1]^2$, with $0 < y$.
4. *$F(\bullet, x)$ is right continuous if and only if*
$$\widetilde{D}_a(x) \leqslant y \quad \Leftrightarrow \quad a \leqslant F(y, x) \tag{4.4}$$
holds for every $(y, a) \in [0,1]^2$, with $y < 1$.

Proof We will prove the first case of the theorem only, the other cases being similar. Note that, by definition, $F(x, y) \leqslant a$ always implies $y \leqslant C_a(x)$. Suppose that $F(x, \bullet)$ is left continuous and consider arbitrary $(y, a) \in [0,1]^2$, $0 < y$. If $y \leqslant C_a(x)$, then for every $\varepsilon \in \,]0, y]$ we know that $F(x, y-\varepsilon) \leqslant a$. The left continuity of $F(x, \bullet)$ then ensures that $F(x, y) \leqslant a$. Conversely, suppose that Eq. (4.1) holds and that $F(x, \bullet)$ is not left continuous. Then there exists $(y, a) \in [0,1]^2$, $0 < y$, such that $F(x, y - \varepsilon) \leqslant a < F(x, y)$, for every $\varepsilon \in \,]0, y]$. However, by definition we obtain that $y - \varepsilon \leqslant C_a(x)$, for every $\varepsilon \in \,]0, y]$, and therefore $y \leqslant C_a(x)$. Applying Eq. (4.1) leads to the contradiction $F(x, y) \leqslant a$. $\square$

The continuity of F also affects the continuity of its contour lines.

Property 4.5 *Consider an increasing* $[0,1]^2 \to [0,1]$ *function* F *and take arbitrary* $(x,a) \in [0,1]^2$*. If* F *is left continuous, then* $C_a(\bullet)$*,* $\widetilde{C}_a(\bullet)$ *are left continuous and* $C_\bullet(x)$*,* $\widetilde{C}_\bullet(x)$ *are right continuous. If* F *is right continuous, then* $D_a(\bullet)$*,* $\widetilde{D}_a(\bullet)$ *are right continuous and* $D_\bullet(x)$*,* $\widetilde{D}_\bullet(x)$ *are left continuous.*

Proof We only prove those properties invoking the $[0,1]^2 \to [0,1]$ function C. Let F be left continuous. Suppose that there exists a triplet $(x,y,a) \in [0,1]^3$ such that $0 < x$, $0 < y$ and $C_a(x) < y \leqslant C_a(x-\varepsilon)$ for every $\varepsilon \in\,]0,x]$. Applying Eq. (4.1), we then know that $F(x-\varepsilon, y) \leqslant a < F(x,y)$ for every $\varepsilon \in\,]0,x]$. This contradicts the left continuity of F and, hence, C_a must be left continuous. Suppose now that there exists a triplet $(x,y,a) \in [0,1]^3$ such that $0 < y$, $a < 1$ and $C_a(x) < y \leqslant C_{a+\varepsilon}(x)$, for every $\varepsilon \in\,]0,1-a]$. From Eq. (4.1) it then follows that $a < F(x,y) \leqslant a+\varepsilon$, for every $\varepsilon \in\,]0,1-a]$. Taking the limit $\varepsilon \searrow 0$ leads to the contradiction $a < a$. We conclude that $C_\bullet(x)$ is right continuous. □

In order to prove the right continuity of $C_\bullet(x)$, it is sufficient to invoke the left continuity of the partial functions $F(x,\bullet)$ only. However, when proving the left continuity of C_a, also the left continuity of the partial functions $F(\bullet,x)$ is needed. For example, if $F(1,y) = 1$, for every $y \in [0,1]$, and $F(x,y) = 0$, elsewhere, then $C_0(x) = 1$ for every $x \in [0,1[$ and $C_0(1) = 0$. The contour line C_0 is, in contrast to the vertical sections $F(x,\bullet)$, not left continuous. Note that the converse implications of Property 4.5 do not hold. If $F(1,1) = 1$ and $F(x,y) = 0$ elsewhere, then $C_a(x) = \widetilde{C}_a(x) = 1$, for every $(x,a) \in [0,1]^2$. F is not left continuous, although $C_a = \widetilde{C}_a$ and $C_\bullet(x) = \widetilde{C}_\bullet(x)$ are continuous for every $(x,a) \in [0,1]^2$.

Taking a closer look at Eqs. (4.1)–(4.4), it strikes that only the restrictions on y prevent them from being fully interpretable as Galois connections. In the following theorem we figure out under which conditions these restrictions on y become superfluous.

Theorem 4.6 *Consider an increasing* $[0,1]^2 \to [0,1]$ *function* F*. For every* $x \in [0,1]$ *we obtain four groups consisting of four equivalent assertions.*

1. *a)* $F(x,\bullet)$ *is left continuous and fulfills* $F(x,0) = 0$*.*
   b) $(F(x,\bullet), C_\bullet(x))$ *is a Galois connection.*
   c) For every $a \in [0,1]$ *it holds that* $F(x, C_a(x)) \leqslant a$*.*
   d) For every $a \in [0,1]$ *it holds that* $C_a(x) = \max\{t \in [0,1] \mid F(x,t) \leqslant a\}$*.*
2. *a)* $F(x,\bullet)$ *is right continuous and fulfills* $F(x,1) = 1$*.*
   b) $(D_\bullet(x), F(x,\bullet))$ *is a Galois connection.*
   c) For every $a \in [0,1]$ *it holds that* $a \leqslant F(x, D_a(x))$*.*
   d) For every $a \in [0,1]$ *it holds that* $D_a(x) = \min\{t \in [0,1] \mid F(x,t) \geqslant a\}$*.*
3. *a)* $F(\bullet,x)$ *is left continuous and fulfills* $F(0,x) = 0$*.*
   b) $(F(\bullet,x), \widetilde{C}_\bullet(x))$ *is a Galois connection.*
   c) For every $a \in [0,1]$ *it holds that* $F(\widetilde{C}_a(x), x) \leqslant a$*.*
   d) For every $a \in [0,1]$ *it holds that* $\widetilde{C}_a(x) = \max\{t \in [0,1] \mid F(t,x) \leqslant a\}$*.*
4. *a)* $F(\bullet,x)$ *is right continuous and fulfills* $F(1,x) = 1$*.*

b) $(\widetilde{D}_\bullet(x), F(\bullet, x))$ *is a Galois connection.*
c) For every $a \in [0,1]$ *it holds that* $a \leqslant F(\widetilde{D}_a(x), x)$.
d) For every $a \in [0,1]$ *it holds that* $\widetilde{D}_a(x) = \min\{t \in [0,1] \mid F(t,x) \geqslant a\}$.

Proof We will only prove the equivalences in the first group, the other cases being similar. Taking into account Theorem 4.4, assertion 1a will be equivalent with assertion 1b if we can show that the boundary condition $F(x,0) = 0$ is equivalent with $F(x,0) \leqslant a \;\Leftrightarrow\; 0 \leqslant C_a(x)$, for every $a \in [0,1]$. As $0 \leqslant C_a(x)$ is always true, this amounts to the trivial equivalence $F(x,0) = 0 \;\Leftrightarrow\; F(x,0) \leqslant a$, for every $a \in [0,1]$. By definition, $F(x,y) \leqslant a$ always implies $y \leqslant C_a(x)$ and $y < C_a(x)$ always implies $F(x,y) \leqslant a$. Therefore, assertion 1b is satisfied if and only if $y = C_a(x)$ implies $F(x,y) \leqslant a$. The latter is expressed by assertion 1c. It is evident that assertion 1c is also equivalent with assertion 1d. $\square$

4.3 Orthosymmetrical contour lines

For a given couple (Φ, Ψ) of monotone $[0,1] \to [0,1]$ bijections, we will characterize, in terms of contour lines, those increasing $[0,1]^2 \to [0,1]$ functions F that satisfy

$$F(x,y) = \Psi\big(F(\Phi^{-1}(y), \Phi(x))\big)\,, \tag{4.5}$$

for every $(x,y) \in [0,1]^2$. In case $\Phi = \Psi = \mathbf{id}$, the latter expresses the commutativity of F. For $\Phi = \Psi = N$, with N an involutive negator, we obtain Eq. (3.6). Due to the structure of Eq. (4.5) there are, however, some restrictions on the choice of Φ and Ψ. To be compatible with the increasingness of F it is clear that Φ and Ψ must have the same type of monotonicity. Furthermore, applying Eq. (4.5) twice results in $F(x,y) = \Psi(\Psi(F(x,y)))$. We will strengthen this condition and require that Ψ is involutive: $\Psi \circ \Psi = \mathbf{id}$. The observation that the binary aggregation operator $\boldsymbol{C}$ from Theorem 3.15 should reach every element of $[0,1]$ also supports this additional condition on Ψ. The considerations above force us to consider functional equation (4.5) in the following two cases only:

A. Φ is an automorphism ϕ and Ψ is the identity function $\mathbf{id}$.
B. Φ is a strict negator M and Ψ is an involutive negator N.

A. $(\Phi, \Psi) = (\phi, \mathrm{id})$, with ϕ an automorphism

In this case, Eq. (4.5) can be rewritten as

$$F(x,y) = F(\phi^{-1}(y), \phi(x))\,, \tag{4.6}$$

for every $(x,y) \in [0,1]^2$. From the observation that $(\phi^{-1}(y), \phi(x))$ is the ϕ-inverse of the point (x,y), we obtain a geometrical characterization of all increasing $[0,1]^2 \to [0,1]$ functions F satisfying Eq. (4.6). It suffices to define F on $\{(x,y) \in [0,1]^2 \mid y \leqslant \phi(x)\}$ as an arbitrary increasing function. Eq. (4.6) can then be used to uniquely complete F on $\{(x,y) \in [0,1] \mid \phi(x) < y\}$. The increasingness of F is easily verified. The construction entangles the contour lines of the types C_a and $\widetilde{C}_a$ and those of the types D_a and $\widetilde{D}_a$.

Theorem 4.7 *Consider an automorphism ϕ. For an increasing $[0,1]^2 \to [0,1]$ function F satisfying Eq. (4.6) the following assertions hold:*

1. $C_a = \phi \circ \widetilde{C}_a \circ \phi$, *for every $a \in [0,1]$.*
2. $D_a = \phi \circ \widetilde{D}_a \circ \phi$, *for every $a \in [0,1]$.*

Proof We will prove the first assertion, the second one is proven in a similar way. If F satisfies Eq. (4.6) then, by definition, we obtain that

$$\begin{aligned} C_a(x) &= \sup\{t \in [0,1] \mid F(x,t) \leqslant a\} = \sup\{t \in [0,1] \mid F(\phi^{-1}(t), \phi(x)) \leqslant a\} \\ &= \phi(\sup\{s \in [0,1] \mid F(s, \phi(x)) \leqslant a\}) = \phi(\widetilde{C}_a(\phi(x))) \,, \end{aligned}$$

for every $(x,a) \in [0,1]^2$. □

Furthermore, the symmetry contained in Eq. (4.6) manifests itself in the ϕ-orthosymmetry, resp. ϕ^{-1}-orthosymmetry, of the contour lines C_a and D_a, resp. $\widetilde{C}_a$ and $\widetilde{D}_a$.

Theorem 4.8 *Consider an automorphism ϕ. For an increasing $[0,1]^2 \to [0,1]$ function F satisfying Eq. (4.6) the following assertions hold:*

1. $C_a \in Q(C_a, \phi)$, *for every $a \in [0,1]$.*
2. $D_a \in Q(D_a, \phi)$, *for every $a \in [0,1]$.*
3. $\widetilde{C}_a \in Q(\widetilde{C}_a, \phi^{-1})$, *for every $a \in [0,1]$.*
4. $\widetilde{D}_a \in Q(\widetilde{D}_a, \phi^{-1})$, *for every $a \in [0,1]$.*

Proof If F satisfies Eq. (4.6) it always holds that $\widetilde{C}_a = \phi^{-1} \circ C_a \circ \phi^{-1}$ and $\widetilde{D}_a = \phi^{-1} \circ D_a \circ \phi^{-1}$ (Theorem 4.7). Invoking Theorems 1.11 and 1.13, assertion 3 amounts to $C_a \in Q(\phi^{-1} \circ C_a \circ \phi^{-1}, \mathbf{id}) = Q(C_a, \phi)$ and assertion 4 amounts to $D_a \in Q(\phi^{-1} \circ D_a \circ \phi^{-1}, \mathbf{id}) = Q(D_a, \phi)$. It is therefore sufficient to focus on assertions 1 and 2 only. We will present the proof of assertion 1, the proof of assertion 2 being similar. Take arbitrary $a \in [0,1]$. By definition, it holds that

$$\begin{aligned} \underline{C_a}^{\phi}(x) &= \sup\{t \in [0,1] \mid C_a(\phi^{-1}(t)) > \phi(x)\} \,, \\ C_a(x) &= \sup\{t \in [0,1] \mid F(x,t) \leqslant a\} \,, \\ \underline{C_a}_{\phi}(x) &= \sup\{t \in [0,1] \mid C_a(\phi^{-1}(t)) \geqslant \phi(x)\} \,. \end{aligned}$$

Eq. (4.6) guarantees that

$$C_a(\phi^{-1}(t)) > \phi(x) \quad \Rightarrow \quad F(x,t) = F(\phi^{-1}(t), \phi(x)) \leqslant a \quad \Rightarrow \quad C_a(\phi^{-1}(t)) \geqslant \phi(x) \,,$$

which leads to $\underline{C_a}^{\phi} \leqslant C_a \leqslant \underline{C_a}_{\phi}$. It follows from Corollary 2.8 that $C_a \in Q(C_a, \phi)$. □

Unfortunately, the orthosymmetry conditions in Theorem 4.8 are not sufficient for Eq. (4.6) to hold. For example, if $F(x,0) = 0$, for all $x \in [0,1]$, and $F(x,y) = 1$, elsewhere, then F is left continuous but does not fulfill Eq. (4.6) ($F(1,0) = 0 < 1 = F(0,1)$). It is easily verified that in this example all contour lines C_a and D_a, resp. $\widetilde{C}_a$ and $\widetilde{D}_a$, are ϕ-orthosymmetrical, resp. ϕ^{-1}-orthosymmetrical. Nevertheless, for a left- or right-continuous, increasing function F one of the assertions in Theorem 4.7 is sufficient to obtain Eq. (4.6).

Theorem 4.9 *Consider an automorphism ϕ and an increasing $[0,1]^2 \to [0,1]$ function F.*

1. *If F is left continuous, then the following assertions are equivalent:*

   a) *F satisfies Eq.* (4.6).
   b) *$C_a = \phi \circ \widetilde{C}_a \circ \phi$, for every $a \in [0,1]$.*

2. *If F is right continuous, then the following assertions are equivalent:*

   a) *F satisfies Eq.* (4.6).
   b) *$D_a = \phi \circ \widetilde{D}_a \circ \phi$, for every $a \in [0,1]$.*

Proof We will prove the first statement. Let F be a left continuous, increasing $[0,1]^2 \to [0,1]$ function. If F satisfies Eq. (4.6) then assertion 1b follows immediately from Theorem 4.7. Conversely, take F such that assertion 1b holds and suppose that $F(x,y) < F(\phi^{-1}(y), \phi(x))$, for some $(x,y) \in [0,1]^2$. Clearly, either $0 < x$ or $0 < y$. It follows from Eq. (4.1) that $C_{F(x,y)}(\phi^{-1}(y)) < \phi(x)$, if $0 < x$, and from Eq. (4.3) that $\widetilde{C}_{F(x,y)}(\phi(x)) < \phi^{-1}(y)$, if $0 < y$. Since $C_{F(x,y)} = \phi \circ \widetilde{C}_{F(x,y)} \circ \phi$, this leads to $\widetilde{C}_{F(x,y)}(y) < x$, if $0 < x$, and $C_{F(x,y)}(x) < y$, if $0 < y$. By definition, we obtain in both cases the contradiction $F(x,y) < F(x,y)$. Hence, $F(\phi^{-1}(y), \phi(x)) \leqslant F(x,y)$, for every $(x,y) \in [0,1]^2$. From the observation that $F(\phi^{-1}(y), \phi(x)) < F(x,y)$ can be reformulated as $F(u,v) < F(\phi^{-1}(v), \phi(u))$, with $u = \phi^{-1}(y)$ and $v = \phi(x)$, we conclude that $F(x,y) = F(\phi^{-1}(y), \phi(x))$ is fulfilled for every $(x,y) \in [0,1]^2$. □

Note that, without the additional continuity conditions, the equivalences in this theorem are not necessarily satisfied. Define, for example, F on $[0,1[^2 \cup \{(1,0)\}$ as $F(x,y) = 0$ and put $F(x,y) = 1$ elsewhere. Then F is not left continuous and $C_a = \widetilde{C}_a$, for every $a \in [0,1]$. As $C_a(x) \in \{0,1\}$, for every $(x,a) \in [0,1]^2$, it clearly holds that $C_a = \phi \circ \widetilde{C}_a \circ \phi$. However, $F(1,0) = 0 < F(0,1) = 1$ such that Eq. (4.6) is not satisfied. As illustrated in the discussion following Theorem 4.8, to invert Theorem 4.8, besides continuity conditions, also some additional boundary conditions will be required.

Theorem 4.10 *Consider an automorphism ϕ and an increasing $[0,1]^2 \to [0,1]$ function F.*

1. *If F is left continuous and $F(0,1) = F(1,0) = 0$, then the following assertions are equivalent:*

   a) *F satisfies Eq.* (4.6).
   b) *$C_a \in Q(C_a, \phi)$, for every $a \in [0,1]$.*
   c) *$\widetilde{C}_a \in Q(\widetilde{C}_a, \phi^{-1})$, for every $a \in [0,1]$.*

2. *If F is right continuous and $F(0,1) = F(1,0) = 1$, then the following assertions are equivalent:*

a) F *satisfies Eq.* (4.6)*.*
b) $D_a \in Q(D_a, \phi)$*, for every* $a \in [0,1]$*.*
c) $\widetilde{D}_a \in Q(\widetilde{D}_a, \phi^{-1})$*, for every* $a \in [0,1]$*.*

Proof We only prove the first part of the theorem. Let F be a left continuous, increasing $[0,1]^2 \to [0,1]$ function satisfying $F(0,1) = F(1,0) = 0$. Note that in that case the increasingness of F implies that $F(x,0) = F(0,x) = 0$, for every $x \in [0,1]$. From Theorem 4.8 we know that assertion 1a implies assertions 1b and 1c. Assume that $C_a \in Q(C_a, \phi)$, for every $a \in [0,1]$. Then, equivalently, $\phi^{-1} \circ C_a \circ \phi^{-1} \in Q(C_a, \mathbf{id})$, for every $a \in [0,1]$ (Theorem 1.11). The left continuity of F ensures that every C_a and thus also every $\phi^{-1} \circ C_a \circ \phi^{-1}$ is left continuous (Property 4.5). Due to the boundary condition $F(0,1) = 0$ it holds that $C_a(0) = 1$ and $\phi^{-1}(C_a(\phi^{-1}(0))) = \phi^{-1}(C_a(0)) = \phi^{-1}(1) = 1$. Invoking Theorems 1.16 and 1.17 these considerations lead to $\phi^{-1} \circ C_a \circ \phi^{-1} = \underline{C_a}_{\mathbf{id}}$. Since $(F(x,\bullet), C_\bullet(x))$ forms a Galois connection for every $x \in [0,1]$ (Theorem 4.6), we obtain the following chain of equalities:

$$\begin{aligned}\phi^{-1}(C_a(\phi^{-1}(x))) &= \inf\{t \in [0,1] \mid C_a(t) < x\} = \sup\{t \in [0,1] \mid C_a(t) \geqslant x\}\\ &= \sup\{t \in [0,1] \mid F(t,x) \leqslant a\} = \widetilde{C}_a(x)\,,\end{aligned}$$

for every $(x,a) \in [0,1]^2$. We conclude that $\phi^{-1} \circ C_a \circ \phi^{-1} = \widetilde{C}_a$, for every $a \in [0,1]$, and thus F satisfies Eq. (4.6) (Theorem 4.9). In a similar way it can be shown that assertion 1c also implies assertion 1a. □

As F is increasing, the boundary condition $F(0,1) = F(1,0) = 0$ implies that F has absorbing element 0. Otherwise, $F(0,1) = F(1,0) = 1$ ensures that 1 is the absorbing element of F.

Definition 4.11 A $[0,1]^2 \to [0,1]$ function F has an *absorbing element* $a \in [0,1]$ if $F(x,a) = F(a,x) = a$, for every $x \in [0,1]$.

In the literature (see e.g. [7]) the term *annihilator* is also used to refer to an absorbing element.

B. $(\Phi, \Psi) = (M, N)$, with M a strict and N an involutive negator

For this particular choice of Φ and Ψ, Eq. (4.5) reads

$$F(x,y) = F\big(y^{(M^{-1})}, x^M\big)^N\,, \tag{4.7}$$

for every $(x,y) \in [0,1]^2$. Note that by putting $y = x^M$, we obtain $F(x,x^M) = F(x,x^M)^N$. Denoting β the unique fixpoint of N, this leads to $F(x,x^M) = \beta$, for every $x \in [0,1]$. As $(y^{(M^{-1})}, x^M)$ is the M-inverse of the point (x,y), we are able to give a geometrical characterization of all increasing $[0,1]^2 \to [0,1]$ functions F satisfying Eq. (4.7). First, we define F on $\{(x,y) \in [0,1]^2 \mid y < x^M\}$ as an arbitrary increasing function taking values in $[0,\beta]$. Next, we put $F(x,x^M) = \beta$, for every $x \in [0,1]$. Finally, we use Eq. (4.7) to uniquely complete F on $\{(x,y) \in [0,1] \mid x^M < y\}$. Eq. (4.7) enforces some kind of symmetry on F that clearly affects the structure of its contour lines.

Theorem 4.12 *Consider a strict negator M and an involutive negator N with fixpoint β. For an increasing $[0,1]^2 \to [0,1]$ function F satisfying Eq. (4.7) the following assertions hold:*

1. $C_{a^N} = M \circ \widetilde{D}_a \circ M$, *for every* $a \in [0,\beta]$.
2. $D_{a^N} = M \circ \widetilde{C}_a \circ M$, *for every* $a \in [0,\beta]$.
3. $\widetilde{C}_{a^N}$=$M^{-1} \circ D_a \circ M^{-1}$, *for every* $a \in [0,\beta]$.
4. $\widetilde{D}_{a^N} = M^{-1} \circ C_a \circ M^{-1}$, *for every* $a \in [0,\beta]$.

Proof If F satisfies Eq. (4.7) then, by definition, we obtain that

$$\begin{aligned} C_{a^N}(x) &= \sup\{t \in [0,1] \mid F(x,t) \leqslant a^N\} = \sup\bigl\{t \in [0,1] \mid F(t^{(M^{-1})}, x^M)^N \leqslant a^N\bigr\} \\ &= \bigl(\inf\bigl\{s \in [0,1] \mid F(s, x^M) \geqslant a\bigr\}\bigr)^M = \bigl(\widetilde{D}_a(x^M)\bigr)^M , \end{aligned}$$

for every $(x,a) \in [0,1]^2$. This proves the first and the fourth assertion. The other two assertions are proven in a similar way. $\square$

Note that we can merge the first and last assertion and the second and third assertion: $C_{a^N} = M \circ \widetilde{D}_a \circ M$ and $D_{a^N} = M \circ \widetilde{C}_a \circ M$ hold for every $a \in [0,1]$. However, in contrast to Eq. (4.6), the involutive negator N in Eq. (4.7) allows us to consider four assertions (Theorem 4.12) instead of two (Theorem 4.7). Each of these assertions will turn out to be sufficient for Eq. (4.7) to hold provided that F is continuous (see Theorem 4.14). As shown in the following theorem, whenever F satisfies Eq. (4.7), C_{a^N} can be understood as some 'M-inverse function' of D_a and $\widetilde{C}_{a^N}$ as some kind of 'M^{-1}-inverse function' of $\widetilde{D}_a$.

Theorem 4.13 *Consider a strict negator M and an involutive negator N with fixpoint β. For an increasing $[0,1]^2 \to [0,1]$ function F satisfying Eq. (4.7) the following assertions hold:*

1. $C_{a^N} \in Q(D_a, M)$, *for every* $a \in [0,\beta]$.
2. $D_{a^N} \in Q(C_a, M)$, *for every* $a \in [0,\beta]$.
3. $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$, *for every* $a \in [0,\beta]$.
4. $\widetilde{D}_{a^N} \in Q(\widetilde{C}_a, M^{-1})$, *for every* $a \in [0,\beta]$.

Proof From Theorem 1.6 we know that $C_{a^N} \in Q(D_a, M)$ is equivalent with $D_a \in Q(C_{a^N}, M)$ and that $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M)$ is equivalent with $\widetilde{D}_a \in Q(\widetilde{C}_{a^N}, M)$, for every $a \in [0,1]$. Hence, combining assertion 1 with assertion 2 and assertion 3 with assertion 4, it suffices to prove that $C_{a^N} \in Q(D_a, M)$ and $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$, for every $a \in [0,1]$. If F satisfies Eq. (4.7) it always holds that $\widetilde{C}_{a^N} = M^{-1} \circ D_a \circ M^{-1}$and $\widetilde{D}_a = M^{-1} \circ C_{a^N} \circ M^{-1}$ (Theorem 4.12). Invoking Theorems 1.11 and 1.13, $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$ amounts to $D_a \in Q(M^{-1} \circ C_{a^N} \circ M^{-1}, \mathbf{id}) = Q(C_{a^N}, M)$. As $D_a \in Q(C_{a^N}, M)$ is equivalent with $C_{a^N} \in Q(D_a, M)$ (Theorem 1.6), this allows us to focus only on the combined assertion $C_{a^N} \in Q(D_a, M)$, for every $a \in [0,1]$. Take arbitrary $a \in [0,1]$. By definition it holds that

$$\begin{aligned} \overline{D_a}^M(x) &= \sup\bigl\{t \in [0,1] \mid D_a(t^{(M^{-1})}) < x^M\bigr\} , \\ C_{a^N}(x) &= \sup\{t \in [0,1] \mid F(x,t) \leqslant a^N\} , \\ \overline{D_a}_M(x) &= \sup\bigl\{t \in [0,1] \mid D_a(t^{(M^{-1})}) \leqslant x^M\bigr\} . \end{aligned}$$

Eq. (4.7) guarantees that

$$D_a(t^{(M^{-1})}) < x^M \quad \Rightarrow \quad F(x,t)^N = F\big(t^{(M^{-1})}, x^M\big) \geqslant a \quad \Rightarrow \quad D_a(t^{(M^{-1})}) \leqslant x^M ,$$

which leads to $\overline{D_a}^M \leqslant C_{a^N} \leqslant \overline{D_a}_M$. As D_a is decreasing, it follows from Theorem 1.28 that $C_{a^N} \in Q(D_a, M)$. $\square$

Unfortunately, the assertions of Theorem 4.13 are again not sufficient for Eq. (4.7) to hold. For example, if $F(0,0) = 0$ and $F(x,y) = \beta$ elsewhere, then $C_a = \widetilde{C}_a = \mathbf{0}$ whenever $a \in [0,\beta[$, $C_a = \widetilde{C}_a = \mathbf{1}$ whenever $a \in [\beta,1]$, $D_a = \widetilde{D}_a = \mathbf{0}$ whenever $a \in [0,\beta]$ and $D_a = \widetilde{D}_a = \mathbf{1}$ whenever $a \in]\beta,1]$. Clearly, these contour lines satisfy the assertions from Theorems 4.12 and 4.13. However, F can never satisfy Eq. (4.7) as $F(1^{(M^{-1})}, 1^M) = F(0,0) = 0 < \beta = F(1,1)^N$. Also, in this case some additional continuity conditions are required to retrieve Eq. (4.7) from the assertions stated in Theorems 4.12 and 4.13. In contrast to Eq. (4.6), the use of strict negators in Eq. (4.7) prevents F from being solely left or right continuous.

Theorem 4.14 *Consider a strict negator M, an involutive negator N with fixpoint β and an increasing $[0,1]^2 \to [0,1]$ function F. If F is continuous, then the following assertions are equivalent:*

1. *F satisfies Eq. (4.7).*
2. *$C_{a^N} = M \circ \widetilde{D}_a \circ M$, for every $a \in [0,\beta]$.*
3. *$D_{a^N} = M \circ \widetilde{C}_a \circ M$, for every $a \in [0,\beta]$.*
4. *$\widetilde{C}_{a^N} = M^{-1} \circ D_a \circ M^{-1}$, for every $a \in [0,\beta]$.*
5. *$\widetilde{D}_{a^N} = M^{-1} \circ C_a \circ M^{-1}$, for every $a \in [0,\beta]$.*

Proof We will only prove the equivalence between the first two assertions. Due to Theorem 4.12 it suffices to prove that assertion 2 implies assertion 1. Take F such that $C_{a^N} = M \circ \widetilde{D}_a \circ M$ holds for every $a \in [0,\beta]$. In case $F(x,y) \leqslant \beta$, for $(x,y) \in [0,1]^2$ with $x < 1$, we obtain from assertion 2 and Eq. (4.1) that

$$\begin{aligned} F(x,y) = F(x,y) \;\Rightarrow\; \widetilde{D}_{F(x,y)}(y) \leqslant x \;&\Leftrightarrow\; x^M \leqslant C_{F(x,y)^N}(y^{(M^{-1})}) \\ &\Leftrightarrow\; F\big(y^{(M^{-1})}, x^M\big) \leqslant F(x,y)^N . \end{aligned} \tag{4.8}$$

In case $\beta \leqslant F(x,y)$, for $(x,y) \in [0,1]^2$ with $0 < y$, we obtain from assertion 2 and Eq. (4.4) that

$$\begin{aligned} F(x,y) = F(x,y) \;\Rightarrow\; y \leqslant C_{F(x,y)}(x) \;&\Leftrightarrow\; \widetilde{D}_{F(x,y)^N}(x^M) \leqslant y^{(M^{-1})} \\ &\Leftrightarrow\; F(x,y)^N \leqslant F\big(y^{(M^{-1})}, x^M\big) . \end{aligned} \tag{4.9}$$

Take arbitrary $(x,y) \in [0,1]^2$ such that $x < 1$, $0 < y$ and $F(x,y) \leqslant \beta$. It then follows from Eq. (4.8) that $F(y^{(M^{-1})}, x^M) \leqslant F(x,y)^N$. Furthermore, $F(x,y) \leqslant \beta$ implies that $y \leqslant C_\beta(x) = (\widetilde{D}_\beta(x^M))^M$ and, hence, $\beta \leqslant F(y^{(M^{-1})}, x^M)$ (Eq. (4.4)). Denoting $u := y^{(M^{-1})}$ and $v = x^M$ we know that $\beta \leqslant F(u,v)$, $u < 1$ and $0 < v$. Performing Eq. (4.9) results in

$$F\big(y^{(M^{-1})}, x^M\big)^N = F(u,v)^N \leqslant F\big(v^{(M^{-1})}, u^M\big) = F(x,y) .$$

We conclude that $F(x,y) = F(y^{(M^{-1})}, x^M)^N$ is satisfied for those couples $(x,y) \in [0,1]^2$ such that $x < 1$, $0 < y$ and $F(x,y) \leqslant \beta$. In a similar way, performing Eq. (4.8) on the outcome of Eq. (4.9), we get that $F(x,y) = F(y^{(M^{-1})}, x^M)^N$ is satisfied for those couples $(x,y) \in [0,1]^2$ such that $x < 1$, $0 < y$ and $\beta < F(x,y)$. Hence, Eq. (4.7) is fulfilled for every $(x,y) \in [0,1]^2$ with $x < 1$ and $0 < y$. Invoking the continuity of F, M and N, we obtain that Eq. (4.7) also holds whenever $x = 1$ or $y = 0$. $\square$

To invert Theorem 4.13 we need to impose some additional boundary conditions on F. Suppose for example that $F(x,y) = 1$, for every $(x,y) \in [0,1]^2$. Then F is trivially continuous, $C_a = \widetilde{C}_a = \mathbf{0}$ whenever $a \in [0,1[$, $C_1 = \widetilde{C}_1 = \mathbf{1}$ and $D_a = \widetilde{D}_a = \mathbf{0}$ for every $a \in [0,1]$. The assertions of Theorem 4.13 hold but $F(0,0) = 1 > 0 = F(1,1)^N$. Note that these assertions do not force F to satisfy $F(x, x^M) = \beta$, which is necessary for Eq. (4.7) to hold. Simply, requiring that $F(0,1) = F(1,0) = \beta$ counters this deficiency.

Theorem 4.15 *Consider a strict negator M, an involutive negator N with fixpoint β and an increasing $[0,1]^2 \to [0,1]$ function F. If F is continuous, then the following assertions are equivalent:*

1. *F satisfies Eq. (4.7).*
2. *$C_{a^N} \in Q(D_a, M)$, for every $a \in [0,\beta]$, and $F(0,1) = F(1,0) = \beta$.*
3. *$D_{a^N} \in Q(C_a, M)$, for every $a \in [0,\beta]$, and $F(0,1) = F(1,0) = \beta$.*
4. *$\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$, for every $a \in [0,\beta]$, and $F(0,1) = F(1,0) = \beta$.*
5. *$\widetilde{D}_{a^N} \in Q(\widetilde{C}_a, M^{-1})$, for every $a \in [0,\beta]$, and $F(0,1) = F(1,0) = \beta$.*

Proof We will illustrate the equivalence between the first two assertions. For assertion 1 to hold it is always necessary that assertion 2 is satisfied (Theorem 4.13) and $F(x, x^M) = \beta$, for every $x \in [0,1]$. Assume now that $C_{a^N} \in Q(D_a, M)$, for every $a \in [0,\beta]$, and $F(0,1) = F(1,0) = \beta$. Then, equivalently, $M^{-1} \circ C_{a^N} \circ M^{-1} \in Q(D_a, \mathbf{id})$, for every $a \in [0,\beta]$ (Theorem 1.11). The left continuity of F ensures that every C_{a^N} is left continuous (Property 4.5) and thus every $M^{-1} \circ C_{a^N} \circ M^{-1}$ must be right continuous. As $F(0,1) = F(1,0) = \beta$ it holds that $(C_{a^N}(1^{(M^{-1})}))^{(M^{-1})} = (C_{a^N}(0))^{(M^{-1})} = 1^{(M^{-1})} = 0$ and $D_a(1) = 0$, for every $a \in [0,\beta]$. Invoking Theorems 1.16 and 1.17 these considerations lead to $M^{-1} \circ C_{a^N} \circ M^{-1} = \underline{D_a}^{\mathbf{id}}$. Thanks to Eq. (4.2) we obtain the following chain of inequalities:

$$
\begin{aligned}
(C_{a^N}(x^{(M^{-1})}))^{(M^{-1})} &= \sup\{t \in [0,1] \mid D_a(t) > x\} = \inf\{t \in [0,1] \mid D_a(t) \leqslant x\} \\
&= \inf\{t \in [0,1] \mid F(t,x) \geqslant a\} = \widetilde{D}_a(x)\,,
\end{aligned}
$$

for every $(x,a) \in [0,1[\times [0,\beta]$. Moreover, $F(0,1) = F(1,0) = \beta$ implies that $(C_{a^N}(1^{M^{-1}}))^{M^{-1}} = 0 = \widetilde{D}_a(1)$ is fulfilled for every $a \in [0,\beta]$. We conclude that $M^{-1} \circ C_{a^N} \circ M^{-1} = \widetilde{D}_a$ whenever $a \in [0,\beta]$ and thus F satisfies Eq. (4.7) (Theorem 4.14). $\square$

Part II

Rotation-invariant t-norms

CHAPTER 5

A contour view on uninorms

5.1 Introduction

In many mathematical investigations and practical applications, the increasing $[0,1]^2 \to [0,1]$ functions involved must satisfy several additional properties. Associativity, for example, allows to straightforwardly extend a $[0,1]^2 \to [0,1]$ function to a more general $[0,1]^n \to [0,1]$ function, with $n > 2$. The use of commutative $[0,1]^2 \to [0,1]$ functions puts symmetry into the considered process or theory. In multi-criteria decision making for example, this amounts to expressing that all criteria are equally important. Furthermore, introducing some level of satisfaction $e \in [0,1]$ allows to rule out a certain criterion from the global evaluation. Many fields in mathematics also require the existence of such an indentity element. In this chapter we provide new insights into all these properties by examining the contour lines C_a, D_a, $\widetilde{C}_a$ and $\widetilde{D}_a$ instead of the original increasing $[0,1]^2 \to [0,1]$ function F. Conversely, we investigate how properties on contour lines affect the structure of F. Special attention goes to the study of continuous contour lines. The results from this chapter pave the way for better understanding the geometrical structure of left-continuous t-norms (see Chapters 6 and 7).

5.2 Uninorms

Uninorms were introduced by Yager and Rybalov [98] as a generalization of t-norms and t-conorms [51].

Definition 5.1 [98] A *uninorm* U is an increasing $[0,1]^2 \to [0,1]$ function that satisfies the following properties:

(UN1) *Neutral element* $e \in [0,1]$: $U(x,e) = U(e,x) = x$, for every $x \in [0,1]$.
(UN2) *Commutativity*: $U(x,y) = U(x,y)$, for every $(x,y) \in [0,1]^2$.

(UN3) *Associativity*: $U(U(x,y),z) = U(x,U(y,z))$, for every $(x,y,z) \in [0,1]^3$.

Clearly, uninorms are special binary aggregation operators as $F(0,0) \leqslant F(0,e) = 0$ and $1 = F(1,e) \leqslant F(1,1)$. They are important from a practical as well as a theoretical point of view. In multi-criteria decision making, for example, they are used to aggregate the evaluation of alternatives, taking into account some level of satisfaction e [98]. Uninorms with $e \in \,]0,1[$ convert the structures $([0,1], \sup, U)$ and $([0,1], \inf, U)$ into distributive semirings in the sense of Golan [35].

Definition 5.2 [51] A *triangular norm* or shortly *t-norm* T is a uninorm with neutral element $e = 1$. A *triangular conorm* or shortly *t-conorm* S is a uninorm with neutral element $e = 0$.

Schweizer and Sklar [85] originally introduced triangular norms in order to generalize the triangle inequality towards probabilistic metric spaces. Nowadays, they are widely used in fuzzy set theory.

Example 5.3 The three prototypical continuous t-norms are the *minimum operator* $T_{\mathbf{M}}(x,y) = \min(x,y)$, the *algebraic product* $T_{\mathbf{P}}(x,y) = xy$ and the *Lukasiewicz t-norm* $T_{\mathbf{L}}(x,y) = \max(x + y - 1, 0)$. The *nilpotent minimum*

$$T^{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x + y \leqslant 1\,, \\ \min(x,y), & \text{elsewhere}\,, \end{cases}$$

has been introduced by Fodor [26] and is a well-known left-continuous t-norm. The *drastic product*

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2\,, \\ \min(x,y), & \text{elsewhere}\,, \end{cases}$$

is a right-continuous t-norm.

The prototypical continuous t-conorms are the *maximum operator* $S_{\mathbf{M}}(x,y) = \max(x,y)$, the *probabilistic sum* $S_{\mathbf{P}}(x,y) = x + y - x\,y$ and the *Lukasiewicz t-conorm* $S_{\mathbf{L}}(x,y) = \min(x+y, 1)$. Familiar non-continuous t-conorms are

$$S^{\mathbf{nM}}(x,y) = \begin{cases} \max(x,y), & \text{if } x + y < 1\,, \\ 1, & \text{elsewhere}\,, \end{cases}$$

and the *drastic sum*

$$S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \,]0,1]^2\,, \\ \max(x,y), & \text{elsewhere}\,. \end{cases} \qquad \triangle$$

For any given $[0,e] \to [0,1]$ isomorphism σ and $[e,1] \to [0,1]$ isomorphism $\hat{\sigma}$, we can extract from a uninorm U a t-norm T and a t-conorm S such that

$$\big(\forall (x,y) \in [0,e]^2\big)\big(U(x,y) = \sigma^{-1}(T(\sigma(x), \sigma(y)))\big)\,, \tag{5.1}$$

$$(\forall (x,y) \in [e,1]^2)(U(x,y) = \hat{\sigma}^{-1}(S(\hat{\sigma}(x), \hat{\sigma}(y))))\,. \tag{5.2}$$

On the other parts of the unit square it always holds that $T_{\mathbf{M}} \leqslant U \leqslant S_{\mathbf{M}}$ [28]. Furthermore, it always holds that either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$ [28].

Definition 5.4 [28] A uninorm U is called *conjunctive* if $U(0,1) = U(1,0) = 0$. In case $U(0,1) = U(1,0) = 1$ we talk about a *disjunctive uninorm*.

Invoking the increasingness of a uninorm, every conjunctive uninorm necessarily has absorbing element 0 and every disjunctive uninorm has absorbing element 1. Given an automorphism ϕ and a strict negator N, the ϕ-transform of a conjunctive uninorm is always a conjunctive uninorm and its N-transform is a disjunctive uninorm. Similarly, transforming a disjunctive uninorm by means of ϕ or N yields, resp., a disjunctive or conjunctive uninorm. In particular, given t-norm T and a t-conorm S, T_ϕ and S_N are t-norms and T_N and S_ϕ are t-conorms [51].

Important classes of uninorms comprise $U_{\mathbf{min}}$ and $U_{\mathbf{max}}$ [15], the representable uninorms [17, 28] and the idempotent uninorms [12].

Example 5.5 A typical example of a uninorm is the conjunctive, binary $3\,\Pi$–operator E. It is defined by

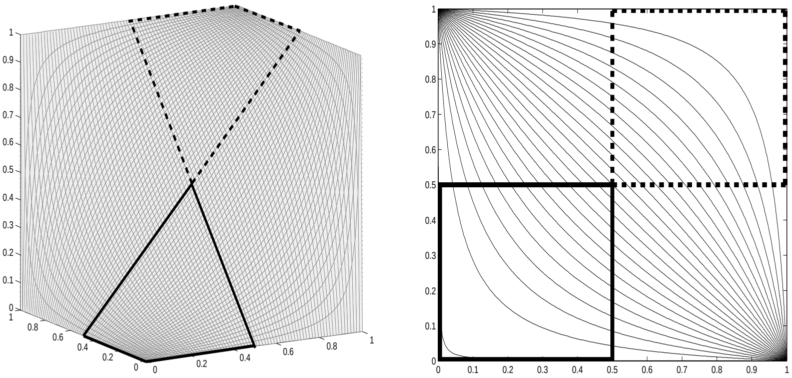
$$E(x,y) = \frac{x\,y}{(1-x)\,(1-y) + x\,y}\,,$$

for every $(x,y) \notin \{(1,0),(0,1)\}$, and $E(0,1) = E(1,0) = 0$. This $3\,\Pi$–operator is left continuous and has neutral element $e = \frac{1}{2}$. Its associativity allows to extend it in a unique way to the $[0,1]^n \to [0,1]$ function from Remarks 3.18. The solid lines in Fig. 5.1 point out $E|_{[0,\frac{1}{2}]^2}$ which is a rescaled and transformed version of the algebraic product. The dashed lines indicate $E|_{[\frac{1}{2},1]^2}$ which is a rescaled and transformed version of the probabilistic sum. Explicitly,

$$\big(\forall (x,y) \in [0,\tfrac{1}{2}]^2\big)\big(E(x,y) = \sigma^{-1}(T_{\mathbf{P}}(\sigma(x),\sigma(y)))\big)\,,$$
$$\big(\forall (x,y) \in [\tfrac{1}{2},1]^2\big)\big(E(x,y) = \hat{\sigma}^{-1}(S_{\mathbf{P}}(\hat{\sigma}(x),\hat{\sigma}(y)))\big)\,,$$

with σ the $[0,\frac{1}{2}] \to [0,1]$ isomorphism defined by $\sigma(x) = x/(1-x)$ and $\hat{\sigma}$ the $[\frac{1}{2},1] \to [0,1]$ isomorphism defined by $\hat{\sigma}(x) = (2\,x-1)/x$. $\triangle$

Dealing with a conjunctive uninorm U, the $[0,1]^2 \to [0,1]$ function C, associated with contour lines of the type C_a (see Section 4.2), can be understood as a generalization of the Boolean implication. In this case C is usually referred to as the *residual implicator* of U and is denoted as I_U [14, 27, 83]. If U is disjunctive, then $J_U := D$ is known as its *residual coimplicator* [11, 14, 27, 83]. Note that $\widehat{C} = C$ and $\widehat{D} = D$, due to the commutativity of U (Theorem 4.7). De Baets and Mesiar [14, 16] proved that a conjunctive uninorm U is left continuous if and only if it satisfies the *generalized modus ponens*: $U(x, I_U(x,a)) \leqslant a$, for every $(x,a) \in [0,1]^2$. Taking into account that $U(x, I_U(x,a)) = U(x, C_a(x))$ the latter turns out to be a very specific case of Theorem 4.6. Dually, if U is disjunctive, the inequality $a \leqslant U(x, D_a(x)) = U(x, J_U(x,a))$, for every $(x,a) \in [0,1]^2$, is equivalent with the right continuity of U (see [14, 16] and, more generaly, Theorem 4.6).



(a) E (b) Contour plot (type C_a) of E

Figure 5.1: The conjunctive, binary $3\,\Pi$–operator E.

5.3 Uninorm properties

Considering increasing $[0,1]^2 \to [0,1]$ functions, we figure out how the existence of a neutral element, commutativity and associativity can be expressed in terms of properties on contour lines. As can be seen from Theorems 4.4 and 4.6, the contour lines, resp. C_a, D_a, $\widetilde{C}_a$, and $\widetilde{D}_a$ are particularly suited to describe increasing $[0,1]^2 \to [0,1]$ functions F that have, respectively, left-continuous partial functions $F(x,\bullet)$, right-continuous partial functions $F(x,\bullet)$, left-continuous partial functions $F(\bullet,x)$ and right-continuous partial functions $F(\bullet,x)$.

A. Neutral element

In the following theorem we investigate, for a fixed $x \in [0,1]$, the conditions $F(x,e) = x$ and $F(e,x) = x$.

Theorem 5.6 *Consider an increasing* $[0,1]^2 \to [0,1]$ *function* F. *For every* $x \in [0,1]$ *the following assertions hold:*

1. *If* $F(x,\bullet)$ *is left continuous, then* $F(x,e) = x$ *is satisfied for some* $e \in \,]0,1]$ *if and only if the equivalence* $e \leqslant C_a(x) \;\Leftrightarrow\; x \leqslant a$ *holds for every* $a \in [0,1]$.
2. *If* $F(x,\bullet)$ *is right continuous, then* $F(x,e) = x$ *is satisfied for some* $e \in [0,1[$ *if and only if the equivalence* $D_a(x) \leqslant e \;\Leftrightarrow\; a \leqslant x$ *holds for every* $a \in [0,1]$.
3. *If* $F(\bullet,x)$ *is left continuous, then* $F(e,x) = x$ *is satisfied for some* $e \in \,]0,1]$ *if and only if the equivalence* $e \leqslant \widetilde{C}_a(x) \;\Leftrightarrow\; x \leqslant a$ *holds for every* $a \in [0,1]$.
4. *If* $F(\bullet,x)$ *is right continuous, then* $F(e,x) = x$ *is satisfied for some* $e \in [0,1[$ *if and only if the equivalence* $\widetilde{D}_a(x) \leqslant e \;\Leftrightarrow\; a \leqslant x$ *holds for every* $a \in [0,1]$.

Proof We prove the first assertion. The necessary condition for $F(x,e) = x$ to hold immediately follows from Eq. (4.1) (take $y = e$). Conversely, if $e \leqslant C_a(x) \;\Leftrightarrow\; x \leqslant a$ holds for every $a \in [0,1]$, then we obtain that $e \leqslant C_x(x)$. Applying Eq. (4.1) leads to $F(x,e) \leqslant x$. In case $F(x,e) < x$, there exists $\varepsilon \in \,]0,x[$ such that $F(x,e) \leqslant x - \varepsilon$. Hence, $e \leqslant C_{x-\varepsilon}(x)$, which is equivalent with the contradiction $x \leqslant x - \varepsilon$. $\square$

In the above theorem there are some restrictions on e. The first assertion, for example, deals with $e \in]0,1]$ only. For $e = 0$ the equivalence between $F(x,0) = x$ and $0 \leqslant C_a(x) \Leftrightarrow x \leqslant a$, for every $a \in [0,1]$, reduces to $F(x,0) = x \Leftrightarrow x = 0$. The latter is incorrect. For example, it does not hold for $F = S_{\mathbf{M}}$. A left-continuous (resp. right-continuous) increasing function F will have a neutral element $e \in]0,1]$ (resp. $e \in [0,1[$) if and only if the equivalences in assertions 1 and 3 (resp. assertions 2 and 4) are fulfilled for every $x \in [0,1]$. In this way conditions on different types of contour lines get paired. The next theorem will enable us to express the existence of a neutral element in terms of a single type of contour line.

Theorem 5.7 *Consider an increasing $[0,1]^2 \to [0,1]$ function F. For every $e \in [0,1]$ the following assertions hold:*

1. *If $F(e,\bullet)$ is left continuous, then $F(e,x) = x$ is satisfied for every $x \in [0,1]$ if and only if $C_a(e) = a$ holds for every $a \in [0,1]$.*
2. *If $F(e,\bullet)$ is right continuous, then $F(e,x) = x$ is satisfied for every $x \in [0,1]$ if and only if $D_a(e) = a$ holds for every $a \in [0,1]$.*
3. *If $F(\bullet,e)$ is left continuous, then $F(x,e) = x$ is satisfied for every $x \in [0,1]$ if and only if $\widetilde{C}_a(e) = a$ holds for every $a \in [0,1]$.*
4. *If $F(\bullet,e)$ is right continuous, then $F(x,e) = x$ is satisfied for every $x \in [0,1]$ if and only if $\widetilde{D}_a(e) = a$ holds for every $a \in [0,1]$.*

Proof We prove the first assertion. In case $F(e,x) = x$ is satisfied for every $x \in [0,1]$ then by definition $x \leqslant C_x(e)$ and $C_x(e) < y$ whenever $x < y$. Hence, $a = C_a(e)$, for every $a \in [0,1]$. Conversely, suppose that the latter is satisfied, then, Eq. (4.1) states that $F(e,x) \leqslant x$ and $y < F(e,x)$, for every $x \in]0,1]$ and $y \in [0,x[$. We conclude that $F(e,x) = x$, for every $x \in]0,1]$, and due to the increasingness of F also for $x = 0$. $\square$

Combining Theorems 5.6 and Theorems 5.7 yields the following characterization of neutral elements. From the definition of a neutral element it trivially follows that an increasing function has at most one neutral element.

Corollary 5.8 *Consider an increasing $[0,1]^2 \to [0,1]$ function F. For every $e \in [0,1]$ the following statements hold:*

1. *If F is left continuous and $e \in]0,1]$ then the following assertions are equivalent:*

   a) *F has neutral element e.*
   b) *$e \leqslant C_a(x) \Leftrightarrow x \leqslant a$ and $C_a(e) = a$ hold for every $(x,a) \in [0,1]^2$.*
   c) *$e \leqslant \widetilde{C}_a(x) \Leftrightarrow x \leqslant a$ and $\widetilde{C}_a(e) = a$ hold for every $(x,a) \in [0,1]^2$.*

2. *If F is right continuous and $e \in [0,1[$ then the following assertions are equivalent:*

   a) *F has neutral element e.*
   b) *$D_a(x) \leqslant e \Leftrightarrow a \leqslant x$ and $D_a(e) = a$ hold for every $(x,a) \in [0,1]^2$.*
   c) *$\widetilde{D}_a(x) \leqslant e \Leftrightarrow a \leqslant x$ and $\widetilde{D}_a(e) = a$ hold for every $(x,a) \in [0,1]^2$.*

B. Commutativity

As pointed out before, an increasing $[0,1]^2 \to [0,1]$ function F is by definition commutative if it satisfies the functional equation Eq. (4.6) for $\phi = \mathbf{id}$. In Section 4.3 we have interrelated the commutativity of F with the **id**-orthosymmetry of its contour lines. The following theorem presents some alternative mathematical formulations of **id**-orthosymmetry that will be important for our further work. Recall that for a commutative function F there only exist two types of contour lines as $\widetilde{C}_a = C_a$ and $\widetilde{D}_a = D_a$, for every $a \in [0,1]$ (Theorem 4.7). For this reason we only consider here contour lines of the types C_a and D_a.

Theorem 5.9 *Consider an increasing $[0,1]^2 \to [0,1]$ function F.*

1. *If F is left continuous and $F(0,1) = F(1,0) = 0$, then the following assertions are equivalent:*

   a) *F is commutative.*
   b) *All contour lines C_a are* **id***-orthosymmetrical.*
   c) *$C_a(x) < y \;\Leftrightarrow\; C_a(y) < x$, for every $(x,y,a) \in [0,1]^3$.*
   d) *$\mathbf{id} \leqslant C_a \circ C_a$, for every $a \in [0,1]$.*

2. *If F is right continuous and $F(0,1) = F(1,0) = 1$, then the following assertions are equivalent:*

   a) *F is commutative.*
   b) *All contour lines D_a are* **id***-orthosymmetrical.*
   c) *$y < D_a(x) \;\Leftrightarrow\; x < D_a(y)$, for every $(x,y,a) \in [0,1]^3$.*
   d) *$D_a \circ D_a \leqslant \mathbf{id}$, for every $a \in [0,1]$.*

Proof We present the proof for the first set of equivalent assertions. The equivalence between assertions 1a and 1b follows immediately from Theorem 4.10. As $F(0,1) = 0$, then $C_a(0) = 1$, for every $a \in [0,1]$, and C_a is **id**-orthosymmetrical if and only if $C_a(x) = \underline{C_a}_{\mathbf{id}}(x) = \inf\{t \in [0,1] \mid C_a(t) < x\}$ holds for every $x \in [0,1]$ (combine Theorems 1.16 and 1.17 with Properties 4.2 and 4.5). The latter is also equivalent with $C_a(x) < y \;\Leftrightarrow\; C_a(y) < x$, for every $(x,y) \in [0,1]^2$. From this equivalence, putting $y = C_a(x)$, it immediately follows that $\mathbf{id} \leqslant C_a \circ C_a$. □

Generalizing the involutive negators studied in [87], Esteva and Domingo [21] use the term *weak negation* to refer to a decreasing $[0,1] \to [0,1]$ function f satisfying $f(1) = 0$ and $\mathbf{id} \leqslant f \circ f$. They showed that weak negations are always left continuous and characterize them as, what we call, **id**-orthosymmetrical, left-continuous $[0,1] \to [0,1]$ functions that map 1 to 0. De Baets [12] calls a $[0,1] \to [0,1]$ function f that satisfies $f \circ f \leqslant \mathbf{id}$ *sub-involutive*. f is *super-involutive* if $\mathbf{id} \leqslant f \circ f$. Decreasing sub-involutive functions were used to describe conjunctive, left-continuous, idempotent uninorms, whereas decreasing super-involutive functions allow to characterize all disjunctive, right-continuous, idempotent uninorms [12].

C. Associativity

Assuming some continuity and boundary conditions, we can also use contour lines to express the associativity of F.

Theorem 5.10 *For every increasing $[0,1]^2 \to [0,1]$ function F the following assertions hold:*

1. *If $F(x,\bullet)$ is left continuous for every $x \in [0,1]$ and $F(1,0)=0$, then F is associative if and only if*
$$C_a(F(x,y)) = C_{C_a(x)}(y) \tag{5.3}$$
holds for every $(x,y,a) \in [0,1]^3$.
2. *If $F(x,\bullet)$ is right continuous for every $x \in [0,1]$ and $F(0,1)=1$, then F is associative if and only if*
$$D_a(F(x,y)) = D_{D_a(x)}(y) \tag{5.4}$$
holds for every $(x,y,a) \in [0,1]^3$.
3. *If $F(\bullet,x)$ is left continuous for every $x \in [0,1]$ and $F(0,1)=0$, then F is associative if and only if*
$$\widetilde{C}_a(F(x,y)) = \widetilde{C}_{\widetilde{C}_a(y)}(x) \tag{5.5}$$
holds for every $(x,y,a) \in [0,1]^3$.
4. *If $F(\bullet,x)$ is right continuous for every $x \in [0,1]$ and $F(1,0)=1$, then F is associative if and only if*
$$\widetilde{D}_a(F(x,y)) = \widetilde{D}_{\widetilde{D}_a(y)}(x) \tag{5.6}$$
holds for every $(x,y,a) \in [0,1]^3$.

Proof We prove the first assertion. Recall that the boundary condition $F(1,0)=0$ is equivalent with $F(x,0)=0$, for every $x \in [0,1]$. This proof makes extensive use of the first group of equivalent assertions in Theorem 4.6. If F is associative, then we know that

$$C_a(F(x,y)) = \sup\{t \in [0,1] \mid F(F(x,y),t) \leqslant a\} = \sup\{t \in [0,1] \mid F(x,F(y,t))) \leqslant a\}\,,$$

for every $(x,y,a) \in [0,1]^3$. Because $F(x,F(y,t)) \leqslant a$ is equivalent with $F(y,t) \leqslant C_a(x)$, we can rewrite this equality as follows:

$$C_a(F(x,y)) = \sup\{t \in [0,1] \mid F(y,t) \leqslant C_a(x)\} = C_{C_a(x)}(y)\,.$$

Conversely, if Eq. (5.3) holds, we need to prove that $F(F(x,y),z) = F(x,F(y,z))$, for every $(x,y,z) \in [0,1]^3$. Since

$$C_{C_{F(F(x,y),z)}(x)}(y) = C_{F(F(x,y),z)}(F(x,y)) \geqslant z\,,$$

we obtain that $F(y,z) \leqslant C_{F(F(x,y),z)}(x)$ and, hence, $F(x,F(y,z)) \leqslant F(F(x,y),z)$. In case $F(x,F(y,z)) < F(F(x,y),z)$, then it follows that $C_{F(x,F(y,z))}(F(x,y)) < z$. Applying Eq. (5.3) yields $C_{C_{F(x,F(y,z))}(x)}(y) < z$ and thus $C_{F(x,F(y,z))}(x) < F(y,z)$. Finally, we obtain the contradiction $F(x,F(y,z)) < F(x,F(y,z))$. $\square$

The continuity and boundary conditions are indispensable in the proof of the above theorem. For example, consider the increasing function F defined by $F(x,1) = \frac{1}{2}$, for every $x \in [0,1]$, and $F(x,y) = 0$, elsewhere. The partial functions $F(x,\bullet)$ are not left continuous, and for every $a \in [0,1]$ it holds that $C_a = \mathbf{1}$. Eq. (5.3) is then trivially fulfilled although F is not associative (e.g. $F(F(1,1),1) = F(\frac{1}{2},1) = \frac{1}{2} > 0 = F(1,\frac{1}{2}) = F(1,F(1,1))$). To illustrate the importance of the boundary conditions, consider the increasing function F defined by $F(1,y) = 1$, for every $y \in [0,1]$, and $F(x,y) = 0$, elsewhere. All partial functions $F(x,\bullet)$ are continuous but $F(1,0) = 1$. It is easily verified that F is associative. However, $C_{\frac{1}{2}}(F(1,0)) = C_{\frac{1}{2}}(1) = 0 < 1 = C_0(0) = C_{C_{\frac{1}{2}}(1)}(0)$, which contradicts Eq. (5.3).

Dealing with a commutative F, Eq. (5.3) also implies that $C_{C_a(x)}(y) = C_{C_a(y)}(x)$ is satisfied for every $(x,y,a) \in [0,1]^3$. This property can also be used to express associativity. The commutativity of F allows us to consider contour lines of the types C_a and D_a only.

Theorem 5.11 *For every commutative, increasing $[0,1]^2 \to [0,1]$ function F the following assertions hold:*

1. *If F is left continuous and $F(1,0) = 0$, then F is associative if and only if*

$$C_{C_a(x)}(y) = C_{C_a(y)}(x) \tag{5.7}$$

holds for every $(x,y,a) \in [0,1]^3$.

2. *If F is right continuous and $F(0,1) = 1$, then F is associative if and only if*

$$D_{D_a(x)}(y) = D_{D_a(y)}(x) \tag{5.8}$$

holds for every $(x,y,a) \in [0,1]^3$.

Proof Assume that F is left continuous and $F(1,0) = 0$. If F is associative, then Eq. (5.7) follows immediately from Eq. (5.3). Conversely, suppose that Eq. (5.7) is satisfied. If F is commutative but not associative, there exists a triplet $(x,y,z) \in [0,1]^3$ such that

$$F(y,F(x,z)) = F(F(x,z),y) < F(x,F(z,y)) = F(x,F(y,z)).$$

Consider $a \in \,]F(y,F(x,z)), F(x,F(y,z))[$. From Theorem 4.6 it then follows that $F(x,z) \leqslant C_a(y)$ and $C_a(x) < F(y,z)$. Applying Theorem 4.6 a second time leads to $z \leqslant C_{C_a(y)}(x)$ and $C_{C_a(x)}(y) < z$. We obtain the contradiction $C_{C_a(x)}(y) < C_{C_a(y)}(x)$. $\square$

Note that the commutativity of F plays a key role in the above theorem. For example, define a non-commutative F by $F(x,0) = 0$, for every $x \in [0,1]$, and $F(x,y) = x$, elsewhere. Although F is associative, left continuous and satisfies $F(1,0) = 0$, it holds that $C_{C_{\frac{1}{2}}(1)}(\frac{1}{2}) = C_0(\frac{1}{2}) = 0 < 1 = C_1(1) = C_{C_{\frac{1}{2}}(\frac{1}{2})}(1)$.

To conclude, we involve all characterization results from this section to obtain the following properties for uninorms. These properties are essential for the description of rotation-invariant t-norms (see Chapter 6).

Theorem 5.12 *Consider a uninorm U.*

1. *If U is left continuous and conjunctive, then*

   a) $C_{C_a(U(x,y))}(z) = C_{C_a(U(x,z))}(y)$, *for every* $(x, y, z, a) \in [0, 1]^4$*;*
   b) $U(x, y) \leqslant C_a(z) \;\Leftrightarrow\; U(x, z) \leqslant C_a(y)$, *for every* $(x, y, z, a) \in [0, 1]^4$*.*

2. *If U is right continuous and disjunctive, then*

   a) $D_{D_a(U(x,y))}(z) = D_{D_a(U(x,z))}(y)$, *for every* $(x, y, z, a) \in [0, 1]^4$*;*
   b) $D_a(z) \leqslant U(x, y) \;\Leftrightarrow\; D_a(y) \leqslant U(x, z)$, *for every* $(x, y, z, a) \in [0, 1]^4$*.*

Proof We prove the first group of properties. Consider arbitrary $(x, y, z, a) \in [0, 1]^4$. As U is a uninorm it clearly holds that $C_a(U(U(x, y), z)) = C_a(U(U(x, z), y))$. Applying Eq. (5.3) immediately leads to assertion 1a. Furthermore, let e be the neutral element of U then the conjunctivity of U (*i.e.* $U(1, 0) = 0$) prevents that $e = 0$. From Theorem 5.6 it then follows that $e \leqslant C_{C_a(U(x,y))}(z)$ is equivalent with $z \leqslant C_a(U(x, y))$. Similarly, $e \leqslant C_{C_a(U(x,z))}(y)$ is equivalent with $y \leqslant C_a(U(x, z))$. Taking into account Theorem 5.9 it follows from assertion 1a that $U(x, y) \leqslant C_a(z)$ is equivalent with $U(x, z) \leqslant C_a(y)$. □

5.4 Uninorms that have a continuous contour line

Depending on the continuity of the partial functions $U(x, \bullet)$ and $U(\bullet, x)$ of a uninorm U, its contour lines fulfill several of the properties stated in the previous section. Uninorms can have discontinuous as well as continuous contour lines. For example, as can be seen in Fig. 5.1(b) all contour lines C_a, with $a \in]0, 1]$, of the conjunctive, binary $3\,\Pi$–operator E are continuous:

$$C_a(x) = \frac{a\,(1 - x)}{x\,(1 - a) + a\,(1 - x)}\,, \tag{5.9}$$

for every $x \in [0, 1]$ such that $\min(x, a) < 1$, and $C_1(1) = 1$. The contour line C_0 however, is discontinuous: $C_0(0) = 1$ and $C_0(x) = 0$ whenever $x \in]0, 1]$. So far it has not been revealed how the continuity of its contour lines affects the structure of the uninorm. The continuous contour lines of a left- or right-continuous uninorm are now characterized in the following way:

Theorem 5.13 *Consider a uninorm U with neutral element $e \in [0, 1]$. The following statements hold:*

1. *If U is left continuous and conjunctive, then, for every $a \in [0, 1]$, the following assertions are equivalent:*

   a) C_a *is continuous.*
   b) C_a *is involutive on* $[C_a(1), 1]$.
   c) $U(x, y) = C_a(C_{C_a(x)}(y))$, *for every* $(x, y) \in [0, 1]^2$ *such that* $C_a(U(x, 1)) < y$.
   d) $C_b(x) = C_{C_a(x)}(C_a(b))$, *for every* $(x, b) \in [0, 1] \times [C_a(1), 1]$.
   e) $U(x, y) \leqslant z \;\Leftrightarrow\; U(x, C_a(z)) \leqslant C_a(y)$, *for every* $(x, y, z) \in [C_a(1), 1]^3$.

2. *If* U *is right continuous and disjunctive, then, for every* $a \in [0, 1]$, *the following assertions are equivalent:*

   a) D_a *is continuous.*
   b) D_a *is involutive on* $[0, D_a(0)]$.
   c) $U(x, y) = D_a(D_{D_a(x)}(y))$, *for every* $(x, y) \in [0, 1]^2$ *such that* $y < D_a(U(0, x))$.
   d) $D_b(x) = D_{D_a(x)}(D_a(b))$, *for every* $(x, b) \in [0, 1] \times [0, D_a(0)]$.
   e) $z \leqslant U(x, y) \;\Leftrightarrow\; D_a(y) \leqslant U(x, D_a(z))$, *for every* $(x, y, z) \in [0, D_a(0)]^3$.

Proof We prove the equivalence between the first group of assertions. Consider a conjunctive left-continuous uninorm U. Then its neutral element e must belong to $]0, 1]$ and $C_1 = \mathbf{1}$ is the only constant contour line of U. Assertions 1a–1e are trivially fulfilled if $a = 1$. Assume now that $a < 1$. Throughout the proof we will make extensive use of the decreasingness of C_a (property **(D1)**), Corollary 5.8 and Eq. (5.3).

1a⇔1b The commutativity of U implies the **id**-orthosymmetry of C_a (Theorem 5.9) and the boundary condition $U(0, 1) = 0$ is equivalent with $C_a(0) = 1$, for every $a \in [0, 1]$. Taking into account Theorem 2.9, we know that a contour line C_a is continuous if and only if it is involutive on $[C_a(1), C_a(0)] = [C_a(1), 1]$. Since $C_a(C_a(1)) = 1$ also ensures that $C_a(C_a(C_a(1))) = C_a(1)$, it suffices that C_a is involutive on $]C_a(1), 1]$.

1b⇔1c If $y > C_a(U(x, 1)) = C_a(U(1, x)) = C_{C_a(1)}(x)$, then it holds by definition that $U(x, y) > C_a(1)$. Under the assumption that C_a is involutive on $]C_a(1), 1]$ it follows that $U(x, y) = C_a(C_a(U(x, y))) = C_a(C_{C_a(x)}(y))$. Conversely, suppose that assertion 1c holds. Let $x = e$, then $y = U(e, y) = C_a(C_{C_a(e)}(y)) = C_a(C_a(y))$, for every $y > C_a(1)$. We conclude that C_a is involutive on $]C_a(1), 1]$ and hence, assertion 1b is satisfied.

1b⇔1d Consider $(x, b) \in [0, 1] \times [C_a(1), 1]$. If C_a is involutive on $[C_a(1), 1]$, then we obtain that

$$C_b(x) = C_{C_a(C_a(b))}(x) = C_a(U(C_a(b), x)) = C_a(U(x, C_a(b))) = C_{C_a(x)}(C_a(b))\,.$$

Conversely, if assertion 1d holds, then $b = C_b(e) = C_{C_a(e)}(C_a(b)) = C_a(C_a(b))$, for every $b \in [C_a(1), 1]$.

1b⇔1e Take arbitrary $(x, y, z) \in [C_a(1), 1]^3$ and assume that C_a is involutive on $[C_a(1), 1]$. From Theorem 4.6 it follows that $U(x, C_a(z)) \leqslant C_a(y)$ is equivalent with $C_a(z) \leqslant C_{C_a(y)}(x)$. The latter can be rewritten as $C_a(z) \leqslant C_a(U(y, x)) = C_a(U(x, y))$. Whenever $C_a(1) \leqslant U(x, y)$, this inequality is equivalent with $U(x, y) \leqslant z$. However, if $U(x, y) < C_a(1)$, then $C_a(U(x, y)) = 1$ (Theorem 5.9). The inequalities $C_a(z) \leqslant C_a(U(x, y))$ and $U(x, y) \leqslant z$ are in that case trivially fulfilled. Assertion 1e is indeed true. Conversely, suppose that assertion 1e holds. As $U(C_a(1), 1) = U(1, C_a(1)) \leqslant a$ (Theorem 4.6), it holds that $C_a(C_a(1)) = 1$ and therefore $C_a([C_a(1), 1]) = [C_a(1), 1]$. Furthermore, $C_a(1) \leqslant C_a(e) = a$. If $e \leqslant C_a(1)$, we obtain from Theorem 5.9 the contradiction $1 \leqslant C_a(e) = a$. Thus, for every $x \in [C_a(1), 1]$ it holds that both $(x, C_a(x), a)$ and $(C_a(C_a(x)), e, x)$ belong to $[C_a(1), 1]^3$. Applying assertion 1e on $U(x, C_a(x)) \leqslant a$ and

$$U(C_a(C_a(x)), C_a(x)) = U(C_a(x), C_a(C_a(x))) \leqslant a = C_a(e)$$

(Theorem 4.6) results in two inequalities:

$$U(x, C_a(a)) \leqslant C_a(C_a(x)) \quad \text{and} \quad C_a(C_a(x)) = U(C_a(C_a(x)), e) \leqslant x\,.$$

From Corollary 5.8 we know that $e \leqslant C_a(a)$. Weakening the first inequality to $x \leqslant C_a(C_a(x))$, we conclude that $x = C_a(C_a(x))$, for every $x \in [C_a(1), 1]$. $\square$

Adding a single additional condition on a contour line C_a or D_a, we can even give alternative conditions for its continuity.

Theorem 5.14 *Consider a uninorm U with neutral element $e \in [0, 1]$. The following statements hold:*

1. *If U is left continuous and conjunctive, then, for every $a \in [0, 1]$ fulfilling $C_a(a) = e$, C_a is continuous if and only if*

$$C_b(x) = y \ \Leftrightarrow\ U(x, C_a(b)) = C_a(y) \tag{5.10}$$

   holds for every $(x, y, b) \in [C_a(1), 1]^3$ such that $C_b(1) < x$.

2. *If U is right continuous and disjunctive, then, for every $a \in [0, 1]$ fulfilling $D_a(a) = e$, D_a is continuous if and only if*

$$D_b(x) = y \ \Leftrightarrow\ U(x, D_a(b)) = D_a(y) \tag{5.11}$$

   holds for every $(x, y, b) \in [0, D_a(0)]^3$ such that $x < D_b(0)$.

Proof We prove the first statement. Throughout the reasonings we make extensive use of Eq. (5.3). Let U be a left-continuous, conjunctive uninorm such that $C_a(a) = e$, for some $a \in [0, 1]$. If C_a is continuous, then assertion 1c of Theorem 5.13 implies that $U(x, C_a(b)) = U(C_a(b), x) = C_a(C_{C_a(C_a(b))}(x))$, for every $(x, b) \in [0, 1]^2$ such that $C_{C_a(C_a(b))}(1) = C_a(U(C_a(b), 1)) < x$. Taking into account the involutivity of C_a on $[C_a(1), 1]$ (assertion 1b of Theorem 5.13), we immediately obtain Eq. (5.10). Note that $C_a(1) \leqslant C_a(U(C_a(b), x)) = C_b(x)$ follows from the decreasingness of C_a (property **(D1)**). Conversely, suppose that Eq. (5.10) is satisfied. Then also $U(x, C_a(b)) = C_a(C_b(x))$, for every $(x, b) \in [C_a(1), 1]^2$ such that $C_b(1) < x$. Recall that $C_a(1) \leqslant C_a(e) = a$ (property **(D1)** and Corollary 5.8). Putting $b = a$ leads to $U(x, C_a(a)) = C_a(C_a(x))$, for every $x > C_a(1)$. We obtain that C_a is involutive on $]C_a(1), 1]$ by expressing that $C_a(a) = e$. The latter is equivalent with the involutivity of C_a on $[C_a(1), 1]$.

Due to Theorem 5.13, this concludes the proof. □

Note that for $a = 1$ the condition $C_1(1) = e$ can be omitted as $C_1 = \mathbf{1}$ and there does not exist an appropriate $(x, y, b) \in [C_1(1), 1]^3$ such that $1 = C_1(1) = C_b(1) < x = 1$. In a similar way $D_0(0) = e$ is superfluous when dealing with the contour line D_0. However in all other cases the additional conditions are absolutely necessary. For example, consider a left-continuous and conjunctive uninorm U and let $a < 1$. If we want that $U(x, C_a(a)) = C_a(C_a(x))$, for every $x > C_a(1)$, is equivalent with the involutivity of C_a on $]C_a(1), 1]$ then $U(x, C_a(a)) = x$ must hold for every $x \in]C_a(1), 1]$. Because $C_a(1) < e$ (Corollary 5.8) we get that $C_a(a) = U(e, C_a(a)) = e$.

The $3\,\Pi$–operator E is an example of a uninorm satisfying the above theorem. From Eq. (5.9) it is easily obtained that $C_a(a) = \frac{1}{2}$, for every $a \in]0, 1[$. Recall that $\frac{1}{2}$ is indeed the neutral element of E and that all its contour lines C_a, with $a \in]0, 1]$, are continuous.

CHAPTER 6

Left-continuous t-norms

6.1 Introduction

Given the typical block structure of a uninorm (see Eqs. (5.1) and (5.2)), revealing the geometrical structure of uninorms that have a continuous contour line usually involves the study of t-norms or t-conorms that have a continuous contour line. As illustrated in Section 5.2, every t-conorm can be understood as the N-transform of a t-norm, with N some strict negator. Therefore, it is essential to first fully understand how the existence of a continuous contour line affects the geometrical structure of a t-norm.

In most studies dealing with t-norms, it is required that the t-norms in question should be left continuous. In monoidal t-norm based logic (MTL logic), for example, where the implication is defined as the residuum of the conjunction, left-continuous t-norms ensure the definability of the t-norm-based residual implicator [23]. Despite their importance, until recently the knowledge about the structure of left-continuous t-norms was rather limited. Various construction methods have been proposed for creating left-continuous t-norms (see e.g. [10, 41, 47, 51, 74]). Most of these methods start from a known t-norm on which a bunch of operations such as rotations, annihilations, rescalings and embeddings is performed. In other cases, multiple t-norms are merged into a brand new t-norm. Unfortunately, these construction methods were only elaborated to create restricted classes of left-continuous t-norms. By studying the contour lines of a left-continuous t-norm T we will give the description of t-norms a new impetus.

We show that the rotation invariance of a left-continuous t-norm T is equivalent with the continuity and with the involutivity of its contour line C_0. In particular, this contour line coincides with the residual negator of T and, therefore, rotation-invariant t-norms are of great interest to people working on involutive monoidal t-norm based logic (IMTL logic) [22, 56] and fuzzy type theory [72]. Furthermore, we introduce all the necessary machinery for a more profound study of the structure of these rotation-invariant t-norms (see Chapters 7 and 8) and reformulate the known results [42, 43, 45, 46, 47] into our framework. This will further on enable a comparative study between the new and old approaches.

6.2 Continuous t-norms

Dealing with left-continuous t-norms, only the subclass of continuous t-norms has been fully characterized (see e.g. [51]). In particular, this class comprises the three prototypical t-norms: the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$.

Definition 6.1 Let I be a countable index set, $(]a_i, e_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint, open subintervals of $[0,1]$, $(\sigma_i)_{i\in I}$ be a family of isomorphisms ($\sigma_i : [a_i, e_i] \to [0,1]$) and $(T_i)_{i\in I}$ be a family of increasing $[0,1]^2 \to [0,1]$ functions satisfying $T_i \leqslant T_{\mathbf{M}}$. The increasing $[0,1]^2 \to [0,1]$ function T defined by

$$T(x,y) = \begin{cases} \sigma_i^{-1}\left[T_i\left(\sigma_i[x], \sigma_i[y]\right)\right], & \text{if } (x,y) \in [a_i, e_i]^2\,, \\ \min(x,y), & \text{elsewhere}\,, \end{cases}$$

is called the *ordinal sum* of the *summands* $\langle a_i, e_i, \sigma_i, T_i\rangle$, $i \in I$. It is shortly written as $T = (\langle a_i, e_i, \sigma_i, T_i\rangle)_{i\in I}$. In case every isomorphism σ_i equals the *linear rescaling function* ς_i from $[a_i, e_i]$ to $[0,1]$ (*i.e.* $\varsigma_i(x) = (x-a_i)/(e_i-a_i)$, for every $x \in [a_i, e_i]$), we use the notation $T = (\langle a_i, e_i, T_i\rangle)_{i\in I}$.

By means of Theorem 3.3, it is easily verified that

$$\langle a_i, e_i, \sigma_i, T_i\rangle = \left\langle a_i, e_i, (T_i)_{\sigma_i\circ\varsigma_i^{-1}}\right\rangle\,, \tag{6.1}$$

for every $i \in I$. Dealing with a family of t-norms $(T_i)_{i\in I}$, our definition of ordinal sums is therefore equivalent with the definition presented in [51], where the t-norms T_i are always linearly rescaled. However, our approach will prove to be indispensable for the construction of De Morgan triplets (Section 9.2). Because the ϕ-transform T_ϕ, with ϕ an automorphism, of a t-norm T always is a t-norm (see Section 5.2), the right-hand side of Eq. (6.1) is indeed an ordinal sum of t-norms.

Theorem 6.2 *The ordinal sum of a family of t-norms is always a t-norm.*

Proof It is well known (see e.g. [29] and [51]) that, dealing with linear rescaling functions only, the ordinal sum of a family of t-norms is always a t-norm. From Eq.(6.1) it then follows that the latter also holds when dealing with an arbitrary family of rescaling functions. □

In the literature, several other types of ordinal sums using a more general family of increasing $[0,1]^2 \to [0,1]$ functions instead of a family of t-norms have been studied (see e.g. [44], [52] and [84]). However, to describe the whole class of continuous t-norms the above definition is sufficient.

Definition 6.3 [51] A t-norm T is called *Archimedean* if for every $(x, y) \in\,]0,1[^2$ there exists $n \in \mathbb{N}_0$ such that

$$\underbrace{T(\ldots T(T(}_{n \text{ times}}x, x), x) \ldots, x) < y\,.$$

Based on the *diagonal* δ_T of T (*i.e.* $\delta_T(x) := T(x, x)$), we recall in the following theorem a more practical method for determining whether a t-norm is Archimedean or not.

Theorem 6.4 [51] *A t-norm T is Archimedean if and only if $\delta_T(x) < x$, for every $x \in\,]0,1[$ and whenever $\lim_{x \searrow x_0} \delta_T(x) = x_0$ for some $x_0 \in\,]0,1[$, there exists $y_0 \in\,]x_0, 1[$ such that $\delta_T(y_0) = x_0$.*

The Archimedean, continuous t-norms are then nothing else than transformations of the algebraic product $T_{\mathbf{P}}$ or of the Łukasiewicz t-norm $T_{\mathbf{L}}$.

Theorem 6.5 [51] *A continuous t-norm T is Archimedean if and only if there exists an automorphism ϕ such that $T = (T_{\mathbf{P}})_\phi$ or $T = (T_{\mathbf{L}})_\phi$.*

All continuous t-norms are characterized in the following way.

Theorem 6.6 [51] *A t-norm T is continuous if and only if $T = T_{\mathbf{M}}$ or T is the ordinal sum of continuous Archimedean t-norms.*

Furthermore, the Archimedean property forces a left-continuous t-norm to be continuous.

Theorem 6.7 [54] *Every Archimedean, left-continuous t-norm is continuous.*

6.3 The companion and zooms

Every left-continuous, increasing function F that satisfies $F(0, 1) = F(1, 0) = 0$ (*i.e.* that has absorbing element 0) can be fully described by contour lines of the types C_a and $\widetilde{C}_a$ (Theorem 4.6). In particular, it holds that $C_a = \widetilde{C}_a$ whenever F is commutative (Theorem 4.7). As we intend to describe left-continuous t-norms by means of their contour lines, we mainly consider contour lines of the type C_a only. The results from Section 5.3 can be used to express the t-norm properties of F. For a left-continuous t-norm T, taking into account the correspondence between its residual implicator I_T and its contour lines C_a, the contour line C_0 coincides with the *residual negator* N_T, defined by $N_T(x) := I_T(x, 0)$. Furthermore, we recognize some well-known properties among the results of Section 5.3. Equation (5.3) coincides with the *portation law* (*i.e.* $I_T(x, I_T(y, z)) = I_T(T(x, y), z)$, for every $(x, y, z) \in [0, 1]^3$) [40] and Eq. (5.7) coincides with the *exchange principle* (*i.e.* $I_T(x, I_T(y, z)) = I_T(y, I_T(x, z))$, for every $(x, y, z) \in [0, 1]^3$) [11, 15, 16]. The associativity of a left-continuous t-norm is therefore established by the portation law as well as by the exchange principle. Note that contour lines of a (left-)continuous t-norm are also called *level functions* [53]. Dealing with a left-continuous t-norm T, Jenei [48] has provided sufficient conditions on its level functions by which T equals the Łukasiewicz t-norm $T_{\mathbf{L}}$, resp. the algebraic product $T_{\mathbf{P}}$. From now on, we assume full familiarity with the results from Section 4.2.

Unfortunately, contour lines are inadequate to give insight into the geometrical structure of left-continuous t-norms. In this section we present two additional tools that describe F in an alternative way. They will prove to be indispensable for the decomposition and construction of rotation-invariant t-norms. As our focus lies on revealing the (underlying) structure of left-continuous t-norms, we do not give here a full description of the new concepts but merely present those results that are necessary for further use.

The companion

A first useful tool for studying an increasing $[0,1]^2 \to [0,1]$ function F is its *companion* Q

Definition 6.8 Let F be an increasing $[0,1]^2 \to [0,1]$ function. The *companion* of F is the $[0,1]^2 \to [0,1]$ function Q defined by

$$Q(x,y) = \sup\{t \in [0,1] \mid C_t(x) \leqslant y\}\,.$$

The following properties provide better insight into the geometrical structure of Q.

Property 6.9 [64] *The companion Q of an increasing $[0,1]^2 \to [0,1]$ function F satisfies the following properties:*

(E1) *Q is increasing in both arguments.*
(E2) $Q(x,y) = \inf\{F(x,u) \mid u \in \,]y,1]\}$, *with* $\inf\emptyset = 1$.
(E3) $F(x,y) \leqslant Q(x,y)$, *for every* $(x,y) \in [0,1]^2$.
(E4) $Q(x,\bullet)$ *is right continuous for every* $x \in [0,1]$.
(E5) *If F has neutral element 1, then $Q(x,y) \leqslant T_{\mathbf{M}}(x,y)$, for every $(x,y) \in [0,1] \times [0,1[$.*
(E6) *If F is a left-continuous t-norm, then $Q(x,y) < C_a(F(u,v)) \Rightarrow Q(v,y) \leqslant C_a(F(u,x))$, for every $(x,y,u,v,a) \in [0,1]^5$.*

Proof **(E1):** Since C_t is decreasing, the first property is trivially fulfilled.

(E2): It is clear that $Q(x,1) = 1$, for every $x \in [0,1]$. Hence, it suffices to show that

$$\inf\{F(x,u) \mid u \in \,]y,1]\} = \sup\{t \in [0,1] \mid C_t(x) \leqslant y\} = \inf\{t \in [0,1] \mid y < C_t(x)\}\,,$$

for every $(x,y) \in [0,1] \times [0,1[$. Whenever $u \in \,]y,1]$ it holds that $F(x,y) \leqslant F(x,u)$, which leads to $y < C_{F(x,u)}(x)$. We conclude that $\{F(x,u) \mid u \in \,]y,1]\} \subseteq \{t \in [0,1] \mid y < C_t(x)\}$ and hence

$$\inf\{t \in [0,1] \mid y < C_t(x)\} \leqslant \inf\{F(x,u) \mid u \in \,]y,1]\}\,.$$

Suppose now that the above inequality is strict, then there exists a $t < \inf\{F(x,u) \mid u \in \,]y,1]\}$ such that $y < C_t(x)$. Consequently, one can find a $u \in \,]y,1]$ such that $F(x,u) \leqslant t$, a contradiction.

(E3): Clearly, $F(x,1) \leqslant 1 = Q(x,1)$, for every $x \in [0,1]$. Consider arbitrary $(x,y) \in [0,1] \times [0,1[$. From property **(E2)** it follows that $F(x,y) \leqslant Q(x,y)$.

(E4): To prove that $Q(x,\bullet)$ is right continuous for every $x \in [0,1]$, we need to show that for every $(x,y) \in [0,1] \times [0,1[$ and every $\varepsilon \in]0,1]$ there exists a $\delta \in]y,1]$ such that $0 \leqslant Q(x,z) - Q(x,y) < \varepsilon$ whenever $z \in]y,\delta[$. Taking into account property **(E2)** we know that for every $\varepsilon \in]0,1]$ there exists a $\delta \in]y,1]$ such that $0 \leqslant F(x,\delta) - Q(x,y) < \varepsilon$. Moreover, every $z \in]y,\delta[$ fulfills $Q(x,y) \leqslant Q(x,z) \leqslant F(x,\delta)$ and, hence, $0 \leqslant Q(x,z) - Q(x,y) < \varepsilon$.

(E5): Take arbitrary $(x,y) \in [0,1] \times [0,1[$. Then, $y < 1 = C_x(x)$ implies that $Q(x,y) \leqslant x$ and $y < y + \varepsilon = C_{y+\varepsilon}(1) \leqslant C_{y+\varepsilon}(x)$, for every $\varepsilon \in]0, 1-y]$, ensures that $Q(x,y) \leqslant y$.

(E6): Consider arbitrary $(x,y,u,v,a) \in [0,1]^5$. If $Q(x,y) < C_a(F(u,v))$, then it holds by definition that $y < C_{C_a(F(u,v))}(x)$. From Theorem 5.12, we know that $C_{C_a(F(u,v))}(x) = C_{C_a(F(u,x))}(v)$. Hence, $y < C_{C_a(F(u,x))}(v)$ which leads to $Q(v,y) \leqslant C_a(F(u,x))$. $\square$

Property **(E2)** allows to straightforwardly construct the graph of Q (*i.e.* $\{(x,y,Q(x,y)) \mid (x,y) \in [0,1]^2\}$) from the graph of F (*i.e.* $\{(x,y,F(x,y)) \mid (x,y) \in [0,1]^2\}$). First, we need to convert the partial functions $F(x,\bullet)$ into right-continuous functions by adapting the value in the discontinuity points. Further, as $Q(x,1) = 1$ must hold for every $x \in [0,1]$, we replace the set $\{(x,1,F(x,1)) \mid x \in [0,1]\}$ by the set $\{(x,1,1) \mid x \in [0,1]\}$. Clearly, $Q(x,y) = F(x,y)$ whenever $F(x,\bullet)$ is right continuous in $y \in [0,1[$. Note that $Q(x,1) = 1$ and $Q(1,x) = x$ prevent Q from being commutative.

Every left-continuous, increasing binary function F that has absorbing element 0 is totally determined by its companion Q. For a left-continuous t-norm T, the definition of Q is structurally identical to the definition of the residual implicator $I_T (= C)$. Both functions map a pair $(x,y) \in [0,1]^2$ to $\sup\{t \in [0,1] \mid F(x,t) \leqslant y\}$, where $F = T$ when defining $I_T = C$ and $F = I_T = C$ when defining Q.

Zooms

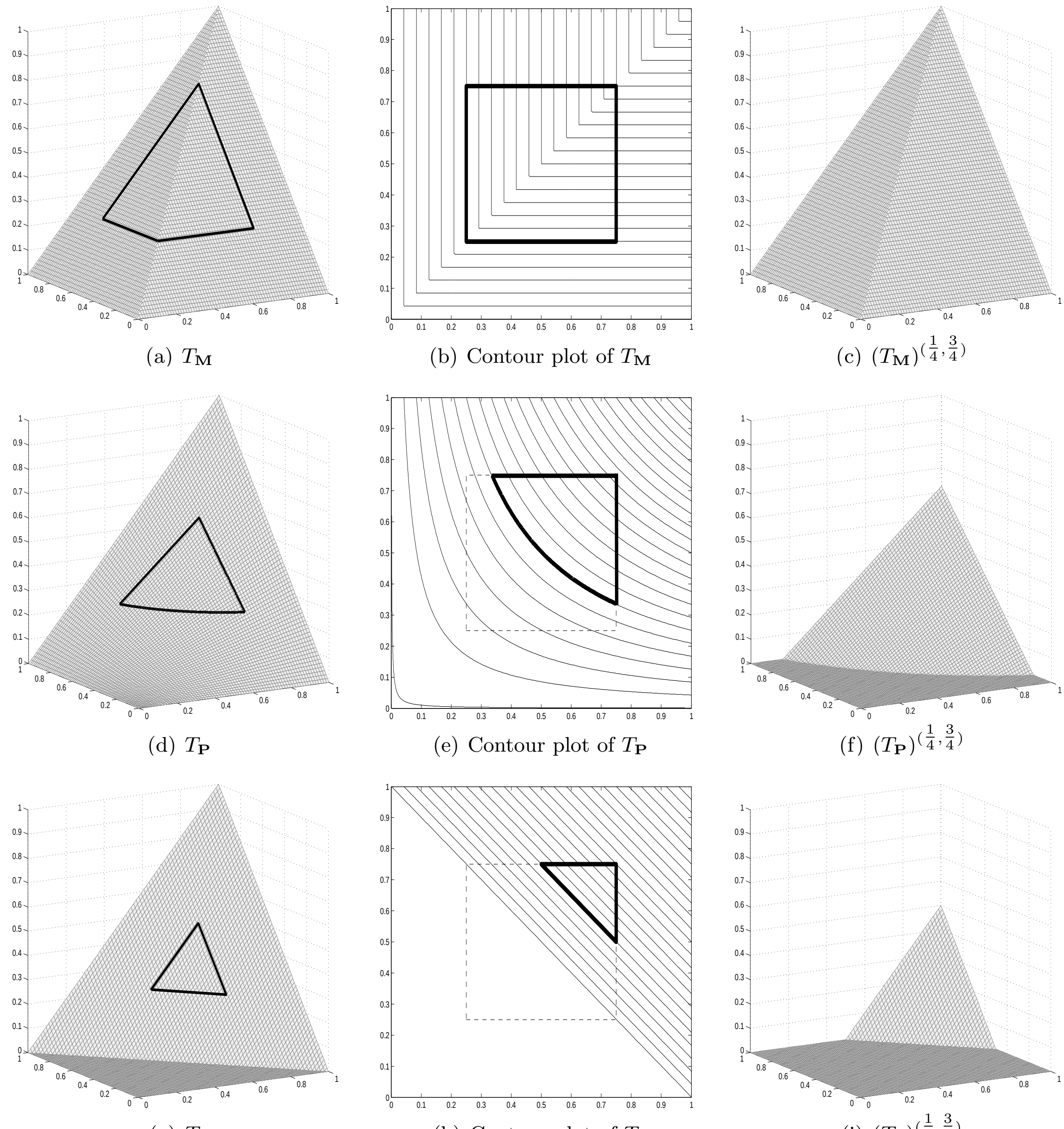
Every increasing $[0,1]^2 \to [0,1]$ function F is trivially described by its associated set of (a,b)-zooms.

Definition 6.10 Let F be an increasing $[0,1]^2 \to [0,1]$ function and take $(a,b) \in [0,1]^2$ such that $a < b$ and $F(b,b) \leqslant b$. Consider an $[a,b] \to [0,1]$ isomorphism σ. The (a,b)-*zoom* of F is the $[0,1]^2 \to [0,1]$ function $F^{(a,b)}$ defined by

$$F^{(a,b)}(x,y) = \sigma\big[\max\big(a, F(\sigma^{-1}[x], \sigma^{-1}[y])\big)\big]\,.$$

If $b = 1$ we simply talk about the a-*zoom* F^a of F.

Geometrically, the graph of $F^{(a,b)}$ is determined by rescaling the set $\{(x,y,F(x,y)) \mid (x,y) \in [a,b]^2 \;\wedge\; a < F(x,y)\}$ (zoom in) into the unit cube (zoom out). Figure 6.1 illustrates this procedure for the three prototypical t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, with $a = \frac{1}{4}$, $b = \frac{3}{4}$ and $\sigma = \varsigma$, where ς is the linear rescaling of $[a,b]$ into $[0,1]$. This linear rescaling function is the prototype of an $[a,b] \to [0,1]$ isomorphism σ. Unless stated differently, we will always use it for our examples.



(a) $T_{\mathbf{M}}$ (b) Contour plot of $T_{\mathbf{M}}$ (c) $(T_{\mathbf{M}})^{(\frac{1}{4},\frac{3}{4})}$

(d) $T_{\mathbf{P}}$ (e) Contour plot of $T_{\mathbf{P}}$ (f) $(T_{\mathbf{P}})^{(\frac{1}{4},\frac{3}{4})}$

(g) $T_{\mathbf{L}}$ (h) Contour plot of $T_{\mathbf{L}}$ (i) $(T_{\mathbf{L}})^{(\frac{1}{4},\frac{3}{4})}$

Figure 6.1: The $(\frac{1}{4},\frac{3}{4})$-zooms of $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$.

Note that whenever $F(b,b) \leqslant a$ the function F is trivially constant: $F^{(a,b)}(x,y) = 0$, for every $(x,y) \in [0,1]^2$.

The boundary condition $F(b,b) \leqslant b$ is absolutely necessary when defining (a,b)-zooms. It ensures that $F(\sigma^{-1}[x], \sigma^{-1}[y]) \leqslant b$, for every $(x,y) \in [0,1]^2$. For $b = 1$ the boundary condition reduces to $F(1,1) \leqslant 1$ such that the a-zoom of F is defined for every $a < 1$. By definition, $F^0 = F_{\sigma^{-1}}$, where $F_{\sigma^{-1}}$ denotes the σ^{-1}-transform of F. Furthermore, all zooms of the (a,b)-zoom $F^{(a,b)}$ of F can be interpreted as zooms of the original function F itself.

Theorem 6.11 *Consider an increasing $[0,1]^2 \to [0,1]$ function F. Take $(a,b,c,d) \in [0,1]^4$, such that $a < b$, $c < d$ and $F(b,b) \leqslant b$. Let σ be an arbitrary $[a,b] \to [0,1]$ isomorphism and $\tilde{\sigma}$ be an arbitrary $[c,d] \to [0,1]$ isomorphism. If $F(\sigma^{-1}[d], \sigma^{-1}[d]) \leqslant \sigma^{-1}[d]$, then $(F^{(a,b)})^{(c,d)} = F^{(\sigma^{-1}[c], \sigma^{-1}[d])}$, where σ is used to compute $F^{(a,b)}$, $\tilde{\sigma}$ is used to compute $(F^{(a,b)})^{(c,d)}$ and $\tilde{\sigma} \circ \sigma$ is used to compute $F^{(\sigma^{-1}[c], \sigma^{-1}[d])}$.*

Proof Clearly, $F(\sigma^{-1}[d], \sigma^{-1}[d]) \leqslant \sigma^{-1}[d]$ implies $F^{(a,b)}(d,d) = \sigma[\max(a, F(\sigma^{-1}[d], \sigma^{-1}[d]))] \leqslant \sigma[\max(a, \sigma^{-1}[d])] = d$. By definition, we immediately obtain

$$\begin{aligned}
(F^{(a,b)})^{(c,d)}(x,y) &= \tilde{\sigma}\left[\max\left(c, F^{(a,b)}\left(\tilde{\sigma}^{-1}[x], \tilde{\sigma}^{-1}[y]\right)\right)\right] \\
&= \tilde{\sigma}\Big[\sigma\Big[\max\Big(\sigma^{-1}[c], \max\big(a, F\big(\sigma^{-1}[\tilde{\sigma}^{-1}[x]], \sigma^{-1}[\tilde{\sigma}^{-1}[y]]\big)\big)\Big)\Big]\Big] \\
&= \tilde{\sigma}\Big[\sigma\Big[\max\Big(\sigma^{-1}[c], F\big(\sigma^{-1}[\tilde{\sigma}^{-1}[x]], \sigma^{-1}[\tilde{\sigma}^{-1}[y]]\big)\Big)\Big]\Big] = F^{(\sigma^{-1}(c), \sigma^{-1}(d))}(x,y)\,,
\end{aligned}$$

for every $(x,y) \in [0,1]^2$ $\square$

Dealing with an increasing function F that satisfies $F \leqslant T_{\mathbf{M}}$ and whose (a,b)-zoom $F^{(a,b)}$ has neutral element 1, we can use the above theorem to give $F^{(a,c)}$ the structure of an ordinal sum for every $c \in \,]b,1]$.

Corollary 6.12 *Consider an increasing $[0,1]^2 \to [0,1]$ function F satisfying $F \leqslant T_{\mathbf{M}}$. Take $(a,b,c) \in [0,1]^2$ such that $a < b < c$. Let $\tilde{\sigma}$ be an arbitrary $[a,b] \to [0,1]$ isomorphism, $\hat{\sigma}$ be an arbitrary $[b,c] \to [0,1]$ isomorphism and σ be an arbitrary $[a,c] \to [0,1]$ isomorphism. If $F^{(a,b)}$ has neutral element 1 then*

$$F^{(a,c)} = \left(\left\langle 0, \sigma[b], \tilde{\sigma} \circ \sigma^{-1}, F^{(a,b)} \right\rangle, \left\langle \sigma[b], 1, \hat{\sigma} \circ \sigma^{-1}, F^{(b,c)} \right\rangle\right)\,, \tag{6.2}$$

where $\tilde{\sigma}$ is used to compute $F^{(a,b)}$, $\hat{\sigma}$ is used to compute $F^{(b,c)}$ and σ is used to compute $F^{(a,c)}$.

Proof The boundary condition $F \leqslant T_{\mathbf{M}}$ allows us to use Theorem 6.11. It follows that $(F^{(a,c)})^{(0,\sigma[b])} = (F^{(a,c)})^{(\sigma[a],\sigma[b])} = F^{(a,b)}$ and $(F^{(a,c)})^{(\sigma[b],1)} = (F^{(a,c)})^{(\sigma[b],\sigma[c])} = F^{(b,c)}$, where $\tilde{\sigma} \circ \sigma^{-1}$ is used to compute $(F^{(a,c)})^{(0,\sigma[b])}$ and $\hat{\sigma} \circ \sigma^{-1}$ is used to compute $(F^{(a,c)})^{(\sigma[b],1)}$. In particular,

$$F^{(a,c)}(x,y) = \max\Big(0, F^{(a,c)}(x,y)\Big) = \sigma\left[\tilde{\sigma}^{-1}\left[(F^{(a,c)})^{(0,\sigma[b])}\big(\tilde{\sigma}[\sigma^{-1}[x]], \tilde{\sigma}[\sigma^{-1}[y]]\big)\right]\right]$$
$$= \sigma\left[\tilde{\sigma}^{-1}\left[F^{(a,b)}\big(\tilde{\sigma}[\sigma^{-1}[x]], \tilde{\sigma}[\sigma^{-1}[y]]\big)\right]\right], \tag{6.3}$$

for every $(x,y) \in [0,\sigma(b)]^2$. As $F^{(a,b)}$ has neutral element 1 the latter implies that

$$x = \sigma\big[\tilde{\sigma}^{-1}\big[F^{(a,b)}\big(\tilde{\sigma}[\sigma^{-1}[x]], 1\big)\big]\big] = F^{(a,c)}(x, \sigma[b])$$
$$x = \sigma\left[\tilde{\sigma}^{-1}\left[F^{(a,b)}\big(1, \tilde{\sigma}[\sigma^{-1}[x]]\big)\right]\right] = F^{(a,c)}(\sigma[b], x)$$

for every $x \in [0,\sigma(b)]$. Furthermore, since $F \leqslant T_{\mathbf{M}}$ it holds that

$$x = F^{(a,c)}(x,\sigma[b]) \leqslant F^{(a,c)}(x,y) \leqslant \sigma\left[\max\big(a, \min\big(\sigma^{-1}[x], \sigma^{-1}[y]\big)\big)\right] = \min(x,y) = x \tag{6.4}$$
$$x = F^{(a,c)}(\sigma[b],x) \leqslant F^{(a,c)}(y,x) \leqslant \sigma\left[\max\big(a, \min\big(\sigma^{-1}[y], \sigma^{-1}[x]\big)\big)\right] = \min(x,y) = x \tag{6.5}$$

for every $(x,y) \in [0,\sigma(b)] \times [\sigma(b),1]$. Invoking the equality $F^{(a,c)}(\sigma[b],\sigma[b]) = \sigma[b]$ and Theorem 6.11, we can compute $F^{(a,c)}|_{[\sigma(b),1]^2}$:

$$F^{(a,c)}(x,y) = \max\Big(\sigma[b], F^{(a,c)}(x,y)\Big) = \sigma\left[\hat{\sigma}^{-1}\left[(F^{(a,c)})^{(\sigma[b],1)}\big(\hat{\sigma}[\sigma^{-1}[x]], \hat{\sigma}[\sigma^{-1}[y]]\big)\right]\right]$$
$$= \sigma\left[\hat{\sigma}^{-1}\left[F^{(b,c)}\big(\hat{\sigma}[\sigma^{-1}[x]], \hat{\sigma}[\sigma^{-1}[y]]\big)\right]\right], \tag{6.6}$$

for every $(x,y) \in [\sigma(b),1]^2$. Combining Eqs. (6.3)–(6.6) yields Eq. 6.2 (Definition 6.1). □

Note that $F \leqslant T_{\mathbf{M}}$ prevents $F^{(a,b)}$ from having a neutral element $e < 1$. Indeed $1 = F^{(a,b)}(1,e) = \tilde{\sigma}[\max(a, F(b, \tilde{\sigma}^{-1}[e]))] \leqslant e$. For an arbitrary increasing function F, its (a,b)-zoom $F^{(a,b)}$ is totally determined by $F|_{[a,b]^2}$. Its contour lines and companion can be computed from the contour lines and companion of F. In case $F^{(a,b)}$ has neutral element 1, there exists a straightforward relationship between its contour lines and those of the original function F.

Property 6.13 *Consider an increasing* $[0,1]^2 \to [0,1]$ *function* F. *Take* $(a,b) \in [0,1]^2$, *such that* $a < b$ *and* $F(b,b) \leqslant b$. *Let* σ *be an arbitrary* $[a,b] \to [0,1]$ *isomorphism. If the* (a,b)-*zoom* $F^{(a,b)}$ *has contour lines* $C_d^{(a,b)}$ *and companion* $Q^{(a,b)}$, *then the following properties hold:*

(F1) $F^{(a,b)}$ *is increasing in both arguments.*
(F2) $Q^{(a,b)}(x,y) = \sigma[Q(\sigma^{-1}[x], \sigma^{-1}[y])]$, *for every* $(x,y) \in [0,1]^2$ *such that* $C_0^{(a,b)}(x) \leqslant y < 1$.
(F3) *If* F *is left continuous, then* $F^{(a,b)}$ *is left continuous.*
(F4) $C_d^{(a,b)}(x) = \sigma[C_{\sigma^{-1}[d]}(\sigma^{-1}[x])]$ *holds if*

- **(F4a)** $b = 1$, $F^{(a,1)}(1,0) = 0$ *and* $(x,d) \in [0,1]^2$;
- **(F4b)** $F^{(a,b)}$ *has neutral element* 1 *and* $(x,d) \in [0,1]^2$ *such that* $d < x$.

(F5) *If* F *is associative and* $\max(F(a,b), F(b,a)) \leqslant a$, *then* $F^{(a,b)}$ *is also associative.*

Proof (F1): The increasingness of $F^{(a,b)}$ is an immediate consequence of the increasingness of F.

(F2): Consider arbitrary $(x,y) \in [0,1]^2$ satisfying $C_0^{(a,b)}(x) \leqslant y < 1$. Using property **(E2)**, $Q^{(a,b)}(x,y)$ is totally determined as follows

$$\begin{aligned} Q^{(a,b)}(x,y) &= \inf\{F^{(a,b)}(x,t) \mid t \in \,]y,1]\} \\ &= \inf\{\sigma[\max(a, F(\sigma^{-1}[x], \sigma^{-1}[t]))] \mid t \in \,]y,1]\} \\ &= \inf\{\sigma[\max(a, F(\sigma^{-1}[x], s))] \mid s \in \,]\sigma^{-1}(y), b]\} \\ &= \sigma[\inf\{\max(a, F(\sigma^{-1}[x], s)) \mid s \in \,]\sigma^{-1}(y), b]\}]\,. \end{aligned}$$

Since $C_0^{(a,b)}(x) \leqslant y$ implies that $\sigma[a] = 0 < F^{(a,b)}(x,t)$, for every $t \in \,]y,1]$, it holds that $a < F(\sigma^{-1}[x], s)$ whenever $s \in \,]\sigma^{-1}(y), b]$. Therefore,

$$\begin{aligned} Q^{(a,b)}(x,y) &= \sigma[\inf\{F(\sigma^{-1}[x], s) \mid s \in \,]\sigma^{-1}(y), b]\}] \\ &= \sigma[\inf\{F(\sigma^{-1}[x], s) \mid s \in \,]\sigma^{-1}(y), 1]\}] = \sigma[Q(\sigma^{-1}[x], \sigma^{-1}[y])]\,, \end{aligned}$$

which finishes the proof.

(F3): As $F^{(a,b)}$ is composed of F and several increasing continuous functions, the continuity of F is passed on to $F^{(a,b)}$.

(F4a): Comparing the contour lines of F and $F^{(a,1)}$ we obtain that

$$\begin{aligned} C_d^{(a,1)}(x) &= \sup\{t \in [0,1] \mid F^{(a,1)}(x,t) \leqslant d\} \\ &= \sup\{t \in [0,1] \mid \sigma[\max(a, F(\sigma^{-1}[x], \sigma^{-1}[t]))] \leqslant d\} \\ &= \sup\{t \in [0,1] \mid F(\sigma^{-1}[x], \sigma^{-1}[t]) \leqslant \sigma^{-1}[d]\} \\ &= \sigma[\sup\{s \in [a,1] \mid F(\sigma^{-1}[x], s) \leqslant \sigma^{-1}[d]\}]\,, \end{aligned}$$

for every $(x,d) \in [0,1]^2$. As $F(\sigma^{-1}[x], a) \leqslant F(1,a) \leqslant \sigma^{-1}[F^{(a,1)}(1,0)] = a \leqslant \sigma^{-1}[d]$, we can rewrite the latter as

$$C_d^{(a,1)}(x) = \sigma[\sup\{s \in [0,1] \mid F(\sigma^{-1}[x], s) \leqslant \sigma^{-1}[d]\}] = \sigma[C_{\sigma^{-1}[d]}(\sigma^{-1}[x])]\,.$$

(F4b): If $F^{(a,b)}$ has neutral element 1, then $\max(a, F(\sigma^{-1}[x], b)) = \max(a, F(b, \sigma^{-1}[x])) = \sigma^{-1}[x]$, for every $x \in [0,1]$. The latter is clearly equivalent with $F(u,b) = F(b,u) = u$, for every $u \in \,]a,b]$. Take arbitrary $(x,d) \in [0,1]^2$ satisfying $d < x$. Following the reasonings in the proof of property **(F4a)**, we know that

$$C_d^{(a,b)}(x) = \sigma[\sup\{s \in [a,b] \mid F(\sigma^{-1}[x], s) \leqslant \sigma^{-1}[d]\}]\,.$$

As $F(\sigma^{-1}[x], a) \leqslant F(b,a) \leqslant a \leqslant \sigma^{-1}[d]$ and $\sigma^{-1}[d] < \sigma^{-1}[x] = F(\sigma^{-1}[x], b)$, we can rewrite the latter as

$$C_d^{(a,b)}(x) = \sigma[\sup\{s \in [0,1] \mid F(\sigma^{-1}[x], s) \leqslant \sigma^{-1}[d]\}] = \sigma[C_{\sigma^{-1}[d]}(\sigma^{-1}[x])]\,.$$

(F5): Explicitly, we need to prove that

$$\max\big(a, F\big(\max(a, F(\sigma^{-1}[x], \sigma^{-1}[y])), \sigma^{-1}[z]\big)\big) = \max\big(a, F\big(\sigma^{-1}[x], \max(a, F(\sigma^{-1}[y], \sigma^{-1}[z]))\big)\big)$$

holds for every $(x, y, z) \in [0, 1]^3$. Taking into account the increasingness of F we can rewrite this equality as follows:

$$\begin{aligned}\max\big(a, F(a, \sigma^{-1}[z]), F\big(F(\sigma^{-1}[x], \sigma^{-1}[y]), \sigma^{-1}[z]\big)\big)& \\ = \max\big(a, F(\sigma^{-1}[x], a), F\big(\sigma^{-1}[x], F(\sigma^{-1}[y], \sigma^{-1}[z])\big)\big)&\,.\end{aligned}$$

The latter is always fulfilled since F is associative, $F(a, \sigma^{-1}[z]) \leqslant F(a, b) \leqslant a$ and $F(\sigma^{-1}[x], a) \leqslant F(b, a) \leqslant a$. $\square$

In accordance to Definition 6.10 we will usually denote the contour lines of $F^a (= F^{(a,1)})$ by $C^a (= C^{(a,1)})$ and its companion by $Q^a (= Q^{(a,1)})$. Zooms are extremely suited to study an increasing function F that satisfies $F \leqslant T_{\mathbf{M}}$. The restrictions $F(b, b) \leqslant b$ (Definition 6.10), $F(1, a) \leqslant a$ (property **(F4b)**) and $\max(F(a, b), F(b, a)) \leqslant a$ (property **(F5)**) are then trivially fulfilled.

Definition 6.14 [42] A *t-subnorm* T is an associative, commutative, increasing $[0, 1]^2 \to [0, 1]$ function that satisfies $T \leqslant T_{\mathbf{M}}$.

Clearly, all t-norms are t-subnorms. The monotonicity and neutral element e of a uninorm U imply that $\mathbf{id} \leqslant U(\bullet, 1)$ and $\mathbf{id} \leqslant U(1, \bullet)$. Therefore, U can only be a t-subnorm if $\mathbf{id} = U(\bullet, 1) = U(1, \bullet)$ and, hence, U must be a t-norm. Due to its boundary condition we can construct all (a, b)-zooms $(a < b)$ of every t-subnorm. Moreover, all these (a, b)-zooms are t-subnorms as well.

Corollary 6.15 *Consider $(a, b) \in [0, 1]^2$ such that $a < b$. The (a, b)-zoom of a t-subnorm is a t-subnorm and the a-zoom of a t-norm is a t-norm.*

Proof Let σ be the $[a, b] \to [0, 1]$ isomorphism used to construct the (a, b)-zoom of a t-subnorm T. The increasingness of $T^{(a,b)}$ follows from property **(F1)**. The commutativity of $T^{(a,b)}$ and T^a is easily verified. Furthermore, as $T \leqslant T_{\mathbf{M}}$ it holds that

$$T^{(a,b)}(x, y) \leqslant \sigma\left[\max\left(a, \min(\sigma^{-1}[x], \sigma^{-1}[y])\right)\right] = \sigma\left[\min(\sigma^{-1}[x], \sigma^{-1}[y])\right] = T_{\mathbf{M}}(x, y)\,,$$

for every $(x, y) \in [0, 1]^2$. Taking into account property **(F5)** it now follows that $T^{(a,b)}$ is a t-subnorm. In case $b = 1$ and T is a t-norm, $\sigma[1] = 1$ is obviously the neutral element of T^a. $\square$

Example 6.16 All three $(\frac{1}{4}, \frac{3}{4})$-zooms in Fig. 6.1 are t-subnorms. In contrast to Fig. 6.1(c), Figs. 6.1(f) and 6.1(i) are are not t-norms. In general, every (a, b)-zoom of the minimum operator $T_{\mathbf{M}}$ equals $T_{\mathbf{M}}$ itself. However, no (a, b)-zoom, with $b < 1$, of the algebraic product $T_{\mathbf{P}}$ or of the Łukasiewicz t-norm $T_{\mathbf{L}}$ can be a t-norm. The latter follows from the observation that the (a, b)-zoom $T^{(a,b)}$ of a t-subnorm T has neutral element 1 whenever $T(x, b) = T(b, x) = x$, for every $x \in\,]a, b]$. Dealing with $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$ this only occurs for $b = 1$. $\triangle$

6.4 A continuous contour line

Equipped with contour lines, the companion and zooms, we can now shed new light on the overall structure of left-continuous t-norms and in particular of rotation-invariant t-norms. Property 4.5 states that the contour lines of a left-continuous t-norm T must be left continuous. Conversely, one could wonder how the continuity of a contour line C_a of T affects the structure of the t-norm itself. T-norms can have continuous as well as discontinuous contour lines. For example, as can be seen in Fig. 6.1(e), every contour line but C_0 of the algebraic product $T_{\mathbf{P}}$ is continuous. The Łukasiewicz t-norm $T_{\mathbf{L}}$ is an example of a t-norm having only continuous contour lines (see Fig. 6.1(h)). Every contour line but C_1 of the minimum operator $T_{\mathbf{M}}$ is discontinuous (see Fig. 6.1(b)). Merging Theorems 5.13 and 5.14 yields five assertions expressing the continuity of a contour line C_a. Considering the companion Q of T, we can extend this set to seven assertions ensuring the continuity of C_a.

Theorem 6.17 *For a left-continuous t-norm T, the following assertions are equivalent:*

(G1) *C_a is continuous.*
(G2) *C_a is involutive on $[a, 1]$.*
(G3) *$T(x, y) = C_a(C_{C_a(x)}(y))$, for every $(x, y) \in [0, 1]^2$ fulfilling $C_a(x) < y$.*
(G4) *$C_b(x) = C_{C_a(x)}(C_a(b))$, for every $(x, b) \in [0, 1] \times [a, 1]$.*
(G5) *$T(x, y) \leqslant z \;\Leftrightarrow\; T(x, C_a(z)) \leqslant C_a(y)$, for every $(x, y, z) \in [a, 1]^3$.*
(G6) *$C_b(x) = y \;\Leftrightarrow\; T(x, C_a(b)) = C_a(y)$, for every $(x, y, b) \in [a, 1]^3$ such that $b < x$.*
(G7) *$Q(x, y) < C_a(z) \;\Leftrightarrow\; Q(x, z) < C_a(y)$, for every $(x, y, z) \in [a, 1] \times [a, 1[^2$.*
(G8) *$Q(x, y) = C_a(D_{C_a(y)}(x))$, for every $(x, y) \in [0, 1]^2$ such that $C_a(x) \leqslant y < 1$.*

Proof Taking into account that T has neutral element 1, the equivalence between assertions **(G1)**–**(G5)** is obtained from Theorem 5.13. The neutral element also ensures that $C_a(a) = 1$ (Corollary 5.8). Theorem 5.14 then establishes the equivalence between assertions **(G1)** and **(G6)**. This leaves us to prove that also assertions **(G7)** and **(G8)** ensure the continuity of C_a.

(G2)⇔(G7): Due to the symmetry of assertion **(G7)** in the variables y and z, it suffices to prove the equivalence from left to right. Suppose that C_a is involutive on $[0, a]$ and $Q(x, y) < C_a(z)$ holds for some $(x, y, z) \in [a, 1] \times [a, 1[^2$. Taking into account the continuity of C_a and property **(E2)**, we know that there exists $\varepsilon \in]0, \min(1-y, 1-z)[$ such that $T(x, y+\varepsilon) < C_a(z+\varepsilon)$. Theorem 5.12 ensures that $T(x, z + \varepsilon) \leqslant C_a(y + \varepsilon)$. Since every involutive decreasing function on $[a, 1]$ is necessarily strictly decreasing we get from property **(E2)** that $Q(x, z) \leqslant T(x, z+\varepsilon) \leqslant C_a(y + \varepsilon) < C_a(y)$. Conversely, if the equivalence $Q(x, y) < C_a(z) \;\Leftrightarrow\; Q(x, z) < C_a(y)$ holds for every $(x, y, z) \in [a, 1] \times [a, 1[^2$, then in particular $y = Q(1, y) < C_a(C_a(y)) \;\Leftrightarrow\; C_a(y) = Q(1, C_a(y)) < C_a(y)$, for every $y \in]a, 1[$. We conclude that $C_a(C_a(y)) \leqslant y$, for every $y \in]a, 1[$ ($C_a(y) < 1$ due to Corollary 5.8). From Theorem 5.9 we also know that $y \leqslant C_a(C_a(y))$, for every $y \in [0, 1]$. Hence, $C_a(C_a(y)) = y$, for every $y \in]a, 1[$. Note that $C_a(C_a(a)) = C_a(1) = a$ and $C_a(C_a(1)) = 1$ always hold (Corollary 5.8).

(G2)⇔(G8): Suppose that C_a is involutive on $[0,a]$ and consider arbitrary $(x,y) \in [0,1]^2$ fulfilling $C_a(x) \leqslant y < 1$. From assertion **(G4)** and Theorem 5.10 it then follows that

$$\begin{aligned} Q(x,y) &= \sup\{t \in [a,1] \mid C_t(x) \leqslant y\} = \sup\{t \in [a,1] \mid C_{C_a(x)}(C_a(t)) \leqslant y\} \\ &= C_a(\inf\{s \in [a,1] \mid C_{C_a(x)}(s) \leqslant y\}) = C_a(\inf\{s \in [a,1] \mid C_a(T(x,s)) \leqslant y\}) \,. \end{aligned}$$

Because $y < 1$, $C_a(T(x,s)) \leqslant y$ can only hold when $a < T(x,s)$ (Corollary 5.8). Also, $a = C_a(1) \leqslant C_a(x) \leqslant y$. Hence, applying assertion **(G2)**, the inequalities $C_a(T(x,s)) \leqslant y$ and $C_a(y) \leqslant T(x,s)$ become equivalent. Note that $C_a(y) \leqslant T(x,s)$ implies $a \leqslant s$ as $a \leqslant C_a(y)$ and $T(x,s) \leqslant s$. Therefore,

$$Q(x,y) = C_a(\inf\{s \in [0,1] \mid C_a(y) \leqslant T(x,s)\}) = C_a(D_{C_a(y)}(x)) \,.$$

Conversely, suppose that $Q(x,y) = C_a(D_{C_a(y)}(x))$ holds for every $(x,y) \in [0,1]^2$ such that $C_a(x) \leqslant y < 1$. Putting $x = 1$ leads to $y = Q(1,y) = C_a(D_{C_a(y)}(1)) = C_a(C_a(y))$, for every $y \in [a,1[$. Invoking Corollary 5.8 it also holds that $1 = C_a(a) = C_a(C_a(1))$. □

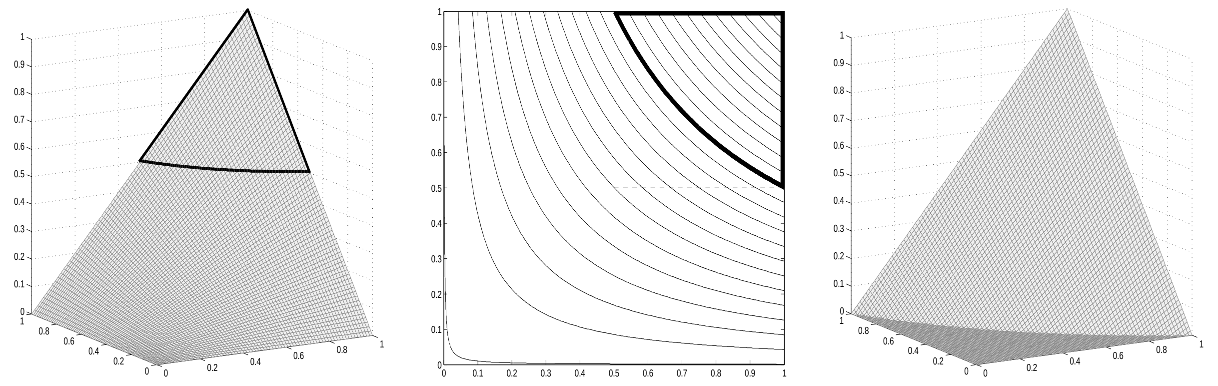
In case $a = 0$, it is easily verified that the additional conditions $C_0(x) < y$ in assertion **(G3)**, $b < x$ in assertion **(G6)**, $\max(y,z) < 1$ in assertion **(G7)** and $C_0(x) \leqslant y < 1$ in assertion **(G8)** can be omitted. To obtain the latter, the involutivity of C_0 is at our disposal. Moreover, as $C = I_T$, we recognize among the assertions of Theorem 6.17 three known properties. Assertion **(G4)** is referred to as the *contrapositive symmetry of* I_T: $I_T(x,y) = I_T(N_T(y), N_T(x))$, for every $(x,y) \in [0,1]^2$ [15, 40]. Assertion **(G5)** expresses the *rotation invariance of* T w.r.t. N_T: $T(x,y) \leqslant z \;\Leftrightarrow\; T(y, N_T(z)) \leqslant N_T(x)$, for every $(x,y,z) \in [0,1]^3$ [40]. Assertion **(G6)** is known as the *self quasi-inverse property of* T: $I_T(x,y) = z \;\Leftrightarrow\; T(x, N_T(y)) = N_T(z)$, for every $(x,y,z) \in [0,1]^3$ [40]. Note that assertion **(G3)** is closely related to the portation law. Jenei [40] has shown that, for a left-continuous t-norm, the involutivity of the residual negator $N_T = C_0$ is equivalent with the self quasi-inverse property of T and with the rotation invariance of T w.r.t. N_T. He has also proved that the involutivity of N_T implies the contrapositive symmetry of I_T.

Theorem 6.18 *If a left-continuous t-norm T has a continuous contour line C_a, with $a \in [0,1[$, then its a-zoom T^a has a continuous contour line C_0^a.*

Proof The proof follows immediately from property **(F4a)**. □

To better comprehend the structure of t-norms that have a continuous contour line C_a, with $a \in [0,1[$, we thus need to focus first on the structure of left-continuous t-norms that have a continuous contour line C_0. In the following chapters we will extensively study these t-norms.

Example 6.19 Recall that the algebraic product $T_{\mathbf{P}}$ has only one discontinuous contour line: C_0. Therefore every t-norm $(T_{\mathbf{P}})^a$, with $a \in\,]0,1[$, has a continuous contour line C_0^a. Moreover, due to the continuity of $T_{\mathbf{P}}$, every $(T_{\mathbf{P}})^a$ must also be continuous. Let σ be the linear rescaling function ς of $[a,1]$ into $[0,1]$. From Fig. 6.2(c) it is observed that $(T_{\mathbf{P}})^a$ is a transformed Łukasiewicz t-norm. Explicitly, $(T_{\mathbf{P}})^a(x,y) = (T_{\mathbf{L}})_{\phi_a}(x,y)$ holds for every $(x,y) \in [0,1]^2$ and with ϕ_a the automorphism defined by $\phi_a(x) = 1 - \ln(\varsigma^{-1}(x))/\ln(a)$. The family t-norms $((T_{\mathbf{L}})_{\phi_a})_{a \in\,]0,1[}$ totally determines the algebraic product $T_{\mathbf{P}}$. △



(a) $T_{\mathbf{P}}$ (b) Contour plot of $T_{\mathbf{P}}$ (c) $(T_{\mathbf{P}})^{\frac{1}{2}}$

Figure 6.2: The $\frac{1}{2}$-zoom of $T_{\mathbf{P}}$.

Note also that the contour line C_0 determines the set of zero divisors of T.

Definition 6.20 A number $x \in]0,1[$ is a *zero divisor* of a t-norm T if there exists $y \in]0,1[$ such that $T(x,y) = 0$.

Hence, $x \in]0,1[$ is a zero-divisor of a t-norm T if and only if $0 < C_0(x)$. In case T is rotation invariant, the involutivity of C_0 (assertion **(G2)**) ensures that every $x \in]0,1[$ is a zero-divisor.

6.5 Rotation-invariant t-norms

Definition 6.21 [40] Let N be an involutive negator. An increasing $[0,1]^2 \to [0,1]$ function F is called *rotation invariant w.r.t.* N if for every $(x,y,z) \in [0,1]^3$ it holds that

$$F(x,y) \leqslant z \quad \Leftrightarrow \quad F(y,z^N) \leqslant x^N\,. \tag{6.7}$$

This property has been first described by Fodor [25]. Jenei emphasized its geometrical interpretation by referring to it as the rotation invariance of T w.r.t. N [40].

Property 6.22 *Consider an involutive negator N and an increasing $[0,1]^2 \to [0,1]$ function F. If F is rotation invariant w.r.t. N, then the following properties hold:*

(H1) *F is left continuous.*
(H2) *If F is a t-subnorm then $N \leqslant C_0$.*
(H3) *If F is a t-norm then $N = C_0$.*

Proof The first and last property have been proven by Jenei [40]. If a t-subnorm F is rotation invariant w.r.t. N then $F(x^N, 0^N) = F(x^N, 1) \leqslant x^N$ implies that $F(x, x^N) \leqslant 0$. The latter leads to $x^N \leqslant C_0(x)$, for every $x \in [0,1]$. $\square$

Dealing with a t-norm T that is rotation invariant w.r.t. an involutive negator N, Eq. (6.7) then coincides with assertion **(G5)**, where $a = 0$. Moreover, it becomes superfluous to mention the negator N explicitly. For a left-continuous t-norm T, its rotation invariance is also equivalent with the continuity of the contour line C_0 (Theorem 6.17). Herein lies the true meaning of the rotation invariance property. Supported by the above considerations, we briefly call a t-norm *rotation invariant* if it is left continuous and has a continuous contour line C_0. Note that the continuity of C_0 does not imply the left continuity of T. For example, converting the left-continuous nilpotent minimum $T^{\mathbf{nM}}$ into a right-continuous $[0,1]^2 \to [0,1]$ function, we obtain the right-continuous t-norm $\widetilde{T^{\mathbf{nM}}}$, defined by

$$\widetilde{T^{\mathbf{nM}}}(x,y) = \begin{cases} 0, & \text{if } x+y<1\,, \\ \min(x,y), & \text{elsewhere}\,. \end{cases}$$

The contour line C_0 of $\widetilde{T^{\mathbf{nM}}}$ coincides with the standard negator $\mathcal{N}$ and is, hence, continuous.

Studying rotation-invariant t-norms, Jenei provided a real breakthrough by introducing his rotation and rotation-annihilation construction [42, 43, 46, 47] which he uses to describe all 'decomposable' rotation-invariant t-norms [45].

Definition 6.23 [45] Let T be a rotation-invariant t-norm and β be the fixpoint of its contour line C_0. T is called *decomposable* if its set of *decomposition points*

$$D_T = \{t \in [\beta,1[\mid (\forall x \in [\beta,t])(C_0(t) \leqslant Q(x, C_0(x)))\}$$

is not empty. If $\beta \in D_T$, then T is called *totally decomposable*.

Jenei [45] refers to the $[0,1] \to [0,1]$ function defined by $Q(x, C_0(x)) = Q(x, N_T(x))$ as the *skeleton function* χ_T of T. Not being familiar with contour lines he used property **(E2)** to define this skeleton function. By means of property **(E5)** and assertions **(G2)** and **(G7)**, we can equivalently describe D_T as follows:

$$D_T = \{t \in [\beta,1[\mid (\forall x \in [\beta,t])(Q(x,t) = x)\}\,. \tag{6.8}$$

Since T has neutral element 1, Property **(E2)** then implies that $T(x,y) = \min(x,y)$, for every $(x,y) \in ([\beta,\alpha] \times]\alpha,1]) \cup (]\alpha,1] \times [\beta,\alpha])$, with $\alpha \in D_T$.

Example 6.24 The nilpotent minimum $T^{\mathbf{nM}}$ is an example of a totally decomposable, rotation-invariant t-norm. Its contour line C_0 coincides with the standard negator $\mathcal{N}$ and is obviously continuous. Because $\beta = \frac{1}{2}$ and $Q^{\mathbf{nM}}(x,t) = \min(x,t)$ whenever $(x,t) \in [\frac{1}{2},1[^2$ it follows that $D_{T^{\mathbf{nM}}} = [\frac{1}{2},1[$. The Łukasiewicz t-norm $T_{\mathbf{L}}$ is an example of a non-decomposable, rotation-invariant t-norm. Its contour line C_0 is also given by the standard negator $\mathcal{N}$ but $Q_{\mathbf{L}}(\frac{1}{2},t) = T_{\mathbf{L}}(\frac{1}{2},t) = t - \frac{1}{2} < \frac{1}{2}$, for every $t \in [\frac{1}{2},1[$. $\triangle$

Although Jenei did not consider contour lines, the companion or zooms, his work concerning rotation-invariant t-norms has given the impulse for our investigations. To make his and our approach comparable we now reformulate his results in our framework, hereby anticipating the formulation of our theorems in Chapters 7 and 8. As a welcome side effect, this whole translation process allows a very concise formulation of Jenei's rotation and rotation-annihilation construction.

A. Decomposition

Consider a decomposable, rotation-invariant t-norm T and fix $\alpha \in D_T$. In the domain $[0,1]^2$ of T we partition the set $\mathcal{D} := \{(x,y) \in [0,1]^2 \mid C_0(x) < y\}$ into six parts as depicted in Fig. 6.3:

$$\begin{aligned}
\mathcal{D}_{\mathrm{A}} &:= \{(x,y) \in \,]\alpha,1]^2\}\,,\\
\mathcal{D}_{\mathrm{B}} &:= \{(x,y) \in \,]0,C_0(\alpha)] \times \,]\alpha,1] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{C}} &:= \{(x,y) \in \,]\alpha,1] \times \,]0,C_0(\alpha)] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{D}} &:= \{(x,y) \in \,]C_0(\alpha),\alpha]^2 \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{E}} &:= \,]C_0(\alpha),\alpha] \times \,]\alpha,1]\,,\\
\mathcal{D}_{\mathrm{F}} &:= \,]\alpha,1] \times \,]C_0(\alpha),\alpha]\,.
\end{aligned}$$

Note that in case α equals the fixpoint β of C_0 (T must then be totally decomposable), areas $\mathcal{D}_{\mathrm{D}}$, $\mathcal{D}_{\mathrm{E}}$ and $\mathcal{D}_{\mathrm{F}}$ are empty. Based on this partition and anticipating on Theorem 7.1, we obtain the following theorem that summarizes multiple results of Jenei [45].

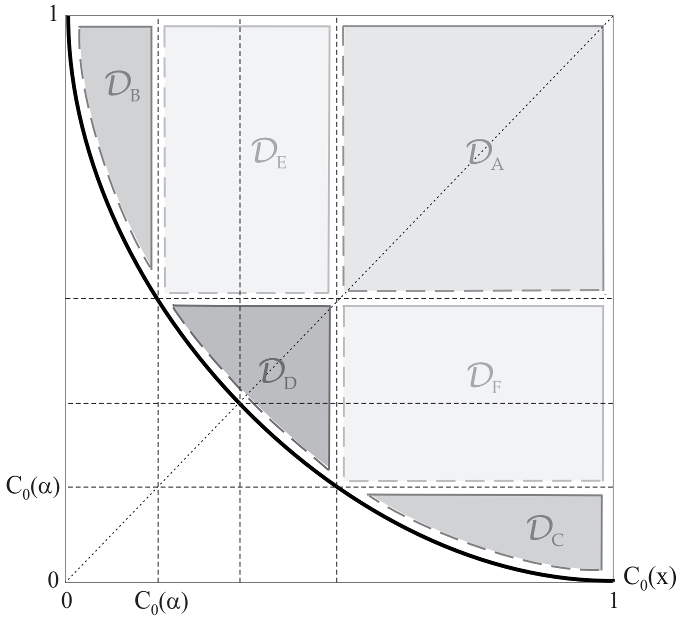


Figure 6.3: The partition $\mathcal{D} = \mathcal{D}_{\mathrm{A}} \cup \mathcal{D}_{\mathrm{B}} \cup \mathcal{D}_{\mathrm{C}} \cup \mathcal{D}_{\mathrm{D}} \cup \mathcal{D}_{\mathrm{E}} \cup \mathcal{D}_{\mathrm{F}}$.

Theorem 6.25 [45] *Consider a decomposable, rotation-invariant t-norm T. Let β be the fix-point of C_0 and take $\alpha \in D_T$. Consider an $[\alpha, 1] \to [0,1]$ isomorphism $\hat{\sigma}$ and, in case $\alpha \neq \beta$, a $[C_0(\alpha), \alpha] \to [0,1]$ isomorphism $\breve{\sigma}$. Then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_b$) and, in case $\alpha \neq \beta$, there also exists a left-continuous t-subnorm $\breve{T}$ that is rotation invariant w.r.t. $\breve{\sigma} \circ C_0 \circ \breve{\sigma}^{-1}$ such that*

$$T(x,y) = \begin{cases} \hat{\sigma}^{-1}\left[\widehat{T}(\hat{\sigma}[x], \hat{\sigma}[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{A}}\,, \\ C_0\left(\hat{\sigma}^{-1}\left[\widehat{C}_{\hat{\sigma}[C_0(x)]}(\hat{\sigma}[y])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{B}}\,, \\ C_0\left(\hat{\sigma}^{-1}\left[\widehat{C}_{\hat{\sigma}[C_0(y)]}(\hat{\sigma}[x])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{C}}\,, \\ \breve{\sigma}^{-1}\left[\breve{T}(\breve{\sigma}[x], \breve{\sigma}[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{D}}\,, \\ \min(x,y), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{E}} \cup \mathcal{D}_{\mathrm{F}}\,, \\ 0, & \text{if } (x,y) \notin \mathcal{D}\,. \end{cases} \tag{6.9}$$

In particular, $\widehat{T} = T^{\alpha}$, $\breve{T} = T^{(C_0(\alpha),\alpha)}$ (if $\alpha \neq \beta$) and the following properties hold:

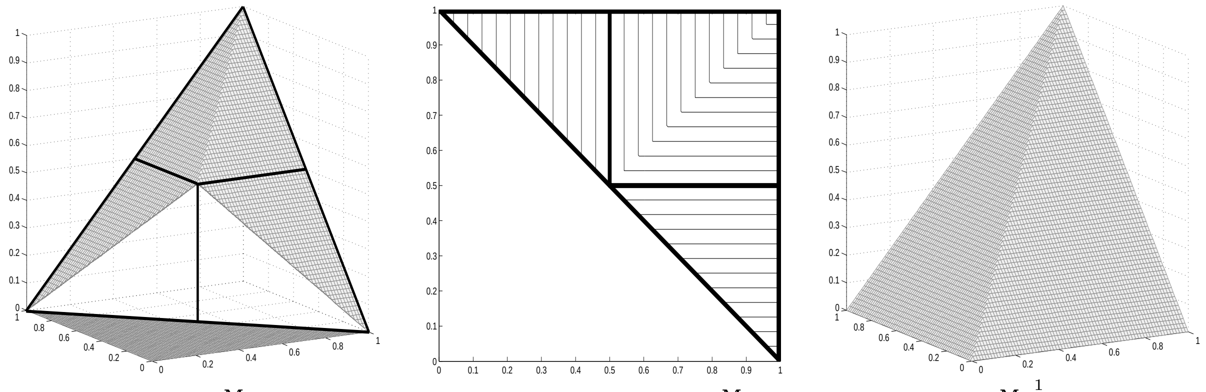
(I1) *If $\alpha = \beta$, then there exists $d \in [0,1[$ such that $\widehat{C}_0(x) = d$ whenever $x \in\,]0,d]$ and $\widehat{C}_0(x) = 0$ whenever $x \in\,]d,1]$.*

(I2) *If $\alpha \neq \beta$ and $\widehat{T}$ has zero divisors, then $\breve{T}$ is a t-norm.*

In this theorem we assume that $\hat{\sigma}$ is used to compute the α-zoom T^{α} of T and, in case $\alpha \neq \beta$, $\breve{\sigma}$ is used to compute the $(C_0(\alpha), \alpha)$-zoom $T^{(C_0(\alpha),\alpha)}$ of T. If D_T is not a singleton, multiple choices for α are possible. Jenei [45] talks about the *maximal decomposition* if the smallest decomposition point is considered. In this case area $\mathcal{D}_{\mathrm{A}}$ gets maximized. Jenei [45] has showed that the smallest decomposition point always exists (*i.e.* $\inf D_T \in D_T$). Note that Figure 6.4 depicts the maximal decomposition of the nilpotent minimum ($\alpha = \beta = \frac{1}{2}$).

Taking into account property **(H2)** it is not difficult to verify that Eq. (6.9) can never be satisfied for multiple left-continuous t-norms $\widehat{T}$ or multiple left-continuous t-subnorms $\breve{T}$ that are rotation invariant w.r.t. $\breve{\sigma} \circ C_0 \circ \breve{\sigma}^{-1}$. For $\widehat{T} = T^{\alpha}$ and $\breve{T} = T^{(C_0(\alpha),\alpha)}$ (if $\alpha \neq \beta$), Eq. (6.9) even easily follows from our previous results:

- $\mathbf{[0,1]} \setminus \boldsymbol{\mathcal{D}}$ **:** The definition of C_0 ensures that $T(x,y) = 0$ whenever $(x,y) \notin \mathcal{D}$.
- $\boldsymbol{\mathcal{D}}_{\mathbf{E}} \cup \boldsymbol{\mathcal{D}}_{\mathbf{F}}$ **:** From the discussion of Eq. (6.8) we know that $T(x,y) = \min(x,y)$, for every $(x,y) \in (\mathcal{D}_{\mathrm{E}} \cup \mathcal{D}_{\mathrm{F}}) \cap [\beta,1]^2$. Hence, $C_a(x) = a$, for every $(x,a) \in\,]\alpha,1] \times [\beta,\alpha[$. Invoking the involutivity of C_0 (assertion **(G2)**), for every $(x,y) \in \mathcal{D}_{\mathrm{E}} \setminus [\beta,1]^2$ it holds that $C_0(x) \in\,]\beta,\alpha[$ and $y \in\,]\alpha,1]$. Consequently, $C_{C_0(x)}(y) = C_0(x)$. Due to the involutivity of C_0 and assertion **(G3)** the latter can be rewritten as $T(x,y) = x = \min(x,y)$, for every $(x,y) \in \mathcal{D}_{\mathrm{E}} \setminus [\beta,1]^2$. As T is commutative and $(x,y) \in \mathcal{D}_{\mathrm{F}} \Leftrightarrow (y,x) \in \mathcal{D}_{\mathrm{E}}$, we finally conclude that $T(x,y) = \min(x,y)$, for every $(x,y) \in \mathcal{D}_{\mathrm{E}} \cup \mathcal{D}_{\mathrm{F}}$.
- $\boldsymbol{\mathcal{D}}_{\mathbf{A}}$ **:** From the preceding result it follows that $\alpha = T(x,\alpha) \leqslant T(x,y)$, for every $(x,y) \in \mathcal{D}_{\mathrm{A}}$, which leads to $T^{\alpha}(x,y) = \hat{\sigma}[\max(\alpha, T(\hat{\sigma}^{-1}[x], \hat{\sigma}^{-1}[y]))] = \hat{\sigma}[T(\hat{\sigma}^{-1}[x], \hat{\sigma}^{-1}[y])]$, for



(a) $T^{\mathbf{nM}}$ (b) Contour plot of $T^{\mathbf{nM}}$ (c) $(T^{\mathbf{nM}})^{\frac{1}{2}} = T_{\mathbf{M}}$

Figure 6.4: Decomposition of $T^{\mathbf{nM}}$.

every $(x,y) \in \,]0,1]^2$. Rewriting T in terms of T^α the latter yields $T|_{\mathcal{D}_\mathrm{A}}$. Corollary 6.15 states that T^α is a t-norm and property **(F3)** yields the left continuity of T^α.

- $\boldsymbol{\mathcal{D}_\mathbf{B} \cup \mathcal{D}_\mathbf{C}}$ **:** For every $(x,y) \in \mathcal{D}_\mathrm{B}$ it holds that $(C_0(x), y) \in [\alpha,1]^2$ and for every $(x,y) \in \mathcal{D}_\mathrm{C}$ it holds that $(C_0(y), x) \in [\alpha,1]^2$. Since $C_b(z) = \hat{\sigma}^{-1}[C^\alpha_{\hat{\sigma}[b]}(\hat{\sigma}[z])]$, for every $(z,b) \in [\alpha,1]^2$ (property **(F4a)**), we obtain that $C_{C_0(x)}(y) = \hat{\sigma}^{-1}[C^\alpha_{\hat{\sigma}[C_0(x)]}(\hat{\sigma}[y])]$ whenever $(x,y) \in \mathcal{D}_\mathrm{B}$ and $C_{C_0(y)}(x) = \hat{\sigma}^{-1}[C^\alpha_{\hat{\sigma}[C_0(y)]}(\hat{\sigma}[x])]$ whenever $(x,y) \in \mathcal{D}_\mathrm{C}$. Using assertion **(G3)** and the commutativity of T, this allows us to express $T|_{\mathcal{D}_\mathrm{B} \cup \mathcal{D}_\mathrm{C}}$ in terms of the contour lines C^α_b of T^α.
- $\boldsymbol{\mathcal{D}_\mathbf{D}}$ **:** Further, if $\alpha \neq \beta$ we know from the definition of D_T and property **(E2)** that $C_0(\alpha) \leqslant Q(x, C_0(x)) \leqslant T(x,t)$, for every $(x,t) \in [\beta,\alpha] \times \,]C_0(x),\alpha]$. For every $(t,x) \in \,]C_0(\alpha),\beta[\, \times \,]C_0(t),\alpha]$ it holds that $(x,t) \in \,]\beta,\alpha] \times \,]C_0(x),\beta[$ due to the involutivity of C_0. Invoking the commutativity of F, we obtain that $C_0(\alpha) \leqslant T(t,x)$. Hence, $C_0(\alpha) \leqslant T(x,y)$, for every $(x,y) \in \mathcal{D}_\mathrm{D}$, which leads to $T^{(C_0(\alpha),\alpha)}(x,y) = \breve{\sigma}[T(\breve{\sigma}^{-1}[x], \breve{\sigma}^{-1}[y])]$, for every $(x,y) \in \,]0,1]^2$ satisfying $C_0(\breve{\sigma}^{-1}[x]) < \breve{\sigma}^{-1}[y]$. Rewriting T in terms of $T^{(C_0(\alpha),\alpha)}$ yields $T|_{\mathcal{D}_\mathrm{D}}$. Recall that $T^{(C_0(\alpha),\alpha)}$ must be a left-continuous t-subnorm (property **(F3)**, Corollary 6.15 and property **(F3)**). Moreover, as T is rotation invariant w.r.t. its contour line C_0 (property **(H1)**), C_0 is involutive, $T^{(C_0(\alpha),\alpha)}(x,y) = 0$ whenever $\breve{\sigma}^{-1}[y] \leqslant C_0(\breve{\sigma}^{-1}[x])$ and $T^{(C_0(\alpha),\alpha)}(x,y) = \breve{\sigma}[T(\breve{\sigma}^{-1}[x], \breve{\sigma}^{-1}[y])]$ whenever $C_0(\breve{\sigma}^{-1}[x]) < \breve{\sigma}^{-1}[y]$, it is a trivial exercise to verify that $T^{(C_0(\alpha),\alpha)}$ is rotation invariant w.r.t. $\breve{\sigma} \circ C_0 \circ \breve{\sigma}^{-1}$ (Definition 6.21).

B. Construction

Jenei [42, 43, 46, 47] also used Eq. (6.9) to construct rotation-invariant t-norms. In our framework, to ensure that the $[0,1]^2 \to [0,1]$ function T given by Eq. (6.9) is well defined, we need to assume the following setting:

- $\widehat{T}$: an arbitrary left-continuous t-norm (with contour lines $\widehat{C}_b$).

- $\breve{T}$: an arbitrary left-continuous t-subnorm.
- C_0: an arbitrary involutive negator with fixpoint β.
- α: an arbitrary number in $[\beta, 1[$.
- $\hat{\sigma}$: an arbitrary $[\alpha, 1] \to [0, 1]$ isomorphism.
- $\breve{\sigma}$: an arbitrary $[C_0(\alpha), \alpha] \to [0, 1]$ isomorphism (if $\alpha \neq \beta$).

Theorem 6.26 [42, 43] *If, in case $\alpha \neq \beta$, $\breve{T}$ is rotation invariant w.r.t. $\breve{\sigma} \circ C_0 \circ \breve{\sigma}^{-1}$, then the $[0,1]^2 \to [0,1]$ function T defined by Eq.* (6.9) *is a rotation-invariant t-norm if and only if $\widehat{T}$ and $\breve{T}$ satisfy properties* **(I1)** *and* **(I2)**. *In this case $\alpha \in D_T$, $T^{\alpha} = \widehat{T}$ and $T^{(C_0(\alpha),\alpha)} = \breve{T}$ (if $\alpha \neq \beta$).*

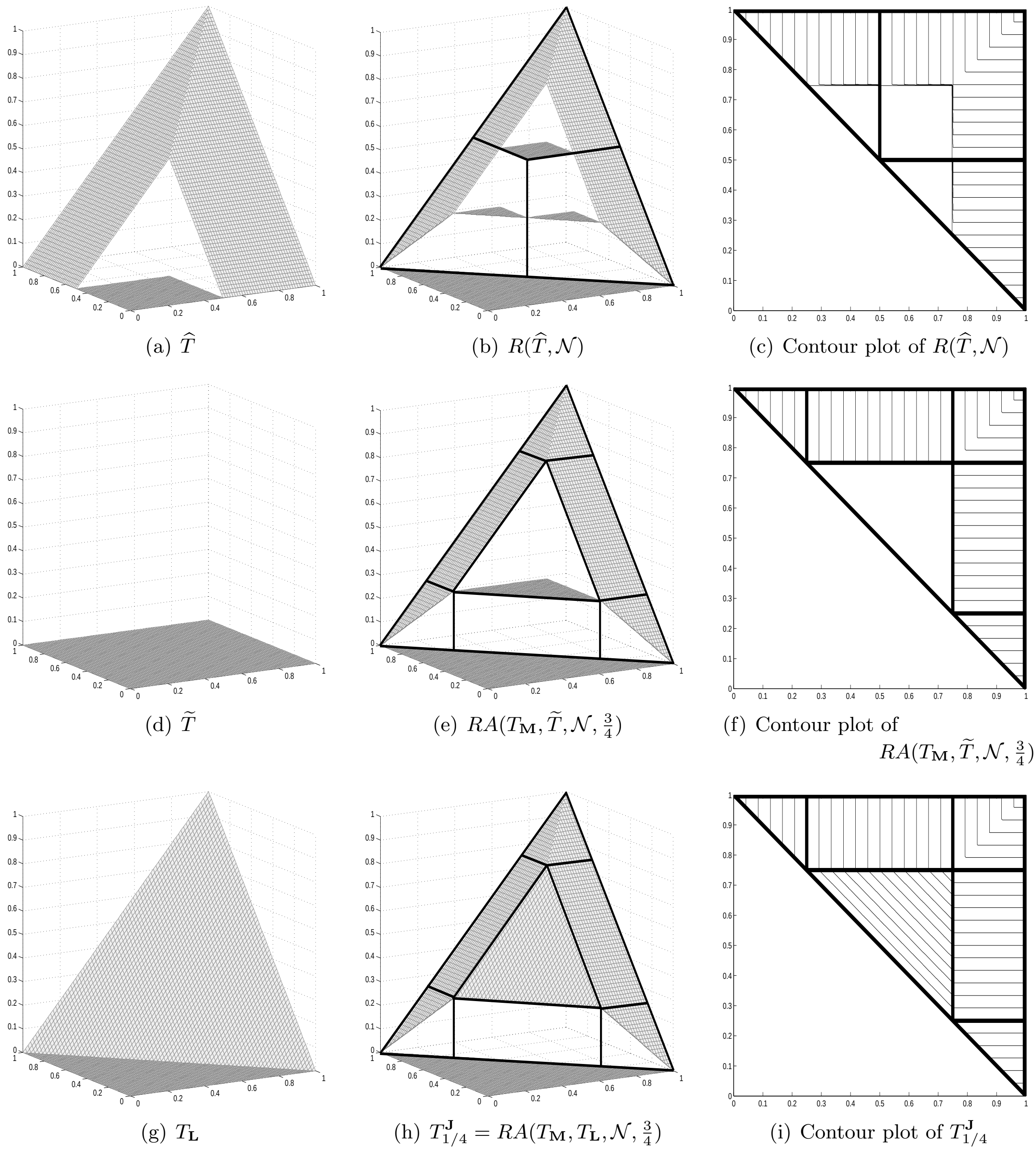
Note that Eq. (6.9) does not require a t-subnorm $\breve{T}$ whenever $\alpha = \beta$. Jenei [42, 46, 47] then uses the term *rotation construction* to denote the method described by Theorem 6.26. If $\alpha \neq \beta$ he refers to it as the *rotation-annihilation construction* [43, 46, 47]. As will become clear from the following example, for this naming Jenei has been inspired by the geometrical interpretation of both constructions. We will briefly use $R(\widehat{T}, C_0)$ to denote a t-norm obtained by the rotation construction and $RA(\widehat{T}, \breve{T}, C_0, \alpha)$ to denote a t-norm obtained by the rotation-annihilation construction.

Example 6.27 Figure 6.5 illustrates both the rotation and rotation-annihilation construction. The bold black lines visualize the partition $\mathcal{D} = \mathcal{D}_{\mathrm{A}} \cup \mathcal{D}_{\mathrm{B}} \cup \mathcal{D}_{\mathrm{C}} \cup \mathcal{D}_{\mathrm{D}} \cup \mathcal{D}_{\mathrm{E}} \cup \mathcal{D}_{\mathrm{F}}$. As in our previous examples we only work with linear rescaling functions.

The increasing $[0,1]^2 \to [0,1]$ function $\widehat{T}$ in Fig. 6.1(a) is obtained from $T_{\mathbf{M}}$ by lowering its values on $[0, \frac{1}{2}]^2$ to zero. It is easily verified that $\widehat{T}$ is a t-norm satisfying property **(I1)** (with $d = \frac{1}{2}$). Theorem 6.26 now ensures that $R(\widehat{T}, \mathcal{N})$ is also a t-norm. $R(\widehat{T}, \mathcal{N})|_{\mathcal{D}_{\mathrm{B}}}$ is geometrically obtained by rotating $R(\widehat{T}, \mathcal{N})|_{\mathcal{D}_{\mathrm{A}}}$ 120 degrees to the left around the axis $\{(x, y, z) \in [0, 1]^2 \mid y = x \ \wedge \ z = 1 - x\}$. Similarly, rotating $R(\widehat{T}, \mathcal{N})|_{\mathcal{D}_{\mathrm{A}}}$ 120 degrees to the right around this axis yields $R(\widehat{T}, \mathcal{N})|_{\mathcal{D}_{\mathrm{C}}}$.

The two t-subnorms in Figs. 6.5(d) and 6.5(g) are obviously rotation invariant w.r.t. $\mathcal{N}$. The minimum operator $T_{\mathbf{M}}$ has no zero-divisors and, hence, applying Theorem 6.26 (with $\widehat{T} = T_{\mathbf{M}}$, $C_0 = \mathcal{N}$ and $\alpha = \frac{3}{4}$) yields the t-norms in Figs. 6.5(e) and 6.5(h). Denote $T := RA(T_{\mathbf{M}}, \widetilde{T}, \mathcal{N}, \frac{3}{4})$. Geometrically, $T|_{\mathcal{D}_{\mathrm{D}}}$ is a rescaled and annihilated version of $\widetilde{T}$. In this context *annihilation* means that some parts of the graph of $\widetilde{T}$ have been lowered. It strikes that $T|_{\mathcal{D}_{\mathrm{B}}}$ and $T|_{\mathcal{D}_{\mathrm{C}}}$ are again determined by, resp., the left and right rotation of $T|_{\mathcal{D}_{\mathrm{A}}}$. Also $RA(T_{\mathbf{M}}, T_{\mathbf{L}}, \mathcal{N}, \frac{3}{4})$ is obtained by performing a double rotation and an annihilation on rescalings of, resp., $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$. The t-norm $RA(T_{\mathbf{M}}, T_{\mathbf{L}}, \mathcal{N}, \frac{3}{4})$ is a member of the Jenei t-norm family $(T^{\mathbf{J}}_{\lambda})_{\lambda \in [0,1/2]}$,

$$T^{\mathbf{J}}_{\lambda}(x, y) = \begin{cases} 0, & \text{if } x + y \leqslant 1\,, \\ x + y - 1 + \lambda, & \text{if } x + y > 1 \text{ and } (x, y) \in \,]\lambda, 1 - \lambda]^2\,, \\ \min(x, y), & \text{elsewhere}\,. \end{cases} \tag{6.10}$$



(a) $\widehat{T}$ (b) $R(\widehat{T},\mathcal{N})$ (c) Contour plot of $R(\widehat{T},\mathcal{N})$

(d) $\widetilde{T}$ (e) $RA(T_{\mathbf{M}},\widetilde{T},\mathcal{N},\frac{3}{4})$ (f) Contour plot of $RA(T_{\mathbf{M}},\widetilde{T},\mathcal{N},\frac{3}{4})$

(g) $T_{\mathbf{L}}$ (h) $T^{\mathbf{J}}_{1/4} = RA(T_{\mathbf{M}},T_{\mathbf{L}},\mathcal{N},\frac{3}{4})$ (i) Contour plot of $T^{\mathbf{J}}_{1/4}$

Figure 6.5: The rotation and rotation-annihilation construction.

This family consists of rotation-invariant t-norms, obtained by applying the rotation-annihilation construction on $T_{\mathbf{M}}$ (rotation) and $T_{\mathbf{L}}$ (annihilation) [41]. Note that $T_0^{\mathbf{J}} = T_{\mathbf{L}}$ and $T_{1/2}^{\mathbf{J}} = T^{\mathbf{nM}}$. $\triangle$

In general, if $C_0 = \mathcal{N}$, the graphs of the t-norms obtained by either the rotation construction or the rotation-annihilation construction remain in some sense invariant under a left and right rotation. In case C_0 differs from the standard negator some reshaping may occur during the rotation process (see Section 8.3).

CHAPTER 7

Decomposing rotation-invariant t-norms

7.1 Introduction

Despite all efforts, the class of rotation-invariant t-norms is not yet fully understood. The decomposition method presented by Jenei [45] only acts on decomposable, rotation-invariant t-norms. Being non-decomposable, the Łukasiewicz t-norm $T_{\mathbf{L}}$ falls outside this setting. Furthermore, there are as many decompositions of a decomposable, rotation-invariant t-norm T as there are decomposition points. To obtain a standard method, Jenei [45] uses the smallest decomposition point α for decomposing T. If T is decomposable but not totally decomposable, Jenei [45] expresses T in terms of its contour line C_0, and two zooms: T^α and $T^{(C_0(\alpha),\alpha)}$. The t-subnorm $T^{(C_0(\alpha),\alpha)}$ is rotation invariant w.r.t. a rescaled part of C_0. In contrast to t-norms, there is, however, little information concerning rotation-invariant t-subnorms. Finally, we would like to point out once again that we rephrased Jenei's results (Section 6.5) into our framework based on contour lines, the companion and zooms. The original formulation of his work is far more complex.

We present a more natural procedure for decomposing a given rotation-invariant t-norm T. Except for a single rescaling function, the new method does not permit any degree of freedom and rewrites T in terms of its contour line C_0 and β-zoom T^β, with β the unique fixpoint of C_0. Depending on the structure of the contour line C_0^β of T^β, the method allows either a full or a partial decomposition of T. Our approach is capable of decomposing the Łukasiewicz t-norm $T_{\mathbf{L}}$ and provides also a better insight into the structure of the algebraic product $T_{\mathbf{P}}$. Throughout the reasonings, we assume full familiarity with the results from Section 4.2.

7.2 Decomposition revisited

Let T be a rotation-invariant t-norm and β be the unique fixpoint of C_0. First, we repartition the area $\mathcal{D} = \{(x,y) \in [0,1]^2 \mid C_0(x) < y\}$ into four parts as pictured in Figure 7.1:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &:= \{(x,y) \in \,]\beta,1]^2 \mid C_\beta(x) < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &:= \{(x,y) \in \,]0,\beta] \times]\beta,1] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &:= \{(x,y) \in \,]\beta,1] \times]0,\beta] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &:= \{(x,y) \in \,]\beta,1[^2 \mid y \leqslant C_\beta(x)\}\,.
\end{aligned}$$

In contrast to Fig. 6.3 this partition exists for any rotation-invariant t-norm and not just for the decomposable ones. Furthermore, it allows no degree of freedom. Besides the contour line C_0, it is based on the (fixed) contour line C_β and not on an arbitrary decomposition point $\alpha \in D_T$. Due to the left continuity of T it is obvious that $T(x,y) = 0$ holds for every $(x,y) \notin \mathcal{D}$. As will become clear, area $\mathcal{D}_{\mathrm{I}}$ is crucial in the decomposition of rotation-invariant t-norms.

Theorem 7.1 *Consider a rotation-invariant t-norm T. Let σ be an arbitrary $[\beta,1] \to [0,1]$ isomorphism with β the fixpoint of C_0. Then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$) such that*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,,\\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,,\\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,,\\ 0, & \text{if } (x,y) \notin \mathcal{D}\,. \end{cases} \tag{7.1}$$

In particular, $\widehat{T} = T^\beta$.

Proof First we prove that Eq. (7.1) holds for $\widehat{T} = T^\beta$. By definition, $\sigma^{-1}[T^\beta(\sigma[x],\sigma[y])] = \max(\beta, T(x,y))$, for every $(x,y) \in [\beta,1]^2$. If in particular $(x,y) \in \mathcal{D}_{\mathrm{I}}$, then $\beta < T(x,y)$ which leads to $T(x,y) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[y])]$. The rotation invariance of T implies that $T(x,y) = T(y,x) = C_0(C_{C_0(x)}(y)) = C_0(C_{C_0(y)}(x))$, for every $(x,y) \in \mathcal{D}$ (assertion **(G3)**). To obtain Eq. (7.1) for $\widehat{T} = T^\beta$, it suffices to recall that $C_b(z) = \sigma^{-1}[C^\beta_{\sigma[b]}(\sigma[z])]$, for every $(z,b) \in [\beta,1]^2$ (property **(F4a)**). Note that $(C_0(x),y) \in [\beta,1]^2$ whenever $(x,y) \in \mathcal{D}_{\mathrm{II}}$ and that $(C_0(y),x) \in [\beta,1]^2$ whenever $(x,y) \in \mathcal{D}_{\mathrm{III}}$. The last case, $T(x,y) = 0$, for $(x,y) \notin \mathcal{D}$, is trivially fulfilled.

We now need to prove that every left-continuous t-norm $\widehat{T}$ satisfying Eq. (7.1) must equal T^β. For every $(x,y) \in \mathcal{D}_{\mathrm{I}}$ it holds that $T(x,y) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[y])] = \sigma^{-1}[\widehat{T}(\sigma[x],\sigma[y])]$. Invoking property **(F4a)** and denoting $\sigma[x]$ and $\sigma[y]$ by, resp., u and v, this implies that $\widehat{T}(u,v) = T^\beta(u,v)$, for every $(u,v) \in \,]0,1]^2$ such that $C_0^\beta(u) < v$. Moreover, it follows from property **(F4a)**, Eqs. (5.3) and (7.1) and the involutivity of C_0 (assertion **(G2)**) that $\sigma^{-1}[C_0^\beta(\sigma[x])] = C_\beta(x) = C_{C_0(\beta)}(x) =$

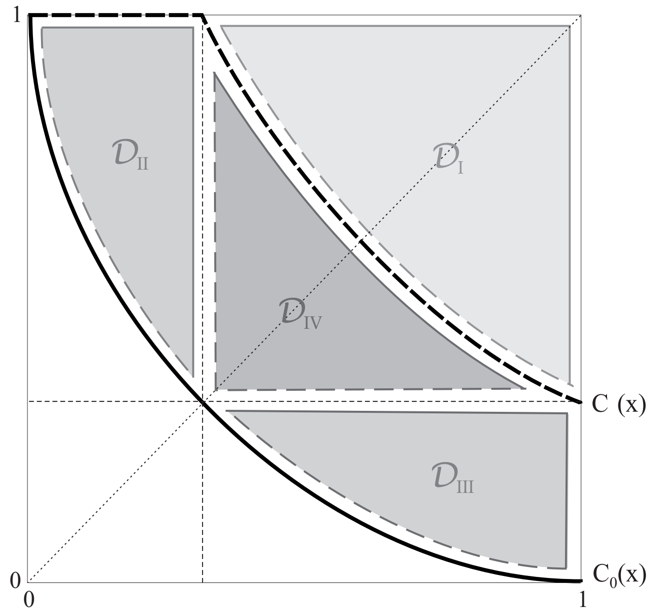
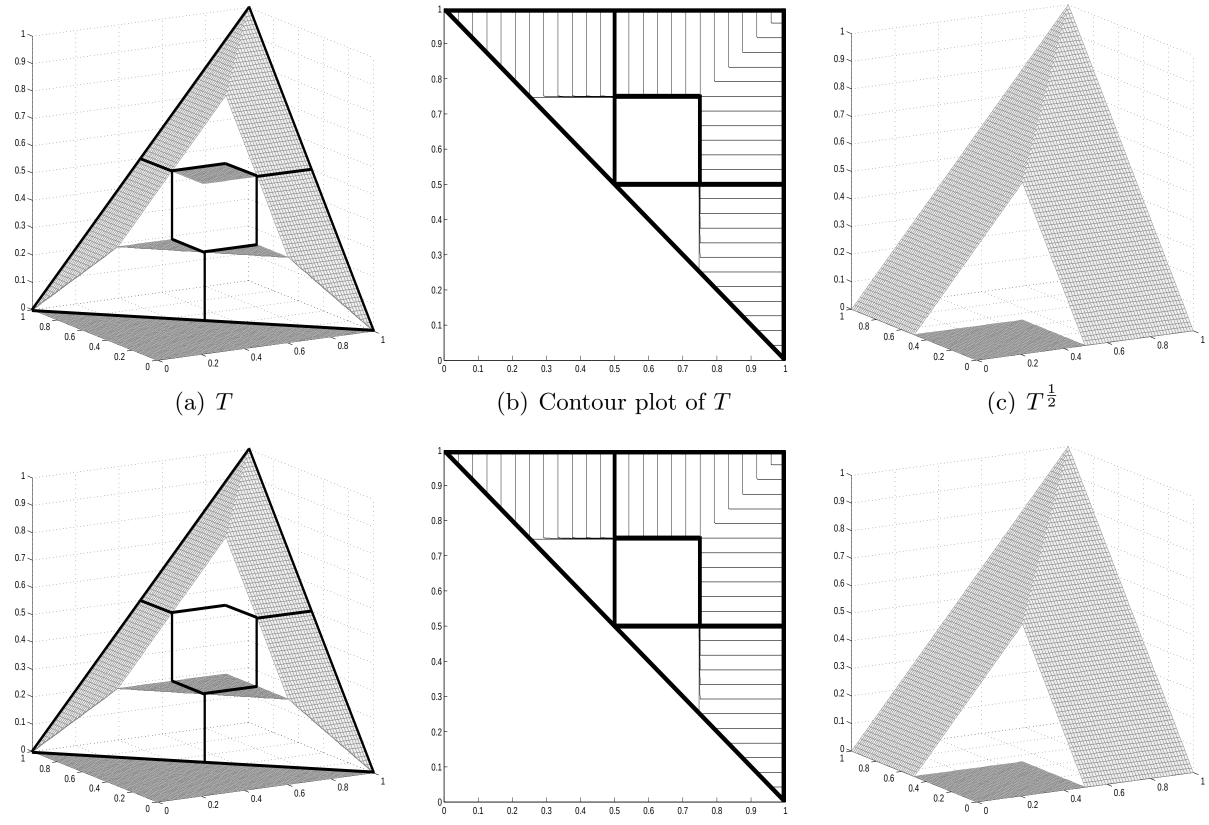


Figure 7.1: The partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$.

$C_0(T(\beta, x)) = \sigma^{-1}[\widehat{C}_0(\sigma[x])$, for every $x \in \,]\beta, 1]$. Denoting $\sigma[x]$ by u yields that $\widehat{C}_0(u) = C_0^{\beta}(u)$, for every $u \in \,]0, 1]$. By definition, we obtain that $\widehat{T}(u, v) = T^{\beta}(u, v) = 0$, for every $(u, v) \in \,]0, 1]^2$ such that $v \leqslant C_0^{\beta}(u)$. Taking into account that $\widehat{T}(u, v) = T^{\beta}(u, v) = 0$ always holds whenever $u = 0$ or $v = 0$, we conclude that $\widehat{T} = T^{\beta}$. $\square$

$T|_{\mathcal{D}_{\mathrm{I}}}$ is a rescaled version of $T^{\beta}|_{\mathcal{D}^{\beta}}$, where $\mathcal{D}^{\beta} = \{(x, y) \in [0, 1]^2 \mid C_0^{\beta}(x) < y\}$. Once $T|_{\mathcal{D}_{\mathrm{I}}}$ is known, it can be used to construct $T|_{\mathcal{D}_{\mathrm{II}}}$, which in turn can be used to construct $T|_{\mathcal{D}_{\mathrm{III}}}$. As can be seen from the proof of the Theorem 7.1, we use σ to compute the β-zoom T^{β} of T.

Example 7.2 Figure 7.2 depicts our decomposition of two different rotation-invariant t-norms T and $\breve{T}$. Both t-norms are constructed from the nilpotent minimum $T^{\mathbf{nM}}$ by lowering its values on the area $\mathcal{E} = \{(x, y) \in [0, 1]^2 \mid 1 < x + y \,\wedge\, \max(x, y) \leqslant \frac{3}{4}\}$. In particular, for every $(x, y) \in \mathcal{E}$ it holds that $T(x, y) = \frac{1}{2}$ whenever $\min(x, y) > \frac{1}{2}$ and $T(x, y) = \frac{1}{4}$ elsewhere. For $\breve{T}$ it holds that $\breve{T}(x, y) = \frac{1}{4}$, for every $(x, y) \in \mathcal{E}$. The bold black lines in the figures indicate the partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$. T and $\breve{T}$ satisfy $C_0 = \breve{C}_0 = \mathcal{N}$, $\beta = \frac{1}{2}$ and $T^{\frac{1}{2}} = \breve{T}^{\frac{1}{2}}$ ($\sigma = \breve{\sigma} = \varsigma$, with ς the linear rescaling function). For such a pair of t-norms we know from Theorem 7.1 that $T|_{\mathcal{D}_{\mathrm{I}}} = \breve{T}|_{\mathcal{D}_{\mathrm{I}}}$, $T|_{\mathcal{D}_{\mathrm{II}}} = \breve{T}|_{\mathcal{D}_{\mathrm{II}}}$ and $T|_{\mathcal{D}_{\mathrm{III}}} = \breve{T}|_{\mathcal{D}_{\mathrm{III}}}$. In Examples 6.27 and Fig. 6.5 we have illustrated that T can be constructed by the rotation construction of Jenei and $\breve{T}$ by means of his rotation-annihilation construction. Consequently, these t-norms are also decomposable (in the sense of Jenei [45]): $D_T = \{\frac{1}{2}\} \cup [\frac{3}{4}, 1[$ and $D_{\breve{T}} = [\frac{3}{4}, 1[$. $\triangle$



(a) T (b) Contour plot of T (c) $T^{\frac{1}{2}}$

(d) $\breve{T}$ (e) Contour plot of $\breve{T}$ (f) $\breve{T}^{\frac{1}{2}}$

Figure 7.2: Decomposition of two left-continuous t-norms T and $\breve{T}$ for which $T^{\frac{1}{2}} = \breve{T}^{\frac{1}{2}}$.

Geometrically, $T|_{\mathcal{D}_{\mathrm{II}}}$ is determined by rotating $T|_{\mathcal{D}_{\mathrm{I}}}$ 120 degrees to the left around the axis $\{(x,y,z) \in [0,1]^3 \mid y = x \,\wedge\, z = 1-x\}$. Similarly, rotating $T|_{\mathcal{D}_{\mathrm{I}}}$ 120 degrees to the right around this axis determines $T|_{\mathcal{D}_{\mathrm{III}}}$. As will be illustrated in Fig. 7.3, these rotations sometimes have to be reshaped to fit into the areas $\mathcal{D}_{\mathrm{II}}$ and $\mathcal{D}_{\mathrm{III}}$, respectively. The contour lines C_0 and C_β are responsible for this reshaping.

If T^β has no zero divisors, then area $\mathcal{D}_{\mathrm{IV}}$ is empty and Eq. (7.1) totally determines T. Since, in this case $C_0^\beta(x) = 0$, for every $x \in \,]0,1]$, it holds that $T(\beta, x) = C_0(\sigma^{-1}[C_0^\beta(\sigma[x])]) = \beta$, for every $x \in \,]\beta, 1]$. Taking the limit $x \searrow \beta$, we obtain that $Q(\beta,\beta) = \beta$ (property **(E2)**). The t-norm T must be totally decomposable (in the sense of Jenei [45]) and Eq. (7.1) coincides with Eq. (6.9) ($\alpha = \beta$).

7.3 Two continuous contour lines

For a rotation-invariant t-norm T, Theorem 7.1 expresses how its contour line C_0 together with its associated t-norm T^β totally fixes $T|_{\mathcal{D}_{\mathrm{I}}}$, $T|_{\mathcal{D}_{\mathrm{II}}}$ and $T|_{\mathcal{D}_{\mathrm{III}}}$. As illustrated in Fig. 7.2, $T|_{\mathcal{D}_{\mathrm{IV}}}$ is in general not uniquely determined by C_0 and T^β. Examining numerous examples, we noticed that the filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is uniquely fixed whenever besides C_0 also C_β is continuous. The following property gives a first clue how to express $T|_{\mathcal{D}_{\mathrm{IV}}}$ in terms of C_0 and T^β.

Property 7.3 *Consider a rotation-invariant t-norm T for which C_β is continuous, with β the unique fixpoint of C_0. Then*

$$C_0(T(x,y)) < T(C_\beta(x-\varepsilon), C_\beta(y-\varepsilon)) \tag{7.2}$$

holds for every $(x,y) \in \,]\beta,1]^2$ and every $\varepsilon \in \,]0, \min(x-\beta, y-\beta)]$.

Proof Suppose that the converse holds: $T(C_\beta(x-\varepsilon), C_\beta(y-\varepsilon)) \leqslant C_0(T(x,y))$, for some $(x,y) \in \,]\beta,1]^2$ and some $\varepsilon \in \,]0, \min(x-\beta, y-\beta)]$. The latter implies that

$$T\big(C_\beta(x-\varepsilon), T(x, T(y, C_\beta(y-\varepsilon)))\big) = T\big(T(C_\beta(x-\varepsilon), C_\beta(y-\varepsilon)), T(x,y)\big) = 0\,.$$

From the involutivity of C_0 on $[0,1]$ and the involutivity of C_β on $[\beta,1]$ (assertion **(G2)**), we know that $C_0(C_\beta(x-\varepsilon)) < C_0(C_\beta(x))$. Assertion **(G3)** yields that $C_0(C_\beta(x)) = C_0(C_{C_0(\beta)}(x)) = T(\beta,x) = T(x,\beta)$. Therefore,

$$T(x, T(y, C_\beta(y-\varepsilon))) \leqslant C_0(C_\beta(x-\varepsilon)) < C_0(C_\beta(x)) = T(x,\beta)\,.$$

This inequality can only hold when $T(y, C_\beta(y-\varepsilon)) < \beta$. We obtain the contradiction $C_\beta(y-\varepsilon) \leqslant C_\beta(y)$. $\square$

If a rotation-invariant t-norm T has a continuous contour line C_β, then the involutivity of C_β on $[\beta,1]$ (assertion **(G2)**) ensures that

$$\begin{aligned}
\{(C_\beta(x), C_\beta(y)) \mid (x,y) \in \mathcal{D}_{\mathrm{IV}}\} &= \{(C_\beta(x), C_\beta(y)) \mid (x,y) \in \,]\beta,1[^2 \;\wedge\; y \leqslant C_\beta(x)\} \\
&= \{(C_\beta(x), C_\beta(y)) \mid (x,y) \in \,]\beta,1[^2 \;\wedge\; x \leqslant C_\beta(y)\} \\
&= \{(u,v) \mid (C_\beta(u), C_\beta(v)) \in \,]\beta,1[^2 \;\wedge\; C_\beta(u) \leqslant v\} \\
&= \{(u,v) \in \,]\beta,1[^2 \mid C_\beta(u) \leqslant v\}\,.
\end{aligned} \tag{7.3}$$

Denoting $\{(x,y) \in \,]\beta,1[^2 \mid C_\beta(x) \leqslant y\}$ as $\overline{\mathcal{D}}_{\mathrm{I}}$, Eq. (7.2) establishes for $(x,y) \in \mathcal{D}_{\mathrm{IV}}$ and $\varepsilon \in \,]0, \min(x-\beta, y-\beta)[$ a link between $T|_{\mathcal{D}_{\mathrm{IV}}}$ and $T|_{\overline{\mathcal{D}}_{\mathrm{I}}}$. In the following theorem we will refine this connection and express $T|_{\mathcal{D}_{\mathrm{IV}}}$ in terms of $Q|_{\overline{\mathcal{D}}_{\mathrm{I}}}$.

Theorem 7.4 *Consider a rotation-invariant t-norm T for which C_β is continuous, with β the unique fixpoint of C_0. Then $T(x,y) = C_0(Q(C_\beta(x), C_\beta(y)))$ holds for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$.*

Proof As C_0 is involutive, showing that $Q(C_\beta(x), C_\beta(y)) = C_0(T(x,y))$ holds for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$, will prove the theorem. Throughout the proof we will make extensive use of the involutivity of C_0 and C_β (assertion **(G2)**). Also, $C_0(C_\beta(x)) = T(x,\beta)$, for every $x \in]\beta, 1]$, (assertion **(G3)**), will be frequently used. Furthermore, it should be noted that the orthosymmetry of C_β (Theorem 5.9) ensures that $(x,y) \in \mathcal{D}_{\mathrm{IV}} \Leftrightarrow (y,x) \in \mathcal{D}_{\mathrm{IV}}$. We distinguish four subproblems.

I. $\boldsymbol{Q(C_\beta(x), C_\beta(y)) \leqslant C_0(T(x,y))}$, for every $\boldsymbol{(x,y) \in \mathcal{D}_{\mathrm{IV}}}$

By definition $\beta \leqslant Q(x, C_\beta(x))$, for every $x \in [0,1]$. Take $(x,y) \in \mathcal{D}_{\mathrm{IV}}$, then $T(x,y) \leqslant \beta$ and $\beta \leqslant Q(y, C_\beta(y)) \leqslant Q(C_\beta(x), C_\beta(y))$ (property **(E1)**). If $Q(C_\beta(x), C_\beta(y)) = \beta$, it holds that $Q(C_\beta(x), C_\beta(y)) = \beta = C_0(\beta) \leqslant C_0(T(x,y))$. Suppose now that $\beta < Q(C_\beta(x), C_\beta(y))$. Applying assertion **(G5)** to

$$T\big(C_\beta(x), C_0(C_0(T(x,y)))\big) = T(T(y,x), C_\beta(x)) = T(y, T(x, C_\beta(x))) \leqslant T(y,\beta) = C_0(C_\beta(y))$$

leads to $T(C_\beta(x), C_\beta(y)) \leqslant C_0(T(x,y))$. From property **(E3)** we know that $T(C_\beta(x), C_\beta(y)) \leqslant Q(C_\beta(x), C_\beta(y))$. In case $T(C_\beta(x), C_\beta(y)) = Q(C_\beta(x), C_\beta(y))$ we immediately obtain that $Q(C_\beta(x), C_\beta(y)) \leqslant C_0(T(x,y))$. If $T(C_\beta(x), C_\beta(y)) < Q(C_\beta(x), C_\beta(y))$, the definition of Q ensures that $C_k(C_\beta(x)) = C_\beta(y)$ holds for every $k \in [\max(\beta, T(C_\beta(x), C_\beta(y))), Q(C_\beta(x), C_\beta(y))[$. As $Q(C_\beta(x), C_\beta(y)) \leqslant C_\beta(x)$ (property **(E5)**), such a k fulfills $k < C_\beta(x)$. From assertion **(G3)** we obtain that

$$T(C_\beta(k), C_\beta(x)) = C_\beta\big(C_{C_\beta(C_\beta(k))}(C_\beta(x))\big) = C_\beta(C_k(C_\beta(x))) = C_\beta(C_\beta(y)) = y\,.$$

Note that we could invoke the involutivity of C_β as $\beta < y$ and $\beta \leqslant k$. The above equalities allow to bound $T(y,k)$ from above:

$$\begin{aligned} T(y,k) &= T\big(T(C_\beta(k), C_\beta(x)), k\big) = T\big(C_\beta(x), T(C_\beta(k), k)\big) \\ &\leqslant T(C_\beta(x), \beta) = C_0(C_\beta(C_\beta(x))) = C_0(x)\,. \end{aligned}$$

Taking the limit $k \nearrow Q(C_\beta(x), C_\beta(y))$ leads to $T(y, Q(C_\beta(x), C_\beta(y))) \leqslant C_0(x)$. It now suffices to apply assertion **(G5)** a second time to conclude that $T(x,y) \leqslant C_0(Q(C_\beta(x), C_\beta(y)))$.

II. $\boldsymbol{Q(x, C_\beta(x)) = C_0(T(x, C_\beta(x)))}$, for every $\boldsymbol{x \in]\beta, 1[}$

For every $x \in]\beta, 1[$ it holds that $(C_\beta(x), x) \in \mathcal{D}_{\mathrm{IV}}$. Invoking Part **I** we know that $Q(x, C_\beta(x)) \leqslant C_0(T(C_\beta(x), x)) = C_0(T(x, C_\beta(x)))$. Suppose that $Q(x, C_\beta(x)) < C_0(T(x, C_\beta(x)))$, for some $x \in]\beta, 1[$. In this case property (**E2**) ensures that there exists $a \in]C_\beta(x), 1]$ such that $T(x,a) \in [Q(x, C_\beta(x)), C_0(T(x, C_\beta(x)))[$. Taking into account that $T(x, C_\beta(x)) \leqslant \beta < x$, it follows from assertion (**G3**) that $T(C_0(T(x, C_\beta(x))), x) = C_0(C_{T(x, C_\beta(x))}(x))$. Combining $C_\beta(x) \leqslant C_{T(x, C_\beta(x))}(x)$ (definition of contour lines) and $T(x, C_\beta(x)) \leqslant \beta$, we know that $C_{T(x, C_\beta(x))}(x) = C_\beta(x)$ and, hence, $T(C_0(T(x, C_\beta(x))), x) = C_0(C_\beta(x)) = T(x,\beta)$. Consequently,

$$T\big(T(x,a), C_0(T(x, C_\beta(x)))\big) = T\big(a, T(x, C_0(T(x, C_\beta(x))))\big) = T(a, T(x,\beta)) = T(T(x,a),\beta)\,.$$

Because $\beta \leqslant Q(x, C_\beta(x)) \leqslant T(x,a) < C_0(T(x, C_\beta(x)))$, it holds that

$$T(T(x,a),\beta) \leqslant T\big(C_0(T(x,C_\beta(x))),\beta\big) \leqslant T\big(C_0(T(x,C_\beta(x))),T(x,a)\big) = T(T(x,a),\beta)\,.$$

We conclude that $T(T(x,a),\beta) = T(C_0(T(x,C_\beta(x))),\beta)$. This equality can be rewritten as $C_0(C_\beta(T(x,a))) = C_0(C_\beta(C_0(T(x,C_\beta(x)))))$. As C_0 is involutive on $[0,1]$ and C_β is involutive on $[\beta,1]$ this leads to the contradiction $T(x,a) = C_0(T(x,C_\beta(x)))$. Therefore, $Q(x,C_\beta(x)) = C_0(T(x,C_\beta(x)))$ holds for every $x \in]\beta,1[$.

III. Q is commutative on $\overline{\mathcal{D}}_{\mathrm{I}}$

Reconsidering Eq. (7.3), we must prove that $Q(C_\beta(x),C_\beta(y)) = Q(C_\beta(y),C_\beta(x))$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. Take $(x,y) \in \mathcal{D}_{\mathrm{IV}}$ and suppose that $Q(C_\beta(x),C_\beta(y)) \neq Q(C_\beta(y),C_\beta(x))$. Without loss of generality we can assume that $Q(C_\beta(x),C_\beta(y)) < Q(C_\beta(y),C_\beta(x))$. From Part **II** we know that

$$Q(x,C_\beta(x)) = C_0(T(x,C_\beta(x))) = C_0(T(C_\beta(x),x)) = C_0\big(T(C_\beta(x),C_\beta(C_\beta(x)))\big) = Q(C_\beta(x),x)$$

and therefore $y < C_\beta(x)$. Consider $t \in]Q(C_\beta(x),C_\beta(y)),Q(C_\beta(y),C_\beta(x))[$. Combining Part **I** with Property 7.3, it follows that $t < T(C_\beta(x-\varepsilon),C_\beta(y-\varepsilon))$, for every $\varepsilon \in]0,\min(x-\beta,y-\beta)]$. As $\beta \leqslant Q(C_\beta(x),C_\beta(y)) < t$, we can apply assertion **(G5)** and rewrite the latter inequality as $y-\varepsilon < T(C_\beta(t),C_\beta(x-\varepsilon))$. Due to the strict decreasingness of C_β on $[\beta,1]$, taking the limit $\varepsilon \searrow 0$ ensures that $C_\beta(x-\varepsilon) \searrow C_\beta(x)$ such that it follows from property **(E2)** that $y \leqslant Q(C_\beta(t),C_\beta(x))$. Furthermore, since $(y,t) \in]\beta,1[^2$, we know that $Q(C_\beta(x),C_\beta(y)) < t$ is equivalent with $Q(C_\beta(x),C_\beta(t)) < y$ (assertion **(G7)**). Summarizing the above reasoning leads to

$$Q(C_\beta(x),C_\beta(t)) < y \leqslant Q(C_\beta(t),C_\beta(x))\,,$$

for every $t \in]Q(C_\beta(x),C_\beta(y)),Q(C_\beta(y),C_\beta(x))[$. Fix t and take $u \in]Q(C_\beta(x),C_\beta(t)),y[\subseteq]Q(C_\beta(x),C_\beta(t)),Q(C_\beta(t),C_\beta(x))[$. Then $\beta < t < y \leqslant C_\beta(x)$ such that $(x,t) \in \mathcal{D}_{\mathrm{IV}}$. Repeating the procedure described above, we obtain that

$$Q(C_\beta(x),C_\beta(u)) < t \leqslant Q(C_\beta(u),C_\beta(x))\,.$$

For every $v \in]\max(Q(C_\beta(x),C_\beta(y)),Q(C_\beta(x),C_\beta(u))),t[$ it holds by definition of Q that $C_\beta(u) < C_v(C_\beta(x))$. Furthermore, as $v < t \leqslant Q(C_\beta(u),C_\beta(x))$, we also know that $C_v(C_\beta(u)) \leqslant C_\beta(x)$. Due to the orthosymmetry of C_β (Theorem 5.9), both inequalities $C_\beta(u) < C_v(C_\beta(x))$ and $C_v(C_\beta(u)) \leqslant C_\beta(x)$ can be equivalently written as $C_\beta(C_v(C_\beta(x))) < u$ and $x \leqslant C_\beta(C_v(C_\beta(u)))$. Applying assertion **(G3)** results in $T(C_\beta(v),C_\beta(x)) < u$ and $x \leqslant T(C_\beta(v),C_\beta(u))$. Note that this assertion is indeed applicable since

$$\beta \leqslant Q(C_\beta(x),C_\beta(y)) < v < Q(C_\beta(u),C_\beta(x)) \leqslant \min(C_\beta(u),C_\beta(x))\,.$$

Furthermore, we know that $T(C_\beta(v),C_\beta(x)) < u$ implies $T(C_\beta(v),C_\beta(u)) \leqslant x$ (assertion **(G5)**). In combination with $x \leqslant T(C_\beta(v),C_\beta(u))$, this leads to $T(C_\beta(v),C_\beta(u)) = x$. In a similar way it follows from $Q(C_\beta(x), C_\beta(y)) < v < t < Q(C_\beta(y), C_\beta(x))$ that $T(C_\beta(v), C_\beta(y)) = x$. In particular, we obtain that

$$T(x, y) = T(T(C_\beta(u), y), C_\beta(v)) = T(T(C_\beta(y), y), C_\beta(v)).$$

Moreover, as $t < Q(C_\beta(y), C_\beta(x)) \leqslant C_\beta(x)$, it holds that $\beta \leqslant Q(t, C_\beta(t)) \leqslant Q(C_\beta(x), C_\beta(t)) < u < y$ and hence, $C_\beta(y) < C_\beta(u)$. The latter inequality implies that $T(C_\beta(y), y) \leqslant \beta < T(C_\beta(u), y)$ and therefore also $T(\beta, C_\beta(v)) = T(x, y)$, for every $v \in \,]\max(Q(C_\beta(x), C_\beta(y)), Q(C_\beta(x), C_\beta(u))), t[$. Taking into account property **(E2)** and the involutivity of C_β on $[\beta, 1]$, we get the following chain of equalities:

$$\begin{aligned} Q(\beta, C_\beta(t)) &= \inf\{T(\beta, s) \mid s \in]C_\beta(t), 1]\} = \inf\{T(\beta, C_\beta(v)) \mid C_\beta(v) \in]C_\beta(t), 1]\} \\ &= \inf\{T(\beta, C_\beta(v)) \mid v \in [\beta, t[\} = T(x, y). \end{aligned}$$

Since $C_\beta(t) < 1$, we know from property **(E3)** that

$$T(C_\beta(t), \beta) = T(\beta, C_\beta(t)) \leqslant Q(\beta, C_\beta(t)) = T(x, y).$$

Taking the limit $t \searrow Q(C_\beta(x), C_\beta(y))$ of the latter inequality results in

$$C_0\big(Q(C_\beta(x), C_\beta(y))\big) = T\big(C_\beta(Q(C_\beta(x), C_\beta(y))), \beta\big) \leqslant T(x, y).$$

Recall from Part **I** that $T(x, y) \leqslant C_0(Q(C_\beta(y), C_\beta(x)))$. It must hold that $Q(C_\beta(y), C_\beta(x)) \leqslant Q(C_\beta(x), C_\beta(y))$, which contradicts our assumption $Q(C_\beta(x), C_\beta(y)) < Q(C_\beta(y), C_\beta(x))$.

IV. $Q(C_\beta(x), C_\beta(y)) = C_0(T(x, y))$, for every $(x, y) \in \mathcal{D}_{\mathrm{IV}}$

Considering Parts **I** and **II** it suffices to prove the inequality $C_0(T(x, y)) \leqslant Q(C_\beta(x), C_\beta(y))$, for every $(x, y) \in \mathcal{D}_{\mathrm{IV}}$ satisfying $y < C_\beta(x)$. Fix such a pair (x, y) and suppose that the converse holds, *i.e.* $Q(C_\beta(x), C_\beta(y)) < C_0(T(x, y))$. From property **(E6)** and Part **II** it follows that $Q(y, C_\beta(y)) \leqslant C_0(T(x, C_\beta(x))) = Q(x, C_\beta(x))$. Due to the commutativity of Q on $\overline{\mathcal{D}}_{\mathrm{I}}$ (Part **III**), we can interchange the role of x and y in the above inequality. Hence, $Q(x, C_\beta(x)) = Q(y, C_\beta(y))$. The right continuity of $Q(C_\beta(x), \bullet)$ (property **(E4)**) and the continuity of C_β on $[\beta, 1]$ ensure the existence of a $\alpha \in]\beta, y[$ such that

$$Q(C_\beta(x), C_\beta(y)) \leqslant Q(C_\beta(x), C_\beta(t)) < C_0(T(x, y)) \leqslant C_0(T(x, t)),$$

for every $t \in \,]\alpha, y[$. In particular, $(C_\beta(x), C_\beta(t)) \in \overline{\mathcal{D}}_{\mathrm{I}}$ and $Q(C_\beta(x), C_\beta(t)) < C_0(T(x, t))$. Repeating the above reasoning, it then follows that $Q(x, C_\beta(x)) = Q(t, C_\beta(t))$, for every $t \in \,]\alpha, y[$. Furthermore, considering property **(E6)** and Part **II**, $Q(C_\beta(x), C_\beta(t)) < C_0(T(x, y))$ implies that $Q(y, C_\beta(t)) \leqslant Q(x, C_\beta(x))$. Because $C_\beta(y) < C_\beta(t)$, $Q(x, C_\beta(x)) = Q(y, C_\beta(y))$ is also a lower bound of $Q(y, C_\beta(t))$. We conclude that $Q(y, C_\beta(t)) = Q(x, C_\beta(x))$, for every $t \in \,]\alpha, y[$. Fix $t \in \,]\alpha, y[$. Then, for every $u \in \,]t, y[$ it follows from Part **II**, property **(E2)** and property **(E3)** that

$$\begin{aligned} Q(x, C_\beta(x)) = Q(t, C_\beta(t)) = Q(C_\beta(t), t) &\leqslant T(C_\beta(t), u) \\ &\leqslant Q(u, C_\beta(t)) \leqslant Q(y, C_\beta(t)) = Q(x, C_\beta(x)). \end{aligned}$$

Note that $T(u, C_\beta(t)) = Q(x, C_\beta(x)$ is equivalent with $C_t(u) = C_\beta(Q(x, C_\beta(x)))$ (assertion **(G3)**) and therefore $T(C_\beta(Q(x, C_\beta(x))), u) \leqslant t$. As this latter inequality holds for every $u \in \,]t, y[$, it implies that also $Q(C_\beta(Q(x, C_\beta(x))), u) \leqslant t$ (property **(E2)**). Furthermore, as $u \in \,]t, y[\, \subset \,]\alpha, y[$, we already know that $Q(u, C_\beta(u)) = Q(x, C_\beta(x))$. This equality is equivalently rewritten as $C_\beta(C_\beta(Q(x, C_\beta(x)))) = Q(u, C_\beta(u))$. Note that $C_\beta(u) < C_\beta(t)$ implies that $\beta < T(u, C_\beta(t)) = Q(x, C_\beta(x))$ and hence, $C_\beta(Q(x, C_\beta)) < 1$ (Corollary 5.8). Applying assertion **(G7)** to $Q(u, C_\beta(u)) = C_\beta(C_\beta(Q(x, C_\beta(x))))$ then leads to $u = C_\beta(C_\beta(u)) \leqslant Q(u, C_\beta(Q(x, C_\beta(x))))$. Summarizing the previous results, we obtain the following chain of inequalities:

$$Q\big(C_\beta(Q(x, C_\beta(x))), u\big) \leqslant t < u \leqslant Q(u, C_\beta(Q(x, C_\beta(x)))) \, . \tag{7.4}$$

As $\beta < Q(x, C_\beta(x)) = T(u, C_\beta(t)) \leqslant u < 1$ and $C_\beta(C_\beta(Q(x, C_\beta(x)))) = Q(u, C_\beta(u)) \leqslant u$, the pair $(C_\beta(Q(x, C_\beta(x))), u)$ must belong to $\overline{\mathcal{D}}_{\mathrm{I}}$. Eq. (7.4) therefore contradicts the commutativity of Q on $\overline{\mathcal{D}}_{\mathrm{I}}$ (Part **III**). Our assumption $Q(C_\beta(x), C_\beta(y)) < C_0(T(x, y))$ can never hold. $\square$

Due to property **(E2)**, $Q|_{\overline{\mathcal{D}}_{\mathrm{I}}}$ is totally determined by $T|_{\mathcal{D}_{\mathrm{I}}}$. Taking into account Theorem 7.1, this means that we can rewrite $T|_{\mathcal{D}_{\mathrm{IV}}}$ in terms of Q^β, the companion of T^β. Combining Theorems 7.1 and 7.4 yields the following decomposition of left-continuous t-norms T that have continuous contour lines C_0 and C_β, where β is the unique fixpoint of C_0.

Theorem 7.5 *Consider a rotation-invariant t-norm T for which C_β is continuous, with β the unique fixpoint of C_0. Let σ be an arbitrary $[\beta, 1] \to [0, 1]$ isomorphism. Then there exists a rotation-invariant t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$ and companion $\widehat{Q}$) such that*

$$T(x, y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x], \sigma[y])\right], & \text{if } (x, y) \in \mathcal{D}_{\mathrm{I}} \, , \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \text{if } (x, y) \in \mathcal{D}_{\mathrm{II}} \, , \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \text{if } (x, y) \in \mathcal{D}_{\mathrm{III}} \, , \\ C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[x]), \widehat{C}_0(\sigma[y])\big)\right]\right), & \text{if } (x, y) \in \mathcal{D}_{\mathrm{IV}} \, , \\ 0, & \text{if } (x, y) \notin \mathcal{D} \, . \end{cases} \tag{7.5}$$

In particular, $\widehat{T} = T^\beta$ and $\widehat{Q}$ must be commutative on $[0, 1[^2$.

Proof Invoking Theorem 7.1, it suffices to prove that T^β is rotation invariant, that Q^β is commutative on $]0, 1[^2$ and that $T(x, y) = C_0(\sigma^{-1}[Q^\beta(C_0^\beta(\sigma[x]), C_0^\beta(\sigma[y]))])$, for every $(x, y) \in \mathcal{D}_{\mathrm{IV}}$. Take such a pair $(x, y) \in \mathcal{D}_{\mathrm{IV}}$. From properties **(F4a)** and **(F2)** we know that $C_\beta(u) = \sigma^{-1}[C_0^\beta(\sigma[u])]$ holds for every $u \in [\beta, 1]$ and that $Q(u, v) = \sigma^{-1}[Q^\beta(\sigma[u], \sigma[v])]$ holds for every $(u, v) \in [\beta, 1[^2$ satisfying $C_\beta(u) \leqslant v$. Furthermore, $(C_\beta(x), C_\beta(y)) \in \,]\beta, 1[^2$ and $C_\beta(C_\beta(y)) =$

$y \leqslant C_\beta(x)$. Invoking Theorem 7.4, we obtain that

$$\begin{aligned} T(x,y) &= C_0\big(Q(C_\beta(x), C_\beta(y))\big) = C_0\left(\sigma^{-1}\left[Q^\beta\big(\sigma[C_\beta(x)], \sigma[C_\beta(y)]\big)\right]\right) \\ &= C_0\left(\sigma^{-1}\left[Q^\beta\big(C_0^\beta(\sigma[x]), C_0^\beta(\sigma[y])\big)\right]\right) . \end{aligned}$$

Furthermore, as C_β is continuous it is clear that $C_0^\beta = \sigma \circ C_\beta \circ \sigma^{-1}$ is also continuous. Hence, T^β must be rotation invariant (assertion **(G5)**). As T is commutative, C_0 is involutive (asertion **(G2)**) and $(x,y) \in \mathcal{D}_{\mathrm{IV}} \;\Leftrightarrow\; (y,x) \in \mathcal{D}_{\mathrm{IV}}$ (Theorem 5.9), it must hold that $Q^\beta(C_0^\beta(\sigma[x]), C_0^\beta(\sigma[y])) = Q^\beta(C_0^\beta(\sigma[y]), C_0^\beta(\sigma[x]))$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. Taking into account the involutivity of C_0^β (assertion **(G2)**) and denoting $C_0^\beta(\sigma[x])$ and $C_0^\beta(\sigma[y])$ as, resp., u and v, the latter expresses that $Q^\beta(u,v) = Q^\beta(v,u)$, for every $(u,v) \in \,]0,1[^2$ satisfying $C_0^\beta(u) \leqslant v$. Obviously, $Q^\beta(u,v) = Q^\beta(v,u) = 0$ whenever $v < C_0^\beta(u)$. Recall also that $Q^\beta(0,v) = 0$ and $0 < C_0^\beta(u)$, for every $(u,v) \in [0,1[^2$ (involutivity of C_0^β). This concludes the proof. $\square$

Note that the continuity of C_β entails the rotation invariance of T^β. This observation enables us to rewrite the formulas for $T|_{\mathcal{D}_{\mathrm{II}}}$ and $T|_{\mathcal{D}_{\mathrm{III}}}$ (Eq. (7.5)) as formulas containing $T^\beta (= \widehat{T})$ explicitly.

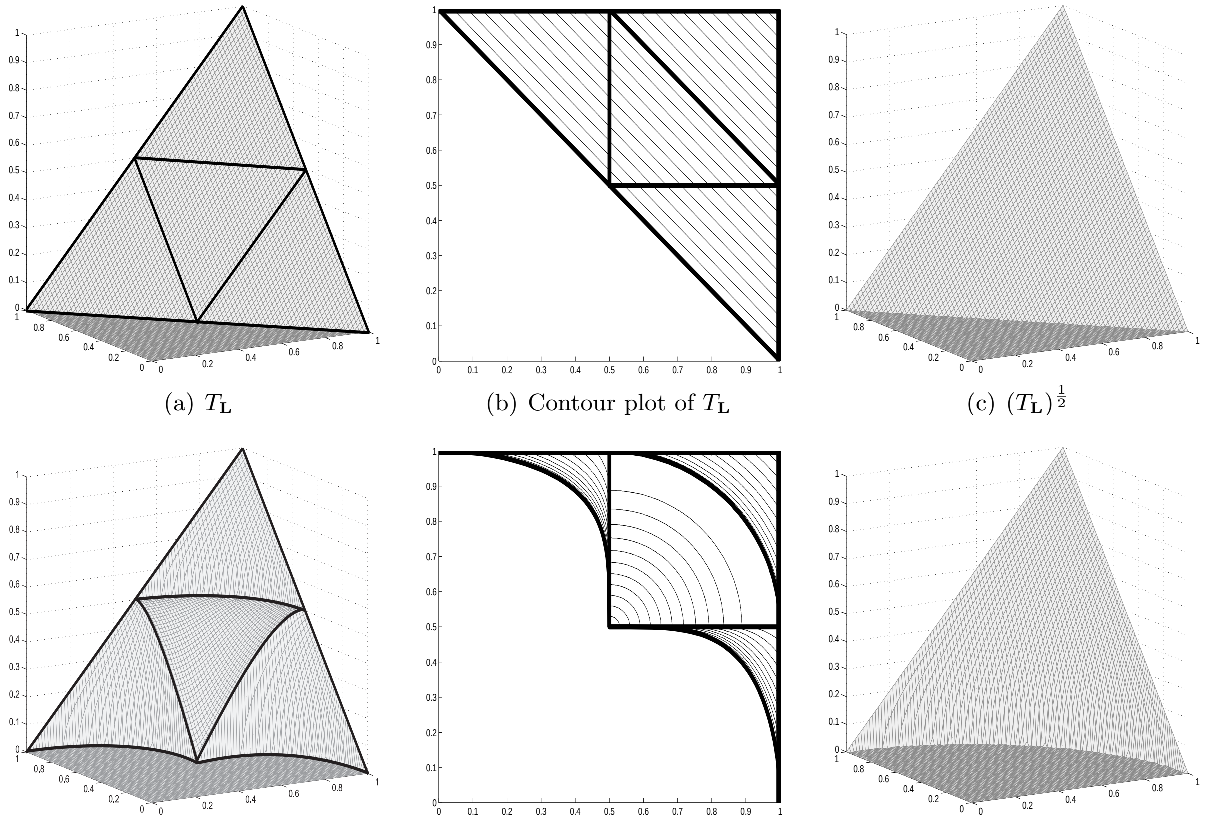
Corollary 7.6 *Consider a rotation-invariant t-norm T for which C_β is continuous, with β the unique fixpoint of C_0. Let σ be an arbitrary $[\beta,1] \to [0,1]$ isomorphism. Then there exists a rotation-invariant t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$ and companion $\widehat{Q}$) such that*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x], \sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_0\left(\widehat{T}\big(\widehat{C}_0(\sigma[C_0(x)]), \sigma[y]\big)\right)\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_0\left(\widehat{T}\big(\sigma[x], \widehat{C}_0(\sigma[C_0(y)])\big)\right)\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[x]), \widehat{C}_0(\sigma[y])\big)\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,, \\ 0, & \text{if } (x,y) \notin \mathcal{D}\,. \end{cases} \tag{7.6}$$

In particular, $\widehat{T} = T^\beta$ and $\widehat{Q}$ must be commutative on $[0,1[^2$.

Proof Follows immediately from Theorem 7.5 and assertions **(G2)** and **(G3)**. $\square$

We now dispose of a method for decomposing a rotation-invariant t-norm T that has a continuous contour line C_β. The knowledge of the contour line C_0 and of the t-norm T^β totally determines T. Moreover, we are able to give a full geometrical interpretation of the structure of T. The geometrical construction of $T|_{\mathcal{D}_{\mathrm{II}}}$ and $T|_{\mathcal{D}_{\mathrm{III}}}$ remains intact: these parts are determined by a transformed left rotation, resp. right rotation, of $T|_{\mathcal{D}_{\mathrm{I}}}$. As can be seen in Figure 7.3(a), area $T|_{\mathcal{D}_{\mathrm{IV}}}$ is obtained by rotating $T|_{\mathcal{D}_{\mathrm{I}}}$ 180 degrees to the front around the axis $\{(x,y,z) \in [0,1]^3 \mid x + y = 1 + \beta \;\wedge\; z = \beta\}$. Depending on the structure of C_0 and C_β some additional reshaping may occur.



(a) $T_{\mathbf{L}}$ (b) Contour plot of $T_{\mathbf{L}}$ (c) $(T_{\mathbf{L}})^{\frac{1}{2}}$

(d) $(T_{\mathbf{L}})_\phi$ (e) Contour plot of $(T_{\mathbf{L}})_\phi$ (f) $((T_{\mathbf{L}})_\phi)^{\frac{1}{2}}$

Figure 7.3: Decomposition of the Łukasiewicz t-norm $T_{\mathbf{L}}$ and $(T_{\mathbf{L}})_\phi$.

Example 7.7 Theorem 7.5 is applicable to the Łukasiewicz t-norm $T_{\mathbf{L}}$. Figure 7.3(a) depicts its decomposition. In particular, it holds that $C_0 = \mathcal{N}$, $\beta = \frac{1}{2}$ and $(T_{\mathbf{L}})^{\frac{1}{2}} = T_{\mathbf{L}}$. As usual the bold black lines indicate the partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$ and we use a linear rescaling function. These new insights into the structure of the Łukasiewicz t-norm $T_{\mathbf{L}}$ als help us to better understand the structure of the algebraic product $T_{\mathbf{P}}$ as every $(T_{\mathbf{P}})^a$, with $a \in]0,1[$, is a transformed Łukasiewicz t-norm $T_{\mathbf{L}}$ (see Example 6.19).

Let ϕ be the automorphism defined by

$$\phi(x) = \begin{cases} \frac{1}{2} - \sqrt{\frac{1}{4} - x^2}, & \text{if } x \leqslant \frac{1}{2}\,, \\ \frac{1}{2} + \frac{1}{2}\,(2\,x - 1)^2, & \text{elsewhere}\,. \end{cases}$$

Transforming $T_{\mathbf{L}}$ by means of ϕ yields a t-norm $(T_{\mathbf{L}})_\phi$ (see Section 5.2). Such a transformation does not affect the continuity of the contour lines. Hence, $(T_{\mathbf{L}})_\phi$ is also rotation invariant with $(C_\phi)_0 = \phi^{-1} \circ \mathcal{N} \circ \phi$, $\beta_\phi = \phi^{-1}(\frac{1}{2}) = \frac{1}{2}$ and $((T_{\mathbf{L}})_\phi)^{\frac{1}{2}} = (T_{\mathbf{L}}^{\frac{1}{2}})_{\varsigma\circ\phi\circ\varsigma^{-1}} = (T_{\mathbf{L}})_{\varsigma\circ\phi\circ\varsigma^{-1}}$, where ς is the linear rescaling function from $[\frac{1}{2},1]$ to $[0,1]$. The order-preserving $[0,1] \to [0,1]$ bijection $\varsigma \circ \phi \circ \varsigma^{-1}$ transforms $T_{\mathbf{L}}$ into $((T_{\mathbf{L}})_\phi)^{\frac{1}{2}}$. As this transformation preserves the continuity of the contour lines, we conclude that $((T_{\mathbf{L}})_\phi)^{\frac{1}{2}}$ must be a rotation-invariant t-norm. Therefore, $(T_{\mathbf{L}})_\phi$ can be fully decomposed by means of Eqs. (7.5) and (7.6). $\triangle$

7.4 Full decomposition

To conclude this chapter, we merge and slightly extend our decomposition results: Theorems 7.1 and 7.5. Instead of requiring the continuity of C_β as in Theorem 7.5, we decompose those rotation-invariant t-norms whose contour line C_β is continuous on the interval $]\beta, 1]$ only. The following theorem is crucial to insert the results from Theorem 7.5 into the new setting.

Theorem 7.8 *Consider a left-continuous t-norm T and take $a \in [0,1]$ such that $a < \alpha := \inf\{t \in [0,1] \mid C_a(t) = a\}$. Then the following assertions are equivalent:*

(J1) *C_a is continuous on $]a,1]$.*
(J2) *C_a is involutive on $]a,\alpha[$.*
(J3) *$C_a(]a,\alpha[) =]a,\alpha[$.*
(J4) *$T^{(a,\alpha)}$ is a rotation-invariant t-norm.*

Proof Let $\tilde{\sigma}$ be the $[a,\alpha] \to [0,1]$ isomorphism that is used to construct $T^{(a,\alpha)}$.

(J1)⇒(J4): We already know from property **(F3)** and Corollary 6.15 that $T^{(a,\alpha)}$ is a left-continuous t-subnorm. $T^{(a,\alpha)}$ will be a t-norm if we can show that it has neutral element 1. Explicitly, $T^{(a,\alpha)}(x,1) = \tilde{\sigma}[\max(a, T(\tilde{\sigma}^{-1}[x], \alpha))] = x$ needs to be fulfilled for every $x \in [0,1]$. This is equivalent with $T(u,\alpha) = u$, for every $u \in]a,\alpha]$. We prove the latter. The continuity of C_a and Corollary 5.8 ensure that then $[a, \lim_{x \searrow a} C_a(x)[= [C_a(1), \lim_{x \searrow a} C_a(x)[\subseteq C_a(]a,1])$. Suppose that $\lim_{x \searrow a} C_a(x) < \alpha$, then there exists $\varepsilon \in]0,\alpha[$ such that $C_a(x) < \alpha - \varepsilon$, for every $x \in]a,1]$. Due to the orthosymmetry of C_a (Theorem 5.9) it holds that $a = C_a(1) \leqslant C_a(\alpha - \varepsilon) < x$. Taking the limit $x \searrow a$, this leads to $C_a(\alpha - \varepsilon) = a$, which contradicts the definition of α. We conclude that $[a,\alpha[\subseteq C_a(]a,1])$. Since $a < \alpha$, assertion **(J1)** also ensures that $C_a(\alpha) = a$. Therefore, $C_{C_a(\alpha)}(x) = C_a(x)$, for every $x \in]a,1]$, and it follows from Eq. (5.7) that also $C_{C_a(x)}(\alpha) = C_a(x)$. Invoking that $[a,\alpha[\subseteq C_a(]a,1])$, this implies that $C_y(\alpha) = y$, for every $y \in [a,\alpha[$. If $a < y$, then $T(\alpha,y) < y$ ensures the existence of $\varepsilon \in]0, y-a]$ such that $T(\alpha,y) < y - \varepsilon$. This leads to the contradiction $y \leqslant C_{y-\varepsilon}(\alpha) = y - \varepsilon$ and we conclude that $T(y,\alpha) = T(\alpha,y) = y$, for every $y \in]a,\alpha[$. The left continuity of T guarantees that also $T(\alpha,\alpha) = \alpha$.

$T^{(a,\alpha)}$ will be rotation invariant if its contour line $C_0^{(a,\alpha)}$ is continuous (assertions **(G1)** and **(G5)**). As $C_0^{(a,\alpha)}$ is decreasing, it will be continuous if it reaches every element of $[0,1]$. By definition it holds that $C_0^{(a,\alpha)}(0) = 1$ such that it suffices to show the inclusion $[0,1[\subseteq C_0^{(a,\alpha)}(]0,1])$. For arbitrary $y \in [a,\alpha[$, we know from the discussion above that there exists $x \in]a,1]$ such that $C_a(x) = y$. Due to $C_a(\alpha) = a$, it even suffices to consider $x \in]a,\alpha]$. Denoting $\tilde{\sigma}[x]$ and

$\tilde{\sigma}[y]$ by, resp., u and v, we obtain that for every $v \in [0,1[$ there exists $u \in]0,1]$ such that $C_0^{(a,\alpha)}(u) = \tilde{\sigma}[C_a(\tilde{\sigma}^{-1}[u])] = v$ (property **(F4b)**).

(J4)⇒(J2): Since property **(F4b)** yields $C_a(x) = \tilde{\sigma}^{-1}[C_0^{(a,\alpha)}(\tilde{\sigma}[x])]$, for every $x \in]a,\alpha]$, the involutivity of $C_0^{(a,\alpha)}$ (assertion **(G2)**) immediately implies the involutivity of C_a on $]a,\alpha[$.

(J2)⇒(J3): By definition of α and Corollary 5.8, it holds that $a < C_a(x)$, for every $x \in]a,\alpha[$. Furthermore, suppose that $\alpha < C_a(x)$, for some $x \in]a,\alpha[$, then $\alpha + \varepsilon < C_a(x)$, with $\varepsilon \in]0, C_a(x) - \alpha[$. Invoking once again the definition of α, we obtain the contradiction $x = C_a(C_a(x)) \leqslant C_a(\alpha + \varepsilon) = a$. Therefore, $C_a(]a,\alpha[) \subseteq]a,\alpha[$. Due to the involutivity of C_a it then also holds that $]a,\alpha[\subseteq C_a(]a,\alpha[)$.

(J3)⇒(J1): Combining assertion **(J3)** with the decreasingness of C_a, it follows that C_a is continuous on $]a,\alpha[$. The left continuity of C_a then ensures that $C_a(\alpha) = \lim_{x \nearrow \alpha} C_a(x) = a$. Because also $C_a(1) = a$ (Corollary 5.8), we immediately obtain assertion **(J1)**. □

Theorem 7.9 *Consider a rotation-invariant t-norm T for which C_β is continuous on $]\beta,1]$, with β the unique fixpoint of C_0. Let σ be an arbitrary $[\beta,1] \to [0,1]$ isomorphism. Then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$ and companion $\widehat{Q}$) such that $\widehat{C}_0$ is continuous on $]0,1]$ and*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[x]),\widehat{C}_0(\sigma[y])\big)\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,, \\ 0, & \text{if } (x,y) \notin \mathcal{D}\,. \end{cases} \tag{7.7}$$

In particular, $\widehat{T} = T^\beta$ and $\widehat{Q}$ must be commutative on $[0,\hat{\alpha}[^2$, with $\hat{\alpha} = \inf\{t \in [0,1] \mid \widehat{C}_0(t) = 0\}$.

Proof Taking into account Theorem 7.1, it suffices to prove that C_0^β is continuous on $]0,1]$, that Q^β is commutative on $[0,\hat{\alpha}[^2$ and that $T(x,y) = C_0(\sigma^{-1}[Q^\beta(C_0^\beta(\sigma[x]), C_0^\beta(\sigma[y]))])$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. As $C_0^\beta = \sigma \circ C_\beta \circ \sigma^{-1}$ (property **(F4a)**), it immediately follows from the continuity of C_β on $]\beta,1]$ that C_0^β is continuous on $]0,1]$. Suppose that $\hat{\alpha} = 0$, then $C_\beta(x) = \sigma^{-1}[C_0^\beta(\sigma[x])] = \sigma^{-1}(0) = \beta$, for every $x \in]\beta,1]$. Hence, $\mathcal{D}_{\mathrm{IV}} = \emptyset$ and the theorem holds. Assume now that $0 < \hat{\alpha}$. We first show that $\breve{T} := T^{(C_0(\alpha),\alpha)}$, with $\alpha := \inf\{t \in [0,1] \mid C_\beta(t) = \beta\}$, is a rotation-invariant t-norm satisfying Theorem 7.5. Next, we translate the properties of $\breve{T}$ into properties of T. The $[C_0(\alpha),\alpha] \to [0,1]$ isomorphism $\breve{\sigma}$ is used to construct $\breve{T}$. Its contour lines and companion are denoted by, resp., $\breve{C}_a$ and $\breve{Q}$.

I. $\breve{T}$ is a rotation-invariant t-norm

Invoking Corollary 5.8 and property **(F4a)** it follows that

$$\begin{aligned}\alpha &= \inf\{t \in [\beta,1] \mid C_\beta(t) = \beta\} = \inf\{\sigma^{-1}[s] \in [\beta,1] \mid C_0^\beta(s) = \sigma[\beta]\}\\ &= \sigma^{-1}[\inf\{s \in [0,1] \mid C_0^\beta(s) = 0\}] = \ \sigma^{-1}[\hat{\alpha}]\end{aligned}$$

and, hence, $\beta < \alpha$. From the proof of Theorem 7.8 we know that $T(\alpha,x) = T(x,\alpha) = x$, for every $x \in]\beta,\alpha]$. Consequently, $C_x(\alpha) = x$, for every $x \in]\beta,\alpha[$. Note that also $C_\beta(\alpha) = \beta$ as C_β is continuous on $]\beta,1]$ and $\beta < \alpha$. Hence, $C_0(C_{C_0(x)}(\alpha)) = C_0(C_0(x)) = x$, for every $x \in]C_0(\alpha),\beta]$. Applying assertion **(G3)** this leads to $T(\alpha,x) = T(x,\alpha) = x$, for every $x \in]C_0(\alpha),\beta]$. Equation (5.3) and the observation that $T(\alpha,x) = T(x,\alpha) = x$ holds for every $x \in]C_0(\alpha),\alpha]$ ensure that $C_{C_0(\alpha)}(x) = C_0(T(\alpha,x)) = C_0(x)$, for every $x \in]C_0(\alpha),\alpha]$. Invoking the involutivity of C_0 (assertion **(G2)**) and $C_{C_0(\alpha)}(\alpha) = C_0(\alpha) = C_{C_0(\alpha)}(1)$ (Corollary 5.8), this leads to $\inf\{t \in [0,1] \mid C_{C_0(\alpha)}(t) = C_0(\alpha)\} = \alpha$. Taking into account the continuity of C_0 (assertion (**G1**)), it now follows that $C_{C_0(\alpha)}$ is continuous on $]C_0(\alpha),1]$. Assertion **(J4)** states that $\breve{T} := T^{(C_0(\alpha),\alpha)}$ must be a rotation-invariant t-norm.

II. $\breve{C}_{\breve{\sigma}[\beta]}$ is continuous, with $\breve{\sigma}[\beta]$ the fixpoint of $\breve{C}_0$

From property **(F4b)** and the proof of Part **I** we obtain that $\breve{C}_0(x) = \breve{\sigma}[C_{C_0(\alpha)}(\breve{\sigma}^{-1}[x])] = \breve{\sigma}[C_0(\breve{\sigma}^{-1}[x])]$, for every $x \in]0,1]$. Trivially, $\breve{C}_0(0) = 1 = \breve{\sigma}[C_0(C_0(\alpha))] = \breve{\sigma}[C_0(\breve{\sigma}^{-1}[0])]$ and we conclude that $\breve{C}_0 = \breve{\sigma} \circ C_0 \circ \breve{\sigma}^{-1}$. It is easily verified that $\breve{\sigma}[\beta]$ is the unique fixpoint of $\breve{C}_0$. Because contour lines are decreasing and $\breve{C}_{\breve{\sigma}[\beta]}(1) = \breve{\sigma}[\beta] < 1 = \breve{C}_{\breve{\sigma}[\beta]}(\breve{\sigma}[\beta])$ (Corollary 5.8), the continuity of $\breve{C}_{\breve{\sigma}[\beta]}$ is implied by $\breve{C}_{\breve{\sigma}[\beta]}(]\breve{\sigma}(\beta),1[) =]\breve{\sigma}(\beta),1[$. The latter follows immediately from $\breve{C}_{\breve{\sigma}[\beta]}[x] = \breve{\sigma}[C_\beta(\breve{\sigma}^{-1}[x])]$, for every $x \in]\breve{\sigma}(\beta),1]$ (property **(F4b)**), and the fact that $C_\beta(]\beta,\alpha[) =]\beta,\alpha[$ (assertion **(J3)**).

III. $\max(C_0(\alpha),T(x,y)) = C_{C_0(\alpha)}(Q(C_\beta(x),C_\beta(y)))$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$

From Parts **I** and **II** we know that Theorem 7.4 is applicable. For every $(u,v) \in]\breve{\sigma}(\beta),1[^2$ satisfying $v \leqslant \breve{C}_{\breve{\sigma}[\beta]}(u)$ it then holds that $\breve{T}(u,v) = \breve{C}_0(\breve{Q}(\breve{C}_{\breve{\sigma}[\beta]}(u),\breve{C}_{\breve{\sigma}[\beta]}(v)))$. Denote $\breve{\sigma}^{-1}[u]$ by x and $\breve{\sigma}^{-1}[v]$ by y. Applying **(F2)** and **(F4b)** the former can be rewritten as $\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y]) = \breve{C}_0(\breve{\sigma}[Q(C_\beta(x),C_\beta(y))])$ whenever $(x,y) \in]\beta,\alpha[^2$ satisfies $y \leqslant C_\beta(x)$. Note that from Corollary 5.8 and Part **II** it indeed follows that $\breve{C}_{\breve{\sigma}[\beta]}(u) < 1$, $\breve{C}_{\breve{\sigma}[\beta]}(v) < 1$ and $\breve{C}_0(\breve{C}_{\breve{\sigma}[\beta]}(u)) \leqslant \breve{C}_0(\breve{\sigma}[\beta]) = \breve{\sigma}[\beta] \leqslant \breve{C}_{\breve{\sigma}[\beta]}(v)$. The involutivity of $\breve{C}_0$ (assertion **(G2)**) ensures that

$$0 < \breve{\sigma}[\beta] = \breve{C}_0(\breve{\sigma}[\beta]) \leqslant \breve{C}_0(\breve{T}(u,v)) = \breve{C}_0(\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y])) = \breve{\sigma}[Q(C_\beta(x),C_\beta(y))]\,. \tag{7.8}$$

Therefore, we can invoke property **(F4b)** a second time, which leads to $\breve{\sigma}^{-1}[\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y])] = C_{C_0(\alpha)}(Q(C_\beta(x),C_\beta(y)))$, for every $(x,y) \in]\beta,\alpha[^2$ such that $y \leqslant C_\beta(x)$. Invoking the definition of $\breve{T} = T^{(C_0(\alpha),\alpha)}$, we conclude that $\max(C_0(\alpha),T(x,y)) = C_{C_0(\alpha)}(Q(C_\beta(x),C_\beta(y)))$. Because $C_\beta(\alpha) = \beta$, the pair (x,y) must belong to area $\mathcal{D}_{\mathrm{IV}}$.

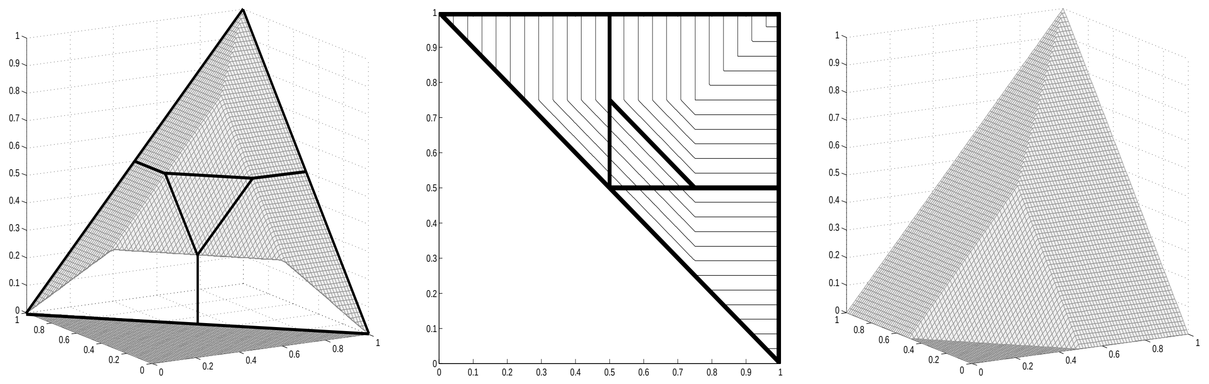
IV. $\boldsymbol{T(x,y) = C_0(\sigma^{-1}[Q^{\beta}(C_0^{\beta}(\sigma[x]), C_0^{\beta}(\sigma[y]))])}$**, for every** $\boldsymbol{(x,y) \in \mathcal{D}_{\mathrm{IV}}}$

Recall from the proof of Part **I** that $C_{C_0(\alpha)}(x) = C_0(x)$, for every $x \in \,]C_0(\alpha), \alpha]$. Furthermore, it follows from Eq. (7.8), $C_\beta(]\beta, \alpha[) = \,]\beta, \alpha[$ (assertion **(J3)**) and property **(E5)** that

$$C_0(\alpha) < \beta \leqslant Q(C_\beta(x), C_\beta(y)) \leqslant C_\beta(x) < \alpha\,,$$

for every $(x,y) \in \,]\beta, \alpha[^2$ satisfying $y \leqslant C_\beta(x)$. Therefore, we obtain $C_{C_0(\alpha)}(Q(C_\beta(x), C_\beta(y))) = C_0(Q(C_\beta(x), C_\beta(y))) > C_0(\alpha)$ and Part **III** can be rewritten as $T(x,y) = C_0(Q(C_\beta(x), C_\beta(y)))$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. Following the reasonings in the proof of Theorem 7.5 and taking into account that C_β is involutive on $]\beta, \alpha[$ (assertion **(B2)**), $C_\beta(]\beta, \alpha[) = \,]\beta, \alpha[$ (assertion **(B3)**) and $\sigma(\alpha) = \hat{\alpha}$, it is not difficult to show that $T(x,y) = C_0(\sigma^{-1}[Q^{\beta}(C_0^{\beta}(\sigma[x]), C_0^{\beta}(\sigma[y]))])$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$, that C_0^{β} is continuous on $]0,1]$ and that Q^{β} is commutative on $[0, \hat{\alpha}[^2$. $\square$

For $\hat{\alpha} = 0$, we get that $\alpha = \sigma^{-1}[\hat{\alpha}] = \beta$ and $\mathcal{D}_{\mathrm{IV}} = \emptyset$. The above theorem then coincides with Theorem 7.1. In case $\hat{\alpha} = 1$, then $\alpha = 1$ such that C_β must be involutive on $]\beta, 1[$ (assertion **(J2)**) and, hence, also on $[\beta, 1]$ as $\widehat{C}_\beta(\beta) = 1$ and $\widehat{C}_\beta(1) = \beta$ (Corollary 5.8). Theorem 7.9 now coincides with Theorem 7.5. Taking a closer look at the proof of Theorem 7.9, it strikes that, in case $\hat{\alpha} \in \,]0,1[$, $T(\bullet, \alpha)|_{]C_0(\alpha),\alpha]} = \mathbf{id}|_{]C_0(\alpha),\alpha]}$ implies that $Q(x, \alpha) = x$, for every $x \in \,]C_0(\alpha), \alpha]$ (property **(E2)**). From Eq. (6.8) we then know that T is decomposable in the sense of Jenei [45] and that $\alpha \in D_T$. Considering α as a decomposition point it holds that $\mathcal{D}_{\mathrm{IV}} \subset \mathcal{D}_{\mathrm{D}}$. In Fig. 7.4 we depict our decomposition of the Jenei t-norm $T^{\mathbf{J}}_{1/4}$. We use a linear rescaling function ς to compute $(T^{\mathbf{J}}_{1/4})^{\frac{1}{2}}$. It is easily verified that $\hat{\alpha} = \frac{1}{2}$ and $\alpha = \varsigma^{-1}(\hat{\alpha}) = \frac{3}{4}$ is a decomposition point of $T^{\mathbf{J}}_{1/4}$ (see also Figs. 6.5(h) and 6.5(i)). Geometrically, the filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is obtained by rotating $T|_{\mathcal{D}_{\mathrm{I}} \cap]\beta,\alpha]^2}$ 180 degrees to the front around the axis $\{(x,y,z) \in [0,1]^3 \mid x + y = \beta + \alpha \,\wedge\, z = \beta\}$. The contour lines C_0 and C_β can cause some additional reshaping.



(a) $T^{\mathbf{J}}_{1/4}$ (b) Contour plot of $T^{\mathbf{J}}_{1/4}$ (c) $(T^{\mathbf{J}}_{1/4})^{\frac{1}{2}}$

Figure 7.4: Decomposition of the Jenei t-norm $T^{\mathbf{J}}_{1/4}$.

CHAPTER 8

The triple rotation method

8.1 Introduction

Reversing the full decomposition method from Section 7.4 yields a tool for constructing rotation-invariant t-norms. Given an involutive negator N and a left-continuous t-norm T whose contour line C_0 is continuous on $]0,1]$, we build a rotation-invariant t-norm from a rescaled version of T and its left, right and front rotation. Depending on the involutive negator N and the set of zero divisors of T, some reshaping of the rescaled T may occur during the rotation process. There is, however, one important restriction: the companion Q of T must be commutative on $]0,\alpha[^2$, with $\alpha = \inf\{t \in [0,1] \mid C_0(t) = 0\}$.

Unfortunately, the mathematical proofs supporting our construction method are quite elaborated and technical. Therefore, we have assembled them in Section 8.2. As in the previous chapter, we assume full familiarity with the results from Section 4.2. In Section 8.3 we reformulate the construction tool in a more straightforward way and illustrate it by means of numerous examples. Also, its applicability and limitations are briefly addressed.

8.2 Mathematical approach

Straightforwardly transforming the full decomposition from Theorem 7.9 into a tool for constructing rotation-invariant t-norms, may lead to some notational flaws. To ensure that the $[0,1]^2 \to [0,1]$ function T from Eq. (7.7) is well defined, we need to assume the following setting:

- $\widehat{T}$: an arbitrary left-continuous t-norm with contour lines $\widehat{C}_a$ and companion $\widehat{Q}$;
- C_0: an arbitrary involutive negator with fixpoint β;
- σ: an arbitrary $[\beta,1] \to [0,1]$ isomorphism;

- C_β: the decreasing $[0,1] \to [0,1]$ function defined by $C_\beta(x) = 1$ whenever $x \in [0, \beta[$ and $C_\beta(x) = \sigma^{-1}[\widehat{C}_0(\sigma[x])]$ whenever $x \in [\beta, 1]$.

By definition, $C_\beta(\beta) = \sigma^{-1}[\widehat{C}_0(0)] = 1$, $C_\beta(x) < \sigma^{-1}[\widehat{C}_0(0)] = 1$, for every $x \in]\beta, 1]$, and $C_\beta(1) = \sigma^{-1}[\widehat{C}_0(1)] = \beta$. Areas $\mathcal{D}_{\mathrm{I}}$, $\mathcal{D}_{\mathrm{II}}$, $\mathcal{D}_{\mathrm{III}}$ and $\mathcal{D}_{\mathrm{IV}}$ in the domain of T are therefore well defined. Note that, at this point, C_0 and C_β do not have an interpretation in terms of contour lines. To avoid this confusion we reformulate our construction tool in Section 8.2 by means of a more robust terminology. However, as the proofs in the present section frequently use results from Chapters 6 and 7, we opt to work with the original notations first. For the sake of brevity, the above setting will not be recalled in the formulation of the theorems and properties.

As Eq. (7.7) comprises Eq. (7.1), we first examine the properties of an increasing $[0,1]^2 \to [0,1]$ function T that is defined on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$ by Eq. (7.1). This equation largely fixes the monotonicity, continuity and commutativity of T. Furthermore, it pinpoints its absorbing and neutral element.

Property 8.1 *If a $[0,1]^2 \to [0,1]$ function T is defined on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$ by Eq. (7.1), then the following properties hold:*

(K1) *T is increasing on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$.*
(K2) *T is left continuous on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$.*
(K3) *T has absorbing element 0 and neutral element 1.*
(K4) *T is commutative on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$.*

Proof (K1): The increasingness of $\widehat{T}$, the decreasingness of its contour lines and the increasingness of the partial functions $\widehat{C}_\bullet(x)$ ensure that T is increasing on $[0,1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$ and on $\mathcal{D}_{\mathrm{I}}$. Furthermore, $(x,y) \in \mathcal{D}_{\mathrm{I}}$ whenever $(x,y) \in]\beta,1]^2$ and $\widehat{C}_0(\sigma[x]) < \sigma[y]$. The latter ensures that $0 < \widehat{T}(\sigma[x], \sigma[y])$ from which it follows that $\beta < T(x,y)$, for $(x,y) \in \mathcal{D}_{\mathrm{I}}$. Otherwise, the inequality $T(x,y) \leqslant \beta$ holds for every $(x,y) \in [0,1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$ since $T(\beta,\beta) = 0$ and

$$\left.\begin{array}{r} T(\beta,x) \\ T(x,\beta) \end{array}\right| = C_0\big(\sigma^{-1}[\widehat{C}_0(\sigma[x])]\big) \leqslant C_0(\sigma^{-1}[0]) = C_0(\beta) = \beta\,,$$

for every $x \in]\beta, 1]$. We conclude T is indeed increasing on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$.

(K2): Clearly, T is left continuous on $[0,1]^2 \setminus \mathcal{D}$. The left continuity of T on $\mathcal{D}_{\mathrm{I}}$ trivially follows from the left continuity of $\widehat{T}$ and from the continuity of σ. Moreover, for every $(x,a) \in [0,1]^2$, $\widehat{C}_a(x)$ is left continuous in x and right continuous in a (Properties 4.5). As C_0 and σ are monotone bijections, we conclude that T is also left continuous on $\mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}}$.

(K3): As $T(0,1) = T(1,0) = 0$ it follows from property **(K1)** that 0 is the absorbing element of T. Take now arbitrary $x \in]0, \beta]$. The strict decreasingness of C_0 ensures that $(x,1) \in \mathcal{D}_{\mathrm{II}}$ and $C_0(1) = 0$ ensures that $(1,x) \in \mathcal{D}_{\mathrm{III}}$. Therefore, Corollary 5.8 ensures that

$$\left.\begin{array}{r} T(x,1) \\ T(1,x) \end{array}\right| = C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(1)\right]\right) = C_0\left(\sigma^{-1}\left[\sigma[C_0(x)]\right]\right) = C_0(C_0(x)) = x\,.$$

If $x \in]\beta, 1]$, then $C_\beta(x) < 1$ implies that $(x, 1) \in \mathcal{D}_{\mathrm{I}}$ and $C_\beta(1) = \beta$ implies that also $(1, x) \in \mathcal{D}_{\mathrm{I}}$. We obtain that

$$\left.\begin{array}{l} T(x,1) \\ T(1,x) \end{array}\right| = \sigma^{-1}\left[\widehat{T}(\sigma[x], 1)\right] = \sigma^{-1}[\sigma[x]] = x\,.$$

We conclude that 1 is indeed the neutral element of T.

(K4): Due to the involutivity of C_0 it holds that $(x, y) \notin \mathcal{D} \;\Leftrightarrow\; (y, x) \notin \mathcal{D}$ and $(x, y) \in \mathcal{D}_{\mathrm{II}} \;\Leftrightarrow\; (y, x) \in \mathcal{D}_{\mathrm{III}}$. The commutativity of T on $[0, 1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$ then follows immediately from Eq. (7.1). Suppose now that $(x, y) \in \mathcal{D}_{\mathrm{I}}$, *i.e.* $(x, y) \in]\beta, 1]^2$ and $\widehat{C}_0(\sigma[x]) < \sigma[y]$. The orthosymmetry of $\widehat{C}_0$ ensures that $\widehat{C}_0(\sigma[y]) < \sigma[x]$ (Theorem 5.9). Hence, $(y, x) \in \mathcal{D}_{\mathrm{I}}$. The commutativity of T on $\mathcal{D}_{\mathrm{I}}$ is then implied by the commutativity of $\widehat{T}$. $\square$

Remark that property **(K4)** implies that necessarily $(x, y) \in \mathcal{D}_{\mathrm{IV}} \;\Leftrightarrow\; (y, x) \in \mathcal{D}_{\mathrm{IV}}$. Requiring that T is increasing on $\mathcal{D}_{\mathrm{IV}}$ and satisfies $T(x, C_\beta(x)) \leqslant \beta$, for every $x \in]\beta, 1[$, the decreasing functions C_0 and C_β can be interpreted as contour lines of T. From that moment we use the standard notation C_a to denote all contour lines of T. Moreover, in this case the associativity property manifests itself in terms of Eq. (5.3).

Property 8.2 *If a $[0, 1]^2 \to [0, 1]$ function T is defined on $[0, 1]^2 \setminus \mathcal{D}_{IV}$ by Eq. (7.1), is increasing and satisfies $T(x, C_\beta(x)) \leqslant \beta$, for every $x \in]\beta, 1[$, then the following properties hold:*

(K5) *C_0 and C_β are contour lines of T.*
(K6) *$T^\beta = \widehat{T}$.*
(K7) *$C_a(T(x, y)) = C_{C_a(x)}(y) = C_{C_a(y)}(x)$ holds for every*

- **(K7a)** *$(x, y) \in [0, 1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$ and $a = 0$;*
- **(K7b)** *$(x, y) \in [0, 1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$ and $a \in]0, \beta[$;*
- **(K7c)** *$(x, y) \in [0, 1]^2$ and $a \in [\beta, 1]$.*

Proof It is trivial to see that Property 8.1 applies to T. Properties **(K1)**–**(K4)** will be frequently used throughout this proof. Note also that the existence of the neutral element 1, combined with the increasingness of T, ensures that $C_a(1) = a$ and $C_a(x) = 1$, for every $(x, a) \in [0, 1]^2$ such that $x \leqslant a$.

(K5)&(K6): Whenever $(x, y) \in \mathcal{D}_{\mathrm{II}}$, it holds that $\widehat{C}_{\sigma[C_0(x)]}(\sigma[y]) < 1$ (Corollary 5.8). From Eq. (7.1), we then obtain that $0 < T(x, y)$. Due to the commutativity of T on $[0, 1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$, this inequality also holds for every $(x, y) \in \mathcal{D}_{\mathrm{III}}$. As $T(x, y) = 0$, for every $(x, y) \notin \mathcal{D}$, the involutive negator C_0 is indeed a contour line of T. Next, we attribute a similar interpretation to the function C_β. Recall from the proof of property **(K1)** that $\beta < T(x, y)$ holds for every $(x, y) \in \mathcal{D}_{\mathrm{I}}$ and $T(x, y) \leqslant \beta$ holds for every $(x, y) \in [0, 1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$. In combination with $T(x, C_\beta(x)) \leqslant \beta$, for every $x \in]\beta, 1[$, and taking into account the increasingness of T, we obtain that C_β is indeed a contour line of T and that $T^\beta = \widehat{T}$.

(K7c): Consider $(x, y) \in \mathcal{D}_{\mathrm{I}}$ and let $a \in [\beta, 1]$. Taking into account that $\widehat{C}_b(\widehat{T}(u, v)) = \widehat{C}_{\widehat{C}_b(u)}(v)$ holds for every $(u, v, b) \in [0, 1]^3$ (Theorem 5.10) and $\widehat{C}_b = \sigma \circ C_{\sigma^{-1}[b]} \circ \sigma^{-1}$ holds for every $b \in [0, 1]$

(properties **(K6)** and **(F4a)**), we can derive the following chain of equalities:

$$\begin{aligned}
C_a(T(x,y)) &= C_a\left(\sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right]\right) = \sigma^{-1}\left[\widehat{C}_{\sigma[a]}\left(\widehat{T}(\sigma[x],\sigma[y])\right)\right] \\
&= \sigma^{-1}\left[\widehat{C}_{\widehat{C}_{\sigma[a]}(\sigma[x])}(\sigma[y])\right] = \sigma^{-1}\left[\sigma\left[C_{\sigma^{-1}[\widehat{C}_{\sigma[a]}(\sigma[x])]}(y)\right]\right] = C_{C_a(x)}(y)\,.
\end{aligned}$$

The commutativity of T on $\mathcal{D}_{\mathrm{I}}$ ensures that also $C_a(T(x,y)) = C_{C_a(y)}(x)$. Whenever $(x,y) \in [0,1]^2 \setminus \mathcal{D}_{\mathrm{I}}$ it holds that $T(x,y) \leqslant \beta$ (property **(K5)**). Since it also holds that $(y,x) \in [0,1]^2 \setminus \mathcal{D}_{\mathrm{I}}$ (proof of property **(K4)**), we obtain $\max(T(x,y),T(y,x)) \leqslant \beta \leqslant a$. Hence, $y \leqslant C_a(x)$ and $x \leqslant C_a(y)$. It then follows that $C_a(T(x,y)) = C_{C_a(x)}(y) = C_{C_a(y)}(x) = 1$.

(K7a): From properties **(K6)** and **(F4a)** we know that $C_x(y) = \sigma^{-1}[\widehat{C}_{\sigma[x]}(\sigma[y])]$ holds for every $(x,y) \in [\beta,1]^2$. Take arbitrary $(x,y) \in \mathcal{D}_{\mathrm{II}}$. Then $(C_0(x),y) \in [\beta,1]^2$ and it follows from Eq. (7.1) that $C_0(T(x,y)) = C_{C_0(x)}(y)$. We now have to verify that also $C_0(T(x,y)) = C_{C_0(y)}(x)$ holds. It follows from the decreasingness of $C_{C_0(x)}$ that $C_0(x) = C_{C_0(x)}(1) \leqslant C_{C_0(x)}(y)$. Due to the increasingness of $C_\bullet(x)$ we know that $C_0(x) \leqslant C_{C_0(y)}(x)$. Taking into account that $C_0(x) < C_0(0) = 1$, we obtain that

$$\begin{aligned}
C_{C_0(x)}(y) &= \sup\{t \in \,]C_0(x),1] \mid T(y,t) \leqslant C_0(x)\}\,, \\
C_{C_0(y)}(x) &= \sup\{t \in \,]C_0(x),1] \mid T(x,t) \leqslant C_0(y)\}
\end{aligned}$$

(with $\sup\emptyset = C_0(x)$). Because $t \in \,]C_0(x),1]$ expresses that $(x,t) \in \mathcal{D}_{\mathrm{II}}$, we can use Eq. (7.1) to rewrite $T(x,t)$. Recall that $\widehat{T} = T^\beta$ (property **(K6)**).

$$\begin{aligned}
C_{C_0(y)}(x) &= \sup\bigl\{t \in \,]C_0(x),1] \mid C_0\bigl(\sigma^{-1}\bigl[\widehat{C}_{\sigma[C_0(x)]}(\sigma[t])\bigr]\bigr) \leqslant C_0(y)\bigr\} \\
&= \sup\bigl\{t \in \,]C_0(x),1] \mid \widehat{C}_{\sigma[C_0(x)]}(\sigma[t]) \geqslant \sigma[y]\bigr\} \\
&= \sup\{t \in \,]C_0(x),1] \mid \widehat{T}(\sigma[t],\sigma[y]) \leqslant \sigma[C_0(x)]\} \\
&= \sup\{t \in \,]C_0(x),1] \mid \sigma^{-1}[\widehat{T}(\sigma[y],\sigma[t])] \leqslant C_0(x)\} \\
&= \sup\{t \in \,]C_0(x),1] \mid \max(\beta,T(y,t)) \leqslant C_0(x)\} \\
&= \sup\{t \in \,]C_0(x),1] \mid T(y,t) \leqslant C_0(x)\} = C_{C_0(x)}(y) = C_0(T(x,y))\,.
\end{aligned}$$

The commutativity of T on $\mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}}$ ensures that property **(K7a)** is also valid for every $(x,y) \in \mathcal{D}_{\mathrm{III}}$.

Now take arbitrary $(x,y) \in \mathcal{D}_{\mathrm{I}}$. Considering the commutativity of T on $\mathcal{D}_{\mathrm{I}}$ it suffices to prove the equality $C_0(T(x,y)) = C_{C_0(x)}(y)$. Note that $C_\beta(x) < y$ and $C_0(y) < \beta$. Because $(x,\beta) \in \mathcal{D}_{\mathrm{III}}$, it follows from the above discussion that $C_{C_0(x)}(\beta) = C_{C_0(\beta)}(x) = C_\beta(x) < y$. Hence, $C_0(x) < T(\beta,y) = T(y,\beta)$ from which we conclude that $C_{C_0(x)}(y) \leqslant \beta$. Recall that $C_0(y) \leqslant C_{C_0(x)}(y)$. In this case $C_{C_0(x)}(y)$ can be expressed explicitly in the following way:

$$C_{C_0(x)}(y) = \sup\{t \in \,]C_0(y),\beta] \mid T(y,t) \leqslant C_0(x)\} = \sup\{t \in \,]C_0(y),\beta] \mid x \leqslant C_0(T(y,t))\}$$

(with $\sup\emptyset = C_0(y)$). Because $t \in\,]C_0(y), \beta]$ states that $(y,t) \in \mathcal{D}_{\text{III}}$, the latter can be rewritten as follows:

$$\begin{aligned} C_{C_0(x)}(y) &= \sup\bigl\{t \in\,]C_0(y),\beta] \mid x \leqslant \sigma^{-1}\bigl[\widehat{C}_{\sigma[C_0(t)]}(\sigma[y])\bigr]\bigr\} \\ &= \sup\{t \in\,]C_0(y),\beta] \mid \widehat{T}(\sigma[y],\sigma[x]) \leqslant \sigma[C_0(t)]\} \\ &= \sup\bigl\{t \in\,]C_0(y),\beta] \mid t \leqslant C_0\bigl(\sigma^{-1}[\widehat{T}(\sigma[x],\sigma[y])]\bigr)\bigr\} \\ &= \sup\{t \in\,]C_0(y),\beta] \mid t \leqslant C_0(T(x,y))\} = C_0(T(x,y))\,. \end{aligned}$$

Note that in the last step properties **(K3)** and **(K5)** ensure that $\beta < T(x,y) \leqslant y$, which leads to $C_0(y) \leqslant C_0(T(x,y)) < \beta$.

For every $(x,y) \in [0,1]^2 \setminus \mathcal{D}$ it holds that $y \leqslant C_0(x)$ and $T(x,y) = 0$. Invoking We immediately obtain that $C_{C_0(x)}(y) = C_0(T(x,y)) = 1$. From the commutativity of T on $[0,1]^2 \setminus \mathcal{D}$ (proof of property **(K4)**) it then follows that also $C_0(T(x,y)) = C_{C_0(y)}(x) = 1$.

(K7b): Take arbitrary $(x,y) \in \mathcal{D}_{\text{II}}$. Then $T(x,y) \leqslant x \leqslant \beta$ and for every $a \in [T(x,y),\beta] = [T(y,x),\beta]$ it holds that $y \leqslant C_a(x)$ and $x \leqslant C_a(y)$. In this case $C_a(T(x,y)) = C_{C_a(x)}(y) = C_{C_a(y)}(x) = 1$. Now let $a = C_0(C_0(a)) < T(x,y) \leqslant \beta$. Then $(C_0(a), T(x,y)) \in \mathcal{D}_{\text{III}}$ and applying property **(K7a)** twice leads to

$$C_a(T(x,y)) = C_{C_0(C_0(a))}(T(x,y)) = C_{C_0(T(x,y))}(C_0(a)) = C_{C_{C_0(x)}(y)}(C_0(a))\,. \tag{8.1}$$

Moreover, as $C_0(x) \in [\beta,1[$ it follows from property **(K7c)** that $C_a(T(x,y)) = C_{C_{C_0(x)}(C_0(a))}(y)$. Note that $C_0(C_0(a)) = a < T(x,y) \leqslant x \leqslant \beta$ from which we can derive that also $(C_0(a),x) \in \mathcal{D}_{\text{III}}$. Hence, $C_{C_0(x)}(C_0(a)) = C_a(x)$ (property **(K7a)**) and we conclude that $C_a(T(x,y)) = C_{C_a(x)}(y)$. This leaves us to prove the equality $C_{C_a(y)}(x) = C_a(T(x,y))$. Clearly, $C_0(x) \leqslant C_{C_a(y)}(x)$ such that

$$C_{C_a(y)}(x) = \sup\{t \in\,]C_0(x),1] \mid T(x,t) \leqslant C_a(y)\} \tag{8.2}$$

(with $\sup\emptyset = C_0(x)$). Since $a < T(x,y) \leqslant T(\beta,y) = C_0(C_0(T(\beta,y)))$ and $(\beta,y) \in \mathcal{D}_{\text{II}}$, it holds that $a < C_0(C_\beta(y))$ (property **(K7a)**). The latter is equivalent with $C_\beta(y) < C_0(a)$ and thus $(y, C_0(a)) \in \mathcal{D}_{\text{I}}$. From property **(K7a)** it then follows that $C_a(y) = C_0(T(y,C_0(a)))$. Furthermore, for every $t \in\,]C_0(x),1]$ it holds that $T(x,t) = C_0(C_{C_0(x)}(t))$ (property **(K7a)**). Eq. (8.2) can then be rewritten as follows

$$\begin{aligned} C_{C_a(y)}(x) &= \sup\{t \in\,]C_0(x),1] \mid C_0(C_{C_0(x)}(t)) \leqslant C_0(T(y,C_0(a)))\} \\ &= \sup\{t \in\,]C_0(x),1] \mid T(y,C_0(a)) \leqslant C_{C_0(x)}(t)\}\,. \end{aligned}$$

Recall that $(y,C_0(a)) \in \mathcal{D}_{\text{I}}$ and $(C_0(x),t) \in [\beta,1]^2$, for every $t \in\,]C_0(x),1]$. Taking into account properties **(K6)** and **(F4a)**, this allows us to rewrite the above expression as follows:

$$C_{C_a(y)}(x) = \sup\bigl\{t \in\,]C_0(x),1] \mid \sigma^{-1}\bigl[\widehat{T}\bigl(\sigma[y],\sigma[C_0(a)]\bigr)\bigr] \leqslant \sigma^{-1}\bigl[\widehat{C}_{\sigma[C_0(x)]}(\sigma[t])\bigr]\bigr\}$$

Applying Theorem 5.12 and invoking properties **(K6)** and **(F4a)** leads to the following chain of equalities:

$$\begin{aligned} C_{C_a(y)}(x) &= \sup\big\{t \in \,]C_0(x), 1] \mid \sigma^{-1}\big[\widehat{T}\big(\sigma[C_0(a)], \sigma[t]\big)\big] \leqslant \sigma^{-1}\big[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\big]\big\} \\ &= \sup\{t \in \,]C_0(x), 1] \mid \max(\beta, T(C_0(a), t)) \leqslant C_{C_0(x)}(y)\}\,. \end{aligned}$$

Because $\beta \leqslant C_0(x) \leqslant C_0(T(x,y)) = C_{C_0(x)}(y)$ and considering Eq. (8.1), the latter reduces to

$$C_{C_a(y)}(x) = \sup\{t \in \,]C_0(x), 1] \mid T(C_0(a), t) \leqslant C_{C_0(x)}(y)\} = C_{C_{C_0(x)}(y)}(C_0(a)) = C_a(T(x,y))\,.$$

Note that indeed $C_0(x) = C_{C_0(x)}(1) \leqslant C_{C_0(x)}(y) = C_{C_{C_0(x)}(y)}(1) \leqslant C_{C_{C_0(x)}(y)}(C_0(a))$. The commutativity of T on $\mathcal{D}_{\text{II}} \cup \mathcal{D}_{\text{III}}$ implies that property **(K7b)** also holds for every $(x,y) \in \mathcal{D}_{\text{III}}$.

Furthermore, for arbitrary $(x,y) \in [0,1]^2 \setminus \mathcal{D}$ we know that $y \leqslant C_0(x) \leqslant C_a(x)$, $x \leqslant C_0(y) \leqslant C_a(y)$ and $T(x,y) = 0 \leqslant a$, for every $a \in [0,\beta]$. Hence, $C_{C_a(x)}(y) = C_{C_a(y)}(x) = C_a(T(x,y)) = 1$. $\square$

Suppose now that $\mathcal{D}_{\text{IV}} = \emptyset$, then $\mathcal{D}_{\text{I}} = \,]\beta, 1]^2$ and Eq. (7.1) determines T on the whole unit square $[0,1]^2$. By definition we know that $C_\beta(x) = \beta$, for every $x \in \,]\beta, 1]$, is equivalent with $\widehat{C}_0(x) = 0$, for every $x \in \,]0,1]$. Hence, $\mathcal{D}_{\text{IV}} = \emptyset$ holds if and only if $\widehat{T}$ has no zero divisors.

Theorem 8.3 *If $\widehat{T}$ has no zero divisors, then the $[0,1]^2 \to [0,1]$ function T defined by Eq. (7.1) is a rotation-invariant t-norm satisfying $T^\beta = \widehat{T}$.*

Proof Property 8.1 is applicable and due to the non-existence of area $\mathcal{D}_{\text{IV}}$ we immediately conclude that T is increasing, left continuous, has neutral element 1 and is commutative. Therefore, $T(x, C_\beta(x)) = T(x,\beta) \leqslant T(1,\beta) = \beta$ holds for every $x \in \,]\beta, 1]$. Property 8.2 then provides the following characteristics: $\beta < T(x,y) \Leftrightarrow (x,y) \in \mathcal{D}_{\text{I}}$, $T^\beta = \widehat{T}$ and $C_a(T(x,y)) = C_{C_a(x)}(y) = C_{C_a(y)}(x)$ holds for every $(x,y) \in [0,1]^2$ if $a \in \{0\} \cup [\beta, 1]$ and for every $(x,y) \in [0,1]^2 \setminus \mathcal{D}_{\text{I}}$ if $a \in \,]0,\beta[$. Take $a \in \,]0,\beta[$ and $(x,y) \in \mathcal{D}_{\text{I}}$. Denote $C_0(a)$ as z, then also $(y,z) \in \mathcal{D}_{\text{I}}$. Hence, $T(x,y) \in \,]\beta,1]$ and $T(y,z) \in \,]\beta,1]$ such that $(T(x,y),z) \in \mathcal{D}_{\text{I}}$ and $(x, T(y,z)) \in \mathcal{D}_{\text{I}}$. Making use of property **(K7a)**, Eq. (7.1) and the associativity of $\widehat{T}$, we obtain the following chain of equalities:

$$\begin{aligned} C_a(T(x,y)) &= C_{C_0(z)}(T(x,y)) = C_0(T(T(x,y),z)) \\ &= C_0\big(\sigma^{-1}\big[\widehat{T}\big(\widehat{T}(\sigma[x], \sigma[y]), \sigma[z]\big)\big]\big) = C_0\big(\sigma^{-1}\big[\widehat{T}\big(\sigma[x], \widehat{T}(\sigma[y], \sigma[z])\big)\big]\big) \\ &= C_0(T(x, T(y,z))) = C_{C_0(T(y,z))}(x) = C_{C_{C_0(z)}(y)}(x) = C_{C_a(y)}(x)\,. \end{aligned}$$

Taking into account the commutativity of T, we conclude that $C_a(T(x,y)) = C_{C_a(x)}(y) = C_{C_a(y)}(x)$ holds for every $(x,y,a) \in [0,1]^3$. The associativity of T now immediately follows from Theorem 5.10. Therefore, T is a left-continuous t-norm. Theorem 6.17 states that T must be rotation invariant as its contour line C_0 is involutive. $\square$

Note that in case $\mathcal{D}_{\mathrm{IV}} = \emptyset$ (*i.e.* $\widehat{T}$ has no zero-divisors), property **(I1)** is trivially satisfied and the construction method from the previous theorem coincides with the rotation construction of Jenei (Theorem 6.26). His approach is slightly more general, as he also allows t-norms $\widehat{T}$ whose set of zero-divisors fills up a sub-square of $[0,1]^2$ (property **(I1)**). However, working with our partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$, the zero divisors of $\widehat{T}$ determine area $\mathcal{D}_{\mathrm{IV}}$. Starting from Eq. (7.1) the question remains how to define $T|_{\mathcal{D}_{\mathrm{IV}}}$ when $\mathcal{D}_{\mathrm{IV}} \neq \emptyset$. Recall from Fig. 7.2 that the filling-in of this area is not always uniquely fixed by $\widehat{T}$ and C_0. In case the contour line $\widehat{C}_0$ of $\widehat{T}$ is continuous on $]0,1]$, however, Theorem 7.9 states that there exists at most one appropriate choice for $T|_{\mathcal{D}_{\mathrm{IV}}}$. Recall that Theorem 7.9 originates from merging Theorem 7.5 with Theorem 7.8. For the construction process, we use a similar approach. First we invert Theorem 7.5 into a construction theorem. That result is then used to invert Theorem 7.9.

Examining Eq. (7.5) more carefully, we have shown in Theorem 7.5 that the companion $\widehat{Q}$ of the β-zoom $\widehat{T}(= T^{\beta})$ of T must be commutative on $[0,1[^2$. When constructing rotation-invariant t-norms, this property restricts the possible choices for $\widehat{T}$. In the following theorem we present three assertions, each establishing the commutativity of $\widehat{Q}$ on $[0,1[^2$.

Theorem 8.4 *For every rotation-invariant t-norm T the following assertions are equivalent:*

(L1) *Q is commutative on $[0,1[^2$.*
(L2) *$T(x,y) < T(x+\varepsilon, y+\varepsilon)$, for every $(x,y,\varepsilon) \in\,]0,1[^3$ satisfying $C_0(x) < y$ and $\varepsilon \leqslant 1 - \max(x,y)$.*
(L3) *$Q\left(C_0(x), C_0(T(y,z))\right) = Q\left(C_0(z), C_0(T(y,x))\right)$, for every $(x,y,z) \in [0,1]^3$ satisfying $C_0(y) < \min(x,z)$.*

Proof (L1)$\Leftrightarrow$(L2): Consider a rotation-invariant t-norm T whose companion Q is commutative on $[0,1[^2$. Suppose that there exist $(x,y,\varepsilon) \in\,]0,1[^3$ satisfying $C_0(x) < y$ and $\varepsilon \leqslant 1 - \max(x,y)$ such that $T(x,y) = T(x+\varepsilon, y+\varepsilon)$. Denote $z := C_0(T(x,y))$. Then, $z \in\,]0,1[$ as $0 < T(x,y) < 1$. The increasingness and commutativity of T ensure that

$$C_0(z) = T(y,x) = T(y+\delta, x+\varepsilon) = T(y+\varepsilon, x+\varepsilon)\,,$$

for every $\delta \in\,]0, \min(\varepsilon, 1-z)]$. Considering the involutivity of C_0 (assertion **(G2)**) and applying assertion **(G5)** on $T(y+\delta, x+\varepsilon) = C_0(z)$ and on $C_0(z+\delta) < T(y,x)$ leads to

$$T(z, y+\delta) = T(y+\delta, z) \leqslant C_0(x+\varepsilon) < C_0(x) < T(y, z+\delta)\,,$$

for every $\delta \in\,]0, \min(\varepsilon, 1-z)]$. From property **(E2)** we obtain the contradiction $Q(z,y) \leqslant C_0(x+\varepsilon) < C_0(x) \leqslant Q(y,z)$. Hence, assertion **(L2)** is implied by assertion **(L1)**.

Conversely, let T be a rotation-invariant t-norm satisfying assertion **(L2)** and suppose that $Q(x,y) < Q(y,x)$, for some $(x,y) \in [0,1[^2$. Note that in particular $(x,y) \in\,]0,1[^2$ as $Q(t,0) = Q(0,t) = 0$, for every $t \in [0,1[$. Furthermore, the involutivity of C_0 and property **(E2)** ensure the existence of a couple $(z,\varepsilon) \in\,]0,1[^2$ such that

$$Q(x,y) \leqslant T(x, y+\varepsilon) \leqslant C_0(z+\varepsilon) < C_0(z) < Q(y,x)\,.$$

Clearly, $\varepsilon \leqslant 1 - \max(y, z)$. Invoking property **(E2)** once more, $Q(y, x)$ can be bounded from above by $T(y, x + \delta)$, for any $\delta \in]0, 1 - x]$. Next, we apply assertion **(G5)** to the inequalities $T(x, y + \varepsilon) \leqslant C_0(z + \varepsilon)$ and $C_0(z) < T(y, x + \delta)$. This results in

$$C_0(x + \delta) < T(y, z) \leqslant T(y + \varepsilon, z + \varepsilon) \leqslant C_0(x)\,,$$

for every $\delta \in]0, 1 - x]$. Taking the limit $\delta \searrow 0$ yields the contradiction $T(y, z) = T(y + \varepsilon, z + \varepsilon)$. Note that indeed $C_0(z) < Q(y, x) \leqslant y$ (property **(E5)**).

(L1)⇔(L3): In case $y = 1$, assertion **(L3)** states that $Q\left(C_0(x), C_0(z)\right) = Q\left(C_0(z), C_0(x)\right)$ whenever $(x, z) \in]0, 1]^2$. Denoting $u := C_0(x)$ and $v := C_0(z)$, the latter expresses the commutativity of Q on $[0, 1[^2$, *i.e.* $Q\left(u, v\right) = Q\left(v, u\right)$ for every $(u, v) \in [0, 1[^2$. Hence, we only need to prove that assertion **(L1)** implies assertion **(L3)**. Suppose that assertion **(L1)** is true and assertion **(L3)** is false. We may assume that $Q\left(C_0(x), C_0(T(y, z))\right) < Q\left(C_0(z), C_0(T(y, x))\right)$, for some $(x, y, z) \in [0, 1]^3$ satisfying $C_0(y) < \min(x, z)$. This particular triplet (x, y, z) fulfills $0 < \min(z, T(y, x))$ and hence, $\max(C_0(z), C_0(T(y, x)) < 1$, implying that $Q(C_0(z), C_0(T(y, x)) = Q(C_0(T(y, x), C_0(z))$. Now take arbitrary

$$t \in \;\big]Q\big(C_0(x), C_0(T(y, z))\big), Q\big(C_0(T(y, x)), C_0(z)\big)\big[\,. \tag{8.3}$$

Using the definition of Q, t must satisfy the following inequalities:

$$C_0(T(y, z)) < C_t(C_0(x)) \quad \wedge \quad C_t(C_0(T(y, x))) \leqslant C_0(z)\,.$$

Due to the involutivity of C_0 and assertion **(G3)**, we can rewrite the above inequalities in the following way

$$T(C_0(t), C_0(x)) = C_0(C_t(C_0(x))) < T(y, z)\,, \tag{8.4}$$

$$z \leqslant C_0\big(C_t(C_0(T(y, x)))\big) = T\big(C_0(t), C_0(T(y, x))\big)\,. \tag{8.5}$$

Applying assertion **(G5)**, Eq. (8.4) is equivalent with

$$C_0(z) < T\big(y, C_0(T(C_0(t), C_0(x)))\big)\,. \tag{8.6}$$

Also, from assertion **(G5)** and $T(C_0(t), C_0(x)) = T(C_0(t), C_0(x))$ we obtain that

$$T\big(C_0(t), C_0(T(C_0(t), C_0(x)))\big) \leqslant x\,.$$

Consequently,

$$\begin{aligned} T(C_0(t), C_0(z)) &\leqslant T\big(C_0(t), T\big(y, C_0(T(C_0(t), C_0(x)))\big)\big) \\ &= T\big(y, T\big(C_0(t), C_0(T(C_0(t), C_0(x)))\big)\big) \leqslant T(y, x)\,. \end{aligned} \tag{8.7}$$

Applying assertion **(G5)** once more on the left- and right-hand side of the above chain of inequalities, we obtain a lower bound for z: $T(C_0(t), C_0(T(y, x))) \leqslant z$. In combination with

Eq. (8.5), we conclude that $z = T(C_0(t), C_0(T(y,x)))$. Substituting this expression for z in Eq. (8.6) and using the involutivity of C_0 and assertion **(G3)** leads to the following inequality:

$$C_{T(y,x)}(C_0(t)) = C_0\big(T\big(C_0(t), C_0(T(y,x))\big)\big) < T\big(y, C_0(T(C_0(t), C_0(x)))\big)\,.$$

Based on the definition of contour lines, this inequality implies that

$$T(y,x) < T\big(C_0(t), T\big(y, C_0(T(C_0(t), C_0(x)))\big)\big)\,,$$

which contradicts Eq. (8.7). $\square$

If the companion $\widehat{Q}$ of a rotation-invariant t-norm $\widehat{T}$ is commutative on $[0,1[^2$, Eq. (7.5) can be invoked to construct a rotation-invariant t-norm that has contour line C_0 and β-zoom $\widehat{T}$.

Theorem 8.5 *If $\widehat{T}$ is rotation invariant, then the $[0,1]^2 \to [0,1]$ function T defined by Eq. (7.5) is a rotation-invariant t-norm if and only if $\widehat{Q}$ is commutative on $[0,1[^2$. In this case $T^\beta = \widehat{T}$.*

Proof As in the proof of Theorem 8.3, Properties 8.1 and 8.2 will provide most of the t-norm properties of T. Because T is on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$ also given by Eq. (7.1), Property 8.1 is immediately applicable. The rotation invariance of $\widehat{T}$ implies that its contour line $\widehat{C}_0$ is involutive on $[0,1]$ (assertion **(G2)**). By definition, the involutivity of $\widehat{C}_0$ implies that C_β is involutive on $[\beta,1]$. Therefore, $(x, C_\beta(x)) \in \mathcal{D}_{\mathrm{IV}}$ and $T(x, C_\beta(x)) \leqslant C_0(\sigma^{-1}[0]) = C_0(\beta) = \beta$, for every $x \in\,]\beta,1[$. To show the validity of Property 8.2 we have to prove that T is increasing. Property **(K1)** and the increasingness of $\widehat{Q}$ (property **(E1)**) imply the increasingness of T on $[0,1]^2 \setminus \mathcal{D}_{\mathrm{IV}}$ and on $\mathcal{D}_{\mathrm{IV}}$. We only have to verify what happens on the borders between areas $[0,1]^2 \setminus (\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}})$ and $\mathcal{D}_{\mathrm{IV}}$ and between areas $\mathcal{D}_{\mathrm{I}}$ and $\mathcal{D}_{\mathrm{IV}}$. Recall from the proof of property **(K1)** that $T(x,y) > \beta$ whenever $(x,y) \in \mathcal{D}_{\mathrm{I}}$. Together with $T(x, C_\beta(x)) \leqslant \beta$, for every $x \in\,]\beta,1[$, this implies that T is increasing on $\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}}$. Making use of $\widehat{Q}|_{[0,1[^2} \leqslant T_{\mathbf{M}}|_{[0,1[^2}$ (property **(E5)**) and the fact that $(x,y) \in \mathcal{D}_{\mathrm{IV}} \;\Leftrightarrow\; (y,x) \in \mathcal{D}_{\mathrm{IV}}$ (property **(K4)**), we get that

$$\left.\begin{array}{c} T(x,\beta) \\ T(\beta,x) \end{array}\right| = C_0\big(\sigma^{-1}[\widehat{C}_0(\sigma[x])]\big) \leqslant \left|\begin{array}{c} T(x,y) \\ T(y,x) \end{array}\right.\,,$$

for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. We conclude that T is indeed increasing on $[0,1]^2$.

By definition, the continuity of $\widehat{C}_0$ (assertion **(G1)**) is passed on to C_β. Property **(K5)** then ensures that C_β is a continuous contour line of T. If T is a rotation-invariant t-norm, it follows from Theorem 7.5 that $\widehat{Q}$ is commutative on $[0,1[^2$. Conversely, assume that $\widehat{Q}$ is commutative on $[0,1[^2$. We now need to show that T is a rotation-invariant t-norm. The left continuity of T follows immediately from property **(K2)**, the continuity of C_0, the involutivity of $\widehat{C}_0$ and the right continuity of $\widehat{Q}$. Note that $\widehat{Q}$ is indeed right continuous as it is right continuous in its second argument (property **(E4)**) and commutative on $[0,1[^2$. This commutativity of $\widehat{Q}$ combined with property **(K4)** implies the commutativity of T. Property **(K3)** states that 1 is the neutral element of T and property **(K6)** states that $T^\beta = \widehat{T}$. Unfortunately, the associativity of T cannot be straightforwardly obtained. We must show that $C_a(T(x,y)) = C_{C_a(x)}(y)$ holds for every $(x, y, a) \in [0, 1]^3$ (Theorem 5.10). Considering properties **(K7a)**–**(K7c)**, it suffices to prove that $C_a(T(x, y)) = C_{C_a(x)}(y)$ holds for every $(x, y) \in \mathcal{D}_{\mathrm{IV}}$ whenever $a = 0$ and for every $(x, y) \in \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}} =]\beta, 1]^2$ whenever $a \in]0, \beta[$. Like in the proofs of Properties 8.1 and 8.2, the key is to translate the properties of the t-norm $\widehat{T}$ into properties of T. In this respect, we make extensive use of Property 6.13 and Corollary 6.15.

Take $(x, y) \in \mathcal{D}_{\mathrm{IV}}$. We will first prove that $C_0(T(x, y)) = C_{C_0(x)}(y)$. From $y \leqslant C_\beta(x)$ we derive that $T(x, \beta) = C_0(C_\beta(x)) \leqslant C_0(y)$ (assertions **(G2)** and **(G3)**). Consequently, $\beta \leqslant C_{C_0(y)}(x)$. Furthermore, as $y \in]\beta, 1[$, we get that $C_0(y) < \beta$. The latter implies that $C_{C_0(y)}(x) \leqslant C_\beta(x)$ and we obtain that

$$\begin{aligned} C_{C_0(y)}(x) &= \sup\{t \in]\beta, C_\beta(x)] \mid T(x, t) \leqslant C_0(y)\} \\ &= \sup\big\{t \in]\beta, C_\beta(x)] \mid C_0\big(\sigma^{-1}\big[\widehat{Q}\big(\widehat{C}_0(\sigma[x]), \widehat{C}_0(\sigma[t])\big)\big]\big) \leqslant C_0(y)\big\} \\ &= \sup\big\{t \in]\beta, C_\beta(x)] \mid \sigma[y] \leqslant \widehat{Q}\big(\widehat{C}_0(\sigma[x]), \widehat{C}_0(\sigma[t])\big)\big\}\,, \end{aligned}$$

(with $\sup \emptyset = \beta$). Taking into account assertion **(G7)** and the involutivity of $\widehat{C}_0$ this leads to

$$\begin{aligned} C_{C_0(y)}(x) &= \sup\big\{t \in]\beta, C_\beta(x)] \mid \sigma[t] \leqslant \widehat{Q}\big(\widehat{C}_0(\sigma[x]), \widehat{C}_0(\sigma[y])\big)\big\} \\ &= \sup\big\{t \in]\beta, C_\beta(x)] \mid t \leqslant \sigma^{-1}\big[\widehat{Q}\big(\sigma[C_\beta(x)], \sigma[C_\beta(y)]\big)\big]\big\} \\ &= \sigma^{-1}\big[\widehat{Q}\big(\sigma[C_\beta(x)], \sigma[C_\beta(y)]\big)\big] = C_0(T(x, y))\,. \end{aligned}$$

Note that indeed $\sigma^{-1}[\widehat{Q}(\sigma[C_\beta(x)], \sigma[C_\beta(y)])] \in [\beta, C_\beta(x)]$ as $\sigma^{-1}[0] = \beta$ and $\widehat{Q}|_{[0,1[^2} \leqslant T_{\mathbf{M}}|_{[0,1[^2}$. Due to the commutativity of T on $\mathcal{D}_{\mathrm{IV}}$, we obtain that $C_0(T(x, y)) = C_{C_0(x)}(y) = C_{C_0(y)}(x)$ holds for every $(x, y) \in \mathcal{D}_{\mathrm{IV}}$. In combination with property **(K7a)** we conclude that the latter chain of equalities must even hold for every $(x, y) \in [0, 1]^2$.

Take arbitrary $a \in]0, \beta[$ and $(x, y) \in]\beta, 1]^2$. As $C_0(T(x, y)) = C_{C_0(x)}(y)$ holds for every $(x, y) \in [0, 1]^2$ we get that $C_a(T(x, y)) = C_{C_0(C_0(a))}(T(x, y)) = C_0(T(C_0(a), T(x, y)))$ and

$$C_{C_a(x)}(y) = C_{C_{C_0(C_0(a))}(x)}(y) = C_{C_0(T(C_0(a), x))}(y) = C_0\big(T(T(C_0(a), x), y)\big)\,.$$

Hence, proving that $C_a(T(x, y)) = C_{C_a(x)}(y)$ holds becomes then equivalent with proving that $T(C_0(a), T(x, y)) = T(T(C_0(a), x), y)$. If we denote $(C_0(a), x, y)$ as (x', y', z'), the latter will be satisfied if $T(T(x', y'), z') = T(x', T(y', z'))$ holds for arbitrary $(x', y', z') \in]\beta, 1]^3$. To prove this we will make extensive use of property **(K5)** and the involutivity of C_β on $[\beta, 1]$. Take $(x, y, z) \in]\beta, 1]^3$. Suppose that $C_\beta(T(x, y)) < z$. Then, because $C_\beta(T(x, y)) < 1$, it must hold that $\beta < T(x, y)$ (Corollary 5.8), meaning that $(x, y) \in \mathcal{D}_{\mathrm{I}}$. From Eq. (7.5) and Theorem 5.12 it then follows that

$$\begin{aligned} C_\beta(T(x, y)) < z \;&\Leftrightarrow\; C_\beta(z) < T(x, y) \;\Leftrightarrow\; \widehat{C}_0(\sigma[z]) < \widehat{T}(\sigma[x], \sigma[y]) \\ &\Leftrightarrow\; \widehat{C}_0(\sigma[x]) < \widehat{T}(\sigma[y], \sigma[z]) \;\Leftrightarrow\; C_\beta(x) < T(y, z)\,. \end{aligned} \tag{8.8}$$

Note that indeed $(y,z) \in \mathcal{D}_{\mathrm{I}}$ since $C_\beta(y) \leqslant C_\beta(T(x,y)) < z$. In particular, Eq. (8.8) expresses that $\{(T(x,y),z),(x,T(y,z))\} \subset \mathcal{D}_{\mathrm{I}}$:

$$T(T(x,y),z) = \sigma^{-1}\big[\widehat{T}\big(\widehat{T}(\sigma[x],\sigma[y]),\sigma[z]\big)\big] = \sigma^{-1}\big[\widehat{T}\big(\sigma[x],\widehat{T}(\sigma[y],\sigma[z])\big)\big] = T(x,T(y,z))\,.$$

Taking into account Eq. (8.8) and the chain of equivalences

$$\begin{aligned} T(T(x,y),z) = 0 \;&\Leftrightarrow\; (T(x,y),z) \in [0,1]^2 \setminus \mathcal{D} \;\Leftrightarrow\; z \leqslant C_0(T(x,y)) = C_{C_0(x)}(y) \\ &\Leftrightarrow\; T(y,z) \leqslant C_0(x) \;\Leftrightarrow\; (x,T(y,z)) \in [0,1]^2 \setminus \mathcal{D} \;\Leftrightarrow\; T(x,T(y,z)) = 0\,, \end{aligned}$$

we only need to show that $T(T(x,y),z) = T(x,T(y,z))$ holds for every $(x,y,z) \in \,]\beta,1]^3$ fulfilling $C_0(T(x,y)) < z \leqslant C_\beta(T(x,y))$. Note that

$$\begin{aligned} C_0(T(x,y)) < z \leqslant C_\beta(T(x,y)) \;&\Leftrightarrow\; (T(x,y),z) \in \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{IV}} \\ &\Leftrightarrow\; (x,T(y,z)) \in \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}} \;\Leftrightarrow\; C_0(x) < T(y,z) \leqslant C_\beta(x)\,. \end{aligned}$$

For such a triplet (x,y,z) we distinguish four cases:

I. $\boldsymbol{(x,y) \in \mathcal{D}_{\mathrm{I}} \quad\wedge\quad (y,z) \in \mathcal{D}_{\mathrm{I}}}$

The location of (x,y) and (y,z) in the domain of T implies that $\beta < \min(T(x,y),T(y,z))$, leading to $\{(T(x,y),z),(x,T(y,z))\} \subset \mathcal{D}_{\mathrm{IV}}$. Invoking the commutativity of T on $\mathcal{D}_{\mathrm{I}}$ and $\mathcal{D}_{\mathrm{IV}}$, we need to prove that $T(z,T(y,x)) = T(x,T(y,z))$, with $\{(z,T(y,x)),(x,T(y,z))\} \subset \mathcal{D}_{\mathrm{IV}}$:

$$\begin{aligned} T(z,T(y,x)) &= C_0\left(\sigma^{-1}\left[\widehat{Q}\left(\widehat{C}_0(\sigma[z]),\widehat{C}_0(\widehat{T}(\sigma[y],\sigma[x]))\right)\right]\right)\,; \\ T(x,T(y,z)) &= C_0\left(\sigma^{-1}\left[\widehat{Q}\left(\widehat{C}_0(\sigma[x]),\widehat{C}_0(\widehat{T}(\sigma[y],\sigma[z]))\right)\right]\right)\,. \end{aligned}$$

Denote $u := \sigma[x]$, $v := \sigma[y]$ and $w := \sigma[z]$. Then $\{(y,x),(y,z)\} \subset \mathcal{D}_{\mathrm{I}}$ becomes equivalent with $\widehat{C}_0(v) < \min(u,w)$ and assertion **(L3)** implies that

$$\begin{aligned} T(z,T(y,x)) &= C_0\left(\sigma^{-1}\left[\widehat{Q}\left(\widehat{C}_0(w),\widehat{C}_0(\widehat{T}(v,u))\right)\right]\right) \\ &= C_0\left(\sigma^{-1}\left[\widehat{Q}\left(\widehat{C}_0(u),\widehat{C}_0(\widehat{T}(v,w))\right)\right]\right) = T(x,T(y,z))\,. \end{aligned}$$

II. $\boldsymbol{(x,y) \in \mathcal{D}_{\mathrm{I}} \quad\wedge\quad (y,z) \in \mathcal{D}_{\mathrm{IV}}}$

Like in the previous case it holds that $(T(x,y),z) \in \mathcal{D}_{\mathrm{IV}}$. Furthermore, from $T(y,z) \leqslant \beta$ it follows that $(x,T(y,z)) \in \mathcal{D}_{\mathrm{III}}$. Denoting $u := \sigma[x]$, $v := \sigma[y]$ and $w := \sigma[z]$, $(x,y) \in \mathcal{D}_{\mathrm{I}}$ becomes equivalent with $\widehat{C}_0(u) < v$, $(y,z) \in \mathcal{D}_{\mathrm{IV}}$ becomes equivalent with $0 < v \leqslant \widehat{C}_0(w) < 1$ and

$$\begin{aligned} T(T(x,y),z) &= C_0\left(\sigma^{-1}\left[\widehat{Q}\left(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(w)\right)\right]\right)\,; \\ T(x,T(y,z)) &= C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(T(y,z))]}(\sigma[x])\right]\right) = C_0\left(\sigma^{-1}\left[\widehat{C}_{\widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))}(u)\right]\right)\,. \end{aligned}$$

It now suffices to prove that $\widehat{Q}\big(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(w)\big) = \widehat{C}_{\widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))}(u)$. As $\widehat{C}_\bullet(u)$ is increasing, we know that $\widehat{C}_0(u) \leqslant \widehat{C}_{\widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))}(u)$, which leads to

$$\widehat{C}_{\widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))}(u) = \sup\{t \in]\widehat{C}_0(u),1] \mid \widehat{T}(u,t) \leqslant \widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))\} \tag{8.9}$$

(with $\sup\emptyset = \widehat{C}_0(u)$). Observe that $0 < \widehat{T}(u,v)$ and $0 \leqslant \widehat{C}_0(u) < \min(v,t)$ whenever $t \in]\widehat{C}_0(u),1]$. Next, we rewrite Eq. (8.9) using the involutivity of $\widehat{C}_0$ and resp. assertion **(G7)**, assertion **(L3)**, the commutativity of $\widehat{Q}$ on $[0,1[^2$ and once again assertion **(G7)**:

$$\begin{aligned}
\widehat{C}_{\widehat{Q}(\widehat{C}_0(v),\widehat{C}_0(w))}(u) &= \sup\big\{t \in]\widehat{C}_0(u),1] \mid w \leqslant \widehat{Q}\big(\widehat{C}_0(v),\widehat{C}_0(\widehat{T}(u,t))\big)\big\}\\
&= \sup\big\{t \in]\widehat{C}_0(u),1] \mid w \leqslant \widehat{Q}\big(\widehat{C}_0(t),\widehat{C}_0(\widehat{T}(u,v))\big)\big\}\\
&= \sup\big\{t \in]\widehat{C}_0(u),1] \mid w \leqslant \widehat{Q}\big(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(t)\big)\big\}\\
&= \sup\big\{t \in]\widehat{C}_0(u),1] \mid t \leqslant \widehat{Q}\big(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(w)\big)\big\}\\
&= \widehat{Q}\big(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(w)\big)\,.
\end{aligned}$$

Note that indeed $\widehat{C}_0(u) \leqslant \widehat{Q}(\widehat{C}_0(\widehat{T}(u,v)),\widehat{C}_0(w))$. For $u = 1$ this is trivial. In case $u < 1$, this inequality is obtained by applying assertion **(G7)** on $\widehat{T}(u,v) \leqslant \widehat{T}(u,\widehat{C}_0(w)) = \widehat{T}(\widehat{C}_0(w),u) \leqslant \widehat{Q}(\widehat{C}_0(w),u)$ (property **(E3)**) and taking into account the involutivity of $\widehat{C}_0$ and the commutativity of $\widehat{Q}$ on $[0,1[^2$.

III. $(x,y) \in \mathcal{D}_{\mathrm{IV}} \quad\wedge\quad (y,z) \in \mathcal{D}_{\mathrm{I}}$

In this case $(T(x,y),z) \in \mathcal{D}_{\mathrm{II}}$ and $(x,T(y,z)) \in \mathcal{D}_{\mathrm{IV}}$. Invoking the commutativity of T on $\mathcal{D}_{\mathrm{I}}$, on $\mathcal{D}_{\mathrm{II}}\cup\mathcal{D}_{\mathrm{III}}$ and on $\mathcal{D}_{\mathrm{IV}}$, we need to prove that $T(z,T(y,x)) = T(T(z,y),x)$, where $(z,T(y,x)) \in \mathcal{D}_{\mathrm{III}}$ and $(T(z,y),x) \in \mathcal{D}_{\mathrm{IV}}$. The latter follows immediately from case **II**.

IV. $(x,y) \in \mathcal{D}_{\mathrm{IV}} \quad\wedge\quad (y,z) \in \mathcal{D}_{\mathrm{IV}}$

In this case $(T(x,y),z) \in \mathcal{D}_{\mathrm{II}}$ and $(x,T(y,z)) \in \mathcal{D}_{\mathrm{III}}$. Invoking the commutativity of T on $\mathcal{D}_{\mathrm{II}}\cup\mathcal{D}_{\mathrm{III}}$ and on $\mathcal{D}_{\mathrm{IV}}$, we need to prove that $T(T(x,y),z) = T(T(z,y),x)$, where $\{(T(x,y),z),(T(z,y),x)\} \subset \mathcal{D}_{\mathrm{II}}$. Denoting $u := \sigma[x]$, $v := \sigma[y]$ and $w := \sigma[z]$, then $\{(x,y),(y,z)\} \subset \mathcal{D}_{\mathrm{IV}}$ becomes equivalent with $0 < \max(u,w) \leqslant \widehat{C}_0(v) < 1$ and

$$\begin{aligned}
T(T(x,y),z) &= C_0\left(\sigma^{-1}\left[\widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w)\right]\right)\,;\\
T(T(z,y),x) &= C_0\left(\sigma^{-1}\left[\widehat{C}_{\widehat{Q}(\widehat{C}_0(w),\widehat{C}_0(v))}(u)\right]\right)\,.
\end{aligned}$$

It then suffices to show that $\widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w) = \widehat{C}_{\widehat{Q}(\widehat{C}_0(w),\widehat{C}_0(v))}(u)$. Clearly, $\widehat{C}_0(w) \leqslant \widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w)$ and

$$\widehat{T}(w,\widehat{C}_0(u)) \leqslant \widehat{T}(\widehat{C}_0(v),\widehat{C}_0(u)) = \widehat{T}(\widehat{C}_0(u),\widehat{C}_0(v)) \leqslant \widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))$$

(property **(E3)**) implies that $\widehat{C}_0(u) \leqslant \widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w)$. Since $\max(\widehat{C}_0(u), \widehat{C}_0(w)) = \widehat{C}_0(\min(u,w))$, we obtain that

$$\widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w) = \sup\{t \in \,]\widehat{C}_0(\min(u,w)), 1] \mid \widehat{T}(w,t) \leqslant \widehat{Q}(\widehat{C}_0(u), \widehat{C}_0(v))\} \tag{8.10}$$

(with $\sup \emptyset = \widehat{C}_0(\min(u,w))$). Taking into account the involutivity of $\widehat{C}_0$ and successively applying assertions **(G7)**, **(L3)** and once again **(G7)**, enables us to rewrite Eq. (8.10) in the following way:

$$\begin{aligned}
\widehat{C}_{\widehat{Q}(\widehat{C}_0(u),\widehat{C}_0(v))}(w) &= \sup\big\{t \in \,]\widehat{C}_0(\min(u,w)), 1] \mid v \leqslant \widehat{Q}\big(\widehat{C}_0(u), \widehat{C}_0(\widehat{T}(t,w))\big)\big\} \\
&= \sup\big\{t \in \,]\widehat{C}_0(\min(u,w)), 1] \mid v \leqslant \widehat{Q}\big(\widehat{C}_0(w), \widehat{C}_0(\widehat{T}(t,u))\big)\big\} \\
&= \sup\{t \in \,]\widehat{C}_0(\min(u,w)), 1] \mid \widehat{T}(u,t) \leqslant \widehat{Q}(\widehat{C}_0(w), \widehat{C}_0(v))\} \\
&= \widehat{C}_{\widehat{Q}(\widehat{C}_0(w),\widehat{C}_0(v))}(u)\,.
\end{aligned}$$

From $\widehat{C}_0(u) \leqslant \widehat{C}_{\widehat{Q}(\widehat{C}_0(w),\widehat{C}_0(v))}(u)$ and $\widehat{T}(u, \widehat{C}_0(w)) \leqslant \widehat{T}(\widehat{C}_0(w), \widehat{C}_0(v)) \leqslant \widehat{Q}(\widehat{C}_0(w), \widehat{C}_0(v))$ (property **(E3)**) it indeed follows that $\widehat{C}_0(\min(u,w)) \leqslant \widehat{C}_{\widehat{Q}(\widehat{C}_0(w),\widehat{C}_0(v))}(u)$.

Summarizing the above reasonings, we conclude that T satisfies all t-norm properties, is left continuous and $T^\beta = \widehat{T}$. Furthermore, we showed that C_0 is a continuous contour line of T (property **(K5)**). Hence, it follows from assertion **(G1)** that T must be a rotation-invariant t-norm. $\square$

Finally, we use Theorem 8.5 to invert Theorem 7.9 into a construction theorem. This procedure yields a single theorem covering both Theorems 8.3 and 8.5. The commutativity of $\widehat{Q}$ on some half-open sub-square of $[0,1]^2$ is required.

Theorem 8.6 *If the contour line $\widehat{C}_0$ of $\widehat{T}$ is continuous on $]0,1]$, then the $[0,1]^2 \to [0,1]$ function T defined by Eq.* (7.7) *is a rotation-invariant t-norm if and only if $\widehat{Q}$ is commutative on $[0,\hat{\alpha}[^2$, with $\hat{\alpha} = \inf\{t \in [0,1] \mid \widehat{C}_0(t) = 0\}$. In this case, $T^\beta = \widehat{T}$.*

Proof In case $\hat{\alpha} = 0$, then $\widehat{T}$ has no zero-divisors and the continuity of $\widehat{C}_0$ on $]0,1]$ is trivially fulfilled. This theorem then coincides with Theorem 8.3. From now on we assume that $0 < \hat{\alpha}$. Due to the correspondence between Eqs. (7.7) and (7.1), it is clear that Property 8.1 applies to T. From assertion **(J2)** we know that $\widehat{C}_0$ is involutive on $]0,\hat{\alpha}[$. Then, by definition, C_β must be involutive on $]\beta, \sigma^{-1}(\hat{\alpha})[$ and $C_\beta(x) = \sigma^{-1}[0] = \beta$, for every $x \in [\sigma^{-1}(\hat{\alpha}), 1]$. We shortly denote $\sigma^{-1}[\hat{\alpha}]$ by α ($\beta < \alpha$). The above observations imply that $(x,y) \in \mathcal{D}_{\mathrm{IV}}$ if and only if $(x,y) \in \,]\beta,\alpha[^2$ such that $y \leqslant C_\beta(x)$. Note that $\beta < C_\beta(x)$, for every $x \in \,]\beta,\alpha[$. It then follows from Eq. (7.7) that $T(x, C_\beta(x)) \leqslant C_0(\sigma^{-1}[0]) = \beta$, for every $x \in \,]\beta,\alpha[$ and $T(x, C_\beta(x)) = T(x,\beta) \leqslant \beta$, for every $x \in [\alpha, 1]$ (property **(K3)**). Similar reasonings as those used in the first paragraph of the proof of Theorem 8.5 yield the increasingness of T. We conclude that Property 8.2 also applies to T.

Property **(K5)** states that C_β is a contour line of T. Furthermore, C_β is continuous on $]\beta, 1]$. If T is a rotation-invariant t-norm, it follows from Theorem 7.9 that $\widehat{Q}$ is commutative on $[0, \hat{\alpha}[^2$. Conversely, assume that $\widehat{Q}$ is commutative on $[0, \hat{\alpha}[^2$. We then need to show that T is a rotation-invariant t-norm. As in Theorem 8.5, the left continuity, commutativity and the neutral element of T follow immediately from Properties 8.1 and 8.2, the involutivity of $\widehat{C}_0$ on $]0, \hat{\alpha}[$ and the properties of $\widehat{Q}$. Furthermore, property **(K6)** states that $T^\beta = \widehat{T}$. To retrieve the associativity of T it is enough to prove that $C_a(T(x,y)) = C_{C_a(x)}(y)$ holds for every $(x,y) \in \mathcal{D}_{\text{IV}}$ whenever $a = 0$ and for every $(x,y) \in \mathcal{D}_{\text{I}} \cup \mathcal{D}_{\text{IV}} =]\beta, 1]^2$ whenever $a \in]0, \beta[$ (Eq. 5.3 and properties **(K7a)**–**(K7c)**). To this end we will show that $T^{(C_0(\alpha),\alpha)}$ satisfies Theorem 8.5, and then translate the associativity properties of $T^{(C_0(\alpha),\alpha)}$ to properties of T. Table 8.1 gives an overview of the zooms involved in the translation procedure. We distinguish 5 consecutive subproblems.

Table 8.1: Zooms used in the proof of Theorem 8.6.

T-norm	Rescaling function	Contour lines	Companion
$\widehat{T} := T^\beta$	$\sigma : [\beta, 1] \to [0, 1]$	$\widehat{C}_a$	$\widehat{Q}$
$\breve{T} := T^{(C_0(\alpha),\alpha)}$	$\breve{\sigma} : [C_0(\alpha), \alpha] \to [0, 1]$	$\breve{C}_a$	$\breve{Q}$
$\overline{T} := \widehat{T}^{(0,\hat{\alpha})}$	$\bar{\sigma} : [0, \hat{\alpha}] \to [0, 1]$	$\overline{C}_a$	$\overline{Q}$

I. $T(x,y) = \min(x,y)$, for every $(x,y) \in (]C_0(\alpha), \alpha] \times [\alpha, 1]) \cup ([\alpha, 1], \times]C_0(\alpha), \alpha])$

The continuity of $\widehat{C}_0$ on $]0, 1]$ implies that $\overline{T}$ is a rotation-invariant t-norm (assertion **(J4)**). By definition, $\overline{T}(x,y) = \bar{\sigma}[\widehat{T}(\bar{\sigma}^{-1}[x], \bar{\sigma}^{-1}[y])]$. Hence, $\overline{T}$ has neutral element 1 if and only if $\widehat{T}(x, \hat{\alpha}) = \widehat{T}(\hat{\alpha}, x) = x$, for every $x \in \bar{\sigma}^{-1}([0,1]) = [0, \hat{\alpha}]$. Because $\widehat{T}$ is a t-norm, this leads to $\widehat{T}(x,y) = \min(x,y)$, for every $(x,y) \in ([0,\hat{\alpha}] \times [\hat{\alpha}, 1]) \cup ([\hat{\alpha}, 1] \times [0, \hat{\alpha}])$. Recall that $(x,y) \in \mathcal{D}_{\text{I}}$ whenever $(x,y) \in (]\beta, \alpha] \times [\alpha, 1]) \cup ([\alpha, 1] \times]\beta, \alpha])$. From Eq. (7.7) and $\alpha = \sigma^{-1}[\hat{\alpha}]$ it follows for such a pair (x,y) that $T(x,y) = \min(x,y)$. Moreover, $\widehat{T}(x,y) = \min(x,y)$, for every $(x,y) \in ([0,\hat{\alpha}] \times [\hat{\alpha}, 1]) \cup ([\hat{\alpha}, 1] \times [0, \hat{\alpha}])$, also implies that $\widehat{C}_a(x) = a$ whenever $(x,a) \in [\hat{\alpha}, 1] \times [0, \hat{\alpha}[$. Consider arbitrary $(x,y) \in]C_0(\alpha), \beta] \times [\alpha, 1]$. Then $(x,y) \in \mathcal{D}_{\text{II}}$, $\sigma[C_0(x)] \in [0, \hat{\alpha}[$ and $\sigma[y] \in [\hat{\alpha}, 1]$. Consequently, $T(x,y) = C_0(\sigma^{-1}[\sigma[C_0(x)]]) = x = \min(x,y)$. Due to the commutativity of T the latter also holds if $(x,y) \in [\alpha, 1] \times]C_0(\alpha), \beta]$.

II. $C_a(x) = C_0(x)$, for every $(x,a) \in]C_0(\alpha), \alpha] \times [0, C_0(\alpha)]$

Because $C_0(x) \leqslant C_a(x) \leqslant C_{C_0(\alpha)}(x)$, it is enough to prove that $C_{C_0(\alpha)}(x) = C_0(x)$, for every $x \in]C_0(\alpha), \alpha]$. The latter will be satisfied if we can show that $C_0(\alpha) < T(x,y)$, for every $(x,y) \in]C_0(\alpha), \alpha] \times]C_0(x), \alpha[$. Invoking the involutivity of C_0 and the increasingness, commutativity and left continuity of T, it even suffices to prove that $C_0(\alpha) < T(x,y)$, for every $(x,y) \in]C_0(\alpha), \beta] \times]C_0(x), \alpha[$. For such a pair (x,y) it clearly holds that $(x,y) \in \mathcal{D}_{\text{II}}$, $\sigma[C_0(x)] \in [0, \hat{\alpha}[$ and $\sigma[y] \in]\sigma(C_0(x)), \hat{\alpha}[$. Since $\sigma[C_0(x)] < \sigma[y] = \widehat{T}(\sigma[y], \hat{\alpha})$ (proof of Part **I**), it must hold that $\widehat{C}_{\sigma[C_0(x)]}(\sigma[y]) < \hat{\alpha}$. Therefore, $C_0(\alpha) = C_0(\sigma^{-1}[\hat{\alpha}]) < T(x,y)$.

III. $\breve{T}$ is a rotation-invariant t-norm

We first show that $\breve{T}$ has neutral element 1. From Part **I** it follows that

$$\left.\begin{aligned}\breve{T}(x,1) &= \breve{\sigma}\big[\max\big(C_0(\alpha), T(\breve{\sigma}^{-1}[x],\alpha)\big)\big]\\ \breve{T}(1,x) &= \breve{\sigma}\big[\max\big(C_0(\alpha), T(\alpha,\breve{\sigma}^{-1}[x])\big)\big]\end{aligned}\right| = \breve{\sigma}[\breve{\sigma}^{-1}[x]] = x\,,$$

for every $x \in\,]0,1]$. The increasingness of $\breve{T}$ (property **(F1)**) ensures that also $\breve{T}(0,1) = \breve{T}(1,0) = 0$. We conclude that 1 is indeed the neutral element of $\breve{T}$. The increasingness and commutativity of T are by definition passed on to $\breve{T}$. To prove the associativity of $\breve{T}$, we invoke Theorem 8.5. In this respect it is necessary to show that $\breve{T}$ and Eq. (7.5) structurally coincide.

From property **(K5)** we know that C_0 and C_β are contour lines of T. Furthermore, invoking Part **II** and the definition of C_β, property **(F4b)** implies the following formulae:

$$\breve{C}_0(x) = \breve{\sigma}\left[C_{C_0(\alpha)}(\breve{\sigma}^{-1}[x])\right] = \breve{\sigma}\left[C_0(\breve{\sigma}^{-1}[x])\right]\,, \tag{8.11}$$

$$\breve{C}_{\breve{\sigma}[\beta]}(y) = \breve{\sigma}\left[C_\beta(\breve{\sigma}^{-1}[y])\right] = \breve{\sigma}\left[\sigma^{-1}\left[\widehat{C}_0\left(\sigma\left[\breve{\sigma}^{-1}[y]\right]\right)\right]\right]\,, \tag{8.12}$$

for every $x \in\,]0,1]$ and every $y \in\,]\breve{\sigma}(\beta),1]$. Because C_0 is involutive, Eq. (8.11) ensures that $\breve{C}_0$ is involutive on $]0,1[$. Furthermore, $\breve{C}_0(0) = 1$ and $\breve{C}_0(1) = 0$. Therefore, $\breve{C}_0$ is involutive, with fixpoint $\breve{\sigma}[\beta]$. Equations (8.11) and (8.12) also yield the inclusions below.

$$\begin{aligned}
\breve{\mathcal{D}}_{\mathrm{I}} &:= \{(x,y) \in\,]\breve{\sigma}(\beta),1]^2 \mid \breve{C}_{\breve{\sigma}[\beta]}(x) < y\} \subseteq \{(x,y) \in [0,1]^2 \mid (\breve{\sigma}^{-1}[x],\breve{\sigma}^{-1}[y]) \in \mathcal{D}_{\mathrm{I}}\}\,,\\
\breve{\mathcal{D}}_{\mathrm{II}} &:= \{(x,y) \in\,]0,\breve{\sigma}(\beta)] \times]\breve{\sigma}(\beta),1] \mid \breve{C}_0(x) < y\} \subseteq \{(x,y) \in [0,1]^2 \mid (\breve{\sigma}^{-1}[x],\breve{\sigma}^{-1}[y]) \in \mathcal{D}_{\mathrm{II}}\}\,,\\
\breve{\mathcal{D}}_{\mathrm{III}} &:= \{(x,y) \in\,]\breve{\sigma}(\beta),1] \times]0,\breve{\sigma}(\beta)] \mid \breve{C}_0(x) < y\} \subseteq \{(x,y) \in [0,1]^2 \mid (\breve{\sigma}^{-1}[x],\breve{\sigma}^{-1}[y]) \in \mathcal{D}_{\mathrm{III}}\}\,,\\
\breve{\mathcal{D}}_{\mathrm{IV}} &:= \{(x,y) \in\,]\breve{\sigma}(\beta),1[^2 \mid y \leqslant \breve{C}_{\breve{\sigma}[\beta]}(x)\} \subseteq \{(x,y) \in [0,1]^2 \mid (\breve{\sigma}^{-1}[x],\breve{\sigma}^{-1}[y]) \in \mathcal{D}_{\mathrm{IV}}\}\,.
\end{aligned}$$

As $\breve{T}$ has neutral element 1 areas $\breve{\mathcal{D}}_{\mathrm{I}}$, $\breve{\mathcal{D}}_{\mathrm{II}}$, $\breve{\mathcal{D}}_{\mathrm{III}}$ and $\breve{\mathcal{D}}_{\mathrm{IV}}$ partition area $\breve{\mathcal{D}} := \{(x,y) \in [0,1]^2 \mid \breve{C}_0(x) < y\}$. By definition, it holds that $\breve{T}(x,y) = 0$, for every $(x,y) \in [0,1]^2 \setminus \breve{\mathcal{D}}$. If $(x,y) \in \breve{\mathcal{D}}$, then $0 < \breve{T}(x,y)$ such that $\breve{T}(x,y) = \breve{\sigma}[T(\breve{\sigma}^{-1}[x],\breve{\sigma}^{-1}[y])]$. Recall that T is defined by Eq. (7.7) and, hence,

$$\breve{T}(x,y) = \begin{cases}
\breve{\sigma}\left[\sigma^{-1}\left[\widehat{T}\big(\sigma[\breve{\sigma}^{-1}[x]],\sigma[\breve{\sigma}^{-1}[y]]\big)\right]\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\mathrm{I}}\,,\\
\breve{\sigma}\left[C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(\breve{\sigma}^{-1}[x])]}(\sigma[\breve{\sigma}^{-1}[y]])\right]\right)\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\mathrm{II}}\,,\\
\breve{\sigma}\left[C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(\breve{\sigma}^{-1}[y])]}(\sigma[\breve{\sigma}^{-1}[x]])\right]\right)\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\mathrm{III}}\,,\\
\breve{\sigma}\left[C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[\breve{\sigma}^{-1}[x]]),\widehat{C}_0(\sigma[\breve{\sigma}^{-1}[y]])\big)\right]\right)\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\mathrm{IV}}\,,\\
0, & \text{if } (x,y) \notin \breve{\mathcal{D}}\,.
\end{cases} \tag{8.13}$$

Next, we express the contour line C_0 in Eq. (8.13) in terms of $\breve{C}_0$. From the involutivity of C_0 and $0 < \breve{T}(x,y)$ whenever $(x,y) \in \breve{\mathcal{D}}$, it follows that $C_0(\breve{\sigma}^{-1}[\breve{T}(x,y)]) < C_0(C_0(\alpha)) = \alpha$. If in particular $(x,y) \in \breve{\mathcal{D}}_{\text{II}} \cup \breve{\mathcal{D}}_{\text{III}} \cup \breve{\mathcal{D}}_{\text{IV}}$, then also $y \leqslant \breve{C}_{\breve{\sigma}[\beta]}(x)$ from which we obtain that $C_0(\alpha) < \beta \leqslant C_0(\breve{\sigma}^{-1}[\breve{T}(x,y)])$. By means of Eq. (8.11) we are now able to rewrite Eq. (8.13) in the following way:

$$\breve{T}(x,y) = \begin{cases} \breve{\sigma}\left[\sigma^{-1}\left[\widehat{T}\big(\sigma[\breve{\sigma}^{-1}[x]], \sigma[\breve{\sigma}^{-1}[y]]\big)\right]\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{I}}\,, \\ \breve{C}_0\left(\breve{\sigma}\left[\sigma^{-1}\left[\widehat{C}_{\sigma[\breve{\sigma}^{-1}[\breve{C}_0(x)]]}(\sigma[\breve{\sigma}^{-1}[y]])\right]\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{II}}\,, \\ \breve{C}_0\left(\breve{\sigma}\left[\sigma^{-1}\left[\widehat{C}_{\sigma[\breve{\sigma}^{-1}[\breve{C}_0(y)]]}(\sigma[\breve{\sigma}^{-1}[x]])\right]\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{III}}\,, \\ \breve{C}_0\left(\breve{\sigma}\left[\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[\breve{\sigma}^{-1}[x]]), \widehat{C}_0(\sigma[\breve{\sigma}^{-1}[y]])\big)\right]\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{IV}}\,, \\ 0, & \text{if } (x,y) \notin \breve{\mathcal{D}}\,. \end{cases} \tag{8.14}$$

Finally, we express $\widehat{T}$, $\widehat{C}$ and $\widehat{Q}$ in terms of the $(0,\hat{\alpha})$-zoom $\overline{T}$, $\overline{C}$ and $\overline{Q}$. By definition, $\widehat{T}(x,y) = \bar{\sigma}^{-1}[\overline{T}(\bar{\sigma}[x], \bar{\sigma}[y])]$, for every $(x,y) \in [0,\hat{\alpha}]^2$. From property **(F4b)** it follows that $\widehat{C}_a(x) = \bar{\sigma}^{-1}[\overline{C}_{\bar{\sigma}[a]}(\bar{\sigma}[x])]$, for every $(x,a) \in [0,\hat{\alpha}]^2$ such that $a < x$. Property **(F2)** then implies that $\widehat{Q}(x,y) = \bar{\sigma}^{-1}[\overline{Q}(\bar{\sigma}[x], \bar{\sigma}[y])]$, for every $(x,y) \in\,]0,\hat{\alpha}[^2$ such that $\widehat{C}_0(x) \leqslant y$. These three properties allow us to remove $\widehat{T}$, $\widehat{C}$ and $\widehat{Q}$ from Eq. (8.14). To not overload the notation, we briefly use γ to denote the $[\breve{\sigma}(\beta),1] \to [0,1]$ isomorphism $\bar{\sigma} \circ \sigma \circ \breve{\sigma}^{-1}$. Note that the involutivity of $\breve{C}_0$ ensures that $\breve{C}_0(x) < y$ is equivalent with $\breve{C}_0(y) < x$. For every $(x,y) \in \breve{\mathcal{D}}_{\text{IV}}$, $\widehat{C}_0(]0,\hat{\alpha}[) =]0,\hat{\alpha}[$ (assertion **(J3)**) ensures that $(\widehat{C}_0(\sigma[\breve{\sigma}^{-1}[x]]), \widehat{C}_0(\sigma[\breve{\sigma}^{-1}[y]])) \in\,]0,\hat{\alpha}[^2$ and Eq. (8.12) enables us to derive $\widehat{C}_0(\widehat{C}_0(\sigma[\breve{\sigma}^{-1}[x]])) \leqslant \widehat{C}_0(\sigma[\breve{\sigma}^{-1}[y]])$ from $y \leqslant \breve{C}_{\breve{\sigma}[\beta]}(x)$. At last, we get that

$$\breve{T}(x,y) = \begin{cases} \gamma^{-1}\left[\overline{T}(\gamma[x], \gamma[y])\right], & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{I}}\,, \\ \breve{C}_0\left(\gamma^{-1}\left[\overline{C}_{\gamma[\breve{C}_0(x)]}(\gamma[y])\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{II}}\,, \\ \breve{C}_0\left(\gamma^{-1}\left[\overline{C}_{\gamma[\breve{C}_0(y)]}(\gamma[x])\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{III}}\,, \\ \breve{C}_0\left(\gamma^{-1}\left[\overline{Q}\big(\overline{C}_0(\gamma[x]), \overline{C}_0(\gamma[y])\big)\right]\right), & \text{if } (x,y) \in \breve{\mathcal{D}}_{\text{IV}}\,, \\ 0, & \text{if } (x,y) \notin \breve{\mathcal{D}}\,. \end{cases} \tag{8.15}$$

Equation (8.15) is structurally identical to Eq. (7.5). Moreover, $\overline{T}$ is a rotation-invariant t-norm (assertion **(J4)**), $\breve{C}_0$ is an involutive negator with fixpoint $\breve{\sigma}[\beta]$ and γ is a $[\breve{\sigma}(\beta),1] \to [0,1]$ isomorphism. From Eq. (8.12) and $\widehat{C}_0(x) = \bar{\sigma}^{-1}[\overline{C}_0(\bar{\sigma}[x])]$, for every $x \in\,]0,\hat{\alpha}]$, it follows that $\breve{C}_{\breve{\sigma}[\beta]}(x) = \gamma^{-1}[\overline{C}_0(\gamma[x])]$, for every $x \in]\breve{\sigma}(\beta),1]$. Since $\breve{T}$ and $\overline{T}$ have neutral element 1 we also know (Corollary 5.8) that $\breve{C}_{\breve{\sigma}[\beta]}(\breve{\sigma}[\beta]) = 1 = \gamma^{-1}[\overline{C}_0(\gamma[\breve{\sigma}[\beta]])]$ and $\breve{C}_{\breve{\sigma}[\beta]}(x) = 1$, for every $x \in [0,\breve{\sigma}(\beta)[$. Finally, the commutativity of $\widehat{Q}$ on $[0,\hat{\alpha}[^2$ yields the commutativity of $\overline{Q}$ on $[0,1[^2$. Indeed, the correspondence between $\widehat{Q}$ and $\overline{Q}$, between $\widehat{C}_0$ and $\overline{C}_0$, and the involutivity of $\overline{C}_0$

(assertion **(G2)**) ensure that $\overline{Q}(x,y) = \overline{Q}(y,x)$, for every $(x,y) \in]0,1[^2$ satisfying $\overline{C}_0(x) \leqslant y$. Note that trivially $\overline{Q}(x,y) = \overline{Q}(y,x) = 0$, for every $(x,y) \in [0,1[^2$ satisfying $y < \overline{C}_0(x)$. We conclude from Theorem 8.5 that $\breve{T}$ must be a rotation-invariant t-norm.

IV. $C_0(T(x,y)) = C_{C_0(x)}(y)$, for every $(x,y) \in \mathcal{D}_{\mathrm{IV}}$

Consider $(x,y) \in \mathcal{D}_{\mathrm{IV}}$. Recall that $(x,y) \in \mathcal{D}_{\mathrm{IV}}$ if and only if $(x,y) \in]\beta,\alpha[^2$ and $y \leqslant C_\beta(x)$. Then $C_{C_0(\alpha)}(x) = C_0(x) < \beta < y$ (Part **II**). The latter inequality is equivalent with $C_0(\alpha) < T(x,y) \leqslant x < \alpha$. By definition, $\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y]) = \breve{\sigma}[\max(C_0(\alpha),T(x,y))] = \breve{\sigma}[T(x,y)]$. From property **(F4b)** we know that $C_{C_0(x)}(y) = \breve{\sigma}^{-1}[\breve{C}_{\breve{\sigma}[C_0(x)]}(\breve{\sigma}[y])]$. Invoking also Eq. (8.11), these observations allow us to rewrite $C_0(T(x,y)) = C_{C_0(x)}(y)$ as

$$\breve{\sigma}^{-1}[\breve{C}_0(\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y]))] = \breve{\sigma}^{-1}[\breve{C}_{\breve{C}_0(\breve{\sigma}[x])}(\breve{\sigma}[y])]\,.$$

Since $\breve{T}$ is a left-continuous t-norm (Part **III**), it follows from Eq. (5.3) that this latter equality is always satisfied.

V. $C_a(T(x,y)) = C_{C_a(x)}(y)$, for every $(x,y) \in]\beta,1]^2$ and every $a \in]0,\beta[$

As $y \leqslant C_a(x)$ is equivalent with $T(x,y) \leqslant a$, it follows from Corollary 5.8 that $C_a(T(x,y)) = 1 = C_{C_a(x)}(y)$. From now on we assume that $C_a(x) < y$. Depending on the values of x, y and a we distinguish the following cases:

1. If $x \in]\beta,\alpha[$ and $a \in [0,C_0(\alpha)]$, then $C_a(x) = C_0(x)$ and $C_{C_0(\alpha)}(x) = C_0(x) < \beta < y$ (Part **II**). Therefore, $C_0(\alpha) < T(x,y) \leqslant x < \alpha$. and applying Part **II** once more results in $C_a(T(x,y)) = C_0(T(x,y))$. Due to property **(K7a)** and part **IV**, $C_0(T(x,y)) = C_{C_0(x)}(y)$ always holds.

2. If $\max(x,y) \in]\beta,\alpha[$ and $a \in]C_0(\alpha),\beta[$, then property **(F4b)** and Eq. (5.3) imply that

$$C_{C_a(x)}(y) = C_{\breve{\sigma}^{-1}[\breve{C}_{\breve{\sigma}[a]}(\breve{\sigma}[x])]}(y) = \breve{\sigma}^{-1}\left[\breve{C}_{\breve{C}_{\breve{\sigma}[a]}(\breve{\sigma}[x])}(\breve{\sigma}[y])\right] = \breve{\sigma}^{-1}\left[\breve{C}_{\breve{\sigma}[a]}\big(\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y])\big)\right]\,.$$

   Because $a < T(x,y)$ ($C_a(x) < y$), we know that $\breve{T}(\breve{\sigma}[x],\breve{\sigma}[y]) = \breve{\sigma}[\max(C_0(\alpha),T(x,y))] = \breve{\sigma}[T(x,y)]$. Hence, $C_{C_a(x)}(y) = \breve{\sigma}^{-1}[\breve{C}_{\breve{\sigma}[a]}(\breve{\sigma}[T(x,y)])]$. Invoking property **(F4b)** once more lead to $C_{C_a(x)}(y) = C_a(T(x,y))$.

3. If $x \in]\beta,\alpha[$, $y \in [\alpha,1]$ and $a \in]C_0(\alpha),\beta[$, then $T(x,y) = x$ (Part **I**) and $C_0(\alpha) \leqslant C_{C_0(\alpha)}(x) = C_0(x) \leqslant C_a(x)$ (Corollary 5.8 and Part **II**). From the first paragraph of the proof we know that $C_\beta(]\beta,\alpha[) =]\beta,\alpha[$. Therefore, $C_a(x) \leqslant C_\beta(x) < \alpha$. From Part **I** and $C_a(x) \in [C_0(\alpha),\alpha[$ it follows that $C_{C_a(x)}(y) = C_a(x) = C_a(T(x,y))$.

4. If $x \in [\alpha,1]$ and $y \in]\beta,\alpha[$, then $T(x,y) = y$ (Part **I**). In case $a \in]0,C_0(\alpha)]$, then $C_a(x) \leqslant C_a(\alpha) = C_0(\alpha)$ (Part **II**). As $\max(a,C_a(x)) \in [0,C_0(\alpha)]$ we know from Part **II** that $C_a(T(x,y)) = C_a(y) = C_0(y) = C_{C_a(x)}(y)$. Otherwise, for $a \in]C_0(\alpha),\beta[$ we have that $C_a(x) = a$ (Part **I**) which leads to $C_a(T(x,y)) = C_a(y) = C_{C_a(x)}(y)$.

5. If $\min(x,y) \in [\alpha,1]$ and $a \in\,]0, C_0(\alpha)]$, then $(x,y,C_0(a)) \in [\alpha,1]^3$, $\alpha = T(\alpha,\alpha) \leqslant \min(T(x,y), T(C_0(a),x)$ (Part **I**). Recall that every $(u,v) \in [\alpha,1]^2$ must belong to $\mathcal{D}_{\mathrm{I}}$. Invoking property **(K7a)**, Eq. (7.7) and the associativity of $\widehat{T}$, we immediately get that

$$\begin{aligned} C_a(T(x,y)) &= C_{C_0(C_0(a))}(T(x,y)) = C_0\big(T\big(C_0(a), T(x,y)\big)\big) \\ &= C_0\big(\sigma^{-1}\big[\widehat{T}\big(\sigma[C_0(a)], \widehat{T}(\sigma[x],\sigma[y])\big)\big]\big) = C_0\big(\sigma^{-1}\big[\widehat{T}\big(\widehat{T}(\sigma[C_0(a)],\sigma[x]),\sigma[y]\big)\big]\big) \\ &= C_0\big(T\big(T(C_0(a),x),y\big)\big) = C_{C_0(T(C_0(a),x))}(y) = C_{C_{C_0(C_0(a))}(x)}(y) = C_{C_a(x)}(y)\,. \end{aligned}$$

6. If $\min(x,y) \in [\alpha,1]$ and $a \in\,]C_0(\alpha),\beta[$, then Part **I** implies that $C_a(x) = a = C_a(y)$, $\alpha = T(\alpha,\alpha) \leqslant T(x,y)$ and, hence, also $C_a(T(x,y)) = a$. The equality $C_a(T(x,y)) = a = C_{C_a(x)}(y)$ then trivially holds.

From the reasonings above, we conclude that T is a left-continuous t-norm that satisfies $T^\beta = \widehat{T}$ and whose contour line C_0 is involutive. Hence, T is a rotation-invariant t-norm (assertion **(G2)**). $\square$

Suppose that $\widehat{C}_0$ is indeed continuous on $]0,1]$. If $\hat{\alpha} = 0$, then $\widehat{T}$ has no zero-divisors and Theorem 8.6 coincides with Theorem 8.3. For $\hat{\alpha} = 1$ we know from property **(J2)**, $\widehat{C}_0(0) = 1$ and $\widehat{C}_0(1) = 0$ that $\widehat{C}_0$ must be involutive. In this case Theorem 8.6 coincides with Theorem 8.5. Assume now that $\hat{\alpha} \in\,]0,1[$. The correspondence between C_β and $\widehat{C}_0$ yields that C_β is continuous on $]\beta,1]$. Therefore, Theorem 7.9 is applicable on any t-norm T procured by Theorem 8.6. In the discussion succeeding Theorem 7.9 we have illustrated that T is then decomposable in the sense of Jenei [45] ($\alpha \in D_T$). Furthermore, Theorem 6.26 states that such a t-norm can always be reconstructed by means of the rotation-annihilation construction of Jenei [47]. In the setting of the present section and the proof of Theorem 8.6, the rotation-annihilation construction requires the prior knowledge of the involutive negator C_0, the left-continuous t-norm $\widehat{T}^{\hat{\alpha}}(= (T^\beta)^{\hat{\alpha}} = T^\alpha$ (Theorem 6.11)) and the rotation-invariant t-norm $\breve{T}(= T^{(C_0(\alpha),\alpha)})$. For our approach it is, however, enough to consider only C_0 and $\widehat{T}(= T^\beta)$.

To conclude this section we briefly discuss the class of left-continuous t-norms $\widehat{T}$ that are appropriate for Theorem 8.6 to hold in case $\hat{\alpha} \in\,]0,1[$. There are only two requirements: The contour line $\widehat{C}_0$ of $\widehat{T}$ must be continuous on $]0,1]$ and the companion $\widehat{Q}$ must be commutative on $[0,\hat{\alpha}[^2$. From property **(J4)** it follows that the continuity condition on $\widehat{C}_0$ is equivalent with $\widehat{T}^{(0,\hat{\alpha})}$ being a rotation-invariant t-norm. Invoking Corollary 6.12, $\widehat{T}$ must be an ordinal sum:

$$\widehat{T} = \left(\left\langle 0,\hat{\alpha},\bar{\sigma},\widehat{T}^{(0,\hat{\alpha})}\right\rangle, \left\langle \hat{\alpha},1,\hat{\sigma},\widehat{T}^{\hat{\alpha}}\right\rangle\right)\,, \tag{8.16}$$

where $\bar{\sigma}$ is the $[0,\hat{\alpha}] \to [0,1]$ isomorphism used to compute $T^{(0,\hat{\alpha})}$ and $\hat{\sigma}$ the $[\hat{\alpha},1] \to [0,1]$ isomorphism used to compute $T^{\hat{\alpha}}$. By means of property **(E2)** it is not difficult to verify that the commutativity of $\widehat{Q}$ on $[0,\hat{\alpha}[^2$ is equivalent with the commutativity of $\widehat{Q}^{(0,\hat{\alpha})}$ on $[0,1[^2$. Consequently, the left-continuous t-norms $\widehat{T}$ ($\hat{\alpha} \in\,]0,1[$) on which Theorem 8.6 is applicable are exactly those ordinal sums $(\langle 0,\hat{\alpha},\bar{\sigma},T_1\rangle, \langle\hat{\alpha},1,\hat{\sigma},T_2\rangle)$, with $\hat{\alpha} \in\,]0,1[$, T_1 a rotation-invariant t-norm whose companion is commutative on $[0,1[^2$ and T_2 an arbitrary left-continuous t-norm. Recall that an ordinal sum of t-norms is always a t-norm (Theorem 6.2).

8.3 Visualization

This final section visualizes the construction methods from Section 8.2. We first reformulate and summarize the requisite mathematical results. Although the notations used in Chapter 7 and Section 8.2 turned out to be extremely convenient for proving and comprehending the theorems and properties, they impede a smooth formulation of a practical 'construction tool'. As C_0 and C_β initially do not have a contour line interpretation, we denote them here as, resp. N and M. The setting of Chapter 7 and Section 8.2 then transforms as follows:

1. *T: an arbitrary left-continuous t-norm (with contour lines C_a and companion Q) such that C_0 is continuous on $]0,1]$ and Q is commutative on $[0,\alpha[^2$, with $\alpha = \inf\{t \in [0,1] \mid C_0(t) = 0\}$;*
2. *N: an arbitrary involutive negator with fixpoint β;*
3. *σ: an arbitrary $[\beta,1] \to [0,1]$ isomorphism;*
4. *M: the decreasing $[0,1] \to [0,1]$ function defined by $x^M = 1$ whenever $x \in [0,\beta[$ and by $x^M = \sigma^{-1}[C_0(\sigma[x])]$ whenever $x \in [\beta,1]$;*
5. *$\mathcal{D}$: the area $\{(x,y) \in [0,1]^2 \mid x^N < y\} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$, with*

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y) \in]\beta,1]^2 \mid x^M < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y) \in]0,\beta]\times\,]\beta,1] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y) \in]\beta,1]\times\,]0,\beta] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y) \in]\beta,1[^2 \mid y \leqslant x^M\}\,.
\end{aligned}$$

Note that the choice of T, N and σ fixes M and $\mathcal{D}$. Considering Theorems 7.9 and 8.6, we then obtain the following tool for constructing rotation-invariant t-norms:

The $[0,1]^2 \to [0,1]$ function $R3(T,N)$ defined by

$$R3(T,N)(x,y) = \begin{cases}
\sigma^{-1}\left[T(\sigma[x],\sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[x^N]}(\sigma[y])\right]\right)^N, & \text{if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[y^N]}(\sigma[x])\right]\right)^N, & \text{if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,,\\
\left(\sigma^{-1}\left[Q\big(C_0(\sigma[x]),C_0(\sigma[y])\big)\right]\right)^N, & \text{if } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,,\\
0, & \text{if } (x,y) \notin \mathcal{D}\,.
\end{cases} \tag{8.17}$$

is a rotation-invariant t-norm. Furthermore, $R3(T,N)$ is the only left-cotninuous t-norm that has N as a contour line ($a = 0$) and that has β-zoom $R3(T,N)^\beta = T$.

In Chapter 7 we showed that $R3(T,N)|_{\mathcal{D}_{\mathrm{II}}}$ and $R3(T,N)|_{\mathcal{D}_{\mathrm{III}}}$ are determined by the (transformed) left and right rotation of $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$ around the axis through the points $(0,0,1)$ and $(1,1,0)$. $R3(T,N)|_{\mathcal{D}_{\mathrm{IV}}}$ is determined by the (transformed) front rotation of $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}\cap]\beta,\sigma^{-1}(\alpha)]^2}$ around the axis through the points $(\beta,\sigma^{-1}[\alpha],\beta)$ and $(\sigma^{-1}[\alpha],\beta,\beta)$. Note also that $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$ is a rescaled version of 'the non-zero part' of T. Inspired by these geometrical observations, we briefly call $R3(T,N)$ the *triple rotation of T based on N*. The construction method itself is referred to as the *triple rotation method*. Repeatedly performing the triple rotation method based on a fixed involutive negator N is briefly denoted as follows:

$$R3^n(T,N) := \underbrace{R3(\ldots R3(R3(}_{n\text{ times}}T,\underbrace{N),N)\ldots,N)}_{n\text{ times}}\,.$$

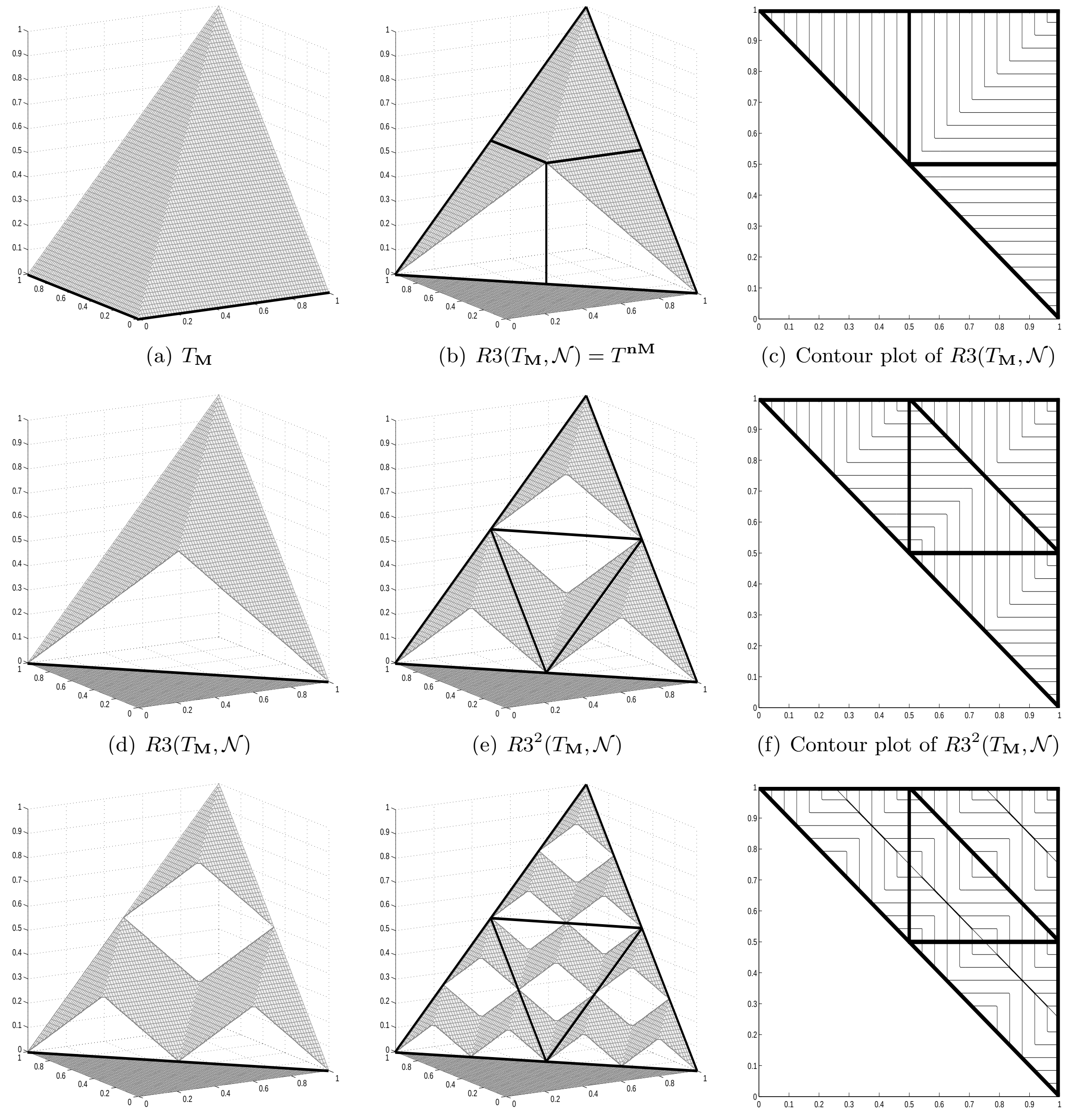
In case T is rotation invariant, assertions **(G2)** and **(G3)** allow us to rewrite Eq. (8.17) in a more feasible form (cf Corollary 7.6):

$$R3(T,N)(x,y) = \begin{cases} \sigma^{-1}\left[T(\sigma[x],\sigma[y])\right], & \text{if } (x,y)\in\mathcal{D}_{\mathrm{I}}\,,\\ \left(\sigma^{-1}\left[C_0\left(T\left(C_0(\sigma[x^N]),\sigma[y]\right)\right)\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{II}}\,,\\ \left(\sigma^{-1}\left[C_0\left(T\left(\sigma[x],C_0(\sigma[y^N])\right)\right)\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{III}}\,,\\ \left(\sigma^{-1}\left[Q\left(C_0(\sigma[x]),C_0(\sigma[y])\right)\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{IV}}\,,\\ 0, & \text{if } (x,y)\notin\mathcal{D}\,. \end{cases} \tag{8.18}$$

For the following examples we use the linear rescaling function $\varsigma : x \mapsto (x-\beta)/(1-\beta)$. Any other rescaling function will entail a transformation of the procured t-norm.

In Fig. 8.1, we apply the triple rotation method to the minimum $T_{\mathbf{M}}$. The triple rotation $R3(T_{\mathbf{M}},\mathcal{N})$ of $T_{\mathbf{M}}$ based on the standard negator $\mathcal{N}$ coincides with the nilpotent minimum $T^{\mathbf{nM}}$. The companion of $T^{\mathbf{nM}}$ is clearly commutative on $[0,1[^2$. Hence, the triple rotation $R3(T^{\mathbf{nM}},\mathcal{N}) = R3^2(T_{\mathbf{M}},\mathcal{N})$ of $T^{\mathbf{nM}}$ based on the standard negator $\mathcal{N}$ is a rotation-invariant t-norm once again. As the companion of this latter t-norm is also commutative on $[0,1[^2$ we can perform a third triple rotation. The bold black lines in Figs. 8.1(a), 8.1(d) and 8.1(g) indicate the contour lines obtained by intersecting these t-norms with a plane that has height 0. The bold black lines in all other subfigures visualize the partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}}\cup\mathcal{D}_{\mathrm{II}}\cup\mathcal{D}_{\mathrm{III}}\cup\mathcal{D}_{\mathrm{IV}}$.

Similarly to Fig. 8.1, we performed in Fig. 8.2 the triple rotation method on the algebraic product $T_{\mathbf{P}}$. The t-norms depicted in Figs. 8.2(b), 8.2(e) and 8.2(h) have a finite number of discontinuity points. Figures. 8.1(b) and 8.2(b) can also be constructed by means of the rotation construction of Jenei [42, 46]. As indicated in Section 8.2, it holds in general that $R3(T,N) = R(T,N)$ whenever T has no zero-divisors. On the other hand, Figs. 8.1(e), 8.1(h), 8.2(e) and 8.2(h) visualize t-norms that cannot be described by the rotation construction nor by the rotation-annihilation construction of Jenei [43, 45, 47]. His constructions only result in decomposable (in the sense of Jenei [45]) t-norms. The four t-norms depicted here are clearly



(a) $T_{\mathbf{M}}$

(b) $R3(T_{\mathbf{M}},\mathcal{N}) = T^{\mathbf{nM}}$

(c) Contour plot of $R3(T_{\mathbf{M}},\mathcal{N})$

(d) $R3(T_{\mathbf{M}},\mathcal{N})$

(e) $R3^2(T_{\mathbf{M}},\mathcal{N})$

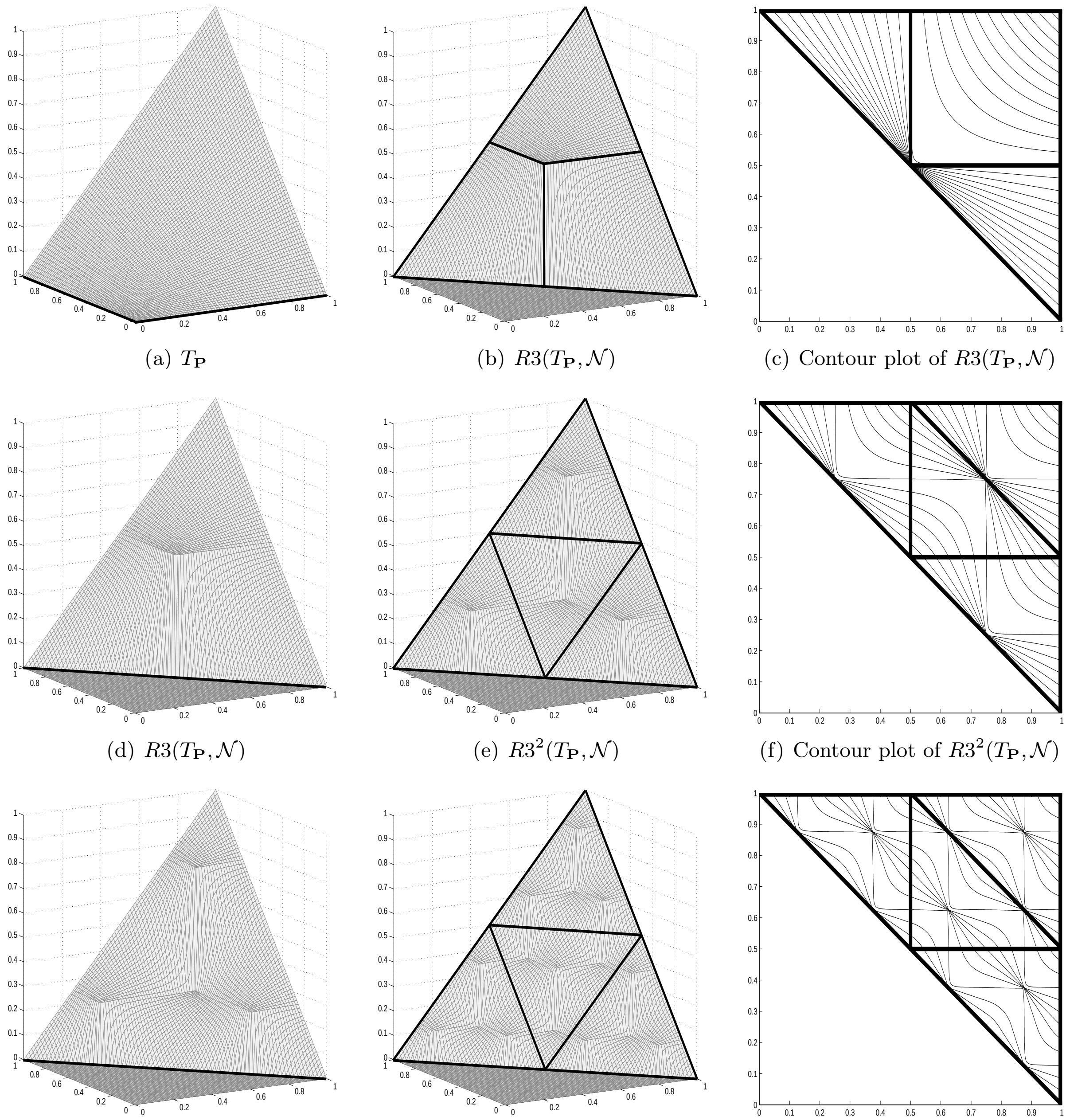
(f) Contour plot of $R3^2(T_{\mathbf{M}},\mathcal{N})$

(g) $R3^2(T_{\mathbf{M}},\mathcal{N})$

(h) $R3^3(T_{\mathbf{M}},\mathcal{N})$

(i) Contour plot of $R3^3(T_{\mathbf{M}},\mathcal{N})$

Figure 8.1: The triple rotation of $T_{\mathbf{M}}$ based on $\mathcal{N}$.



(a) $T_{\mathbf{P}}$ (b) $R3(T_{\mathbf{P}},\mathcal{N})$ (c) Contour plot of $R3(T_{\mathbf{P}},\mathcal{N})$

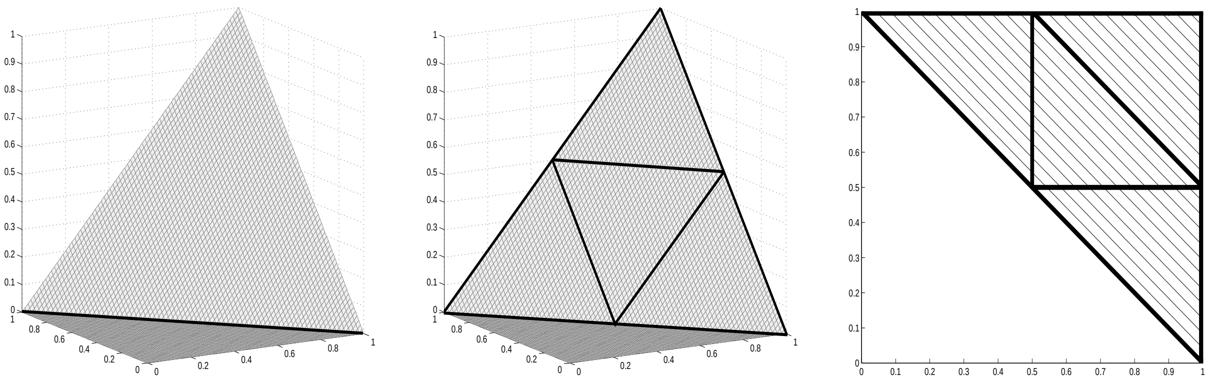
(d) $R3(T_{\mathbf{P}},\mathcal{N})$ (e) $R3^2(T_{\mathbf{P}},\mathcal{N})$ (f) Contour plot of $R3^2(T_{\mathbf{P}},\mathcal{N})$

(g) $R3^2(T_{\mathbf{P}},\mathcal{N})$ (h) $R3^3(T_{\mathbf{P}},\mathcal{N})$ (i) Contour plot of $R3^3(T_{\mathbf{P}},\mathcal{N})$

Figure 8.2: The triple rotation of $T_{\mathbf{P}}$ based on $\mathcal{N}$.

non-decomposable: their companions Q_{R3} always satisfy the inequality $Q_{R3}(\frac{1}{2}, t) < \frac{1}{2}$, for every $t \in [\frac{1}{2}, 1[$.

The triple rotation $R3(T_{\mathbf{L}}, \mathcal{N})$ of the Łukasiewicz t-norm $T_{\mathbf{L}}$ yields the Łukasiewicz t-norm $T_{\mathbf{L}}$ once again (see Fig. 8.3). Therefore, $R3^n(T_{\mathbf{L}}, \mathcal{N}) = T_{\mathbf{L}}$ holds for every $n \in \mathbb{N} \setminus \{0\}$.



(a) $T_{\mathbf{L}}$ (b) $R3(T_{\mathbf{L}}, \mathcal{N}) = T_{\mathbf{L}}$ (c) Contour plot of $R3(T_{\mathbf{L}}, \mathcal{N})$

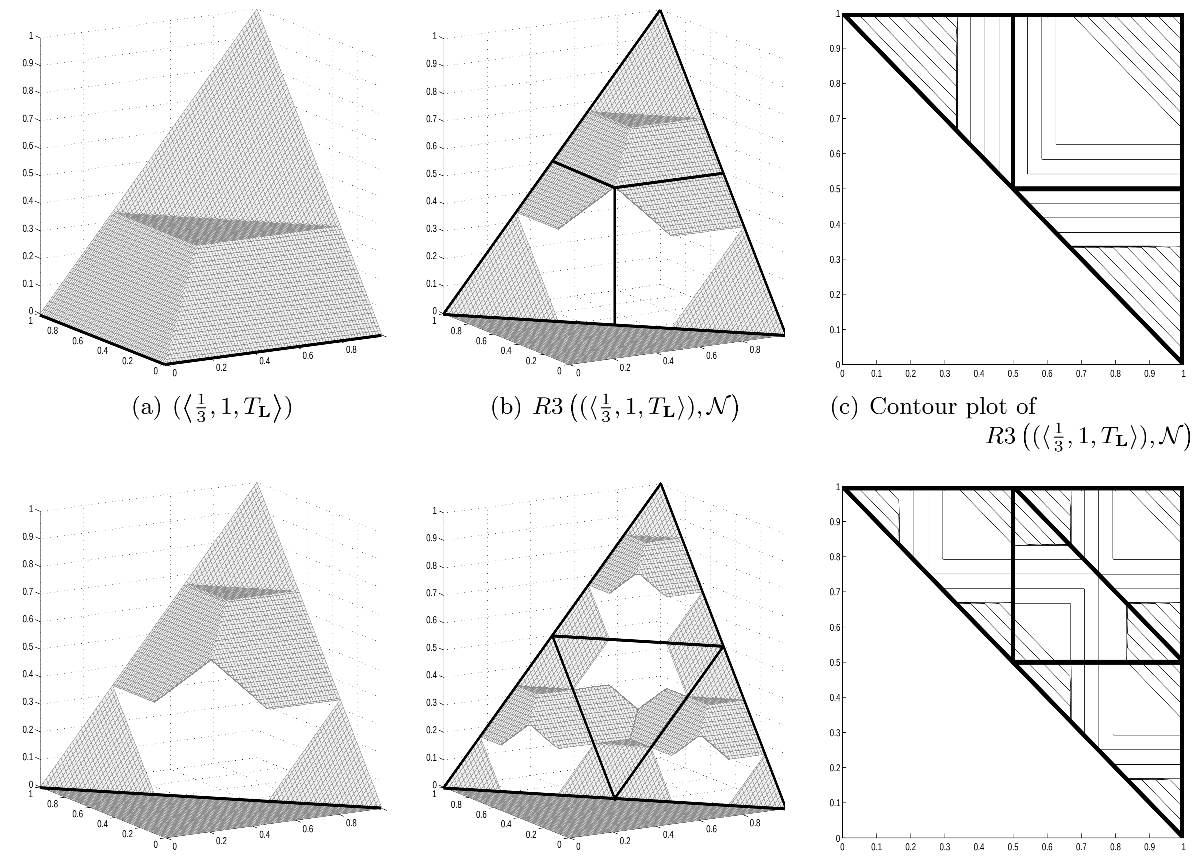
Figure 8.3: The triple rotation of $T_{\mathbf{L}}$ based on $\mathcal{N}$.

For an arbitrary left-continuous t-norm T, the triple rotation method, however, does not always yield a t-norm. Fig. 8.4 depicts the triple rotation of the ordinal sum $(\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle)$ [51]. It is clear that $R3\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ is a rotation-invariant t-norm. Its left continuity, however, prevents its companion Q_{R3} from being commutative on $[0, 1[^2$. For example, $Q_{R3}(\frac{1}{3}, \frac{2}{3}) = 0 < \frac{1}{3} = Q_{R3}(\frac{2}{3}, \frac{1}{3})$. The latter prevents $F := R3^2\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ from being commutative and associative:

$$F\left(\frac{4}{6}, \frac{5}{6}\right) = \frac{1}{3} < \frac{1}{2} = F\left(\frac{5}{6}, \frac{4}{6}\right) ;$$
$$F\left(F\left(\frac{4}{6}, \frac{5}{6}\right), \frac{4}{6}\right) = 0 < \frac{1}{6} = F\left(\frac{4}{6}, F\left(\frac{5}{6}, \frac{4}{6}\right)\right) .$$

Note that if the companion Q of a rotation-invariant t-norm T is not commutative on $[0, 1[^2$, there can never exist a rotation-invariant t-norm with β-zoom T (see Theorem 7.5). For our example this means that it is impossible to construct a rotation-invariant t-norm with contour line $\mathcal{N}$ and $\frac{1}{2}$-zoom $R3\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$.

Figures. 8.1–8.3 illustrate that if the involutive negator N and the contour line C_0 of T both equal the standard negator $\mathcal{N}$, then $R3(T, \mathcal{N})|_{\mathcal{D}_{\mathrm{II}}}$, $R3(T, \mathcal{N})|_{\mathcal{D}_{\mathrm{III}}}$ and $R3(T, \mathcal{N})|_{\mathcal{D}_{\mathrm{IV}}}$ are as good as perfect rotations of $R3(T, \mathcal{N})|_{\mathcal{D}_{\mathrm{I}}}$. There does not occur any reshaping. Dealing with an arbitrary involutive negator N and a contour line $C_0 \neq \mathcal{N}$, the left rotation, right rotation and front rotation of $R3(T, N)|_{\mathcal{D}_{\mathrm{I}}}$ have to be reshaped to fit into the areas $\mathcal{D}_{\mathrm{II}}$, $\mathcal{D}_{\mathrm{III}}$ and $\mathcal{D}_{\mathrm{IV}}$ respectively. The involutive negator N and the contour line C_0 of T are responsible for this



(a) $(\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle)$ (b) $R3\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ (c) Contour plot of $R3\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$

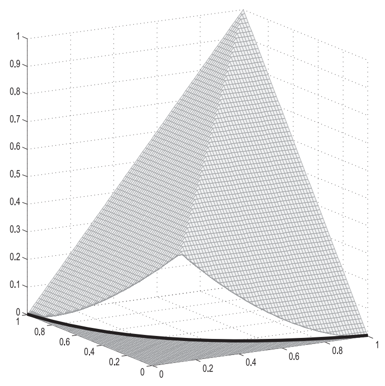
(d) $R3\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ (e) $R3^2\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ (f) Contour plot of $R3^2\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$

Figure 8.4: The triple rotation of $(\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle)$ based on $\mathcal{N}$. $R3^2\left((\langle \frac{1}{3}, 1, T_{\mathbf{L}} \rangle), \mathcal{N}\right)$ is **not** a t-norm.

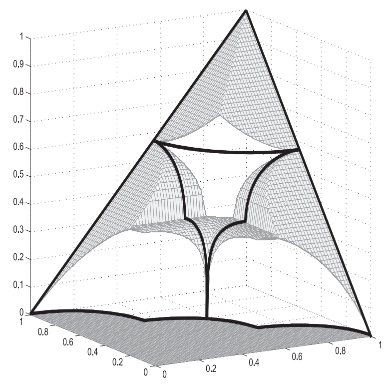
reshaping. Figure 8.5 illustrates this phenomenon for the involutive negator N^* defined by

$$x^{N^*} = \begin{cases} \frac{2}{3} + \sqrt{\frac{1}{9} - x^2}, & \text{if } x \in [0, \frac{1}{3}]\,, \\ \frac{1}{3} + \sqrt{\frac{1}{9} - (x - \frac{1}{3})^2}, & \text{if } x \in [\frac{1}{3}, \frac{2}{3}]\,, \\ \sqrt{\frac{1}{9} - (x - \frac{2}{3})^2}, & \text{if } x \in [\frac{2}{3}, 1]\,. \end{cases} \tag{8.19}$$

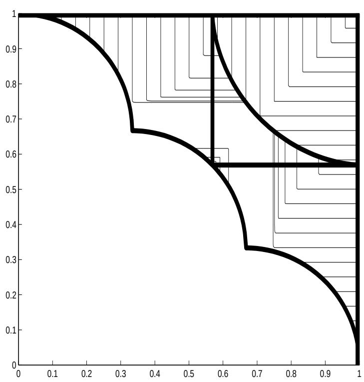
We applied the triple rotation method based on N^* to the ϕ-transforms of the triple rotations $R3(T_{\mathbf{M}}, \mathcal{N})$, $R3(T_{\mathbf{P}}, \mathcal{N})$ and $R3(T_{\mathbf{L}}, \mathcal{N})$, with ϕ the automorphism defined by $\phi(x) := x^{3/5}$. Note that in general $R3(T, N)(\bullet, \beta) = N \circ M = R3(T, N)(\beta, \bullet)$. Therefore, the t-norms $R3(T, N^*)$ visualized in Figs. 8.5(b), 8.5(e) and 8.5(h) have identical partial functions $R3(T, N^*)(\bullet, \beta) = R3(T, N^*)(\beta, \bullet)$, with $\beta = \frac{1}{3} + \frac{1}{\sqrt{18}}$ the fixpoint of N^*. Indeed, their associated functions M ($C_0 = \mathcal{N}_\phi$) and $N = N^*$ are identical.



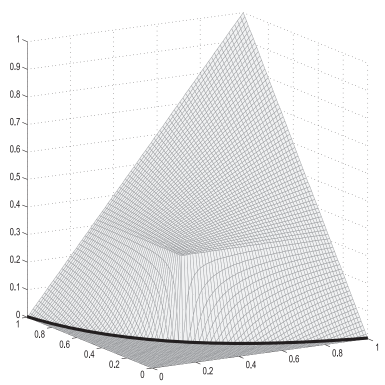
(a) $R3(T_{\mathbf{M}},\mathcal{N})_\phi$



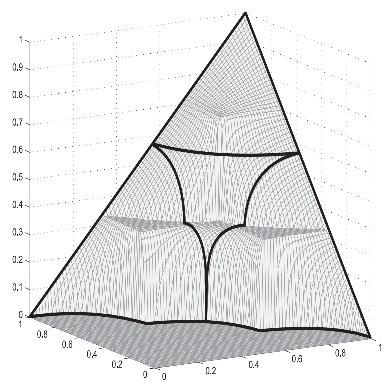
(b) $R3(R3(T_{\mathbf{M}},\mathcal{N})_\phi, N^*)$



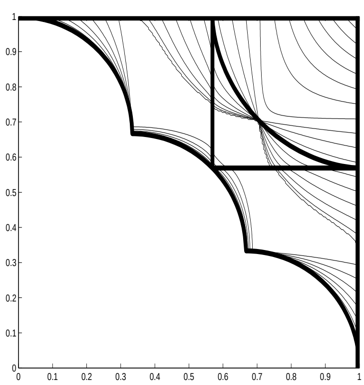
(c) Contour plot of $R3(R3(T_{\mathbf{M}},\mathcal{N})_\phi, N^*)$



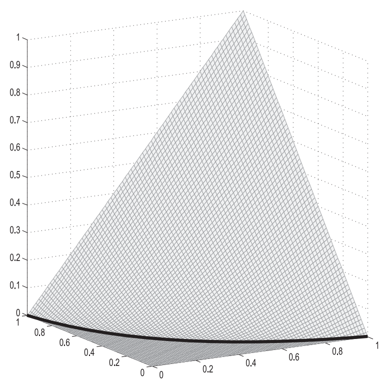
(d) $R3(T_{\mathbf{P}},\mathcal{N})_\phi$



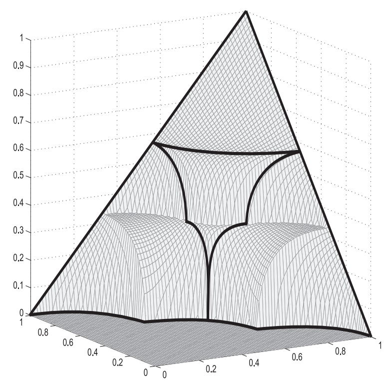
(e) $R3(R3(T_{\mathbf{P}},\mathcal{N})_\phi, N^*)$



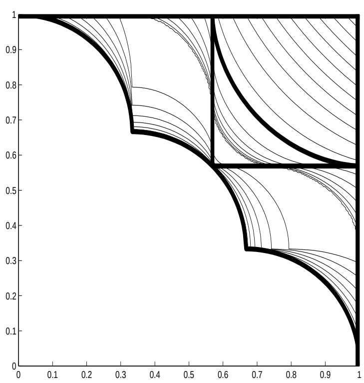
(f) Contour plot of $R3(R3(T_{\mathbf{P}},\mathcal{N})_\phi, N^*)$



(g) $R3(T_{\mathbf{L}},\mathcal{N})_\phi$

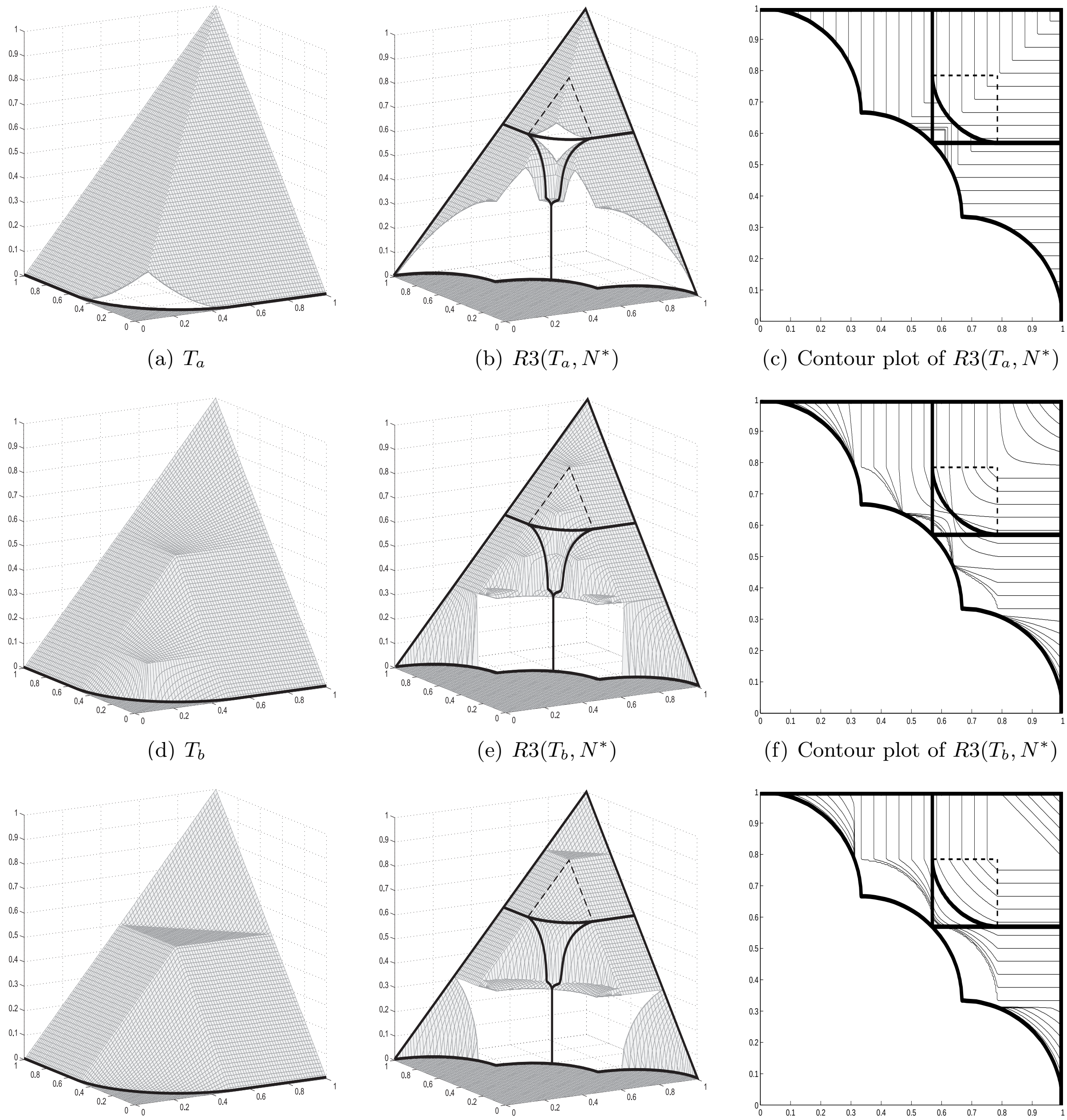


(h) $R3(R3(T_{\mathbf{L}},\mathcal{N})_\phi, N^*)$



(i) Contour plot of $R3(R3(T_{\mathbf{L}},\mathcal{N})_\phi, N^*)$

Figure 8.5: The triple rotation of $R3(T_{\mathbf{M}},\mathcal{N})_\phi$, $R3(T_{\mathbf{P}},\mathcal{N})_\phi$ and $R3(T_{\mathbf{L}},\mathcal{N})_\phi$ based on N^*.



(a) T_a (b) $R3(T_a, N^*)$ (c) Contour plot of $R3(T_a, N^*)$

(d) T_b (e) $R3(T_b, N^*)$ (f) Contour plot of $R3(T_b, N^*)$

(g) T_c (h) $R3(T_c, N^*)$ (i) Contour plot of $R3(T_c, N^*)$

Figure 8.6: The triple rotation of T_a, T_b and T_c based on N^*.

So far, we have only presented examples of the ‘triple rotation’ of left-continuous t-norms T that either have no zero-divisors ($\alpha = 0$) or are rotation invariant ($\alpha = 1$). As discussed at the end of Section 8.2, if $\alpha \in]0, 1[$, then T is necessarily an ordinal sum of a rotation-invariant t-norm whose companion is commutative on $[0, 1[^2$ and an arbitrary left-continuous t-norm. In Fig. 8.6 we present the triple rotation of the ordinal sums

$$\begin{aligned} T_a &:= \left(\left\langle 0, \tfrac{1}{2}, R3(T_{\mathbf{M}}, \mathcal{N})_\phi \right\rangle, \left\langle \tfrac{1}{2}, 1, T_{\mathbf{M}} \right\rangle\right) \\ T_b &:= \left(\left\langle 0, \tfrac{1}{2}, R3(T_{\mathbf{P}}, \mathcal{N})_\phi \right\rangle, \left\langle \tfrac{1}{2}, 1, T_{\mathbf{P}} \right\rangle\right) \\ T_c &:= \left(\left\langle 0, \tfrac{1}{2}, R3(T_{\mathbf{L}}, \mathcal{N})_\phi \right\rangle, \left\langle \tfrac{1}{2}, 1, T_{\mathbf{L}} \right\rangle\right) , \end{aligned}$$

based on the involutive negator N^*. Note that here $\alpha = \frac{1}{2}$. For the t-norms $R3(T, N^*)$ visualized in Figs. 8.6(b), 8.6(e) and 8.6(h) it clearly holds that $R3(T, N^*)|_{\mathcal{D}_{\mathrm{IV}}}$ can be understood as a reshaped front rotation of $R3(T, N^*)|_{\mathcal{D}_{\mathrm{I}} \cap]\beta, \varsigma^{-1}(\frac{1}{2})]^2}$, with β the fixpoint of N^*. The dashed lines in the figures indicate the area $\mathcal{D}_{\mathrm{I}} \cap]\beta, \varsigma^{-1}(\frac{1}{2})]^2$. Being based on the same functions M and N, the three t-norms $R3(T, N^*)$ (with $T \in \{T_a, T_b, T_c\}$) have identical partial functions $R3(T, N^*)(\bullet, \beta) = R3(T, N^*)(\beta, \bullet)$. Finally, it can also be observed that their zooms $(R3(T, N^*))^{((\varsigma^{-1}[\frac{1}{2}])^{N^*}, \varsigma^{-1}[\frac{1}{2}])}$ are rotation-invariant t-norms, obtained by performing the triple rotation method on the rotation-invariant t-norms $(R3(T, N^*))^{(\beta, \varsigma^{-1}[\frac{1}{2}])} = T^{(0, \frac{1}{2})}$. For this latter construction the involutive negator $\breve{\varsigma} \circ N^* \circ \breve{\varsigma}^{-1}$ is used, with $\breve{\varsigma}$ the linear rescaling function from $[(\varsigma^{-1}(\frac{1}{2}))^{N^*}, \varsigma^{-1}(\frac{1}{2})]$ to $[0, 1]$. The theoretical reasonings supporting these latter observations can be found in the proof of Theorem 8.6.

Part III

Fuzzified normal forms

CHAPTER 9

Facts and figures

9.1 Introduction

A Boolean expression is an expression involving variables each of which can take either the value true or false. These variables are combined using Boolean operations such as conjunction ($\wedge$), disjunction ($\vee$) and negation ($'$). It is common knowledge that each Boolean function can be represented by a well-formed formula (wff) in Boolean propositional logic. Moreover, there are two special forms, the disjunctive and conjunctive normal form, which are of great interest, for each of these forms defines the Boolean function in a unique way.

In fuzzy logic, it is generally accepted to work with t-norms and t-conorms. Fuzzifying the Boolean normal forms of a given (Boolean) wff by interpreting $\wedge$ as a t-norm T, $\vee$ as a t-conorm S and $'$ as an involutive negator N leads to what Türkşen calls disjunctive and conjunctive fuzzy normal forms [88, 89]. However, given their origin, we prefer and insist to talk about fuzzified normal forms. In general, the disjunctive and conjunctive fuzzified normal forms are $[0,1]^n \to [0,1]$ functions, with $n \in \mathbb{N}_0$. They are sometimes used as a kind of standard fuzzification procedure. The reason for this lies in the observation that the crisp concepts themselves are often mathematically expressed by means of their disjunctive or conjunctive normal form. For example, when constructing fuzzy preference structures (P, I, J), researchers have made intensive use of the disjunctive fuzzified normal forms of the original crisp binary relations expressing preference (P), indifference (I) and incomparability (J) [13].

Until now, little is known about the relationships between the fuzzified normal forms. All attention so far has focused on their comparability, in particular for $\{0,1\}^2 \to \{0,1\}$ functions [5, 34, 88, 89, 94, 95]. We contribute to the existing knowledge on this comparability. As in the crisp case the De Morgan laws are (sometimes) used to establish a link between the disjunction, the conjunction and the negation. The chapter starts with a brief survey on De Morgan triplets.

We do not intend to give here a full description of all relationships between the two types of fuzzified normal forms. We merely present some remarkable results that provide more insight into their true nature. In this way a framework is created in which a family of rotation-invariant t-norms surfaces as a solution of a system of functional equations (Chapter 10).

9.2 De Morgan triplets

For a t-norm T, a t-conorm S and two strict negators N_1 and N_2, the two *laws of De Morgan* [27] are given by

$$S(x,y)^{N_1} = T(x^{N_1}, y^{N_1})\,, \tag{9.1}$$
$$T(x,y)^{N_2} = S(x^{N_2}, y^{N_2})\,, \tag{9.2}$$

for every $(x,y) \in [0,1]^2$. In particular, Eq. (9.1) expresses that $S = T_{N_1}$ and Eq. (9.2) expresses that $T = S_{N_2}$. If a triplet (T, S, N_1) satisfies Eq. (9.1) then (T, S, N_1^{-1}) satisfies Eq. (9.2). Conversely, whenever a triplet (T, S, N_2) satisfies Eq. (9.2) then (T, S, N_2^{-1}) satisfies Eq. (9.1). Dealing with an involutive negator N, a triplet (T, S, N) satisfies Eq. (9.1) with $N_1 = N$ if and only if it satisfies Eq. (9.2) with $N_2 = N$.

Definition 9.1 [27] A *De Morgan triplet* (T, S, N) consists of a t-norm T, a t-conorm S and a strict negator N such that Eq. (9.1) is satisfied with $N_1 = N$.

Given a t-norm T and a strict negator N, Eq. (9.1) can be used to construct the unique t-conorm S that forms a De Morgan triplet with T and N. Some basic De Morgan triplets are $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$, $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$ and $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$. Note that for a (left-)continuous t-norm T, the t-conorm in a De Morgan triplet must be (right) continuous. Moreover, if T is continuous we talk about a *continuous De Morgan triplet* (T, S, N).

Transforming a De Morgan triplet (T, S, N) by means of a triplet (ϕ, ψ, ϑ) of automorphisms does not necessarily yield a De Morgan triplet $(T_\phi, S_\psi, N_\vartheta)$. In the following theorem we look for those triplets (ϕ, ψ, ϑ) that preserve the De Morgan property (Eq. (9.1)).

Theorem 9.2 *Consider a De Morgan triplet (T, S, N) and a triplet (ϕ, ψ, ϑ) of automorphisms. Then $(T_\phi, S_\psi, N_\vartheta)$ is a De Morgan triplet if and only if T is γ-invariant, with*

$$\gamma := N \circ \psi \circ \vartheta^{-1} \circ N^{-1} \circ \vartheta \circ \phi^{-1}\,. \tag{9.3}$$

Proof Before we start we would like to point out that $(N^{-1})_\theta = (N_\theta)^{-1}$. We briefly use the notation N_θ^{-1}. By definition, $(T_\phi, S_\psi, N_\vartheta)$ satisfies the first De Morgan law (Eq. (9.1)) if and only if

$$\left(\psi^{-1}\left[S\left(\psi[x], \psi[y]\right)\right]\right)^{N_\vartheta} = \left(S_\psi(x,y)\right)^{N_\vartheta} = T_\phi\left(x^{N_\vartheta}, y^{N_\vartheta}\right) = \phi^{-1}\left[T\left(\phi[x^{N_\vartheta}], \phi[y^{N_\vartheta}]\right)\right]$$

holds for every $(x, y) \in [0,1]^2$. Denote $u := \phi[x^{N_\vartheta}]$ and $v := \phi[y^{N_\vartheta}]$, then the above expression is equivalent with

$$\begin{aligned} T(u,v) &= \phi\left[\left(\psi^{-1}\left[S\left(\psi\left[(\phi^{-1}[u])^{(N_\vartheta^{-1})}\right], \psi\left[(\phi^{-1}[v])^{(N_\vartheta^{-1})}\right]\right)\right]\right)^{N_\vartheta}\right] \\ &= \phi \circ N_\vartheta \circ \psi^{-1}\left[S\left(\psi \circ N_\vartheta^{-1} \circ \phi^{-1}[u], \psi \circ N_\vartheta^{-1} \circ \phi^{-1}[v]\right)\right] , \end{aligned}$$

for every $[u, v] \in [0,1]^2$. As (T, S, N) is a De Morgan triplet and $\gamma = N \circ \psi \circ N_\vartheta^{-1} \circ \phi^{-1}$, the latter can be rewritten as

$$\begin{aligned} T(u,v) &= \phi \circ N_\vartheta \circ \psi^{-1}\left[\left(T\left((\psi \circ N_\vartheta^{-1} \circ \phi^{-1}[u])^N, (\psi \circ N_\vartheta^{-1} \circ \phi^{-1}[v])^N\right)\right)^{(N^{-1})}\right] \\ &= \gamma^{-1}[T(\gamma(u), \gamma(v))] = T_\gamma(u, v) , \end{aligned}$$

for every $(u, v) \in [0,1]^2$. This finishes the proof. □

It is well known that the minimum operator $T_{\mathbf{M}}$ is the only t-norm that is invariant under all automorphisms γ. Therefore, any transformation $((T_{\mathbf{M}})_\phi, (S_{\mathbf{M}})_\psi, \mathcal{N}_\vartheta)$ of $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$ is a De Morgan triplet. If $\phi = \psi = \vartheta$ then, necessarily, $\gamma = \mathbf{id}$ and (T_ϕ, S_ϕ, N_ϕ) will be a De Morgan triplet if and only if (T, S, N) is a De Morgan triplet (see also [30]). In general, given a De Morgan triplet (T, S, N) and two automorphisms ϕ and ϑ, we can always select ψ such that also $(T_\phi, S_\psi, N_\vartheta)$ satisfies the first De Morgan law. It suffices to take $\gamma = \mathbf{id}$ and solve Eq. (9.3) for ψ: $\psi = N^{-1} \circ \phi \circ \vartheta^{-1} \circ N \circ \vartheta = N^{-1} \circ \phi \circ N_\vartheta$.

Let T be the ordinal sum $(\langle a_i, e_i, \sigma_i, T_i\rangle)_{i\in I}$ and consider a strict negator N. Then S will form a De Morgan triplet with T and N if

$$\begin{aligned} S(x,y) &= \begin{cases} \left(\sigma_i^{-1}\left[T_i\left(\sigma_i[x^N], \sigma_i[y^N]\right)\right]\right)^{(N^{-1})}, & \text{if } (x^N, y^N) \in [a_i, e_i]^2 , , \\ (\min(x^N, y^N))^{(N^{-1})}, & \text{elsewhere} , \end{cases} \\ &= \begin{cases} (\sigma_i)_N^{-1}\left[(T_i)_N\left((\sigma_i)_N[x], (\sigma_i)_N[y]\right)\right], & \text{if } (x, y) \in [e_i^{(N^{-1})}, a_i^{(N^{-1})}]^2 , \\ \max(x, y), & \text{elsewhere} , \end{cases} \end{aligned} \tag{9.4}$$

holds for every $(x, y) \in [0,1]^2$. Inspired by this observation we can dualize our definition of ordinal sum.

Definition 9.3 Let I be a countable index set, $(]e_i, a_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, $(\sigma_i)_{i\in I}$ a family of isomorphisms ($\sigma_i : [e_i, a_i] \to [0,1]$) and $(S_i)_{i\in I}$ a family of increasing $[0,1]^2 \to [0,1]$ functions satisfying $S_{\mathbf{M}} \leqslant S_i$. The increasing $[0,1]^2 \to [0,1]$ function S defined by

$$S(x,y) = \begin{cases} \sigma_i^{-1}\left[S_i\left(\sigma_i[x], \sigma_i[y]\right)\right], & \text{if } (x, y) \in [e_i, a_i]^2 , \\ \max(x, y), & \text{elsewhere} , \end{cases}$$

is called the *ordinal sum* of the *summands* $\langle e_i, a_i, \sigma_i, S_i\rangle$, $i \in I$. It is shortly written as $S = (\langle e_i, a_i, \sigma_i, S_i\rangle)_{i\in I}$. In case every isomorphism σ_i equals the linear rescaling function ς_i from $[e_i, a_i]$ to $[0, 1]$, we use the notation $S = (\langle e_i, a_i, S_i\rangle)_{i\in I}$.

Note that this definition complements our previous definition of ordinal sums (Definition 6.1) as no increasing $[0,1]^2 \to [0,1]$ function F can satisfy both $F \leqslant T_{\mathbf{M}}$ and $S_{\mathbf{M}} \leqslant F$. Thanks to Definition 9.3 we can more compactly rewrite Eq. (9.4) as

$$S = \left(\left\langle e_i^{(N^{-1})}, a_i^{(N^{-1})}, (\sigma_i)_N, (T_i)_N \right\rangle\right)_{i\in I} . \tag{9.5}$$

Such an elegant formulation is not possible if we only consider linear rescalings of T_i and S_i. Indeed, $(\varsigma_i)_N$ is rarely a linear rescaling of $[(e_i)^{(N^{-1})}, (a_i)^{(N^{-1})}]$ into $[0, 1]$. For example, let $a_i = 0$, $e_i = \frac{1}{\sqrt{2}}$ and define N by $x^N = \sqrt{1 - x^2}$. It then follows that

$$(\varsigma_i)_N\left(\tfrac{3}{4}\right) = \sqrt{\tfrac{1}{8}} \neq \frac{3\frac{\sqrt{2}}{4} - 1}{\sqrt{2} - 1} = \frac{\frac{3}{4} - e_i^{(N^{-1})}}{a_i^{(N^{-1})} - e_i^{(N^{-1})}} .$$

In the trivial cases $[a_i, e_i] = [0, 1]$ and $N = \mathcal{N}$, however, $(\varsigma_i)_N$ linearly 'rescales' $[e_i^{(N^{-1})}, a_i^{(N^{-1})}]$. Nevertheless, taking into account that

$$\langle a_i, e_i, \sigma_i, S_i\rangle = \left\langle a_i, e_i, \varsigma_i, (S_i)_{\sigma_i \circ \varsigma_i^{-1}} \right\rangle$$

holds for every $i \in I$, it is technically possible to rewrite Eq. (9.5) as an ordinal sum that uses linear rescaling functions only. Unfortunately, this procedure will yield a more complex formula.

9.3 Fuzzified normal forms of $\{0,1\}^2 \to \{0,1\}$ functions

In the Boolean algebra $(\{0,1\}, \vee, \wedge, ', 0, 1)$ every $\{0,1\}^2 \to \{0,1\}$ function F can be represented by its *disjunctive* ($D_{\mathcal{B}}(F)$) and *conjunctive* ($C_{\mathcal{B}}(F)$) *normal form*. There exist exactly sixteen different $\{0,1\}^2 \to \{0,1\}$ functions F. Their Boolean normal forms are listed in Table 9.1.

One can fuzzify the Boolean normal forms by replacing $(\wedge, \vee, ')$ by a triplet (T, S, N), with T a t-norm, S a t-conorm and N an involutive negator. For each $\{0,1\}^2 \to \{0,1\}$ function F its *disjunctive* and *conjunctive fuzzified normal form* are $[0,1]^2 \to [0,1]$ functions. We denote them by $D_{\mathcal{F}}(F)$ and $C_{\mathcal{F}}(F)$, respectively. In order to be unequivocally defined, these fuzzified normal forms should indeed be constructed by means of an involutive negator N. Involving a strict negator N in the fuzzification process, it would remain unclear which occurrences of $'$ have to be replaced by N and which by N^{-1}. Note also that the associativity of T and S allows to extend them in a unique way to $[0,1]^n \to [0,1]$ functions ($n \geqslant 2$):

$$T(x_1, \ldots, x_n) := T(\ldots T(T(x_1, x_2), x_3) \ldots, x_n) \tag{9.6}$$

$$S(x_1, \ldots, x_n) := S(\ldots S(S(x_1, x_2), x_3) \ldots, x_n) . \tag{9.7}$$

Table 9.1: Disjunctive and conjunctive Boolean normal forms for the 16 different $\{0,1\}^2 \to \{0,1\}$ functions

No	$D_\mathcal{B}(F) = C_\mathcal{B}(F)$	Concept
1	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = 1$	Complete affirmation
2	$0 = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$	Complete negation
3	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) = x \vee y$	Disjunction
4	$x' \wedge y' = (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$	Conjunctive negation
5	$(x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y') = x' \vee y'$	Incompatibility
6	$x \wedge y = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y)$	Conjunction
7	$(x \wedge y) \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y$	Implication
8	$x \wedge y' = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y')$	Non-implication
9	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y') = x \vee y'$	Inverse implication
10	$x' \wedge y = (x \vee y) \wedge (x' \vee y) \wedge (x' \vee y')$	Non-inverse implication
11	$(x \wedge y) \vee (x' \wedge y') = (x \vee y') \wedge (x' \vee y)$	Equivalence
12	$(x \wedge y') \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y')$	Exclusion
13	$(x \wedge y) \vee (x \wedge y') = (x \vee y) \wedge (x \vee y')$	Affirmation
14	$(x' \wedge y) \vee (x' \wedge y') = (x' \vee y) \wedge (x' \vee y')$	Negation
15	$(x \wedge y) \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y)$	Affirmation
16	$(x \wedge y') \vee (x' \wedge y') = (x \vee y') \wedge (x' \vee y')$	Negation

Table 9.2 lists the 16 disjunctive and conjunctive fuzzified normal forms retrieved from the Boolean normal forms in Table 9.1. Until now most authors have restricted themselves to fuzzified normal forms of $\{0,1\}^2 \to \{0,1\}$ functions. In Section 9.3 we will give a more formal definition of fuzzified normal forms based on an arbitrary $\{0,1\}^n \to \{0,1\}$ function F, with $n \in \mathbb{N}_0$.

The main point of study so far has been the relationship between $D_\mathcal{F}(F)$ and $C_\mathcal{F}(F)$. On the one hand, Bilgiç [5] showed that $D_\mathcal{F}(F)$ can never equal $C_\mathcal{F}(F)$ for every $\{0,1\}^2 \to \{0,1\}$ function F. On the other hand, Türkşen [88, 89] discovered that some particular triplets (T, S, N) ensure that

$$D_\mathcal{F}(F)(x,y) \leqslant C_\mathcal{F}(F)(x,y)\,, \tag{9.8}$$

for every $\{0,1\}^2 \to \{0,1\}$ function F and every $(x,y) \in [0,1]^2$. We will use the shorthand $D_\mathcal{F} \leqslant_2 C_\mathcal{F}$ to express Eq. (9.8).

Theorem 9.4 [5] *For any De Morgan triplet* (T, S, N) *with involutive negator* N*,* $D_\mathcal{F} \leqslant_2 C_\mathcal{F}$

Table 9.2: Disjunctive and conjunctive fuzzified normal forms for the 16 different $\{0,1\}^2 \to \{0,1\}$ functions

No	$D_{\mathcal{F}}(F)$	$C_{\mathcal{F}}(F)$
1	$S\big(T(x,y),T(x,y^N),T(x^N,y),T(x^N,y^N)\big)$	1
2	0	$T\big(S(x,y),S(x,y^N),S(x^N,y),S(x^N,y^N)\big)$
3	$S\big(T(x,y),T(x,y^N),T(x^N,y)\big)$	$S(x,y)$
4	$T(x^N,y^N)$	$T\big(S(x,y^N),S(x^N,y),S(x^N,y^N)\big)$
5	$S\big(T(x^N,y),T(x,y^N),T(x^N,y^N)\big)$	$S(x^N,y^N)$
6	$T(x,y)$	$T\big(S(x,y),S(x,y^N),S(x^N,y)\big)$
7	$S\big(T(x,y),T(x^N,y),T(x^N,y^N)\big)$	$S(x^N,y)$
8	$T(x,y^N)$	$T\big(S(x,y),S(x,y^N),S(x^N,y^N)\big)$
9	$S\big(T(x,y),T(x,y^N),T(x^N,y^N)\big)$	$S(x,y^N)$
10	$T(x^N,y)$	$T\big(S(x,y),S(x^N,y),S(x^N,y^N)\big)$
11	$S\big(T(x,y),T(x^N,y^N)\big)$	$T\big(S(x,y^N),S(x^N,y)\big)$
12	$S\big(T(x,y^N),T(x^N,y)\big)$	$T\big(S(x,y),S(x^N,y^N)\big)$
13	$S\big(T(x,y),T(x,y^N)\big)$	$T\big(S(x,y),S(x,y^N)\big)$
14	$S\big(T(x^N,y),T(x^N,y^N)\big)$	$T\big(S(x^N,y),S(x^N,y^N)\big)$
15	$S\big(T(x,y),T(x^N,y)\big)$	$T\big(S(x,y),S(x^N,y)\big)$
16	$S\big(T(x,y^N),T(x^N,y^N)\big)$	$T\big(S(x,y^N),S(x^N,y^N)\big)$

is equivalent with the following system of inequalities:

$$S\left(T(x,y),T(x,y^N),T(x^N,y)\right) \leqslant S(x,y)\,, \tag{9.9}$$

$$S\left(T(x,y),T(x^N,y^N)\right) \leqslant T\left(S(x,y^N),S(x^N,y)\right)\,, \tag{9.10}$$

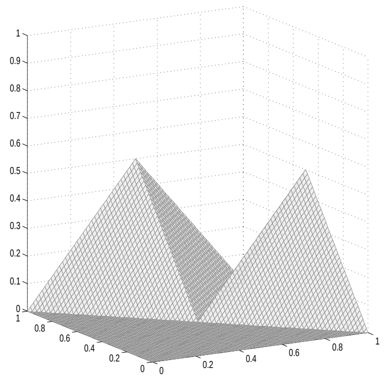
$$S\left(T(x,y),T(x,y^N)\right) \leqslant T\left(S(x,y),S(x,y^N)\right)\,, \tag{9.11}$$

for every $(x,y) \in [0,1]^2$.

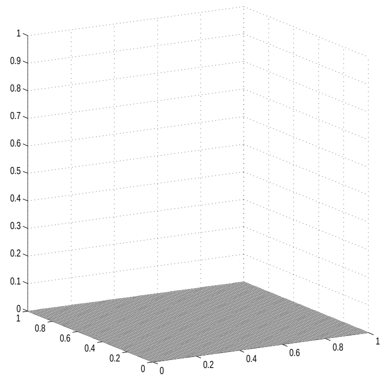
Inequalities (9.9)–(9.11) were obtained from rows 3, 11 and 13 of Table 9.2. They correspond to the $\{0,1\}^2 \to \{0,1\}$ functions modeling Disjunction, Equivalence and Affirmation. For the three De Morgan triplets $(T_{\mathbf{M}},S_{\mathbf{M}},\mathcal{N})$, $(T_{\mathbf{P}},S_{\mathbf{P}},\mathcal{N})$ and $(T_{\mathbf{L}},S_{\mathbf{L}},\mathcal{N})$ it readily follows that Eqs. (9.9)–(9.11) are true. Consequently, $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$. Figure 9.1 depicts for these triplets the difference between their conjunctive and disjunctive fuzzified normal form.

C. and E. Walker [94] pointed out that if $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$ is satisfied for a De Morgan triplet, then it holds for all isomorphic De Morgan triplets.

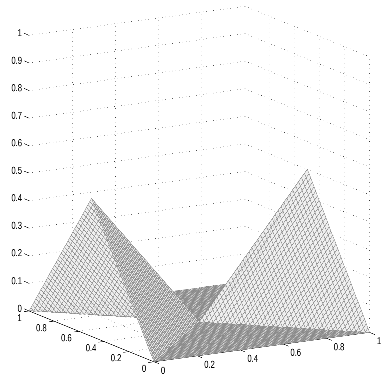
Theorem 9.5 [94] *If $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$ holds for some De Morgan triplet (T,S,N) with involutive negator N, then it also holds for all De Morgan triplets (T_ϕ,S_ϕ,N_ϕ), with ϕ an arbitrary automorphism.*



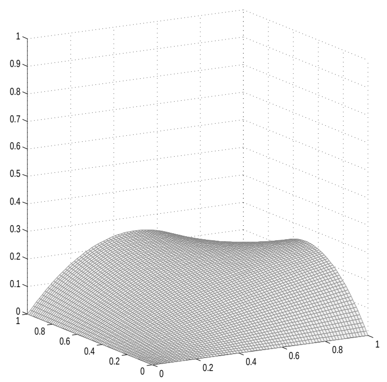
(a) $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$; Disjunction



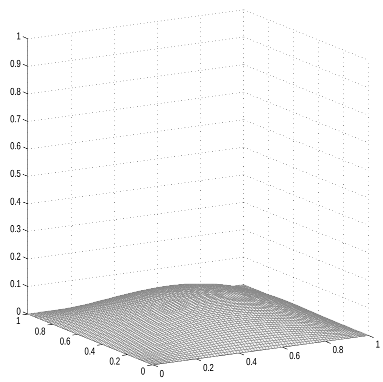
(b) $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$; Equivalence



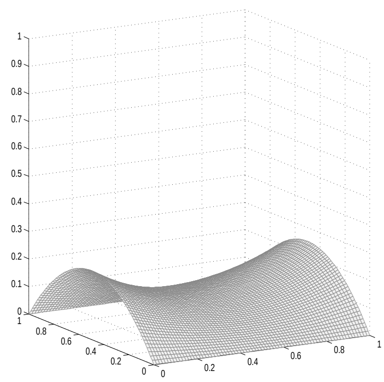
(c) $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$; Affirmation



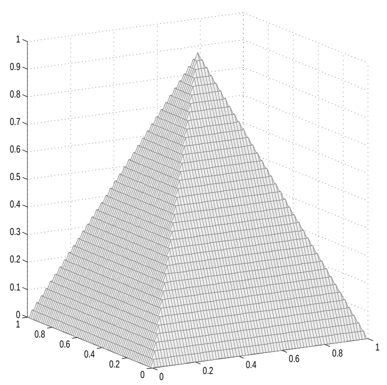
(d) $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$; Disjunction



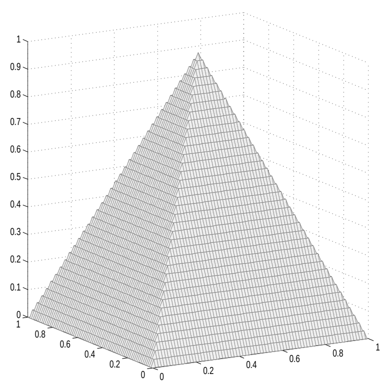
(e) $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$; Equivalence



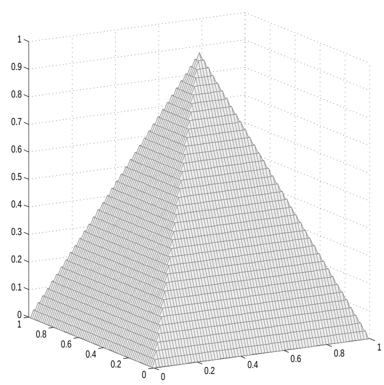
(f) $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$; Affirmation



(g) $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$; Disjunction



(h) $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$; Equivalence



(i) $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$; Affirmation

Figure 9.1: $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ for the Disjunction (Eq. (9.9)), the Equivalence (Eq. (9.10)) and the Affirmation (Eq. (9.11)).

C. and E. Walker [94, 95] also illustrated that (even for De Morgan triplets containing a continuous Archimedean t-norm) a more general transformation $(T_\phi, S_\psi, N_\theta)$ of the De Morgan triplet (T, S, N) does not necessarily preserve the inequality $D_\mathcal{F} \leqslant_2 C_\mathcal{F}$.

Example 9.6 Transforming the De Morgan triplets $(T_\mathbf{P}, S_\mathbf{P}, \mathcal{N})$ and $(T_\mathbf{L}, S_\mathbf{L}, \mathcal{N})$ by means of the triplet of automorphisms $(\mathbf{id}, \mathcal{N} \circ \mathcal{N}_\phi, \phi)$, with

$$\phi(x) := \begin{cases} \frac{7-\sqrt[4]{2401-10000\,x^4}}{14}, & \text{if } x \leqslant \frac{7}{10}\,, \\ \frac{6-\left(\sqrt[4]{3}-\sqrt[4]{10\,x-7}\right)^4}{6}, & \text{elsewhere}\,, \end{cases} \tag{9.12}$$

yields, resp., the continuous De Morgan triplets $(T_\mathbf{P}, (S_\mathbf{P})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi) = (T_\mathbf{P}, (T_\mathbf{P})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$ and $(T_\mathbf{L}, (S_\mathbf{L})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi) = (T_\mathbf{L}, (T_\mathbf{L})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$. Unfortunately, as illustrated in Figs. 9.2(a)–9.2(f) this transformation does not preserve the inequality $D_\mathcal{F} \leqslant_2 C_\mathcal{F}$. Note that, although it is not visible, also $D_\mathcal{F}(F) \nleqslant C_\mathcal{F}(F)$ in Fig. 9.2(d).

Even continuous De Morgan triplets $(T, S, \mathcal{N})$ based on an ordinal sum $T = (\langle a_i, e_i, T_i\rangle)_{i\in I}$, where every $T_i \in \{T_\mathbf{P}, T_\mathbf{L}\}$, do not necessarily yield $D_\mathcal{F} \leqslant_2 C_\mathcal{F}$. Figures 9.2(g)–9.2(i) visualize this for the De Morgan triplet $(T, T_\mathcal{N}, \mathcal{N})$, with T the ordinal sum $(\langle 0, \frac{1}{3}, T_\mathbf{P}\rangle, \langle \frac{1}{3}, 1, T_\mathbf{L}\rangle)$. $\triangle$

Let us now focus on inequalities Eqs. (9.9)–(9.11). The question arises whether some of these inequalities can be turned into equalities, as in the Boolean case.

Proposition 9.7 *Consider a De Morgan triplet (T, S, N) with involutive negator N. Equality in Eq.* (9.9) *can never hold for all $(x, y) \in [0, 1]^2$.*

Proof Suppose the converse: $S(T(x, y), T(x, y^N), T(x^N, y)) = S(x, y)$ holds for every $(x, y) \in [0, 1]^2$. Let β be the unique fixpoint of N. For $(x, y) = (\beta, 1)$, we obtain that $S(\beta, \beta) = 1$. Because of the De Morgan law (Eq. (9.1)) it then holds that $T(\beta, \beta) = 0$. This leads to the contradiction $0 = S\big(T(\beta, \beta), T(\beta, \beta), T(\beta, \beta)\big) = S(\beta, \beta) = 1$. $\square$

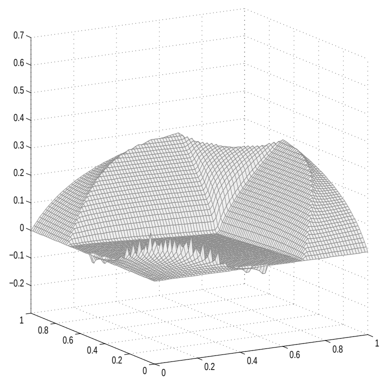
Proposition 9.8 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N with fixpoint β. If $T(x, \beta) = x \leqslant x^N = S(x^N, \beta)$, for every $x \in [0, \beta]$, then equality in Eq.* (9.10) *holds for every $(x, y) \in [0, 1]^2$.*

Proof The conditions imposed on T and S can be rewritten as

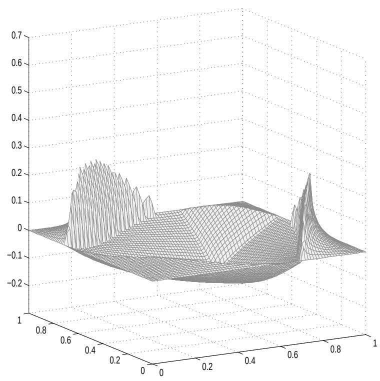
$$\begin{array}{ll} T(x, y) \in [0, \beta] \text{ and } S(x, y) \in [0, \beta], & \text{if } (x, y) \in [0, \beta]^2 \\ T(x, y) \in [\beta, 1] \text{ and } S(x, y) \in [\beta, 1], & \text{if } (x, y) \in [\beta, 1]^2 \\ T(x, y) = \min(x, y) \text{ and } S(x, y) = \max(x, y), & \text{if } (\min(x, y), \max(x, y)) \in [0, \beta] \times [\beta, 1]\,. \end{array} \tag{9.13}$$

In case $(x, y) \in [0, \beta]^2$, it holds that

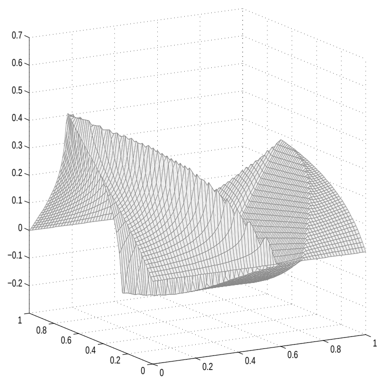
$$S\left(T(x, y), T(x^N, y^N)\right) = \max\left(T(x, y), T(x^N, y^N)\right) = T(x^N, y^N)$$



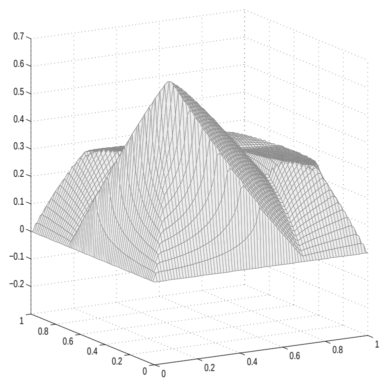
(a) $(T_{\mathbf{P}}, (T_{\mathbf{P}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Disjunction



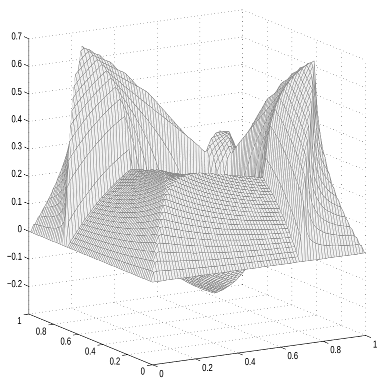
(b) $(T_{\mathbf{P}}, (T_{\mathbf{P}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Equivalence



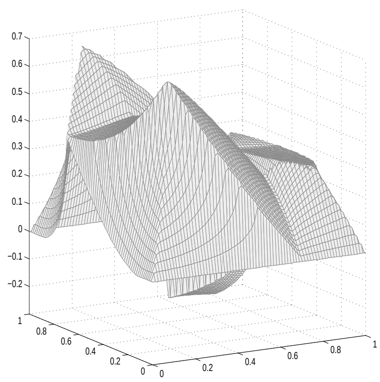
(c) $(T_{\mathbf{P}}, (T_{\mathbf{P}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Affirmation



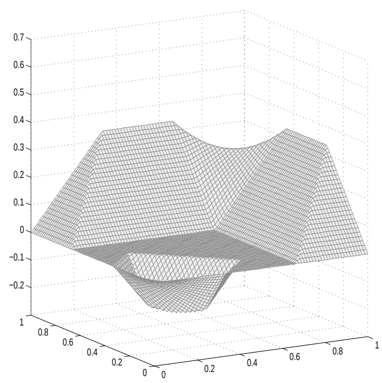
(d) $(T_{\mathbf{L}}, (T_{\mathbf{L}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Disjunction



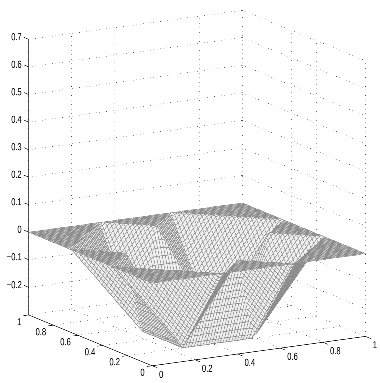
(e) $(T_{\mathbf{L}}, (T_{\mathbf{L}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Equivalence



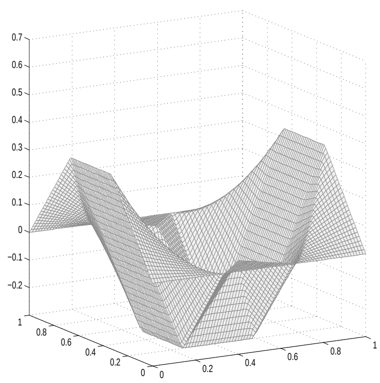
(f) $(T_{\mathbf{L}}, (T_{\mathbf{L}})_{\mathcal{N}_\phi}, \mathcal{N}_\phi)$; Affirmation



(g) $(T, T_{\mathcal{N}}, \mathcal{N})$; Disjunction



(h) $(T, T_{\mathcal{N}}, \mathcal{N})$; Equivalence



(i) $(T, T_{\mathcal{N}}, \mathcal{N})$; Affirmation

Figure 9.2: $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ for the Disjunction (Eq. (9.9)), the Equivalence (Eq. (9.10)) and the Affirmation (Eq. (9.11)).

and

$$T\left(S(x,y^N),S(x^N,y)\right)=T\left(\max(x,y^N),\max(x^N,y)\right)=T(x^N,y^N)\,.$$

In the same way one can prove the equality in case $(x,y)\in[0,\beta]\times[\beta,1]$, $(x,y)\in[\beta,1]\times[0,\beta]$ and $(x,y)\in[\beta,1]^2$. $\square$

Proposition 9.9 *Consider a triplet (T,S,N) consisting of a t-norm T, a t-conorm S and an involutive negator N. Equality in Eq. (9.11) can never hold for all $(x,y)\in[0,1]^2$.*

Proof Suppose the converse: $S(T(x,y),T(x,y^N)) = T(S(x,y),S(x,y^N))$ holds for every $(x,y)\in[0,1]^2$. Let β be the fixpoint of N. For $(x,y)=(0,\beta)$, resp., $(x,y)=(1,\beta)$, we obtain that $T(\beta,\beta)=0$, resp., $S(\beta,\beta)=1$. This leads to the contradiction $0=S(T(\beta,\beta),T(\beta,\beta))=T(S(\beta,\beta),S(\beta,\beta))=1$. $\square$

Let (T,S,N) be a De Morgan triplet with involutive negator N. Propositions 9.7 and 9.9 imply that equality for all $(x,y)\in[0,1]^2$ cannot occur in Eqs. (9.9) and (9.11). Taking into account Eqs. (9.1), (9.13) and Corollary 6.15, the conditions of Proposition 9.8 express that $T^{(0,\beta)}$ must be a t-norm. Therefore, Eq. (9.10) becomes an equality if T is an ordinal sum $(\langle 0,\beta,\sigma_1,T_1\rangle,\langle\beta,1,\sigma_2,T_2\rangle)$, with T_1 and T_2 two t-norms (Corollary 6.12). Figure 9.1(b) illustrates this phenomenon for the triplet $(T_{\mathbf{M}},S_{\mathbf{M}},\mathcal{N})$ ($\beta=\frac{1}{2}$ and $(T_{\mathbf{M}})^{(0,\frac{1}{2})}=T_{\mathbf{M}}$). From Theorem 6.6 it follows that the $(0,\beta)$-zoom $T^{(0,\beta)}$ of a continuous t-norm T will be a t-norm if and only if β is an idempotent element of T.

9.4 Fuzzified normal forms of $\{0,1\}^n\to\{0,1\}$ functions

Consider $n\in\mathbb{N}_0$. Also every $\{0,1\}^n\to\{0,1\}$ function F can be represented by its disjunctive and conjunctive (Boolean) normal form. Counting all $\{0,1\}^n\to\{0,1\}$ functions, we know that there are $2^{(2^n)}$ different disjunctive and $2^{(2^n)}$ different conjunctive normal forms.

Definition 9.10 Let $n\in\mathbb{N}_0$ and consider the Boolean algebra $(\{0,1\},\vee,\wedge,',0,1)$. The *disjunctive* and *conjunctive normal forms* of a $\{0,1\}^n\to\{0,1\}$ function F are given by

$$D_{\mathcal{B}}(F)(x_1,...,x_n)=\bigvee_{F(e_1,...,e_n)=1} x_1^{e_1}\wedge ...\wedge x_n^{e_n}\,, \tag{9.14}$$

$$C_{\mathcal{B}}(F)(x_1,...,x_n)=\bigwedge_{F(e_1,...,e_n)=0} x_1^{e_1'}\vee ...\vee x_n^{e_n'}\,, \tag{9.15}$$

where $x^e=x$ if $e=1$ and $x^e=x'$ if $e=0$.

As in the binary case, replacing $(\wedge,\vee,')$ by a triplet (T,S,N), with T a t-norm, S a t-conorm and N an involutive negator, results in a straightforward fuzzification of Eqs. (9.14) and (9.15). For this procedure we invoke Eqs. (9.6) and (9.7). By convention, we put $T(x):=x$ and $S(x):=x$, for every $x\in[0,1]$. Taking into account that T has neutral element 1 and S has neutral element 0, we obtain the definition below. To make the formulae readable, the vector notation $\vec{x}$ is used to denote the n-tuples $(x_1,\ldots,x_n) \in [0,1]^n$.

Definition 9.11 Let $n \in \mathbb{N}_0$ and consider a triplet (T,S,N) consisting of a t-norm T, a t-conorm S and an involutive negator N. The *disjunctive* and *conjunctive fuzzified normal forms* of a $\{0,1\}^n \to \{0,1\}$ function F are given by

$$D_{\mathcal{F}}(F)(\vec{x}) = S\left\{F(\vec{e})\ T(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\}, \tag{9.16}$$

$$C_{\mathcal{F}}(F)(\vec{x}) = T\left\{\left((1-F(\vec{e}))\ S\left(\vec{x}^{(\vec{e}^{\vec{0}})}\right)^N\right)^N \,\middle|\, \vec{e} \in \{0,1\}^n\right\}, \tag{9.17}$$

where $\vec{x} \in [0,1]^n$, $\vec{0} = (0,\ldots,0)$, $\vec{x}^{\vec{e}} = (x_1^{e_1},\ldots,x_n^{e_n})$, $x^e = x$ if $e = 1$ and $x^e = x^N$ if $e = 0$.

Clearly, Eq. (9.16) coincides with Eq. (9.14), and Eq. (9.17) coincides with Eq. (9.15), whenever $\vec{x} \in \{0,1\}^n$. In case (T,S,N) is a De Morgan triplet, Eq. (9.17) can be rewritten as

$$\begin{aligned}
C_{\mathcal{F}}(F)(\vec{x}) &= \left(S\left\{(1-F(\vec{e}))\ T\left(\left(\vec{x}^{(\vec{e}^{\vec{0}})}\right)^{\vec{0}}\right) \,\middle|\, \vec{e} \in \{0,1\}^n\right\}\right)^N \\
&= \left(S\left\{(1-F(\vec{e}))\ T\left(\vec{x}^{((\vec{e}^{\vec{0}})^{\vec{0}})}\right) \mid \vec{e} \in \{0,1\}^n\right\}\right)^N \\
&= \left(S\left\{(1-F(\vec{e}))\ T(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\}\right)^N .
\end{aligned} \tag{9.18}$$

In view of Section 9.3 it remains an intriguing problem to figure out which triplets (T,S,N) ensure that $D_{\mathcal{F}}(F)(\vec{x}) \leqslant C_{\mathcal{F}}(F)(\vec{x})$ is satisfied for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$. We use $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ to express the latter inequality. It is easily verified that $C_{\mathcal{F}}(f)(x) - D_{\mathcal{F}}(f)(x) \in \{0, T(x,x^N), 1-S(x^N,x)\}$ in case $n=1$. Hence, $D_{\mathcal{F}} \leqslant_1 C_{\mathcal{F}}$ is satisfied for every triplet (T,S,N) with involutive negator N. As illustrated in Section 9.3 this is not true when working in 2 dimensions. For a given triplet (T,S,N) and $n \in \mathbb{N}_0$ it is therefore not guaranteed that $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ implies $D_{\mathcal{F}} \leqslant_{n+1} C_{\mathcal{F}}$. Nevertheless, the converse implication is always fulfilled.

Theorem 9.12 *Consider a triplet (T,S,N) consisting of a t-norm T, a t-conorm S and an involutive negator N. If $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ holds for some $n \in \mathbb{N}_0$, then $D_{\mathcal{F}} \leqslant_m C_{\mathcal{F}}$ is satisfied for every $m \in \mathbb{N}_0$ such that $m \leqslant n$.*

Proof It suffices to prove that $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ implies $D_{\mathcal{F}} \leqslant_{n-1} C_{\mathcal{F}}$, for every $n \geqslant 2$. Consider an arbitrary $\{0,1\}^{n-1} \to \{0,1\}$ function F and suppose that $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$. Define the $\{0,1\}^n \to \{0,1\}$ function G as follows:

$$G(e_1,\ldots,e_n) = \begin{cases} 0, & \text{if } e_n = 0\,,\\ F(e_1,\ldots,e_{n-1}), & \text{elsewhere}\,. \end{cases}$$

Since T has neutral element 1 and S has neutral element 0, we obtain from Eq. (9.16) that

$$\begin{aligned}D_{\mathcal{F}}(G)(x_1,\ldots,x_{n-1},1) &= S\left\{G(e_1,\ldots,e_{n-1},1)\ T(x_1^{e_1},\ldots,x_{n-1}^{e_{n-1}},1) \mid (e_1,\ldots,e_{n-1}) \in \{0,1\}^{n-1}\right\}\\ &= S\left\{F(e_1,\ldots,e_{n-1})\ T(x_1^{e_1},\ldots,x_{n-1}^{e_{n-1}}) \mid (e_1,\ldots,e_{n-1}) \in \{0,1\}^{n-1}\right\}\\ &= D_{\mathcal{F}}(F)(x_1,\ldots,x_{n-1}),\end{aligned}$$

for every $(x_1,\ldots,x_{n-1}) \in [0,1]^2$. Recall that S has absorbing element 1. It then follows in a similar way from Eq. (9.17) that $C_{\mathcal{F}}(G)(x_1,\ldots,x_{n-1},1) = C_{\mathcal{F}}(F)(x_1,\ldots,x_{n-1})$, for every $(x_1,\ldots,x_{n-1}) \in [0,1]^2$. Taking into account that $D_{\mathcal{F}}(G) \leqslant C_{\mathcal{F}}(G)$ this concludes the proof. $\square$

This theorem puts the results from Example 9.6 and Figs 9.1 and 9.2 in a new light. On the one hand, the De Morgan triplets $(T_{\mathbf{P}},(T_{\mathbf{P}})_{\mathcal{N}_\phi},\mathcal{N}_\phi)$ and $(T_{\mathbf{L}},(T_{\mathbf{L}})_{\mathcal{N}_\phi},\mathcal{N}_\phi)$, with ϕ the automorphism defined by Eq. (9.12), never yield the inequality $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$, for some $n \in \mathbb{N}_0 \setminus \{1\}$. On the other hand, the De Morgan triplets $(T_{\mathbf{M}},S_{\mathbf{M}},\mathcal{N})$, $(T_{\mathbf{P}},S_{\mathbf{P}},\mathcal{N})$ and $(T_{\mathbf{L}},S_{\mathbf{L}},\mathcal{N})$ ensure that $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$. It is not inconceivable that, when dealing with these triplets, $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ is also satisfied for dimensions $n \geqslant 2$. In the following propositions we further explore this conjecture. We use the notation $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ to denote that $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ holds for every $n \in \mathbb{N}_0$.

Proposition 9.13 *$D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for the triplet $(T_{\mathbf{M}},S_{\mathbf{M}},\mathcal{N})$.*

Proof Take arbitrary $n \in \mathbb{N}_0$. As $(T_{\mathbf{M}},S_{\mathbf{M}},\mathcal{N})$ is a De Morgan triplet we express $C_{\mathcal{F}}(F)(\vec{x})$ by means of Eq. (9.18). We have to prove that

$$\begin{aligned}&0 \leqslant C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})\\ &\quad= \left(\max\left\{(1-F(\vec{e}))\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\}\right)^{\mathcal{N}} - \max\left\{F(\vec{e})\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\}\end{aligned}$$

holds for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$. The above inequality is equivalent with

$$\max\left\{F(\vec{e})\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} + \max\left\{(1-F(\vec{e}))\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} \leqslant 1\,. \tag{9.19}$$

Case 1: Suppose there exists an index $i \in \{1,...,n\}$ such that $x_i = \frac{1}{2}$. Consequently $x_i^{e_i} = \frac{1}{2}$, for all $e_i \in \{0,1\}$, and

$$\max\left\{F(\vec{e})\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} + \max\left\{(1-F(\vec{e}))\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} \leqslant \tfrac{1}{2} + \tfrac{1}{2} = 1\,.$$

Case 2: If for every index i it holds that $x_i \neq \frac{1}{2}$, then there exists a unique n-tuple $\vec{\varepsilon}$ for which $\frac{1}{2} < x_k^{\varepsilon_k} = \min(\vec{x}^{\vec{\varepsilon}})$. For each n-tuple $\vec{e} \neq \vec{\varepsilon}$ one can find an index $j \in \{1,\ldots,n\}$ such that $x_j^{e_j} = x_j^{(\varepsilon_j^0)} = (x_j^{\varepsilon_j})^{\mathcal{N}} \leqslant (x_k^{\varepsilon_k})^{\mathcal{N}} < \frac{1}{2}$ and hence, $\min(\vec{x}^{\vec{e}}) \leqslant (x_k^{\varepsilon_k})^{\mathcal{N}}$. Consequently we distinguish the following cases:

$$\max\left\{F(\vec{e})\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} + \max\left\{(1-F(\vec{e}))\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} \leqslant x_k^{\varepsilon_k} + (x_k^{\varepsilon_k})^{\mathcal{N}} = 1\,,$$

whenever $F(\vec{\varepsilon}) = 1$ and

$$\max\left\{F(\vec{e})\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\}+\max\left\{(1-F(\vec{e}))\min(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\right\} \leqslant (x_k^{\varepsilon_k})^{\mathcal{N}}+x_k^{\varepsilon_k} = 1\,,$$

whenever $F(\vec{\varepsilon}) = 0$. $\square$

Next, we want to prove a similar theorem for the triplet $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$. We need to recall first the definition of a multiset.

Definition 9.14 A *multiset* is a set-like object where the elements can occur more than once. A multiset can be formally defined as a pair (A, m) where A is some set and m is an $A \to \mathbb{N}_0$ function that associates to each $a \in A$ its multiplicity in the multiset. The set A is called the *underlying set of elements.*

The following set-theoretical result will be crucial to compute a lower bound for $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$.

Lemma 9.15 *For every multiset (A, m) with $A \subset [0,1]$ and $|(A,m)| \in \mathbb{N}_0$ it holds that*

$$1 - \sum_{x\in(A,m)} x \leqslant \prod_{x\in(A,m)} (1-x)\,. \tag{9.20}$$

Proof If $|(A,m)| = 1$, Eq. (9.20) is a trivial equality. We continue the proof by induction on the cardinality of (A, m). Suppose that Eq. (9.20) holds for all multisets that contain only elements from $[0,1]$ and have cardinality $n \geqslant 1$. Let (A, m) be an arbitrary multiset with $A \subset [0,1]$ and $|(A,m)| = n+1$. Take arbitrary $\alpha \in (A, m)$. Then

$$\begin{aligned} 1 - \sum_{x\in(A,m)} x &\leqslant 1 + \alpha \times \sum_{x\in(A,m)\setminus\{\alpha\}} x - \sum_{x\in(A,m)} x = (1-\alpha)\times\left(1 - \sum_{x\in(A,m)\setminus\{\alpha\}} x\right) \\ &\leqslant (1-\alpha)\times \prod_{x\in(A,m)\setminus\{\alpha\}}(1-x) = \prod_{x\in(A,m)}(1-x)\,. \end{aligned}$$

Hence, Eq. (9.20) also holds for every multiset (A, m) with $A \subset [0,1]$ and $|(A,m)| = n+1$. This finishes the proof. $\square$

Proposition 9.16 *$D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for the triplet $(T_{\mathbf{P}}, S_{\mathbf{P}}, \mathcal{N})$.*

Proof Take arbitrary $n \in \mathbb{N}_0$. Recall that $T_{\mathbf{P}}(\vec{x}) = \prod_{i=1}^n x_i$ and $S_{\mathbf{P}}(\vec{x}) = 1 - \prod_{i=1}^n (1-x_i)$, for every $\vec{x} \in [0,1]^n$. Based on Eqs.(9.16) and (9.18), $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ becomes equivalent with

$$\left(1 - \prod_{\vec{e}\in\{0,1\}^n}\left(1 - F(\vec{e})\, T_{\mathbf{P}}(\vec{x}^{\vec{e}})\right)\right) + \left(1 - \prod_{\vec{e}\in\{0,1\}^n}\left(1 - (1-F(\vec{e}))\, T_{\mathbf{P}}(\vec{x}^{\vec{e}})\right)\right) \leqslant 1\,, \tag{9.21}$$

for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$. All numbers $F(\vec{e})\,T(\vec{x}^{\vec{e}})$ and $(1-F(\vec{e}))\,T(\vec{x}^{\vec{e}})$, with $\vec{e} \in \{0,1\}^n$, constitute a multiset for which the underlying set of elements belongs to $[0,1]$ and that has cardinality $|\{0,1\}^n| = 2^n$. Applying Lemma 9.15 we obtain that

$$\sum_{\vec{e}\in\{0,1\}^n} F(\vec{e})\,T_{\mathbf{P}}(\vec{x}^{\vec{e}}) + \sum_{\vec{e}\in\{0,1\}^n} (1-F(\vec{e}))\,T_{\mathbf{P}}(\vec{x}^{\vec{e}}) = \sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{P}}(\vec{x}^{\vec{e}})$$

forms an upper bound for the the left-hand side of Eq. (9.21). To conclude the proof, we show that $\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{P}}(\vec{x}^{\vec{e}}) = 1$. By regrouping the terms of the summation we get

$$\begin{aligned}\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{P}}(\vec{x}^{\vec{e}}) &= \sum_{\vec{e}\in\{0,1\}^n} \prod_{i=1}^{n} x_i^{e_i} = x_n \times \left(\sum_{\vec{\varepsilon}\in\{0,1\}^{n-1}} \prod_{i=1}^{n-1} x_i^{\varepsilon_i} \right) + x_n^{\mathcal{N}} \times \left(\sum_{\vec{\varepsilon}\in\{0,1\}^{n-1}} \prod_{i=1}^{n-1} x_i^{\varepsilon_i} \right)\\ &= \sum_{\vec{\varepsilon}\in\{0,1\}^{n-1}} \prod_{i=1}^{n-1} x_i^{\varepsilon_i}\,.\end{aligned}$$

Repeating this procedure $(n-1)$ times results in $\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{P}}(\vec{x}^{\vec{e}}) = x_1 + 1 - x_1 = 1$. $\square$

As is shown in the next theorem, also for the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ we can prove that $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$.

Proposition 9.17 *$D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$.*

Proof Take arbitrary $n \in \mathbb{N}_0$. Recall that $T_{\mathbf{L}}(\vec{x}) = \max((\sum_{i=1}^{n} x_i) - (n-1), 0)$ and $S_{\mathbf{L}}(\vec{x}) = \min(\sum_{i=1}^{n} x_i, 1)$, for every $\vec{x} \in [0,1]^n$. Again, taking into account Eqs. (9.16) and (9.18), we get that

$$\begin{aligned}&C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})\\ &\quad= 1 - \min\left(\sum_{\vec{e}\in\{0,1\}^n} (1-F(\vec{e}))\,T_{\mathbf{L}}(\vec{x}^{\vec{e}}), 1 \right) - \min\left(\sum_{\vec{e}\in\{0,1\}^n} F(\vec{e})\,T_{\mathbf{L}}(\vec{x}^{\vec{e}}), 1 \right)\\ &\quad= 1 - \min\left(\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\vec{e}}), 1 + \sum_{\vec{e}\in\{0,1\}^n} F(\vec{e})\,T_{\mathbf{L}}(\vec{x}^{\vec{e}}), 1 + \sum_{\vec{e}\in\{0,1\}^n} (1-F(\vec{e}))\,T_{\mathbf{L}}(\vec{x}^{\vec{e}}), 2 \right),\end{aligned} \tag{9.22}$$

for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$. For $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ to hold it is then necessary and sufficient that

$$\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\vec{e}}) \leqslant 1\,. \tag{9.23}$$

The proof of Eq. (9.23) goes by induction on the dimension n.

Case 1: For $n = 1$, we get $\sum_{e\in\{0,1\}} T_{\mathbf{L}}(x_1^e) = x_1 + x_1^{\mathcal{N}} = 1$.

Case 2: If $n = 2$, we rewrite the left-hand side of Eq. (9.23) as

$$\begin{aligned}\sum_{\vec{e}\in\{0,1\}^2} T_{\mathbf{L}}(\vec{x}^{\,\vec{e}}) &= \max(x_1 + x_2 - 1, 0) + \max(1 - x_1 - x_2, 0) + \max(x_1 - x_2, 0) + \max(x_2 - x_1, 0)\\ &= \max(2\,x_1 - 1, 2\,x_2 - 1, 1 - 2\,x_1, 1 - 2\,x_2) \leqslant 1\,.\end{aligned}$$

Case 3: Consider now the case $n \geqslant 3$ and suppose that Eq. (9.23) holds for dimension $n-2$. Let $\vec{x} \in [0,1]^n$, then for each $\vec{\varepsilon} \in \{0,1\}^{n-2}$ we denote

$$G_{\vec{\varepsilon}}(\vec{x}) := n - 2 - \sum_{i=1}^{n-2} x_i^{\varepsilon_i}\,.$$

It follows that

$$\begin{aligned}\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\,\vec{e}}) = \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \Big(&\max(x_{n-1} + x_n - G_{\vec{\varepsilon}}(\vec{x}) - 1, 0) + \max(2 - x_{n-1} - x_n - G_{\vec{\varepsilon}}(\vec{x}) - 1, 0)\\ &+ \max(1 - x_{n-1} + x_n - G_{\vec{\varepsilon}}(\vec{x}) - 1, 0) + \max(x_{n-1} + 1 - x_n - G_{\vec{\varepsilon}}(\vec{x}) - 1, 0)\Big)\,,\end{aligned}$$

which can also be written as

$$\begin{aligned}\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\,\vec{e}}) = \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \Big(&-4\,G_{\vec{\varepsilon}}(\vec{x}) + \max(x_{n-1} + x_n - 1, G_{\vec{\varepsilon}}(\vec{x})) + \max(1 - x_{n-1} - x_n, G_{\vec{\varepsilon}}(\vec{x}))\\ &+ \max(-x_{n-1} + x_n, G_{\vec{\varepsilon}}(\vec{x})) + \max(x_{n-1} - x_n, G_{\vec{\varepsilon}}(\vec{x}))\Big)\,.\end{aligned}$$

Taking into account that $G_{\vec{\varepsilon}}(\vec{x}) \geqslant 0$, for any $\vec{\varepsilon} \in \{0,1\}^{n-2}$, leads to

$$\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\,\vec{e}}) = \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \Big(-2\,G_{\vec{\varepsilon}}(\vec{x}) + \max(|x_{n-1} + x_n - 1|, G_{\vec{\varepsilon}}(\vec{x})) + \max(|x_{n-1} - x_n\,|, G_{\vec{\varepsilon}}(\vec{x}))\Big)\,.$$

We then combine the three terms from the latter summation. Since

$$|x_{n-1} + x_n - 1| + |x_{n-1} - x_n| \leqslant 1$$

we finally get that

$$\begin{aligned}\sum_{\vec{e}\in\{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\,\vec{e}}) &\leqslant \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \max(1 - 2\,G_{\vec{\varepsilon}}(\vec{x}), 1 - G_{\vec{\varepsilon}}(\vec{x}), 0) \leqslant \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \max(1 - G_{\vec{\varepsilon}}(\vec{x}), 0)\\ &= \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} \max\left(\left(\sum_{i=1}^{n-2} x_i^{\varepsilon_i}\right) - (n-3), 0\right) = \sum_{\vec{\varepsilon}\in\{0,1\}^{n-2}} T_{\mathbf{L}}(\vec{x}^{\,\vec{\varepsilon}})\,.\end{aligned}$$

By invoking the induction hypothesis, this completes the proof. □

Studying the proof of the above proposition more carefully it strikes that we can compute $C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})$ explicitly from Eqs. (9.22) and (9.23).

Corollary 9.18 *Consider $n \in \mathbb{N}_0$ For the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ it holds that:*

$$C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) = 1 - \sum_{\vec{e} \in \{0,1\}^n} T_{\mathbf{L}}(\vec{x}^{\vec{e}}), \tag{9.24}$$

for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$.

The right-hand side of Eq. (9.24) does not depend on the Boolean function F. This independence can also be recognized in Figs. 9.1(g)–9.1(i). In two dimensions and working with the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$, the difference between both fuzzified normal forms always seems to be a pyramid. We will study this phenomenon more profoundly in Chapter 10.

As in the binary case (Theorem 9.5), transforming a De Morgan triplet (T, S, N) by means of an automorphism ϕ does not affect the (in)validity of the inequality $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$.

Theorem 9.19 *Consider an automorphism ϕ. $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$, with $n \in \mathbb{N}_0$, holds for some triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N if and only if it holds for the triplet (T_ϕ, S_ϕ, N_ϕ).*

Proof Consider an arbitrary automorphism ϕ and an arbitrary $\{0,1\}^n \to \{0,1\}$ function F. In order to distinguish the normal forms constructed from the triplet (T, S, N) and those corresponding to the triplet (T_ϕ, S_ϕ, N_ϕ), we denote the normal forms constructed from (T_ϕ, S_ϕ, N_ϕ) by $D^\phi_{\mathcal{F}}(F)$ and $C^\phi_{\mathcal{F}}(F)$. It now suffices to prove that $D_{\mathcal{F}}(F) \leqslant_n C_{\mathcal{F}}(F)$ holds if and only if $D^\phi_{\mathcal{F}}(F) \leqslant_n C^\phi_{\mathcal{F}}(F)$. Note that $D_{\mathcal{F}}(F) \leqslant_n C_{\mathcal{F}}(F)$ is satisfied if and only if

$$(D_{\mathcal{F}}(F))_\phi(\vec{x}) = \phi^{-1}[D_{\mathcal{F}}(F)(\phi[x_1], ..., \phi[x_n])] \leqslant_n \phi^{-1}[C_{\mathcal{F}}(F)(\phi[x_1], ..., \phi[x_n])] = (C_{\mathcal{F}}(F))_\phi(\vec{x}),$$

holds for every $\vec{x} \in [0,1]^n$. We now prove that $D^\phi_{\mathcal{F}}(F) = (D_{\mathcal{F}}(F))_\phi$ and $C^\phi_{\mathcal{F}}(F) = (C_{\mathcal{F}}(F))_\phi$. By definition,

$$\begin{aligned} D^\phi_{\mathcal{F}}(F)(\vec{x}) &= S_\phi \left\{ F(\vec{e})\ T_\phi(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n \right\} \\ &= \phi^{-1}\left[S\left\{ \phi\left[F(\vec{e})\ \phi^{-1}\left[T(\phi[x_1^{e_1}], \ldots, \phi[x_n^{e_n}]) \right] \right] \mid \vec{e} \in \{0,1\}^n \right\} \right], \end{aligned}$$

with $x_i^{e_i} = x_i$ if $e_i = 1$ and $x_i^{e_i} = x_i^{N_\phi}$ if $e_i = 0$. As $F(\vec{e}) \in \{0,1\}$, $x_i = \phi^{-1}[\phi[x_i]]$ and $x_i^{N_\phi} = \phi^{-1}[(\phi[x])^N]$ we can rewrite the above equation as follows

$$\begin{aligned} D^\phi_{\mathcal{F}}(F)(\vec{x}) &= \phi^{-1}\left[S\left\{ F(\vec{e})\ \phi\left[\phi^{-1}\left[T\left(\phi[\phi^{-1}[\phi[x_1]^{e_1}]], \ldots, \phi[\phi^{-1}[\phi[x_n]^{e_n}]] \right) \right] \right] \mid \vec{e} \in \{0,1\}^n \right\} \right] \\ &= \phi^{-1}\left[S\left\{ F(\vec{e})\ T(\phi[x_1]^{e_1}, \ldots, \phi[x_n]^{e_n}) \mid \vec{e} \in \{0,1\}^n \right\} \right], \end{aligned}$$

with $\phi[x_i]^{e_i} = \phi[x_i]$ if $e_i = 1$ and $\phi[x_i]^{e_i} = (\phi[x_i])^N$ if $e_i = 0$. We conclude that $D^\phi_{\mathcal{F}}(F) = (D_{\mathcal{F}}(F))_\phi$. In a similar way it is shown that $C^\phi_{\mathcal{F}}(F) = (C_{\mathcal{F}}(F))_\phi$. $\square$

In combination with Propositions 9.13, 9.16 and 9.17, and invoking Theorem 9.2, the above theorem straightforwardly yields the following corollary.

Corollary 9.20 *Consider an automorphism ϕ. $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for all triplets $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N}_\phi)$, $((T_{\mathbf{P}})_\phi, (S_{\mathbf{P}})_\phi, \mathcal{N}_\phi)$ and $((T_{\mathbf{L}})_\phi, (S_{\mathbf{L}})_\phi, \mathcal{N}_\phi)$.*

Taking into account the characterization of involutive negators by Trillas (Theorem 3.2), this corollary states in particular that $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ is satisfied for every triplet $(T_{\mathbf{M}}, S_{\mathbf{M}}, N)$ with involutive negator N.

Remarks 9.21

1. In case we work with the triplet $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N})$, the inequality $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ also follows from the work of Gehrke *et al.* [34]. For a finite set of propositional variables, they showed that the evaluation of a well-formed formula w in the propositional logic over the Kleene algebra $([0,1], T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N}, 0, 1)$ is comprised in the interval $[D_{\mathcal{F}}(F), C_{\mathcal{F}}(F)]$, with F the Boolean function obtained by evaluating w in the Boolean algebra $(\{0,1\}, \vee, \wedge, ', 0, 1)$. Their findings are largely due to the idempotence of the conjunction and disjunction involved and the distributivity of the disjunction over the conjunction [34]. Considering general t-norms T and t-conorms S, we loose this idempotence and distributivity ($T_{\mathbf{M}}$ is the only idempotent t-norm and $(T_{\mathbf{M}}, S_{\mathbf{M}})$ is the only distributive pair [51]). Due to the lack of both properties, a generalization of the approach of Gehrke *et al.* [34] is unrealistic.
2. As fuzzified normal forms rarely coincide (Propositions 9.7–9.9) they cannot be seen as true normal forms in a $[0,1]$-valued algebra $([0,1], T_{\mathbf{M}}, S_{\mathbf{M}}, T, S, N, 0, 1)$ where $T_{\mathbf{M}}$ and $S_{\mathbf{M}}$ are the lattice operators defining the order on $[0,1]$ and the t-norm T, the t-conorm S and involutive negator N are used to model, resp., disjunction, conjunction and negation. Depending on what is deemed crucial to the concept of normal forms, different approaches have been proposed to generalize the classical Boolean normal forms to normal forms in algebras based on the distributive lattice $([0,1], T_{\mathbf{M}}, S_{\mathbf{M}})$. Gehrke *et al.* [34] focus merely on the underlying propositional logic. They use pairwisely incomparable join and meet irreducibles to construct disjunctive and conjunctive normal forms. Truth table methods were developed to recover these normal forms. Perfilieva intensively studied normal forms as real standard function representations in BL-algebras [77, 78, 79, 80]. She defines (infinite) disjunctive and conjunctive normal forms based on the lattice operators and aggregation operator of the BL-algebra. She also uses a fuzzy equivalence relation to incorporate the neighbourhood of points. Discretising these infinite normal forms finally transforms them into (more manageable) approximations of the original function. Truth table methods are not available for this approach.

CHAPTER 10

Rotation-invariant t-norms solving a system of functional equations

10.1 Introduction

Knowing that $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for a triplet (T,S,N) consisting of a t-norm T, a t-conorm S and an involutive negator N, it remains an intriguing problem, from a mathematical point of view, to understand to what extent $C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})$ depends on the $\{0,1\}^n \to \{0,1\}$ function F. Inspired by Corollary 9.18, we wonder for which triplets it holds that $C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})$ is only a function of the variable $\vec{x} \in [0,1]^n$ (*i.e.* independent of the $\{0,1\}^n \to \{0,1\}$ function F). The latter amounts to solving a system of functional equations for T, S and N.

First, we equivalently rewrite the system of functional equations as a system consisting of only 3 functional equations. Given its complexity, additional assumptions on the triplet (T,S,N) are needed to actually solve this (reduced) system. We use the first De Morgan law (Eq. (9.1)) to compress the system into a single functional equation. Imposing some additional continuity conditions on the partial functions $T(x,\bullet)$ allows us to characterize multiple solution triplets, each containing a rotation-invariant t-norm. For this purpose our results from Chapters 7 and 8 are essential.

10.2 A system of functional equations

Let $n \in \mathbb{N}_0$. By definition, the difference between both fuzzified normal forms is given by

$$\begin{aligned} &C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) \\ &\quad = T\left\{ \left((1-F(\vec{e}))\, S\left(\vec{x}^{(\vec{e}^{\vec{0}})}\right)^{N} \right)^{N} \;\middle|\; \vec{e} \in \{0,1\}^n \right\} - S\{F(\vec{e})\, T(\vec{x}^{\vec{e}}) \mid \vec{e} \in \{0,1\}^n\}\,, \end{aligned} \tag{10.1}$$

for any $\{0,1\}^n \to \{0,1\}$ function F and any $\vec{x} \in [0,1]^n$. For a fixed Boolean function F, we use $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ to denote the $[0,1]^n \to [0,1]$ function determined by Eq. (10.1). The shortening CD_n is used when the difference between both fuzzified normal forms is independent of F. The triplets (T, S, N) for which CD_n exists, are solutions of the system of functional equations, obtained by putting, for every $\vec{x} \in [0,1]^n$, the $2^{(2^n)}$ different expressions Eq. (10.1) on a par (one for every Boolean function F).

As a first step in our search for suitable triplets (T, S, N), we have to narrow the class of t-norms to the subclass fulfilling the law of contradiction w.r.t. N. Similarly, we are forced to consider t-conorms that fulfill the law of the excluded middle w.r.t. N.

Definition 10.1 [81] Let N be a strict negator. A t-norm T fulfills the *law of contradiction* w.r.t. N if $T(x, x^N) = 0$ holds for every $x \in [0,1]$. A t-conorm S fulfills the *law of the excluded middle* w.r.t. N if $S(x, x^N) = 1$ holds for every $x \in [0,1]$.

Fodor and Roubens [27] gave the following characterization for continuous t-norms satisfying the law of contradiction and continuous t-conorms satisfying the law of the excluded middle.

Theorem 10.2 [27] *Consider a strict negator N. A continuous t-norm T fulfills the law of contradiction w.r.t. N if and only if there exists an automorphism ϕ such that $T = (T_{\mathbf{L}})_\phi$ and $N \leqslant \mathcal{N}_\phi$. A continuous t-conorm S fulfills the law of the excluded middle w.r.t. N if and only if there exists an automorphism ψ such that $S = (S_{\mathbf{L}})_\psi$ and $\mathcal{N}_\psi \leqslant N$.*

In general, for a left-continuous t-norm T the law of contradiction expresses that $N \leqslant C_0$ (with C_0 a contour line of T). For a right-continuous t-conorm S the law of the excluded middle holds if and only if $D_1 \leqslant N$ (with D_1 a contour line of S) (Theorem 4.6).

Theorem 10.3 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N. Let $n \in \mathbb{N}_0$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F, then T fulfills the law of contradiction w.r.t. N and S fulfills the law of the excluded middle w.r.t. N.*

Proof Consider arbitrary $x \in [0,1]$ and let $\vec{x} = (x, 1, ..., 1)$. Then $T(\vec{x}^{\vec{e}}) = 0$ and $S(\vec{x}^{(\vec{e}^{\vec{0}})}) = 1$ if $\vec{e}$ differs from $(0, 1, ..., 1)$ and from $(1, 1, ..., 1)$. For this particular $\vec{x}$, the system of functional equations defined by Eq. (10.1) reduces to

$$\begin{aligned}
CD_n(\vec{x}) &= S(x, 0, ..., 0) - T(x, 1, ..., 1) = 0 \\
&= S(x^N, 0, ..., 0) - T(x^N, 1, ..., 1) = 0 \\
&= T\left(S(x, 0, ..., 0), S(x^N, 0, ..., 0)\right) = T(x, x^N) \\
&= 1 - S\left(T(x^N, 1, ..., 1), T(x, 1, ..., 1)\right) = 1 - S(x, x^N)\,.
\end{aligned}$$

Therefore $S(x, x^N) = 1$ and $T(x, x^N) = 0$. As x has been arbitrarily chosen in $[0,1]$ this finishes the proof. $\square$

In the particular case $n = 1$, the necessary condition in the previous theorem also provides a sufficient condition.

Corollary 10.4 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N. Let $n = 1$. Then $C_{\mathcal{F}}(f) - D_{\mathcal{F}}(f)$ is independent of the $[0,1] \to [0,1]$ function f if and only if T fulfills the law of contradiction w.r.t. N and S fulfills the law of the excluded middle w.r.t. N.*

Proof Follows immediately from Theorem 10.3 and the observation that $C_{\mathcal{F}}(f)(x) - D_{\mathcal{F}}(f)(x)$ can only take as values 0, $T(x, x^N)$ or $1 - S(x, x^N)$. $\square$

As a second step, taking into account some well-chosen n-tuples $\vec{x}$, we are able to derive from Eq. (10.1) a system consisting of only three functional equations. This reduction does not affect the set of solutions.

Theorem 10.5 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N with fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. Then $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F if and only if for all $\vec{x} \in [0, \beta]^n$, $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$, the following expressions are equal to each other*

$$S(x_1, \ldots, x_{n-1}, x_n) - T(x_1^N, \ldots, x_{n-1}^N, x_n)\,, \tag{10.2}$$

$$S(x_1, \ldots, x_{n-1}, x_n^N) - T(x_1^N, \ldots, x_{n-1}^N, x_n^N)\,, \tag{10.3}$$

$$T\left(S(x_1, \ldots, x_{n-1}, x_n), S(x_1, \ldots, x_{n-1}, x_n^N)\right)\,, \tag{10.4}$$

$$1 - S\left(T(x_1^N, \ldots, x_{n-1}^N, x_n^N), T(x_1^N, \ldots, x_{n-1}^N, x_n)\right)\,. \tag{10.5}$$

Proof We denote the index set $\{1, \ldots, n\}$ by I. Suppose that $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F. From Theorem 10.3 we already know that $T(x, y) = 0$ if $y \leqslant x^N$ and that $S(x, y) = 1$ if $x^N \leqslant y$. In particular, consider an arbitrary vector $\vec{x} \in [0, \beta]^n$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, then $0 \leqslant T(x_i, x_j) \leqslant T(\beta, \beta) = 0$ and $0 \leqslant T(x_i, x_j^N) \leqslant T(x_j, x_j^N) = 0$, for any $(i, j) \in I^2$, $i < j$. Analogously, it also holds that $S(x_i^N, x_j^N) = 1$ and that $S(x_i^N, x_j) = 1$, for any $(i, j) \in I^2$, $i < j$. Since 0, resp. 1, is the absorbing element for t-norms, resp. t-conorms, there exist only two n-tuples, namely $\vec{e} = (0, \ldots, 0, 0)$ and $\vec{e} = (0, \ldots, 0, 1)$, for which $T(\vec{x}^{\vec{e}})$ might differ from 0 or $S\big(\vec{x}^{(\vec{e}^{\vec{0}})}\big)$ might differ from 1. Consequently, the system of functional equations defined by Eq. (10.1) implies that $CD_n(\vec{x}) = (10.2) = (10.3) = (10.4) = (10.5)$.

Conversely, suppose $(10.2) = (10.3) = (10.4) = (10.5)$ holds for every $\vec{x} \in [0, \beta]^n$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Consider now an arbitrary $\{0,1\}^n \to \{0,1\}$ function F and an arbitrary vector $\vec{x} \in [0,1]^n$. We first rewrite $C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})$ as $C_{\mathcal{F}}(G)(\vec{y}) - D_{\mathcal{F}}(G)(\vec{y})$, with G a $\{0,1\}^n \to \{0,1\}$ function determined by F and $\vec{x}$, and with $\vec{y}$ an element of $[0, \beta]^n$, determined by $\vec{x}$ and satisfying $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$. One can always find an n-tuple $\vec{\varepsilon} \in \{0,1\}^n$ such that $x_i^{\varepsilon_i} \leqslant \beta$, for every index i. Further, let $\iota : I \to I : i \mapsto \iota(i) = k$ be a permutation such that $y_i := x_k^{\varepsilon_k}$ is the ith smallest value in $\vec{x}^{\vec{\varepsilon}}$. Remark that $x_i = (x_i^{\varepsilon_i})^{\varepsilon_i} = (y_{\iota^{-1}(i)})^{\varepsilon_i}$, for any index i. Consequently,

$$\begin{aligned} T(x_1^{e_1}, \ldots, x_n^{e_n}) &= T\left(\left((y_{\iota^{-1}(1)})^{\varepsilon_1}\right)^{e_1}, \ldots, \left((y_{\iota^{-1}(n)})^{\varepsilon_n}\right)^{e_n}\right) \\ &= T\left(\left(y_1^{\varepsilon_{\iota(1)}}\right)^{e_{\iota(1)}}, \ldots, \left(y_n^{\varepsilon_{\iota(n)}}\right)^{e_{\iota(n)}}\right) \\ &= T\left(y_1^{\left((\varepsilon_{\iota(1)})^{e_{\iota(1)}}\right)}, \ldots, y_n^{\left((\varepsilon_{\iota(n)})^{e_{\iota(n)}}\right)}\right) \end{aligned}$$

and analogously

$$S\left(x_1^{(e_1^0)}, \ldots, x_n^{(e_n^0)}\right) = S\left(y_1^{\left(\left((\varepsilon_{\iota(1)})^{e_{\iota(1)}}\right)^0\right)}, \ldots, y_n^{\left(\left((\varepsilon_{\iota(n)})^{e_{\iota(n)}}\right)^0\right)}\right),$$

for any n-tuple $\vec{e} \in \{0,1\}^n$. Next, denote for every index i the exponent $(\varepsilon_{\iota(i)})^{e_{\iota(i)}}$ by ξ_i, then

$$T(x_1^{e_1}, \ldots, x_n^{e_n}) = T(y_1^{\xi_1}, \ldots, y_n^{\xi_n}) \quad \text{and} \quad S\left(x_1^{(e_1^0)}, \ldots, x_n^{(e_n^0)}\right) = S\left(y_1^{(\xi_1^0)}, \ldots, y_n^{(\xi_n^0)}\right).$$

Relying on the meaning of the exponential notation, we can rewrite e_i as

$$e_i = ((e_i)^{\varepsilon_i})^{\varepsilon_i} = ((\varepsilon_i)^{e_i})^{\varepsilon_i} = (\xi_{\iota^{-1}(i)})^{\varepsilon_i},$$

for any index i. Remark that, because $\vec{\varepsilon}$ is a fixed n-tuple, if $\vec{e}$ passes through $\{0,1\}^n$, also $\vec{\xi}$ will reach every element of $\{0,1\}^n$. Finally, we introduce a new $\{0,1\}^n \to \{0,1\}$ function G, based on the function F and the fixed n-tuple $\vec{\varepsilon}$:

$$G(\xi_1, \ldots, \xi_n) := F\left((\xi_{\iota^{-1}(1)})^{\varepsilon_1}, \ldots, (\xi_{\iota^{-1}(n)})^{\varepsilon_n}\right) = F(e_1, \ldots, e_n).$$

Applying the previous results, we obtain:

$$\begin{aligned} &C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) \\ &\quad = T\left\{\left((1 - G(\vec{\xi}))\, S\left(\vec{y}^{(\vec{\xi}^{\vec{0}})}\right)^N\right)^N \;\Big|\; \vec{\xi} \in \{0,1\}^n\right\} - S\left\{G(\vec{\xi})\, T\left(\vec{y}^{\vec{\xi}}\right) \;\Big|\; \vec{\xi} \in \{0,1\}^n\right\}, \end{aligned} \tag{10.6}$$

Moreover, due to the construction and our special choice of the n-tuple $\vec{\varepsilon}$, we also know that $\vec{y} \in [0,\beta]^n$ and that $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$. Therefore (10.2) = (10.3) = (10.4) = (10.5) will hold for the vector $\vec{y}$. Note that, for every vector $(0, \ldots, 0, x)$, with $x \in [0, \beta]$, this system of functional equalities reduces to $0 = T(x, x^N) = 1 - S(x^N, x)$. Denoting x^N by y we also obtain that $0 = T(y^N, y) = 1 - S(y, y^N)$, for every $y \in [\beta, 1]$. Therefore, T satisfies the law of contradiction w.r.t. N and S satisfies the law of the excluded middle w.r.t. N. The latter allows us to reduce Eq. (10.6) to one of the following equalities

$$\begin{aligned} C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) &= S(y_1, \ldots, y_{n-1}, y_n) - T(y_1^N, \ldots, y_{n-1}^N, y_n), \\ C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) &= S(y_1, \ldots, y_{n-1}, y_n^N) - T(y_1^N, \ldots, y_{n-1}^N, y_n^N), \\ C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) &= T\left(S(y_1, \ldots, y_{n-1}, y_n), S(y_1, \ldots, y_{n-1}, y_n^N)\right), \\ C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x}) &= 1 - S\left(T(y_1^N, \ldots, y_{n-1}^N, y_n^N), T(y_1^N, \ldots, y_{n-1}^N, y_n)\right), \end{aligned}$$

depending on the Boolean function F. The right-hand of the latter four equalities are assumed to be equal to each other. Consequently, the difference $C_{\mathcal{F}}(F)(\vec{x}) - D_{\mathcal{F}}(F)(\vec{x})$ between both fuzzified normal forms is in fact a single expression in the variable $\vec{y}$. Because $\vec{y}$ depends only on $\vec{x}$, this completes the proof. $\square$

For a De Morgan triplet $(T, S, \mathcal{N})$ containing the standard negator $\mathcal{N}$, the system of functional equations (10.2) = (10.3) = (10.4) = (10.5) can be replaced by a single functional equation:

$$S\big(T(x_1^{\mathcal{N}}, \ldots, x_{n-1}^{\mathcal{N}}, x_n^{\mathcal{N}}), T(x_1^{\mathcal{N}}, \ldots, x_{n-1}^{\mathcal{N}}, x_n)\big) = T(x_1^{\mathcal{N}}, \ldots, x_{n-1}^{\mathcal{N}}, x_n) + T(x_1^{\mathcal{N}}, \ldots, x_{n-1}^{\mathcal{N}}, x_n^{\mathcal{N}})\,.$$

For this functional equality it is even sufficient to consider two dimensions only.

Theorem 10.6 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N with fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. If T fulfills the law of contradiction w.r.t. N, then the equality*

$$S\left(T(x_1^N, \ldots, x_{n-1}^N, x_n^N), T(x_1^N, \ldots, x_{n-1}^N, x_n)\right) = T(x_1^N, \ldots, x_{n-1}^N, x_n) + T(x_1^N, \ldots, x_{n-1}^N, x_n^N) \tag{10.7}$$

holds for every $\vec{x} \in [0, \beta]^n$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, if and only if it holds for $n = 2$.

Proof Suppose that Eq. (10.7) holds for every $\vec{x} \in [0, \beta]^n$, with $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. In particular, for $\vec{x} = (0, \ldots, 0, x_{n-1}, x_n)$, we immediately get the desired result:

$$S\left(T(x_{n-1}^N, x_n^N), T(x_{n-1}^N, x_n)\right) = T(x_{n-1}^N, x_n) + T(x_{n-1}^N, x_n^N)\,.$$

Conversely, suppose that Eq. (10.7) holds in the binary case and that $n > 2$. Consider an arbitrary vector $\vec{x} \in [0, \beta]^n$, with $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. If $T(x_1^N, \ldots, x_{n-1}^N) \leqslant x_n^N$, then $0 \leqslant T(x_1^N, \ldots, x_{n-1}^N, x_n) \leqslant T(x_n^N, x_n) = 0$ and consequently Eq. (10.7) is trivially true. On the other hand, if $x_n^N < T(x_1^N, \ldots, x_{n-1}^N)$, then $0 \leqslant u := T(x_1^N, \ldots, x_{n-1}^N)^N < x_n \leqslant \beta$. Making use of our assumption, we know that

$$S\left(T\left(u^N, x_n^N\right), T(u^N, x_n)\right) = T(u^N, x_n) + T(u^N, x_n^N)\,.$$

Replacing u by its explicit form, we retrieve Eq. (10.7). $\square$

When considering a De Morgan triplet $(T, S, \mathcal{N})$, Eq. (10.7) ($n = 2$) provides a necessary and sufficient condition for the existence of the function CD_n.

Corollary 10.7 *Consider a De Morgan triplet $(T, S, \mathcal{N})$. Let $n \in \mathbb{N}$, $n > 1$. Then $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F if and only if*

$$S\left(T(x^{\mathcal{N}}, y^{\mathcal{N}}), T(x^{\mathcal{N}}, y)\right) = T(x^{\mathcal{N}}, y) + T(x^{\mathcal{N}}, y^{\mathcal{N}}) \tag{10.8}$$

holds for every $(x, y) \in [0, \frac{1}{2}]^2$, $x \leqslant y$.

Proof For such a triplet (10.2) = (10.3) = (10.4) = (10.5) is equivalent with Eq. (10.7) ($N = \mathcal{N}$). Combining Theorems 10.3, 10.5 and 10.6 leads to the desired result. Note hereby that whenever a De Morgan triplet $(T, S, \mathcal{N})$ satisfies Eq. (10.8), then S must fulfill the law of the excluded middle w.r.t. $\mathcal{N}$ (put $x = 0$). Due to the first De Morgan law (Eq. (9.1)), the latter ensures that T fulfills the law of contradiction w.r.t. $\mathcal{N}$. $\square$

10.3 Solutions

Figures 9.1(g)–9.1(i) visualize that the continuous De Morgan triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ solves the system of functional equations generated by Eq. (10.1) (Corollary 9.18). The question arises now whether this is the only appropriate De Morgan triplet or whether there exist also other solution triplets.

Consider the equalities (10.2) = (10.3) = (10.4) = (10.5). If we put $x_1 = \ldots = x_{n-2} = 0$ and denote x_{n-1} by x and x_n by y, we can extract a more manageable necessary condition for the existence of CD_n: equality must hold between the following expressions

$$S(x, y) - T(x^N, y)\,, \tag{10.9}$$

$$S(x, y^N) - T(x^N, y^N)\,, \tag{10.10}$$

$$T\left(S(x, y), S(x, y^N)\right)\,, \tag{10.11}$$

$$1 - S\left(T(x^N, y^N), T(x^N, y)\right)\,, \tag{10.12}$$

for any $(x, y) \in [0, \beta]^2$, $x \leqslant y$. Taking a closer look at these expressions we see that the t-norm T, the t-conorm S and the involutive negator N get entangled. Moreover, despite all efforts, the geometrical structure of t-norms and t-conorms is not yet fully understood. Without any further assumptions it becomes as good as impossible to further tackle the system of functional equations.

As a first restriction, we focus on De Morgan triplets only. If we are able to derive from the reduced system (10.9) = (10.10) = (10.11) = (10.12) that $N = \mathcal{N}$, then we are sure, in view of Corollary 10.7, that the De Morgan triplet $(T, S, \mathcal{N})$ solves the original system of functional equations generated by Eq. (10.1). Studying more profoundly Eqs. (10.11) and (10.12), it strikes that the range of T (and hence also the range of S) will determine to a considerable extent for which $x \in [0, 1]$ it holds that $x^N = x^{\mathcal{N}}$. More precisely, if CD_n exists and if the range of the *diagonal* δ_T of T (*i.e.* $\delta_T(x) := T(x, x)$) is sufficiently large, then N must be the standard negator.

Lemma 10.8 *Consider a De Morgan triplet (T, S, N) with involutive negator N that has fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0, 1\}^n \to \{0, 1\}$ function F and $]\beta, 1] \subseteq \delta_T([0, 1])$, then $N = \mathcal{N}$.*

Proof Due to the first De Morgan law, Eq. (10.11) equals $(S(T(x^N, y^N), T(x^N, y)))^N$. Because (10.11) = (10.12), it must hold that

$$\left(S\left(T(x^N, y^N), T(x^N, y)\right)\right)^N = 1 - S\left(T(x^N, y^N), T(x^N, y)\right) , \tag{10.13}$$

for every $(x, y) \in [0, \beta]^2$, $x \leqslant y$. Let $y = x$. If we take into account that T fulfills the law of contradiction and denote x^N as u, then Eq. (10.13) reduces to $\delta_T(u)^N = T(u, u)^N = 1 - T(u, u) = 1 - \delta_T(u)$, for every $u \in [\beta, 1]$. Moreover, since $]\beta, 1] \subseteq \delta_T([0, 1])$ and $T(\beta, \beta) = 0 \leqslant \beta$, we know that δ_T reaches on $]\beta, 1]$ every value of $]\beta, 1]$. This leads to $x^N = 1 - x$, for every $x \in]\beta, 1]$. If $x \in [0, \beta[$, it follows that $x^N \in]\beta, 1]$ and therefore $x = (x^N)^N = 1 - x^N$. The continuity of N ensures that also $\beta = \beta^N = 1 - \beta$, from which it follows that $\beta = \frac{1}{2}$. $\square$

Having a closer look at (10.10) = (10.11), for a De Morgan triplet $(T, S, \mathcal{N})$ one could notice that T structurally resembles the Łukasiewicz t-norm $T_{\mathbf{L}}$ in the points $(S(x, y), S(x, y^N))$, with $(x, y) \in [0, \beta]^2$ and $x \leqslant y$: $T(S(x, y), S(x, y^{\mathcal{N}})) = S(x, y^{\mathcal{N}}) + S(x, y) - 1$. In case T is rotation invariant with $C_0 = \mathcal{N}$ and has a continuous Archimedean t-norm as a-zoom ($a \in [\frac{1}{2}, 1[$), then the upper part of T must equal the Łukasiewicz t-norm $T_{\mathbf{L}}$.

Lemma 10.9 *Consider a De Morgan triplet $(T, S, \mathcal{N})$ with left-continuous t-norm T. Let $n \in \mathbb{N}$, $n > 1$ and take $a \in [\frac{1}{2}, 1[$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F, $C_0 = \mathcal{N}$ and T^a is a continuous Archimedean t-norm, then $T(x, y) = T_{\mathbf{L}}(x, y)$, for every $(x, y) \in]a, 1]^2$ satisfying $C_a(x) < y$.*

Proof We partition the domain $[0, 1]^2$ of T as depicted in Fig. 7.1. The requirement $C_0 = \mathcal{N}$ implies that T must be rotation invariant (assertion **(G2)**). Therefore, invoking Eq. (5.3), $T(x, y) = x + y - 1$ holds for every $(x, y) \in]a, 1]^2$ satisfying $C_a(x) < y$ if and only if

$$C_b(x) = C_{C_0(b^{\mathcal{N}})}(x) = C_0(T(x, b^{\mathcal{N}})) = T(x, b^{\mathcal{N}})^{\mathcal{N}} = 1 - x + b$$

is fulfilled for every $(x, b) \in]a, 1] \times [0, a^{\mathcal{N}}[$ such that $a < T(x, b^{\mathcal{N}})$ (*i.e.* $C_a(x) < b^{\mathcal{N}}$). Assertion **(G5)** implies that $a < T(x, b^{\mathcal{N}})$ is equivalent with $b < T(x, a^{\mathcal{N}})$. Hence, it suffices to show that $C_b(x) = b + 1 - x$, for every $(x, b) \in]a, 1] \times [0, T(x, a^{\mathcal{N}})[$. The latter is true if $T(x, y) = T_{\mathbf{L}}(x, y)$, for every $(x, y) \in \mathcal{D}_{\mathrm{III}}$ such that $y \leqslant a^{\mathcal{N}}$.

Firstly, we show that for every $(b, d) \in]0, 1[^2$ satisfying $b < d$, there always exists $(x, y) \in]b, 1[^2$ such that $y < x$, $T^a(x, y) = b$ and $C_y^a(x) = d$. From the observation that either $T^a = (T_{\mathbf{P}})_\phi$ or $T^a = (T_{\mathbf{L}})_\phi$, for some automorphism ϕ (Theorem 6.5), we derive that, for every $(x, y, b) \in [0, 1]^3$ satisfying $0 < b < x$, $T^a(x, y) = b$ is equivalent with $C_b^a(x) = y$ and that $C_b^a(x) = b$ can only occur for $x = 1$. Note hereby that always $C_b^a(x) < 1$ (Corollary 5.8). Take now arbitrary $(b, d) \in]0, 1[^2$ such that $b < d$ then $C_b^a(d) \in]b, 1[$. The continuity of T^a ensures the existence of $x \in]b, 1[$ for which $C_b^a(d) = T^a(x, x)$ and hence $T^a(x, T^a(x, d)) = T^a(d, T^a(x, x)) = b$. Defining $y := T^a(x, d)$ allows us to rewrite the latter as $T^a(x, y) = b$. From $C_b^a(d) = T^a(x, x) < T^a(x, 1) = x$ it follows that $0 < b < T^a(x, d) = y$. Furthermore, $y = T^a(x, d) < T^a(x, 1) = x$. Hence, $b < y < x$ and $T^a(d, x) = y$ is equivalent with $C_y^a(x) = d$.

Secondly, we prove that $T(x,y) = T_{\mathbf{L}}(x,y)$, for every $(x,y) \in \mathcal{D}_{\rm II} \cup \mathcal{D}_{\rm III}$ such that $\min(x,y) \leqslant a^{\mathcal{N}}$. We present the proof for $(x,y) \in \mathcal{D}_{\rm II}$. The case $(x,y) \in \mathcal{D}_{\rm III}$ then follows immediately from the commutativity of T. Let σ be the $[a,1] \to [0,1]$ isomorphism used to construct the a-zoom T^a of T. Take arbitrary $(x,y) \in \mathcal{D}_{\rm II}$ such that $x = \min(x,y) < a^{\mathcal{N}}$ and $y < 1$ (*i.e.* $x \in \,]0, a^{\mathcal{N}}[$ and $y \in \,]x^{\mathcal{N}}, 1[$). Then, $(\sigma[x^{\mathcal{N}}], \sigma[y]) \in \,]0,1[^2$ and $\sigma[x^{\mathcal{N}}] < \sigma[y]$. From the previous paragraph we know that there exists $(u,v) \in \,]\sigma(x^{\mathcal{N}}), 1[^2$ such that $v < u$, $T^a(u,v) = \sigma[x^{\mathcal{N}}]$ and $C^a_v(u) = \sigma[y]$. Taking into account property **(F4a)** and introducing $x_1 := (\sigma^{-1}[u])^{\mathcal{N}}$ and $y_1 := (\sigma^{-1}[v])^{\mathcal{N}}$, it follows that $(x_1, y_1) \in \,]0, a^{\mathcal{N}}[^2$, $x_1 < y_1$, $\sigma[T(x_1^{\mathcal{N}}, y_1^{\mathcal{N}})] = T^a(u,v) = \sigma[x^{\mathcal{N}}]$ and $C_{y_1^{\mathcal{N}}}(x_1^{\mathcal{N}}) = \sigma^{-1}[C^a_v(u)] = y$. Clearly, $T(x_1^{\mathcal{N}}, y_1^{\mathcal{N}}) = x^{\mathcal{N}}$ and from assertion **(G3)** we obtain that $T(x_1^{\mathcal{N}}, y_1) = C_0(C_{C_0(y_1)}(x_1^{\mathcal{N}})) = (C_{y_1^{\mathcal{N}}}(x_1^{\mathcal{N}}))^{\mathcal{N}} = y^{\mathcal{N}}$. To conclude, it now suffices to express (10.10) = (10.11):

$$T(x,y) = T\left(T(x_1^{\mathcal{N}}, y_1^{\mathcal{N}})^{\mathcal{N}}, T(x_1^{\mathcal{N}}, y_1)^{\mathcal{N}}\right) = T(x_1^{\mathcal{N}}, y_1)^{\mathcal{N}} - T(x_1^{\mathcal{N}}, y_1^{\mathcal{N}}) = x + y - 1\,.$$

Note that (x,y) has been arbitrarily chosen in $\mathcal{D}_{\rm II}$ and fulfills $x < a^{\mathcal{N}}$ and $y < 1$. As 1 is the neutral element of T and T is left continuous, this finishes the proof. □

Note that for every De Morgan triplet $(T, S, \mathcal{N})$ satisfying this lemma it necessarily holds that $C_a(x) = \min(a + 1 - x, 1)$. As indicated by Lemmata 10.8 and 10.9, the first De Morgan law is inadequate to fully solve the system of functional equations generated by Eq. (10.1). Additional continuity conditions are required to sift out those triplets for which CD_n exists. Given the importance of left-continuous t-norms (see Section 6.1), it is not so restrictive to invoke only those De Morgan triplets based on a left-continuous t-norm T and on an involutive negator with fixpoint β. Despite all our efforts in Chapters 6–8, the structure of such a left-continuous t-norm is, however, not yet fully understood. Therefore, also other continuity conditions are needed. In particular, we invoke two properties that allow us to involve our results from Chapters 7 and 8: $T(\bullet, \beta)$ is continuous on $[\beta, 1]$ or $T(\bullet, \beta)$ is continuous on $]\beta, 1]$.

A. $T(\bullet, \beta)$ is continuous on $[\beta, 1]$

From Corollary 9.18 we know that the De Morgan triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ solves the system of functional equations generated by Eq. (10.1). Clearly, $\beta = \frac{1}{2}$ is the fixpoint of $\mathcal{N}$, $T_{\mathbf{L}}$ is continuous and, hence, also the partial function $T_{\mathbf{L}}(\bullet, \frac{1}{2})$ is continuous. In order to compute all De Morgan triplets (T, S, N) that solve the system of functional equations and are based on a left-continuous t-norm T whose partial function $T(\bullet, \beta)$ is continuous on $[\beta, 1]$ and on an involutive negator N with fixpoint β, we need to consider two lemmata first.

In Section 6.2 we drew attention to the importance of the Archimedean property as a transitional means from left continuity to continuity (Theorem 6.7). In the first lemma, we show that for those triplets (T, S, N) that fulfill (10.9) = (10.10) = (10.11) = (10.12) it is enough that $T(\bullet, \beta)$ and $S(\bullet, \beta)$ are continuous on, resp., $[\beta, 1]$ and $[0, \beta]$, to obtain the Archimedean property.

Lemma 10.10 *Consider a triplet (T, S, N) consisting of a t-norm T, a t-conorm S and an involutive negator N with fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of* *the $\{0,1\}^n \to \{0,1\}$ function F and the partial functions $T(\bullet,\beta)$ and $S(\bullet,\beta)$ are continuous on, resp., $[\beta,1]$ and $[0,\beta]$, then T is Archimedean.*

Proof We know already that (10.9) = (10.10) = (10.11) = (10.12) must hold for every $(x,y) \in [0,\beta]^2$, $x \leqslant y$. Putting $y=\beta$ in (10.9) = (10.11) leads to

$$T(S(x,\beta),S(x,\beta)) = S(x,\beta) - T(x^N,\beta)\,, \tag{10.14}$$

for every $x \in [0,\beta]$. Remark that (10.9) = (10.12) implies that if $T(x^N,\beta)=0$, then necessarily $S(x,\beta)=1$. Consequently, $T(S(x,\beta),S(x,\beta)) < S(x,\beta)$, if $S(x,\beta) \neq 1$. Since $S(\bullet,\beta)$ is continuous on $[0,\beta]$ and S fulfills the law of the excluded middle w.r.t. N (Theorem 10.3), $S(x,\beta)$ reaches every value of $[\beta,1]$ when x goes through $[0,\beta]$. Therefore, if we denote $S(x,\beta)$ by u, the above inequality implies that $\delta_T(u) = T(u,u) < u$, for every $u \in [\beta,1[$. The t-norm T fulfills the law of contradiction w.r.t. N (Theorem 10.3) such that $0 = \delta_T(x) < x$, for every $x \in]0,\beta]$. Since $T(\bullet,\beta)$ is continuous on $[\beta,1]$, $S(\bullet,\beta)$ is continuous on $[0,\beta]$ and $T(\beta,\beta) = 0 < 1 = S(\beta,\beta)$ (Theorem 10.3), it also follows from Eq. (10.14) that $\delta_T([\beta,1]) = [0,1]$. We conclude that $\delta_T(x) < x$, for every $x \in]0,1[$ and that δ_T reaches every element of $[0,1]$. Both properties finally imply the Archimedean property (Theorem 6.4). $\square$

Note that for a De Morgan triplet (T,S,N) with involutive negator N, the continuity condition on S in the previous lemma is a trivial consequence of the continuity condition on T. The second lemma partially solves (10.9) = (10.10) = (10.11) = (10.12) when dealing with triplets (T,S,N) consisting of a continuous t-norm T, a continuous t-conorm S and an involutive negator N. The solution triplets are necessarily transformations of the known solution $(T_{\mathbf{L}},S_{\mathbf{L}},\mathcal{N})$.

Lemma 10.11 *Consider a triplet (T,S,N) consisting of a continuous t-norm T, a continuous t-conorm S and an involutive negator N. Let $n \in \mathbb{N}$, $n > 1$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F, then there exist two automorphisms ϕ and ψ such that $T = (T_{\mathbf{L}})_\phi$, $S = (S_{\mathbf{L}})_\psi$ and $\mathcal{N}_\phi = \mathcal{N}_\psi = N$.*

Proof Because T fulfills the law of contradiction w.r.t. N and S fulfills the law of the excluded middle w.r.t. N (Theorem 10.3), there exist two automorphisms ϕ and ψ such that $T = (T_{\mathbf{L}})_\phi$, $S = (S_{\mathbf{L}})_\psi$ and $\mathcal{N}_\psi \leqslant N \leqslant \mathcal{N}_\phi$ (Theorem 10.2). It is then sufficient to prove that these latter inequalities are, in fact, equalities.

Let β be the fixpoint of N and consider arbitrary $y \in]0,\beta]$ and $x \in]0,y]$. Then it holds that

$$\begin{gathered} T(x^N,y^N) = \phi^{-1}\left[\max\left(\phi[x^N]+\phi[y^N]-1,0\right)\right] < y^N\,, \\ y^N < \psi^{-1}\left[\min\left(\psi[x]+\psi[y^N],1\right)\right] = S(x,y^N)\,. \end{gathered}$$

Due to the existence of the function CD_n, it holds for such x and y that (10.10) = (10.11) = (10.12). The above strict inequalities imply that

$$\begin{aligned} 0 &< S(x,y^N) - T(x^N,y^N) \\ &= T(S(x,y),S(x,y^N)) = \phi^{-1}\left[\max\left(\phi[S(x,y)]+\phi[S(x,y^N)]-1,0\right)\right] \\ &= 1 - S(T(x^N,y^N),T(x^N,y)) = 1 - \psi^{-1}\left[\min\left(\psi[T(x^N,y^N)]+\psi[T(x^N,y)],1\right)\right]\,. \end{aligned}$$

Due to the strict positivity of the expressions above, we can eliminate max and min:

$$0 < \phi^{-1}\left[\phi[S(x,y)] + \phi[S(x,y^N)] - 1\right] = 1 - \psi^{-1}\left[\psi[T(x^N,y^N)] + \psi[T(x^N,y)]\right] .$$

Thanks to the continuity of (T,S,N) we can take the limit for $x \searrow 0$. This results in

$$0 \leqslant \phi^{-1}[\phi[y] + \phi[y^N] - 1] = 1 - \psi^{-1}[\psi[y^N] + \psi[y]] \leqslant 1 - \psi^{-1}[\psi[y^{\mathcal{N}_\psi}] + \psi[y]] = 0\,,$$

or, equivalently, $y^{\mathcal{N}_\phi} = y^N = y^{\mathcal{N}_\psi}$. Remark that the variable y was arbitrarily chosen in $]0,\beta]$. If $y \in [\beta,1[$, then $y^N \in]0,\beta]$ and $[y^N]^{\mathcal{N}_\phi} = [y^N]^N = [y^N]^{\mathcal{N}_\psi}$. As all negators involved are involutive we get that $y^{\mathcal{N}_\phi} = y^N = y^{\mathcal{N}_\psi}$, for all $y \in]0,1[$. Including the trivial cases $y=0$ and $y=1$ finally leads to the desired result. $\square$

In the following theorem we show that successively executing the previous lemmata on a De Morgan triplet with left-continuous t-norm T and involutive negator N yields a unique triplet solving the system of functional equations: $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$.

Theorem 10.12 *Consider a De Morgan triplet (T,S,N) with left-continuous t-norm T and involutive negator N that has fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. If $T(\bullet,\beta)$ is continuous on $[\beta,1]$, then $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F if and only if $(T,S,N) = (T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$.*

Proof When working with the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ the difference between both fuzzified normal forms is of course independent of the Boolean function F (Corollary 9.18). Conversely, suppose that CD_n exists. As $S(x,\beta) = T(x^N,\beta)^N$, for every $x \in [0,1]$ (Eq. (9.1)), it follows from the involutivity of N and the continuity of $T(\bullet,\beta)$ on $[\beta,1]$ that $S(\bullet,\beta)$ is continuous on $[0,\beta]$. Applying Lemma 10.10, we conclude that T is Archimedean. In combination with the left continuity of T we can even state that T must be continuous (Theorem 6.7). Hence, $\delta_T([0,1]) = [0,1]$ and the first De Morgan law (Eq. (9.1)) ensures that also S is continuous. Invoking Lemmata 10.11 and 10.8, and Theorem 3.3, it follows that there exists an automorphism ϕ such that $T = (T_{\mathbf{L}})_\phi$, $N = \mathcal{N} = \mathcal{N}_\phi$ and $S = ((T_{\mathbf{L}})_\phi)_N = ((T_{\mathbf{L}})_\phi)_{\mathcal{N}_\phi} = (T_{\mathbf{L}})_{\mathcal{N}\circ\phi} = (S_{\mathbf{L}})_\phi$. In this case $\beta = \frac{1}{2}$. Note that $T^{\frac{1}{2}}$ is continuous and $T^{\frac{1}{2}}(x,x) = \sigma[\max(\frac{1}{2}, T(\sigma^{-1}[x],\sigma^{-1}[x]))] < \sigma[\max(\frac{1}{2}, T(\sigma^{-1}[x],\sigma^{-1}[1]))] = x$, for every $x \in]0,1[$. Hence, $T^{\frac{1}{2}}$ is Archimedean (Theorem 6.4). Furthermore, the contour line C_0 of $T = (T_{\mathbf{L}})_\phi$ is given by $\mathcal{N}_\phi = \mathcal{N}$. Lemma 10.9 is applicable. We get that $T(x,y) = T_{\mathbf{L}}(x,y)$, for every $(x,y) \in]\frac{1}{2},1]^2$ satisfying $C_{\frac{1}{2}}(x) < y$. As there only exists at most one left-continuous t-norm that has contour line $\mathcal{N}$ $(a=0)$ and that has $\frac{1}{2}$-zoom $T_{\mathbf{L}}$ (Theorem 7.5) it necessarily holds that $T = T_{\mathbf{L}}$. $\square$

Let N be an involutive negator with fixpoint β. We briefly figure out on which De Morgan triplets (T,S,N) the above theorem is applicable (*i.e.* for which left-continuous t-norms T the partial function $T(\bullet,\beta)$ is continuous on $[\beta,1]$). By definition, all partial functions of a continuous t-norm are continuous. Hence, $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ is the only continuous De Morgan triplet (T,S,N) solving the system of functional equations generated by Eq. (10.1) (Theorem 10.12). Furthermore, for a left-continuous t-norm T, the law of contradiction expresses that $N \leqslant C_0$. In the case of equality, then T is necessarily rotation invariant (assertion **(G2)**) and $T(x,\beta) = T(\beta,x) = C_0(C_{C_0(\beta)}(x)) = C_0(C_\beta(x))$, for every $x \in [0,1]$ (assertion **(G3)**). The left-continuity of T ensures that $T(\beta,\beta) = 0$ (Theorem 10.3). Therefore, invoking the involutivity of C_0, $T(\bullet,\beta)$ is continuous on $[\beta,1]$ if and only if C_β is continuous on $[\beta,1]$. The latter is equivalent with the continuity of C_0^β (property **(F4a)**) which expresses the rotation invariance of T^β (assertion **(G1)**). We conclude that the partial function $T(\bullet,\beta)$ of a rotation-invariant t-norm T, with $C_0 = N$, is continuous on $[\beta,1]$ if and only if T is the triple rotation $T = R3(\widehat{T},N)$ based on N of a rotation-invariant t-norm $\widehat{T}$ (Section 8.3). From Theorem 10.12 it then follows that $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N}) = (R3(T_{\mathbf{L}},\mathcal{N}), R3(T_{\mathbf{L}},\mathcal{N})_{\mathcal{N}}, \mathcal{N})$ is the only member in this class of De Morgan triplets $(R3(\widehat{T},N), R3(\widehat{T},N)_N, N)$, with $\widehat{T}$ a rotation-invariant t-norm and N an involutive negator, that solves the system of functional equations generated by Eq. (10.1).

B. $T(\bullet,\beta)$ is continuous on $]\beta,1]$

In the previous section we already considered t-norms T whose partial function $T(\bullet,\beta)$ is continuous on $[\beta,1]$. Therefore, it suffices now to consider only t-norms T whose partial function $T(\bullet,\beta)$ is continuous on $]\beta,1]$ but not on $[\beta,1]$. Invoking property **(E2)** and $T(\beta,\beta) = 0$ (Theorem 10.3), the latter can only occur if $0 < Q(\beta,\beta)$.

Lemma 10.13 *Consider a triplet (T,S,N) consisting of a t-norm T, a t-conorm S and an involutive negator N with fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. If $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F, $0 < Q(\beta,\beta)$ and the partial functions $T(\bullet,\beta)$ and $S(\bullet,\beta)$ are continuous on, resp., $]\beta,1]$ and $[0,\beta[$, then $Q(\beta,\beta) = \beta$.*

Proof Suppose the converse, *i.e.* $Q(\beta,\beta) \neq \beta$. Then necessarily $Q(\beta,\beta) < \beta$ due to property **(E5)**. Using equality (10.10)=(10.11) with $y = \beta$, we get

$$S(x,\beta) - T(x^N,\beta) = T\left(S(x,\beta), S(x,\beta)\right) ,$$

for every $x \in [0,\beta[$. The continuity of the partial functions $T(\bullet,\beta)$ and $S(\bullet,\beta)$ on, resp., $]\beta,1]$ and $[0,\beta[$, then yields that $T\left(S(x,\beta),S(x,\beta)\right)$ reaches for $x \in [0,\beta[$ every element of $[0,\omega[$, with $\omega = \lim_{x\nearrow\beta} S(x,\beta) - T(x^N,\beta)$. Because $\beta \leqslant S(x,\beta)$ and $\lim_{x\nearrow\beta} T(x^N,\beta) = \lim_{x\nearrow\beta} T(\beta,x^N) = Q(\beta,\beta) < \beta$ (property **(E2)**), it holds that $0 < \omega$. Hence, there exists $x \in [0,\beta[$ such that $0 < T\left(S(x,\beta),S(x,\beta)\right) < Q(\beta,\beta)$. Denoting $S(x,\beta)$ by y and invoking that $T(\beta,\beta) = 0$ (Theorem 10.3), we conclude that $y \in]\beta,1]$ and $T(y,y) < Q(\beta,\beta)$. From property **(E2)** however, we also know that $Q(\beta,\beta) \leqslant T(\beta,y) \leqslant T(y,y)$, a contradiction. $\square$

Similarly as for Lemma 10.10, the continuity condition on S can be omitted when considering De Morgan triplets with an involutive negator. If T is left continuous and $C_0 = N$, then the continuity of $T(\bullet,\beta)$ on $]\beta,1]$ is due to $T(x,\beta) = T(\beta,x) = C_0(C_\beta(x))$ (assertion **(G3)**) and the involutivity of C_0 equivalent with the continuity of C_β on $]\beta,1]$. Hence, rotation-invariant t-norms satisfying $C_0 = N$ and whose partial function $T(\bullet,\beta)$ is continuous on $]\beta,1]$ are exactly the triple-rotations based on N that have been discussed in Section 8.3. In case $0 < Q(\beta,\beta)$, then the previous lemma states that necessarily $Q(\beta,\beta)=\beta$ which is equivalent with $C_0(C_\beta(x)) = C_0(T(\beta,x)) = C_0(\beta)=\beta$, for every $x \in\,]\beta,1]$ (assertion **(G3)** and property **(E2)**). The latter is on its turn equivalent with $C_\beta(x)=\beta$, for every $x\in\,]\beta,1]$, which expresses that T^β has no zero-divisors. From Section 8.3 it then follows that the companion Q of a rotation-invariant t-norm T, with $C_0 = N$, satisfies $Q(\beta,\beta)=\beta$ if and only if T is the triple-rotation $R3(\widehat{T},N)$ based on N of a left-continuous t-norm $\widehat{T}$ that has no zero-divisors. Note that since $Q(\beta,\beta)=\beta$ implies $T(x,\beta)=\beta$, for every $x\in\,]\beta,1]$, we immediately obtain the required continuity for the partial functions $T(\bullet,\beta)$.

Our characterization in Subsection **A** was largely due to the continuity of T and Lemma 10.9. Unfortunately, if $0 < Q(\beta,\beta)$, then T is necessarily discontinuous and it becomes doubtful whether Lemma 10.9 remains applicable. To solve (10.9) = (10.10) = (10.11) = (10.12) we have to impose some further restrictions on the t-norms considered. In particular, we assume that the partial functions $T(\bullet,x)$ are continuous on $]x^N,1]$ whenever $x\in\,]0,\beta]$ and on $[x,1]$ whenever $x\in\,]\beta,1[$. Note that these continuity conditions comprise the continuity of $T(\bullet,\beta)$ on $]\beta,1]$. They are visualized in Figure 10.1.

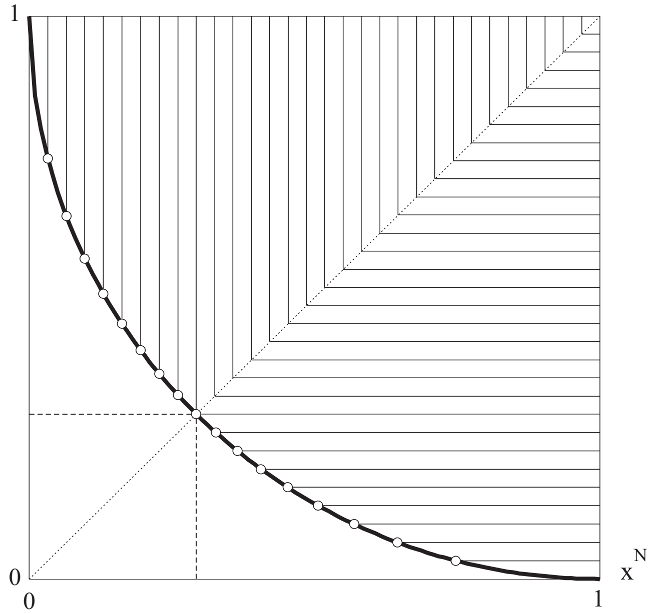


Figure 10.1: Domain of a left-continuous t-norm T where the continuity conditions are indicated by the horizontal and vertical lines

For a left-continuous t-norm T that has contour line $C_0 = N$, the proposed continuity conditions, in combination with the typical geometrical structure for rotation-invariant t-norms (Eq. (7.1)), are quite restrictive and ensure the continuity of the β-zoom T^β.

Lemma 10.14 *Consider a rotation-invariant t-norm T, with β the unique fixpoint of its contour line C_0. Then T^β is continuous if and only if the partial functions $T(\bullet, x)$ are continuous on $]C_0(x), 1]$, whenever $x \in]0, \beta]$, and on $[x, 1]$, whenever $x \in]\beta, 1[$.*

Proof Let σ be the $[\beta, 1] \to [0, 1]$ isomorphism used to construct T^β. It follows from Eq. (7.1) that, for every $x \in]0, \beta]$, $T(\bullet, x) = T(x, \bullet)$ is continuous on $]C_0(x), 1]$ if and only if, for every $x \in]0, \beta]$, $C^\beta_{\sigma[C_0(x)]}(\sigma[y])$ is continuous in y whenever $y \in]C_0(x), 1]$. Denoting $\sigma[C_0(x)]$ by u the latter expresses that, for every $u \in [0, 1[$, C^β_u must be continuous on $]u, 1]$. Therefore, we need to prove that T^β is continuous if and only if each partial function $T(x, \bullet)$, $x \in]\beta, 1[$, is continuous on $[x, 1]$ and each contour line C^β_u, $u \in [0, 1[$, is continuous on $]u, 1]$.

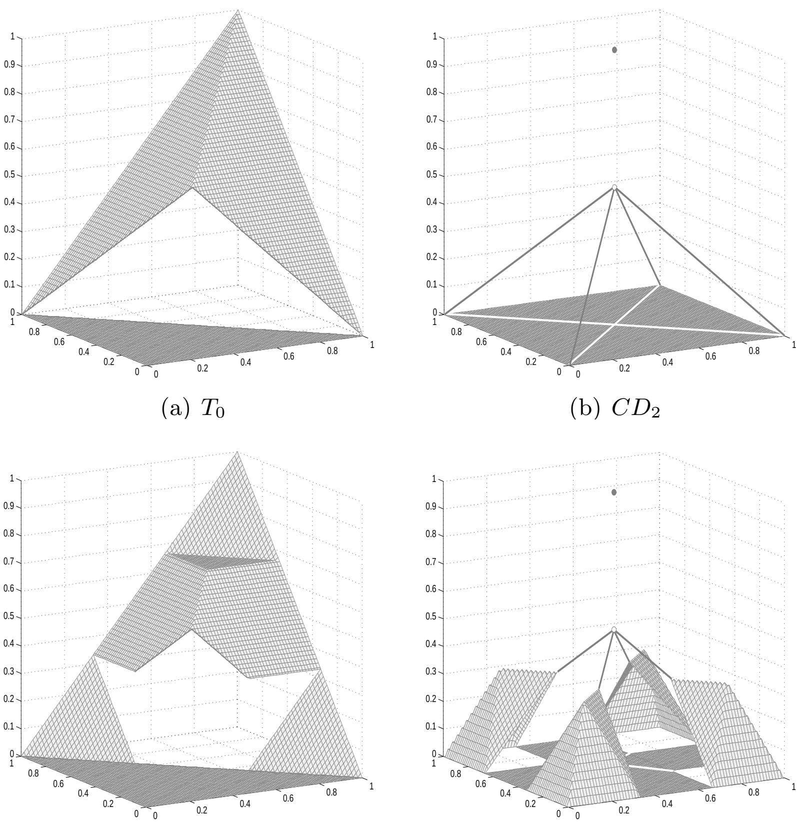
If T^β is continuous, then we know from the characterization of continuous t-norms (Theorem 6.6) that each contour line C^β_u, $u \in [0, 1[$, is continuous on $]u, 1]$. Furthermore, $Q^\beta(x, y) = T^\beta(x, y)$, for every $(x, y) \in [0, 1]^2$ such that $y < 1$, and $Q^\beta(x, C^\beta_0(x)) = 0$, for every $x \in]0, 1]$. As C^β_0 is continuous on $]0, 1]$, it follows from property **(F4a)** that C_β must be continuous on $]\beta, 1]$. Hence, Eq. (7.7) holds. Take arbitrary $x \in]\beta, 1[$. Then, for $y \in]C_\beta(x), 1]$, $T(x, y) = \sigma^{-1}[T^\beta(\sigma[x], \sigma[y])]$ is continuous in y and, for $y \in]\beta, C_\beta(x)]$, $T(x, y) = C_0(\sigma^{-1}[T^\beta(C^\beta_0(\sigma[x]), C^\beta_0(\sigma[y]))])$ is also continuous in y. Note that due to properties **(F4a)** and **(F2)** it holds that $T(x, C_\beta(x)) = T(x, \sigma^{-1}[C^\beta_0(\sigma[x])]) = C_0(\sigma^{-1}[0]) = \beta$ and

$$Q(x, C_\beta(x)) = \sigma^{-1}[Q^\beta(\sigma[x], \sigma[C_\beta(x)])] = \sigma^{-1}[T^\beta(\sigma[x], C^\beta_0(\sigma[x]))] = \sigma^{-1}[0] = \beta\,.$$

Invoking property **(E2)**, we conclude that $T(\bullet, x) = T(x, \bullet)$ is continuous on $]\beta, 1]$ and thus also on $[x, 1]$.

Conversely, assume that each partial function $T(x, \bullet)$, $x \in]\beta, 1[$, is continuous on $[x, 1]$ and that each contour line C^β_u, $u \in [0, 1[$, is continuous on $]u, 1]$. Recall that by definition, $T^\beta(v, w) = \sigma[\max(\beta, T(\sigma^{-1}[v], \sigma^{-1}[w]))]$, for every $(v, w) \in [0, 1]^2$. Take arbitrary $v \in]0, 1[$. From the continuity of $T(\sigma^{-1}[v], \bullet)$ on $[\sigma^{-1}(v), 1]$, it then follows that $T^\beta(v, \bullet)$ is continuous on $[v, 1]$. Clearly, $T^\beta(0, \bullet)$ and $T^\beta(1, \bullet)$ are continuous on $[0, 1]$. To illustrate the continuity of T^β, it then suffices to show that $T^\beta(v, \bullet)$, $v \in]0, 1[$, is continuous on $[0, v]$. Because of the left continuity of T^β (property **(F3)**) it is enough to prove the right continuity of $T^\beta(v, \bullet)$ on $[0, v[$. Suppose the converse, then there exist $(v, w) \in [0, 1[^2$ such that $w < v$ and $T^\beta(v, w) < Q^\beta(v, w)$ (property **(E2)**). Because $0 = T^\beta(v, 0) \leqslant Q^\beta(v, 0) \leqslant \min(v, 0) = 0$ (properties **(E3)** and **(E5)**), we may assume that $0 < w$. By definition, it holds for every $a \in]T^\beta(v, w), Q^\beta(v, w)[$ that $w \leqslant C^\beta_a(v) \leqslant w$. As $T^\beta(\bullet, w)$ is continuous on $[w, 1]$ and $Q^\beta(v, w) \leqslant w$ (property **(E5)**), we are sure that there exists $z \in]v, 1[$ such that $T^\beta(z, w) = a$ and, hence, $w \leqslant C^\beta_a(z) \leqslant C^\beta_a(v) = w$. Finally, we use the orthosymmetry of C^β_a (Theorem 5.9) to retrieve from $w = C^\beta_a(z) = C^\beta_a(v) < w + \varepsilon$ that $C^\beta_a(w + \varepsilon) < v < z \leqslant C^\beta_a(w)$, for every $\varepsilon \in]0, 1 - w]$. We conclude that C^β_a is not continuous in w. Since $a < Q^\beta(v, w) \leqslant w$, this contradicts the continuity of C^β_a on $]a, 1]$. □

Figure 10.2 presents two rotation-invariant t-norms that have a continuous β-zoom. As can be seen in Fig. 10.2(c), the continuity of the β-zoom does not ensure the continuity of the t-norm



(a) T_0

(b) CD_2

(c) $T_{\frac{1}{3}}$

(d) CD_2

Figure 10.2: CD_2 for two De Morgan triplets $(T_\lambda, S_\lambda, \mathcal{N})$.

on area $\mathcal{D} = \{(x, y) \in [0,1]^2 \mid C_0(x) < y\}$. Both t-norms belong to the same family of t-norms $(T_\lambda)_{\lambda \in [0,\frac{1}{2}[}$. For $\lambda \in]0, \frac{1}{2}[$, T_λ is the triple rotation $R3((\langle 1-2\,\lambda, 1, T_{\mathbf{L}}\rangle), \mathcal{N})$ of the ordinal sum $(\langle 1-2\,\lambda, 1, T_{\mathbf{L}}\rangle)$ and based on the standard negator $\mathcal{N}$. As for the considered ordinal sum, a linear rescaling function is used to construct T_λ. The limit case T_0 is the triple rotation of $\lim_{\lambda \searrow 0}(\langle 1-2\,\lambda, 1, T_{\mathbf{L}}\rangle) = T_{\mathbf{M}}$ based on $\mathcal{N}$. Therefore, $T_0 = R3(T_{\mathbf{M}}, \mathcal{N}) = T^{\mathbf{nM}}$ (see Fig. 8.1(b)). Note that for $\lambda \in]0, \frac{1}{2}[$, the ordinal sum $(\langle 1-2\,\lambda, 1, T_{\mathbf{L}}\rangle)$ has no zero-divisors. Hence, each t-norm in this family can be fully decomposed by means of Theorem 7.1. A more explicit formulation for T_λ, with $\lambda \in [0, \frac{1}{2}[$, can be found below:

$$T_\lambda(x, y) = \begin{cases} 0, & \text{if } x + y \leqslant 1\,, \\ \min(x, y), & \text{if } x + y > 1 \quad \wedge \quad \min(x, y) \in]\lambda, 1-\lambda]\,, \\ x + y - 1, & \text{if } x + y > 1 \quad \wedge \quad \big(x + y > 2 - \lambda \quad \vee \quad \min(x, y) \in [0, \lambda]\big)\,, \\ 1 - \lambda, & \text{if } x + y \leqslant 2 - \lambda \quad \wedge \quad \min(x, y) \in]1-\lambda, 1]\,. \end{cases}$$

The family of dual t-conorms $(S_\lambda)_{\lambda\in[0,\frac{1}{2}[} = ((T_\lambda)_{\mathcal{N}})_{\lambda\in[0,\frac{1}{2}[}$ is given by

$$S_\lambda(x,y) = \begin{cases} 1, & \text{if } x+y \geqslant 1\,, \\ \max(x,y), & \text{if } x+y<1 \quad\wedge\quad \max(x,y)\in\ [\lambda,1-\lambda[\,, \\ x+y, & \text{if } x+y<1 \quad\wedge\quad \big(x+y<\lambda \quad\vee\quad \max(x,y)\in[1-\lambda,1]\big)\,, \\ \lambda, & \text{if } x+y\geqslant\lambda \quad\wedge\quad \max(x,y)\in\ [0,\lambda[\,. \end{cases}$$

By construction $(T_\lambda, S_\lambda, \mathcal{N})$ fulfills the De Morgan laws for every $\lambda\in[0,\frac{1}{2}[$.

Proposition 10.15 *Let $n\in\mathbb{N}$, $n>1$. For every De Morgan triplet $(T_\lambda,S_\lambda,\mathcal{N})$, with $\lambda\in[0,\frac{1}{2}[$, it holds that $C_{\mathcal{F}}(F)-D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n\to\{0,1\}$ function F.*

Proof It is sufficient (Corollary 10.7) to show that

$$S_\lambda\left(T_\lambda(x^{\mathcal{N}},y),T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})\right) = T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})\,, \tag{10.15}$$

for every $\lambda\in[0,\frac{1}{2}[$ and every $(x,y)\in[0,\frac{1}{2}]^2$, $x\leqslant y$. We distinguish two cases:

1. $\boldsymbol{T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})\geqslant 1}$

   Since $T_\lambda\leqslant T_{\mathbf{M}}$ it holds that $T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})\leqslant y+y^{\mathcal{N}}=1$. Consequently, we obtain that $T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})=1$, in which case Eq. (10.15) follows immediately from the definition of S_λ.

2. $\boldsymbol{T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})< 1}$

   For $y=x$ it holds that $T_\lambda(x^{\mathcal{N}},y)=0$ and therefore Eq. (10.15) is trivially fulfilled. If $x<y$, then $1<x^{\mathcal{N}}+y\leqslant x^{\mathcal{N}}+y^{\mathcal{N}}$. Suppose that $\lambda<y$, then $y\in\,]\lambda,1-\lambda[$ and $y^{\mathcal{N}}\in\,]\lambda,1-\lambda[$. Thus

   $$\lambda<\min(x^{\mathcal{N}},y)=y\leqslant y^{\mathcal{N}}=\min(x^{\mathcal{N}},y^{\mathcal{N}})<1-\lambda\,.$$

   Invoking the definition of T_λ, it holds that $T_\lambda(x^{\mathcal{N}},y)=\min(x^{\mathcal{N}},y)=y$ and $T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})=\min(x^{\mathcal{N}},y^{\mathcal{N}})=y^{\mathcal{N}}$. Finally, we sum these expressions side by side and get the contradiction $T_\lambda(x^{\mathcal{N}},y)+T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})=y+y^{\mathcal{N}}=1$. We conclude that $y\leqslant\lambda$ and $1-\lambda\leqslant y^{\mathcal{N}}\leqslant x^{\mathcal{N}}$. By definition of T_λ it then follows that $1-\lambda\leqslant T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}})$ and thus $\max(T_\lambda(x^{\mathcal{N}},y),T_\lambda(x^{\mathcal{N}},y^{\mathcal{N}}))\in[1-\lambda,1]$, in which case Eq. (10.15) is automatically fulfilled (see definition of S_λ). □

Figures 10.2(b) and 10.2(d) illustrate for $n=2$ the difference CD_2 between both fuzzified normal forms when dealing with the De Morgan triplets $(T_0,S_0,\mathcal{N})$ and $(T_{\frac{1}{3}},S_{\frac{1}{3}},\mathcal{N})$.

Theorem 10.16 *Consider a De Morgan triplet (T,S,N), with T a left-continuous t-norm and N an involutive negator with fixpoint β. Let $n\in\mathbb{N}$, $n>1$. If $0<Q(\beta,\beta)$ and the partial functions $T(\bullet,x)$ are continuous on $]x^N,1]$, whenever $x\in\,]0,\beta]$, and on $[x,1]$, whenever $x\in\,]\beta,1[$, then $C_{\mathcal{F}}(F)-D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n\to\{0,1\}$ function F if and only if $(T,S,N)=(T_\lambda,S_\lambda,\mathcal{N})$, for some $\lambda\in[0,\frac{1}{2}[$.*

Proof Due to Proposition 10.15 we only need to prove that every De Morgan triplet (T, S, N) fulfilling the conditions of the theorem and for which CD_n exists must belong to the family $((T_\lambda, S_\lambda, \mathcal{N}))_{\lambda\in[0,\frac{1}{2}[}$. To improve the readability of the proof, we distinguish six consecutive subproblems.

I. $T(x,y) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[y])]$, for every $(x,y) \in [\beta,1]^2 \setminus \{(\beta,\beta)\}$

Let σ be the $[\beta,1] \to [0,1]$ isomorphism that is used to construct T^β. By definition, $\max(\beta, T(x,y)) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[y])]$, for every $(x,y) \in [\beta,1]^2$. Hence, $T(x,y) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[y])]$, will hold for every $(x,y) \in [\beta,1]^2 \setminus \{(\beta,\beta)\}$ if and only if $T(x,\beta) = T(\beta,x) = \beta$, for every $x \in]\beta,1]$. We need to prove that $Q(\beta,\beta) = \beta$ (property **(E2)**). The latter follows immediately from Lemma 10.13.

II. $C_0 = N = \mathcal{N}$

Because T satisfies the law of contradiction w.r.t. N (Theorem 10.3), we know that $N \leqslant C_0$. Suppose that $N < C_0$ then there exists a couple $(x,y) \in [0,1]^2$ fulfilling $x^N < y$ and such that $T(x,y) = 0$ (*i.e.* $y \leqslant C_0(x)$). Without loss of generality, we may assume that $x \leqslant \beta$ as T is commutative and $\beta \leqslant T(x,y)$ whenever $(x,y) \in]\beta,1]^2$ (Part **I**). Furthermore, consider arbitrary $y_1 \in]y^N, x[$. The monotonicity of T implies that $T(u,v) = 0$, for every $(u,v) \in [0,x] \times [0,y]$. In particular, $T(y_1,y) = 0$. The continuity conditions on T then ensure the existence of $x_1 \in]0, y^N[$ for which $T(x_1^N, y_1) \in [y^N, y_1[$ and $T(x_1^N, y_1^N) \in [x^N, y_1^N]$ or for which $T(x_1^N, y_1) \in [y^N, y_1]$ and $T(x_1^N, y_1^N) \in [x^N, y_1^N[$. Indeed, we know that there exists $x_2 \in]0, y^N[$ such that $T(y_1, x_2^N) = y^N$. If $T(x_2^N, y_1^N) \in [x^N, y_1^N]$, we put $x_1 = x_2$. Else, in case $T(x_2^N, y_1^N) < x^N$, we know that there exists $x_3 \in]0, x_2[$ such that $T(x_3^N, y_1^N) = x^N$. It is then sufficient to put $x_1 = x_3$. Expressing that (10.10) = (10.11) leads to the contradiction

$$0 = y_1^N - y_1^N < T(x_1^N, y_1)^N - T(x_1^N, y_1^N) = T\left(T(x_1^N, y_1^N)^N, T(x_1^N, y_1)^N\right) \leqslant T(x,y) = 0\,.$$

Hence, $C_0 = N$ such that T is a rotation-invariant t-norm (assertion **(G2)**). The continuity conditions on the partial functions $T(\bullet, x)$ yield that the β-zoom T^β of T is continuous (Lemma 10.14). From Part **I** it then follows that T is continuous on $]\beta,1]^2$. In particular, δ_T is continuous on $]\beta,1]$ and $\delta_T(x) = T(x,x) = \sigma^{-1}[T^\beta(\sigma[x],\sigma[x])]$, for every $x \in]\beta,1]$. The diagonal $T^\beta(u,u)$ is continuous and reaches on $]0,1]$ every element of $]0,1]$. Therefore, $]\beta,1] = \sigma^{-1}(]0,1]) \subseteq \delta_T(]\beta,1])$ and from Lemma 10.8 it follows that $N = \mathcal{N}$ and $\beta = \frac{1}{2}$

III. $Q(y, y^{\mathcal{N}}) \in \{0, y\}$, for all $y \in]0, \frac{1}{2}]$

For $y = \frac{1}{2}$ this property follows from Part **I**. Take $y \in]0, \frac{1}{2}[$. Property **(E5)** ensures that always $Q(y, y^{\mathcal{N}}) \leqslant y$. Suppose that $Q(y, y^{\mathcal{N}}) \in]0, y[$. We consider the following cases:

1. If $T(y^{\mathcal{N}}, y^{\mathcal{N}}) < y^{\mathcal{N}}$, then $B(x) := T(x^{\mathcal{N}}, y^{\mathcal{N}})^{\mathcal{N}}$ is continuous on $[0,y]$ and reaches on this interval every number in $[y, T(y^{\mathcal{N}}, y^{\mathcal{N}})^{\mathcal{N}}]$. Analogously, since $Q(y, y^{\mathcal{N}}) < y$, $A(x) := T(x^{\mathcal{N}}, y)^{\mathcal{N}}$ is continuous on $[0, y[$ and reaches on $[0, y[$ every number in $[y^{\mathcal{N}}, Q(y, y^{\mathcal{N}})^{\mathcal{N}}[$ (property **(E2)**). From (10.10) = (10.11) it follows that $T(A(x), B(x)) = A(x) - B(x)^{\mathcal{N}}$ is continuous on $[0, y[$ and reaches on this interval every number in $[0, Q(y, y^{\mathcal{N}})^{\mathcal{N}} -$

$T(y^{\mathcal{N}}, y^{\mathcal{N}})[$. Take $x_1 \in [0, y[$ such that $0 < T(A(x_1), B(x_1)) < Q(y, y^{\mathcal{N}})$. Because $\mathcal{N} = C_0$ (Part **II**) we know that $A(x_1)^{\mathcal{N}} < B(x_1)$. From Part **I** it follows that $\frac{1}{2} \leqslant T(y^{\mathcal{N}}, y^{\mathcal{N}}) \leqslant B(x_1)^{\mathcal{N}}$. Therefore, the continuity of $T(\bullet, B(x_1))$ on $]B(x_1)^{\mathcal{N}}, 1]$ ensures the existence of $x_2 \in [0, A(x_1)^{\mathcal{N}}[\subset [0, B(x_1)[$ such that $T(x_2^{\mathcal{N}}, B(x_1)) < Q(y, y^{\mathcal{N}})$. Since $y \leqslant B(x_1)$ this leads to $T(y, x_2^{\mathcal{N}}) = T(x_2^{\mathcal{N}}, y) < Q(y, y^{\mathcal{N}})$. Taking into account that $y^{\mathcal{N}} \leqslant A(x_1) < x_2^{\mathcal{N}}$ this latter inequality contradicts property **(E2)**.

2. If $T(y^{\mathcal{N}}, y^{\mathcal{N}}) = y^{\mathcal{N}}$, we know that $B(x) = y$, for every $x \in [0, y]$. Like in the previous case, we also know that $A(x)$ reaches on $[0, y[$ every number in $[y^{\mathcal{N}}, Q(y, y^{\mathcal{N}})^{\mathcal{N}}[$. We conclude from (10.10) = (10.11) that $T(y, u) = u - y^{\mathcal{N}}$, for every $u \in [y^{\mathcal{N}}, Q(y, y^{\mathcal{N}})^{\mathcal{N}}[$. Taking into account property **(E2)** this leads to the contradiction
$$0 < Q(y, y^{\mathcal{N}}) = \lim_{u \searrow y^{\mathcal{N}}} T(y, u) = \lim_{u \searrow y^{\mathcal{N}}} u - y^{\mathcal{N}} = 0\,.$$

IV. $Q(y, y^{\mathcal{N}}) = y$ implies $\delta_T(y^{\mathcal{N}}) = y^{\mathcal{N}}$, for all $y \in]0, \frac{1}{2}[$

Recall that $Q(y, y^{\mathcal{N}}) = y$ implies that $x < y = T(x^{\mathcal{N}}, y)$, for all $x \in [0, y[$ (property **(E2)**). Applying assertion **(G5)** on this inequality results in $y^{\mathcal{N}} = C_0(y) < T(x^{\mathcal{N}}, C_0(x)) = T(x^{\mathcal{N}}, x^{\mathcal{N}})$, for every $x \in [0, y[\subset [0, \frac{1}{2}[$. As discussed in Part **II**, the diagonal δ_T of T is continuous on $]\frac{1}{2}, 1]$. Taking the limit $x \nearrow y$ leads to $y^{\mathcal{N}} \leqslant T(y^{\mathcal{N}}, y^{\mathcal{N}}) \leqslant y^{\mathcal{N}}$.

V. $\exists \lambda \in [0, \frac{1}{2}]$: $\delta_T(x) = x$, for all $x \in]\frac{1}{2}, \lambda^{\mathcal{N}}]$, and $\delta_T(x) < x$, for all $x \in]\lambda^{\mathcal{N}}, 1[$

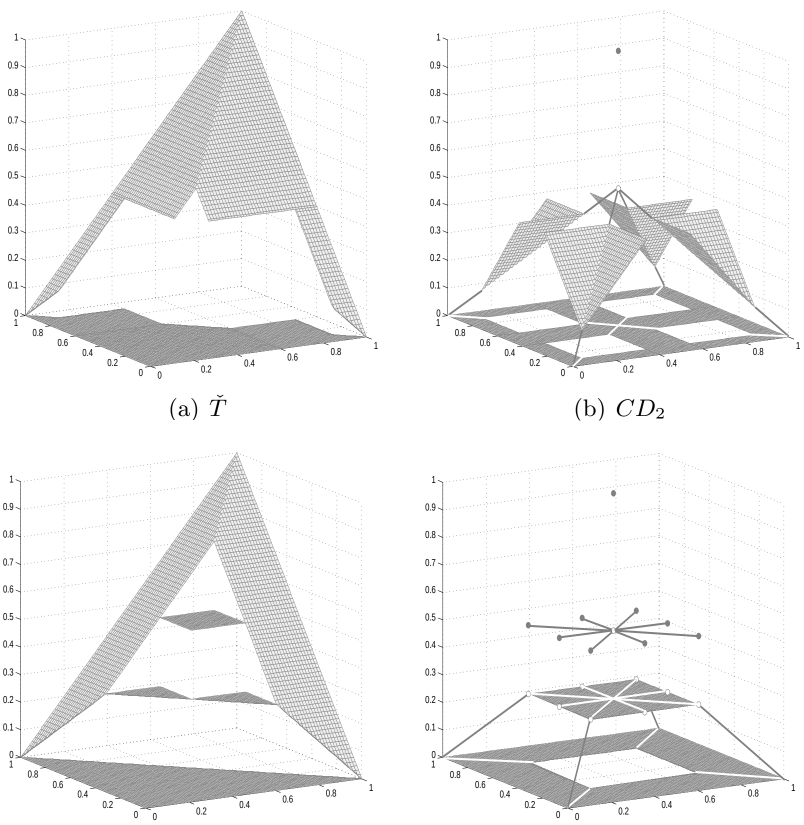
We prove that that whenever $\delta_T(x) = x$, for some $x \in]\frac{1}{2}, 1[$, then also $\delta_T(y) = y$, for every $y \in]\frac{1}{2}, x[$. The number λ is then defined by $\lambda^{\mathcal{N}} = \sup\{x \in]\frac{1}{2}, 1[\mid \delta_T(x) = x\}$ (with $\sup \emptyset = \frac{1}{2}$). Note that the left continuity of T ensures that $\delta_T(\lambda^{\mathcal{N}}) = \lambda^{\mathcal{N}}$ whenever $\frac{1}{2} < \lambda^{\mathcal{N}}$. Suppose now that $\delta_T(x) = x$, for some $x \in]\frac{1}{2}, 1[$. From Part **I** it then follows that $x = T(x, x) = \sigma^{-1}[T^{\frac{1}{2}}(\sigma[x], \sigma[x])]$ and hence, $T^{\frac{1}{2}}(\sigma[x], \sigma[x]) = \sigma[x]$, with $\sigma[x] \in]0, 1[$. The ordinal sum structure of $T^{\frac{1}{2}}$ (Lemma 10.14 and Theorem 6.6) fixes $T^{\frac{1}{2}}(u, \sigma[x]) = u$, for every $u \in [0, \sigma(x)]$. Translating this result for $T^{\frac{1}{2}}$ back to a property on T, we get that $T(y, x) = \sigma^{-1}[T^{\frac{1}{2}}(\sigma[y], \sigma[x])] = y$, for every $y \in [\frac{1}{2}, x]$ (Part **I**). Take arbitrary $y \in]\frac{1}{2}, x[$ then $y < z = T(z, x) = T(x, z)$, for every $z \in]y, x[$. Applying Theorem 5.12 on this inequality results in $z^{\mathcal{N}} < T(x, y^{\mathcal{N}}) \leqslant y^{\mathcal{N}}$. We obtain that $T(x, y^{\mathcal{N}}) = y^{\mathcal{N}}$ by taking the limit $z \searrow y$. Since $T(y^{\mathcal{N}}, x) \leqslant Q(y^{\mathcal{N}}, x)$ (property **(E3)**), we get that $y^{\mathcal{N}} \leqslant Q(y^{\mathcal{N}}, x)$. Recall that $C_0 = \mathcal{N}$ (Part **II**) which allows us to apply assertion **(G7)** on the above inequality: $0 < x^{\mathcal{N}} \leqslant Q(y^{\mathcal{N}}, y) = Q(y^{\mathcal{N}}, (y^{\mathcal{N}})^{\mathcal{N}})$. Because $Q(y^{\mathcal{N}}, (y^{\mathcal{N}})^{\mathcal{N}})$ can only take values in $\{0, y^{\mathcal{N}}\}$ (Part **III**), it necessarily holds that $Q(y^{\mathcal{N}}, (y^{\mathcal{N}})^{\mathcal{N}}) = y^{\mathcal{N}}$. Finally, we use Part **IV** and obtain $\delta_T(y) = y$, for every $y \in]\frac{1}{2}, x[$.

VI. $T = T_\lambda$, for some $\lambda \in [0, \frac{1}{2}[$

Invoking the rotation invariance of T (Theorem 7.1), Parts **I** and **V**, and the increasingness of T, we need to prove that $T(x, y) = x + y - 1$, for every $(x, y) \in]\lambda^{\mathcal{N}}, 1]$ satisfying $C_{\lambda^{\mathcal{N}}}(x) < y$ (*i.e.* $\lambda^{\mathcal{N}} < T(x, y)$). Note that this rules out $\lambda = \frac{1}{2}$ as this would imply that $T^{\frac{1}{2}} = T_{\mathbf{L}}$

which contradicts Part **I** due to Eq. (7.1). Recall from Part **II** that $C_0 = N = \mathcal{N}$. Furthermore, $T^{\frac{1}{2}}(x,x) = \sigma[T(\sigma^{-1}[x],\sigma^{-1}[x])] < x$, for every $x \in \,]\sigma(\lambda^{\mathcal{N}}),1[$ (Parts **I** and **V**). As $T^{\frac{1}{2}}$ is continuous (Lemma 10.14), the latter yields that $T^{\frac{1}{2}}|_{[\sigma(\lambda^{\mathcal{N}}),1]^2}$ is a rescaled continuous Archimedean t-norm (Theorems 6.6 and 6.4). Lemma 10.9 is applicable with $a = \lambda^{\mathcal{N}}$. We get that $T(x,y) = x+y-1$, for every $(x,y) \in \,]\lambda^{\mathcal{N}},1]$ satisfying $C_{\lambda^{\mathcal{N}}}(x) < y$. $\square$

In view of Lemma 10.14, one could wonder why we used in the above theorem the rather complex continuity conditions on the partial functions $T(\bullet,x)$ instead of the continuity of T^{β}. However, in the previous theorem we neither assumed the rotation invariance of T, nor that $C_0 = N$. Therefore, Lemma 10.14 is not straightforwardly applicable. Figure 10.3 illustrates that there exist De Morgan triplets $(T,S,\mathcal{N})$ that solve the system of functional equations but contain a left-continuous t-norm T that is not rotation invariant and has a continuous $\frac{1}{2}$-zoom $T^{\frac{1}{2}}$. The t-norm $\check{T}$ from Fig. 10.3(a) is obtained from the minimum operator $T_{\mathbf{M}}$ by lowering its values on the area $\{(x,y) \in [0,1]^2 \mid x+y \leqslant 0 \ \vee \ (\min(x,y) \leqslant \frac{2}{5} \ \wedge \ \max(x,y) \leqslant \frac{9}{10})\}$ to zero.



(a) $\check{T}$ (b) CD_2

(c) T (d) CD_2

Figure 10.3: CD_2 for the De Morgan triplets $(\check{T},\check{S},\mathcal{N})$ and $(T,S,\mathcal{N})$.

Finally, we would like to point out once more that we focused in Subsections **A** and **B** on those solution triplets that are based on the t-norms described in Section 8.3. The system of functional equations can also be solved in case the partial functions $T(\bullet, \beta)$ are not continuous on $]\beta, 1]$. An example of such a solution can be found in Fig. 10.3. The mathematical formulation of the t-norm T depicted in Fig. 10.3(c) has been stated in Example 7.2. However, as a complete characterization of all left-continuous t-norms is still lacking and given the incompatibility of such solutions with our previous results, the study of these solutions is left for further research.

Appendices

APPENDIX A

Summary

Fuzzy logic and fuzzy set theory make extensive use of monotone $[0,1] \to [0,1]$ and monotone $[0,1]^2 \to [0,1]$ functions. On the one hand, increasing $[0,1]^2 \to [0,1]$ functions such as t-norms and t-conorms, are used as straightforward generalizations of the Boolean conjunction and disjunction (see e.g. [19, 36, 51, 71]). On the other hand, decreasing $[0,1] \to [0,1]$ functions are used to generalize the Boolean negation (see e.g. [19, 51, 71]). Most attention goes by far to the study of $[0,1]^2 \to [0,1]$ functions and their properties. Monotone $[0,1] \to [0,1]$ functions are more elementary. In this dissertation we have used these functions to describe monotone $[0,1]^2 \to [0,1]$ functions. The achieved results contribute to a better insight into the structure of left-continuous t-norms. Such insights are essential for various studies involving t-norms.

A.1 Inverses of monotone functions

Reflections are not always apt to define the inverse of a curve w.r.t. a given monotone $[0,1] \to [0,1]$ bijection Φ. Therefore, we have introduced the Φ-*inverse* of a set $A \subseteq [0,1]^2$: $A^\Phi := \{(x,y) \in [0,1]^2 \mid (\Phi^{-1}(y), \Phi(x)) \in A\}$. Geometrically, A^Φ is the set of those vertices that constitute the fourth point of a rectangle with sides parallel to the axes, that has two 2 vertices on the graph of Φ and has one vertex belonging to A. It is clear that $(A^\Phi)^\Phi = A$ and $A^{\mathbf{id}} = A^{-1}$, with **id** the identity function.

As for the classical inverse, the Φ-inverse of a monotone $[0,1] \to [0,1]$ function f is again a $[0,1] \to [0,1]$ function if and only if f is bijective. Largely extending the approach of Schweizer and Sklar [85] we have associated to each monotone function f a set $Q(f,\Phi)$ of 'Φ-inverse' functions. This set consists of all monotone $[0,1] \to [0,1]$ functions whose completion is the Φ-inverse of the completion of f. The *completion* of a monotone function f is a continuous increasing or decreasing line that reaches every element in the unit interval and is constructed from the graph of f by simply adding vertical segments. For a constant function f the set $Q(f,\Phi)$ contains the functions constructed from the increasing completion of f as well as those constructed from the decreasing completion of f. For a monotone $[0,1] \to [0,1]$ bijection Ψ it clearly holds that $Q(\Psi,\Phi) = \{\Psi^\Phi\}$.

Theorem *The sets $Q(f,\Phi)$ and $Q(f,\mathbf{id})$ are isomorphic whenever Φ is increasing and antimorphic whenever Φ is decreasing. In particular, for every $g \in Q(f,\Phi)$ there exists a unique $h \in Q(f,\mathbf{id})$ such that $g = \Phi \circ h \circ \Phi$.*

Increasing $[0,1] \to [0,1]$ bijections are also kown as *automorphisms*. Decreasing bijections are referred to as *strict negators*. The isomorphy/antimorphy between $Q(f,\mathbf{id})$ and $Q(f,\Phi)$ allows for a straightforward conversion of the properties of $Q(f,\mathbf{id})$ into those of $Q(f,\Phi)$. Hence, it suffices to study the set $Q(f,\mathbf{id})$ only. To describe the members of $Q(f,\mathbf{id})$ mathematically, we use four $[0,1] \to [0,1]$ functions:

$$\overline{f}^{\mathbf{id}}(x) = \sup\{t \in [0,1] \mid f(t) < x\} \qquad \underline{f}^{\mathbf{id}}(x) = \sup\{t \in [0,1] \mid f(t) > x\}$$
$$\overline{f}_{\mathbf{id}}(x) = \inf\{t \in [0,1] \mid f(t) > x\} \qquad \underline{f}_{\mathbf{id}}(x) = \inf\{t \in [0,1] \mid f(t) < x\}\,.$$

Both functions $\overline{f}^{\mathbf{id}}$ and $\overline{f}_{\mathbf{id}}$ are increasing. The functions $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ are decreasing. In case $f(0) < f(1)$, resp. $f(1) < f(0)$, the function $\overline{f}^{\mathbf{id}}$, resp. $\underline{f}^{\mathbf{id}}$, is known as the *pseudo-inverse* $f^{(-1)}$ of f [51]. Depending on the type of monotonicity of f, (some of) the functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ or $\underline{f}_{\mathbf{id}}$ constitute the boundaries of $Q(f,\mathbf{id})$. Furthermore, if they belong to $Q(f,\mathbf{id})$ they can also be sifted out of that set on the basis of their continuity.

Finally, the characteristic properties of the classical inverse show up in the set $Q(f,\mathbf{id})$:

Theorem *For every $g \in Q(f,\mathbf{id})$ the following assertions hold:*

1. *$f \in Q(g,\mathbf{id})$.*
2. *$g \circ f = \mathbf{id}$ if and only if f is injective.*
3. *$f \circ g = \mathbf{id}$ if and only if f is surjective.*

A.2 Orthosymmetry of monotone functions

Generalizing the classical notion of symmetry, we call a set $A \subseteq [0,1]^2$ Φ-*symmetrical* if it coincides with its Φ-inverse, *i.e.* $(\Phi^{-1}(y),\Phi(x)) \in A \Leftrightarrow (x,y) \in A$. Unfortunately, when dealing with monotone $[0,1] \to [0,1]$ functions f only bijections can coincide with their Φ-inverse. To overcome this problem we have generalized the classical concept of symmetry by invoking the set $Q(f,\Phi)$. We call a monotone $[0,1] \to [0,1]$ function f Φ-*orthosymmetrical* if $f \in Q(f,\Phi)$. Considering the geometrical construction of $Q(f,\Phi)$, it is clear that f is Φ-orthosymmetrical if and only if its completion is Φ-symmetrical. There exist only two Φ-orthosymmetrical, constant functions: $\mathbf{0}$ and $\mathbf{1}$.

Theorem *If f is non-constant and Φ-orthosymmetrical, then either $f = \Phi$ or f and Φ have opposite types of monotonicity.*

In contrast to Φ-symmetry, Φ-orthosymmetry admits the following limit property.

Theorem *The limit of a pointwisely converging sequence of Φ-orthosymmetrical, monotone $[0,1] \to [0,1]$ functions $(f_n)_{n\in\mathbb{N}}$ is always a Φ-orthosymmetrical, monotone $[0,1] \to [0,1]$ function.*

The (ortho)symmetry of monotone $[0,1] \to [0,1]$ bijections has some characteristic properties providing us with better insights into the structure of automorphisms and strict negators. As $Q(\Psi,\Phi) = \{\Psi^\Phi\}$, Φ-symmetry and Φ-orthosymmetry coincide when dealing with monotone $[0,1] \to [0,1]$ bijections. Explicitly, a monotone $[0,1] \to [0,1]$ bijection Ψ is Φ-symmetrical if and only if $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$, or equivalently $\Phi = \Psi \circ \Phi^{-1} \circ \Psi$, which expresses the Ψ-symmetry of Φ. We say that Φ and Ψ form a *symmetrical pair* $\{\Phi, \Psi\}$. *Involutive negators* (*i.e.* involutive strict negators) are exactly those decreasing $[0,1] \to [0,1]$ bijections that form a symmetrical pair with the identity function **id**.

Theorem *A monotone $[0,1] \to [0,1]$ bijection Ψ is Φ-symmetrical if and only if $\Psi = \Phi$ or there exists an involutive negator N such that $\Psi = \Phi \circ N$.*

Based on this theorem it has been possible to reveal the (ortho)symmetrical aspects of the following historical, mathematical result.

Theorem

1. *For every strict negator N there exist three involutive negators N_1, N_2 and N_3 such that $N = N_1 \circ N_2 \circ N_3$.*
2. *For every automorphism ϕ there exist four involutive negators N_1, N_2, N_3 and N_4 such that $\phi = N_1 \circ N_2 \circ N_3 \circ N_4$.*

The set of monotone $[0,1] \to [0,1]$ bijections can be partitioned into four subsets: monotone bijections composed of one, two, three or four involutive negators. Every involutive negator trivially generates itself. All other strict negators are always composed of exactly three involutive negators. Automorphisms composed of two involutive negators must have some kind of alternating behaviour w.r.t. some fixpoint. An automorphism that has no alternating behaviour is always composed of four involutive negators. We have presented a simple method for constructing an appropriate sequence of involutive negators generating a given monotone $[0,1] \to [0,1]$ bijection Φ. This sequence is not uniquely determined.

A.3 Invariance of monotone functions

Φ-orthosymmetry plays a crucial role in the study of Φ-invariant, monotone $[0,1]^n \to [0,1]$ functions. Let Φ be some monotone $[0,1] \to [0,1]$ bijection. Then a monotone $[0,1]^n \to [0,1]$ function F is said to be Φ-*invariant* if $F_\Phi = F$, with $F_\Phi(x_1,\ldots,x_n) := \Phi^{-1}(F(\Phi(x_1),\ldots,\Phi(x_n)))$ (see e.g. [7]). In this context it suffices to study the invariance of increasing functions only. Given a bijection Φ, it remains an intriguing problem how to characterize all Φ-invariant, increasing $[0,1]^n \to [0,1]$ functions. A first subset of solutions consists of those functions that are invariant under all monotone $[0,1] \to [0,1]$ bijections.

Theorem *An increasing $[0,1]^n \to [0,1]$ function F is invariant under all automorphisms if and only if there exists an increasing $[0,1]^n \to [0,1]$ function G such that $F_N = G$ holds for every involutive negator N. In this case it also holds that $F_N = G$, for every strict negator N.*

Hence, an increasing $[0,1]^n \to [0,1]$ function F is invariant under all monotone $[0,1] \to [0,1]$ bijections if and only if it is invariant under all involutive negators.

We have also proposed a wide class of methods for characterizing N-invariant, increasing $[0,1]^n \to [0,1]$ functions, with N some fixed involutive negator. All of these methods invoke a $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ that allows a characterization in the following sense:

A $[0,1]^n \to [0,1]$ function F is increasing and N-invariant if and only if there exists an increasing $[0,1]^n \to [0,1]$ function G such that

$$F(x_1,\ldots,x_n) = \boldsymbol{\mathcal{C}}\big(G(x_1,\ldots,x_n), G_N(x_1,\ldots,x_n)\big) \tag{A.1}$$

holds for every $(x_1,\ldots,x_n) \in [0,1]^n$.

We then say that $\boldsymbol{\mathcal{C}}$ *enables a full characterization* of all N-invariant, increasing $[0,1]^n \to [0,1]$ functions.

Theorem *$\boldsymbol{\mathcal{C}}$ enables a full characterization of all N-invariant, increasing $[0,1]^n \to [0,1]$ functions if and only if the following assertions hold:*

1. *$\boldsymbol{\mathcal{C}}$ is increasing.*
2. *$\boldsymbol{\mathcal{C}}(x,y) = \boldsymbol{\mathcal{C}}(y^N, x^N)^N$ holds for every $(x,y) \in [0,1]^2$.*
3. *The graph of $\boldsymbol{\mathcal{C}}$ contains an increasing (w.r.t. the three space coordinates) curve whose Z-coordinate reaches every number of $[0,1]$.*

The above class of characterizations (one for every choice of $\boldsymbol{\mathcal{C}}$) comprises two known methods for characterizing self-dual aggregation operators [7, 33]. A similar approach for describing all Φ-invariant, increasing $[0,1]^n \to [0,1]$ functions, with Φ a non-involutive monotone $[0,1] \to [0,1]$ bijection, cannot be expected. To conclude we have shown the following two properties.

Theorem *$F = \boldsymbol{\mathcal{C}}_F$ holds for every N-invariant, increasing $[0,1]^n \to [0,1]$ function F if and only if $\boldsymbol{\mathcal{C}}$ is idempotent.*

Theorem *If $n > 1$, then the arithmetic mean $\mathbf{M}$ is the only increasing $[0,1]^2 \to [0,1]$ function $\boldsymbol{\mathcal{C}}$ that enables a full characterization of all $\mathcal{N}$-invariant, increasing $[0,1]^n \to [0,1]$ functions and for which $(\boldsymbol{\mathcal{C}},\mathcal{N})$ preserves shift invariance.*

Note that $\mathcal{N}$ denotes the *standard negator* ($x^{\mathcal{N}} = 1 - x$) and that $(\boldsymbol{\mathcal{C}}, N)$ *preserves shift-invariance* if the right-hand side of Eq. (A.1) is shift-invariant for every shift-invariant, increasing $[0,1]^n \to [0,1]$ function G.

A.4 Traces of orthosymmetry

Each increasing $[0,1]^2 \to [0,1]$ function F is totally determined by its *horizontal cuts* (*i.e.* the intersections of its graph by planes parallel to the domain $[0,1]^2$). The contour lines of F are those $[0,1] \to [0,1]$ functions determining the upper, lower, right or left limits of these horizontal cuts. We associate with an increasing $[0,1]^2 \to [0,1]$ function F four types of *contour lines* ($a \in [0,1]$):

$$\begin{aligned}
C_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(x,t) \leqslant a\}\,,\\
D_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(x,t) \geqslant a\}\,,\\
\widetilde{C}_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(t,x) \leqslant a\}\,,\\
\widetilde{D}_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(t,x) \geqslant a\}\,.
\end{aligned}$$

All contour lines are decreasing. Depending on the continuity of F, different types of contour lines form a Galois connection with F. Contour lines of the type C_a or $\widetilde{C}_a$ are extremely suited to describe left-continuous, increasing $[0,1]^2 \to [0,1]$ functions. Contour lines of the type D_a or $\widetilde{D}_a$ are used to describe right-continuous, increasing $[0,1]^2 \to [0,1]$ functions.

For a given couple (Φ, Ψ) of monotone $[0,1] \to [0,1]$ bijections, we have characterized, in terms of contour lines, those increasing $[0,1]^2 \to [0,1]$ functions F that satisfy

$$F(x,y) = \Psi\big(F(\Phi^{-1}(y), \Phi(x))\big)\,, \tag{A.2}$$

for every $(x,y) \in [0,1]^2$. In case $\Phi = \Psi = \mathbf{id}$, the latter expresses the commutativity of F. For $\Phi = \Psi = N$, with N an involutive negator, we retrieve the second assertion from the second theorem in Section **A3**. There are, however, some restrictions on the choice of Φ and Ψ. Both bijections must have the same type of monotonicity and $F(x,y) = \Psi(\Psi(F(x,y)))$ must always be satisfied. We have strengthened the latter condition and required that Ψ is involutive. The following cases are to be distinguished:

A. Φ is an automorphism ϕ and Ψ is the identity function $\mathbf{id}$.

B. Φ is a strict negator M and Ψ is an involutive negator N, with fixpoint β.

Theorem *The following characterizations hold:*

1. *If F is left continuous and $F(0,1) = F(1,0) = 0$, then the following assertions are equivalent:*

   a) F satisfies Eq. (A.2)*, with $\Phi = \phi$ and $\Psi = \mathbf{id}$.*
   b) $C_a \in Q(C_a, \phi)$, for every $a \in [0,1]$.
   c) $\widetilde{C}_a \in Q(\widetilde{C}_a, \phi^{-1})$, for every $a \in [0,1]$.

2. *If F is right continuous and $F(0,1) = F(1,0) = 1$, then the following assertions are equivalent:*

a) F *satisfies Eq.* (A.2)*, with* $\Phi = \phi$ *and* $\Psi = \mathbf{id}$.
b) $D_a \in Q(D_a, \phi)$*, for every* $a \in [0,1]$.
c) $\widetilde{D}_a \in Q(\widetilde{D}_a, \phi^{-1})$*, for every* $a \in [0,1]$.

Hence, Eq. (A.2), with $\Phi = \phi$ and $\Psi = \mathbf{id}$, expresses the ϕ-orthosymmetry, resp., ϕ^{-1}-orthosymmetry, of the contour lines C_a and D_a, resp., $\widetilde{C}_a$ and $\widetilde{D}_a$. As shown in the following theorem, whenever F satisfies Eq. (A.2), with $\Phi = M$ and $\Psi = N$, then C_{a^N} can be understood as some 'M-inverse function' of D_a and $\widetilde{C}_{a^N}$ as some kind of 'M^{-1}-inverse function' of $\widetilde{D}_a$.

Theorem *If* F *is continuous, then the following assertions are equivalent:*

1. F *satisfies Eq.* (A.2)*, with* $\Phi = M$ *and* $\Psi = N$.
2. $C_{a^N} \in Q(D_a, M)$*, for every* $a \in [0,\beta]$*, and* $F(0,1) = F(1,0) = \beta$.
3. $D_{a^N} \in Q(C_a, M)$*, for every* $a \in [0,\beta]$*, and* $F(0,1) = F(1,0) = \beta$.
4. $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$*, for every* $a \in [0,\beta]$*, and* $F(0,1) = F(1,0) = \beta$.
5. $\widetilde{D}_{a^N} \in Q(\widetilde{C}_a, M^{-1})$*, for every* $a \in [0,\beta]$*, and* $F(0,1) = F(1,0) = \beta$.

A.5 A contour view on uninorms

Examining the contour lines C_a, D_a, $\widetilde{C}_a$ and $\widetilde{D}_a$ of a *uninorm* U (*i.e.* an associative, commutative, increasing $[0,1]^2 \to [0,1]$ function that has a neutral element) instead of the uninorm itself, we have given the description of uninorms a new impetus. Let F be an increasing $[0,1]^2 \to [0,1]$ function.

Theorem *If* F *is left continuous and* $F(0,1) = F(1,0) = 0$*, then the following characterizations hold:*

1. F *has neutral element* $e \in \,]0,1]$ *if and only if* $e \leqslant C_a(x) \Leftrightarrow x \leqslant a$ *and* $C_a(e) = a$ *hold for every* $(x,a) \in [0,1]^2$.
2. F *is commutative if and only if* $C_a(x) < y \Leftrightarrow C_a(y) < x$ *holds for every* $(x,y,a) \in [0,1]^3$.
3. F *is associative if and only if* $C_a(F(x,y)) = C_{C_a(x)}(y)$ *holds for every* $(x,y,a) \in [0,1]^3$.

Also contour lines of the type $\widetilde{C}_a$ can be used to characterize the neutral element, the commutativity and the associativity of a left-continuous, increasing $[0,1]^2 \to [0,1]$ function F. For right-continuous functions F satisfying $F(0,1) = F(1,0) = 1$ contour lines of the types D_a or $\widetilde{D}_a$ have to be used. The characterization of the commutativity in the above theorem is equivalent with the $\mathbf{id}$-orthosymmetry of the contour lines. If F is commutative, then $\widetilde{C}_a = C_a$ and $\widetilde{D}_a = D_a$, for every $a \in [0,1]$.

A uninorm U that satisfies $U(0,1) = U(1,0) = 0$ is called *conjunctive* [28]. If $U(0,1) = U(1,0) = 1$, then U is called *disjunctive*.

Theorem *For a left-continuous, conjunctive uninorm* U *it holds for every* $(x,y,z,a) \in [0,1]^4$ *that*

$$U(x,y) \leqslant C_a(z) \;\Leftrightarrow\; U(x,z) \leqslant C_a(y)\,.$$

We have also investigated how properties on contour lines affect the structure of U. Special attention goes to the study of continuous contour lines.

Theorem *For a left-continuous, conjunctive uninorm U the following assertions are equivalent:*

1. C_a *is continuous.*
2. C_a *is involutive on* $[C_a(1), 1]$.
3. $U(x,y) = C_a(C_{C_a(x)}(y))$, *for every* $(x,y) \in [0,1]^2$ *such that* $C_a(U(x,1)) < y$.
4. $C_b(x) = C_{C_a(x)}(C_a(b))$, *for every* $(x,b) \in [0,1] \times [C_a(1), 1]$.
5. $U(x,y) \leqslant z \;\Leftrightarrow\; U(x, C_a(z)) \leqslant C_a(y)$, *for every* $(x,y,z) \in [C_a(1), 1]^3$.

Similar results hold for right-continuous uninorms. In that case one has to use contour lines of the type D_a.

A.6 Left-continuous t-norms

For a left-continuous *t-norm* T (*i.e.* a left-continuous uninorm with neutral element 1), each contour line C_a equals the partial function $I_T(\bullet, a)$ of the *residual implicator* I_T (see e.g. [27]). In particular, the contour line C_0 coincides with the *residual negator* $N_T = I_T(\bullet, 0)$. In this case $(a = 0)$ the fifth assertion in the last theorem of the Section **A5** expresses the *rotation invariance* of T w.r.t. the contour line C_0. The rotation invariance of a t-norm has been defined originally w.r.t. an arbitrary involutive negator N [25, 40]. Furthermore, if a t-norm T is rotation invariant w.r.t. an involutive negator N, then T is necessarily left continuous and $C_0 = N_T = N$ [40]. In view of the last theorem in Section **A5**, we briefly talk about a *rotation-invariant t-norm* if it is left continuous and has a continuous contour line C_0.

Unfortunately, contour lines are inadequate to give insight into the geometrical structure of rotation-invariant t-norms. Also the companion and zooms are indispensable for the decomposition and construction of rotation-invariant t-norms.

The companion

The *companion* Q of an increasing $[0,1]^2 \to [0,1]$ function F is the $[0,1]^2 \to [0,1]$ function defined by

$$Q(x,y) = \sup\{t \in [0,1] \mid C_t(x) \leqslant y\} .$$

We have shown that $Q(x,y) = \inf\{F(x,u) \mid u \in \,]y,1]\}$. This property allows to straightforwardly construct the graph of Q from the graph of F. Clearly, $Q(x,y) = F(x,y)$ whenever $F(x,\bullet)$ is right continuous in $y \in [0,1[$. Every left-continuous, increasing $[0,1]^2 \to [0,1]$ function F that has absorbing element 0 is totally determined by its companion Q.

Zooms

An increasing $[0,1]^2 \to [0,1]$ function F can also be described by its associated set of zooms. Take $(a,b) \in [0,1]^2$ such that $a < b$ and $F(b,b) \leqslant b$. Consider an $[a,b] \to [0,1]$ isomorphism σ.

The (a,b)-*zoom* $F^{(a,b)}$ of F is the $[0,1]^2 \to [0,1]$ function defined by

$$F^{(a,b)}(x,y) = \sigma\left[\max\left(a, F(\sigma^{-1}[x], \sigma^{-1}[y])\right)\right] .$$

If $b=1$ we briefly talk about the a-*zoom* F^a of F. In this case the boundary condition $F(1,1) \leqslant 1$ is always true and, hence, the a-zoom of F is defined for every $a<1$. The graph of $F^{(a,b)}$ is in some sense obtained by rescaling the set $\{(x,y,F(x,y)) \mid (x,y) \in [a,b]^2 \ \wedge\ a < F(x,y)\}$ (zoom in) into the unit cube (zoom out).

Zooms are extremely suited to study increasing functions F satisfying $F(x,y) \leqslant \min(x,y)$, for every $(x,y) \in [0,1]^2$. The restriction $F(b,b) \leqslant b$ then trivially holds. By definition, a *t-subnorm* is a $[0,1]^2 \to [0,1]$ function F satisfying all uninorm properties but the neutral element. Instead $F(x,y) \leqslant \min(x,y)$ must hold for every $(x,y) \in [0,1]^2$ [47].

Theorem *Consider $(a,b) \in [0,1]^2$ such that $a<b$. Then the (a,b)-zoom of a t-subnorm is a t-subnorm and the a-zoom of a t-norm is a t-norm.*

Equipped with contour lines, the companion and zooms, we have been able to concisely reformulate the rotation and rotation-annihilation construction of Jenei [47]. Furthermore, we have illustrated how his decomposition methods [45] can be straightforwardly retrieved from our results.

A.7 Decomposing rotation-invariant t-norms

Despite all efforts, the class of rotation-invariant t-norms is not yet fully understood. The decomposition method presented by Jenei [45] only applies to very specific rotation-invariant t-norms. The Łukasiewicz t-norm, for example, falls outside this setting. We have introduced a more natural procedure for decomposing T. Based on a new partition of the domain of the t-norm T, we express T in terms of its contour line C_0 and β-zoom T^β, with β the unique fixpoint of C_0.

Let T be a rotation-invariant t-norm and β be the unique fixpoint of C_0. We partition the area $\mathcal{D} = \{(x,y) \in [0,1]^2 \mid C_0(x) < y\}$ into four parts:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y) \in \,]\beta,1]^2 \mid C_\beta(x) < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y) \in \,]0,\beta] \times]\beta,1] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y) \in \,]\beta,1] \times]0,\beta] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y) \in \,]\beta,1[^2 \mid y \leqslant C_\beta(x)\}\,.
\end{aligned}$$

Theorem *Let σ be an arbitrary $[\beta,1] \to [0,1]$ isomorphism. If the contour line C_β of T is continuous on $]\beta,1]$, then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$ and* *companion $\widehat{Q}$) such that $\widehat{C}_0$ is continuous on* $]0,1]$ *and*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right], & \textit{if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \textit{if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \textit{if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[x]),\widehat{C}_0(\sigma[y])\big)\right]\right), & \textit{if } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,, \\ 0, & \textit{if } (x,y) \notin \mathcal{D}\,. \end{cases}$$

In particular, $\widehat{T} = T^\beta$ and $\widehat{Q}$ must be commutative on $[0,\hat{\alpha}[^2$, *with* $\hat{\alpha} = \inf\{t \in [0,1] \mid \widehat{C}_0(t) = 0\}$.

The isomorphism σ must also be used to compute the β-zoom T^β of T. The decomposition on $[0,1] \setminus \mathcal{D}_{\mathrm{IV}}$ is valid for every rotation-invariant t-norm T. The filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is, however, not always uniquely determined.

Geometrically, $T|_{\mathcal{D}_{\mathrm{I}}}$ is a rescaled version of $T^\beta|_{\mathcal{D}^\beta}$, with $\mathcal{D}^\beta = \{(x,y) \in [0,1]^2 \mid 0 < T^\beta(x,y)\}$. $T|_{\mathcal{D}_{\mathrm{II}}}$ is determined by rotating $T|_{\mathcal{D}_{\mathrm{I}}}$ 120 degrees to the left around the axis $\{(x,y,z) \in [0,1]^2 \mid y = x \ \wedge \ z = 1-x\}$. Similarly, rotating $T|_{\mathcal{D}_{\mathrm{I}}}$ 120 degrees to the right around this axis determines $T|_{\mathcal{D}_{\mathrm{III}}}$. In the above theorem the filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is obtained by rotating $T|_{\mathcal{D}_{\mathrm{I}} \cap\,]\beta,\sigma^{-1}(\hat{\alpha})]^2}$ 180 degrees to the front around the axis $\{(x,y,z) \in [0,1]^3 \mid x+y = \beta + \sigma^{-1}[\hat{\alpha}] \ \wedge \ z = \beta\}$. In case C_β is continuous it holds that $\hat{\alpha} = 1$. Note that the rotations sometimes have to be reshaped to fit into the areas $\mathcal{D}_{\mathrm{II}}$, $\mathcal{D}_{\mathrm{III}}$ and $\mathcal{D}_{\mathrm{IV}}$, respectively. The contour lines C_0 and $\widehat{C}_0$ cause this reshaping.

A.8 The triple rotation method

Next, we have transformed our decomposition method into a straightforward construction tool for rotation-invariant t-norms. The presented results largely comprise the rotation and rotation-annihilation construction of Jenei [47]. We assume the following setting:

- T: an arbitrary left-continuous t-norm (with contour lines C_a and companion Q) such that C_0 is continuous on $]0,1]$ and Q is commutative on $[0,\alpha[^2$, with $\alpha = \inf\{t \in [0,1] \mid C_0(t) = 0\}$;
- N: an arbitrary involutive negator with fixpoint β;
- σ: an arbitrary $[\beta,1] \to [0,1]$ isomorphism;
- M: the decreasing $[0,1] \to [0,1]$ function defined by $x^M = 1$ whenever $x \in [0,\beta[$ and by $x^M = \sigma^{-1}[C_0(\sigma[x])]$ whenever $x \in [\beta,1]$;

- $\mathcal{D}$: the area $\{(x,y)\in[0,1]^2 \mid x^N < y\} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$, with

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y)\in\,]\beta,1]^2 \mid x^M < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y)\in\,]0,\beta]\times]\beta,1] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y)\in\,]\beta,1]\times]0,\beta] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y)\in\,]\beta,1[^2 \mid y \leqslant x^M\}\,.
\end{aligned}$$

Theorem *The* $[0,1]^2 \to [0,1]$ *function* $R3(T,N)$ *defined by*

$$R3(T,N)(x,y) = \begin{cases}
\sigma^{-1}\left[T(\sigma[x],\sigma[y])\right], & \text{if } (x,y)\in\mathcal{D}_{\mathrm{I}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[x^N]}(\sigma[y])\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{II}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[y^N]}(\sigma[x])\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{III}}\,,\\
\left(\sigma^{-1}\left[Q\big(C_0(\sigma[x]),C_0(\sigma[y])\big)\right]\right)^N, & \text{if } (x,y)\in\mathcal{D}_{\mathrm{IV}}\,,\\
0, & \text{if } (x,y)\notin\mathcal{D}\,,
\end{cases}$$

is a rotation-invariant t-norm. Furthermore, $R3(T,N)$ *is the only left-continuous t-norm that has* N *as a contour line (*$a=0$*) and that has* β*-zoom* $R3(T,N)^\beta = T$.

As for our decomposition, $R3(T,N)|_{\mathcal{D}_{\mathrm{II}}}$, $R3(T,N)|_{\mathcal{D}_{\mathrm{III}}}$ and $R3(T,N)|_{\mathcal{D}_{\mathrm{IV}}}$ are determined by the (transformed) left, right and front rotation of $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$. Moreover, $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$ is a rescaled version of 'the non-zero part' of T. Inspired by these geometrical observations, we have briefly called $R3(T,N)$ the *triple rotation of* T *based on* N. The construction method itself is referred to as the *triple rotation method.* For the triple rotation method to yield a t-norm it is absolutely necessary that the companion Q of T is commutative on $[0,\alpha[^2$.

A.9 Facts and figures on fuzzified normal forms

In Section **A10** we invoke our knowledge on the structure of rotation-invariant t-norms to solve a system of functional equations. The present section sets out the framework in which the system of functional equations surfaces.

In the Boolean algebra $(\{0,1\},\vee,\wedge,',0,1)$ every $\{0,1\}^n \to \{0,1\}$ function F can be represented by its *disjunctive* ($D_{\mathcal{B}}(F)$) and *conjunctive* ($C_{\mathcal{B}}(F)$) *normal form.* In fuzzy logic, it is generally accepted to work with t-norms and *t-conorms* (t-conorms are uninorms that have neutral element 0). Fuzzifying the Boolean normal forms of F by interpreting $\wedge$ as a t-norm T, $\vee$ as a t-conorm S and $'$ as an involutive negator N yields two $[0,1]^n \to [0,1]$ functions: the *disjunctive* ($D_{\mathcal{F}}(F)$) and *conjunctive* ($C_{\mathcal{F}}(F)$) *fuzzified normal form* of F [88, 89]. These fuzzified normal forms can rarely be considered as true normal forms in an extended logic or algebra. However, they are sometimes used as a kind of standard fuzzification procedure for crisp concepts.

The main point of study so far has been the relationship between $D_{\mathcal{F}}(F)$ and $C_{\mathcal{F}}(F)$. On the one hand, Bilgiç [5] has shown that $D_{\mathcal{F}}(F)$ can never equal $C_{\mathcal{F}}(F)$ for every $\{0,1\}^2 \to \{0,1\}$ function F. On the other hand, Türkşen [88, 89] has discovered that some particular triplets (T, S, N) ensure that

$$D_{\mathcal{F}}(F)(x,y) \leqslant C_{\mathcal{F}}(F)(x,y)\,,$$

for every $\{0,1\}^2 \to \{0,1\}$ function F and every $(x,y) \in [0,1]^2$. The shorthand $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$ is used to express the latter. In general, $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ expresses that $D_{\mathcal{F}}(F)(\vec{x}) \leqslant C_{\mathcal{F}}(F)(\vec{x})$ holds for every $\{0,1\}^n \to \{0,1\}$ function F and every $\vec{x} \in [0,1]^n$.

Theorem *Consider an automorphism ϕ. $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$, with $n \in \mathbb{N}_0$, holds for some triplet (T, S, N) if and only if it holds for the triplet (T_ϕ, S_ϕ, N_ϕ).*

Theorem *If $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ holds for some $n \in \mathbb{N}_0$, then $D_{\mathcal{F}} \leqslant_m C_{\mathcal{F}}$ is satisfied for every $m \in \mathbb{N}_0$ such that $m \leqslant n$.*

We use the notation $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ to denote that $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ holds for every $n \in \mathbb{N}_0$. In the following proposition we investigate $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ for triplets based on one of the three prototypical continuous t-norms: the minimum operator $T_{\mathbf{M}}(x,y) = \min(x,y)$, the algebraic product $T_{\mathbf{P}}(x,y) = x\,y$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. Dually, the three prototypical continuous t-conorms are the maximum operator $S_{\mathbf{M}}(x,y) = \max(x,y)$, the probabilistic sum $S_{\mathbf{P}}(x,y) = x + y - x\,y$ and the Łukasiewicz t-conorm $S_{\mathbf{L}}(x,y) = \min(x+y,1)$.

Propositie *Consider an automorphism ϕ. $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ holds for all triplets $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N}_\phi)$, $((T_{\mathbf{P}})_\phi, (S_{\mathbf{P}})_\phi, \mathcal{N}_\phi)$ and $((T_{\mathbf{L}})_\phi, (S_{\mathbf{L}})_\phi, \mathcal{N}_\phi)$.*

The inequality $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ does not hold in general. We have illustrated that even the transformed triplets $(T_{\mathbf{P}}, (S_{\mathbf{P}})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi)$ and $(T_{\mathbf{L}}, (S_{\mathbf{L}})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi)$ do not necessarily yield $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$.

A.10 Rotation-invariant t-norms solving a system of functional equations

Investigating the inequalities $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ we noticed that the difference between $C_{\mathcal{F}}(F)$ en $D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F in case we work with the triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$. There exist $2^{(2^n)}$ different expressions $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ (one for every Boolean function F). To find those triplets (T, S, N) for which $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of F, we need to solve the system of functional equations, obtained by putting on a par all expressions for $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$.

Theorem *Consider a triplet (T, S, N) with involutive negator N that has fixpoint β. Let $n \in \mathbb{N}$, $n > 1$. Then $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ is independent of the $\{0,1\}^n \to \{0,1\}$ function F if and only if* *for all* $\vec{x} \in [0,\beta]^n$*,* $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$*, the following expressions are equal to each other*

$$\begin{gathered}S(x_1,\ldots,x_{n-1},x_n) - T(x_1^N,\ldots,x_{n-1}^N,x_n)\,,\\ S(x_1,\ldots,x_{n-1},x_n^N) - T(x_1^N,\ldots,x_{n-1}^N,x_n^N)\,,\\ T\left(S(x_1,\ldots,x_{n-1},x_n),S(x_1,\ldots,x_{n-1},x_n^N)\right)\,,\\ 1 - S\left(T(x_1^N,\ldots,x_{n-1}^N,x_n^N),T(x_1^N,\ldots,x_{n-1}^N,x_n)\right)\,.\end{gathered}$$

In case we work with a *De Morgan triplet* (T,S,N) (*i.e.* $S = T_N$) and $N = \mathcal{N}$, it suffices to solve a single functional equation.

Theorem *Consider a De Morgan triplet* $(T,S,\mathcal{N})$*. Let* $n \in \mathbb{N}$*,* $n > 1$*. Then* $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ *is independent of the* $\{0,1\}^n \to \{0,1\}$ *function* F *if and only if*

$$S\left(T(x^{\mathcal{N}},y^{\mathcal{N}}),T(x^{\mathcal{N}},y)\right) = T(x^{\mathcal{N}},y) + T(x^{\mathcal{N}},y^{\mathcal{N}})$$

holds for every $(x,y) \in [0,\frac{1}{2}]^2$*,* $x \leqslant y$*.*

To solve the original system of functional equations we need to impose some additional continuity conditions on T, S and N. In particular we consider De Morgan triplets (T,S,N) based on a left-continuous t-norm T and an involutive negator N. Furthermore, we assume that the partial functions $T(\bullet,\beta)$ of T are continuous on $]\beta,1]$. In case $C_0 = N$, these t-norms are exactly the t-norms constructed by means of the triple rotation method.

Theorem *Consider a De Morgan triplet* (T,S,N)*, with* T *a left-continuous t-norm and* N *an involutive negator with fixpoint* β*. Let* $n \in \mathbb{N}$*,* $n > 1$*. If* $T(\bullet,\beta)$ *is continuous on* $[\beta,1]$*, then* $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ *is independent of the* $\{0,1\}^n \to \{0,1\}$ *function* F *if and only if* $(T,S,N) = (T_{\mathbf{L}},S_{\mathbf{L}},\mathcal{N})$*.*

If $0 < Q(\beta,\beta)$ (*i.e.* $T(\bullet,\beta)$ is discontinuous in β), we have shown that necessarily $Q(\beta,\beta) = \beta$. Unfortunately, without any further restrictions, a similar straightforward solution as in the previous theorem cannot be expected.

Theorem *Consider a De Morgan triplet* (T,S,N)*, with* T *a left-continuous t-norm and* N *an involutive negator with fixpoint* β*. Let* $n \in \mathbb{N}$*,* $n > 1$*. If* $0 < Q(\beta,\beta)$ *and the partial functions* $T(\bullet,x)$ *are continuous on* $]x^N,1]$*, whenever* $x \in]0,\beta]$*, and on* $[x,1]$*, whenever* $x \in]\beta,1[$*, then* $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ *is independent of the* $\{0,1\}^n \to \{0,1\}$ *function* F *if and only if* $(T,S,N) = (T_\lambda,S_\lambda,\mathcal{N})$*, for some* $\lambda \in [0,\frac{1}{2}[$*.*

For $\lambda \in]0,\frac{1}{2}[$, T_λ is the triple rotation $R3((\langle 1-2\,\lambda,1,T_{\mathbf{L}}\rangle),\mathcal{N})$ of the ordinal sum $(\langle 1-2\,\lambda,1,T_{\mathbf{L}}\rangle)$ based on the standard negator $\mathcal{N}$. The *ordinal sum* $(\langle 1-2\,\lambda,1,T_{\mathbf{L}}\rangle)$ is on $[1-2\,\lambda,1]^2$ defined as the linear rescaling of $T_{\mathbf{L}}$. Elsewhere, $(\langle 1-2\,\lambda,1,T_{\mathbf{L}}\rangle)$ equals the minimum operator. Note that there exist, however, also other triplets solving the system of functional equations.

APPENDIX B

Samenvatting

De vaaglogica en de theorie van de vaagverzamelingen maken veelvuldig gebruik van monotone functies. Stijgende $[0,1]^2 \to [0,1]$ functies zoals driehoeksnormen en driehoeksconormen vervagen de Boolese conjunctie en disjunctie (cf. [19, 36, 51, 71]). Dalende $[0,1] \to [0,1]$ functies worden daarentegen gebruikt om de Boolese negatie te veralgemenen (cf. [19, 51, 71]). De studie van monotone $[0,1]^2 \to [0,1]$ functies en hun eigenschappen geniet veruit de meeste aandacht. Monotone $[0,1] \to [0,1]$ functies zijn op zich meer elementair. We hebben ze in dit proefschrift gebruikt om monotone $[0,1]^2 \to [0,1]$ functies te beschrijven. De verkregen resultaten dragen in grote mate bij tot diepere inzichten in de structuur van linkscontinue driehoeksnormen. Dergelijke inzichten zijn onontbeerlijk voor driehoeksnorm-gerelateerd onderzoek.

B.1 Inverse monotone functies

Het is meestal niet mogelijk om d.m.v. spiegelingen een curve te inverteren m.b.t. een gegeven monotone $[0,1] \to [0,1]$ bijectie Φ. We definiëren daarom als volgt de Φ-*inverse* van een verzameling $A \subseteq [0,1]^2$: $A^\Phi := \{(x,y) \in [0,1]^2 \mid (\Phi^{-1}(y), \Phi(x)) \in A\}$. De verzameling A^Φ bestaat uit die punten die het vierde hoekpunt uitmaken van een rechthoek met zijden parallel met de assen, met twee hoekpunten gelegen op de grafiek van Φ en met een hoekpunt behorende tot A. In het bijzonder geldt er dat $(A^\Phi)^\Phi = A$ en $A^{\mathbf{id}} = A^{-1}$, waarbij **id** de identieke afbeelding voorstelt.

De Φ-inverse van een monotone functie f is niet noodzakelijk een functie. We kunnen echter met elke monotone functie f een verzameling $Q(f, \Phi)$ associëren van 'Φ-inverse' functies. Deze verzameling bevat alle monotone $[0,1] \to [0,1]$ functies wiens vervollediging de Φ-inverse is van een vervollediging van f. Met de *vervollediging* van een monotone functie f bedoelen we een continue stijgende of dalende uitbreiding van f die elk punt van het eenheidsinterval $[0,1]$ bereikt. Het volstaat om aan de grafiek van f vertikale segmenten toe te voegen. Voor constante functies f gebruiken we voor de constructie van $Q(f,\Phi)$ zowel de stijgende als de dalende vervollediging van f. Voor een monotone $[0,1] \to [0,1]$ bijectie Ψ geldt er in het bijzonder dat $Q(\Psi,\Phi) = \{\Psi^\Phi\}$. Het werk van Schweizer and Sklar [85] ligt aan de basis van onze constructie.

Stelling *$Q(f,\Phi)$ is isomorf met $Q(f,\mathbf{id})$ indien Φ stijgend is en $Q(f,\Phi)$ is antimorf met $Q(f,\mathbf{id})$ indien Φ dalend is. In het bijzonder bestaat er voor elke $g \in Q(f,\Phi)$ een unieke $h \in Q(f,\mathbf{id})$ zodat $g = \Phi \circ h \circ \Phi$.*

Een stijgende $[0,1] \to [0,1]$ bijectie wordt een *automorfisme* genoemd. Een *strikte negator* is een dalende $[0,1] \to [0,1]$ bijectie. Dankzij de bovenstaande stelling kunnen we de eigenschappen van $Q(f,\mathbf{id})$ rechtstreeks vertalen naar eigenschappen van $Q(f,\Phi)$. We gebruiken de volgende functies om de elementen van $Q(f,\mathbf{id})$ wiskundig te beschrijven:

$$\begin{aligned} \overline{f}^{\mathbf{id}}(x) &= \sup\{t \in [0,1] \mid f(t) < x\} & \qquad \underline{f}^{\mathbf{id}}(x) &= \sup\{t \in [0,1] \mid f(t) > x\} \\ \overline{f}_{\mathbf{id}}(x) &= \inf\{t \in [0,1] \mid f(t) > x\} & \qquad \underline{f}_{\mathbf{id}}(x) &= \inf\{t \in [0,1] \mid f(t) < x\}\,. \end{aligned}$$

Zowel $\overline{f}^{\mathbf{id}}$ als $\overline{f}_{\mathbf{id}}$ zijn stijgend. Daarentegen zijn $\underline{f}^{\mathbf{id}}$ en $\underline{f}_{\mathbf{id}}$ steeds dalend. De functies $\overline{f}^{\mathbf{id}}$ en $\underline{f}^{\mathbf{id}}$ zijn beter bekend als de *pseudo-inverse* $f^{(-1)}$ van f als $f(0) < f(1)$, resp., $f(1) < f(0)$ [51]. Het type monotoniteit van f bepaalt welke functies $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ of $\underline{f}_{\mathbf{id}}$ de grenzen van $Q(f,\mathbf{id})$ uitmaken. Omgekeerd kunnen deze functies, indien ze tot $Q(f,\mathbf{id})$ behoren, ook op basis van continuïteitseigenschappen uit $Q(f,\mathbf{id})$ gefilterd worden.

Tot slot zijn ook de eigenschappen van de klassieke inverse van toepassing op $Q(f,\mathbf{id})$.

Stelling *Elke $g \in Q(f,\mathbf{id})$ voldoet aan de volgende beweringen:*

1. *$f \in Q(g,\mathbf{id})$.*
2. *$g \circ f = \mathbf{id}$ als en slechts als f injectief is.*
3. *$f \circ g = \mathbf{id}$ als en slechts als f surjectief is.*

B.2 Orthosymmetrie van monotone functies

De beschouwingen uit de voorgaande sectie maken het mogelijk om ook klassieke symmetrie te veralgemenen. We noemen een verzameling $A \subseteq [0,1]^2$ *Φ-symmetrisch* als ze samenvalt met haar Φ-inverse, *i.e.* $(\Phi^{-1}(y),\Phi(x)) \in A \Leftrightarrow (x,y) \in A$. Merk op dat bijecties de enige monotone $[0,1] \to [0,1]$ functies zijn die Φ-symmetrisch kunnen zijn. Een monotone $[0,1] \to [0,1]$ functie f wordt *Φ-orthosymmetrisch* genoemd als $f \in Q(f,\Phi)$. Het is duidelijk dat f orthosymmetrisch is als en slechts als haar vervollediging Φ-symmetrisch is. $\mathbf{0}$ en $\mathbf{1}$ zijn de enige Φ-orthosymmetrische, constante $[0,1] \to [0,1]$ functies.

Stelling *Onderstel f niet constant en Φ-orthosymmetrisch, dan is $f = \Phi$ of de monotoniteit van f is tegengesteld aan de monotoniteit van Φ.*

De volgende limietstelling geldt enkel voor Φ-orthosymmetrie en niet voor Φ-symmetrie.

Stelling *Elke puntsgewijs convergerende rij van Φ-orthosymmetrische, monotone* $[0,1] \to [0,1]$ *functies* $(f_n)_{n\in\mathbb{N}}$ *convergeert steeds naar een Φ-orthosymmetrische, monotone* $[0,1] \to [0,1]$ *functie.*

Door (ortho)symmetrische, monotone $[0,1] \to [0,1]$ bijecties te bestuderen, hebben we ook talrijke inzichten verworven betreffende de meetkundige structuur van automorfismen en strikte negatoren. Vermits $Q(\Psi,\Phi) = \{\Psi^\Phi\}$, volstaat het de Φ-symmetrie van monotone $[0,1] \to [0,1]$ bijecties te bestuderen. Een dergelijke bijectie Ψ is Φ-symmetrisch als en slechts als $\Psi = \Phi \circ \Psi^{-1} \circ \Phi$. Deze gelijkheid is equivalent met $\Phi = \Psi \circ \Phi^{-1} \circ \Psi$ en drukt dus ook de Ψ-symmetrie van Φ uit. We zeggen dat Φ en Ψ een *symmetrisch paar* $\{\Phi,\Psi\}$ vormen. *Involutieve negatoren* (*i.e.* involutieve strikte negatoren) vormen samen met de identieke afbeelding **id** een symmetrisch paar.

Stelling *Een monotone* $[0,1] \to [0,1]$ *bijectie Ψ is Φ-symmetrisch als en slechts als $\Psi = \Phi$ of $\Psi = \Phi \circ N$, met N een involutieve negator.*

Dankzij deze stelling is het mogelijk gebleken de (ortho)symmetrische aspecten van de volgende historische, wiskundige stelling te belichten.

Stelling

1. *Een strikte negator N kan steeds geschreven worden als de samenstelling van drie involutieve negatoren.*
2. *Een automorfisme ϕ kan steeds geschreven worden als de samenstelling van vier involutieve negatoren.*

We kunnen de verzameling van alle monotone $[0,1] \to [0,1]$ bijecties dus partitioneren in vier deelverzamelingen: monotone bijecties die een samenstelling van één, twee, drie of vier involutieve negatoren. Elke involutieve negator genereert op triviale wijze zichzelf. Alle andere strikte negatoren zijn steeds samengesteld uit drie involutieve negatoren. Ook de automorfismen kunnen in twee groepen ingedeeld worden. Enerzijds zijn er de automorfismen die zich alternerend gedragen t.o.v. een fixpunt. Deze automorfismen worden gegenereerd door twee involutieve negatoren. Alle andere automorfismen zijn steeds samengesteld uit vier involutieve negatoren. We hebben rechttoe, rechtaan methodes uitgewerkt om een rij involutieve negatoren te construeren die een gegeven monotone $[0,1] \to [0,1]$ bijectie Φ genereert.

B.3 Invariante monotone functies

Φ-orthosymmetrie is van cruciaal belang voor de studie van Φ-invariante, monotone $[0,1]^n \to [0,1]$ functies. Zij Φ een monotone $[0,1] \to [0,1]$ bijectie. Dan noemen we een monotone $[0,1]^n \to [0,1]$ functie F *Φ-invariant* als $F_\Phi = F$, waarbij $F_\Phi(x_1,\ldots,x_n) := \Phi^{-1}(F(\Phi(x_1),\ldots,\Phi(x_n)))$ [7]. In deze context volstaat het om enkel stijgende functies te bestuderen. Ons doel bestond erin om, voor een gegeven bijectie Φ, alle Φ-invariante, stijgende $[0,1]^n \to [0,1]$ functies te karakteriseren. Functies die invariant zijn onder alle monotone bijecties vormen een eerste belangrijke klasse van oplossingen.

Stelling *Een stijgende $[0,1]^n \to [0,1]$ functie F is invariant onder alle automorfismen als en slechts als er een stijgende $[0,1]^n \to [0,1]$ functie G bestaat zodat $F_N = G$, voor alle involutieve negatoren N. In het bijzonder is $F_N = G$ ook voldaan voor elke strikte negator N.*

Bijgevolg is een stijgende $[0,1]^n \to [0,1]$ functie F invariant onder alle monotone $[0,1] \to [0,1]$ bijecties als en slechts als ze invariant is onder alle involutieve negatoren.

We hebben een uitgebreide klasse methodes geïntroduceerd die alle N-invariante, stijgende $[0,1]^n \to [0,1]$ functies karakteriseren. Hierbij is N een vooraf vastgelegde involutieve negator. Al onze methodes maken gebruik van een $[0,1]^2 \to [0,1]$ functie $\mathcal{C}$. De karakterisatie verloopt telkens als volgt:

Een $[0,1]^n \to [0,1]$ functie F is stijgend en N-invariant als en slechts als er een stijgende $[0,1]^n \to [0,1]$ functie G bestaat waarvoor

$$F(x_1,\ldots,x_n) = \mathcal{C}\big(G(x_1,\ldots,x_n), G_N(x_1,\ldots,x_n)\big)\,, \tag{B.1}$$

voor alle $(x_1,\ldots,x_n) \in [0,1]^n$.

We zeggen kortweg dat $\mathcal{C}$ een *volledige karakterisatie* van alle N-invariante, stijgende $[0,1]^n \to [0,1]$ functies mogelijk maakt.

Stelling *$\mathcal{C}$ maakt de karakteristie van alle N-invariante, stijgende $[0,1]^n \to [0,1]$ functies mogelijk als en slechts als*

1. *$\mathcal{C}$ stijgend is.*
2. *$\mathcal{C}(x,y) = \mathcal{C}(y^N, x^N)^N$, voor alle $(x,y) \in [0,1]^2$.*
3. *De grafiek van $\mathcal{C}$ bevat een een stijgende (m.b.t. de drie ruimtecoördinaten) kromme waarvan de Z-coördinaat elk getal in het interval $[0,1]$ bereikt.*

De bovenstaande stelling omvat onder meer twee gekende technieken om zelfduale aggregatieoperatoren te bestuderen [7, 33]. De involutiviteit van N is de sleutel voor onze aanpak. Er bestaan geen gelijkaardige karakterisaties voor Φ-invariante, stijgende $[0,1]^n \to [0,1]$ functies, waarbij Φ een niet-involutieve bijectie is. Tot slot hebben we nog kort de volgende twee eigenschappen besproken.

Stelling *$F = \mathcal{C}_F$ geldt voor elke N-invariante, stijgende $[0,1]^n \to [0,1]$ functie F als en slechts als $\mathcal{C}$ idempotent is.*

Stelling *Zij $n > 1$, dan is is het rekenkundig gemiddelde $\mathbf{M}$ de enige stijgende $[0,1]^2 \to [0,1]$ functie $\mathcal{C}$ die een karakterisatie van alle $\mathcal{N}$-invariante, stijgende $[0,1]^n \to [0,1]$ functies mogelijk maakt en waarvoor $(\mathcal{C},\mathcal{N})$ verschuivingsinvariantie bewaart.*

In de bovenstaande stelling duidt $\mathcal{N}$ de *standaard negator* aan ($x^{\mathcal{N}} = 1-x$). Bovendien bewaart $(\mathcal{C}, N)$ *verschuivingsinvariantie* als de rechterzijde van Vgl. (B.1) verschuivingsinvariant is voor elke verschuivingsinvariante, stijgende $[0,1]^n \to [0,1]$ functie G.

B.4 Sporen van orthosymmetry

Elke stijgende $[0,1]^2 \to [0,1]$ functie F wordt volledig bepaald door haar *horizontale snedes* (*i.e.* de doorsnijdingen van haar grafiek met vlakken die evenwijdig zijn met het domein $[0,1]^2$). De *contourlijnen* van F bepalen de boven-, onder-, rechter- en linkergrenzen van haar horizontale snedes. Ze worden wiskundig als volgt gedefineerd ($a \in [0,1]$):

$$\begin{aligned}
C_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(x,t) \leqslant a\}\,, \\
D_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(x,t) \geqslant a\}\,, \\
\widetilde{C}_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(t,x) \leqslant a\}\,, \\
\widetilde{D}_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(t,x) \geqslant a\}\,.
\end{aligned}$$

Contourlijnen zijn steeds dalend. Afhankelijk van de continuïteit van F vormen welbepaalde types contourlijnen samen met F een Galois connectie. Linkscontinue, stijgende functies worden het best beschreven a.d.h.v. contourlijnen van het type C_a of $\widetilde{C}_a$. Contourlijnen van het type D_a of $\widetilde{D}_a$ zijn dan weer beter geschikt om rechtscontinue, stijgende functies te beschrijven.

We hebben contourlijnen gebruikt om, voor een gegeven stel monotone $[0,1] \to [0,1]$ bijecties (Φ, Ψ), alle stijgende $[0,1]^2 \to [0,1]$ functies F te bepalen die voor alle $(x,y) \in [0,1]^2$ voldoen aan

$$F(x,y) = \Psi\big(F(\Phi^{-1}(y), \Phi(x))\big)\,. \tag{B.2}$$

Het is duidelijk dat deze vergelijking het begrip commutativiteit ($\Phi = \Psi = \mathbf{id}$) veralgemeent. Daarenboven herleidt ze zich voor $\Phi = \Psi = N$, met N een involutieve negator, tot de tweede voorwaarde in de tweede stelling uit Sectie **B3**. Merk op dat Vgl. (B.2) de keuze van Φ en Ψ beperkt. Beide bijecties moeten immers hetzelfde type monotoniteit bezitten en er moet steeds gelden dat $F(x,y) = \Psi(\Psi(F(x,y)))$. Involutieve bijecties Ψ voldoen steeds aan deze laatste restrictie. We onderscheiden twee gevallen:

A. Φ is een automorfisme ϕ en Ψ is gelijk aan de identieke afbeelding **id**.

B. Φ is een strikte negator M en Ψ is een involutieve negator N met fixpunt β.

Stelling *De volgende karakteristies gelden:*

1. *Zij F linkscontinu en $F(0,1) = F(1,0) = 0$, dan zijn de onderstaande beweringen equivalent.*
   a) *F voldoet aan Vgl.* (B.2), *met $\Phi = \phi$ en $\Psi = \mathbf{id}$.*
   b) *$C_a \in Q(C_a, \phi)$, voor alle $a \in [0,1]$.*
   c) *$\widetilde{C}_a \in Q(\widetilde{C}_a, \phi^{-1})$, voor alle $a \in [0,1]$.*

2. *Zij F rechtscontinu en $F(0,1) = F(1,0) = 1$, dan zijn de onderstaande beweringen equivalent.*
   a) *F voldoet aan Vgl.* (B.2), *met $\Phi = \phi$ en $\Psi = \mathbf{id}$.*

b) $D_a \in Q(D_a, \phi)$, *voor alle* $a \in [0,1]$.
c) $\widetilde{D}_a \in Q(\widetilde{D}_a, \phi^{-1})$, *voor alle* $a \in [0,1]$.

Bijgevolg drukt Vgl. (B.2) voor $\Phi = \phi$ en $\Psi = \mathbf{id}$ de ϕ-orthosymmetrie, resp., de ϕ^{-1}-orthosymmetrie, uit van de contourlijnen C_a en D_a, resp., $\widetilde{C}_a$ en $\widetilde{D}_a$. Indien $\Phi = M$ en $\Psi = N$ en F voldoet aan Vgl. (B.2) dan kan men C_{a^N} opvatten als een 'M-inverse functie' van D_a en $\widetilde{C}_{a^N}$ als een 'M^{-1}-inverse functie' van $\widetilde{D}_a$.

Stelling *Zij F continu, dan zijn de onderstaande beweringen equivalent.*

1. *F voldoet aan Vgl.* (B.2), *met* $\Phi = M$ *en* $\Psi = N$.
2. $C_{a^N} \in Q(D_a, M)$, *voor alle* $a \in [0,\beta]$, *en* $F(0,1) = F(1,0) = \beta$.
3. $D_{a^N} \in Q(C_a, M)$, *voor alle* $a \in [0,\beta]$, *en* $F(0,1) = F(1,0) = \beta$.
4. $\widetilde{C}_{a^N} \in Q(\widetilde{D}_a, M^{-1})$, *voor alle* $a \in [0,\beta]$, *en* $F(0,1) = F(1,0) = \beta$.
5. $\widetilde{D}_{a^N} \in Q(\widetilde{C}_a, M^{-1})$, *voor alle* $a \in [0,\beta]$, *en* $F(0,1) = F(1,0) = \beta$.

B.5 Een contourkijk op uninormen

Contourlijnen kunnen ook gebruikt worden om op een alternatieve manier de karakteristieke eigenschappen van uninormen (*i.e.* associatieve, commutatieve, stijgende $[0,1]^2 \to [0,1]$ functies die een neutraal element bezitten) te bestuderen. Als vertrekpunt beschouwen we opnieuw een stijgende $[0,1]^2 \to [0,1]$ functie F.

Stelling *Zij F linkscontinu en* $F(0,1) = F(1,0) = 0$, *dan zijn de onderstaande beweringen equivalent.*

1. *F heeft een neutraal element* $e \in \,]0,1]$ *als en slechts als* $e \leqslant C_a(x) \Leftrightarrow x \leqslant a$ *en* $C_a(e) = a$, *voor alle* $(x,a) \in [0,1]^2$.
2. *F is commutatief als en slechts als* $C_a(x) < y \Leftrightarrow C_a(y) < x$, *voor alle* $(x,y,a) \in [0,1]^3$.
3. *F is associatief als en slechts als* $C_a(F(x,y)) = C_{C_a(x)}(y)$, *voor alle* $(x,y,a) \in [0,1]^3$.

Ook contourlijnen van het type $\widetilde{C}_a$ kunnen gebruikt worden om het neutraal element, de commutativiteit en de associativiteit van een linkscontinue functie F te karakteriseren. Rechtscontinue functies F die voldoen aan $F(0,1) = F(1,0) = 1$ kunnen beschreven worden a.d.h.v. contourlijnen van het type D_a of $\widetilde{D}_a$. De karakterisatie van de commutativiteit in de bovenstaande stelling is tevens equivalent met de **id**-orthosymmetry van de contourlijnen. Als F commutatief is, dan geldt er noodzakelijk dat $C_a = \widetilde{C}_a$ en $D_a = \widetilde{D}_a$, voor alle $a \in [0,1]$.

Een uninorm U wordt *conjunctief* genoemd als $U(0,1) = U(1,0) = 0$ [28]. U is *disjunctief* indien $U(0,1) = U(1,0) = 1$.

Stelling *Elke linkscontinue, conjunctieve uninorm U voldoet aan* $U(x,y) \leqslant C_a(z) \Leftrightarrow U(x,z) \leqslant C_a(y)$, *voor alle* $(x,y,z,a) \in [0,1]^4$.

Tot slot hebben we onderzocht in welke mate eigenschappen van contourlijnen de algemene structuur van een uninorm beïnvloeden. Hierbij hebben we in het bijzonder aandacht besteed aan de structurele gevolgen van continue contourlijnen.

Stelling *De onderstaande beweringen zijn equivalent voor elke linkscontinue, conjunctieve uninorm U.*

1. *C_a is continu.*
2. *C_a is involutief over $[C_a(1), 1]$.*
3. *$U(x, y) = C_a(C_{C_a(x)}(y))$, voor alle $(x, y) \in [0, 1]^2$ waarvoor $C_a(U(x, 1)) < y$.*
4. *$C_b(x) = C_{C_a(x)}(C_a(b))$, voor alle $(x, b) \in [0, 1] \times [C_a(1), 1]$.*
5. *$U(x, y) \leqslant z \;\Leftrightarrow\; U(x, C_a(z)) \leqslant C_a(y)$, voor alle $(x, y, z) \in [C_a(1), 1]^3$.*

Er gelden gelijkaardige resultaten voor rechtscontinue, disjunctieve uninormen. Hierbij dienen contourlijnen van het type D_a gebruikt te worden.

B.6 Linkscontinue driehoeksnormen

De contourlijnen C_a van een linkscontinue *driehoeksnorm* T (*i.e.* een linkscontinue uninorm met neutraal element 1) vallen samen met de partiële afbeeldingen $I_T(\bullet, a)$ van de *residuele implicator* I_T (cf. [27]). Bijgevolg is de contourlijn C_0 niets anders dan de *residuele negator* $N_T = I_T(\bullet, 0)$. In dit laatste geval $(a = 0)$ drukt de vijfde bewering uit de laatste stelling van Sectie **(B5)** de *rotatie-invariantie* van T t.o.v. zijn contourlijn C_0 uit. Oorspronkelijk werd de rotatie-invariantie van een driehoeksnorm echter gedefinieerd t.o.v. een willekeurige involutieve negator N [25, 40]. Een dergelijke driehoeksnorm T is noodzakelijkerwijze linkscontinu en $C_0 = N_T = N$ [40]. Rekening houdend met de laatste stelling uit Sectie **B5**, noemen we een driehoeksnorm kortweg *rotatie-invariant* als hij linkscontinu is en als zijn contourlijn C_0 continu is.

Contourlijnen alleen verschaffen ons echter onvoldoende inzicht in de meetkundige structuur van rotatie-invariante driehoeksnormen. De kompaan en zooms leveren de nodige bijkomende informatie om rotatie-invariante driehoeksnormen verder te ontleden en te (re)construeren.

De kompaan

De *kompaan* Q van een stijgende $[0, 1]^2 \to [0, 1]$ functie F is de $[0, 1]^2 \to [0, 1]$ functie gedefinieerd door

$$Q(x, y) = \sup\{t \in [0, 1] \mid C_t(x) \leqslant y\}\,.$$

Er geldt dat $Q(x, y) = \inf\{F(x, u) \mid u \in\,]y, 1]\}$. Het is duidelijk dat $Q(x, y) = F(x, y)$ wanneer $F(x, \bullet)$ rechtscontinu is in $y \in [0, 1[$. Elke linkscontinue, stijgende $[0, 1]^2 \to [0, 1]$ functie F die 0 als absorberend element bezit, wordt volledig bepaald door haar kompaan Q.

Zooms

Een stijgende $[0,1]^2 \to [0,1]$ functie F kan eveneens beschreven worden a.d.h.v. haar geassocieerde verzameling zooms. Neem willekeurig $(a,b) \in [0,1]^2$ zodat $a < b$ en $F(b,b) \leqslant b$. Beschouw een $[a,b] \to [0,1]$ isomorfisme σ. De (a,b)-*zoom* $F^{(a,b)}$ van F is de $[0,1]^2 \to [0,1]$ functie gedefinieerd door

$$F^{(a,b)}(x,y) = \sigma\left[\max\left(a, F(\sigma^{-1}[x], \sigma^{-1}[y])\right)\right] .$$

Indien $b = 1$, dan noteren we $F^{(a,1)}$ kortweg als F^a, de a-*zoom* van F. De randvoorwaarde $F(1,1) \leqslant 1$ is dan steeds voldaan zodat de a-zoom van F gedefinieerd is voor alle $a < 1$. De grafiek van $F^{(a,b)}$ wordt volledig bepaald door de herschaling van de verzameling $\{(x,y,F(x,y)) \mid (x,y) \in [a,b]^2 \ \wedge\ a < F(x,y)\}$ (zoom in) naar de eenheidskubus (zoom out).

Zooms zijn uitermate geschikt om stijgende functies F te bestuderen die voor alle $(x,y) \in [0,1]^2$ voldoen aan $F(x,y) \leqslant \min(x,y)$. De voorwaarde $F(b,b) \leqslant b$ is in dit geval triviaal voldaan. Driehoeksnormen en subdriehoeksnormen zijn voorbeelden van dergelijke functies. Een *subdriehoeksnorm* F is een $[0,1]^2 \to [0,1]$ functie die op het neutraal element na alle uninormeigenschappen bezit en waarvoor $F(x,y) \leqslant \min(x,y)$, voor alle $(x,y) \in [0,1]^2$ [47].

Stelling *Beschouw $(a,b) \in [0,1]^2$ zodat $a < b$. De (a,b)-zoom van een subdriehoeksnorm is steeds een subdriehoeksnorm en de a-zoom van een driehoeksnorm is steeds een driehoeksnorm.*

Contourlijnen, de kompaan en zooms maken het mogelijk de rotatie en rotatie-annihilatie constructie van Jenei [47] uiterst compact te herformuleren. Daarenboven volgen de decompostiemethodes van Jenei [45] zo goed als rechtstreeks uit onze resultaten.

B.7 Decompositie van rotatie-invariante driehoeksnormen

Ondanks alle inspanningen kunnen we de klasse der rotatie-invariante driehoeksnormen nog niet ten volle doorgronden. De bestaande decomposities [45] zijn enkel toepasbaar op enkele heel specieke rotatie-invariante driehoeksnormen. Driehoeksnormen zoals de Łukasiewicz driehoeksnorm vallen volledig buiten dit kader. Door het domein van een rotatie-invariante driehoeksnorm T op een alternatieve en meer natuurlijke manier te partitioneren, is het echter mogelijk T zo goed als volledig te beschrijven op basis van zijn contourlijn C_0 en zijn β-zoom T^β. Hierbij is β het unieke fixpunt van C_0.

Zij T een rotatie-invariante driehoeksnorm en β het unieke fixpunt van C_0. We partitioneren het gebied $\mathcal{D} = \{(x,y) \in [0,1]^2 \mid C_0(x) < y\}$ in vier deelgebieden:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y) \in]\beta,1]^2 \mid C_\beta(x) < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y) \in]0,\beta]\times\,]\beta,1] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y) \in]\beta,1]\times\,]0,\beta] \mid C_0(x) < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y) \in]\beta,1[^2 \mid y \leqslant C_\beta(x)\}\,.
\end{aligned}$$

Stelling *Zij σ een willekeurig $[\beta,1]\to[0,1]$ isomorfisme. Als de contourlijn C_β van T continu is over $]\beta,1]$, dan bestaat er een linkscontinue driehoeksnorm $\widehat{T}$ (met contourlijnen $\widehat{C}_a$ en kompaan $\widehat{Q}$) waarvoor $\widehat{C}_0$ continu is op $]0,1]$en*

$$T(x,y)=\begin{cases}\sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right], & \textit{als } (x,y)\in\mathcal{D}_{\mathrm{I}}\,,\\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \textit{als } (x,y)\in\mathcal{D}_{\mathrm{II}}\,,\\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \textit{als } (x,y)\in\mathcal{D}_{\mathrm{III}}\,,\\ C_0\left(\sigma^{-1}\left[\widehat{Q}\big(\widehat{C}_0(\sigma[x]),\widehat{C}_0(\sigma[y])\big)\right]\right), & \textit{als } (x,y)\in\mathcal{D}_{\mathrm{IV}}\,,\\ 0, & \textit{als } (x,y)\notin\mathcal{D}\,.\end{cases}$$

In het bijzonder is $\widehat{T}=T^\beta$ en is $\widehat{Q}$ commutatief op $[0,\hat{\alpha}[^2$, waarbij $\hat{\alpha}=\inf\{t\in[0,1]\mid\widehat{C}_0(t)=0\}$.

Het isomorfisme σ in de voorgaande stelling tevens gebruikt moet worden om T^β te construeren. Voorts is onze decompositie over het gebied $[0,1]\setminus\mathcal{D}_{\mathrm{IV}}$ geldig voor elke rotatie-invariante driehoeksnorm T. De invulling van het gebied $\mathcal{D}_{\mathrm{IV}}$ ligt echter niet altijd op een unieke wijze vast.

Meetkundig gezien is $T|_{\mathcal{D}_{\mathrm{I}}}$ als het ware een herschaalde versie van $T^\beta|_{\mathcal{D}^\beta}$, met $\mathcal{D}^\beta=\{(x,y)\in[0,1]^2\mid 0<T^\beta(x,y)\}$. $T|_{\mathcal{D}_{\mathrm{II}}}$ bepaalt men door $T|_{\mathcal{D}_{\mathrm{I}}}$ over een hoek van 120 graden naar links te roteren rond de as $\{(x,y,z)\in[0,1]^2\mid y=x\ \wedge\ z=1-x\}$. Op een gelijkaardige wijze wordt $T|_{\mathcal{D}_{\mathrm{III}}}$ bepaald door een rechtse rotatie van $T|_{\mathcal{D}_{\mathrm{I}}}$ over een hoek van 120 graden rond dezelfde as. De invulling van het gebied $\mathcal{D}_{\mathrm{IV}}$ wordt in de bovenstaande stelling verkregen door $T|_{\mathcal{D}_{\mathrm{I}}\cap\,]\beta,\sigma^{-1}(\hat{\alpha})]^2}$ 180 graden voorwaarts te roteren rond de as $\{(x,y,z)\in[0,1]^3\mid x+y=\beta+\sigma^{-1}[\hat{\alpha}]\ \wedge\ z=\beta\}$. Indien C_β continu is, dan geldt er dat $\hat{\alpha}=1$. In sommige gevallen moeten de bovenstaande rotaties bijkomend vervormd worden om in de desbetreffende gebieden te passen. De contourlijnen C_0 en $\widehat{C}_0$ zijn verantwoordelijk voor deze vervormingen.

B.8 Drievoudige rotatie

Tot slot hebben we onze decompositiemethode omgezet naar een handige constructietool. Deze tool omvat grotendeels de rotatie- en rotatie-annihilatiemethode van Jenei [47]. De opzet is als volgt:

- T: een willekeurige linkscontinue driehoeksnorm (met contourlijnen C_a en kompaan Q) waarvoor C_0 continu is over $]0,1]$ en Q commutatief is over $[0,\alpha[^2$, met $\alpha=\inf\{t\in[0,1]\mid C_0(t)=0\}$;
- N: een willekeurige involutieve negator met fixpunt β;
- σ: een willekeurig $[\beta,1]\to[0,1]$ isomorfisme;
- M: de dalende $[0,1]\to[0,1]$ functie gedefinieerd door $x^M=1$ als $x\in[0,\beta[$ en door $x^M=\sigma^{-1}[C_0(\sigma[x])]$ als $x\in[\beta,1]$;

- $\mathcal{D}$: het gebied $\{(x,y) \in [0,1]^2 \mid x^N < y\} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$ waarbij

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y) \in]\beta,1]^2 \mid x^M < y\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y) \in]0,\beta] \times]\beta,1] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y) \in]\beta,1] \times]0,\beta] \mid x^N < y\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y) \in]\beta,1[^2 \mid y \leqslant x^M\}\,.
\end{aligned}$$

Stelling *De* $[0,1]^2 \to [0,1]$ *functie* $R3(T,N)$ *gedefinieerd door*

$$R3(T,N)(x,y) = \begin{cases}
\sigma^{-1}\left[T(\sigma[x],\sigma[y])\right], & \textit{als } (x,y) \in \mathcal{D}_{\mathrm{I}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[x^N]}(\sigma[y])\right]\right)^N, & \textit{als } (x,y) \in \mathcal{D}_{\mathrm{II}}\,,\\
\left(\sigma^{-1}\left[C_{\sigma[y^N]}(\sigma[x])\right]\right)^N, & \textit{als } (x,y) \in \mathcal{D}_{\mathrm{III}}\,,\\
\left(\sigma^{-1}\left[Q\big(C_0(\sigma[x]),C_0(\sigma[y])\big)\right]\right)^N, & \textit{als } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,,\\
0, & \textit{als } (x,y) \notin \mathcal{D}\,,
\end{cases}$$

is een rotatie-invariante driehoeksnorm. Daarenboven is $R3(T,N)$ *de enige linkscontinue driehoeksnorm met* N *als contourlijn (*$a=0$*) en waarvoor* $R3(T,N)^\beta = T$.

Net als bij de decompositie worden $R3(T,N)|_{\mathcal{D}_{\mathrm{II}}}$, $R3(T,N)|_{\mathcal{D}_{\mathrm{III}}}$ en $R3(T,N)|_{\mathcal{D}_{\mathrm{IV}}}$ bepaald door de (vervormde) linkse, rechtse en voorwaartse rotatie van $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$. Merk op dat $R3(T,N)|_{\mathcal{D}_{\mathrm{I}}}$ een herschaling is van het 'niet-nul-deel' van T. Geïnspireerd door deze meetkundige observaties, hebben we $R3(T,N)$ de *drievoudige rotatie van* T *gebaseerd op* N genoemd. De constructiemethode op zich heeft de naam *drievoudige rotatiemethode* gekregen. Deze methode levert slechts een driehoeksnorm op indien de kompaan Q van T commutatief is over $[0,\alpha[^2$.

B.9 Vervaagde normaalvormen

In Sectie **B10** gebruiken we onze kennis betreffende de structuur van rotatie-invariante driehoeksnormen om een stelsel functionele vergelijkingen op te lossen. De huidige sectie schetst het kader waarin het stelsel functionele vergelijkingen gesitueerd kan worden.

Elke $\{0,1\}^n \to \{0,1\}$ functie F in de Boolese algebra $(\{0,1\},\vee,\wedge,',0,1)$ wordt op een unieke wijze gerepresenteerd door haar *disjunctieve* ($D_{\mathcal{B}}(F)$) en *conjunctieve* ($C_{\mathcal{B}}(F)$) *normaalvorm.* De vaaglogica is grotendeels gebaseerd op het gebruik van driehoeksnormen en *driehoeksconormen* (driehoeksconormen zijn uninormen die 0 als neutraal element hebben). De Boolese normaalvormen van F kunnen rechttoe, rechtaan vervaagd worden door $\wedge$ te vervangen door een driehoeksnorm T, $\vee$ door een driehoeksconorm S en $'$ door een involutieve negator N. We verkrijgen zo twee $[0,1]^n \to [0,1]$ functies: de *disjunctieve* ($D_{\mathcal{F}}(F)$) en *conjunctieve* ($C_{\mathcal{F}}(F)$) *vervaagde normaalvorm* van F [88, 89]. Deze vervaagde normaalvormen kunnen echter meestal niet geïnterpreteerd worden als echte normaalvormen in een veralgemeende logica of algebra.

Desalniettemin worden ze dikwijls gebruikt als een soort van standaard vervagingsprocedure voor scherpe concepten.

Het onderzoek betreffende de vervaagde normaalvormen richt(te) zich hoofdzakelijk op de onderlinge verbanden tussen $D_{\mathcal{F}}(F)$ and $C_{\mathcal{F}}(F)$. Bilgiç [5] heeft aangetoond dat $D_{\mathcal{F}}(F)$ nooit met $C_{\mathcal{F}}(F)$ kan samenvallen voor alle $\{0,1\}^2 \to \{0,1\}$ functies F. Türkşen [88, 89] heeft dan weer ontdekt dat sommige tripletten (T,S,N) verzekeren dat

$$D_{\mathcal{F}}(F)(x,y) \leqslant C_{\mathcal{F}}(F)(x,y)\,,$$

voor alle $\{0,1\}^2 \to \{0,1\}$ functies F en alle $(x,y) \in [0,1]^2$. Deze laatste eigenschap noteren we kortweg als $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$. We gebruiken $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ om uit te drukken dat $D_{\mathcal{F}}(F)(\vec{x}) \leqslant C_{\mathcal{F}}(F)(\vec{x})$ geldig is voor alle $\{0,1\}^n \to \{0,1\}$ functies F en alle $\vec{x} \in [0,1]^n$.

Stelling *Beschouw een automorfisme ϕ. De ongelijkheid $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$, met $n \in \mathbb{N}_0$, geldt voor een triplet (T,S,N) als en slechts als ze geldt voor het triplet (T_ϕ, S_ϕ, N_ϕ).*

Stelling *Als er een $n \in \mathbb{N}_0$ bestaat waarvoor $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ geldt, dan geldt ook $D_{\mathcal{F}} \leqslant_m C_{\mathcal{F}}$ voor elke $m \in \mathbb{N}_0$ waarvoor $m \leqslant n$.*

De ongelijkheid $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ drukt uit dat $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ geldt voor alle $n \in \mathbb{N}_0$. In de volgende propositie onderzoeken we $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ voor tripletten gebaseerd op één van de volgende driehoeksnormen: het minimum $T_{\mathbf{M}}(x,y) = \min(x,y)$, het product $T_{\mathbf{P}}(x,y) = x\,y$ en de Łukasiewicz driehoeksnorm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. De corresponderende driehoeksconormen zijn het maximum $S_{\mathbf{M}}(x,y) = \max(x,y)$, de probabilistische som $S_{\mathbf{P}}(x,y) = x+y-x\,y$ en de Łukasiewicz driehoeksconorm $S_{\mathbf{L}}(x,y) = \min(x+y,1)$.

Propositie *Beschouw een automorfisme ϕ. De ongelijkheid $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ geldt voor alle tripletten $(T_{\mathbf{M}}, S_{\mathbf{M}}, \mathcal{N}_\phi)$, $((T_{\mathbf{P}})_\phi, (S_{\mathbf{P}})_\phi, \mathcal{N}_\phi)$ en $((T_{\mathbf{L}})_\phi, (S_{\mathbf{L}})_\phi, \mathcal{N}_\phi)$.*

De ongelijkheid $D_{\mathcal{F}} \leqslant C_{\mathcal{F}}$ is evenwel niet algemeen geldig. We hebben aangetoond dat zelfs getransformeerde tripletten van de vorm $(T_{\mathbf{P}}, (S_{\mathbf{P}})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi)$ en $(T_{\mathbf{L}}, (S_{\mathbf{L}})_{\mathcal{N}\circ\mathcal{N}_\phi}, \mathcal{N}_\phi)$ niet garanderen dat $D_{\mathcal{F}} \leqslant_2 C_{\mathcal{F}}$.

B.10 Rotatie-invariante driehoeksnormen als oplossingen van een stelsel functionele vergelijkingen

Door $D_{\mathcal{F}} \leqslant_n C_{\mathcal{F}}$ te onderzoeken, hebben we ontdekt dat het verschil tussen $C_{\mathcal{F}}(F)$ en $D_{\mathcal{F}}(F)$ onafhankelijk is van de $\{0,1\}^n \to \{0,1\}$ functie F indien we het triplet $(T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$ gebruiken bij de berekening van de vervaagde normaalvormen. Er bestaan $2^{(2^n)}$ verschillende $\{0,1\}^n \to \{0,1\}$ functies F en bijgevolg ook evenveel uitdrukkingen voor $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$. Om dus die tripletten (T,S,N) te bepalen waarvoor $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ onafhankelijk wordt van F, moeten we het stelsel functionele vergelijkingen oplossen dat ontstaat door all uitdrukkingen voor $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ aan elkaar gelijk te stellen.

Stelling *Beschouw een triplet (T, S, N) en onderstel β het fixpunt van de involutieve negator N. Zij $n \in \mathbb{N}$, $n > 1$. Dan is $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ onafhankelijk van de $\{0,1\}^n \to \{0,1\}$ functie F als en slechts als de onderstaande uitdrukkingen voor alle $\vec{x} \in [0,\beta]^n$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, aan elkaar gelijk zijn*

$$\begin{gathered} S(x_1,\ldots,x_{n-1},x_n) - T(x_1^N,\ldots,x_{n-1}^N,x_n)\,, \\ S(x_1,\ldots,x_{n-1},x_n^N) - T(x_1^N,\ldots,x_{n-1}^N,x_n^N)\,, \\ T\left(S(x_1,\ldots,x_{n-1},x_n), S(x_1,\ldots,x_{n-1},x_n^N)\right)\,, \\ 1 - S\left(T(x_1^N,\ldots,x_{n-1}^N,x_n^N), T(x_1^N,\ldots,x_{n-1}^N,x_n)\right)\,. \end{gathered}$$

Indien we werken met een *De Morgan triplet* (T, S, N) (*i.e.* $S = T_N$) waarbij $N = \mathcal{N}$, dan reduceert het stelsel functionele vergelijkingen zelfs tot één enkele functionele vergelijking.

Stelling *Beschouw een De Morgan triplet $(T, S, \mathcal{N})$. Zij $n \in \mathbb{N}$, $n > 1$. Dan is $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ onafhankelijk van de $\{0,1\}^n \to \{0,1\}$ functie F als en slechts als*

$$S\left(T(x^{\mathcal{N}}, y^{\mathcal{N}}), T(x^{\mathcal{N}}, y)\right) = T(x^{\mathcal{N}}, y) + T(x^{\mathcal{N}}, y^{\mathcal{N}})\,,$$

voor alle $(x, y) \in [0, \frac{1}{2}]^2$, $x \leqslant y$.

Om het oorspronkelijke stelsel functionele vergelijkingen te kunnen oplossen zijn we genoodzaakt om op T, S en N bijkomende continuïteitsvoorwaarden op te leggen. In het bijzonder beschouwen we enkel De Morgan tripletten (T, S, N) die gebaseerd zijn op een linkscontinue driehoeksnorm T en een involutieve negator N. Daarenboven veronderstellen we dat de partiële afbeeldingen $T(\bullet, \beta)$ continu zijn over $]\beta, 1]$. Indien $C_0 = N$, dan vereist deze laatste restrictie dat T d.m.v. de drievoudige rotatiemethode geconstrueerd moet zijn.

Stelling *Beschouw een De Morgan triplet (T, S, N) dat gebaseerd is op een linkscontinue driehoeksnorm T en een involutieve negator N met fixpunt β. Zij $n \in \mathbb{N}$, $n > 1$. Als $T(\bullet, \beta)$ continu is over $[\beta, 1]$, dan is $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ onafhankelijk van de $\{0,1\}^n \to \{0,1\}$ functie F als en slechts als $(T, S, N) = (T_{\mathbf{L}}, S_{\mathbf{L}}, \mathcal{N})$.*

Voor $0 < Q(\beta, \beta)$ (*i.e.* $T(\bullet, \beta)$ is discontinu in β) hebben we aangetoond dat noodzakelijkerwijze $Q(\beta, \beta) = \beta$. Echter, zonder bijkomende voorwaarden was het vooralsnog niet mogelijk om het stelsel functionele vergelijkingen op te lossen.

Stelling *Beschouw een De Morgan triplet (T, S, N) dat gebaseerd is op een linkscontinue driehoeksnorm T en een involutieve negator N met fixpunt β. Zij $n \in \mathbb{N}$, $n > 1$. Als $0 < Q(\beta, \beta)$ en de functies $T(\bullet, x)$ zijn continu over $]x^N, 1]$ als $x \in]0, \beta]$ en over $[x, 1]$ als $x \in]\beta, 1[$, dan is $C_{\mathcal{F}}(F) - D_{\mathcal{F}}(F)$ onafhankelijk van de $\{0,1\}^n \to \{0,1\}$ functie F als en slechts als $(T, S, N) = (T_\lambda, S_\lambda, \mathcal{N})$, met $\lambda \in [0, \frac{1}{2}[$.*

De driehoeksnorm T_λ, met $\lambda \in [0, \frac{1}{2}[$,wordt gedefinieerd als de drievoudige rotatie $R3((\langle 1 - 2\,\lambda, 1, T_{\mathbf{L}}\rangle), \mathcal{N})$ van de ordinale som $(\langle 1 - 2\,\lambda, 1, T_{\mathbf{L}}\rangle)$. Deze *ordinale som* $(\langle 1 - 2\,\lambda, 1, T_{\mathbf{L}}\rangle)$ is op $[1 - 2\,\lambda, 1]^2$ gedefinieerd als de lineaire herschaling van $T_{\mathbf{L}}$. Elders valt $(\langle 1 - 2\,\lambda, 1, T_{\mathbf{L}}\rangle)$ samen met het minimum. Merk op dat er nog andere tripletten bestaan die het stelsel functionele vergelijkingen oplossen.

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